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Convergence rate for the incompressible limit of nonlinear diffusion–advection equations

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Abstract. The incompressible limit of nonlinear diffusion equations of porous medium type has attracted a lot of attention in recent years, due to its ability to link the weak formulation of cell-population models to free boundary problems of Hele-Shaw type. Although a vast literature is available on this singular limit, little is known on the convergence rate of the solutions. In this work, we compute the convergence rate in a negative Sobolev norm and, upon interpolating with BV-uniform bounds, we deduce a convergence rate in appropriate Lebesgue spaces.

1. Introduction

We consider the nonlinear drift–diffusion equation

$$\frac{\partial n}{\partial t} - \nabla \cdot (n\nabla p + n\nabla V) = ng, \tag{1}$$

posed on $(0,T) \times \mathbb{R}^d$, $d \geq 2$, where n describes a population density and $p = p(n)$ is the density-dependent pressure. The reaction term on the right-hand side represents the population growth rate $g = g(t,x)$, while $V = V(t,x)$ is a chemical concentration. The pressure is assumed to be a known increasing function of the density. We consider the following two representative examples:

$$p_\gamma = P_\gamma(n) := \frac{\gamma}{\gamma - 1} n^{\gamma-1}, \quad \gamma > 1, \tag{2}$$

and

$$p_\varepsilon = P_\varepsilon(n) := \varepsilon \frac{n}{1-n}, \qquad \varepsilon > 0. \tag{3}$$

We are concerned with calculating the rate at which solutions to (1) converge to the so-called *incompressible* (or *stiff pressure*) limit, as described below. More precisely we prove the following results.

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Theorem 1.1 (Convergence rate in $\dot{H}^{-1}$). *Assume* (A-L^1data)*,* (A-drift)*, and* (A-reaction) *(for $d=2$) and* (A-reaction$'$) *or* (A-reaction$''$) *(for $d\geq 3$). For $d=2$ assume additionally* (A-2D)*. Then, for all $T>0$, there exists a unique function $n_\infty \in C([0,T);L^1(\mathbb{R}^d))$ such that the sequence n_γ (resp. n_ε) converges, as $\gamma\to\infty$ (resp. $\varepsilon\to 0$), to n_∞ strongly in $L^\infty(0,T;\dot{H}^{-1}(\mathbb{R}^d))$ with the rate*

$$\sup_{t\in[0,T]} \|n_\gamma(t)-n_\infty(t)\|_{\dot{H}^{-1}(\mathbb{R}^d)} \leq \frac{C(T)}{\gamma^{1/2}} + \|n_\gamma^0 - n_\infty^0\|_{\dot{H}^{-1}(\mathbb{R}^d)}.$$

Theorem 1.2 (Convergence rate in $L^{4/3}$). *Under the assumptions of Theorem* 1.1*, and additionally* (A-BVdata)*,* (A-BVdrift) *and $g\in \mathrm{BV}((0,T)\times\mathbb{R}^d)$, we also have $n_\infty \in \mathrm{BV}((0,T)\times\mathbb{R}^d)$ and*

$$\sup_{t\in[0,T]} \|n_\gamma(t)-n_\infty(t)\|_{L^{4/3}(\mathbb{R}^d)} \leq \frac{C(T)}{\gamma^{1/4}} + \|n_\gamma^0 - n_\infty^0\|^{1/2}_{\dot{H}^{-1}(\mathbb{R}^d)}. \tag{4}$$

Theorem 1.3. *Under the assumptions of Theorem* 1.1*, there exists a function $p_\infty \in L^\infty((0,T)\times\mathbb{R}^d)$ such that, after extracting a subsequence, the sequence p_γ converges to p_∞ weakly* in $L^\infty((0,T)\times\mathbb{R}^d)$ and the relation*

$$p_\infty(1-n_\infty)=0 \tag{5}$$

holds almost everywhere in $(0,T)\times\mathbb{R}^d$.

The above graph relation between the limit pressure and density is well known in the literature. In particular, when considering tumor growth models it implies that saturation holds in the region where there is a positive pressure, which is usually referred to as the region occupied by the tumor. Here we provide a new proof that does not require strong convergence of the density or the pressure.

In fact, the limit n_∞ satisfies (together with a limit pressure, p_∞) a free-boundary-type problem, discussed briefly below, and the question of passing to this limiting problem has been vastly addressed in literature. Our contribution is to provide a new proof together with a convergence rate.

