Ann. Inst. H. Poincaré
Anal. Non Linéaire 40 (2023), 531–555
DOI 10.4171/AIHPC/50

Rigidity of closed CSL submanifolds in the unit sphere

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Abstract. We are concerned with the rigidity of contact stationary Legendrian (CSL) submanifolds, critical points of the volume functional of Legendrian submanifolds in a Sasakian manifold, whose Euler–Lagrange equation is a third-order elliptic PDE. We obtain several optimal rigidity theorems for closed CSL submanifolds in the unit sphere by utilizing the maximum principle together with Simons' identity. In particular, we proved that a closed CSL submanifold $M^n \subset \mathbb{S}^{2n+1}$ is a totally geodesic sphere or a Calabi 2-torus if $|\mathbf{B}|^2 \leq \frac{4(n-1)}{n} + \frac{3n-2}{n^2}|\mathbf{H}|^2$, where $\mathbf{B}$ and $\mathbf{H}$ are the second fundamental form and the mean curvature vector, respectively. Moreover, an example shows that this assumption is optimal.

1. Introduction

1.1. CSL submanifolds in a Sasakian manifold

Let $(\bar{M}^{2n+1}, \bar{\alpha}, \bar{g}_{\bar{\alpha}}, \bar{J})$ be a $(2n+1)$-dimensional contact metric manifold with contact structure $\bar{\alpha}$, associated metric $\bar{g}_{\bar{\alpha}}$ and almost complex structure $\bar{J}$. Assume that (M, g) is an n-dimensional compact Legendrian submanifold of $\bar{M}^{2n+1}$ with the metric g induced from $\bar{g}_{\bar{\alpha}}$. The volume of M is defined by $V(M) := \int_M \mathrm{d}\mu_g$, where $\mathrm{d}\mu_g$ is the volume form of g. A *contact stationary Legendrian submanifold* (briefly, CSL submanifold) of $\bar{M}^{2n+1}$ is a Legendrian submanifold of $\bar{M}^{2n+1}$ which is a stationary point of V with respect to contact deformations. In other words, a CSL submanifold is a stationary point of variation of the volume functional among Legendrian submanifolds. The Euler–Lagrange equation for a CSL submanifold M is [5, 18]

$$\operatorname{div}_g(\bar{J}\mathbf{H}) = 0,$$

where div_g is the divergence operator with respect to g and $\mathbf{H}$ is the mean curvature vector of M in $\bar{M}^{2n+1}$.

Remark 1.1. The notion of a CSL submanifold was first defined by Iriyeh [18] and Castro et al. [5] independently, using the names "Legendrian minimal Legendrian submanifold" and "contact minimal Legendrian submanifold", respectively. In this paper we prefer to use the name "CSL submanifold".

2020 Mathematics Subject Classification. Primary 53C24; Secondary 53C40.
Keywords. Gap theorem, contact stationary Legendrian submanifolds.

The study of CSL submanifolds was motivated by the study of Hamiltonian minimal Lagrangian (briefly, HSL) submanifolds, which was first studied by Oh [39, 40]. An HSL submanifold in a Kähler manifold is a Lagrangian submanifold which is a stationary point of the volume functional under Hamiltonian deformations. By [42], Legendrian submanifolds in a Sasakian manifold $\bar{M}^{2n+1}$ can be seen as links of Lagrangian submanifolds in the cone $C\bar{M}^{2n+1}$, which is a Kähler manifold with proper metric and complex structure. In fact, a close relation between CSL submanifolds and HSL submanifolds was found by Iriyeh [18] and Castro et al. [5]. Precisely, they independently proved that $C(M)$ is an HSL submanifold in $\mathbb{C}^n$ ($n \geq 2$) if and only if M is a CSL submanifold in $\mathbb{S}^{2n-1}$ and M is a CSL submanifold in $\mathbb{S}^{2n+1}$ ($n \geq 1$) if and only if $\Pi(M)$ is an HSL submanifold in $\mathbb{CP}^n$, where $\Pi\colon \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ is the Hopf fibration.

From the definition we see that CSL submanifolds are natural generalizations of minimal Legendrian submanifolds. The study of (nonminimal) CSL submanifolds of $\mathbb{S}^{2n+1}$ is a relatively recent endeavor. For $n = 1$, by [18], CSL curves in $\mathbb{S}^3$ are the so-called (p, q) curves discovered by Schoen and Wolfson [43], where p, q are relatively prime integers. For $n = 2$, since a harmonic 1-form on a 2-sphere must be trivial, CSL 2-spheres in $\mathbb{S}^5$ must be minimal and so must be the equatorial 2-spheres by Yau's result [51]. There are a lot of contact stationary doubly periodic surfaces from $\mathbb{R}^2$ to $\mathbb{S}^5$ by lifting Hélein and Romon's examples [15] and more CSL surfaces (mainly tori) were constructed in [4, 14, 16, 18, 34–36, 38] etc. And for general dimensions, examples were constructed in [3, 7, 11, 12, 20, 24, 37, 40] etc.

1.2. Gap phenomenon of closed minimal submanifolds in the unit sphere

In the theory of minimal submanifolds, the following Simons' integral inequality and pinching theorem due to Simons [45], Lawson [22] and Chern et al. [9] are well known.

Theorem A (Simons, Lawson, Chern–do Carmo–Kobayashi). *Let M^n be a compact minimal submanifold in a unit sphere $\mathbb{S}^{n+p}$ and* $\mathbf{B}$ *the second fundamental form of M in $\mathbb{S}^{n+p}$. Then we have*

$$\int_M |\mathbf{B}|^2\Big(\frac{n}{2-\frac{1}{p}} - |\mathbf{B}|^2\Big)\,\mathrm{d}\mu \leq 0.$$

In particular, if $0 \leq |\mathbf{B}|^2 \leq \frac{n}{2-\frac{1}{p}}$, then either $|\mathbf{B}|^2 = 0$ or $|\mathbf{B}|^2 = \frac{n}{2-\frac{1}{p}}$ and M is the Clifford hypersurface or the Veronese surface in $\mathbb{S}^4$.

Remark 1.2. Usually we call the number $\frac{n}{2-\frac{1}{p}}$ the first gap of minimal submanifolds in a sphere because Chern conjectured that the set of numbers which are the constant square length of the second fundamental form of compact minimal submanifolds in a sphere is discrete.

From the classification of compact minimal submanifolds with $|\mathbf{B}|^2 = \frac{n}{2-\frac{1}{p}}$ we see that the first gap is not optimal except when $p = 1$ or $n = p = 2$. It is interesting to sharpen the first gap for other cases of higher codimension. In this direction there are many studies by several authors (see [1, 13, 44]) and finally Li–Li [25] and Chen–Xu [8] independently proved the following theorem.

Theorem B (Li–Li, Chen–Xu). *Let M^n be a compact minimal submanifold in a unit sphere. Assuming that $|\mathbf{B}|^2 \leq \frac{2n}{3}$, then either $|\mathbf{B}|^2 = 0$ and M is totally geodesic or $n = 2, |\mathbf{B}|^2 = \frac{4}{3}$ and M is the Veronese surface in $\mathbb{S}^4$.*

Remark 1.3. Theorem B was generalized slightly by Lu [31].

Since Legendrian submanifolds are a special class of submanifolds, one may hope to solve the first gap problem for this class of submanifolds. This was done when $n = 2$ (see [50]).

Theorem C (Yamaguchi–Kon–Miyahara). *If Σ is a closed minimal Legendrian surface of the unit sphere $\mathbb{S}^5$ and $0 \leq |\mathbf{B}|^2 \leq 2$, then $|\mathbf{B}|^2$ is identically 0 or 2.*

Remark 1.4. Theorem C is inspired by Yau's Lagrangian version of this result (see [51, Theorem 7]).

The higher-dimensional case of this problem largely remains open and one may see [10, 49] for related results.

1.3. Main results

Besides efforts made to obtain the existence of CSL submanifolds, it is also of interest to understand the properties of these examples. See [17, 21, 41] for progress in this direction. To understand the geometry of CSL submanifolds, and inspired by the first gap problem of closed minimal submanifolds in the unit sphere, Luo [32, 33] studied the first gap problem of CSL surfaces and obtained the following result.

Theorem D (Luo). *Let Σ be a closed contact stationary Legendrian surface in $\mathbb{S}^5$. Assuming that $0 \leq |\mathbf{B}|^2 \leq 2$, then Σ is either totally geodesic or $|\mathbf{B}|^2 = 2$ and Σ is a flat minimal Legendrian torus.*

Remark 1.5. A flat minimal Legendrian torus in $\mathbb{S}^5$ must be a generalized Clifford torus, which also is a minimal Calabi torus as stated in the appendix. For details we refer to [14, p. 853].

The study of the first gap problem for submanifolds satisfying a fourth-order quasi-elliptic nonlinear equation was first carried out by Li: in [26–28], several gap theorems for Willmore submanifolds in a sphere are proved.

