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Existence results in large-strain magnetoelasticity

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Abstract. We investigate variational problems in large-strain magnetoelasticity, in both the static and the quasistatic settings. The model contemplates a mixed Eulerian–Lagrangian formulation: while deformations are defined on the reference configuration, magnetizations are defined on the deformed set in the actual space. In the static setting, we establish the existence of minimizers. In particular, we provide a compactness result for sequences of admissible states with equi-bounded energies which gives the convergence of the composition of magnetizations with deformations. In the quasistatic setting, we consider a notion of dissipation which is frame-indifferent and we show that the incremental minimization problem is solvable. Then we propose a regularization of the model in the spirit of gradient polyconvexity and we prove the existence of energetic solutions for the regularized model.

1. Introduction

In this paper we study a variational model for magnetoelastic materials at large strains and we provide existence results for optimal configurations, in both the static and the quasistatic settings.

The theory of Brown [9] (see also [13–15,29]) is based on the assumption that equilibrium configurations of a magnetoelastic body are given by minimizers of an energy functional that depends on the deformation $\boldsymbol{y}\colon \Omega \to \mathbb{R}^3$ and on the magnetization $\boldsymbol{m}\colon \boldsymbol{y}(\Omega) \to \mathbb{S}^2$, where $\Omega \subset \mathbb{R}^3$ represents the reference configuration of the body and $\mathbb{S}^2$ denotes the unit sphere in $\mathbb{R}^3$. The fact that magnetizations are sphere valued resembles the constraint of magnetic saturation which, up to normalization, reads $|\boldsymbol{m}| = 1$ in $\boldsymbol{y}(\Omega)$. The magnetoelastic energy functional is defined, for $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m})$, by setting

$$E(\boldsymbol{q}) := \int_\Omega W(\nabla \boldsymbol{y}, \boldsymbol{m} \circ \boldsymbol{y})\,\mathrm{d}\boldsymbol{x} + \alpha \int_{\boldsymbol{y}(\Omega)} |\nabla \boldsymbol{m}|^2\,\mathrm{d}\boldsymbol{\xi} + \frac{\mu_0}{2} \int_{\mathbb{R}^3} |\nabla \zeta_{\boldsymbol{m}}|^2\,\mathrm{d}\boldsymbol{\xi}. \tag{1.1}$$

Here, W denotes a nonlinear, frame-indifferent, magnetostrictive energy density and the corresponding integral in (1.1) represents the elastic energy. The second term in (1.1) is termed exchange energy, penalizing spatial changes of $\boldsymbol{m}$; $\alpha > 0$ is the exchange constant. The third contribution in (1.1) encodes the magnetostatic energy and favors divergence-free states of the magnetization; $\mu_0 > 0$ is the permeability of the vacuum. In particular,

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the stray-field potential $\zeta_{\boldsymbol{m}}\colon \mathbb{R}^3 \to \mathbb{R}$ is defined as a weak solution of the magnetostatic Maxwell equation [9, 15, 29]:

$$\Delta\zeta_{\boldsymbol{m}} = \operatorname{div}(\chi_{\boldsymbol{y}(\Omega)}\boldsymbol{m}) \quad \text{in } \mathbb{R}^3.$$

The magnetoelastic energy functional usually includes additional terms, such as the anisotropy energy and the asymmetric exchange energy, also known as Dzyaloshinsky–Moriya interaction (DMI) energy [16,45]. For $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m})$, these two terms are respectively given by

$$E^{\mathrm{ani}}(\boldsymbol{q}) := \int_{\boldsymbol{y}(\Omega)} \phi(\boldsymbol{m})\,\mathrm{d}\boldsymbol{\xi}, \quad E^{\mathrm{DMI}}(\boldsymbol{q}) := \kappa \int_{\boldsymbol{y}(\Omega)} \operatorname{curl}\boldsymbol{m}\cdot\boldsymbol{m}\,\mathrm{d}\boldsymbol{\xi}. \tag{1.2}$$

The first term takes into account the magnetocrystalline anisotropy; the function $\phi\colon \mathbb{S}^2 \to \mathbb{R}$ is continuous and nonnegative, and vanishes only on a finite set of directions, the easy axis, along which magnetizations tend to align themselves spontaneously. The DMI energy is linked with the possible lack of centrosymmetry in the crystalline structure. The sign of the parameter $\kappa \in \mathbb{R}$ is not prescribed. According to its value, this energy term alone would be minimized by configurations satisfying $\operatorname{curl}\boldsymbol{m} = \pm\boldsymbol{m}$, or equivalently, $\pm\boldsymbol{m} = -\Delta\boldsymbol{m}$. Altogether, the sum of the symmetric and DMI exchange-energy terms is optimized by helical fields $\boldsymbol{m}$ describing a rotation of constant frequency κ orthogonal to one of the coordinate axes, and rotating clockwise or counterclockwise according to the sign of κ [11, 37, 42].

The two energy contributions in (1.2) do not introduce additional mathematical challenges in our analysis as the corresponding functionals are continuous with respect to the topology considered. Therefore, these energy terms will be neglected and we will consider the magnetoelastic energy functional in (1.1). The same holds for applied loads which, in the first instance, will also be neglected. These comprise applied body and surface forces and external magnetic fields.

A peculiar feature of the energy in (1.1) is its mixed Eulerian–Lagrangian structure. Whereas the elastic energy is evaluated on the reference configuration, and is hence Lagrangian, in fact all magnetic contributions are set on the actual deformed set, thus being Eulerian. Therefore, the problem needs to be formulated in a class of admissible states such that the deformed set $\boldsymbol{y}(\Omega)$ corresponding to each deformation $\boldsymbol{y}$ can be suitably interpreted.

The existence of minimizers for the functional in (1.1) was first proven in [51] for nonsimple materials, and then in [3, 34] for simple materials under the constraint of incompressibility. In [34], the authors also studied quasistatic evolutions driven by time-dependent applied loads and a rate-independent dissipation, and established the existence of energetic solutions. Subsequently, in [4], the existence of minimizers for (1.1) was obtained for compressible materials under weak growth assumptions on the elastic energy density. The analysis in this case becomes quite technical because of the possible discontinuity of deformations. A further extension of this result, contemplating an even larger class of admissible deformations, was obtained in [28]. Actually, the analysis in [3, 28] concerns nematic elastomers; in that case, the variable $\boldsymbol{m}$ represents the nematic director and the magnetostatic energy term is dropped. However, from the mathematical point of view, the problem is substantially the same as that of magnetoelasticity.

To complete our review on magnetoelasticity, we also mention a few recent works dealing with the analysis of magnetoelastic thin films. In particular, in [33] magnetoelastic plates and their corresponding quasistatic evolutions are studied within the framework of linearized elasticity in a purely Lagrangian setting. A large-strain analysis of magnetoelastic plates has been initiated in [35], under a priori constraints on the Jacobian determinant of deformations (see also [36, 38] for numerical results). The membrane regime for non-simple materials has been recently investigated in [12], whereas von Kármán theories starting from a nonlinear model have been identified in [7] in the case of incompressible materials.

This paper is subdivided into two parts. The first part concerns the analysis of the variational model in the static case. Our setting is comparable with that in [34], except that here we are not restricted to incompressible materials. In particular, the coercivity properties of the elastic energy density ensure that admissible deformations are continuous. The main contribution in the first part consists in proving existence of equilibrium configurations for the magnetoelastic energy in (1.1). A simplified version of this result reads as follows. We refer to Theorem 3.2 for the precise statement and assumptions.

Theorem 1.1 (Existence of minimizers). *Assume that the elastic energy density W is continuous, p-coercive with $p > 3$, blows up under extreme compressions and is polyconvex in its first argument. Assume also that interpenetration of matter is prevented. Then the magnetoelastic energy in* (1.1) *admits a minimizer.*

As already mentioned, the existence of minimizers for the functional in (1.1) has already been established, even for larger classes of admissible deformations, in [3, 4, 28]. However, here the result is proven in a more direct way by arguing similarly to [34]. The main point is the compactness of sequences of admissible states with equi-bounded energies achieved in Proposition 3.4. In particular, we prove the convergence of the composition of magnetizations with deformations. From this, the lower semicontinuity of the elastic energy, which represents the most problematic term, is obtained by a standard application of the classical Eisen selection lemma [17].

The convergence of the compositions of magnetizations with deformations constitutes one of the main novelties of our paper and is going to be fundamental for the analysis in the quasistatic setting. Note that this is a very delicate issue: indeed, contrary to deformations, magnetizations may be discontinuous. In [34], the convergence of compositions follows easily from the fact that deformations are volume preserving. Instead, in [4] this issue is circumvented by working in the deformed configuration and by exploiting the weak convergence of inverse deformations together with their Jacobian minors.

The techniques employed in our analysis require careful study of the geometry of the deformed set and of fine and invertibility properties of admissible deformations. Essential tools are given by the topological degree [20] and by refined versions of the area formula and the change-of-variable formula [18, 24]. In Lemma 2.1 we see that the deformed set $\boldsymbol{y}(\Omega)$ can be replaced with a suitable open subset of it which has full measure. This is necessary in order to be able to give a precise meaning to the gradient of magnetizations appearing in the expression of the exchange energy.

The proof of the convergence of the compositions of magnetizations with deformations combines three main ingredients: the convergences of the deformed sets, the equi-integrability of the Jacobian determinants of inverse deformations, and two classical results of measure theory, namely the Egorov and Lusin theorems. For a similar approach relying on the equi-integrability of Jacobian determinants of inverse deformations, we refer to [23]. In that paper, the desired equi-integrability property follows from some a priori control on the distortion of admissible deformations obtained by imposing specific growth conditions on the elastic energy density, while here this property is deduced from the singular behavior of the elastic energy density in response to extreme compressions. This allows us to work with a more natural class of admissible deformations, which are not necessarily homeomorphisms.

In the second part of our paper, we study quasistatic evolutions driven by the energy functional in (1.1), complemented by the work of time-dependent applied loads determined by external body forces, surface forces, and magnetic fields, and a rate-independent dissipation. Our analysis is set within the theory of rate-independent processes [44] with the notion of energetic solution.

Our setting is again similar to the one in [34], but a key difference is our definition of dissipation distance. In [34], this is simply defined as the distance in L^1 between the compositions of magnetizations with deformations. Here, instead, this is constructed by introducing a dissipative variable, the Lagrangian magnetization, which is obtained as the pull-back of the magnetization to the reference configuration. More precisely, for $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{Q}$, this is given by

$$\mathcal{Z}(\boldsymbol{q}) := (\operatorname{adj} \nabla \boldsymbol{y}) \boldsymbol{m} \circ \boldsymbol{y}, \tag{1.3}$$

where the adjugate matrix simply denotes the transpose of the cofactor matrix. Then the dissipation distance $\mathcal{D}\colon \mathcal{Q} \times \mathcal{Q} \to [0, +\infty)$ is defined as

$$\mathcal{D}(\boldsymbol{q}, \hat{\boldsymbol{q}}) := \int_\Omega |\mathcal{Z}(\boldsymbol{q}) - \mathcal{Z}(\hat{\boldsymbol{q}})| \, \mathrm{d}\boldsymbol{x}. \tag{1.4}$$

We observe that this dissipation is frame indifferent, i.e. rigid motions do not dissipate energy. This is because the Lagrangian magnetization in (1.3) is an objective quantity. To see this, let $\boldsymbol{T}\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a rigid motion of the form $\boldsymbol{T}(\boldsymbol{\xi}) := \boldsymbol{Q}\boldsymbol{\xi} + \boldsymbol{c}$ for every $\boldsymbol{\xi} \in \mathbb{R}^3$, where $\boldsymbol{Q} \in SO(3)$ and $\boldsymbol{c} \in \mathbb{R}^3$. The admissible state $\tilde{\boldsymbol{q}} = (\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{m}}) \in \mathcal{Q}$ obtained from $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{Q}$ by superposition with $\boldsymbol{T}$ is defined by setting $\tilde{\boldsymbol{y}} := \boldsymbol{T} \circ \boldsymbol{y}$ and $\tilde{\boldsymbol{m}} := \boldsymbol{Q}(\boldsymbol{m} \circ \boldsymbol{T}^{-1})$. Thus, $\mathcal{Z}(\tilde{\boldsymbol{q}}) = \mathcal{Z}(\boldsymbol{q})$. This observation demonstrates that the dissipation introduced in (1.4) is selective enough to only account for the part of the magnetic reorientation that corresponds to changes of the geometry of the deformed sets.

We mention that (1.3) is not the only possible way to pull back the magnetization to the reference configuration as an objective quantity. Our choice preserves the flux of magnetizations through closed surfaces. Another possible choice, preserving the circulation of magnetizations along closed loops, consists in replacing the matrix $\operatorname{adj}\nabla \boldsymbol{y}$ in (1.3) with $(\nabla \boldsymbol{y})^\top$. Both quantities have been considered in different contexts (see [48–50] for the flux-preserving pull-back and [30, 52] for the circulation-preserving pull-back). Here, the pull-back in (1.3) is preferred as it appears naturally while rewriting the magnetostatic energy as an integral on the reference configuration [48].

The compactness established in Proposition 3.4 ensures that the dissipation distance in (1.4) is lower semicontinuous on the sublevel sets of the total energy. Specifically, this follows combining the weak continuity of the Jacobian cofactor with the convergence of the compositions of magnetizations with deformations. As a consequence, the incremental minimization problem is solvable for each fixed partition of the time interval. Nevertheless, the existence of energetic solutions is out of reach in this framework. Roughly speaking, this is because the dissipation distance is not continuous on the sublevel sets of the total energy. Such a situation is quite common for large-strain theories (see [40] for an example in finite plasticity).

Therefore, in the last part of the paper, we resort to a regularized counterpart to the functional in (1.1), which is obtained by augmenting the magnetoelastic energy by the total variation of the Jacobian cofactor of the deformation. Namely, for every $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{Q}$, we consider the regularized energy functional

$$\widetilde{E}(\boldsymbol{q}) := E(\boldsymbol{q}) + |D(\operatorname{cof}\nabla \boldsymbol{y})|(\Omega). \tag{1.5}$$

This brings us to the theory of nonsimple materials initiated by Toupin [55, 56] and later extended by many authors (see, for instance, [2, 44, 50]). The idea is to assume that the stored energy density depends also on higher-order gradients of the deformation. More regularity allows us to work in a stronger topology and to gain the continuity of the dissipation distance on the sublevel sets of the regularized total energy. Here, we apply a fairly weak concept of nonsimple materials introduced in [5] under the name of gradient polyconvex materials (see also [32]). Indeed, in view of (1.5), we only need to assume that $\operatorname{cof}\nabla \boldsymbol{y} \in BV(\Omega; \mathbb{R}^{3\times 3})$.

Our second main result asserts the existence of energetic solutions for the regularized model. We present a simplified statement below and we refer to Theorem 4.6 for its precise formulation.

Theorem 1.2 (Existence of energetic solutions). *Under the same assumptions as Theorem 1.1, there exists an energetic solution for the regularized model determined by the energy in* (1.5)*, complemented by time-dependent applied loads, and the dissipation in* (1.4)*.*

The existence of energetic solutions is proved by time discretization following the well-established scheme introduced in [22] (see also [44]). Thus, the compactness of time-discrete solutions is achieved by appealing to some version of the Helly selection theorem, here provided by Lemma 4.10. We stress that the existence of time-discrete solutions is available even in the absence of the regularization introduced in (1.5) (see Proposition 4.3), and that this is only needed to construct time-continuous solutions.

Note that taking the same definition of dissipation distance as in [34], simply given by the distance in L^1 between the compositions of magnetizations with deformations, we would be able to establish the existence of energetic solutions without resorting to any regularization.

To summarize, the novelty of our analysis is twofold. First, we prove the compactness of the compositions of magnetizations with deformations for sequences of admissible states with equi-bounded energies. This extends the compactness result obtained in [34] for incompressible materials to compressible ones. Moreover, this provides a more direct proof of the existence of minimizers for the functional in (1.1) for compressible materials compared to those available in [3, 4]. Note that, unlike in [51], no higher-order term is included in the magnetoelastic energy in the static setting. Second, in the quasistatic setting, we consider a more realistic notion of dissipation and we do not restrict ourselves to incompressible materials [34]. Solutions of the incremental minimization problem are shown to exist without resorting to any regularization. Finally, the existence of energetic solutions is achieved by including an additional energy term controlling the derivatives of the Jacobian cofactor of deformations only, instead of the full Hessian matrix of deformations [51].

We remark that the choice to limit ourselves to the case of continuous deformations is taken just for convenience. We do not see substantial obstacles in extending our arguments to more general classes of possibly discontinuous deformations for which cavitation is excluded, like those considered in [4, 28], with the help of the techniques that have already been developed in these settings. Also, the global injectivity of admissible deformations is assumed in view of its physical interpretation, i.e. to avoid the interpenetration of matter, but this does not seem to be crucial for the analysis. It might be possible to achieve the same results without this assumption by relying on the local invertibility results available in the literature [4, 20, 28] in combination with suitable covering arguments.

We mention that, more recently, we have been able to extend the present analysis by imposing the magnetic saturation constraint in the more realistic form $|\boldsymbol{m} \circ \boldsymbol{y}| \det \nabla \boldsymbol{y} = 1$ in Ω (see [9, 29, 51]). We refer to [8] for more details.

The paper is organized as follows. In Section 2 we recall some preliminary results on the invertibility of Sobolev functions. Section 3 is devoted to the analysis in the static setting including the proof of Theorem 1.1. Finally, Section 4 describes the quasistatic problem and contains the proof of Theorem 1.2.

2. Preliminaries

In this section we collect some results regarding the invertibility of Sobolev maps with supercritical integrability. Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. We consider maps in $W^{1,p}(\Omega;\mathbb{R}^3)$ with $p > 3$. Any such map admits a representative in $C^0(\overline{\Omega};\mathbb{R}^3)$ which has the Lusin property (N) [41, Corollary 1], i.e. it maps sets of zero Lebesgue measure to sets of zero Lebesgue measure. Henceforth, *we will always tacitly consider this representative*. In this case, the image of measurable sets is measurable and the area formula holds [41, Corollary 2 and Theorem 2]. As a consequence, if the Jacobian determinant is different from zero almost everywhere, then the map also has the Lusin property (N^{-1}), i.e. the preimage of every set with zero Lebesgue measure has zero Lebesgue measure.

Let $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^3)$. To make up for the fact that $\boldsymbol{y}(\Omega)$ might not be open, even if $\det \nabla \boldsymbol{y} > 0$ almost everywhere, we introduce the deformed configuration, which is defined as $\Omega^{\boldsymbol{y}} := \boldsymbol{y}(\Omega) \setminus \boldsymbol{y}(\partial\Omega)$. To prove that this set is actually open, we employ the topological degree. Recall that the degree of $\boldsymbol{y}$ on Ω is a continuous map $\deg(\boldsymbol{y},\Omega,\cdot)\colon \mathbb{R}^3 \setminus \boldsymbol{y}(\partial\Omega) \to \mathbb{Z}$. For its definition and main properties, we refer to [19].