Motivation and previous works. Models like (1) are well known and commonly employed in a variety of applications, for instance in bio-mathematical modeling of living tissue. In the case $V=0$, $g=0$, it is well known that if the pressure satisfies the power law (2), then (1) is actually the porous medium equation

$$\frac{\partial n_\gamma}{\partial t} - \Delta n_\gamma^\gamma = 0, \tag{6}$$

whose well-understood properties (e.g. regularizing effects) facilitate the analysis notably. The other choice of the pressure, given by (3), is well known in kinetic theory of dense gases where the short-distance interactions between particles are strongly repulsive. In this spirit it has been used in models describing collective motion or congested traffic flow;

see [2, 3, 12, 19, 24]. Despite having a singularity when the population density reaches its maximum value (here standardized to 1), this choice of pressure gives rise to a tissue growth model with similar properties – indeed, the crucial a priori estimates are the same and the limiting free-boundary-type problem is almost identical. A difference is that the singularity in the pressure prevents the cell densities ever rising above the maximum value 1. Taking advantage of these similarities, we shall henceforth index the solution of (1) by γ, $n = n_\gamma$, and consider the singular limit $\gamma \to \infty$. Each of the assumptions and properties we discuss below has its natural ε-analogue by putting $\varepsilon = 1/\gamma$.

Let us recall that the study of the incompressible limit has a long history and it has been investigated for many different models related to (1). The first result on the limit $\gamma \to \infty$ was obtained for the classical porous medium equation (6). The most interesting difference from the case with a non-trivial reaction term is that the free boundary problem arising in the limit turns out to be stationary. In fact, as proven in [5] the limit density n_∞ is independent of time. This result can be intuitively explained by noticing that the degenerate diffusivity of (1), namely $\gamma n^{\gamma-1}$, converges to 0 if $n < 1$, while it tends to infinity in the regions where $n > 1$. Therefore, while there is no motion in the regions where the density is below 1, where the solution lies above this level it tends to collapse instantaneously; cf. [16]. In the absence of reaction terms and, hence, of any evolution process in the Hele-Shaw problem, the limit pressure turns out to be constantly equal to 0, $p_\infty \equiv 0$.

Introducing non-trivial Dirichlet boundary conditions drastically changes the behavior of the limiting free boundary problem. In fact, the limit pressure no longer vanishes and this triggers the evolution of the interface in accordance with Darcy's law (which states that the velocity of the free boundary is proportional to the pressure gradient). This problem was addressed in [15], where the authors study the incompressible limit of the porous medium equation defined in $[0,\infty) \times \Omega$, where Ω is a compact subset of $\mathbb{R}^d$, and the pressure satisfies $p(t,x) = f(t,x)$ on $\partial\Omega$ for some $f(t,x) \geq 0$. In the absence of Dirichlet boundary data, i.e. $f \equiv 0$, and for Ω large enough, the problem is actually the same as in [5] and it still holds that $n_\infty = n_\infty(x)$ as well as $p_\infty \equiv 0$. On the other hand, if one imposes the pressure to be strictly positive somewhere on $\partial\Omega$, i.e. $f \not\equiv 0$, then the pressure gradient no longer vanishes and the dynamics of the limit problem is governed by Darcy's law.

The same non-stationary effect, although due to different dynamics, is produced by a non-trivial reaction process. The incompressible limit for (1) without convective effects, i.e. $V = 0$, and with a pressure-dependent growth rate $g = G(p)$, was first addressed in the seminal paper [25] by Perthame, Quirós, and Vázquez. They prove that it is possible to extract subsequences of n_γ and p_γ which converge in the L^1-norm to functions

$$n_\infty \in C([0,T]; L^1(\mathbb{R}^d)) \cap \mathrm{BV}((0,T) \times \mathbb{R}^d),$$

$$p_\infty \in L^2(0,T; H^1(\mathbb{R}^d)) \cap \mathrm{BV}((0,T) \times \mathbb{R}^d),$$

satisfying the following equation in the sense of distributions on $(0, T) \times \mathbb{R}^d$,

$$\frac{\partial n_\infty}{\partial t} - \Delta p_\infty = n_\infty G(p_\infty),$$

and the relations

$$(1 - n_\infty) p_\infty = 0$$

almost everywhere, as well as

$$p_\infty(\Delta p_\infty + G(p_\infty)) = 0 \tag{7}$$

in the sense of distributions. The last equality is usually referred to as the *complementarity relation* and represents the link between the limit equation and the free boundary problem. In fact, denoting by $\Omega(t) := \{x \in \mathbb{R}^d \mid p_\infty(x,t) > 0\}$ the region occupied by the tumor, from (7) one can see that the pressure satisfies an elliptic equation in the evolving domain $\Omega(t)$ with homogeneous Dirichlet boundary conditions. The free boundary $\partial\Omega(t)$ is moving under Darcy's law, which finally allows the fully geometrical representation of the limit problem to be obtained. A derivation of the velocity law can be found in [25] for initial data given by characteristic functions of bounded sets, although the proof relies on formal arguments. A weak (distributional) and a measure-theoretic interpretation of the free boundary condition have been recovered in [23], while in [20] the same result is achieved through the viscosity solutions approach.