In this paper we aim to study this kind of problem further. We will not only generalize Theorem D in dimension 2, but also prove such a result in higher dimensions. The main results of this manuscript are the following theorems:

Theorem 1.1. *Suppose M^n $(n \geq 2)$ is a closed contact stationary Legendrian submanifold of $\mathbb{S}^{2n+1}$ and*

$$|\mathbf{B}|^2 \leq \frac{(n-1)(n+2)}{n} + \frac{n^2+3n-2}{2n^2}|\mathbf{H}|^2 - \frac{(n-1)(n-2)|\mathbf{H}|\sqrt{4n+|\mathbf{H}|^2}}{2n^2}. \quad (1.1)$$

(1) *If $n = 2$, then M is either totally geodesic or a Calabi torus as stated in the appendix.*

(2) *If $n \geq 3$, then M is either minimal or a Calabi product Legendrian immersion of a totally geodesic Legendrian immersion and a point, as stated in the appendix.*

As a consequence, we obtain the following theorem:

Theorem 1.2. *If M^n $(n \geq 3)$ is a closed contact stationary Legendrian submanifold of $\mathbb{S}^{2n+1}$ and*

$$|\mathbf{B}|^2 \leq \frac{4(n-1)}{n} + \frac{3n-2}{n^2}|\mathbf{H}|^2, \quad (1.2)$$

then M is totally geodesic.

Remark 1.6. According to examples of Calabi product Legendrian immersion of a totally geodesic Legendrian immersion and a point, we see that both Theorem 1.1 and Theorem 1.2 are optimal.

By a similar argument we can obtain similar results for Hamiltonian minimal submanifolds in $\mathbb{CP}^n$, which we would like to omit here.

The rest of this paper will be organized as follows. In Section 2 we give some preliminaries on the Sasakian geometry, CSL submanifolds in a sphere and prove several important lemmas which will be useful in the remaining sections. In Sections 3 and 4 we give a complete proof of Theorem 1.1. Actually in Section 3 we get stronger results in the surface case. Theorem 1.2 is proved in Section 5, where we also prove more results and propose several conjectures. In the appendix, we state the examples which are not only used in the statement of our theorems, but also illustrate that both Theorem 1.1 and Theorem 1.2 are optimal.

2. Preliminaries

In this section we recall some basic material from contact geometry and submanifold geometry. For more information we refer to [2, 48].

2.1. Contact manifolds

Definition 2.1. A contact manifold $\bar{M}$ is an odd-dimensional manifold with a 1-form $\bar{\alpha}$ such that $\bar{\alpha} \wedge (d\bar{\alpha})^n \neq 0$, where $\dim \bar{M} = 2n + 1$.

Assume now that $(\bar{M}, \bar{\alpha})$ is a given contact manifold of dimension $2n+1$. Then $\bar{\alpha}$ defines a $2n$-dimensional vector bundle over $\bar{M}$, where the fiber at each point $p \in \bar{M}$ is given by

$$\bar{\xi}_p = \ker \bar{\alpha}_p.$$

Since $\bar{\alpha} \wedge (\mathrm{d}\bar{\alpha})^n$ defines a volume form on $\bar{M}$, we see that

$$\bar{\omega} := \mathrm{d}\bar{\alpha}$$

is a closed nondegenerate 2-form on $\bar{\xi} \oplus \bar{\xi}$ and hence it defines a symplectic product on $\bar{\xi}$ such that $(\bar{\xi}, \bar{\omega}|_{\bar{\xi}\oplus\bar{\xi}})$ becomes a symplectic vector bundle. A consequence of this fact is that there exists an almost complex bundle structure

$$\tilde{J} : \bar{\xi} \to \bar{\xi}$$

compatible with $\mathrm{d}\bar{\alpha}$, i.e. a bundle endomorphism satisfying

(1) $\tilde{J}^2 = -\mathrm{id}_{\bar{\xi}}$,

(2) $\mathrm{d}\bar{\alpha}(\tilde{J}X, \tilde{J}Y) = \mathrm{d}\bar{\alpha}(X, Y)$ for all $X, Y \in \bar{\xi}$,

(3) $\mathrm{d}\bar{\alpha}(X, \tilde{J}X) > 0$ for $X \in \bar{\xi} \setminus 0$.

Since $\bar{M}$ is an odd-dimensional manifold, $\bar{\omega}$ must be degenerate on $T\bar{M}$, and so we obtain a line bundle $\bar{\eta}$ over $\bar{M}$ with fibers

$$\bar{\eta}_p := \left\{ V \in T_p\bar{M} \;\middle|\; \bar{\omega}(V, W) = 0 \; \forall W \in \bar{\xi}_p \right\}.$$

Definition 2.2. The Reeb vector field $\bar{\mathbf{R}}$ is the section of $\bar{\eta}$ such that $\bar{\alpha}(\bar{\mathbf{R}}) = 1$.

Thus $\bar{\alpha}$ defines a splitting of $T\bar{M}$ into a line bundle $\bar{\eta}$ with the canonical section $\bar{\mathbf{R}}$ and a symplectic vector bundle $(\bar{\xi}, \bar{\omega}|\bar{\xi} \oplus \bar{\xi})$. We denote the projection along $\bar{\eta}$ by $\bar{\pi}$, i.e.

$$\bar{\pi} : T\bar{M} \to \bar{\xi}, \quad V \mapsto \bar{\pi}(V) := V - \bar{\alpha}(V)\bar{\mathbf{R}}.$$

Using this projection we extend the almost complex structure $\tilde{J}$ to a section $\bar{J} \in \Gamma(T^*\bar{M} \otimes T\bar{M})$ by setting

$$\bar{J}(V) := \tilde{J}(\pi(V))$$

for $V \in T\bar{M}$.

We call $\bar{J}$ an almost complex structure of the contact manifold $\bar{M}$.

Definition 2.3. Let $(\bar{M}, \bar{\alpha})$ be a contact manifold. Then a submanifold M of $(\bar{M}, \bar{\alpha})$ is called an isotropic submanifold if $T_xM \subseteq \bar{\xi}_x$ for all $x \in M$.

For algebraic reasons the dimension of an isotropic submanifold of a $(2n+1)$-dimensional contact manifold cannot be bigger than n.

Definition 2.4. An isotropic submanifold $M \subseteq (\bar{M}, \bar{\alpha})$ of maximal possible dimension n is called a Legendrian submanifold.

2.2. Sasakian manifolds

Let $(\bar{M}, \bar{\alpha})$ be a contact manifold, with the almost complex structure $\bar{J}$ and Reeb field $\bar{\mathbf{R}}$. A Riemannian metric $\bar{g}_{\bar{\alpha}}$ defined on $\bar{M}$ is said to be associated if it satisfies the following three conditions:

(1) $\bar{g}_{\bar{\alpha}}(\bar{\mathbf{R}}, \bar{\mathbf{R}}) = 1$,

(2) $\bar{g}_{\bar{\alpha}}(V, \bar{\mathbf{R}}) = 0$ for all $V \in \bar{\xi}$,

(3) $\bar{\omega}(V, \bar{J}W) = \bar{g}_{\bar{\alpha}}(V, W)$ for all $V, W \in \bar{\xi}$.

We should mention here that on any contact manifold there exists an associated metric on it.

Sasakian manifolds are the odd-dimensional analogue of Kähler manifolds. They are defined as follows.

Definition 2.5. A contact manifold $(\bar{M}, \bar{\alpha})$ with an associated metric $\bar{g}_{\bar{\alpha}}$ is called Sasakian if the cone $C\bar{M}$ equipped with the extended metric $\bar{\bar{g}}$,

$$(C\bar{M}, \bar{\bar{g}}) = (\mathbb{R}_+ \times \bar{M}, \mathrm{d}r^2 + r^2 \bar{g}_{\bar{\alpha}})$$

is Kähler with respect to the following canonical almost complex structure $\bar{\bar{J}}$ on $TC\bar{M} = \mathbb{R} \oplus \langle \bar{\mathbf{R}} \rangle \oplus \bar{\xi}$:

$$\bar{\bar{J}}(r\partial r) = \bar{\mathbf{R}}, \quad \bar{\bar{J}}(\bar{\mathbf{R}}) = -r\partial r.$$

Furthermore, if $\bar{g}_{\bar{\alpha}}$ is Einstein, $\bar{M}$ is called a Sasakian Einstein manifold.

We recall several lemmas which are well known in Sasakian geometry. These lemmas will be used in the next section.

Lemma 2.1. *Let $(\bar{M}, \bar{\alpha}, \bar{g}_{\bar{\alpha}}, \bar{J})$ be a Sasakian manifold. Then*

$$\bar{\nabla}_X \bar{\mathbf{R}} = \bar{J}X$$

for $X, Y \in TM$, where $\bar{\nabla}$ is the Levi-Civita connection on $(\bar{M}, \bar{g}_{\bar{\alpha}})$.

Lemma 2.2. *Let M be a Legendrian submanifold in a Sasakian Einstein manifold $(\bar{M}, \bar{\alpha}, \bar{g}_{\bar{\alpha}}, \bar{J})$. Then the mean curvature form $\bar{\omega}(\mathbf{H}, \cdot)|_M$ defines a closed 1-form on M.*

For a proof of this lemma we refer to [23, Proposition A.2] or [46, Lemma 2.8]. In fact, they proved this result under a weaker assumption that $(\bar{M}, \bar{\alpha}, \bar{g}_{\bar{\alpha}}, \bar{J})$ is a weakly Sasakian Einstein manifold, where weakly Einstein means that $\bar{g}_{\bar{\alpha}}$ is Einstein only when restricted to the contact hyperplane $\ker \bar{\alpha}$.