Lemma 2.1 (Deformed configuration). *Let $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^3)$ be such that $\det \nabla \boldsymbol{y} > 0$ almost everywhere in Ω. Then the deformed configuration $\Omega^{\boldsymbol{y}}$ is an open set that differs from $\boldsymbol{y}(\Omega)$ by at most a set of zero Lebesgue measure. Moreover, $\overline{\Omega^{\boldsymbol{y}}} = \boldsymbol{y}(\overline{\Omega})$ and $\partial\Omega^{\boldsymbol{y}} = \boldsymbol{y}(\partial\Omega)$.*

Proof. We claim that $\Omega^{\boldsymbol{y}} = \{\boldsymbol{\xi} \in \mathbb{R}^3 \setminus \boldsymbol{y}(\partial\Omega) : \deg(\boldsymbol{y},\Omega,\boldsymbol{\xi}) > 0\}$. Once the claim is proved, we deduce that $\Omega^{\boldsymbol{y}}$ is open. Indeed, the set on the right-hand side is open by the continuity of the degree.

Let $\boldsymbol{\xi}_0 \in \mathbb{R}^3 \setminus \boldsymbol{y}(\partial\Omega)$ be such that $\deg(\boldsymbol{y},\Omega,\boldsymbol{\xi}_0) > 0$. Then, by the solvability property of the degree, $\boldsymbol{\xi}_0 \in \boldsymbol{y}(\Omega)$ and, in turn, $\boldsymbol{\xi}_0 \in \Omega^{\boldsymbol{y}}$. Conversely, let $\boldsymbol{\xi}_0 \in \Omega^{\boldsymbol{y}}$. Denote by V the connected component of $\mathbb{R}^3 \setminus \boldsymbol{y}(\partial\Omega)$ containing $\boldsymbol{\xi}_0$ and consider $R > 0$ such that $B(\boldsymbol{\xi}_0, R) \subset\subset V$. Let $\psi \in C^\infty_c(\mathbb{R}^3)$ be such that $\psi \geq 0$, $\operatorname{supp}\psi \subset \overline{B}(\boldsymbol{\xi}_0,R) \subset V$, and $\int_{\mathbb{R}^3} \psi \,\mathrm{d}\boldsymbol{\xi} = 1$. Then, by the integral formula for the degree, we compute

$$\deg(\boldsymbol{y},\Omega,\boldsymbol{\xi}) = \int_\Omega \psi \circ \boldsymbol{y} \det \nabla \boldsymbol{y} \,\mathrm{d}\boldsymbol{x} = \int_{\boldsymbol{y}^{-1}(B(\boldsymbol{\xi}_0,R))} \psi \circ \boldsymbol{y} \det \nabla \boldsymbol{y} \,\mathrm{d}\boldsymbol{x}.$$

As $\psi \circ \boldsymbol{y} > 0$ on $\boldsymbol{y}^{-1}(B(\boldsymbol{\xi}_0,R))$ and $\det\nabla\boldsymbol{y} > 0$ almost everywhere, we obtain $\deg(\boldsymbol{y},\Omega,\boldsymbol{\xi}) > 0$ and this proves the claim.

By the Lusin property (N), we have $\mathcal{L}^3(\boldsymbol{y}(\Omega) \setminus \Omega^{\boldsymbol{y}}) \leq \mathcal{L}^3(\boldsymbol{y}(\partial\Omega)) = 0$. For simplicity, define $U := \boldsymbol{y}^{-1}(\Omega^{\boldsymbol{y}}) = \Omega \setminus \boldsymbol{y}^{-1}(\boldsymbol{y}(\partial\Omega))$. Then $\Omega \setminus U = \boldsymbol{y}^{-1}(\boldsymbol{y}(\partial\Omega))$, so that $\mathcal{L}^3(\Omega \setminus U) = 0$ by the Lusin properties (N) and (N^{-1}). In particular, U is dense in Ω.

We prove that $\overline{\Omega^{\boldsymbol{y}}} = \boldsymbol{y}(\overline{\Omega})$. As $\Omega^{\boldsymbol{y}} \subset \boldsymbol{y}(\Omega)$, we immediately have $\overline{\Omega^{\boldsymbol{y}}} \subset \overline{\boldsymbol{y}(\Omega)} = \boldsymbol{y}(\overline{\Omega})$. Let $\boldsymbol{\xi} \in \boldsymbol{y}(\overline{\Omega})$ and consider $\boldsymbol{x} \in \overline{\Omega}$ such that $\boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{\xi}$. By density, $\overline{U} = \overline{\Omega}$. Thus, there exists $(\boldsymbol{x}_n) \subset U$ such that $\boldsymbol{x}_n \to \boldsymbol{x}$ and, in turn, $\boldsymbol{\xi}_n := \boldsymbol{y}(\boldsymbol{x}_n) \to \boldsymbol{\xi}$. As $(\boldsymbol{\xi}_n) \subset \Omega^{\boldsymbol{y}}$, this yields $\boldsymbol{\xi} \in \overline{\Omega^{\boldsymbol{y}}}$.

Finally, we prove that $\partial\Omega^{\boldsymbol{y}} = \boldsymbol{y}(\partial\Omega)$. This follows combining

$$\partial\Omega^{\boldsymbol{y}} = \overline{\Omega^{\boldsymbol{y}}} \setminus (\Omega^{\boldsymbol{y}})^\circ = \boldsymbol{y}(\overline{\Omega}) \setminus \Omega^{\boldsymbol{y}} = (\boldsymbol{y}(\overline{\Omega}) \setminus \boldsymbol{y}(\Omega)) \cup (\boldsymbol{y}(\overline{\Omega}) \cap \boldsymbol{y}(\partial\Omega)) \subset \boldsymbol{y}(\partial\Omega)$$

and

$$\partial\Omega^{\boldsymbol{y}} = \overline{\Omega^{\boldsymbol{y}}} \cap \overline{\mathbb{R}^3 \setminus \Omega^{\boldsymbol{y}}} = \boldsymbol{y}(\bar{\Omega}) \cap \overline{(\mathbb{R}^3 \setminus \boldsymbol{y}(\Omega)) \cup \boldsymbol{y}(\partial\Omega)} \supset \boldsymbol{y}(\bar{\Omega}) \cap \boldsymbol{y}(\partial\Omega) = \boldsymbol{y}(\partial\Omega). \ \blacksquare$$

The next example clarifies the difference between the sets $\boldsymbol{y}(\Omega)$ and $\Omega^{\boldsymbol{y}}$.

Example 2.2 (Ball's example). The following is inspired by [1, Example 1]. Let $\Omega := (-1,1)^3$ and write $\Omega = \Omega^+ \cup P \cup \Omega^-$, where

$$\Omega^+ := (0,1) \times (-1,1)^2, \quad P := \{0\} \times (-1,1)^2, \quad \Omega^- := (-1,0) \times (-1,1)^2.$$

Define $\boldsymbol{y}\colon \Omega \to \mathbb{R}^3$ by $\boldsymbol{y}(\boldsymbol{x}) := (x_1, x_2, |x_1|x_3)$, where $\boldsymbol{x} = (x_1, x_2, x_3)$. The corresponding deformed set is depicted in Figure 1. Then $\boldsymbol{y} \in W^{1,\infty}(\Omega;\mathbb{R}^3)$ and for every $\boldsymbol{x} \in \Omega \setminus P$ we have

$$\nabla \boldsymbol{y}(\boldsymbol{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_1 x_3/|x_1| & 0 & |x_1| \end{pmatrix}.$$

In particular, $\det \nabla \boldsymbol{y} > 0$ on $\Omega \setminus P$. We have $\boldsymbol{y}(\Omega^+) = V^+$, $\boldsymbol{y}(P) = S$, and $\boldsymbol{y}(\Omega^-) = V^-$, where, for $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$, we set

$$\begin{aligned} V^+ &:= \{\boldsymbol{\xi} \in \mathbb{R}^3 : 0 < \xi_1 < 1,\ -1 < \xi_2 < 1,\ |\xi_3| < \xi_1\}, \\ S &:= \{0\} \times (-1,1) \times \{0\}, \\ V^- &:= \{\boldsymbol{\xi} \in \mathbb{R}^3 : -1 < \xi_1 < 0,\ -1 < \xi_2 < 1,\ |\xi_3| < -\xi_1\}. \end{aligned}$$

Note that $\boldsymbol{y}|_{\Omega \setminus P}$ is injective, but $\boldsymbol{y}$ is not a homeomorphism. Also, $\boldsymbol{y}(\Omega) = V^+ \cup S \cup V^-$ is not open. Instead, $\Omega^{\boldsymbol{y}} = V^+ \cup V^-$, since $S \subset \boldsymbol{y}(\bar{P} \cap \partial\Omega)$, and this set is open. Note also that, while $\boldsymbol{y}(\Omega)$ is necessarily connected, the deformed configuration $\Omega^{\boldsymbol{y}}$ is not.

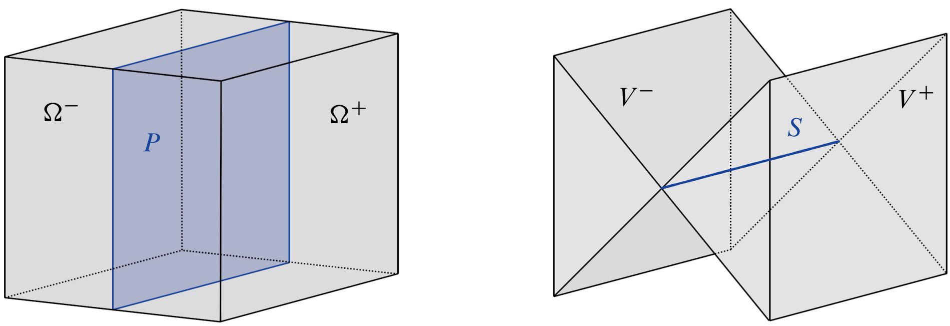

Figure 1. The deformation in Example 2.2.

Remark 2.3 (Topological image). Let $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^3)$. The topological image of $\boldsymbol{y}$ is given by the set $\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega) := \{\boldsymbol{\xi} \in \mathbb{R}^3 \setminus \boldsymbol{y}(\partial\Omega) : \deg(\boldsymbol{y},\Omega,\boldsymbol{\xi}) \neq 0\}$. Note that

$\deg(\boldsymbol{y},\Omega,\boldsymbol{\xi})=0$ for every $\boldsymbol{\xi}\in\mathbb{R}^3\setminus\boldsymbol{y}(\overline{\Omega})$, so that $\mathrm{im_T}(\boldsymbol{y},\Omega)\subset\boldsymbol{y}(\Omega)$. In relation to the problem of invertibility of deformations in elasticity, the topological image was first considered in [53] and then in several other contributions [4,6,25–27,46,47,54]. In Lemma 2.1, we proved that, if $\det\nabla\boldsymbol{y}>0$ almost everywhere, then

$$\Omega^{\boldsymbol{y}}=\mathrm{im_T}(\boldsymbol{y},\Omega)=\left\{\boldsymbol{\xi}\in\mathbb{R}^3\setminus\boldsymbol{y}(\partial\Omega):\deg(\boldsymbol{y},\Omega,\boldsymbol{\xi})>0\right\}.$$

For more information about the topological properties of Sobolev maps with supercritical integrability, we refer to [31].

We now consider the invertibility of Sobolev maps with supercritical integrability. Let $\boldsymbol{y}\in W^{1,p}(\Omega;\mathbb{R}^3)$ with $p>3$ be such that $\det\nabla\boldsymbol{y}>0$ almost everywhere. Assume that $\boldsymbol{y}$ is almost everywhere injective, i.e. there exists a set $X\subset\Omega$ with $\mathscr{L}^3(X)=0$ such that $\boldsymbol{y}|_{\Omega\setminus X}$ is injective. In this case, we can consider the inverse $\boldsymbol{y}|_{\Omega\setminus X}^{-1}\colon\boldsymbol{y}(\Omega\setminus X)\to\Omega\setminus X$. Note that $\mathscr{L}^3(\boldsymbol{y}(X))=0$ by the Lusin property (N). We define the map $\boldsymbol{v}\colon\Omega^{\boldsymbol{y}}\to\mathbb{R}^3$ by setting

$$\boldsymbol{v}(\boldsymbol{\xi}):=\begin{cases}\boldsymbol{y}|_{\Omega\setminus X}^{-1}(\boldsymbol{\xi}) & \text{if } \boldsymbol{\xi}\in\Omega^{\boldsymbol{y}}\setminus\boldsymbol{y}(X),\\ \boldsymbol{a} & \text{if } \boldsymbol{\xi}\in\Omega^{\boldsymbol{y}}\cap\boldsymbol{y}(X),\end{cases}\tag{2.1}$$

where $\boldsymbol{a}\in\mathbb{R}^3$ is arbitrarily fixed. The map $\boldsymbol{v}$ satisfies $\boldsymbol{v}\circ\boldsymbol{y}=\mathbf{id}$ almost everywhere in Ω and $\boldsymbol{y}\circ\boldsymbol{v}=\mathbf{id}$ almost everywhere in $\Omega^{\boldsymbol{y}}$. Since $\boldsymbol{y}$ maps measurable sets to measurable sets, the measurability of $\boldsymbol{v}$ follows. As $\boldsymbol{y}$ has both Lusin properties (N) and (N^{-1}), the map $\boldsymbol{v}$ has the same properties. Moreover, $\boldsymbol{v}\in L^\infty(\Omega^{\boldsymbol{y}};\mathbb{R}^3)$ since $\boldsymbol{v}(\Omega^{\boldsymbol{y}})\subset\Omega\cup\{\boldsymbol{a}\}$ and Ω is bounded.

We remark that the definition of $\boldsymbol{v}$ in (2.1) depends on the choice of the set X where $\boldsymbol{y}$ is not injective and of the value $\boldsymbol{a}\in\mathbb{R}^3$. However, as $\boldsymbol{y}$ has the Lusin property (N), its equivalence class is uniquely determined and coincides with that of the classical inverse $\boldsymbol{y}^{-1}$, where the latter is defined out of a subset of $\boldsymbol{y}(\Omega)$ with zero Lebesgue measure. Hence, with a slight abuse of notation, we will denote this equivalence class of functions defined on $\Omega^{\boldsymbol{y}}$ by $\boldsymbol{y}^{-1}$ and we will refer to it as *the* inverse of $\boldsymbol{y}$.

Remark 2.4 (Ciarlet–Nečas condition). Let $\boldsymbol{y}\in W^{1,p}(\Omega;\mathbb{R}^3)$ be such that $\det\nabla\boldsymbol{y}>0$ almost everywhere. Then $\boldsymbol{y}$ is almost everywhere injective if and only if it satisfies the Ciarlet–Nečas condition [10], which reads

$$\int_\Omega\det\nabla\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}\leq\mathscr{L}^3(\boldsymbol{y}(\Omega)).$$

This equivalence easily follows from the area formula [10, p. 185]. Note that the Ciarlet–Nečas condition is preserved under weak convergence in $W^{1,p}(\Omega;\mathbb{R}^3)$ thanks to the weak continuity of Jacobian minors and the Morrey embedding. As a consequence, given $(\boldsymbol{y}_n)\subset W^{1,p}(\Omega;\mathbb{R}^3)$ such that each $\boldsymbol{y}_n$ is almost everywhere injective with $\det\nabla\boldsymbol{y}_n>0$ almost everywhere, if $\boldsymbol{y}_n\rightharpoonup\boldsymbol{y}$ in $W^{1,p}(\Omega;\mathbb{R}^3)$ for some $\boldsymbol{y}\in W^{1,p}(\Omega;\mathbb{R}^3)$ with $\det\nabla\boldsymbol{y}>0$ almost everywhere, then $\boldsymbol{y}$ is almost everywhere injective. Note that the condition $\det\nabla\boldsymbol{y}>0$ almost everywhere has to be assumed a priori.

The inverse $\boldsymbol{y}^{-1}$ of $\boldsymbol{y}$ turns out to have Sobolev regularity. Note that this makes sense since, by definition, $\boldsymbol{y}^{-1}$ is defined on the deformed configuration $\Omega^{\boldsymbol{y}}$, which is open by Lemma 2.1. The Sobolev regularity of the inverse has been proved for more general classes of deformations, such as in [4, Proposition 5.3], [27, Theorem 9.3], [54, Theorem 4.6], and [53, Theorem 8]. For the convenience of the reader, we recall the proof. Note that here we do not impose any regularity on the boundary.

Proposition 2.5 (Global invertibility). *Let* $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^3)$ *be almost everywhere injective with* $\det \nabla \boldsymbol{y} > 0$ *almost everywhere. Then* $\boldsymbol{y}^{-1} \in W^{1,1}(\Omega^{\boldsymbol{y}};\mathbb{R}^3)$ *with* $\nabla \boldsymbol{y}^{-1} = (\nabla \boldsymbol{y})^{-1} \circ \boldsymbol{y}^{-1}$ *almost everywhere in* $\Omega^{\boldsymbol{y}}$*. Moreover,* $\operatorname{cof} \nabla \boldsymbol{y}^{-1} \in L^1(\Omega^{\boldsymbol{y}};\mathbb{R}^{3\times 3})$ *and* $\det \nabla \boldsymbol{y}^{-1} \in L^1(\Omega^{\boldsymbol{y}})$.