An analogous result regarding the limit $\gamma \to \infty$ has been shown in [19] for the pressure law given by (3). The authors obtain virtually the same limiting problem, the only difference being that the complementarity relation (7) becomes

$$p_\infty^2(\Delta p_\infty + G(p_\infty)) = 0;$$

see [19, Theorem 2.1]. Let us point out that due to uniform estimates in L^∞ the convergence of the sequence of densities is also true in any L^p-space, $p < \infty$.

The Hele-Shaw limit for the porous medium equation including convective effects, cf. (1) with $V \not\equiv 0$, and possibly reaction terms, has attracted a lot of interest as well. Similarly to the driftless case, when passing to the limit $\gamma \to \infty$, the model converges to a free boundary problem where, however, the interface dynamics is no longer driven only by Darcy's law, but also by the external drift, i.e. the normal velocity is given by $-(\nabla p_\infty + \nabla V) \cdot \nu$, where ν is the outward normal direction. The asymptotics as $\gamma \to \infty$ has been addressed both for local and non-local drift, in the absence of reactions; see for instance [1, 9], where the authors adopt techniques relying on the gradient flow structure of the equation. In [21], Kim, Požár, and Woodhouse also include a linear reaction term in the equation and are able to prove convergence to the incompressible limit using viscosity solutions. Recently, in [11] the authors show that the complementarity condition including a drift, i.e.

$$p_\infty(\Delta p_\infty + \Delta V + G(p_\infty)) = 0,$$

holds in the sense of distributions.

In recent years, many other variations of the model at hand have been proposed, together with the analysis of their incompressible limit. We refer the reader to [10] for a model including the effects of nutrients, [17] for the generalization of the driftless model with a non-monotone proliferation term, and [28] for the model including active motion. In order to account for viscoelastic effects, several models propose to use Brinkman's law instead of Darcy's law [26]. Moreover, cross-reaction–diffusion models using Darcy's law, Brinkman's law, or a singular pressure law have attracted a lot of attention as they raise challenging questions both on the existence of solutions and their incompressible limit; see [4, 6, 13, 14, 18, 22].

Our aim is to compute the rate of convergence of the solutions of (1) as $\varepsilon \to 0$ or $\gamma \to \infty$ in (3) or (2) respectively. To the best of our knowledge the only result in this direction is given by Alexander, Kim, and Yao [1] for the porous medium equation including a space-dependent drift. Passing to the incompressible limit, the authors are able to build a link between the Hele-Shaw model and the congested crowd motion model

$$\partial_t n + \nabla \cdot (n \nabla V) = 0 \quad \text{if } n < 1,$$

with the constraint $n \le 1$. To prove the equivalence of the two models, they study the convergence as $\gamma \to \infty$ of the porous medium equation with drift; cf. (1) with $G \equiv 0$. Unlike [25], their approach is based on viscosity solutions. On the one hand, they are able to prove locally uniform convergence of the viscosity solution of (1) to a solution of the Hele-Shaw model. On the other hand, they show the convergence of the porous medium equation with drift to the aforementioned crowd motion model in the 2-Wasserstein distance. Therefore, they prove the equivalence of the two models in the special case of initial data given by "patches", namely $n^0 = \mathbb{1}_{\Omega_0}$ for a compact set Ω_0. In fact, the locally uniform limit holds only for solutions of the form of a characteristic function, while the limit in the 2-Wasserstein metric holds for any bounded initial data, $0 \le n^0 \le 1$ with finite energy and second moment. Moreover, while the local uniform convergence only requires a strict subharmonicity assumption on the drift term, i.e. $V \in C^2(\mathbb{R}^d)$, $\Delta V > 0$, stronger regularity is needed to pass to the 2-Wasserstein limit. More precisely the authors make the following assumptions on $V = V(x)$: there exists $\lambda \in \mathbb{R}$ such that

$$\inf_{x \in \mathbb{R}^d} V(x) = 0, \quad D^2 V(x) \ge \lambda I_d \;\; \forall x \in \mathbb{R}^d, \quad \|\Delta V\|_{L^\infty(\mathbb{R}^d)} \le C.$$

Under these assumptions, they derive the rate of convergence (cf. [1, Theorem 4.2.])

$$\sup_{t \in [0,T]} W_2(n_\gamma(t), n_\infty(t)) \le \frac{C}{\gamma^{1/24}},$$

where C is a positive constant depending on $\int V n^0$, $\|\Delta V\|_\infty$, and T.

The main result of this paper offers an improved polynomial rate of convergence in a negative Sobolev norm and the strong topology of Lebesgue spaces; see Theorems 1.1 and 1.2 above and Corollary 1.7 below. Let us remark that the 2-Wasserstein distance and the $\dot{H}^{-1}$-norm can be bounded by each other when the densities are uniformly bounded away from vacuum: see Appendix A. We refer the reader to [27, Section 5.5.2], and references therein, for further discussion of the equivalence of the two distances.