Lemma 2.3. *Let M be a Legendrian submanifold in a Sasakian manifold $(\bar{M}, \bar{\alpha}, \bar{g}_{\bar{\alpha}}, \bar{J})$ and $\mathbf{B}$ be the second fundamental form of M in $\bar{M}$. Then we have*

$$\bar{g}_{\bar{\alpha}}(\mathbf{B}(X, Y), \bar{\mathbf{R}}) = 0$$

for any $X, Y \in TM$.

In particular, this lemma implies that the mean curvature $\mathbf{H}$ of M is orthogonal to the Reeb field $\bar{\mathbf{R}}$.

Lemma 2.4. *For any $Y, Z \in \ker \bar{\alpha}$, we have*

$$\bar{g}_{\bar{\alpha}}(\bar{\nabla}_X(\bar{J}Y), Z) = \bar{g}_{\bar{\alpha}}(\bar{J}\bar{\nabla}_X Y, Z).$$

A most canonical example of Sasakian Einstein manifolds is the standard odd-dimensional sphere $\mathbb{S}^{2n+1}$.

The standard sphere $\mathbb{S}^{2n+1}$. Let $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ be the Euclidean space with coordinates $(x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1})$ and $\mathbb{S}^{2n+1}$ be the standard unit sphere in $\mathbb{R}^{2n+2}$. Defining

$$\alpha_0 = \frac{1}{2}\sum_{j=1}^{n+1}(x_j\,\mathrm{d}y_j - y_j\,\mathrm{d}x_j),$$

then

$$\bar{\alpha} := \alpha_0|_{\mathbb{S}^{2n+1}}$$

defines a contact 1-form on $\mathbb{S}^{2n+1}$. Assume that g_0 is the standard metric on $\mathbb{R}^{2n+2}$ and J is the standard complex structure of $\mathbb{C}^{n+1}$. Defining

$$\bar{g}_{\bar{\alpha}} = g_0|_{\mathbb{S}^{2n+1}}, \quad \bar{J} = J|_{\mathbb{S}^{2n+1}},$$

then $(\mathbb{S}^{2n+1}, \bar{\alpha}, \bar{g}_{\bar{\alpha}}, \bar{J})$ is a Sasakian Einstein manifold with associated metric $\bar{g}_{\bar{\alpha}}$. Its contact hyperplane is characterized by

$$\ker \bar{\alpha}_x = \{Y \in T_x\mathbb{S}^{2n+1} \mid \langle Y, J\mathbf{x}\rangle = 0\}.$$

2.3. CSL submanifolds in the unit sphere

Assume $\phi: M^n \to \mathbb{S}^{2n+1} \subseteq \mathbb{C}^{n+1}$ is a Legendrian immersion. The shape operator $\mathbf{A}^\nu$ with respect to the normal vector $\nu \in T^\perp M$ is a symmetric operator on the tangent bundle and satisfies the Weingarten equations

$$\langle \mathbf{B}(X, Y), \nu\rangle = \langle \mathbf{A}^\nu(X), Y\rangle \quad \forall X, Y \in TM,\ \nu \in T^\perp M.$$

The Gauss equations, Codazzi equations and Ricci equations are given by

$$\begin{aligned} R(X, Y, Z, W) &= \langle X, Z\rangle\langle Y, W\rangle - \langle X, W\rangle\langle Y, Z\rangle + \langle \mathbf{B}(X, Z), \mathbf{B}(Y, W)\rangle \\ &\quad - \langle \mathbf{B}(X, W), \mathbf{B}(Y, Z)\rangle, \\ (\nabla^\perp_X \mathbf{B})(Y, Z) &= (\nabla^\perp_Y \mathbf{B})(X, Z), \\ R^\perp(X, Y, \mu, \nu) &= \langle \mathbf{A}^\mu(X), \mathbf{A}^\nu(Y)\rangle - \langle \mathbf{A}^\mu(Y), \mathbf{A}^\nu(X)\rangle, \end{aligned}$$

where $X, Y, Z, W \in TM$, $\mu, \nu \in T^\perp M$.

Let $\{e_i\}$ be a local orthonormal frame of M. Then $\{Je_i, J\phi\}$ is a local orthonormal frame of the normal bundle $T^\perp M$, where J is the complex structure of $\mathbb{C}^{n+1}$. Recall that M is CSL if and only if

$$\operatorname{div}(J\mathbf{H}) = 0.$$

It is obvious that M is CSL when M is minimal.

Notice that for all $X, Y, Z \in \Gamma(TM)$, by Lemma 2.1 we see

$$(\nabla_Z^\perp \mathbf{B})(X, Y) = -J(\nabla_Z(J\mathbf{B}))(X, Y) + \langle Z, J\mathbf{B}(X, Y)\rangle J\phi.$$

Thus,

$$\nabla_X^\perp \mathbf{H} = -J\nabla_X(J\mathbf{H}) + \langle X, J\mathbf{H}\rangle J\phi,$$
$$\operatorname{div}(J\mathbf{H}) = -\sum_{i=1}^{n}\langle \nabla_{e_i}^\perp \mathbf{H}, Je_i\rangle.$$

As an immediate consequence, there is no closed nonminimal CMC Legendrian submanifold in $\mathbb{S}^{2n+1}$. Moreover, M is CSL if and only if

$$\sum_{i=1}^{n}\langle \nabla_{e_i}^\perp \mathbf{H}, Je_i\rangle = 0.$$

Set $\sigma_{ijk} := \langle \mathbf{B}(e_i, e_j), Je_k\rangle$ and $\mu_j := \langle \mathbf{H}, Je_j\rangle = \sum_{i=1}^n \sigma_{iij}$ $(1 \le i, j, k \le n)$. Then

$$|\mathbf{B}|^2 = |\sigma|^2, \quad |\mathbf{H}|^2 = |\mu|^2,$$
$$|\nabla^\perp \mathbf{B}|^2 = |\nabla\sigma|^2 + |\sigma|^2, \quad |\nabla^\perp \mathbf{H}|^2 = |\nabla\mu|^2 + |\mu|^2.$$

Moreover, by Lemma 2.4, the Codazzi equation and Lemma 2.2 we have

$$\sigma_{ijk} = \sigma_{jik} = \sigma_{ikj}, \quad \sigma_{ijk,l} = \sigma_{ijl,k},$$
$$\mathrm{d}\mu = 0, \quad \delta\mu = \operatorname{div}(J\mathbf{H}).$$

Therefore we have the following lemma:

Lemma 2.5. *The submanifold M is CSL if and only if μ is a harmonic 1-form if and only if $J\mathbf{H}$ is a harmonic vector field.*

By using the Bochner formula for harmonic vector fields (cf. [19]), we get the following lemma:

Lemma 2.6. *If M is CSL, then*

$$\frac{1}{2}\Delta|\mathbf{H}|^2 = |\nabla(J\mathbf{H})|^2 + \operatorname{Ric}(J\mathbf{H}, J\mathbf{H}).$$

From Lemma 2.6, one can check the following lemma:

Lemma 2.7. *If $\Sigma \subset \mathbb{S}^5$ is CSL and nonminimal, then the zero set of* $\mathbf{H}$ *is isolated and*

$$\Delta \log|\mathbf{H}| = \kappa$$

provided $\mathbf{H} \neq 0$, *where* κ *is the Gauss curvature of* Σ.

We will need the following Simons' identity (cf. [45]; see also [6, 49]).

Lemma 2.8 (Simons' identity). *Assume that M^n is a Legendrian submanifold in $\mathbb{S}^{2n+1}$. Then*

$$\begin{aligned}
\Delta\sigma_{ijk} &:= \sum_l \sigma_{ijk,ll} \\
&= \mu_{i,jk} - \mu_i\delta_{jk} - \mu_j\delta_{ik} + \sum_{s,t}\sigma_{ijt}\sigma_{tks}\mu_s \\
&\quad + (n+1)\sigma_{ijk} + 2\sum_{l,s,t}\sigma_{isl}\sigma_{jlt}\sigma_{kts} - \sum_{l,s,t}\sigma_{tli}\sigma_{tls}\sigma_{jks} \\
&\quad - \sum_{l,s,t}\sigma_{tlj}\sigma_{tls}\sigma_{iks} - \sum_{l,s,t}\sigma_{tlk}\sigma_{tls}\sigma_{ijs}.
\end{aligned} \tag{2.1}$$

Consequently,

$$\Delta\mu_k := \sum_l \mu_{k,ll} = \sum_i \mu_{i,ik} + (n-1)\mu_k + \sum_{s,t}\sigma_{tsk}\mu_t\mu_s - \sum_{l,s,t}\sigma_{tlk}\sigma_{tls}\mu_s.$$

Proof. The Ricci identity yields

$$\sigma_{ijk,lm} = \sigma_{ijk,ml} + \sum_t \sigma_{tjk}R_{tilm} + \sum_t \sigma_{itk}R_{tjlm} + \sum_t \sigma_{ijt}R_{tklm}.$$