Proof. By the Piola identity we have

$$\int_\Omega \operatorname{cof} \nabla \boldsymbol{y} : \nabla \boldsymbol{\zeta} \,\mathrm{d}\boldsymbol{x} = 0 \tag{2.2}$$

for every $\boldsymbol{\zeta} \in C^\infty_c(\Omega;\mathbb{R}^3)$. As $\operatorname{cof} \nabla \boldsymbol{y} \in L^{p/2}(\Omega;\mathbb{R}^{3\times 3})$ and $p/2 > p/(p-1) = p'$ since $p > 3$, by density, this actually holds for $\boldsymbol{\zeta} \in W^{1,p}_0(\Omega;\mathbb{R}^3)$. Let $\varphi \in C^\infty(\overline{\Omega})$ and $\boldsymbol{\psi} \in C^\infty_c(\Omega^{\boldsymbol{y}};\mathbb{R}^3)$. Choosing $\boldsymbol{\zeta} = \varphi \boldsymbol{\psi} \circ \boldsymbol{y}$ in (2.2), after some algebraic manipulation, we obtain the following identity:

$$-\int_\Omega \varphi \operatorname{div} \boldsymbol{\psi} \circ \boldsymbol{y} \det \nabla \boldsymbol{y} \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{\psi} \circ \boldsymbol{y} \otimes \nabla \varphi : \operatorname{cof} \nabla \boldsymbol{y} \,\mathrm{d}\boldsymbol{x}. \tag{2.3}$$

Let $X \subset \Omega$ with $\mathcal{L}^3(X) = 0$ be such that $\boldsymbol{y}|_{\Omega \setminus X}$ is injective. For clarity, let us consider the representative $\boldsymbol{v}$ of $\boldsymbol{y}^{-1}$ in (2.1) and let us fix a representative of $\nabla \boldsymbol{y}$. Set $D := \Omega \setminus (\boldsymbol{y}^{-1}(\boldsymbol{y}(\partial\Omega)) \cup \{\det \nabla \boldsymbol{y} \le 0\} \cup X)$, so that $\boldsymbol{v} = \boldsymbol{y}|_D^{-1}$ on $\boldsymbol{y}(D)$ and $\nabla \boldsymbol{y}$ is invertible on D. Let $\boldsymbol{\Phi} \in C^\infty_c(\Omega^{\boldsymbol{y}};\mathbb{R}^{3\times 3})$ and denote its rows by $\boldsymbol{\Phi}^i = (\Phi^i_1, \Phi^i_2, \Phi^i_3)^\top$, where $i = 1,2,3$. Using the change-of-variable formula, we compute

$$\begin{aligned}
-\int_{\Omega^{\boldsymbol{y}}} \boldsymbol{v} \cdot \operatorname{div} \boldsymbol{\Phi} \,\mathrm{d}\boldsymbol{\xi} &= -\int_{\boldsymbol{y}(D)} \boldsymbol{y}|_D^{-1} \cdot \operatorname{div} \boldsymbol{\Phi} \,\mathrm{d}\boldsymbol{\xi} = -\int_D \boldsymbol{x} \cdot \operatorname{div} \boldsymbol{\Phi} \circ \boldsymbol{y} \det \nabla \boldsymbol{y} \,\mathrm{d}\boldsymbol{x} \\
&= -\int_\Omega \boldsymbol{x} \cdot \operatorname{div} \boldsymbol{\Phi} \circ \boldsymbol{y} \det \nabla \boldsymbol{y} \,\mathrm{d}\boldsymbol{x} \\
&= -\sum_{i=1}^3 \int_\Omega x_i \operatorname{div} \boldsymbol{\Phi}^i \circ \boldsymbol{y} \det \nabla \boldsymbol{y} \,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Then using (2.3) with $\varphi(\boldsymbol{x}) = x_i$ for every $\boldsymbol{x} \in \Omega$ and $\boldsymbol{\psi}(\boldsymbol{\xi}) = \boldsymbol{\Phi}^i(\boldsymbol{\xi})$ for every $\boldsymbol{\xi} \in \Omega^{\boldsymbol{y}}$, we obtain

$$\begin{aligned}
-\int_{\Omega^{\boldsymbol{y}}} \boldsymbol{v} \cdot \operatorname{div} \boldsymbol{\Phi} \,\mathrm{d}\boldsymbol{\xi} &= \sum_{i,j=1}^3 \int_\Omega \Phi^i_j \circ \boldsymbol{y} (\operatorname{cof} \nabla \boldsymbol{y})^j_i \,\mathrm{d}\boldsymbol{x} \\
&= \sum_{i,j=1}^3 \int_\Omega \Phi^i_j \circ \boldsymbol{y} (\operatorname{adj} \nabla \boldsymbol{y})^i_j \,\mathrm{d}\boldsymbol{x}
\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \boldsymbol{\Phi} \circ \boldsymbol{y} : \operatorname{adj} \nabla \boldsymbol{y} \, \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{\Phi} \circ \boldsymbol{y} : (\nabla \boldsymbol{y})^{-1} \det \nabla \boldsymbol{y} \, \mathrm{d}\boldsymbol{x} \\
&= \int_D \boldsymbol{\Phi} \circ \boldsymbol{y} : (\nabla \boldsymbol{y})^{-1} \det \nabla \boldsymbol{y} \, \mathrm{d}\boldsymbol{x} \\
&= \int_{\boldsymbol{y}(D)} \boldsymbol{\Phi} : (\nabla \boldsymbol{y})^{-1} \circ \boldsymbol{y}|_D^{-1} \, \mathrm{d}\boldsymbol{\xi},
\end{aligned}
$$

where, in the last line, we used again the change-of-variable formula. Hence, as $\mathcal{L}^3(\Omega^{\boldsymbol{y}} \setminus \boldsymbol{y}(D)) = 0$, we deduce that $\boldsymbol{v}$ admits a weak gradient with a representative given by

$$
\nabla \boldsymbol{v}(\boldsymbol{\xi}) := \begin{cases} (\nabla \boldsymbol{y})^{-1} \circ \boldsymbol{y}|_D^{-1}(\boldsymbol{\xi}) & \text{if } \boldsymbol{\xi} \in \boldsymbol{y}(D), \\ \boldsymbol{A} & \text{if } \boldsymbol{\xi} \in \Omega^{\boldsymbol{y}} \setminus \boldsymbol{y}(D), \end{cases}
$$

where $\boldsymbol{A} \in \mathbb{R}^{3\times 3}$ is arbitrary. Thanks to the Lusin property (N), the equivalence class of $\nabla \boldsymbol{v}$ is uniquely determined. Moreover, it belongs to $L^1(\Omega^{\boldsymbol{y}}; \mathbb{R}^{3\times 3})$. Indeed, by the change-of-variable formula,

$$
\begin{aligned}
\int_{\Omega^{\boldsymbol{y}}} |\nabla \boldsymbol{v}| \, \mathrm{d}\boldsymbol{\xi} &= \int_{\boldsymbol{y}(D)} |(\nabla \boldsymbol{y})^{-1}| \circ \boldsymbol{y}|_D^{-1} \, \mathrm{d}\boldsymbol{\xi} = \int_D |(\nabla \boldsymbol{y})^{-1}| \det \nabla \boldsymbol{y} \, \mathrm{d}\boldsymbol{x} \\
&= \int_D |\operatorname{adj} \nabla \boldsymbol{y}| \, \mathrm{d}\boldsymbol{x} = \int_\Omega |\operatorname{adj} \nabla \boldsymbol{y}| \, \mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

Thus, $\boldsymbol{v} \in W^{1,1}(\Omega^{\boldsymbol{y}}; \mathbb{R}^3)$. Similarly, using the identity $\operatorname{adj}(\boldsymbol{F}^{-1}) = (\det \boldsymbol{F})^{-1} \boldsymbol{F}$ for every $\boldsymbol{F} \in \mathbb{R}^{3\times 3}_+$, we compute

$$
\begin{aligned}
\int_{\Omega^{\boldsymbol{y}}} |\operatorname{adj} \nabla \boldsymbol{v}| \, \mathrm{d}\boldsymbol{\xi} &= \int_{\boldsymbol{y}(D)} (\det \nabla \boldsymbol{y})^{-1} \circ \boldsymbol{y}|_D^{-1} |\nabla \boldsymbol{y}| \circ \boldsymbol{y}|_D^{-1} \, \mathrm{d}\boldsymbol{\xi} \\
&= \int_D |\nabla \boldsymbol{y}| \, \mathrm{d}\boldsymbol{x} = \int_\Omega |\nabla \boldsymbol{y}| \, \mathrm{d}\boldsymbol{x},
\end{aligned}
$$

while, using the identity $\det(\boldsymbol{F}^{-1}) = (\det \boldsymbol{F})^{-1}$ in $\boldsymbol{F} \in \mathbb{R}^{3\times 3}_+$, we obtain

$$
\int_{\Omega^{\boldsymbol{y}}} \det \nabla \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} = \int_{\boldsymbol{y}(D)} (\det \nabla \boldsymbol{y})^{-1} \circ \boldsymbol{y}|_D^{-1} \, \mathrm{d}\boldsymbol{\xi} = \mathcal{L}^3(D) = \mathcal{L}^3(\Omega).
$$

Therefore, $\operatorname{cof} \nabla \boldsymbol{v} \in L^1(\Omega^{\boldsymbol{y}}; \mathbb{R}^{3\times 3})$ and $\det \nabla \boldsymbol{v} \in L^1(\Omega^{\boldsymbol{y}})$. ∎

Remark 2.6 (Area formula for the inverse). Let $\boldsymbol{y} \in W^{1,p}(\Omega; \mathbb{R}^3)$ be almost everywhere injective with $\det \nabla \boldsymbol{y} > 0$ almost everywhere. Let $X \subset \Omega$ with $\mathcal{L}^3(X) = 0$ be such that $\boldsymbol{y}|_{\Omega \setminus X}$ is injective and let $\boldsymbol{v}$ be the representative of $\boldsymbol{y}^{-1}$ in (2.1). By Proposition 2.5, $\boldsymbol{v} \in W^{1,1}(\Omega^{\boldsymbol{y}}; \mathbb{R}^3)$. Since $\boldsymbol{y}$ has the Lusin property (N^{-1}), the map $\boldsymbol{v}$ has the Lusin property (N). Moreover, $\boldsymbol{v}$ is almost everywhere injective. Thus, we can use the area formula to estimate the measure of preimages of sets via $\boldsymbol{y}$ as follows. For every $F \subset \mathbb{R}^3$, we write $\boldsymbol{y}^{-1}(F) := \{\boldsymbol{x} \in \Omega : \boldsymbol{y}(\boldsymbol{x}) \in F\}$. Let $F \subset \mathbb{R}^3$ be measurable. By (2.1), we have

$\boldsymbol{v}(F) = \boldsymbol{y}^{-1}(F \setminus (\boldsymbol{y}(\partial\Omega) \cup \boldsymbol{y}(X)))$. Then, exploiting both Lusin properties (N) and (N^{-1}) of $\boldsymbol{y}$ and applying the area formula [24, Theorem 2], we compute

$$\mathcal{L}^3(\boldsymbol{y}^{-1}(F)) = \mathcal{L}^3(\boldsymbol{v}(F)) = \int_F \det \nabla \boldsymbol{v} \, \mathrm{d}\boldsymbol{\xi}.$$

3. Static setting

3.1. The mathematical model

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. For $p > 3$ fixed, the class of admissible deformations is given by

$$\mathcal{Y} := \left\{\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^3) : \det \nabla \boldsymbol{y} > 0 \text{ a.e.}, \boldsymbol{y} \text{ a.e. injective}, \boldsymbol{y} = \bar{\boldsymbol{y}} \text{ on } \Gamma\right\}, \tag{3.1}$$

where $\Gamma \subset \partial\Omega$ is $\mathcal{H}^2$-measurable with $\mathcal{H}^2(\Gamma) > 0$ and $\bar{\boldsymbol{y}} \in C^0(\Gamma;\mathbb{R}^3)$.

Example 3.1. Let Ω and $\boldsymbol{y}$ be as in Example 2.2. Given $\Gamma := \{-1, 1\} \times (-1, 1)^2$ and $\bar{\boldsymbol{y}} := \mathbf{id}$, we have $\boldsymbol{y} \in \mathcal{Y}$. In particular, this is a case in which $\mathcal{Y} \neq \emptyset$.

Henceforth, we identify each $\boldsymbol{y} \in \mathcal{Y}$ with its continuous representative and we set $\Omega^{\boldsymbol{y}} := \boldsymbol{y}(\Omega) \setminus \boldsymbol{y}(\partial\Omega)$. Then admissible magnetizations are given by the maps $\boldsymbol{m} \in W^{1,2}(\Omega^{\boldsymbol{y}};\mathbb{S}^2)$. Note that this makes sense as $\Omega^{\boldsymbol{y}}$ is open by Lemma 2.1. Thus, the class of admissible states is defined as

$$\mathcal{Q} := \left\{(\boldsymbol{y}, \boldsymbol{m}) : \boldsymbol{y} \in \mathcal{Y}, \ \boldsymbol{m} \in W^{1,2}(\Omega^{\boldsymbol{y}};\mathbb{S}^2)\right\}. \tag{3.2}$$

We endow the set $\mathcal{Q}$ with the topology that makes the map $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m}) \mapsto (\boldsymbol{y}, \chi_{\Omega^{\boldsymbol{y}}}\boldsymbol{m}, \chi_{\Omega^{\boldsymbol{y}}}\nabla\boldsymbol{m})$ from $\mathcal{Q}$ to $W^{1,p}(\Omega;\mathbb{R}^3) \times L^2(\mathbb{R}^3;\mathbb{R}^3) \times L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})$ a homeomorphism onto its image, where the latter space is equipped with the weak product topology. Hence $\boldsymbol{q}_n \to \boldsymbol{q}$ in $\mathcal{Q}$ if and only if the following convergences hold:

$$\boldsymbol{y}_n \rightharpoonup \boldsymbol{y} \quad \text{in } W^{1,p}(\Omega;\mathbb{R}^3), \tag{3.3}$$

$$\chi_{\Omega^{\boldsymbol{y}_n}}\boldsymbol{m}_n \rightharpoonup \chi_{\Omega^{\boldsymbol{y}}}\boldsymbol{m} \quad \text{in } L^2(\mathbb{R}^3;\mathbb{R}^3), \tag{3.4}$$

$$\chi_{\Omega^{\boldsymbol{y}_n}}\nabla\boldsymbol{m}_n \rightharpoonup \chi_{\Omega^{\boldsymbol{y}}}\nabla\boldsymbol{m} \quad \text{in } L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3}). \tag{3.5}$$

In this case, up to subsequences, we actually have

$$\chi_{\Omega^{\boldsymbol{y}_n}}\boldsymbol{m}_n \to \chi_{\Omega^{\boldsymbol{y}}}\boldsymbol{m} \quad \text{in } L^a(\mathbb{R}^3;\mathbb{R}^3) \text{ for every } 1 \leq a < \infty. \tag{3.6}$$

The energy functional $E\colon \mathcal{Q} \to \mathbb{R}$ is defined, for $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{Q}$, by setting

$$E(\boldsymbol{q}) := \int_\Omega W(\nabla\boldsymbol{y}, \boldsymbol{m} \circ \boldsymbol{y}) \, \mathrm{d}\boldsymbol{x} + \alpha \int_{\Omega^{\boldsymbol{y}}} |\nabla\boldsymbol{m}|^2 \, \mathrm{d}\boldsymbol{\xi} + \frac{\mu_0}{2} \int_{\mathbb{R}^3} |\nabla\zeta_{\boldsymbol{m}}|^2 \, \mathrm{d}\boldsymbol{\xi}. \tag{3.7}$$

The first term represents the elastic energy of the system. Note that, as $\boldsymbol{y}$ satisfies the Lusin property (N^{-1}), the composition $\boldsymbol{m} \circ \boldsymbol{y}$ is measurable and its equivalence class does not depend on the choice of the representative of $\boldsymbol{m}$. The nonlinear elastic energy density $W\colon \mathbb{R}^{3\times 3}_+ \times \mathbb{S}^2 \to [0, +\infty)$ is continuous and satisfies the following two assumptions:

(Coercivity) There exist a constant $K > 0$ and a Borel function $\gamma\colon [0, +\infty) \to [0, +\infty)$ satisfying $\lim_{h\to 0^+} \gamma(h) = +\infty$ such that

$$W(\boldsymbol{F}, \boldsymbol{\lambda}) \geq K|\boldsymbol{F}|^p + \gamma(\det \boldsymbol{F}) \tag{3.8}$$

for every $\boldsymbol{F} \in \mathbb{R}^{3\times 3}_+$ and $\boldsymbol{\lambda} \in \mathbb{S}^2$.

(Polyconvexity) There exists a function $\widehat{W}\colon \mathbb{R}^{3\times 3}_+ \times \mathbb{R}^{3\times 3}_+ \times \mathbb{R}_+ \times \mathbb{S}^2 \to [0, +\infty)$ such that $\widehat{W}(\cdot, \cdot, \cdot, \boldsymbol{\lambda})$ is convex for every $\boldsymbol{\lambda} \in \mathbb{S}^2$ and there holds

$$W(\boldsymbol{F}, \boldsymbol{\lambda}) = \widehat{W}(\boldsymbol{F}, \operatorname{cof} \boldsymbol{F}, \det \boldsymbol{F}, \boldsymbol{\lambda}) \tag{3.9}$$

for every $\boldsymbol{F} \in \mathbb{R}^{3\times 3}_+$ and $\boldsymbol{\lambda} \in \mathbb{S}^2$.

Another standard assumption on the elastic energy density is the one of frame indifference, which reads

$$\forall \boldsymbol{F} \in \mathbb{R}^{3\times 3}_+,\ \forall \boldsymbol{\lambda} \in \mathbb{S}^2,\ \forall \boldsymbol{Q} \in SO(3), \quad W(\boldsymbol{Q}\boldsymbol{F}, \boldsymbol{Q}\boldsymbol{\lambda}) = W(\boldsymbol{F}, \boldsymbol{\lambda}).$$

This assumption is crucial from the physical point of view, as it ensures the objectivity of the magnetoelastic energy. However, this requirement will play no role in our analysis.

The second term in (3.7) is the exchange energy and comprises the parameter $\alpha > 0$. The third term is called magnetostatic energy and involves the function $\zeta_{\boldsymbol{m}}\colon \mathbb{R}^3 \to \mathbb{R}$ which is a weak solution of the magnetostatic Maxwell equation:

$$\Delta \zeta_{\boldsymbol{m}} = \operatorname{div}(\chi_{\Omega^{\boldsymbol{y}}} \boldsymbol{m}) \quad \text{in } \mathbb{R}^3. \tag{3.10}$$

This means that $\zeta_{\boldsymbol{m}}$ belongs to the homogeneous Sobolev space

$$V^{1,2}(\mathbb{R}^3) := \left\{\varphi \in L^2_{\mathrm{loc}}(\mathbb{R}^3) : \nabla\varphi \in L^2(\mathbb{R}^3; \mathbb{R}^3)\right\}$$

and satisfies

$$\forall \varphi \in V^{1,2}(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} \nabla\zeta_{\boldsymbol{m}} \cdot \nabla\varphi \,\mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{R}^3} \chi_{\Omega^{\boldsymbol{y}}} \boldsymbol{m} \cdot \nabla\varphi \,\mathrm{d}\boldsymbol{\xi}.$$

Note that such weak solutions exist and are unique up to additive constants [4, Proposition 8.8], so that their gradient is uniquely defined. The constant $\mu_0 > 0$ denotes the vacuum permeability.

3.2. Compactness and existence of minimizers

The main result of this section is the existence of minimizers of the energy E in (3.7). Recall the definition of the class of admissible states in (3.1)–(3.2).

Theorem 3.2 (Existence of minimizers). *Assume $p > 3$ and $\mathcal{Y} \neq \emptyset$. Suppose that W is continuous and satisfies* (3.8)–(3.9). *Then the functional E admits a minimizer in $\mathcal{Q}$.*

Remark 3.3 (Applied loads). In Theorem 3.2, applied loads can be also taken into account. Let $\boldsymbol{f} \in L^{p'}(\Omega;\mathbb{R}^3)$, $\boldsymbol{g} \in L^{p'}(\Sigma;\mathbb{R}^3)$, where $\Sigma \subset \partial\Omega$ is $\mathcal{H}^2$-measurable and such that $\Gamma \cap \Sigma = \emptyset$, and $\boldsymbol{h} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ represent an applied body force, surface force, and magnetic field, respectively. Then the work of applied loads, which should be subtracted from the magnetoelastic energy, is described by the functional $L\colon \mathcal{Q} \to \mathbb{R}$ given by

$$L(\boldsymbol{q}) := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{y}\,\mathrm{d}\boldsymbol{x} + \int_\Sigma \boldsymbol{g} \cdot \boldsymbol{y}\,\mathrm{d}\mathcal{H}^2 + \int_{\Omega^{\boldsymbol{y}}} \boldsymbol{h} \cdot \boldsymbol{m}\,\mathrm{d}\boldsymbol{\xi},$$

where $\boldsymbol{q} = (\boldsymbol{y},\boldsymbol{m})$. Note that the energy contribution determined by the external magnetic field, usually called Zeemann energy, is described by an Eulerian term. The functional L is evidently continuous with respect to the topology of $\mathcal{Q}$, so that its treatment is trivial.