Preliminaries and assumptions. Throughout this paper we make the following assumptions on the components of the model. Firstly, we assume that (1) is equipped with non-negative initial data n_γ^0 (resp. n_ε^0) such that there is a compact set $K \subset \mathbb{R}^d$ and a function $n_\infty^0 \in L^1(\mathbb{R}^d)$ satisfying

$$\begin{aligned} &\operatorname{supp} n_\gamma^0 \subset K, \\ &p_\gamma^0 = P_\gamma(n_\gamma^0) \in L^\infty(\mathbb{R}^d), 0 \le n_\gamma^0 \in L^1(\mathbb{R}^d), \quad \|n_\gamma^0 - n_\infty^0\|_{L^1(\mathbb{R}^d)} \to 0, \\ &p_\varepsilon^0 = P_\varepsilon(n_\varepsilon^0) \in L^\infty(\mathbb{R}^d),\ 0 \le n_\varepsilon^0 \in L^1(\mathbb{R}^d), \quad \|n_\varepsilon^0 - n_\infty^0\|_{L^1(\mathbb{R}^d)} \to 0. \end{aligned} \tag{A-L^1data}$$

Note in particular that the compact support assumption is needed only in the power law pressure. This is because when the pressure is given by (3) we can achieve our main estimate without a uniform bound for the pressure in L^∞, which is not the case for the power law. Having uniformly compactly supported data allows a maximum principle for the equation satisfied by the pressure to be derived. When additionally specified, we assume further

$$n_\gamma^0 \in \mathrm{BV}(\mathbb{R}^d), \quad \Delta(n_\gamma^0)^\gamma \in L^1(\mathbb{R}^d), \tag{A-BVdata}$$

uniformly in γ. Secondly, the chemical concentration potential V is assumed to satisfy

$$D^2 V \ge \Big(\lambda + \frac{1}{2}\operatorname{tr}(D^2 V)\Big) I_d \quad \text{for some } \lambda \in \mathbb{R}, \tag{A-drift}$$

and additionally

$$\begin{gathered} D^2 V \in L^\infty((0,T)\times\mathbb{R}^d), \quad \nabla V \in L^\infty((0,T)\times\mathbb{R}^d), \\ \nabla\Delta V \in L^1((0,T)\times\mathbb{R}^d). \end{gathered} \tag{A-BVdrift}$$

Thirdly, we assume the proliferation rate $g = g(t,x)$ to be locally integrable and satisfy either

$$g_+ \in L^\infty((0,T)\times\mathbb{R}^d) \quad \text{and} \quad \Delta g \ge 0, \tag{A-reaction}$$

where $f_+ := \max(f,0)$ denotes the positive part of the function, or

$$g_+ \in L^\infty((0,T)\times\mathbb{R}^d) \quad \text{and} \quad (\Delta g)_- \in L^\infty(0,T; L^{d/2}(\mathbb{R}^d)), \quad d \ge 3, \tag{A-reaction$'$}$$

where $f_- := \max(-f,0)$ denotes the negative part of the function, or alternatively

$$g_+ \in L^\infty((0,T)\times\mathbb{R}^d) \quad \text{and} \quad \nabla g \in L^\infty(0,T; L^d(\mathbb{R}^d)), \quad d \ge 3. \tag{A-reaction$''$}$$

Under these assumptions one can derive several crucial uniform estimates for (1).

Lemma 1.4 (A priori estimates). *Under assumption* (A-L^1data) *the family n_γ of solutions to* (1) *satisfies the following bounds, uniformly in γ:*

(1) supp $p_\gamma(t) \subset K(t)$ *for some compact set* $K(t)$,

(2) *there exists a positive constant* $p_M = p_M(T)$ *such that* $0 \le p_\gamma \le p_M$, $0 \le n_\gamma \le (\frac{\gamma-1}{\gamma} p_M)^{\frac{1}{\gamma-1}}$,

(3) $n_\gamma \in L^\infty(0,T; L^1(\mathbb{R}^d))$.

Assuming in addition (A-BVdrift) *we also have* $n_\gamma \in L^\infty(0,T;\mathrm{BV}(\mathbb{R}^d))$. *When the pressure is given by* (3)*, points* (2) *and* (3) *still hold, and moreover* $0 \le n_\varepsilon \le 1$.

These bounds are enough for our purposes. Their proofs are fairly standard and derived in full detail in [11, 17, 19, 25], so we omit them here. Let us point out that to fully justify passing to the incompressible limit $\gamma \to \infty$, one usually needs to derive additional estimates for the time derivative of the population density and the pressure.