Therefore,

$$\begin{aligned}
\Delta\sigma_{ijk} &= \sum_l \sigma_{ijk,ll} \\
&= \sum_l \sigma_{ijl,kl} \\
&= \sum_l \sigma_{ijl,lk} + \sum_{l,t}\sigma_{tjl}R_{tikl} + \sum_{l,t}\sigma_{itl}R_{tjkl} + \sum_{l,t}\sigma_{ijt}R_{tlkl} \\
&= \mu_{i,jk} + \sum_{l,t}\sigma_{tjl}R_{tikl} + \sum_{l,t}\sigma_{itl}R_{tjkl} + \sum_{l,t}\sigma_{ijt}R_{tlkl}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\Delta\sigma_{ijk} &= \mu_{i,jk} + \sum_{l,t}\sigma_{tjl}(\delta_{tk}\delta_{il} - \delta_{tl}\delta_{ik} + \sigma_{tks}\sigma_{ils} - \sigma_{tls}\sigma_{iks}) \\
&\quad + \sum_{l,t}\sigma_{til}(\delta_{tk}\delta_{jl} - \delta_{tl}\delta_{jk} + \sigma_{tks}\sigma_{jls} - \sigma_{tls}\sigma_{jks}) \\
&\quad + \sum_{l,t}\sigma_{ijt}((n-1)\delta_{tk} + \sigma_{tks}\sigma_{lls} - \sigma_{tls}\sigma_{lks})
\end{aligned}$$

$$
\begin{aligned}
&= \mu_{i,jk} + \sigma_{ijk} - \mu_j\delta_{ik} + \sum_{l,s,t}\sigma_{tjl}(\sigma_{tks}\sigma_{ils} - \sigma_{tls}\sigma_{iks}) \\
&\quad + \sigma_{ijk} - \mu_i\delta_{jk} + \sum_{l,s,t}\sigma_{til}(\sigma_{tks}\sigma_{jls} - \sigma_{tls}\sigma_{jks}) \\
&\quad + (n-1)\sigma_{ijk} + \sum_{l,s,t}\sigma_{ijt}(\sigma_{tks}\mu_s - \sigma_{tls}\sigma_{lks}) \\
&= \mu_{i,jk} - \mu_i\delta_{jk} - \mu_j\delta_{ik} + \sum_{s,t}\sigma_{ijt}\sigma_{tks}\mu_s \\
&\quad + (n+1)\sigma_{ijk} + 2\sum_{l,s,t}\sigma_{tjl}\sigma_{tks}\sigma_{ils} - \sum_{l,s,t}\sigma_{tjl}\sigma_{tls}\sigma_{iks} \\
&\quad - \sum_{l,s,t}\sigma_{til}\sigma_{tls}\sigma_{jks} - \sum_{l,s,t}\sigma_{tls}\sigma_{lks}\sigma_{ijt}. \qquad \blacksquare
\end{aligned}
$$

3. Rigidity results for closed CSL surfaces in the unit sphere

In this section we assume $\Sigma \subset \mathbb{S}^5$ is a closed CSL surface.

Lemma 3.1 (Cf. [51]). *If Σ is minimal and not totally geodesic, then the zero set of $\mathbf{B}$ is isolated and*

$$\Delta \log|\mathbf{B}| = 3\kappa$$

provided $\mathbf{B} \neq 0$.

Corollary 3.2. *If $\Sigma \subset \mathbb{S}^5$ is a closed minimal Legendre surface with constant Gauss curvature, then Σ is either totally geodesic or a Calabi torus as stated in the appendix.*

Proof. By the Gauss equation, $2\kappa = 2 - |\mathbf{B}|^2$, we know that $|\mathbf{B}|^2$ is a constant. According to Lemma 3.1, we know that either $\mathbf{B} \equiv 0$ or $\kappa \equiv 0$. Thus Σ is either the totally geodesic sphere or a flat minimal Legendrian torus. ∎

More generally, we have the following proposition:

Proposition 3.3. *Assume that $\Sigma \subset \mathbb{S}^5$ is a closed nonminimal Legendre surface with $\nabla(J\mathbf{H}) = 0$. Then Σ is a Calabi torus as stated in the appendix.*

Proof. Denote $e_1 = \frac{J\mathbf{H}}{|\mathbf{H}|}$ and let $\{e_1, e_2\}$ be the global orthonormal frame of $T\Sigma$. We consider the function $f := 3\sigma_{111} - 2\mu_1 = \sigma_{111} - 2\sigma_{122}$, where $\mu_1 = |\mathbf{H}|$ is a positive constant. The Simons' identity (2.1) gives

$$
\begin{aligned}
\frac{1}{3}\Delta f &= -2\mu_1 + \sum_t \sigma_{11t}^2\mu_1 + 3\sigma_{111} + 2\sum_{l,s,t}\sigma_{1sl}\sigma_{1lt}\sigma_{1ts} - 3\sum_{l,s,t}\sigma_{tl1}\sigma_{tls}\sigma_{11s} \\
&= f + \sum_t \sigma_{11t}^2\mu_1 + 2\sum_{l,s,t}\sigma_{1sl}\sigma_{1lt}\sigma_{1ts} - 3\sum_{l,t}\sigma_{1tl}^2\sigma_{111} - 3\sum_{l,t}\sigma_{1tl}\sigma_{2tl}\sigma_{112}.
\end{aligned}
$$

On one hand, noticing that $0 = \mu_2 = \sigma_{112} + \sigma_{222}$, we have

$$\sum_{l,t} \sigma_{1tl}\sigma_{2tl} = \sigma_{111}\sigma_{211} + 2\sigma_{112}\sigma_{212} + \sigma_{122}\sigma_{222} \\ = (\sigma_{111} + \sigma_{122})\sigma_{112}.$$

On the other hand, assuming that the eigenvalues of the symmetric matrix $(\sigma_{1tl})_{1\le t,l\le 2}$ are c_1, c_2, then

$$\sum_{s,l,t} \sigma_{1sl}\sigma_{1lt}\sigma_{1ts} = c_1^3 + c_2^3 \\ = (c_1 + c_2)(c_1^2 + c_2^2 - c_1c_2) \\ = (\sigma_{111} + \sigma_{122})\Big(\sum_{l,t=1}^{2} \sigma_{1tl}^2 - (\sigma_{111}\sigma_{122} - \sigma_{112}^2)\Big).$$

Thus,

$$2\sum_{l,s,t} \sigma_{1sl}\sigma_{1lt}\sigma_{1ts} - 3\sum_{l,t} \sigma_{1tl}^2\sigma_{111} - 3\sum_{l,t} \sigma_{1tl}\sigma_{2tl}\sigma_{112} \\ = (2\sigma_{122} - \sigma_{111})\sum_{l,t=1}^{2} \sigma_{1tl}^2 - 2\mu_1\sigma_{111}\sigma_{122} - \mu_1\sigma_{112}^2.$$

Therefore,

$$\frac{1}{3}\Delta f = f + \sigma_{111}^2\mu_1 - f\sum_{t,l=1}^{2} \sigma_{1tl}^2 - 2\mu_1\sigma_{111}\sigma_{122} \\ = f\Big[1 + \sigma_{111}\mu_1 - \sum_{t,l} \sigma_{1tl}^2\Big] \\ = f\Delta\mu_1 \\ = 0.$$

Consequently, f is a constant. We conclude that both σ_{111} and σ_{122} are constants. From Lemma 2.7 we see that $\kappa = 0$, which implies from the Gauss equation $2\kappa = 2 + |\mathbf{H}|^2 - |\mathbf{B}|^2$ that $|\mathbf{B}|^2$ is a constant, and we get that $\sigma_{112} = -\sigma_{222}$ is a constant. Up to now, we show that σ is a covariant constant (see also [51]).

We want to show that Σ is a Calabi torus as defined in the appendix.

At a point $p \in \Sigma$, we choose an orthonormal frame $\{e_1, e_2\}$ on $T_p\Sigma$ such that

$$\sigma(e_1, e_1, e_1) = \max_{|X|=1, X\in T_pM} \sigma(X, X, X).$$

Then since

$$f(t) := \sigma(\cos(t)e_1 + \sin(t)e_2, \cos(t)e_1 + \sin(t)e_2, \cos(t)e_1 + \sin(t)e_2)$$

achieves its maximum value at $t = 0$, we see that $f'(0) = 0$, which implies that $\sigma_{112}(p) = 0$. Since $\nabla(J\mathbf{H}) = 0$, we see that the unit smooth orthogonal vector field of $J\mathbf{H}$, say υ, is also parallel. Remember that we have proved σ is a parallel 3-symmetric tensor. Assume that $(e_1, e_2)(p) = D(J\mathbf{H}, \upsilon)(p)$, where D is a constant matrix. Then we extend $\{e_1, e_2\}$ to get a global orthonormal tangent vector frame on Σ by $(E_1, E_2) := D(J\mathbf{H}, \upsilon)$. Moreover, E_1 and E_2 are two unit parallel vector fields on Σ.