We begin by proving a compactness result for sequences of admissible states with equi-bounded energies. In particular, we establish the convergence of compositions of magnetizations with deformations. Recall the function γ introduced in (3.8).

Proposition 3.4 (Compactness). *Let $(\boldsymbol{q}_n) \subset \mathcal{Q}$ with $\boldsymbol{q}_n = (\boldsymbol{y}_n,\boldsymbol{m}_n)$ satisfy*

$$\|\nabla \boldsymbol{y}_n\|_{L^p(\Omega;\mathbb{R}^{3\times3})} \le C, \quad \|\nabla \boldsymbol{m}_n\|_{L^2(\Omega^{\boldsymbol{y}_n};\mathbb{R}^{3\times3})} \le C, \quad \|\gamma(\det\nabla\boldsymbol{y}_n)\|_{L^1(\Omega)} \le C \tag{3.11}$$

for every $n \in \mathbb{N}$. Then there exists $\boldsymbol{q} \in \mathcal{Q}$ with $\boldsymbol{q} = (\boldsymbol{y},\boldsymbol{m})$ such that, up to subsequences, we have $\boldsymbol{q}_n \to \boldsymbol{q}$ in $\mathcal{Q}$ and also

$$\boldsymbol{m}_n \circ \boldsymbol{y}_n \to \boldsymbol{m} \circ \boldsymbol{y} \quad \text{in } L^a(\Omega;\mathbb{R}^3) \text{ for every } 1 \le a < \infty. \tag{3.12}$$

Remark 3.5 (Anisotropy and DMI energies). The crystalline anisotropy and the asymmetric exchange can be easily included in Theorem 3.2 without additional difficulties. The corresponding energy terms are described by the functionals $E^{\mathrm{ani}}\colon \mathcal{Q} \to \mathbb{R}$ and $E^{\mathrm{DMI}}\colon \mathcal{Q} \to \mathbb{R}$ defined, for $\boldsymbol{q} = (\boldsymbol{y},\boldsymbol{m}) \in \mathcal{Q}$, by

$$E^{\mathrm{ani}}(\boldsymbol{q}) := \int_{\Omega^{\boldsymbol{y}}} \phi(\boldsymbol{m})\,\mathrm{d}\boldsymbol{\xi}, \quad E^{\mathrm{DMI}}(\boldsymbol{q}) := \kappa \int_{\Omega^{\boldsymbol{y}}} \operatorname{curl}\boldsymbol{m} \cdot \boldsymbol{m}\,\mathrm{d}\boldsymbol{\xi},$$

where $\phi\colon \mathbb{S}^2 \to \mathbb{R}$ is continuous and $\kappa \in \mathbb{R}$. These two functionals are indeed continuous with respect to the convergences given by Proposition 3.4. The continuity of E^{DMI} is evident from (3.5)–(3.6). The continuity of E^{ani} follows easily from (3.3) and (3.12). By (3.12), we can assume that compositions converge almost everywhere so that, by the dominated convergence theorem, $\phi(\boldsymbol{m}_n \circ \boldsymbol{y}_n) \to \phi(\boldsymbol{m} \circ \boldsymbol{y})$ in $L^a(\Omega)$ for every $1 \le a < \infty$. Then, exploiting the weak convergence of Jacobian determinants, which follows from (3.3), and employing the change of variable formula, we obtain

$$E^{\mathrm{ani}}(\boldsymbol{q}_n) = \int_\Omega \phi(\boldsymbol{m}_n \circ \boldsymbol{y}_n)\det\nabla\boldsymbol{y}_n\mathrm{d}\boldsymbol{x} \to \int_\Omega \phi(\boldsymbol{m} \circ \boldsymbol{y})\det\nabla\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} = E^{\mathrm{ani}}(\boldsymbol{q}).$$

We mention that the continuity of E^{ani} can be also established without relying on (3.12), but exploiting only (3.3) and (3.6) by means of a localization argument based on (3.13)–(3.14).

Proof of Proposition 3.4. For the convenience of the reader, the proof is divided into three steps.

Step 1 *(compactness).* By (3.11), using the Poincaré inequality with boundary terms, we deduce that $(\boldsymbol{y}_n)$ is bounded in $W^{1,p}(\Omega;\mathbb{R}^3)$. Thus, up to subsequences, (3.3) holds for some $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^3)$.

We claim that $\boldsymbol{y} \in \mathcal{Y}$. Thanks to Remark 2.4 and the compactness of the trace operator, we only have to prove that $\det \nabla \boldsymbol{y} > 0$ almost everywhere in Ω. By the weak continuity of Jacobian minors, $\det \nabla \boldsymbol{y}_n \rightharpoonup \det \nabla \boldsymbol{y}$ in $L^{p/3}(\Omega)$. Then, for every $S \subset \Omega$ measurable, we have

$$\int_S \det \nabla \boldsymbol{y} \,\mathrm{d}\boldsymbol{x} = \lim_{n\to\infty} \int_S \det \nabla \boldsymbol{y}_n \,\mathrm{d}\boldsymbol{x} \geq 0,$$

and, given the arbitrariness of S, we deduce that $\det \nabla \boldsymbol{y} \geq 0$ almost everywhere in Ω. By contradiction, suppose that $\det \nabla \boldsymbol{y} = 0$ on a measurable set $S_0 \subset \Omega$ with $\mathcal{L}^3(S_0) > 0$. In this case, up to subsequences, $\det \nabla \boldsymbol{y}_n \to 0$ almost everywhere in S_0, and, taking into account (3.8), we obtain $\gamma(\det \nabla \boldsymbol{y}_n) \to +\infty$ almost everywhere in S_0. Then, by the Fatou lemma, we obtain $\liminf_{n\to\infty} \int_{S_0} \gamma(\det \nabla \boldsymbol{y}_n)\,\mathrm{d}\boldsymbol{x} = +\infty$, which contradicts (3.11). Therefore, $\mathcal{L}^3(S_0) = 0$ and $\det \nabla \boldsymbol{y} > 0$ almost everywhere in Ω.

The compactness of the sequence $(\boldsymbol{q}_n)$ is proved as in [34, Proposition 2.1]. By the Morrey embedding, we have $\boldsymbol{y}_n \to \boldsymbol{y}$ uniformly in Ω. From this, we obtain the following:

$$\forall A \subset\subset \Omega^{\boldsymbol{y}} \text{ open}, \quad A \subset \Omega^{\boldsymbol{y}_n} \text{ for } n \gg 1 \text{ depending on } A, \tag{3.13}$$

$$\forall O \supset\supset \Omega^{\boldsymbol{y}} \text{ open}, \quad O \supset \Omega^{\boldsymbol{y}_n} \text{ for } n \gg 1 \text{ depending on } O. \tag{3.14}$$

To see (3.13), let $A \subset\subset \Omega^{\boldsymbol{y}}$ be open so that $\mathrm{dist}(\partial A; \partial\Omega^{\boldsymbol{y}}) > 0$. Recall that $\partial\Omega^{\boldsymbol{y}} = \boldsymbol{y}(\partial\Omega)$ by Lemma 2.1. Then, for $n \gg 1$ depending on A, we have

$$\|\boldsymbol{y}_n - \boldsymbol{y}\|_{C^0(\bar{\Omega};\mathbb{R}^3)} \leq \mathrm{dist}(\partial A; \boldsymbol{y}(\partial\Omega)).$$

Let $\boldsymbol{\xi} \in A$. We obtain

$$\|\boldsymbol{y}_n - \boldsymbol{y}\|_{C^0(\bar{\Omega};\mathbb{R}^3)} \leq \mathrm{dist}(\boldsymbol{\xi}; \boldsymbol{y}(\partial\Omega)),$$

and, by the stability property of the degree [20, Theorem 2.3, Claim (1)], we deduce $\boldsymbol{\xi} \notin \boldsymbol{y}_n(\partial\Omega)$ and $\deg(\boldsymbol{y}_n,\Omega,\boldsymbol{\xi}) = \deg(\boldsymbol{y},\Omega,\boldsymbol{\xi})$ for $n \gg 1$. As $\deg(\boldsymbol{y},\Omega,\boldsymbol{\xi}) > 0$ by Remark 2.3, the solvability property of the degree [20, Theorem 2.1] gives $\boldsymbol{\xi} \in \Omega^{\boldsymbol{y}_n}$ for $n \gg 1$. This proves (3.13), while (3.14) is immediate.

Let $A \subset\subset \Omega^{\boldsymbol{y}}$ be open with smooth boundary and $n \gg 1$ as in (3.13). From (3.11), we have

$$\int_A |\nabla \boldsymbol{m}_n|^2 \,\mathrm{d}\boldsymbol{\xi} \leq \int_{\Omega^{\boldsymbol{y}_n}} |\nabla \boldsymbol{m}_n|^2 \,\mathrm{d}\boldsymbol{\xi} \leq C \tag{3.15}$$

for every $n \gg 1$. Recalling that magnetizations are sphere valued, we deduce that $(\boldsymbol{m}_n)$ is bounded in $W^{1,2}(A;\mathbb{R}^3)$, so that, up to subsequences, $\boldsymbol{m}_n \rightharpoonup \boldsymbol{m}$ in $W^{1,2}(A;\mathbb{R}^3)$ for some $\boldsymbol{m} \in W^{1,2}(A;\mathbb{R}^3)$. By the Rellich embedding, $\boldsymbol{m}_n \to \boldsymbol{m}$ in $L^2(A;\mathbb{R}^3)$ and, in turn,

$|\boldsymbol{m}| = 1$ almost everywhere in A. The map $\boldsymbol{m} \in W^{1,2}_{\mathrm{loc}}(\Omega^{\boldsymbol{y}};\mathbb{S}^2)$ does not depend on A. In particular, as the right-hand side of (3.15) does not depend on A, we actually have $\boldsymbol{m} \in W^{1,2}(\Omega^{\boldsymbol{y}};\mathbb{S}^2)$. Therefore, $\boldsymbol{q} = (\boldsymbol{y},\boldsymbol{m}) \in \mathcal{Q}$. Moreover, arguing with a sequence (A_j) of open sets with smooth boundaries such that $A_j \subset\subset A_{j+1} \subset\subset \Omega^{\boldsymbol{y}}$ for every $j \in \mathbb{N}$ and $\Omega^{\boldsymbol{y}} = \bigcup_{j=1}^{\infty} A_j$, we select a (not relabeled) subsequence of $(\boldsymbol{m}_n)$ such that

$$\forall A \subset\subset \Omega^{\boldsymbol{y}} \text{ open}, \quad \boldsymbol{m}_n \rightharpoonup \boldsymbol{m} \text{ in } W^{1,2}(A), \boldsymbol{m}_n \to \boldsymbol{m} \text{ almost everywhere in } A. \tag{3.16}$$

We remark that, for every $A \subset\subset \Omega^{\boldsymbol{y}}$ open, the sequence $(\boldsymbol{m}_n) \subset W^{1,2}(A;\mathbb{S}^2)$ is defined only for $n \gg 1$ depending on A.

Step 2 *(Convergence in* $\boldsymbol{\mathcal{Q}}$*).* In order to prove that $\boldsymbol{q}_n \to \boldsymbol{q}$ in $\mathcal{Q}$, we are left to show (3.4) and (3.5). To prove the first claim, we consider $\boldsymbol{\varphi} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$. We need to show that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} (\chi_{\Omega^{\boldsymbol{y}_n}} \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \boldsymbol{m}) \cdot \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x} = 0. \tag{3.17}$$

Let $A, O \subset \mathbb{R}^3$ be open such that $A \subset\subset \Omega^{\boldsymbol{y}} \subset\subset O$. We write

$$\begin{aligned}\int_{\mathbb{R}^3} (\chi_{\Omega^{\boldsymbol{y}_n}} \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \boldsymbol{m}) \cdot \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x} &= \int_{A} (\chi_{\Omega^{\boldsymbol{y}_n}} \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \boldsymbol{m}) \cdot \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x} \\ &\quad + \int_{O\setminus A} (\chi_{\Omega^{\boldsymbol{y}_n}} \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \boldsymbol{m}) \cdot \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x} \\ &\quad + \int_{\mathbb{R}^3\setminus O} (\chi_{\Omega^{\boldsymbol{y}_n}} \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \boldsymbol{m}) \cdot \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{3.18}$$

For the first integral on the right-hand side of (3.18), by (3.13) for $n \gg 1$ we have

$$\int_{A} (\chi_{\Omega^{\boldsymbol{y}_n}} \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \boldsymbol{m}) \cdot \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x} = \int_{A} (\boldsymbol{m}_n - \boldsymbol{m}) \cdot \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x}, \tag{3.19}$$

where, as $n \to \infty$, the right-hand side goes to zero since $\boldsymbol{m}_n \rightharpoonup \boldsymbol{m}$ in $W^{1,2}(A;\mathbb{R}^3)$ by (3.16). Using the Hölder inequality, the second integral on the right-hand side of (3.18) is estimated as

$$\left| \int_{O\setminus A} (\chi_{\Omega^{\boldsymbol{y}_n}} \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \boldsymbol{m}) \cdot \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x} \right| \leq 2\sqrt{\mathscr{L}^3(O \setminus A)} \|\boldsymbol{\varphi}\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}. \tag{3.20}$$

By (3.14), the third integral on the right-hand side of (3.18) equals zero for $n \gg 1$. Therefore, we obtain

$$\limsup_{n\to\infty} \left| \int_{\mathbb{R}^3} (\chi_{\Omega^{\boldsymbol{y}_n}} \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \boldsymbol{m}) \cdot \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x} \right| \leq 2\sqrt{\mathscr{L}^3(O \setminus A)} \|\boldsymbol{\varphi}\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)},$$

from which, letting $O \searrow \overline{\Omega^{\boldsymbol{y}}}$ and $A \nearrow \Omega^{\boldsymbol{y}}$ so that $\mathscr{L}^3(O \setminus A) \to \mathscr{L}^3(\partial\Omega^{\boldsymbol{y}}) = 0$, we deduce (3.17). Here, we used that $\partial\Omega^{\boldsymbol{y}} = \boldsymbol{y}(\partial\Omega)$ by Lemma 2.1 and that $\mathscr{L}^3(\boldsymbol{y}(\partial\Omega)) = 0$ thanks to the Lusin property (N). Thus (3.4) is proved.

For the second claim, we proceed in a similar way. Given $\boldsymbol{\Phi} \in L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})$, we need to show

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} (\chi_{\Omega^{\boldsymbol{y}_n}} \nabla \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \nabla \boldsymbol{m}) : \boldsymbol{\Phi} \, \mathrm{d}\boldsymbol{x} = 0. \tag{3.21}$$

As before, we consider $A, O \subset \mathbb{R}^3$ open with $A \subset\subset \Omega^{\boldsymbol{y}} \subset\subset O$ and we write

$$\begin{aligned}\int_{\mathbb{R}^3} (\chi_{\Omega^{\boldsymbol{y}_n}} \nabla \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \nabla \boldsymbol{m}) : \boldsymbol{\Phi} \, \mathrm{d}\boldsymbol{x} &= \int_{A} (\chi_{\Omega^{\boldsymbol{y}_n}} \nabla \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \nabla \boldsymbol{m}) : \boldsymbol{\Phi} \, \mathrm{d}\boldsymbol{x} \\ &\quad + \int_{O\setminus A} (\chi_{\Omega^{\boldsymbol{y}_n}} \nabla \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \nabla \boldsymbol{m}) : \boldsymbol{\Phi} \, \mathrm{d}\boldsymbol{x} \\ &\quad + \int_{\mathbb{R}^3\setminus O} (\chi_{\Omega^{\boldsymbol{y}_n}} \nabla \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \nabla \boldsymbol{m}) : \boldsymbol{\Phi} \, \mathrm{d}\boldsymbol{x}. \end{aligned} \tag{3.22}$$

For the first integral on the right-hand side of (3.22), by (3.13), for $n \gg 1$ we have

$$\int_{A} (\chi_{\Omega^{\boldsymbol{y}_n}} \nabla \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \nabla \boldsymbol{m}) : \boldsymbol{\Phi} \, \mathrm{d}\boldsymbol{x} = \int_{A} (\nabla \boldsymbol{m}_n - \nabla \boldsymbol{m}) : \boldsymbol{\Phi} \, \mathrm{d}\boldsymbol{x},$$

and, as $n \to \infty$, the right-hand side goes to zero since $\boldsymbol{m}_n \rightharpoonup \boldsymbol{m}$ in $W^{1,2}(A;\mathbb{R}^3)$ by (3.16). Note that the sequence $(\chi_{\Omega^{\boldsymbol{y}_n}} \nabla \boldsymbol{m}_n) \subset L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})$ is bounded by (3.11). Using the Hölder inequality, the second integral on the right-hand side of (3.22) is estimated as

$$\begin{aligned}&\left| \int_{O\setminus A} (\chi_{\Omega^{\boldsymbol{y}_n}} \nabla \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \nabla \boldsymbol{m}) : \boldsymbol{\Phi} \, \mathrm{d}\boldsymbol{x} \right| \\ &\qquad \leq (\|\chi_{\Omega^{\boldsymbol{y}_n}} \nabla \boldsymbol{m}_n\|_{L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})} + \|\chi_{\Omega^{\boldsymbol{y}}} \nabla \boldsymbol{m}\|_{L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})}) \|\boldsymbol{\Phi}\|_{L^2(O\setminus A;\mathbb{R}^{3\times 3})} \\ &\qquad \leq (C + \|\chi_{\Omega^{\boldsymbol{y}}} \nabla \boldsymbol{m}\|_{L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})}) \|\boldsymbol{\Phi}\|_{L^2(O\setminus A;\mathbb{R}^{3\times 3})}.\end{aligned}$$

By (3.14), the third integral on the right-hand side of (3.22) equals zero for $n \gg 1$. Therefore, we obtain

$$\begin{aligned}&\limsup_{n\to\infty} \left| \int_{\mathbb{R}^3} (\chi_{\Omega^{\boldsymbol{y}_n}} \nabla \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \nabla \boldsymbol{m}) : \boldsymbol{\Phi} \, \mathrm{d}\boldsymbol{x} \right| \\ &\qquad \leq (C + \|\chi_{\Omega^{\boldsymbol{y}}} \nabla \boldsymbol{m}\|_{L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})}) \|\boldsymbol{\Phi}\|_{L^2(O\setminus A;\mathbb{R}^{3\times 3})}.\end{aligned}$$

From this, letting $O \searrow \overline{\Omega^{\boldsymbol{y}}}$ and $A \nearrow \Omega^{\boldsymbol{y}}$ so that $\mathscr{L}^3(O \setminus A) \to \mathscr{L}^3(\partial \Omega^{\boldsymbol{y}}) = 0$ and, in turn, $\|\boldsymbol{\Phi}\|_{L^2(O\setminus A;\mathbb{R}^{3\times 3})} \to 0$, we deduce (3.21). Thus also (3.5) is proved.