Remark 1.5 (More general drift term). It is easily seen in the proof of our main results that we do not require the drift velocity to be a gradient. Indeed, one can replace the term $n\nabla V$ in (1) by $nU(t,x)$ with appropriate modifications to the regularity assumptions (A-drift) and (A-BVdrift).

Our approach is to first obtain a rate of convergence in the homogeneous negative Sobolev norm $\dot{H}^{-1}$ and then interpolate with the uniform bound in BV to deduce a convergence rate in Lebesgue spaces. To realize this program we make use of the diffusion structure of the problem and "lift" the Laplacian. More precisely, we define the function φ to be the solution of the following Poisson equation in $(0,T)\times\mathbb{R}^d$,

$$-\Delta\varphi_\gamma = n_\gamma, \tag{8}$$

given by the convolution $\varphi_\gamma = \mathcal{K} \star n_\gamma$, where $\mathcal{K}$ is the fundamental solution of the Laplace equation. Explicitly, for $x \neq 0$,

$$\mathcal{K}(x) = \begin{cases} -\dfrac{1}{2\pi}\ln|x| & \text{for } d = 2, \\ \dfrac{1}{d(d-2)\omega_d}|x|^{2-d} & \text{for } d \ge 3, \end{cases}$$

where ω_d denotes the volume of the unit ball in $\mathbb{R}^d$.

Suppose for now that $d \ge 3$. Then, since $n_\gamma \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, a straightforward application of Young's inequality shows that

$$\varphi_\gamma \in L^p(\mathbb{R}^d) \quad \text{for } p > \frac{d}{d-2}$$

and

$$\nabla\varphi_\gamma \in L^2(\mathbb{R}^d).$$

If $d = 2$, then we do not have $\varphi_\gamma \in L^\infty(\mathbb{R}^2)$ and we cannot apply Young's inequality to deduce square integrability of $\nabla\varphi_\gamma$ (indeed, this is an endpoint case). However, let us point out that, for the power law case, since by Lemma 1.4 (1) solutions are always compactly supported, we can take φ_γ to be the solution of the Poisson equation on $K(T) \subset \mathbb{R}^2$ with homogeneous Dirichlet boundary conditions. In this case, we know that $\nabla\varphi_\gamma \in L^2(K)$.

Under suitable conditions it is possible to infer the L^2-integrability of $\nabla(\varphi_\varepsilon - \varphi_{\varepsilon'})$ in $\mathbb{R}^2$, which is needed for the singular pressure law. In this case, we impose the following additional assumptions:

$$g = g(t), \quad \nabla V \in L^1((0,T) \times \mathbb{R}^2), \quad \int_{\mathbb{R}^2} |x| n_\varepsilon^0 < \infty. \tag{A-2D}$$

The bound on the first moment is propagated in time and guarantees the well-posedness of $\mathcal{K} \star n_\varepsilon$. Taking a space-independent growth rate implies that the difference $n_\varepsilon - n_{\varepsilon'}$ has zero mean for all times. Therefore, we have

$$\int_{\mathbb{R}^2} (n_\varepsilon - n_{\varepsilon'}) = 0, \quad \int_{\mathbb{R}^2} |x|\, |n_\varepsilon - n_{\varepsilon'}| < \infty,$$

from which we conclude that $\nabla(\varphi_\varepsilon - \varphi_{\varepsilon'}) \in L^2(\mathbb{R}^2)$.

Notice that the L^1 convergence of the initial data implies the convergence of $\nabla\varphi_\gamma^0$ to $\nabla\varphi_\infty^0$ in L^2. Moreover, the uniform bounds on n_γ together with the Hardy–Littlewood–Sobolev inequality imply that the convolution $n_\gamma \mapsto \mathcal{K} \star n_\gamma$ is a bounded linear operator from $L^{2d/d+2}$ to L^2. Therefore there is a subsequence $\nabla\varphi_{\gamma_k}$ which converges weakly in L^2 to $\nabla\varphi_\infty$.

Finally, we recall that the gradient $\nabla\varphi$ can be used to represent the $\dot{H}^{-1}$-norm of the function n as

$$\|n(t)\|_{\dot{H}^{-1}(\mathbb{R}^d)} = \|\nabla\varphi(t)\|_{L^2(\mathbb{R}^d)}.$$

Having obtained a convergence rate in the negative norm and assuming additionally the BV bounds provided by Lemma 1.4, we will use the following interpolation inequality, proved (in greater generality) by Cohen et al. [8] (see also [7]) to deduce a rate in the Lebesgue 4/3-norm:

Lemma 1.6 (Interpolation inequality). *There exists a constant $C = C(d,T) > 0$ such that, for all $t \in [0,T]$,*

$$\|n(t)\|_{L^{4/3}(\mathbb{R}^d)} \le C |n(t)|^{1/2}_{\mathrm{BV}(\mathbb{R}^d)} \|\nabla\varphi(t)\|^{1/2}_{L^2(\mathbb{R}^d)}. \tag{9}$$

Thus, Theorem 1.2 is a simple consequence of Theorem 1.1, Lemma 1.6, and the uniform bound in BV provided by Lemma 1.4.