We claim that

$$1 + \sigma_{111}\sigma_{122} - \sigma_{122}^2 = 0. \tag{3.1}$$

Assume that $\{\omega_1, w_2\}$ is the dual of $\{E_1, E_2\}$. Then the connection coefficient ω_{12} of Σ equals zero since E_1 is parallel. Then

$$\begin{aligned} 0 &= d\omega_{12} \\ &= -\omega_{13} \wedge \omega_{32} - \omega_{14} \wedge \omega_{42} + \omega_1 \wedge \omega_2 \\ &= \sum_{j,k} \sigma_{11j}\sigma_{12k}\omega_j \wedge \omega_k + \sum_{j,k} \sigma_{21j}\sigma_{22k}\omega_j \wedge \omega_k + \omega_1 \wedge \omega_2 \\ &= (1 + \sigma_{111}\sigma_{122} - \sigma_{122}^2)\omega_1 \wedge \omega_2. \end{aligned}$$

Due to (3.1), we choose four nonzero constants r_1, r_2, r_3, r_4 such that $r_1^2 + r_2^2 = r_3^2 + r_4^2 = 1$ and

$$\sigma_{111} = \frac{r_2}{r_1} - \frac{r_1}{r_2}, \quad \sigma_{122} = \frac{r_2}{r_1}, \quad \sigma_{222} = \frac{1}{r_1}\Big(\frac{r_4}{r_3} - \frac{r_3}{r_4}\Big).$$

Comparing with the Calabi torus defined by r_1, r_2, r_3, r_4 as stated in the appendix, we know that Σ is locally isometric to the Calabi torus (see also [29, Theorem 1.5]). Since Σ is closed, Σ coincides with the Calabi torus. ∎

Proof of the 2*-dimensional case of Theorem* 1.1. By the Gauss equation and assumption, we have $2\kappa = 2 + |\mathbf{H}|^2 - |\mathbf{B}|^2 \geq 0$. According to Lemma 2.6, since $\mathrm{Ric}(J\mathbf{H}, J\mathbf{H}) = \kappa|\mathbf{H}|^2 \geq 0$, we know that

$$\nabla(J\mathbf{H}) \equiv 0, \quad \kappa|\mathbf{H}|^2 \equiv 0.$$

If M is minimal, then M is either the totally geodesic sphere $\mathbb{S}^2$ or a flat minimal Legendrian torus by Theorem C. If M is not minimal, then the conclusion follows from Proposition 3.3. ∎

4. Rigidity results for closed CSL submanifolds in the unit sphere

In this section we assume M^n $(n \geq 3)$ is a closed CSL submanifold in $\mathbb{S}^{2n+1}$.

Put

$$\sigma_{ijk} := \mathring{\sigma}_{ijk} + \frac{1}{n+2}(\mu_i\delta_{jk} + \mu_j\delta_{ki} + \mu_k\delta_{ij}).$$

Notice that $(\mathring{\sigma}_{ijk})$ is 3-symmetric and is trace-free with any two symbols.

Lemma 4.1. *If at some* $p \in M$,

$$|\mathbf{B}|^2 \leq \frac{(n-1)(n+2)}{n} + \frac{n^2+3n-2}{2n^2}|\mathbf{H}|^2 - \frac{(n-1)(n-2)|\mathbf{H}|\sqrt{4n+|\mathbf{H}|^2}}{2n^2}, \quad (4.1)$$

then at p *we have* $\mathrm{Ric}(J\mathbf{H}, J\mathbf{H}) \geq 0$*. Moreover, if* $\mathbf{H} \neq 0$ *then* $\mathrm{Ric}(J\mathbf{H}, J\mathbf{H}) = 0$ *if and only if*

$$\begin{aligned} \mathbf{B}(J\mathbf{H}, J\mathbf{H}) = \lambda_1|\mathbf{H}|\mathbf{H}, \quad \mathbf{B}(J\mathbf{H}, X) = \lambda_2|\mathbf{H}|JX, \\ \mathbf{B}(X,Y) = \frac{\lambda_2}{|\mathbf{H}|}\langle X, Y\rangle \mathbf{H} \quad \forall X, Y \perp J\mathbf{H}, \end{aligned} \quad (4.2)$$

where λ_1, λ_2 *satisfy*

$$1 + \lambda_1\lambda_2 - \lambda_2^2 = 0.$$

Proof. Without loss of generality, assume $\mathbf{H} \neq 0$ at p. Moreover, assume $\mu_1 = |\mathbf{H}| > 0$ and $\mu_j = 0$ for all $j > 1$. A direct calculation yields

$$\begin{aligned} \mathrm{Ric}_{11} &= n - 1 + \sum_j \sigma_{11j}\mu_j - \sum_{j,k}\sigma_{1jk}^2 \\ &= n - 1 + \frac{n-2}{n+2}\mathring{\sigma}_{111}\mu_1 + \frac{2(n-1)}{(n+2)^2}\mu_1^2 - \sum_{j,k}\mathring{\sigma}_{1jk}^2 \\ &\geq n - 1 - \frac{n-2}{n+2}|\mathring{\sigma}_{111}|\,|\mu_1| + \frac{2(n-1)}{(n+2)^2}\mu_1^2 - \sum_{j,k}\mathring{\sigma}_{1jk}^2. \end{aligned}$$

The equality holds if and only if

$$\mathring{\sigma}_{111} \leq 0. \quad (4.3)$$

Notice that

$$\begin{aligned} \sum_{i,j,k}\mathring{\sigma}_{ijk}^2 &= \mathring{\sigma}_{111}^2 + 3\sum_{j=2}^n\mathring{\sigma}_{11j}^2 + 3\sum_{j=2}^n\mathring{\sigma}_{1jj}^2 + 6\sum_{2\leq j<k\leq n}\mathring{\sigma}_{1jk}^2 + \sum_{i=2}^n\mathring{\sigma}_{iii}^2 \\ &\quad + 3\sum_{2\leq i\neq j\leq n}\mathring{\sigma}_{ijj}^2 + 6\sum_{2\leq i<j<k\leq n}\mathring{\sigma}_{ijk}^2 \\ &\geq \mathring{\sigma}_{111}^2 + 3\sum_{j=2}^n\mathring{\sigma}_{11j}^2 + 3\sum_{j=2}^n\mathring{\sigma}_{1jj}^2 + 6\sum_{2\leq j<k\leq n}\mathring{\sigma}_{1jk}^2 + \frac{3}{n+1}\sum_{i=2}^n\left(\sum_{j=2}^n\mathring{\sigma}_{ijj}\right)^2 \\ &\quad + 6\sum_{2\leq i<j<k\leq n}\mathring{\sigma}_{ijk}^2 \\ &= \mathring{\sigma}_{111}^2 + 3\sum_{j=2}^n\mathring{\sigma}_{1jj}^2 + \frac{3(n+2)}{n+1}\sum_{j=2}^n\mathring{\sigma}_{11j}^2 + 6\sum_{2\leq j<k\leq n}\mathring{\sigma}_{1jk}^2 \\ &\quad + 6\sum_{2\leq i<j<k\leq n}\mathring{\sigma}_{ijk}^2 \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{n+2}{n}\left(\mathring{\sigma}_{111}^2 + \sum_{j=2}^{n} \mathring{\sigma}_{1jj}^2\right) + \frac{3(n+2)}{n+1}\sum_{j=2}^{n}\mathring{\sigma}_{11j}^2 + 6\sum_{2\leq j<k\leq n}\mathring{\sigma}_{1jk}^2 \\
&\quad + 6\sum_{2\leq i<j<k\leq n}\mathring{\sigma}_{ijk}^2 \\
&\geq \frac{n+2}{n}\sum_{j,k}\mathring{\sigma}_{1jk}^2 \\
&\geq \frac{n+2}{n-1}|\mathring{\sigma}_{111}|^2.
\end{aligned}
$$

And the equality holds if and only if (the assumption $n \geq 3$ here is essential)

$$
\begin{aligned}
\mathring{\sigma}_{11j} = 0, \quad \mathring{\sigma}_{1jk} = 0, \quad \mathring{\sigma}_{1jj} &= \mathring{\sigma}_{1kk} \qquad \forall 2 \leq j < k \leq n, \\
\mathring{\sigma}_{ijk} &= 0 \qquad \forall 2 \leq i, j, k \leq n.
\end{aligned}
\tag{4.4}
$$

Therefore, we obtain

$$
\mathrm{Ric}_{11} \geq n - 1 - \frac{n-2}{n+2}\sqrt{\frac{n-1}{n+2}\sum_{i,j,k}\mathring{\sigma}_{ijk}^2}\,|\mu_1| + \frac{2(n-1)}{(n+2)^2}\mu_1^2 - \frac{n}{n+2}\sum_{i,j,k}\mathring{\sigma}_{ijk}^2.
$$

Since assumption (4.1) is equivalent to

$$
n - 1 - \frac{n-2}{n+2}\sqrt{\frac{n-1}{n+2}\sum_{i,j,k}\mathring{\sigma}_{ijk}^2}\,|\mu_1| + \frac{2(n-1)}{(n+2)^2}\mu_1^2 - \frac{n}{n+2}\sum_{i,j,k}\mathring{\sigma}_{ijk}^2 \geq 0,
$$

we complete the proof of the first part.