Step 3 *(Convergence of the compositions).* By Proposition 2.5, $\boldsymbol{y}_n^{-1} \in W^{1,1}(\Omega^{\boldsymbol{y}_n};\mathbb{R}^3)$ with $\det \nabla \boldsymbol{y}_n^{-1} \in L^1(\Omega^{\boldsymbol{y}_n})$ for every $n \in \mathbb{N}$. We claim that, for every open set $A \subset\subset \Omega^{\boldsymbol{y}}$, the sequence $(\det \nabla \boldsymbol{y}_n^{-1}) \subset L^1(A)$ is equi-integrable. To show this, we argue as in [4, Proposition 7.8]. Define $\hat{\gamma}\colon (0,+\infty) \to [0,+\infty)$ by setting $\hat{\gamma}(z) := z\gamma(1/z)$. In this case

$$\begin{aligned}\lim_{z\to+\infty} \frac{\hat{\gamma}(z)}{z} &= \lim_{z\to+\infty} \gamma(1/z) \\ &= \lim_{h\to 0^+} \gamma(h) = +\infty,\end{aligned}$$

where we used (3.8). Using the change-of-variable formula we compute

$$\begin{aligned}\int_{\Omega^{\boldsymbol{y}_n}} \hat{\gamma}(\det \nabla \boldsymbol{y}_n^{-1})\,\mathrm{d}\boldsymbol{\xi} &= \int_{\Omega^{\boldsymbol{y}_n}} \gamma(1/\det \nabla \boldsymbol{y}_n^{-1}) \det \nabla \boldsymbol{y}_n^{-1}\,\mathrm{d}\boldsymbol{\xi} \\ &= \int_{\Omega^{\boldsymbol{y}_n}} \gamma(\det \nabla \boldsymbol{y}_n) \circ \boldsymbol{y}_n^{-1} (\det \nabla \boldsymbol{y}_n)^{-1} \circ \boldsymbol{y}_n^{-1}\,\mathrm{d}\boldsymbol{\xi} \\ &= \int_{\Omega} \gamma(\det \nabla \boldsymbol{y}_n)\,\mathrm{d}\boldsymbol{x},\end{aligned}$$

where the right-hand side is uniformly bounded by (3.11). Thus, the claim follows by the de la Vallée–Poussin criterion [21, Theorem 2.29]. In particular, using the area formula as in Remark 2.6, we deduce

$$\begin{aligned}&\forall A \subset\subset \Omega^{\boldsymbol{y}} \text{ open},\ \forall \varepsilon > 0,\ \exists\, \delta(A,\varepsilon) > 0 : \forall F \subset A \text{ measurable}, \\ &\qquad \mathcal{L}^3(F) < \delta(A,\varepsilon) \Rightarrow \sup_{n\in\mathbb{N}} \mathcal{L}^3(\boldsymbol{y}_n^{-1}(F)) < \varepsilon.\end{aligned} \tag{3.23}$$

We now prove that $\boldsymbol{m}_n \circ \boldsymbol{y}_n \to \boldsymbol{m} \circ \boldsymbol{y}$ in $L^1(\Omega;\mathbb{R}^3)$. Fix $\varepsilon > 0$. Take $A \subset\subset \Omega^{\boldsymbol{y}}$ open such that $\mathcal{L}^3(\Omega \setminus \boldsymbol{y}^{-1}(A)) < \varepsilon$. We compute

$$\begin{aligned}\int_\Omega |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}|\,\mathrm{d}\boldsymbol{x} &= \int_{\Omega \setminus \boldsymbol{y}^{-1}(A)} |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}|\,\mathrm{d}\boldsymbol{x} \\ &\quad + \int_{\boldsymbol{y}^{-1}(A)} |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}|\,\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{3.24}$$

As magnetizations are sphere valued, for every $n \in \mathbb{N}$ the first integral on the right-hand side of (3.24) is bounded by $2\mathcal{L}^3(\Omega \setminus \boldsymbol{y}^{-1}(A)) < 2\varepsilon$. For the second integral on the right-hand side of (3.24), we split it as

$$\begin{aligned}\int_{\boldsymbol{y}^{-1}(A)} |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}|\,\mathrm{d}\boldsymbol{x} &= \int_{\boldsymbol{y}^{-1}(A) \setminus \boldsymbol{y}_n^{-1}(A)} |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}|\,\mathrm{d}\boldsymbol{x} \\ &\quad + \int_{\boldsymbol{y}^{-1}(A) \cap \boldsymbol{y}_n^{-1}(A)} |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}|\,\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{3.25}$$

We claim that $\mathcal{L}^3(\boldsymbol{y}^{-1}(A) \setminus \boldsymbol{y}_n^{-1}(A)) < \varepsilon$ for $n \gg 1$ depending only on ε, so that the second integral on the right-hand side of (3.25) is bounded by 2ε. To see this, let $V \subset \mathbb{R}^3$ be open and such that $A \subset\subset V \subset\subset \Omega^{\boldsymbol{y}}$. In this case, $\boldsymbol{y}(\boldsymbol{y}^{-1}(A)) = A \subset\subset V$ so that, by uniform convergence, $\boldsymbol{y}_n(\boldsymbol{y}^{-1}(A)) \subset V$ for $n \gg 1$ which, in turn, gives $\boldsymbol{y}^{-1}(A) \subset \boldsymbol{y}_n^{-1}(V)$ for $n \gg 1$. Then we have

$$\boldsymbol{y}^{-1}(A) \setminus \boldsymbol{y}_n^{-1}(A) \subset \boldsymbol{y}_n^{-1}(V) \setminus \boldsymbol{y}_n^{-1}(A) = \boldsymbol{y}_n^{-1}(V \setminus A) \tag{3.26}$$

for $n \gg 1$. In particular, let V be chosen such that $\mathcal{L}^3(V \setminus A) < \delta(V,\varepsilon)$ with $\delta(V,\varepsilon) > 0$ given by (3.23). Hence, for $n \gg 1$ depending only on ε, from (3.23) and (3.26), we obtain $\mathcal{L}^3(\boldsymbol{y}^{-1}(A) \setminus \boldsymbol{y}_n^{-1}(A)) < \varepsilon$ and the claim is proved.

To estimate the second integral on the right-hand side of (3.25) we proceed as follows. Henceforth, we will simply write δ in place of $\delta(A,\varepsilon)$, where $\delta(A,\varepsilon)>0$ is given by (3.23). Without loss of generality, we can assume that δ is sufficiently small in order to have $\mathcal{L}^3(\boldsymbol{y}^{-1}(F))<\varepsilon$ for every $F\subset A$ measurable with $\mathcal{L}^3(F)<\delta$. By the Lusin theorem, there exists $K_1\subset A$ compact with $\mathcal{L}^3(A\setminus K_1)<\delta/2$ such that $\boldsymbol{m}|_{K_1}$ is continuous while, by the Egorov theorem, there exists $K_2\subset A$ compact with $\mathcal{L}^3(A\setminus K_2)<\delta/2$ such that $\boldsymbol{m}_n\to\boldsymbol{m}$ uniformly on K_2. Set $K:=K_1\cap K_2$, so that $K\subset A$ is compact and $\mathcal{L}^3(A\setminus K)<\delta$. We have

$$\boldsymbol{y}^{-1}(A)\cap\boldsymbol{y}_n^{-1}(A)\subset(\boldsymbol{y}^{-1}(K)\cap\boldsymbol{y}_n^{-1}(K))\cup\boldsymbol{y}^{-1}(A\setminus K)\cup\boldsymbol{y}_n^{-1}(A\setminus K)$$

so that we estimate the second integral on the right-hand side of (3.25) as

$$\begin{aligned}\int_{\boldsymbol{y}^{-1}(A)\cap\boldsymbol{y}_n^{-1}(A)}&|\boldsymbol{m}_n\circ\boldsymbol{y}_n-\boldsymbol{m}\circ\boldsymbol{y}|\,\mathrm{d}\boldsymbol{x}\\&\leq\int_{\boldsymbol{y}^{-1}(K)\cap\boldsymbol{y}_n^{-1}(K)}|\boldsymbol{m}_n\circ\boldsymbol{y}_n-\boldsymbol{m}\circ\boldsymbol{y}|\,\mathrm{d}\boldsymbol{x}\\&\quad+\int_{\boldsymbol{y}^{-1}(A\setminus K)\cup\boldsymbol{y}_n^{-1}(A\setminus K)}|\boldsymbol{m}_n\circ\boldsymbol{y}_n-\boldsymbol{m}\circ\boldsymbol{y}|\,\mathrm{d}\boldsymbol{x}.\end{aligned}\tag{3.27}$$

For the second integral on the right-hand side of (3.27), we have

$$\begin{aligned}\int_{\boldsymbol{y}^{-1}(A\setminus K)\cup\boldsymbol{y}_n^{-1}(A\setminus K)}&|\boldsymbol{m}_n\circ\boldsymbol{y}_n-\boldsymbol{m}\circ\boldsymbol{y}|\,\mathrm{d}\boldsymbol{x}\\&\leq 2\mathcal{L}^3(\boldsymbol{y}^{-1}(A\setminus K)\cup\boldsymbol{y}_n^{-1}(A\setminus K))\\&<4\varepsilon,\end{aligned}\tag{3.28}$$

where, in the last line, we used (3.23). For the first integral on the right-hand side of (3.27), we have

$$\begin{aligned}\int_{\boldsymbol{y}^{-1}(K)\cap\boldsymbol{y}_n^{-1}(K)}&|\boldsymbol{m}_n\circ\boldsymbol{y}_n-\boldsymbol{m}\circ\boldsymbol{y}|\,\mathrm{d}\boldsymbol{x}\\&\leq\int_{\boldsymbol{y}^{-1}(K)\cap\boldsymbol{y}_n^{-1}(K)}|\boldsymbol{m}_n\circ\boldsymbol{y}_n-\boldsymbol{m}\circ\boldsymbol{y}_n|\,\mathrm{d}\boldsymbol{x}\\&\quad+\int_{\boldsymbol{y}^{-1}(K)\cap\boldsymbol{y}_n^{-1}(K)}|\boldsymbol{m}\circ\boldsymbol{y}_n-\boldsymbol{m}\circ\boldsymbol{y}|\,\mathrm{d}\boldsymbol{x}.\end{aligned}\tag{3.29}$$

Note that, in the previous equation, the composition $\boldsymbol{m}\circ\boldsymbol{y}_n$ is meaningful, at least for $n\gg 1$, since the domain of integration is a subset of $\boldsymbol{y}^{-1}(A)\cap\boldsymbol{y}_n^{-1}(A)$ and $A\subset\Omega^{\boldsymbol{y}_n}$. Given the choice of K, for $n\gg 1$ depending only on ε, we have $\sup_K|\boldsymbol{m}_n-\boldsymbol{m}|<\varepsilon/\mathcal{L}^3(\Omega)$. Thus, we estimate

$$\int_{\boldsymbol{y}^{-1}(K)\cap\boldsymbol{y}_n^{-1}(K)}|\boldsymbol{m}_n\circ\boldsymbol{y}_n-\boldsymbol{m}\circ\boldsymbol{y}_n|\,\mathrm{d}\boldsymbol{x}<\frac{\varepsilon}{\mathcal{L}^3(\Omega)}\mathcal{L}^3(\boldsymbol{y}^{-1}(K)\cap\boldsymbol{y}_n^{-1}(K))<\varepsilon.$$

On the other hand, $\boldsymbol{m}$ is uniformly continuous on K. Hence, there exists $\eta(\varepsilon) > 0$ such that for every $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in K$ with $|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| < \eta(\varepsilon)$ there holds $|\boldsymbol{m}(\xi_1) - \boldsymbol{m}(\boldsymbol{\xi}_2)| < \varepsilon/\mathscr{L}^3(\Omega)$. As a consequence, for $n \gg 1$ such that $\|\boldsymbol{y}_n - \boldsymbol{y}\|_{C^0(\bar{\Omega};\mathbb{R}^3)} < \eta(\varepsilon)$, we obtain

$$\int_{\boldsymbol{y}^{-1}(K)\cap \boldsymbol{y}_n^{-1}(K)} |\boldsymbol{m} \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}| \, \mathrm{d}\boldsymbol{x} < \frac{\varepsilon}{\mathscr{L}^3(\Omega)} \mathscr{L}^3(\boldsymbol{y}^{-1}(K) \cap \boldsymbol{y}_n^{-1}(K)) < \varepsilon.$$

Therefore, combining (3.24)–(3.25) and (3.27)–(3.29), we deduce that

$$\limsup_{n\to\infty} \int_\Omega |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}| \, \mathrm{d}\boldsymbol{x} \le 10\varepsilon.$$

As $\varepsilon > 0$ was arbitrary, this concludes the proof of the convergence of compositions in $L^1(\Omega;\mathbb{R}^3)$. The convergence in $L^a(\Omega;\mathbb{R}^3)$ for every $1 < a < \infty$ follows immediately by extracting a subsequence that converges almost everywhere and by applying the dominated convergence theorem. ∎

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. Let $(\boldsymbol{q}_n) \subset \mathcal{Q}$ with $\boldsymbol{q}_n = (\boldsymbol{y}_n, \boldsymbol{m}_n)$ be a minimizing sequence for E, namely such that $E(\boldsymbol{q}_n) \to \inf_{\mathcal{Q}} E$. In particular, $\sup_{n\in\mathbb{N}} E(\boldsymbol{q}_n) < +\infty$. From (3.8), we deduce (3.11) so that we can apply Proposition 3.4. This gives a (not relabeled) subsequence $(\boldsymbol{q}_n)$ and an admissible state $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{Q}$ such that $\boldsymbol{q}_n \to \boldsymbol{q}$ in $\mathcal{Q}$ and $\boldsymbol{m}_n \circ \boldsymbol{y}_n \to \boldsymbol{m} \circ \boldsymbol{y}$ in $L^1(\Omega;\mathbb{R}^3)$. We claim that

$$E(\boldsymbol{q}) \le \liminf_{n\to\infty} E(\boldsymbol{q}_n), \tag{3.30}$$

so that $\boldsymbol{q}$ is a minimizer of E. We focus on the elastic energy first. We have $\nabla \boldsymbol{y}_n \rightharpoonup \nabla \boldsymbol{y}$ in $L^p(\Omega;\mathbb{R}^{3\times 3})$ and, by the weak continuity of Jacobian minors, we also have $\operatorname{cof} \nabla \boldsymbol{y}_n \rightharpoonup \operatorname{cof} \nabla \boldsymbol{y}$ in $L^{p/2}(\Omega;\mathbb{R}^{3\times 3})$ and $\det \nabla \boldsymbol{y}_n \rightharpoonup \det \nabla \boldsymbol{y}$ in $L^{p/3}(\Omega)$. Moreover, the subsequence can be chosen in order to have $\boldsymbol{m}_n \circ \boldsymbol{y}_n \to \boldsymbol{m} \circ \boldsymbol{y}$ almost everywhere in Ω. Thus, given (3.9), applying [2, Theorem 5.4] we prove that

$$E^{\mathrm{el}}(\boldsymbol{q}) \le \liminf_{n\to\infty} E^{\mathrm{el}}(\boldsymbol{q}_n). \tag{3.31}$$

The lower semicontinuity of the exchange energy is immediate. Indeed, by (3.5) and the lower semicontinuity of the norm, there holds

$$E^{\mathrm{exc}}(\boldsymbol{q}) \le \liminf_{n\to\infty} E^{\mathrm{exc}}(\boldsymbol{q}_n). \tag{3.32}$$

We focus on the magnetostatic energy. Denote by ζ_n a weak solution of the Maxwell equation corresponding to $\boldsymbol{q}_n$. Thus, for every $n \in \mathbb{N}$ and for every $\varphi \in V^{1,2}(\mathbb{R}^3)$, there holds

$$\int_{\mathbb{R}^3} \nabla \zeta_n \cdot \nabla \varphi \, \mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{R}^3} \chi_{\Omega^{\boldsymbol{y}_n}} \boldsymbol{m}_n \cdot \nabla \varphi \, \mathrm{d}\boldsymbol{\xi}. \tag{3.33}$$

Denote by $V^{1,2}(\mathbb{R}^3)/\mathbb{R}$ the quotient of $V^{1,2}(\mathbb{R}^3)$ with respect to constant functions and recall that this is a Hilbert space with inner product given by

$$([\varphi],[\psi]) \mapsto \int_{\mathbb{R}^3} \nabla\varphi \cdot \nabla\psi \,\mathrm{d}\boldsymbol{\xi}.$$

Testing (3.33) with $\varphi = \zeta_n$ and using that $\sup_{n\in\mathbb{N}} \|\chi_{\Omega^{\boldsymbol{y}_n}} \boldsymbol{m}_n\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} < +\infty$ by (3.4), we obtain

$$\sup_{n\in\mathbb{N}} \|[\zeta_n]\|_{V^{1,2}(\mathbb{R}^3)/\mathbb{R}} = \sup_{n\in\mathbb{N}} \|\nabla\zeta_n\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} < +\infty.$$

Therefore, there exists $\zeta \in V^{1,2}(\mathbb{R}^3)$ such that, up to subsequences, we have $[\zeta_n] \rightharpoonup [\zeta]$ in $V^{1,2}(\mathbb{R}^3)/\mathbb{R}$, or equivalently, $\nabla\zeta_n \rightharpoonup \nabla\zeta$ in $L^2(\mathbb{R}^3;\mathbb{R}^3)$. Passing to the limit, as $n\to\infty$, in (3.33), we obtain

$$\int_{\mathbb{R}^3} \nabla\zeta \cdot \nabla\varphi \,\mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{R}^3} \chi_{\Omega^{\boldsymbol{y}}} \boldsymbol{m} \cdot \nabla\varphi \,\mathrm{d}\boldsymbol{\xi},$$

for every $\varphi \in V^{1,2}(\mathbb{R})$. Thus ζ is a weak solution of the Maxwell equation corresponding to $\boldsymbol{q}$ and, in turn, $E^{\mathrm{mag}}(\boldsymbol{q}) = (\mu_0/2)\|\nabla\zeta\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)}$. By the lower semicontinuity of the norm, we conclude

$$E^{\mathrm{mag}}(\boldsymbol{q}) \le \liminf_{n\to\infty} E^{\mathrm{mag}}(\boldsymbol{q}_n). \tag{3.34}$$

Finally, combining (3.31)–(3.32) and (3.34), we get (3.30). ∎

4. Quasistatic setting

4.1. General setting

In this section we study quasistatic evolutions of the model driven by time-dependent applied loads and dissipative effects. The framework is the theory of rate-independent processes [44] with the notion of energetic solutions.