By the usual log-convex interpolation of L^p-norms we readily obtain the following corollary to Theorem 1.2.

Corollary 1.7 (Convergence rate in L^p).

$$\sup_{t\in[0,T]} \|n_\gamma(t) - n_\infty(t)\|_{L^p(\mathbb{R}^d)} \le \frac{C}{\gamma^\alpha},$$

with

$$\alpha := \begin{cases} \dfrac{p-1}{p} & \textit{for } p \in (1, 4/3], \\ \dfrac{1}{3p} & \textit{for } p \in [4/3, \infty). \end{cases}$$

Remark 1.8 (Finite speed of propagation). When one assumes additionally that the initial data have uniformly compact support, then at any later time the support of n_γ is still uniformly contained in a bounded set (this is one of the fundamental properties of the porous medium equation; see [25, Lemma 2.6] and [19, Lemma 3.3] for the model with a non-zero right-hand side). Therefore, one can consider problem (1) to be posed on a bounded subset of $\mathbb{R}^d$ with homogeneous Dirichlet boundary condition. Naturally, our results remain true in this case with the improvement that we obtain a rate $\sim \gamma^{-1/4}$ in any L^p-norm, $1 \le p \le 4/3$. In particular, this covers the case of "patches", i.e. when the initial distribution is given by an indicator function of a compact set, as considered recently in [1].

Plan of the paper. The remainder of the paper is devoted to proving the main theorem. It turns out that the equation can be conveniently trisected and dealt with term by term: the pressure-driven advection, drift, and proliferation are considered separately. Indeed, it is the diffusion term that governs the rate of convergence. The proof is therefore structured as follows. In Sections 2 and 3 we prove the main theorem for the choice of the singular pressure in (3) and the power law pressure in (2) in the absence of reactions and drift. Then in Section 4 we explain how to treat the additional terms.

Notation. Henceforth we shall usually suppress the dependence on time and space of the quantities of interest, only exhibiting the time variable in the final results. Similarly, for the sake of brevity, all space integration should be understood with respect to the d-dimensional Lebesgue measure.

2. Singular pressure law

In this and the following section, to explain the main idea in a simple situation, we ignore the drift and proliferation terms in (1) and consider only the nonlinear diffusion equation

$$\frac{\partial n_\varepsilon}{\partial t} - \nabla \cdot (n_\varepsilon \nabla p_\varepsilon) = 0, \tag{10}$$

assuming now the pressure law as in (3). In this case we can rewrite (10) as

$$\frac{\partial n_\varepsilon}{\partial t} - \Delta H_\varepsilon(n_\varepsilon) = 0,$$

with

$$H_\varepsilon(n_\varepsilon) := \int_0^{n_\varepsilon} s p_\varepsilon'(s)\, \mathrm{d}s = \varepsilon \frac{n_\varepsilon}{1-n_\varepsilon} + \varepsilon \ln(1-n_\varepsilon).$$

Recall that we have the uniform bound $n_\varepsilon < 1$, so that the right-hand side above is well defined with $\ln(1-n_\varepsilon) \le 0$.

Let us take $\varepsilon > \varepsilon' > 0$. We subtract the equation for $n_{\varepsilon'}$ from the equation for n_ε to obtain

$$\frac{\partial(n_\varepsilon - n_{\varepsilon'})}{\partial t} - \Delta(H_\varepsilon(n_\varepsilon) - H_{\varepsilon'}(n_{\varepsilon'})) = 0. \tag{11}$$

Now we pose (8) for both solutions n_ε and $n_{\varepsilon'}$:

$$-\Delta\varphi_\varepsilon = n_\varepsilon, \quad -\Delta\varphi_{\varepsilon'} = n_{\varepsilon'}.$$

Then (11) reads

$$-\Delta\frac{\partial(\varphi_\varepsilon - \varphi_{\varepsilon'})}{\partial t} - \Delta(H_\varepsilon(n_\varepsilon) - H_{\varepsilon'}(n_{\varepsilon'})) = 0,$$

and we test it against $\varphi_\varepsilon - \varphi_{\varepsilon'}$ to derive

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}|\nabla(\varphi_\varepsilon - \varphi_{\varepsilon'})|^2 = \int_{\mathbb{R}^d}(n_\varepsilon - n_{\varepsilon'})(H_{\varepsilon'}(n_{\varepsilon'}) - H_\varepsilon(n_\varepsilon)).$$