If $\mathrm{Ric}(J\mathbf{H}, J\mathbf{H}) = 0$, then (4.4) is equivalent to (4.2) while (4.3) is equivalent to

$$
\lambda_1 \leq \frac{3}{n+2}|\mathbf{H}|.
$$

We conclude that

$$
\lambda_2 = \frac{1}{n-1}(|\mathbf{H}| - \lambda_1) \geq \frac{1}{n+2}|\mathbf{H}| \geq \frac{1}{3}\lambda_1.
$$

According to (4.2),

$$
|\mathbf{B}|^2 = \lambda_1^2 + 3(n-1)\lambda_2^2, \quad |\mathbf{H}|^2 = (\lambda_1 + (n-1)\lambda_2)^2.
$$

Thus,

$$
\sum_{i,j,k}\mathring{\sigma}_{ijk}^2 = |\mathbf{B}|^2 - \frac{3}{n+2}|\mathbf{H}|^2 = \frac{n-1}{n+2}(3\lambda_2 - \lambda_1)^2, \quad \mu_1 = |\mathbf{H}| = \lambda_1 + (n-1)\lambda_2.
$$

Now the equality in (4.1) is equivalent to

$$
\begin{aligned}
&n - 1 - \frac{n-2}{n+2}\sqrt{\frac{n-1}{n+2}\sum_{i,j,k}\mathring{\sigma}_{ijk}^2}\,|\mu_1| + \frac{2(n-1)}{(n+2)^2}\mu_1^2 - \frac{n}{n+2}\sum_{i,j,k}\mathring{\sigma}_{ijk}^2 = 0 \\
&\Leftrightarrow \left(\sqrt{\frac{n+2}{n-1}\sum_{i,j,k}\mathring{\sigma}_{ijk}^2} + \frac{n-2}{2n}|\mu_1|\right)^2 = \frac{(n+2)^2}{4n^2}(\mu_1^2 + 4n)
\end{aligned}
$$

$$\Leftrightarrow \Big((3\lambda_2 - \lambda_1) + \frac{n-2}{2n}(\lambda_1 + (n-1)\lambda_2)\Big)^2 = \frac{(n+2)^2}{4n^2}\big((\lambda_1 + (n-1)\lambda_2)^2 + 4n\big)$$
$$\Leftrightarrow ((n+1)\lambda_2 - \lambda_1)^2 = (\lambda_1 + (n-1)\lambda_2)^2 + 4n,$$

which is equivalent to

$$1 + \lambda_1\lambda_2 - \lambda_2^2 = 0.$$

■

Proof of Theorem 1.1 *when* $n \geq 3$. From Lemma 4.1 we have $\mathrm{Ric}(J\mathbf{H}, J\mathbf{H}) \geq 0$ and Lemma 2.6 implies that $\nabla(J\mathbf{H}) = 0$. Therefore we have either $\mathbf{H} = 0$ or $\mathbf{H} \neq 0$ and the second fundamental form of M in $\mathbf{S}^{2n+1}$ is given by (4.2) where λ_1 and λ_2 are two smooth functions satisfying

$$1 + \lambda_1\lambda_2 - \lambda_2^2 = 0.$$

Now assume $\mathbf{H} \neq 0$. Let $e_1 = \frac{J\mathbf{H}}{|\mathbf{H}|}$. Then

$$\sigma_{111} = \lambda_1, \quad \sigma_{11j} = 0, \quad \sigma_{1jk} = \lambda_2\delta_{jk}, \quad \sigma_{ijk} = 0 \quad \forall 2 \leq i, j, k \leq n.$$

According to Simons' identity (2.1), we have

$$\begin{aligned}
\Delta\lambda_1 &= -2\mu_1 + \sum_t \sigma_{11t}^2\mu_1 + (n+1)\sigma_{111} + 2\sum_{l,s,t}\sigma_{1sl}\sigma_{1lt}\sigma_{1ts} - 3\sum_{l,s,t}\sigma_{tl1}\sigma_{tls}\sigma_{11s}\\
&= -2\mu_1 + \lambda_1^2\mu_1 + (n+1)\lambda_1 + 2(\lambda_1^3 + (n-1)\lambda_2^3) - 3(\lambda_1^2 + (n-1)\lambda_2^2)\lambda_1\\
&= -2(n-1)\lambda_2 + (n-1)\lambda_1^2\lambda_2 + (n-1)\lambda_1 + 2(n-1)\lambda_2^3 - 3(n-1)\lambda_2^2\lambda_1\\
&= (n-1)(\lambda_1 - 2\lambda_2)(1 + \lambda_1\lambda_2 - \lambda_2^2)\\
&= 0.
\end{aligned}$$

Hence λ_1 and λ_2 are constants. Therefore there exist two constants r_1, r_2 such that $r_1^2 + r_2^2 = 1$ and

$$\lambda_1 = \frac{r_2}{r_1} - \frac{r_1}{r_2}, \quad \lambda_2 = \frac{r_2}{r_1}.$$

Comparing with the Calabi product Legendrian immersion of a totally geodesic Legendrian immersion and a point determined by r_1, r_2 as stated in the appendix, we may conclude that M is locally isometric to the Calabi product Legendrian immersion of a totally geodesic Legendrian immersion and a point (see also [29, Theorem 1.5]). Since M is closed, we conclude that M must be a Calabi product Legendrian immersion of a totally geodesic Legendrian immersion and a point. This completes the proof of Theorem 1.1. ■

As an application of Theorem 1.1, we give a proof of Theorem 1.2. Firstly, we prove the following result:

Theorem 4.2. *Suppose M^n $(n \geq 3)$ is a closed CSL submanifold of $\mathbb{S}^{2n+1}$ and for some $\varepsilon > 0$ we have*

$$|\mathbf{B}|^2 \leq \frac{(n-1)(n+2)}{n} - \frac{(n-1)(n-2)\varepsilon}{n} + \Big(\frac{n^2+3n-2}{2n^2} - \frac{(n-1)(n-2)}{4n^2}\Big(\varepsilon + \frac{1}{\varepsilon}\Big)\Big)|\mathbf{H}|^2.$$

(1) *If* $\varepsilon \geq 1$, *then* M *is a minimal Legendrian immersion.*

(2) *If* $0 < \varepsilon < 1$, *then* M *is either a minimal Legendrian immersion or the Calabi product Legendrian immersion of the totally geodesic* $\psi: \mathbb{S}^{n-1} \to \mathbb{S}^{2n-1}$ *and a point.*

Proof. By Young's inequality, for every $\varepsilon > 0$ we have

$$2\sqrt{\mathbf{H}}\sqrt{4n + |\mathbf{H}|^2} \leq \frac{|\mathbf{H}|^2}{\varepsilon} + \varepsilon(4n + |\mathbf{H}|^2).$$

The equality holds if and only if

$$\frac{|\mathbf{H}|^2}{\varepsilon} = \varepsilon(4n + |\mathbf{H}|^2).$$

Therefore, under the assumption, we have (1.1). Moreover, when $\varepsilon \geq 1$, we have a strict inequality.

(1) When $\varepsilon \geq 1$, according to Theorem 1.1, we know that M is minimal.

(2) When $0 < \varepsilon < 1$, according to Theorem 1.1, we know that M is either a minimal Legendrian immersion or the Calabi product Legendrian immersion of the totally geodesic $\psi: \mathbb{S}^{n-1} \to \mathbb{S}^{2n-1}$ and a point. ∎

Theorem 4.3. *Let* M^n $(n \geq 3)$ *be a closed CSL submanifold of* $\mathbb{S}^{2n+1}$ *and*

$$|\mathbf{B}|^2 \leq \frac{2(n+1)}{3} - \frac{n-17}{3(n+3)}|\mathbf{H}|^2.$$

(1) *If* $n = 3$, *then* M *is the totally geodesic Legendrian immersion.*

(2) *If* $n \geq 4$, *then* M *is either the totally geodesic Legendrian immersion or is a non-minimal Calabi product Legendrian immersion of the totally geodesic* $\psi: \mathbb{S}^{n-1} \to \mathbb{S}^{2n-1}$ *and a point.*

Proof. Choose $\varepsilon = \frac{n+3}{3(n-1)}$ $(n \geq 3)$ as in Theorem 4.2. Since $\varepsilon \geq 1$ if and only if $n = 3$, when $n = 3$, M is minimal with $|\mathbf{B}|^2 \leq \frac{2(n+1)}{3}$. It remains to consider the case that $n \geq 3$, M is minimal and

$$|\mathbf{B}|^2 \leq \frac{2(n+1)}{3}.$$

According to Simons' identity (2.1), we have

$$\frac{1}{2}\Delta|\sigma|^2 = |\nabla\sigma|^2 + (n+1)|\sigma|^2 + 2\sum_{i,j,k,l,s,t} \sigma_{isl}\sigma_{jlt}\sigma_{kts}\sigma_{ijk} - 3\sum_{i,j,k,l,s,t} \sigma_{tli}\sigma_{tls}\sigma_{jks}\sigma_{ijk}.$$

Define

$$A_i = (\sigma_{ijk})_{1 \leq j,k \leq n}, \quad 1 \leq i \leq n,$$

then (see [49])

$$\begin{aligned}\frac{1}{2}\Delta|\sigma|^2 &= |\nabla\sigma|^2 + (n+1)|\sigma|^2 - \sum_{i,j}|[A_i, A_j]|^2 - \sum_{i,j}\langle A_i, A_j\rangle^2 \\ &\geq |\nabla\sigma|^2 + (n+1)|\sigma|^2 - \frac{3}{2}\Big(\sum_i |A_i|^2\Big)^2 \\ &= |\nabla\sigma|^2 + (n+1)|\sigma|^2 - \frac{3}{2}|\sigma|^4.\end{aligned}$$

By assumption, we have

$$\frac{1}{2}\Delta|\sigma|^2 \geq |\nabla\sigma|^2.$$

Thus $\sigma \equiv 0$ or $|\sigma| \equiv \frac{2(n+1)}{3}$. Following the same argument as [8] or [25], when $|\sigma| \equiv \frac{2(n+1)}{3}$, we must have $n = 2$, hence the last case cannot happen. Therefore M is totally geodesic. ∎

Proof of Theorem 1.2. Condition (1.2) in Theorem 1.2 is just the case $\varepsilon = 1$ in Theorem 4.2. We conclude that M is minimal and $|\mathbf{B}|^2 \leq \frac{4(n-1)}{n}$. Since $\frac{4(n-1)}{n} \leq \frac{2(n+1)}{3}$ always holds when $n \geq 3$, by Theorem 4.3, we finish the proof. ∎

5. More results and discussions

In this section we will get more results from Theorem 4.2.