We start by describing the general setting. The applied loads are determined by the functions

$$\begin{gathered} \boldsymbol{f} \in C^1([0,T]; L^{p'}(\Omega;\mathbb{R}^3)), \quad \boldsymbol{g} \in C^1([0,T]; L^{p'}(\Sigma;\mathbb{R}^3)), \\ \boldsymbol{h} \in C^1([0,T]; L^2(\mathbb{R}^3;\mathbb{R}^3)), \end{gathered} \tag{4.1}$$

where $\Sigma \subset \partial\Omega$ is $\mathcal{H}^2$-measurable and such that $\Gamma \cap \Sigma = \emptyset$, representing external body forces, surface forces, and magnetic fields, respectively. Define the functional $\mathcal{L}\colon [0,T] \times \mathcal{Q} \to \mathbb{R}$ by setting

$$\mathcal{L}(t,\boldsymbol{q}) := \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} + \int_\Sigma \boldsymbol{g}(t)\cdot\boldsymbol{y}\,\mathrm{d}\mathcal{H}^2 + \int_{\Omega^{\boldsymbol{y}}} \boldsymbol{h}(t)\cdot\boldsymbol{m}\,\mathrm{d}\boldsymbol{\xi}, \tag{4.2}$$

where $\boldsymbol{q} = (\boldsymbol{y},\boldsymbol{m})$. The total energy is given by the functional $\mathcal{E}\colon [0,T]\times\mathcal{Q} \to \mathbb{R}$ defined by

$$\mathcal{E}(t,\boldsymbol{q}) := E(\boldsymbol{q}) - \mathcal{L}(t,\boldsymbol{q}). \tag{4.3}$$

By a repeated application of the Hölder inequality and the Young inequality and using (3.8), we prove

$$\mathcal{E}(t,\boldsymbol{q}) \geq C_0 \|\nabla \boldsymbol{y}\|^p_{L^p(\Omega;\mathbb{R}^{3\times 3})} + \|\gamma(\det \nabla \boldsymbol{y})\|_{L^1(\Omega)} + \alpha \|\nabla \boldsymbol{m}\|^2_{L^2(\Omega^{\boldsymbol{y}};\mathbb{R}^{3\times 3})} - C_1 \tag{4.4}$$

for every $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{Q}$. Here, $C_0(K) > 0$ and $C_1(p, \overline{M}, M_{\boldsymbol{f}}, M_{\boldsymbol{g}}, M_{\boldsymbol{h}}) > 0$ are two constants, where $K > 0$ was introduced in (3.8) and $\overline{M} := \|\bar{\boldsymbol{y}}\|_{L^{p'}(\Sigma;\mathbb{R}^3)}$ takes into account the boundary datum in (3.1). Also, we set

$$M_{\boldsymbol{f}} := \|\boldsymbol{f}\|_{C^0([0,T];L^{p'}(\Omega;\mathbb{R}^3))}, \quad M_{\boldsymbol{g}} := \|\boldsymbol{g}\|_{C^0([0,T];L^{p'}(\Sigma;\mathbb{R}^3))},$$
$$M_{\boldsymbol{h}} := \|\boldsymbol{h}\|_{C^0([0,T];L^2(\mathbb{R}^3;\mathbb{R}^3))}.$$

Note that, from (4.4), we deduce $\inf_{[0,T]\times\mathcal{Q}} \mathcal{E} \geq -C_1$.

Given the regularity of the applied loads, for every $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{Q}$, the map $t \mapsto \mathcal{L}(t, \boldsymbol{q})$ belongs to $C^1([0,T])$. In particular, for every $t \in [0,T]$, we compute

$$\partial_t \mathcal{E}(t,\boldsymbol{q}) = -\partial_t \mathcal{L}(t,\boldsymbol{q}) = -\int_\Omega \dot{\boldsymbol{f}}(t)\cdot \boldsymbol{y}\,\mathrm{d}\boldsymbol{x} - \int_\Sigma \dot{\boldsymbol{g}}(t)\cdot\boldsymbol{y}\,\mathrm{d}\mathcal{H}^2 - \int_{\Omega^{\boldsymbol{y}}} \dot{\boldsymbol{h}}(t)\cdot \boldsymbol{m}\,\mathrm{d}\boldsymbol{\xi}. \tag{4.5}$$

Employing the Hölder inequality and the Young inequality again and exploiting (4.4), we prove the estimate

$$|\partial_t \mathcal{E}(t,\boldsymbol{q})| \leq L(\mathcal{E}(t,\boldsymbol{q}) + M). \tag{4.6}$$

Here, $L(p, K, \overline{M}, L_{\boldsymbol{f}}, L_{\boldsymbol{g}}, L_{\boldsymbol{h}}) > 0$ and $M(p, K, \overline{M}, M_{\boldsymbol{f}}, M_{\boldsymbol{g}}, M_{\boldsymbol{h}}) > 0$ are two constants and we set

$$L_{\boldsymbol{f}} := \|\dot{\boldsymbol{f}}\|_{C^0([0,T];L^{p'}(\Omega;\mathbb{R}^3))}, \quad L_{\boldsymbol{g}} := \|\dot{\boldsymbol{g}}\|_{C^0([0,T];L^{p'}(\Sigma;\mathbb{R}^3))},$$
$$L_{\boldsymbol{h}} := \|\dot{\boldsymbol{h}}\|_{C^0([0,T];L^2(\mathbb{R}^3;\mathbb{R}^3))}.$$

From this, using the Grönwall inequality, we obtain

$$\mathcal{E}(t,\boldsymbol{q}) + M \leq (\mathcal{E}(s,\boldsymbol{q}) + M)e^{L(t-s)} \tag{4.7}$$

for every $\boldsymbol{q} \in \mathcal{Q}$ and $s, t \in [0,T]$ with $s < t$.

As in [50], we introduce the Lagrangian magnetization given, for $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{Q}$, by

$$\mathcal{Z}(\boldsymbol{q}) := (\operatorname{adj} \nabla \boldsymbol{y})\boldsymbol{m} \circ \boldsymbol{y}. \tag{4.8}$$

The dissipation distance $\mathcal{D}\colon \mathcal{Q}\times\mathcal{Q} \to [0,+\infty)$ is defined as

$$\mathcal{D}(\boldsymbol{q},\hat{\boldsymbol{q}}) := \int_\Omega |\mathcal{Z}(\boldsymbol{q}) - \mathcal{Z}(\hat{\boldsymbol{q}})|\,\mathrm{d}\boldsymbol{x}. \tag{4.9}$$

Moreover, the variation of any map $\boldsymbol{q}\colon [0,T] \to \mathcal{Q}$ with respect to $\mathcal{D}$ on the interval $[s,t] \subset [0,T]$ is defined by

$$\operatorname{Var}_{\mathcal{D}}(\boldsymbol{q};[s,t]) := \sup\left\{ \sum_{i=1}^N \mathcal{D}(\boldsymbol{q}(t_i), \boldsymbol{q}(t_{i-1})) : \Pi = (t_0,\dots,t_N) \text{ is a partition of } [s,t] \right\}. \tag{4.10}$$

Here, by a partition of the interval $[s,t]$ we mean any finite ordered set $\Pi = (t_0, \dots, t_N) \subset [0,T]^N$ with $s = t_0 < t_1 < \dots < t_N = t$. Note that in (4.10) each partition can have different cardinality.

Remark 4.1 (Regularity of the applied loads). The regularity assumptions on the applied loads in (4.1) can be relaxed. Indeed, following [44], all the analysis can still be carried out if we just assume

$$\boldsymbol{f} \in W^{1,1}(0,T;L^{p'}(\Omega;\mathbb{R}^3)), \quad \boldsymbol{g} \in W^{1,1}(0,T;L^{p'}(\Sigma;\mathbb{R}^3)),$$
$$\boldsymbol{h} \in W^{1,1}(0,T;L^2(\mathbb{R}^3;\mathbb{R}^3)).$$

Remark 4.2 (Time-dependent boundary conditions). At the current stage, we are not able to treat time-dependent Dirichlet boundary conditions (except for the case in which the boundary datum is given time-by-time by a rigid motion). In particular, the strategy devised in [22] is hindered by the fact that the magnetostatic energy is not differentiable in time. However, time-dependent Dirichlet boundary conditions can be included in the analysis in a relaxed form by removing the boundary condition in (3.1) and by enriching the total energy with the term

$$\boldsymbol{q} \mapsto \int_\Gamma |\boldsymbol{y} - \bar{\boldsymbol{y}}(t)| \, \mathrm{d}\mathcal{H}^2,$$

where $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{Q}$ and $\bar{\boldsymbol{y}} \in C^1([0,T];C^0(\Gamma;\mathbb{R}^3))$ (or just $\bar{\boldsymbol{y}} \in W^{1,1}(0,T;C^0(\Gamma;\mathbb{R}^3))$). From a modeling point of view, in the case in which the material is clamped, this corresponds to also keeping track of deformations of the clamp itself. Additionally, under such relaxed boundary conditions, existence of admissible deformations with finite energy is automatically guaranteed.

The existence of energetic solutions is usually proved in two steps: first, for a given partition of the time interval, one constructs a time-discrete solution by solving the corresponding incremental minimization problem; then one considers the piecewise constant interpolants determined by the time-discrete solutions for a sequence of partitions of vanishing size and, by means of compactness arguments, obtains the desired time-continuous solution.

The first step is addressed by employing the results of Section 3. Let $\Pi = (t_0, \dots, t_N)$ be a partition of $[0,T]$. We consider the incremental minimization problem determined by Π with initial data $\boldsymbol{q}^0 \in \mathcal{Q}$, which reads

$$\begin{aligned} &\text{find } (\boldsymbol{q}^1, \dots, \boldsymbol{q}^N) \in \mathcal{Q}^N \text{ such that each } \boldsymbol{q}^i \text{ is a minimizer} \\ &\quad \text{of } \boldsymbol{q} \mapsto \mathcal{E}(t_i, \boldsymbol{q}) + \mathcal{D}(\boldsymbol{q}^{i-1}, \boldsymbol{q}) \text{ for } i = 1, \dots, N. \end{aligned} \tag{4.11}$$

The next result states the existence of solutions of (4.11) and collects their main properties. Recall the definition of the total energy $\mathcal{E}$ and of the dissipation distance $\mathcal{D}$ in (4.3) and (4.9), respectively.

Proposition 4.3 (Solutions of the incremental minimization problem). *Assume $p > 3$ and $\mathcal{Y} \neq \emptyset$. Suppose that W is continuous and satisfies (3.8)–(3.9), and that the applied loads satisfy (4.1). Let $\Pi = (t_0, \dots, t_N)$ be a partition of $[0,T]$ and let $\boldsymbol{q}^0 \in \mathcal{Q}$. Then the incremental minimization problem (4.11) admits a solution $(\boldsymbol{q}^1, \dots, \boldsymbol{q}^N) \in \mathcal{Q}^N$. Moreover, if $\boldsymbol{q}^0$ is such that*

$$\mathcal{E}(0, \boldsymbol{q}^0) \leq \mathcal{E}(0, \hat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}^0, \hat{\boldsymbol{q}}) \tag{4.12}$$

for every $\hat{\boldsymbol{q}} \in \mathcal{Q}$, then the following holds:

$$\forall i = 1, \dots, N,\ \forall \hat{\boldsymbol{q}} \in \mathcal{Q}, \quad \mathcal{E}(t_i, \boldsymbol{q}^i) \leq \mathcal{E}(t_i, \hat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}^i, \hat{\boldsymbol{q}}), \tag{4.13}$$

$$\forall i = 1, \dots, N, \quad \mathcal{E}(t_i, \boldsymbol{q}^i) - \mathcal{E}(t_{i-1}, \boldsymbol{q}^{i-1}) + \mathcal{D}(\boldsymbol{q}^{i-1}, \boldsymbol{q}^i) \leq \int_{t_{i-1}}^{t_i} \partial_t \mathcal{E}(\tau, \boldsymbol{q}^{i-1})\, \mathrm{d}\tau, \tag{4.14}$$

$$\forall i = 1, \dots, N, \quad \mathcal{E}(t_i, \boldsymbol{q}^i) + M + \sum_{j=1}^{i} \mathcal{D}(\boldsymbol{q}^{j-1}, \boldsymbol{q}^j) \leq (\mathcal{E}(0, \boldsymbol{q}^0) + M) e^{L t_i}. \tag{4.15}$$

Proof. The main point is to prove the existence of solutions of (4.11). Given a solution of (4.11) where $\boldsymbol{q}^0$ satisfies (4.12), then (4.13)–(4.15) are obtained by standard computations as in [43, Theorem 3.2].

It is sufficient to show that, for $\tilde{t} \in [0,T]$ and $\tilde{\boldsymbol{q}} \in \mathcal{Q}$ fixed, the auxiliary functional $\mathcal{F} \colon \mathcal{Q} \to \mathbb{R}$ given by $\mathcal{F}(\boldsymbol{q}) := \mathcal{E}(\tilde{t}, \boldsymbol{q}) + \mathcal{D}(\tilde{\boldsymbol{q}}, \boldsymbol{q})$, admits a minimizer in $\mathcal{Q}$. As $\mathcal{D}$ is positive, from (4.4) we have

$$\mathcal{F}(\boldsymbol{q}) \geq C_0 \|\nabla \boldsymbol{y}\|^p_{L^p(\Omega;\mathbb{R}^{3\times 3})} + \|\gamma(\det \nabla \boldsymbol{y})\|_{L^1(\Omega)} + \alpha \|\nabla \boldsymbol{m}\|^2_{L^2(\Omega^{\boldsymbol{y}};\mathbb{R}^{3\times 3})} - C_1 \tag{4.16}$$

for every $\boldsymbol{q} \in \mathcal{Q}$ with $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m})$. Let $(\boldsymbol{q}_n) \subset \mathcal{Q}$ with $\boldsymbol{q}_n = (\boldsymbol{y}_n, \boldsymbol{m}_n)$ be a minimizing sequence for $\mathcal{F}$, namely such that $\mathcal{F}(\boldsymbol{q}_n) \to \inf_{\mathcal{Q}} \mathcal{F}$. In particular, $\sup_{n \in \mathbb{N}} \mathcal{F}(\boldsymbol{q}_n) < +\infty$, so that (4.16) yields (3.11). By Proposition 3.4, there exists $\boldsymbol{q} \in \mathcal{Q}$ such that, up to subsequences, we have $\boldsymbol{q}_n \to \boldsymbol{q}$ in $\mathcal{Q}$ and $\boldsymbol{m}_n \circ \boldsymbol{y}_n \to \boldsymbol{m} \circ \boldsymbol{y}$ in $L^a(\Omega;\mathbb{R}^3)$ for every $1 \leq a < \infty$. Arguing as in the proof of Theorem 3.2, we prove (3.30) while, exploiting the weak continuity of the trace operator, we get

$$\mathcal{L}(\tilde{t}, \boldsymbol{q}) = \lim_{n\to\infty} \mathcal{L}(\tilde{t}, \boldsymbol{q}_n). \tag{4.17}$$

By the weak continuity of Jacobian minors, $\operatorname{cof} \nabla \boldsymbol{y}_n \rightharpoonup \operatorname{cof} \nabla \boldsymbol{y}$ in $L^{p/2}(\Omega;\mathbb{R}^{3\times 3})$. This, combined with the convergence of $(\boldsymbol{m}_n \circ \boldsymbol{y}_n)$ in $L^{(p/2)'}(\Omega;\mathbb{R}^3)$, yields $\mathcal{Z}(\boldsymbol{q}_n) \rightharpoonup \mathcal{Z}(\boldsymbol{q})$ in $L^1(\Omega;\mathbb{R}^3)$ and, by the lower semicontinuity of the norm, we deduce

$$\mathcal{D}(\tilde{\boldsymbol{q}}, \boldsymbol{q}) \leq \liminf_{n\to\infty} \mathcal{D}(\tilde{\boldsymbol{q}}, \boldsymbol{q}_n). \tag{4.18}$$

Finally, combining (3.30) and (4.17)–(4.18), we obtain

$$\mathcal{F}(\boldsymbol{q}) \leq \liminf_{n\to\infty} \mathcal{F}(\boldsymbol{q}_n),$$

so that $\boldsymbol{q}$ is a minimizer of $\mathcal{F}$. ∎

Unfortunately, in our setting we cannot proceed with the second step of the proof of the existence of energetic solutions. This is due to a lack of compactness in the dissipative variable which is typical of large-strain theories. Therefore, in the next subsection we propose a regularization of the model in the spirit of gradient polyconvexity [5].

4.2. Regularized setting

Henceforth, we regularize the problem as follows. Recalling (3.1), we restrict ourselves to the class of deformations

$$\widetilde{\mathcal{Y}} := \big\{ \boldsymbol{y} \in \mathcal{Y} : \mathrm{cof}\, \nabla \boldsymbol{y} \in BV(\Omega; \mathbb{R}^{3\times 3}) \big\}, \tag{4.19}$$

so that the corresponding class of admissible states is given by

$$\widetilde{\mathcal{Q}} := \big\{ (\boldsymbol{y}, \boldsymbol{m}) : \boldsymbol{y} \in \widetilde{\mathcal{Y}},\ \boldsymbol{m} \in W^{1,2}(\Omega^{\boldsymbol{y}}; \mathbb{S}^2) \big\}. \tag{4.20}$$

Equivalently, in (4.19) we require that the distributional gradient of $\mathrm{cof}\, \nabla \boldsymbol{y}$ is given by a bounded tensor-valued Radon measure $D(\mathrm{cof}\, \nabla \boldsymbol{y}) \in \mathcal{M}_{\mathrm{b}}(\Omega; \mathbb{R}^{3\times 3\times 3})$.

Example 4.4. Let Ω, P, and $\boldsymbol{y}$ be as in Example 2.2 and let Γ and $\bar{\boldsymbol{y}}$ be as in Example 3.1. Then $\boldsymbol{y} \in \widetilde{\mathcal{Y}}$. To see this, for every $\boldsymbol{x} \in \Omega \setminus P$ with $\boldsymbol{x} = (x_1, x_2, x_3)$, we compute

$$\mathrm{cof}\, \nabla \boldsymbol{y}(\boldsymbol{x}) := \begin{pmatrix} |x_1| & 0 & -x_1 x_3/|x_1| \\ 0 & |x_1| & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Set $u(\boldsymbol{x}) := -x_1 x_3/|x_1|$. Then $u \in BV(\Omega)$ since $D_1 u = v \mathcal{H}^2 \llcorner P$, where D_1 denotes the distributional derivative with respect to the first variable and we set $v(\boldsymbol{x}) := 2x_3$. Therefore $\boldsymbol{y} \in \widetilde{\mathcal{Y}}$.

Example 4.5. Define $f\colon [0,1] \to \mathbb{R}$ by setting $f(x) := x^2 \cos^2(\pi/x^2)$ for every $0 < x \leq 1$ and $f(0) := 0$, and let $g\colon [0,1] \to \mathbb{R}$ be given by $g(x) := \int_0^x f(z)\, \mathrm{d}z$. We have $f \in C^0([0,1]) \setminus BV([0,1])$ and $g \in C^1([0,1])$. Moreover, g is strictly increasing and, in turn, injective. Let $\Omega := (0,1)^3$ and define $\boldsymbol{y}\colon \Omega \to \mathbb{R}^3$ by $\boldsymbol{y}(\boldsymbol{x}) := (x_1, x_2, g(x_1) x_3)$, where $\boldsymbol{x} = (x_1, x_2, x_3)$. In this case, $\boldsymbol{y} \in C^1(\overline{\Omega}; \mathbb{R}^3)$ is a homeomorphism and $\det \nabla \boldsymbol{y} > 0$. However,

$$\mathrm{cof}\, \nabla \boldsymbol{y}(\boldsymbol{x}) := \begin{pmatrix} g(x_1) & 0 & -f(x_1) x_3 \\ 0 & g(x_1) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so that $\mathrm{cof}\, \nabla \boldsymbol{y} \notin BV(\Omega; \mathbb{R}^{3\times 3})$. In particular, for $\Gamma := \{1\} \times (0,1)^2$ and $\bar{\boldsymbol{y}} := \mathbf{id}$, there holds $\boldsymbol{y} \in \mathcal{Y} \setminus \widetilde{\mathcal{Y}}$.