We now proceed to estimate the right-hand side. On the set $\{n_\varepsilon > n_{\varepsilon'}\}$ we make use of the non-negativity of $H_\varepsilon(n_\varepsilon)$ and non-positivity of the logarithmic term in $H_{\varepsilon'}(n_{\varepsilon'})$ to write

$$\begin{aligned}\int_{\{n_\varepsilon > n_{\varepsilon'}\}}(n_\varepsilon - n_{\varepsilon'})(H_{\varepsilon'}(n_{\varepsilon'}) - H_\varepsilon(n_\varepsilon)) &\le \varepsilon'\int_{\{n_\varepsilon > n_{\varepsilon'}\}}(n_\varepsilon - n_{\varepsilon'})\frac{n_{\varepsilon'}}{1-n_{\varepsilon'}}\\ &\le \varepsilon'\int_{\{n_\varepsilon > n_{\varepsilon'}\}} n_{\varepsilon'}.\end{aligned}$$

Similarly, on the complementary set $\{n_\varepsilon \le n_{\varepsilon'}\}$ we have

$$\int_{\{n_\varepsilon \le n_{\varepsilon'}\}}(n_\varepsilon - n_{\varepsilon'})(H_{\varepsilon'}(n_{\varepsilon'}) - H_\varepsilon(n_\varepsilon)) \le \varepsilon\int_{\{n_\varepsilon \le n_{\varepsilon'}\}}(n_{\varepsilon'} - n_\varepsilon)\frac{n_\varepsilon}{1-n_\varepsilon} \le \varepsilon\int_{\{n_\varepsilon \le n_{\varepsilon'}\}} n_\varepsilon.$$

Therefore we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}|\nabla(\varphi_\varepsilon - \varphi_{\varepsilon'})|^2 &\le \varepsilon\int_{\{n_\varepsilon \le n_{\varepsilon'}\}} n_\varepsilon + \varepsilon'\int_{\{n_\varepsilon \ge n_{\varepsilon'}\}} n_{\varepsilon'}\\ &\le \varepsilon\|n_\varepsilon(t)\|_{L^1(\mathbb{R}^d)} + \varepsilon'\|n_{\varepsilon'}(t)\|_{L^1(\mathbb{R}^d)},\end{aligned}$$

and since n_ε and $n_{\varepsilon'}$ are uniformly bounded in $L^\infty((0,T), L^1(\mathbb{R}^d))$ with respect to ε and ε', we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}|\nabla(\varphi_\varepsilon - \varphi_{\varepsilon'})(t)|^2 \le C(\varepsilon + \varepsilon').$$

Integrating in time on $[0,t)$ we then have

$$\frac{1}{2}\int_{\mathbb{R}^d}|\nabla(\varphi_\varepsilon - \varphi_{\varepsilon'})(t)|^2 \le Ct(\varepsilon + \varepsilon') + \int_{\mathbb{R}^d}|\nabla(\varphi_\varepsilon - \varphi_{\varepsilon'})(0)|^2.$$

It follows that the sequence $(\nabla\varphi_\varepsilon)_\varepsilon$ converges in the strong topology of $L^\infty((0,T), L^2(\mathbb{R}^d))$ to $\nabla\varphi_\infty$. Consequently, letting $\varepsilon' \to 0$, we deduce the following rate for the convergence $n_\varepsilon \to n_\infty$ in the space $\dot{H}^{-1}(\mathbb{R}^d)$:

$$\|n_\varepsilon(t) - n_\infty(t)\|_{\dot{H}^{-1}(\mathbb{R}^d)} \le C\sqrt{t}\sqrt{\varepsilon} + \|n_\varepsilon^0 - n_\infty^0\|_{\dot{H}^{-1}(\mathbb{R}^d)},$$

where C is a positive constant defined as

$$C = \sqrt{2\sup_{\varepsilon>0}\|n_\varepsilon\|_{L^1((0,T)\times\mathbb{R}^d))}}.$$

Assuming the additional BV bounds for the initial data, we get from Lemma 1.4 that n_ε is uniformly bounded in $L^\infty(0,T;\mathrm{BV}(\mathbb{R}^d))$, and we can use (9) to obtain the rate $\varepsilon^{1/4}$, as announced in (4). Thus Theorems 1.1 and 1.2 are proved in this special case.