Theorem 5.1. *Suppose M^n ($n \geq 3$) is a closed CSL submanifold of $\mathbb{S}^{2n+1}$ and*

$$|\mathbf{B}|^2 \leq \begin{cases} \dfrac{2(n+1)}{3}, & 3 \leq n \leq 16, \\ 2(\sqrt{3n-2} - 1), & n \geq 17. \end{cases}$$

(1) *If $3 \leq n \leq 16$, then M is the totally geodesic Legendrian immersion.*

(2) *If $n \geq 17$, then M is either the totally geodesic Legendrian immersion or the Calabi product Legendrian immersion of the totally geodesic $\psi\colon \mathbb{S}^{n-1} \to \mathbb{S}^{2n-1}$ and a point.*

Proof. Take $\varepsilon = \frac{(n-\sqrt{3n-2})^2}{(n-1)(n-2)}$ as in Theorem 4.2. Noticing that $\varepsilon > 0$, since $n \geq 3$ and $\frac{2(n+1)}{3} < 2(\sqrt{3n-2} - 1)$ when $3 \leq n \leq 16$ we see that $|\mathbf{B}|^2 \leq \frac{2(n+1)}{3} < 2(\sqrt{3n-2} - 1)$, which implies that M is minimal and hence totally geodesic by the same argument as Theorem 4.3. When $n \geq 17$, if M is minimal, then $|\mathbf{B}|^2 \leq 2(\sqrt{3n-2} - 1) \leq \frac{2(n+1)}{3}$ we get M is totally geodesic by the same argument with Theorem 4.3 again. Therefore we complete the proof. ∎

Theorem 5.2. *If M^n $(n \geq 3)$ is a closed CSL submanifold of $\mathbb{S}^{2n+1}$ and*

$$|\mathbf{B}|^2 \leq 2 + \frac{3}{n+1}|\mathbf{H}|^2,$$

then M is a totally geodesic Legendrian immersion.

Proof. Take $\varepsilon = \frac{n+1}{n-1}$ as in Theorem 4.2. Notice that $\varepsilon > 1$ and hence M is minimal. Therefore $|\mathbf{B}|^2 \leq 2 \leq \frac{2(n+1)}{3}$. Then by a similar argument to Theorem 4.3 we complete the proof. ∎

Remark 5.1. Under the assumption $n \geq 4$ and

$$|\mathbf{B}|^2 < 2 + \frac{3}{n+3/2}|\mathbf{H}|^2,$$

Li–Wang [30] proved that the closed simply connected Legendrian submanifold M^n in $\mathbb{S}^{2n+1}$ must be diffeomorphic to $\mathbb{S}^n$. Under the assumption

$$|\mathbf{B}|^2 < \begin{cases} 6 + \dfrac{3}{n+2/3}|\mathbf{H}|^2, & n \geq 5, \\ 6 + \dfrac{3}{4}|\mathbf{H}|^2, & n = 4, \end{cases}$$

Sun–Sun [47] proved that a closed simply connected Legendrian submanifold M^n in $\mathbb{S}^{2n+1}$ must be a topological sphere.

At the end of this paper, we list some conjectures.

Conjecture 1. *If M^n $(n \geq 2)$ is a closed minimal Legendrian submanifold in $\mathbb{S}^{2n+1}$ and*

$$|\mathbf{B}|^2 \leq \frac{(n-1)(n+2)}{n},$$

then M is either totally geodesic or a minimal Calabi product Legendrian immersion of a totally geodesic Legendrian immersion and a point.

This conjecture is equivalent to the following one:

Conjecture 2. *If M^n $(n \geq 2)$ is a closed contact stationary Legendrian submanifold of $\mathbb{S}^{2n+1}$ and*

$$|\mathbf{B}|^2 \leq \frac{(n-1)(n+2)}{n} + \frac{n^2+3n-2}{2n^2}|\mathbf{H}|^2 - \frac{(n-1)(n-2)|\mathbf{H}|\sqrt{4n+|\mathbf{H}|^2}}{2n^2},$$

then M is either totally geodesic or a Calabi product Legendrian immersion of a totally geodesic Legendrian immersion and a point.

From Theorem 1.1, we know that this conjecture is true for $n = 2$.

For the first gap of the length of fundamental form of CSL submanifolds in the unit sphere, motivated by Theorem 5.1, we list a conjecture.

Conjecture 3. *If M^n $(n \geq 2)$ is a closed contact stationary Legendrian submanifold of $\mathbb{S}^{2n+1}$ and*

$$|\mathbf{B}|^2 \leq 2(\sqrt{3n-2}-1),$$

then M is either totally geodesic or a Calabi product Legendrian immersion of a totally geodesic Legendrian immersion and a point.

Theorems D and 5.1 claim that this conjecture is true for $n = 2$ and $n \geq 17$ respectively.

A. Examples

A.1. Calabi tori

For every four nonzero real numbers r_1, r_2, r_3, r_4 with $r_1^2 + r_2^2 = r_3^2 + r_4^2 = 1$, a Calabi torus is a CSL surface in $\mathbf{S}^5$ defined as

$$\begin{aligned} &F\colon \Sigma := \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^5, \\ &(t,s) \mapsto \Big(r_1 r_3 \exp\Big(\sqrt{-1}\Big(\frac{r_2}{r_1}t + \frac{r_4}{r_3}s\Big)\Big), r_1 r_4 \exp\Big(\sqrt{-1}\Big(\frac{r_2}{r_1}t - \frac{r_3}{r_4}s\Big)\Big), \\ &\qquad\qquad r_2 \exp\Big(-\sqrt{-1}\frac{r_1}{r_2}t\Big)\Big). \end{aligned}$$

Denoting

$$\phi_1 = \exp\Big(\sqrt{-1}\Big(\frac{r_2}{r_1}t + \frac{r_4}{r_3}s\Big)\Big), \quad \phi_2 = \exp\Big(\sqrt{-1}\Big(\frac{r_2}{r_1}t - \frac{r_3}{r_4}s\Big)\Big),$$
$$\phi_3 = \exp\Big(-\sqrt{-1}\frac{r_1}{r_2}t\Big),$$

then $F(t,s) = (r_1 r_3 \phi_1, r_1 r_4 \phi_2, r_2 \phi_3)$. Since

$$\begin{aligned} \frac{\partial F}{\partial t} &= \big(\sqrt{-1}r_2 r_3 \phi_1, \sqrt{-1}r_2 r_4 \phi_2, -\sqrt{-1}r_1 \phi_3\big), \\ \frac{\partial F}{\partial s} &= \big(\sqrt{-1}r_1 r_4 \phi_1, -\sqrt{-1}r_1 r_3 \phi_2, 0\big), \end{aligned}$$

the induced metric in Σ is given by

$$g = \mathrm{d}t^2 + r_1^2\, \mathrm{d}s^2.$$

Let $E_1 = \frac{\partial F}{\partial t}$, $E_2 = \frac{1}{r_1}\frac{\partial F}{\partial s}$. Then $\{E_1, E_2, \nu_1 = \sqrt{-1}E_1, \nu_2 = \sqrt{-1}E_2, \mathbf{R} = \sqrt{-1}F\}$ is a local orthonormal frame of $\mathbb{S}^5$ such that $\{E_1, E_2\}$ is a local orthonormal tangent frame and $\mathbf{R}$ is the Reeb field. Direct calculation yields

$$\begin{aligned}
\frac{\partial \nu_1}{\partial t} &= \Big(-\sqrt{-1}\frac{r_2^2 r_3}{r_1}\phi_1, -\sqrt{-1}\frac{r_2^2 r_4}{r_1}\phi_2, -\sqrt{-1}\frac{r_1^2}{r_2}\phi_3\Big),\\
\frac{\partial \nu_1}{\partial s} &= \Big(-\sqrt{-1}\frac{r_2 r_3^2}{r_4}\phi_1, \sqrt{-1}\frac{r_2 r_4^2}{r_3}\phi_2, 0\Big),\\
\frac{\partial \nu_2}{\partial t} &= \Big(-\sqrt{-1}\frac{r_2 r_4}{r_1}\phi_1, \sqrt{-1}\frac{r_2 r_3}{r_1}\phi_2, 0\Big),\\
\frac{\partial \nu_2}{\partial s} &= \Big(-\sqrt{-1}\frac{r_4^2}{r_3}\phi_1, -\sqrt{-1}\frac{r_3^2}{r_4}\phi_2, 0\Big),\\
\frac{\partial \mathbf{R}}{\partial t} &= \Big(-r_2 r_3\phi_1, -r_2 r_4\phi_2, r_1\phi_3\Big),\\
\frac{\partial \mathbf{R}}{\partial s} &= \Big(-r_1 r_4\phi_1, r_1 r_3\phi_2, 0\Big).
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbf{A}^{\nu_1} &= -\Re\langle \mathrm{d}F, \mathrm{d}\nu_1\rangle = \Big(\frac{r_2}{r_1} - \frac{r_1}{r_2}\Big)\,\mathrm{d}t^2 + r_1 r_2\,\mathrm{d}s^2,\\
\mathbf{A}^{\nu_2} &= -\Re\langle \mathrm{d}F, \mathrm{d}\nu_2\rangle = 2r_2\,\mathrm{d}t\,\mathrm{d}s + r_1\Big(\frac{r_4}{r_3} - \frac{r_3}{r_4}\Big)\,\mathrm{d}s^2,\\
\mathbf{A}^{\mathbf{R}} &= 0.
\end{aligned}$$