Recalling (3.7), the regularized magnetoelastic energy $\widetilde{E}\colon \widetilde{\mathcal{Q}} \to \mathbb{R}$ is given by

$$\widetilde{E}(\boldsymbol{q}) := E(\boldsymbol{q}) + |D(\mathrm{cof}\, \nabla \boldsymbol{y})|(\Omega), \tag{4.21}$$

where $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m})$ and $|D(\operatorname{cof} \nabla \boldsymbol{y})|(\Omega)$ denotes the total variation of the measure $D(\operatorname{cof} \nabla \boldsymbol{y})$ over Ω. The corresponding total energy $\widetilde{\mathcal{E}} \colon [0,T] \times \widetilde{\mathcal{Q}} \to \mathbb{R}$ is defined as

$$\widetilde{\mathcal{E}}(t, \boldsymbol{q}) := \widetilde{E}(\boldsymbol{q}) - \mathcal{L}(t, \boldsymbol{q}), \tag{4.22}$$

where $\mathcal{L}$ is given by (4.2). Also, analogously to (4.7), there holds

$$\widetilde{\mathcal{E}}(t, \boldsymbol{q}) + M \leq (\widetilde{\mathcal{E}}(s, \boldsymbol{q}) + M) e^{L(t-s)} \tag{4.23}$$

for every $\boldsymbol{q} \in \widetilde{\mathcal{Q}}$ and $s, t \in [0,T]$ with $s < t$.

The second main result of the paper states the existence of energetic solutions for the regularized model.

Theorem 4.6 (Existence of energetic solutions). *Assume $p > 3$ and $\widetilde{\mathcal{Y}} \neq \emptyset$. Suppose that W is continuous and satisfies* (3.8)–(3.9)*, and that the applied loads satisfy* (4.1)*. Then, for every $\boldsymbol{q}^0 \in \widetilde{\mathcal{Q}}$ satisfying*

$$\forall \hat{\boldsymbol{q}} \in \widetilde{\mathcal{Q}}, \quad \widetilde{\mathcal{E}}(0, \boldsymbol{q}^0) \leq \widetilde{\mathcal{E}}(0, \hat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}^0, \hat{\boldsymbol{q}}), \tag{4.24}$$

there exists an energetic solution $\boldsymbol{q} \colon [0,T] \to \widetilde{\mathcal{Q}}$ of the regularized model which fulfills the initial condition $\boldsymbol{q}(0) = \boldsymbol{q}^0$. Namely, the following stability condition *and* energy balance *hold:*

$$\forall t \in [0,T], \ \forall \hat{\boldsymbol{q}} \in \widehat{\mathcal{Q}}, \quad \widetilde{\mathcal{E}}(t, \boldsymbol{q}(t)) \leq \widetilde{\mathcal{E}}(t, \hat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}(t), \hat{\boldsymbol{q}}), \tag{4.25}$$

$$\begin{aligned} \forall t \in [0,T], \quad & \widetilde{\mathcal{E}}(t, \boldsymbol{q}(t)) + \operatorname{Var}_{\mathcal{D}}(\boldsymbol{q}; [0,t]) \\ & = \widetilde{\mathcal{E}}(0, \boldsymbol{q}^0) + \int_0^t \partial_t \widetilde{\mathcal{E}}(\tau, \boldsymbol{q}(\tau)) \, \mathrm{d}\tau. \end{aligned} \tag{4.26}$$

Given the highly nonconvex character of the energy, we cannot establish any regularity of the solution. However, by applying a suitable version of the measurable selection lemma, the existence of a measurable energetic solution can be ensured.

As already mentioned, the proof Theorem 4.6 proceeds by time discretization. Let $\Pi = (t_0, \dots, t_N)$ be a partition of $[0,T]$. The regularized incremental minimization problem determined by Π with initial data $\boldsymbol{q}^0 \in \mathcal{Q}$ reads

$$\begin{aligned} &\text{find } (\boldsymbol{q}^1, \dots, \boldsymbol{q}^N) \in \widetilde{\mathcal{Q}}^N \text{ such that each } \boldsymbol{q}^i \text{ is a minimizer} \\ &\quad \text{of } \boldsymbol{q} \mapsto \widetilde{\mathcal{E}}(t_i, \boldsymbol{q}) + \mathcal{D}(\boldsymbol{q}^{i-1}, \boldsymbol{q}) \text{ for } i = 1, \dots, N. \end{aligned} \tag{4.27}$$

The next result provides the analogue of Proposition 4.3 in the regularized setting.

Proposition 4.7 (Solutions of the regularized incremental minimization problem). *Assume $p > 3$ and $\widetilde{\mathcal{Y}} \neq \emptyset$. Suppose that W is continuous and satisfies* (3.8)–(3.9)*, and that the applied loads satisfy* (4.1)*. Let $\Pi = (t_0, \dots, t_N)$ be a partition of $[0,T]$ and let $\boldsymbol{q}^0 \in \widetilde{\mathcal{Q}}$.*

Then the incremental minimization problem (4.27) *admits a solution* $(\boldsymbol{q}^1,\dots,\boldsymbol{q}^N)\in\widetilde{\mathcal{Q}}^N$. *Moreover, if* $\boldsymbol{q}^0$ *satisfies* (4.24)*, then the following holds:*

$$\forall i=1,\dots,N,\ \forall\hat{\boldsymbol{q}}\in\widetilde{\mathcal{Q}},\quad \widetilde{\mathcal{E}}(t_i,\boldsymbol{q}^i)\le\widetilde{\mathcal{E}}(t_i,\hat{\boldsymbol{q}})+\mathcal{D}(\boldsymbol{q}^i,\hat{\boldsymbol{q}}), \tag{4.28}$$

$$\forall i=1,\dots,N,\quad \widetilde{\mathcal{E}}(t_i,\boldsymbol{q}^i)-\widetilde{\mathcal{E}}(t_{i-1},\boldsymbol{q}^{i-1})+\mathcal{D}(\boldsymbol{q}^{i-1},\boldsymbol{q}^i)\le\int_{t_{i-1}}^{t_i}\partial_t\widetilde{\mathcal{E}}(\tau,\boldsymbol{q}^{i-1})\,\mathrm{d}\tau, \tag{4.29}$$

$$\forall i=1,\dots,N,\quad \widetilde{\mathcal{E}}(t_i,\boldsymbol{q}^i)+M+\sum_{j=1}^{i}\mathcal{D}(\boldsymbol{q}^{j-1},\boldsymbol{q}^j)\le(\widetilde{\mathcal{E}}(0,\boldsymbol{q}^0)+M)e^{Lt_i}. \tag{4.30}$$

Proof. Again, the main point is to prove the existence of solutions to (4.27). Hence, we show that the auxiliary functional $\widetilde{\mathcal{F}}\colon\widetilde{\mathcal{Q}}\to\mathbb{R}$ defined by $\widetilde{\mathcal{F}}(\boldsymbol{q}):=\widetilde{\mathcal{E}}(\tilde{t},\boldsymbol{q})+\mathcal{D}(\tilde{\boldsymbol{q}},\boldsymbol{q})$, where $\tilde{t}\in[0,T]$ and $\tilde{\boldsymbol{q}}\in\widetilde{\mathcal{Q}}$ are fixed, admits a minimizer in $\widetilde{\mathcal{Q}}$. The proof goes as that of Proposition 4.3. Let $(\boldsymbol{q}_n)\subset\widetilde{\mathcal{Q}}$ with $\boldsymbol{q}_n=(\boldsymbol{y}_n,\boldsymbol{m}_n)$ be a minimizing sequence for $\widetilde{\mathcal{F}}$, namely such that $\widetilde{\mathcal{F}}(\boldsymbol{q}_n)\to\inf_{\widetilde{\mathcal{Q}}}\widetilde{\mathcal{F}}$. We have $\sup_{n\in\mathbb{N}}\widetilde{\mathcal{F}}(\boldsymbol{q}_n)<+\infty$. From this, exploiting the coercivity in (4.4) and applying Proposition 3.4, we find $\boldsymbol{q}=(\boldsymbol{y},\boldsymbol{m})\in\mathcal{Q}$ such that, up to subsequences, the convergences in (3.4)–(3.6) and (3.12) hold true. These allow us to establish (3.30) and (4.17)–(4.18) as in Proposition 4.3. We also deduce that the sequence $(\operatorname{cof}\nabla\boldsymbol{y}_n)$ is bounded in $BV(\Omega;\mathbb{R}^{3\times3})$. Thus, up to subsequences, there hold

$$\operatorname{cof}\nabla\boldsymbol{y}_n\to\boldsymbol{G}\ \text{ in } L^{3/2}(\Omega;\mathbb{R}^{3\times3}),\quad D(\operatorname{cof}\nabla\boldsymbol{y}_n)\overset{*}{\rightharpoonup}D\boldsymbol{G}\ \text{ in } \mathcal{M}_{\mathrm{b}}(\Omega;\mathbb{R}^{3\times3\times3}) \tag{4.31}$$

for some $\boldsymbol{G}\in BV(\Omega;\mathbb{R}^{3\times3})$. By (3.3) and the weak continuity of Jacobian minors, $\boldsymbol{G}=\operatorname{cof}\nabla\boldsymbol{y}$ and, in particular, $\boldsymbol{y}\in\widetilde{\mathcal{Y}}$ and $\boldsymbol{q}\in\widetilde{\mathcal{Q}}$. Moreover, by the lower semicontinuity of the total variation, we obtain

$$|D(\operatorname{cof}\nabla\boldsymbol{y})|(\Omega)\le\liminf_{n\to\infty}|D(\operatorname{cof}\nabla\boldsymbol{y}_n)|(\Omega).$$

This, combined with (3.30) and (4.17)–(4.18), yields

$$\widetilde{\mathcal{F}}(\boldsymbol{q})\le\liminf_{n\to\infty}\widetilde{\mathcal{F}}(\boldsymbol{q}_n)$$

and, in turn, $\boldsymbol{q}$ is a minimizer of $\widetilde{\mathcal{F}}$. ■

Remark 4.8 (Gradient polyconvexity). The regularization introduced in Section 4.2 makes assumption (3.9) superfluous in the proof of Proposition 4.7, as well as in the rest of our analysis. By (4.31), up to subsequences, we have $\operatorname{cof}\nabla\boldsymbol{y}_n\to\operatorname{cof}\nabla\boldsymbol{y}$ almost everywhere. As observed in [5], this entails $\nabla\boldsymbol{y}_n\to\nabla\boldsymbol{y}$ almost everywhere. Indeed, exploiting the identity $\det(\operatorname{cof}\boldsymbol{F})=(\det\boldsymbol{F})^2$ for every $\boldsymbol{F}\in\mathbb{R}^{3\times3}$, we see that $\det\nabla\boldsymbol{y}_n\to\det\nabla\boldsymbol{y}$ almost everywhere. Then, by the formula $\boldsymbol{F}^{-1}=(\det\boldsymbol{F})^{-1}(\operatorname{adj}\boldsymbol{F})$ for every $\boldsymbol{F}\in\mathbb{R}^{3\times3}_+$, we obtain $(\nabla \boldsymbol{y}_n)^{-1} \to (\nabla \boldsymbol{y})^{-1}$ almost everywhere and the claim follows by the continuity of the map $\boldsymbol{F} \mapsto \boldsymbol{F}^{-1}$ on $\mathbb{R}^{3\times 3}_+$. The almost everywhere convergence of $(\nabla \boldsymbol{y}_n)$ and $(\boldsymbol{m}_n \circ \boldsymbol{y}_n)$, which can be assumed by (3.12), entails the lower semicontinuity of the elastic energy in (3.31) by a simple application of the Fatou lemma. Therefore, assumption (3.9) is actually not necessary in the regularized setting.

In the next proposition, we consider the piecewise-constant interpolants corresponding to solutions of (4.27) and we collect their main properties. Recall the definition of variation with respect to $\mathcal{D}$ in (4.10).

Proposition 4.9 (Piecewise-constant interpolants). *Assume $p > 3$ and $\widetilde{\mathcal{Y}} \neq \emptyset$. Suppose that W is continuous and satisfies (3.8)–(3.9), and that the applied loads satisfy (4.1). Let $\Pi = (t_0, \dots, t_N)$ be a partition of $[0,T]$ and let $\boldsymbol{q}^0 \in \widetilde{\mathcal{Q}}$ satisfy (4.24). Let $(\boldsymbol{q}^1, \dots, \boldsymbol{q}^N) \in \widetilde{\mathcal{Q}}^N$ be a solution of the regularized incremental minimization problem (4.27) and define the (right-continuous) piecewise-constant interpolant $\boldsymbol{q}_\Pi \colon [0,T] \to \widetilde{\mathcal{Q}}$ as*

$$\boldsymbol{q}_\Pi(t) := \begin{cases} \boldsymbol{q}^{i-1} & \text{if } t \in [t_{i-1}, t_i) \text{ for some } i = 1, \dots, N, \\ \boldsymbol{q}^N & \text{if } t = T. \end{cases} \tag{4.32}$$

Then the following holds:

$$\forall t \in \Pi,\ \forall \hat{\boldsymbol{q}} \in \widetilde{\mathcal{Q}}, \quad \widetilde{\mathcal{E}}(t, \boldsymbol{q}_\Pi(t)) \leq \widetilde{\mathcal{E}}(t, \hat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}_\Pi(t), \hat{\boldsymbol{q}}), \tag{4.33}$$

$$\begin{aligned} \forall s,t \in \Pi : s < t, \quad & \widetilde{\mathcal{E}}(t, \boldsymbol{q}_\Pi(t)) - \widetilde{\mathcal{E}}(s, \boldsymbol{q}_\Pi(s)) + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_\Pi; [s,t]) \\ & \leq \int_s^t \partial_t \widetilde{\mathcal{E}}(\tau, \boldsymbol{q}_\Pi(\tau))\, \mathrm{d}\tau, \end{aligned} \tag{4.34}$$

$$\begin{aligned} \forall t \in [0,T], \quad & \widetilde{\mathcal{E}}(t, \boldsymbol{q}_\Pi(t)) + M + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_\Pi; [0,t]) \\ & \leq (\widetilde{\mathcal{E}}(0, \boldsymbol{q}^0) + M) e^{Lt}. \end{aligned} \tag{4.35}$$

Proof. Claims (4.33)–(4.34) follow immediately from (4.28)–(4.29), respectively. We prove (4.35). Let $t \in [0,T]$ and let $i \in \{1, \dots, N\}$ be such that $t_{i-1} \leq t < t_i$. In this case we have

$$\boldsymbol{q}_\Pi(t) = \boldsymbol{q}^{i-1}, \quad \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_\Pi; [0,t]) = \sum_{j=1}^{i-1} \mathcal{D}(\boldsymbol{q}^{j-1}, \boldsymbol{q}^j).$$

Thus, using (4.23) and (4.30), we compute

$$\begin{aligned} & \widetilde{\mathcal{E}}(t, \boldsymbol{q}_\Pi(t)) + M + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_\Pi; [0,t]) \\ & \quad \leq (\widetilde{\mathcal{E}}(t_{i-1}, \boldsymbol{q}^{i-1}) + M) e^{L(t - t_{i-1})} + \sum_{j=1}^{i-1} \mathcal{D}(\boldsymbol{q}^{j-1}, \boldsymbol{q}^j) \\ & \quad \leq \left(\widetilde{\mathcal{E}}(t_{i-1}, \boldsymbol{q}_{i-1}) + M + \sum_{j=1}^{i-1} \mathcal{D}(\boldsymbol{q}^{j-1}, \boldsymbol{q}^j) \right) e^{L(t - t_{i-1})} \\ & \quad \leq (\widetilde{\mathcal{E}}(0, \boldsymbol{q}^0) + M) e^{Lt}. \end{aligned} \tag{4.36}$$

∎

In the proof of Theorem 4.6, we will use the following version of the Helly selection principle, which is a special case of [39, Theorem 3.2].

Lemma 4.10 (Helly selection principle). *Let Z be a Banach space and let $\mathcal{K} \subset Z$ be compact. Let $(\boldsymbol{z}_n) \subset \mathrm{BV}([0,T];Z)$ be such that for every $n \in \mathbb{N}$ there holds*

$$\forall t \in [0,T], \quad \boldsymbol{z}_n(t) \in \mathcal{K} \tag{4.37}$$

and

$$\sup_{n\in\mathbb{N}} \mathrm{Var}(\boldsymbol{z}_n;[0,T]) < +\infty. \tag{4.38}$$

Then there exist a subsequence $(\boldsymbol{z}_{n_k})$ and a map $\boldsymbol{z} \in \mathrm{BV}([0,T];Z)$ such that

$$\forall t \in [0,T], \quad \boldsymbol{z}_{n_k}(t) \to \boldsymbol{z}(t) \text{ in } Z. \tag{4.39}$$

The proof of Theorem 4.6 rigorously follows the well-established scheme introduced in [22]. Therefore, we simply show how to lead the argument back to the original scheme. For additional details we refer to [43, Theorem 5.2].

Proof of Theorem 4.6. Following [43, Theorem 5.2], we divide the proof into five steps.

Step 1 (A priori estimates). Let (Π_n) be a sequence of partitions of $[0,T]$ with $\Pi_n = (t_0^n, \dots, t_{N_n}^n)$ such that $|\Pi_n| := \max\{t_i^n - t_{i-1}^n : i = 1,\dots,N_n\} \to 0$ as $n \to \infty$. For every $n \in \mathbb{N}$, by Proposition 4.7, the incremental minimization problem (4.27) determined by Π_n admits a solution and, by Proposition 4.9, the corresponding piecewise-constant interpolant $\boldsymbol{q}_n := \boldsymbol{q}_{\Pi_n}$ with $\boldsymbol{q}_n = (\boldsymbol{y}_n, \boldsymbol{m}_n)$ defined according to (4.32) satisfies

$$\forall t \in \Pi_n,\ \forall \hat{\boldsymbol{q}} \in \widetilde{\mathcal{Q}}, \quad \widetilde{\mathcal{E}}(t,\boldsymbol{q}_n(t)) \leq \widetilde{\mathcal{E}}(t,\hat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}_n(t),\hat{\boldsymbol{q}}), \tag{4.40}$$

$$\begin{aligned}\forall s,t \in \Pi_n : s<t, \quad & \widetilde{\mathcal{E}}(t,\boldsymbol{q}_n(t)) - \widetilde{\mathcal{E}}(s,\boldsymbol{q}_n(s)) + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_n;[s,t]) \\ & \leq \int_s^t \partial_t \widetilde{\mathcal{E}}(\tau,\boldsymbol{q}_n(\tau))\,\mathrm{d}\tau,\end{aligned} \tag{4.41}$$

$$\begin{aligned}\forall t \in [0,T], \quad & \widetilde{\mathcal{E}}(t,\boldsymbol{q}_n(t)) + M + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_n;[0,t]) \\ & \leq (\widetilde{\mathcal{E}}(0,\boldsymbol{q}^0) + M)e^{Lt}.\end{aligned} \tag{4.42}$$

In particular, from (4.42), we deduce that, for every $n \in \mathbb{N}$, there holds

$$\sup_{n\in\mathbb{N}} \left\{ \sup_{t\in[0,T]} \widetilde{\mathcal{E}}(t,\boldsymbol{q}_n(t)) + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_n;[0,T]) \right\} \leq C \tag{4.43}$$

for some constant $C(\boldsymbol{q}^0, M, L, T) > 0$.