Thus

$$\mathbf{H} = \Big(\frac{2r_2}{r_1} - \frac{r_1}{r_2}\Big)\nu_1 + \frac{1}{r_1}\Big(\frac{r_4}{r_3} - \frac{r_3}{r_4}\Big)\nu_2.$$

Moreover, E_1 and E_2 are two parallel tangent vector fields. Under the orthonormal frame $\{E_1, E_2\}$, the second fundamental form can be written as

$$\mathbf{A}^{\nu_1} = \begin{pmatrix} \frac{r_2}{r_1} - \frac{r_1}{r_2} & 0\\ 0 & \frac{r_2}{r_1}\end{pmatrix}, \quad \mathbf{A}^{\nu_2} = \begin{pmatrix} 0 & \frac{r_2}{r_1}\\ \frac{r_2}{r_1} & \frac{1}{r_1}\big(\frac{r_4}{r_3} - \frac{r_3}{r_4}\big)\end{pmatrix}, \quad \mathbf{A}^{\mathbf{R}} = 0.$$

A direct calculation shows that

$$\kappa = 2 + |\mathbf{H}|^2 - |\mathbf{B}|^2 = 0.$$

It is obvious that $J\mathbf{H}$ is parallel. In particular, Σ is CSL. Moreover, F is a minimal Legendrian surface if and only if $r_1 = \pm\frac{\sqrt{6}}{3}$, $r_2 = \pm\frac{\sqrt{3}}{3}$, $r_3 = r_4 = \pm\frac{\sqrt{2}}{2}$. In this case $|\mathbf{B}|^2 = 2$ and the Gauss curvature of F is 0, i.e. F is a flat minimal Legendrian torus.

A.2. Calabi product Legendrian immersions

Let $F = (F^1, F^2, \ldots, F^{n+1})\colon M^n \to \mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ be an isometric immersion. Then F is a Legendrian immersion if and only if

$$\sum_\alpha F_i^\alpha \bar{F}^\alpha = 0 \quad \forall i.$$

Let

$$\begin{aligned}\gamma = (\gamma_1, \gamma_2)\colon \mathbb{S}^1 &\to \mathbb{S}^3,\\ t &\mapsto \Big(r_1 \exp\Big(\sqrt{-1}\frac{r_2}{r_1}t\Big), r_2 \exp\Big(-\sqrt{-1}\frac{r_1}{r_2}t\Big)\Big)\end{aligned}$$

be a Legendre curve where r_1, r_2 are two nonzero constants satisfying $r_1^2 + r_2^2 = 1$. Let $F = (F^1, F^2, \ldots, F^n)\colon M^{n-1} \to \mathbb{S}^{2n-1}$ be a Legendrian immersion. Then $\tilde{F} := (\gamma_1 F, \gamma_2)\colon \widetilde{M} := \mathbb{S}^1 \times M \to \mathbb{S}^{2n+1}$ is a Legendrian immersion. We call $\tilde{F}$ a *Calabi product Legendrian immersion* of F and a point.

The induced metric on $\widetilde{M}$ is given by

$$\tilde{g} = \mathrm{d}t^2 + r_1^2 g,$$

where g is the induced metric on M. Denote

$$\begin{aligned}E_1 &= \Big(\sqrt{-1}r_2 \exp\Big(\sqrt{-1}\frac{r_2}{r_1}t\Big)F, -\sqrt{-1}r_1 \exp\Big(-\sqrt{-1}\frac{r_1}{r_2}t\Big)\Big) = \mathrm{d}\tilde{F}\Big(\frac{\partial}{\partial t}\Big),\\ E_j &= \Big(\exp\Big(\sqrt{-1}\frac{r_2}{r_1}t\Big)\,\mathrm{d}F(e_j), 0\Big) = \frac{1}{r_1}\,\mathrm{d}\tilde{F}(e_j), \quad j = 2, \ldots, n,\end{aligned}$$

where $\{\mathrm{d}F(e_j)\}_{j=2}^n$ is a local orthonormal frame of TM. We obtain a local orthonormal frame $\{E_j\}_{j=1}^n$ of $T\widetilde{M}$. Then $\{\nu_j := \sqrt{-1}E_j, \sqrt{-1}\tilde{F}\}$ is a local orthonormal frame of the normal bundle $T^{\perp}\widetilde{M}$. A direct calculation yields

$$\begin{aligned}\tilde{\mathbf{A}}^{\nu_1} &= -\Re\big\{\big\langle \mathrm{d}\tilde{F}, \mathrm{d}\nu_1\big\rangle\big\} = \Big(\frac{r_2}{r_1} - \frac{r_1}{r_2}\Big)\mathrm{d}t^2 + r_1 r_2 g,\\ \tilde{\mathbf{A}}^{\nu_j} &= -\Re\big\{\big\langle \mathrm{d}\tilde{F}, \mathrm{d}\nu_j\big\rangle\big\} = r_1 \mathbf{A}^{\sqrt{-1}\,\mathrm{d}F(e_j)} + \frac{r_2}{r_1}\,\mathrm{d}t \otimes (E_j)^{\sharp}\\ &\qquad + \frac{r_2}{r_1}(E_j)^{\sharp} \otimes \mathrm{d}t, \quad j = 2, \ldots, n,\\ \tilde{\mathbf{A}}^{\sqrt{-1}\tilde{F}} &= 0.\end{aligned}$$

We obtain that

- $\tilde{F}$ is CSL if and only if F is CSL,
- $\sqrt{-1}\tilde{\mathbf{H}}$ is parallel if and only if $\sqrt{-1}\mathbf{H}$ is parallel,
- $\tilde{F}$ is minimal if and only if F is minimal and $|r_1| = \sqrt{\frac{n}{n+1}}$.

The second fundamental form can be written in matrix form as

$$\tilde{\mathbf{A}}^{\nu_1} = \begin{pmatrix} \frac{r_2}{r_1} - \frac{r_1}{r_2} & 0\\ 0 & \frac{r_2}{r_1}\mathrm{Id}_{(n-1)\times(n-1)}\end{pmatrix}, \quad \tilde{\mathbf{A}}^{\nu_j} = \begin{pmatrix} 0 & \frac{r_2}{r_1}\alpha_j^T\\ \frac{r_2}{r_1}\alpha_j & \frac{1}{r_1}\mathbf{A}^{\sqrt{-1}\,\mathrm{d}F(e_j)}\end{pmatrix},$$
$$\tilde{\mathbf{A}}^{\sqrt{-1}\tilde{\mathbf{F}}} = 0, \quad j = 2, \ldots, n,$$

where

$$\begin{pmatrix}\alpha_2 & \alpha_3 & \cdots & \alpha_n\end{pmatrix} = \mathrm{Id}_{(n-1)\times(n-1)}.$$

Hence,

$$
\begin{aligned}
\widetilde{\mathbf{H}}_{\tilde{g}} &= \Big(\frac{nr_2}{r_1} - \frac{r_1}{r_2}\Big) \otimes \nu_1 + \frac{1}{r_1^2}\mathbf{H}, \\
|\widetilde{\mathbf{H}}|_{\tilde{g}}^2 &= \Big(\frac{nr_2}{r_1} - \frac{r_1}{r_2}\Big)^2 + \frac{1}{r_1^2}|\mathbf{H}|_g^2, \\
|\widetilde{\mathbf{B}}|_{\tilde{g}}^2 &= \Big(\frac{r_2}{r_1} - \frac{r_1}{r_2}\Big)^2 + (n-1)\Big(\frac{r_2}{r_1}\Big)^2 + 2(n-1)\Big(\frac{r_2}{r_1}\Big)^2 + \frac{1}{r_1^2}|\mathbf{B}|_g^2.
\end{aligned}
$$

When M is totally geodesic, a direct calculation yields

$$
|\widetilde{\mathbf{B}}|_{\tilde{g}}^2 \geq \frac{(n-1)(n+2)}{n} + \frac{n^2+3n-2}{2n^2}|\widetilde{\mathbf{H}}|_{\tilde{g}}^2 - \frac{(n-1)(n-2)|\widetilde{\mathbf{H}}_{\tilde{g}}|\sqrt{4n+|\widetilde{\mathbf{H}}_{\tilde{g}}|^2}}{2n^2}.
$$

The equality holds if and only if $|r_1| \leq \sqrt{\frac{n}{n+1}}$ or equivalently $|r_2| \geq \sqrt{\frac{1}{n+1}}$. We also have

$$
|\widetilde{\mathbf{B}}|_{\tilde{g}}^2 - \frac{3n-2}{n^2}|\widetilde{\mathbf{H}}|_{\tilde{g}}^2 - \frac{4(n-1)}{n} = \frac{(n-1)(n-2)}{n^2}\frac{r_1^2}{r_2^2}.
$$

Acknowledgments. The authors would like to thank the anonymous referee for their careful reading and useful comments.

Funding. Yong Luo is supported by the Research Start-Up Funds of Chongqing University of Technology and Chongqing NSF (grant no. cstc2021jcjy-msxmX0443). Linlin Sun is supported by the National Natural Science Foundation of China (grant nos. 11801420, 11971358) and the Hubei Provincial Natural Science Foundation of China (grant no. 2021CFB400).

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Received 16 May 2019; accepted 4 Jan 2022.

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