Step 2 (Selection of subsequences). From (4.4) and (4.43), for every $n \in \mathbb{N}$ and $t \in [0,T]$ we have

$$\begin{aligned} &\|\nabla \boldsymbol{y}_n(t)\|_{L^p(\Omega;\mathbb{R}^{3\times3})} \leq C, && \|\nabla \boldsymbol{m}_n(t)\|_{L^2(\Omega^{\boldsymbol{y}_n(t)};\mathbb{R}^{3\times3})} \leq C, \\ &\|\gamma(\det \nabla \boldsymbol{y}_n(t))\|_{L^1(\Omega)} \leq C, && \|D(\mathrm{cof}\,\nabla \boldsymbol{y}_n(t))\|_{\mathcal{M}_b(\Omega;\mathbb{R}^{3\times3\times3})} \leq C. \end{aligned}$$

This shows that each map of the sequence $(\boldsymbol{q}_n)$ takes values in the set $\widetilde{\mathcal{K}} \subset \widetilde{\mathcal{Q}}$ defined as

$$\widetilde{\mathcal{K}} := \big\{\hat{\boldsymbol{q}} = (\hat{\boldsymbol{y}}, \hat{\boldsymbol{m}}) \in \widetilde{\mathcal{Q}} : \|\nabla \hat{\boldsymbol{y}}\|_{L^p(\Omega;\mathbb{R}^{3\times 3})} \leq C,\ \|\nabla \hat{\boldsymbol{m}}\|_{L^2(\Omega^{\hat{\boldsymbol{y}}};\mathbb{R}^{3\times 3})} \leq C,$$
$$\|\gamma(\det \nabla \hat{\boldsymbol{y}})\|_{L^1(\Omega)} \leq C,\ \|D(\operatorname{cof} \nabla \hat{\boldsymbol{y}})\|_{\mathcal{M}_{\mathrm{b}}(\Omega;\mathbb{R}^{3\times 3\times 3})} \leq C\big\}.$$

Applying Proposition 3.4 and arguing as in the proof of Proposition 4.7, we prove the following:

$$\begin{aligned}
&\text{for every } (\hat{\boldsymbol{q}}_n) \subset \widetilde{\mathcal{K}} \text{ with } \hat{\boldsymbol{q}}_n = (\hat{\boldsymbol{y}}_n, \hat{\boldsymbol{m}}_n) \text{ there exist a subsequence } (\hat{\boldsymbol{q}}_{n_k})\\
&\text{and an admissible state } \hat{\boldsymbol{q}} \in \widetilde{\mathcal{Q}} \text{ with } \hat{\boldsymbol{q}} = (\hat{\boldsymbol{y}}, \hat{\boldsymbol{m}}) \text{ such that } \hat{\boldsymbol{q}}_{n_k} \to \hat{\boldsymbol{q}} \text{ in } \widetilde{\mathcal{Q}},\\
&\qquad \hat{\boldsymbol{m}}_{n_k} \circ \hat{\boldsymbol{y}}_{n_k} \to \hat{\boldsymbol{m}} \circ \hat{\boldsymbol{y}} \qquad \text{in } L^a(\Omega;\mathbb{R}^3) \text{ for every } 1 \leq a < \infty,\\
&\qquad \mathcal{Z}(\hat{\boldsymbol{q}}_{n_k}) \to \mathcal{Z}(\hat{\boldsymbol{q}}) \qquad \text{in } L^1(\Omega;\mathbb{R}^3),\\
&\qquad D(\operatorname{cof} \nabla \hat{\boldsymbol{y}}_{n_k}) \overset{*}{\rightharpoonup} D(\operatorname{cof} \nabla \hat{\boldsymbol{y}}) \qquad \text{in } \mathcal{M}_{\mathrm{b}}(\Omega;\mathbb{R}^{3\times 3\times 3}).
\end{aligned} \tag{4.44}$$

In particular, we stress the strong convergence of $(\mathcal{Z}(\hat{\boldsymbol{q}}_{n_k}))$ in $L^1(\Omega;\mathbb{R}^3)$ which follows from the strong convergence of $(\operatorname{cof} \nabla \hat{\boldsymbol{y}}_{n_k})$ in $L^{3/2}(\Omega;\mathbb{R}^3)$ (see (4.31)) and the strong convergence of $(\boldsymbol{m}_{n_k} \circ \boldsymbol{y}_{n_k})$ in $L^3(\Omega;\mathbb{R}^3)$. Thus, the set

$$\mathcal{K} := \big\{\mathcal{Z}(\hat{\boldsymbol{q}}) : \hat{\boldsymbol{q}} \in \widetilde{\mathcal{K}}\big\}$$

is compact with respect to the strong topology of $L^1(\Omega;\mathbb{R}^3)$. Now consider the sequence $(\boldsymbol{z}_n) \subset BV([0,T];L^1(\Omega;\mathbb{R}^3))$ with $\boldsymbol{z}_n(t) := \mathcal{Z}(\boldsymbol{q}_n(t))$ for every $t \in [0,T]$. Setting $Z = L^1(\Omega;\mathbb{R}^3)$, the sequence $(\boldsymbol{z}_n)$ satisfies (4.37) by construction, as the maps of the sequence $(\boldsymbol{q}_n)$ take values in $\widetilde{\mathcal{K}}$, while (4.38) holds in view of (4.43). Thus, by Lemma 4.10, there exist a subsequence $(\boldsymbol{z}_{n_k})$ and a map $\boldsymbol{z} \in BV([0,T];L^1(\Omega;\mathbb{R}^3))$ such that (4.39) holds.

For every $n \in \mathbb{N}$, define $\vartheta_n\colon [0,T] \to \mathbb{R}$ by setting $\vartheta_n(t) := \partial_t \widetilde{\mathcal{E}}(t, \boldsymbol{q}_n(t))$. By (4.6) and (4.43), the sequence (ϑ_n) is bounded in $L^\infty(0,T)$. Hence, up to subsequences, $\vartheta_n \overset{*}{\rightharpoonup} \vartheta$ in $L^\infty(0,T)$ for some $\vartheta \in L^\infty(0,T)$. If we define $\bar{\vartheta}\colon [0,T] \to \mathbb{R}$ as $\bar{\vartheta}(t) := \limsup_{n\to\infty} \vartheta_n(t)$, then $\bar{\vartheta} \in L^\infty(0,T)$ and, by the Fatou lemma, $\vartheta \leq \bar{\vartheta}$.

Finally, for every fixed $t \in [0,T]$, exploiting (4.44), we select a subsequence $(\boldsymbol{q}_{n_{k_\ell^t}}(t))$ and some $\boldsymbol{q}(t) \in \widetilde{\mathcal{Q}}$ with $\boldsymbol{q}(t) = (\boldsymbol{y}(t), \boldsymbol{m}(t))$ such that

$$\begin{aligned}
\boldsymbol{q}_{n_{k_\ell^t}}(t) &\to \boldsymbol{q}(t) && \text{in } \widetilde{\mathcal{Q}},\\
\boldsymbol{m}_{n_{k_\ell^t}} \circ \boldsymbol{y}_{n_{k_\ell^t}}(t) &\to \boldsymbol{m} \circ \boldsymbol{y}(t) && \text{in } L^a(\Omega;\mathbb{R}^3) \text{ for every } 1 \leq a < \infty,\\
\mathcal{Z}(\boldsymbol{q}_{n_{k_\ell^t}}(t)) &\to \mathcal{Z}(\boldsymbol{q}(t)) && \text{in } L^1(\Omega;\mathbb{R}^{3\times 3}),\\
D(\operatorname{cof} \nabla \boldsymbol{y}_{n_{k_\ell^t}}(t)) &\overset{*}{\rightharpoonup} D(\operatorname{cof} \nabla \boldsymbol{y}(t)) && \text{in } \mathcal{M}_{\mathrm{b}}(\Omega;\mathbb{R}^{3\times 3\times 3}).
\end{aligned} \tag{4.45}$$

The candidate solution $\boldsymbol{q}\colon [0,T] \to \widetilde{\mathcal{Q}}$ is pointwise defined by this procedure. Note that, by (4.39), there holds $\boldsymbol{z}(t) = \mathcal{Z}(\boldsymbol{q}(t))$. Also, we choose the subsequence in (4.45) in order to have $\vartheta_{n_{k_\ell^t}}(t) \to \bar{\vartheta}(t)$.

Step 3 *(Stability of the limiting function).* We claim that $\boldsymbol{q}$ satisfies (4.25). Fix $t \in [0, T]$. Henceforth, for simplicity, we will replace the subscripts n_k and $n_{k^t_\ell}$ by k and k^t_ℓ, respectively. For every $k \in \mathbb{N}$, set $\tau_k(t) := \max\{s \in \Pi_k : s \leq t\}$ and note that $\tau_k(t) \to t$, since $|\Pi_k| \to 0$. Then $\boldsymbol{q}_k(t) = \boldsymbol{q}_k(\tau_k(t))$ so that, by (4.28), we have

$$\forall \hat{\boldsymbol{q}} \in \widetilde{\mathcal{Q}}, \quad \widetilde{\mathcal{E}}(\tau_k(t), \boldsymbol{q}_k(t)) \leq \widetilde{\mathcal{E}}(\tau_k(t), \hat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}_k(t), \hat{\boldsymbol{q}}). \tag{4.46}$$

Recall (4.45). Arguing as in the proof of Proposition 4.7 and exploiting the continuity of the applied loads in (4.1), we obtain

$$\widetilde{\mathcal{E}}(t, \boldsymbol{q}(t)) \leq \liminf_{\ell \to \infty} \widetilde{\mathcal{E}}(\tau_{k^t_\ell}(t), \boldsymbol{q}_{k^t_\ell}(t)). \tag{4.47}$$

Moreover, by the continuity of the applied loads in (4.1), there holds

$$\forall \hat{\boldsymbol{q}} \in \widetilde{\mathcal{Q}}, \quad \widetilde{\mathcal{E}}(\tau_{k^t_\ell}(t), \hat{\boldsymbol{q}}) \to \widetilde{\mathcal{E}}(t, \hat{\boldsymbol{q}}), \tag{4.48}$$

while, as $\boldsymbol{z}_{k^t_\ell}(t) \to \boldsymbol{z}(t)$ in $L^1(\Omega; \mathbb{R}^3)$ and $\boldsymbol{z}(t) = \mathcal{Z}(\boldsymbol{q}(t))$, we have

$$\forall \hat{\boldsymbol{q}} \in \widetilde{\mathcal{Q}}, \quad \mathcal{D}(\boldsymbol{q}_{k^t_\ell}(t), \hat{\boldsymbol{q}}) \to \mathcal{D}(\boldsymbol{q}(t), \hat{\boldsymbol{q}}). \tag{4.49}$$

Hence, combining (4.46)–(4.49), we deduce

$$\begin{aligned}
\widetilde{\mathcal{E}}(t, \boldsymbol{q}(t)) &\leq \liminf_{\ell \to \infty} \widetilde{\mathcal{E}}(\tau_{k^t_\ell}(t), \boldsymbol{q}_{k^t_\ell}(t)) \\
&\leq \liminf_{\ell \to \infty} \left\{ \widetilde{\mathcal{E}}(\tau_{k^t_\ell}(t), \hat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}_{k^t_\ell}(t), \hat{\boldsymbol{q}}) \right\} \\
&= \widetilde{\mathcal{E}}(t, \hat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}(t), \hat{\boldsymbol{q}})
\end{aligned}$$

for every $\hat{\boldsymbol{q}} \in \widetilde{\mathcal{Q}}$, which gives (4.25) for t fixed.

Step 4 *(Upper energy estimate).* We claim that $\boldsymbol{q}$ satisfies the upper energy estimate

$$\forall t \in [0, T], \quad \widetilde{\mathcal{E}}(t, \boldsymbol{q}(t)) + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}; [0, t]) \leq \widetilde{\mathcal{E}}(0, \boldsymbol{q}^0) + \int_0^t \partial_t \widetilde{\mathcal{E}}(\tau, \boldsymbol{q}(\tau))\, \mathrm{d}\tau. \tag{4.50}$$

Recall (4.43). For every $n \in \mathbb{N}$, using (4.23) we obtain

$$\begin{aligned}
\forall s, t \in [0, T], \quad |\widetilde{\mathcal{E}}(t, \boldsymbol{q}_n(t)) - \widetilde{\mathcal{E}}(s, \boldsymbol{q}_n(s))| &\leq (C + M)|e^{L|t-s|} - 1| \\
&=: \rho(t - s),
\end{aligned} \tag{4.51}$$

where $\rho(r) \to 0$, as $r \to 0$.

Fix $t \in [0, T]$, so that $\boldsymbol{q}_n(t) = \boldsymbol{q}_n(\tau_n(t))$ and $\mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_n; [0, t]) = \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_n; [0, \tau_n(t)])$ for every $n \in \mathbb{N}$. Recall the definition of ϑ_n in Step 2. By (4.51) and (4.41) we have

$$\begin{aligned}
\widetilde{\mathcal{E}}(t, \boldsymbol{q}_n(t)) + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_n; [0, t]) &\leq \widetilde{\mathcal{E}}(\tau_n(t), \boldsymbol{q}_n(\tau_n(t))) \\
&\quad + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_n; [0, \tau_n(t)]) + \rho(|\Pi_n|) \\
&\leq \widetilde{\mathcal{E}}(0, \boldsymbol{q}^0) + \int_0^{\tau_n(t)} \vartheta_n(\tau)\, \mathrm{d}\tau + \rho(|\Pi_n|)
\end{aligned} \tag{4.52}$$

for every $n \in \mathbb{N}$. Also, by the lower semicontinuity of the total variation we have

$$\begin{aligned}\mathrm{Var}_{\mathcal{D}}(\boldsymbol{q};[0,t]) = \mathrm{Var}(\boldsymbol{z};[0,t]) &\leq \liminf_{n\to\infty} \mathrm{Var}(\boldsymbol{z}_n;[0,t]) \\ &= \liminf_{n\to\infty} \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_n;[0,t]),\end{aligned} \tag{4.53}$$

as (4.39) holds and $\boldsymbol{z}(s) = \mathcal{Z}(\boldsymbol{q}(s))$ for every $s \in [0,T]$. Then, from (4.47) and (4.52)–(4.53), we deduce

$$\begin{aligned}&\widetilde{\mathcal{E}}(t,\boldsymbol{q}(t)) + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q};[0,t]) \\ &\quad\leq \liminf_{\ell\to\infty}\big\{\widetilde{\mathcal{E}}(t,\boldsymbol{q}_{k^t_\ell}(t)) + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_{k^t_\ell};[0,t])\big\} \\ &\quad\leq \widetilde{\mathcal{E}}(0,\boldsymbol{q}^0) + \liminf_{\ell\to\infty}\left\{\int_0^{\tau_{k^t_\ell}(t)} \vartheta_{k^t_\ell}(\tau)\,\mathrm{d}\tau + \rho(|\Pi_{k^t_\ell}|)\right\} \\ &\quad= \widetilde{\mathcal{E}}(0,\boldsymbol{q}^0) + \int_0^t \vartheta(\tau)\,\mathrm{d}\tau \leq \widetilde{\mathcal{E}}(0,\boldsymbol{q}^0) + \int_0^t \bar{\vartheta}(\tau)\,\mathrm{d}\tau,\end{aligned} \tag{4.54}$$

where, in the last line, we used that $\vartheta_k \overset{*}{\rightharpoonup} \vartheta$ in $L^\infty(0,T)$, $\vartheta \leq \bar{\vartheta}$, and $\rho(|\Pi_k|) \to 0$.

We claim that $\bar{\vartheta}(t) = \partial_t \widetilde{\mathcal{E}}(t,\boldsymbol{q}(t))$ for every $t \in (0,T)$. Fix $t \in (0,T)$. Testing (4.25) with $\boldsymbol{q}_{k^t_\ell}(t)$, we have

$$-\widetilde{\mathcal{E}}(t,\boldsymbol{q}_{k^t_\ell}(t)) \leq -\widetilde{\mathcal{E}}(t,\boldsymbol{q}(t)) + \mathcal{D}(\boldsymbol{q}(t),\boldsymbol{q}_{k^t_\ell}(t))$$

so that, using (4.49), we compute

$$\begin{aligned}\limsup_{\ell\to\infty} \widetilde{\mathcal{E}}(t,\boldsymbol{q}_{k^t_\ell}(t)) &= -\liminf_{\ell\to\infty}\big(-\widetilde{\mathcal{E}}(t,\boldsymbol{q}_{k^t_\ell}(t))\big) \\ &\leq -\liminf_{\ell\to\infty}\big(-\widetilde{\mathcal{E}}(t,\boldsymbol{q}(t)) + \mathcal{D}(\boldsymbol{q}(t),\boldsymbol{q}_{k^t_\ell}(t))\big) \\ &\leq \widetilde{\mathcal{E}}(t,\boldsymbol{q}(t)).\end{aligned}$$

Given (4.47), we conclude that $\widetilde{\mathcal{E}}(t,\boldsymbol{q}_{k^t_\ell}(t)) \to \widetilde{\mathcal{E}}(t,\boldsymbol{q}(t))$. Recalling (4.45), by [43, Proposition 5.6], we have

$$\vartheta_{k^t_\ell}(t) = \partial_t \widetilde{\mathcal{E}}(t,\boldsymbol{q}_{k^t_\ell}(t)) \to \partial_t \widetilde{\mathcal{E}}(t,\boldsymbol{q}(t))$$

and, as $\vartheta_{k^t_\ell}(t) \to \bar{\vartheta}(t)$, we deduce $\bar{\vartheta}(t) = \partial_t \widetilde{\mathcal{E}}(t,\boldsymbol{q}(t))$. Therefore, (4.54) gives (4.50) for fixed t.

Step 5 (Lower energy estimate). Finally, we show that $\boldsymbol{q}$ satisfies

$$\forall t \in [0,T], \quad \widetilde{\mathcal{E}}(t,\boldsymbol{q}(t)) + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q};[0,t]) \geq \widetilde{\mathcal{E}}(0,\boldsymbol{q}^0) + \int_0^t \partial_t \widetilde{\mathcal{E}}(\tau,\boldsymbol{q}(\tau))\,\mathrm{d}\tau, \tag{4.55}$$

which, combined with (4.50), proves (4.26).

Note that, by (4.47), we have $\sup_{t\in[0,T]} \widetilde{\mathcal{E}}(t, \boldsymbol{q}(t)) \le C$. Moreover, the function $t \mapsto \partial_t \widetilde{\mathcal{E}}(t, \boldsymbol{q}(t))$ belongs to $L^\infty(0,T)$, as it coincides with $\bar{\vartheta}$. Hence, by [43, Proposition 5.7], for every $s, t \in [0,T]$ with $s < t$ we have

$$\widetilde{\mathcal{E}}(t, \boldsymbol{q}(t)) + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}; [s,t]) \ge \widetilde{\mathcal{E}}(s, \boldsymbol{q}(s)) + \int_s^t \partial_t \widetilde{\mathcal{E}}(\tau, \boldsymbol{q}(\tau))\,\mathrm{d}\tau,$$

which in turn yields (4.55). ∎

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