3. Power law

Let us now consider (10) with the pressure law given by (2) and demonstrate that the method employed in the previous section remains valid. We now have the porous medium equation

$$\frac{\partial n_\gamma}{\partial t} - \Delta n_\gamma^\gamma = 0.$$

Let us recall that there exists a positive constant p_M such that

$$0 \le \frac{\gamma}{\gamma-1} n_\gamma^{\gamma-1} \le p_M, \quad 0 \le \frac{\gamma'}{\gamma'-1} n_{\gamma'}^{\gamma'-1} \le p_M.$$

Let us define

$$c_\gamma := \left(\frac{\gamma-1}{\gamma}\right)^{\frac{1}{\gamma-1}} p_M^{1/(\gamma-1)} \quad \text{and} \quad \tilde{n}_\gamma := \frac{n_\gamma}{c_\gamma}.$$

Then it immediately follows that $\tilde{n}_\gamma \le 1$ and solves the equation

$$\partial_t \tilde{n}_\gamma - \Delta(c_\gamma^{\gamma-1}\tilde{n}_\gamma^\gamma) = 0.$$

Following the same argument as before, we define φ_γ and $\tilde{\varphi}_\gamma$ by

$$-\Delta\varphi_\gamma = n_\gamma, \quad -\Delta\tilde{\varphi}_\gamma = \tilde{n}_\gamma,$$

i.e. $\tilde{\varphi}_\gamma = \varphi_\gamma / c_\gamma$.

Without loss of generality, we take $1 < \gamma < \gamma'$. Now we subtract the equation for $\tilde{n}_{\gamma'}$ from the equation for $\tilde{n}_\gamma$ to obtain

$$\frac{\partial(\tilde{n}_\gamma - \tilde{n}_{\gamma'})}{\partial t} - \Delta(c_\gamma^{\gamma-1}\tilde{n}_\gamma^\gamma - c_{\gamma'}^{\gamma'-1}\tilde{n}_{\gamma'}^{\gamma'}) = 0. \tag{12}$$

Then from (12) we have

$$-\Delta\frac{\partial(\tilde\varphi_\gamma-\tilde\varphi_{\gamma'})}{\partial t}-\Delta(c_\gamma^{\gamma-1}\tilde n_\gamma^\gamma-c_{\gamma'}^{\gamma'-1}\tilde n_{\gamma'}^{\gamma'})=0,$$

and we test it against $\tilde\varphi_\gamma-\tilde\varphi_{\gamma'}$ to deduce

$$\begin{aligned}\frac12\frac{\mathrm d}{\mathrm dt}\int_{\mathbb R^d}|\nabla(\tilde\varphi_\gamma-\tilde\varphi_{\gamma'})|^2&=\int_{\mathbb R^d}(c_\gamma^{\gamma-1}\tilde n_\gamma^\gamma-c_{\gamma'}^{\gamma'-1}\tilde n_{\gamma'}^{\gamma'})(\tilde n_{\gamma'}-\tilde n_\gamma)\\&\le\int_{\mathbb R^d}c_\gamma^{\gamma-1}\tilde n_\gamma^\gamma(1-\tilde n_\gamma)+\int_{\mathbb R^d}c_{\gamma'}^{\gamma'-1}\tilde n_{\gamma'}^{\gamma'}(1-\tilde n_{\gamma'}),\end{aligned}\tag{13}$$

where the inequality follows from the fact that $\tilde n_\gamma,\tilde n_{\gamma'}\le1$. It is easy to see that for $0\le s\le1$ it holds that $s^\gamma(1-s)\le\frac{s}{\gamma}$. Hence, we have

$$\begin{aligned}\frac12\frac{\mathrm d}{\mathrm dt}\int_{\mathbb R^d}|\nabla(\tilde\varphi_\gamma-\tilde\varphi_{\gamma'})|^2&\le c_\gamma^{\gamma-1}\frac1\gamma\int_{\mathbb R^d}\tilde n_\gamma+c_{\gamma'}^{\gamma'-1}\frac1{\gamma'}\int_{\mathbb R^d}\tilde n_{\gamma'}\\&\le\Big(\frac{\gamma-1}{\gamma}p_M\sup_\gamma\|\tilde n_\gamma(t)\|_{L^1(\mathbb R^d)}\Big)\frac1\gamma\\&\quad+\Big(\frac{\gamma'-1}{\gamma'}p_M\sup_{\gamma'}\|\tilde n_{\gamma'}(t)\|_{L^1(\mathbb R^d)}\Big)\frac1{\gamma'}\\&\le C\Big(\frac1\gamma+\frac1{\gamma'}\Big),\end{aligned}$$

where in the last inequality we used the fact that by Lemma 1.4, n_γ is uniformly bounded in $L^\infty(0,T;L^1(\mathbb R^d))$. Finally, we remove the scaling using the triangle inequality:

$$\begin{aligned}&\frac13\|\nabla(\varphi_\gamma-\varphi_{\gamma'})(t)\|^2_{L^2(\mathbb R^d)}\\&\quad\le\|\nabla(\varphi_\gamma-\tilde\varphi_\gamma)(t)\|^2_{L^2(\mathbb R^d)}+\|\nabla(\tilde\varphi_{\gamma'}-\varphi_{\gamma'})(t)\|^2_{L^2(\mathbb R^d)}+\|\nabla(\tilde\varphi_\gamma-\tilde\varphi_{\gamma'})(t)\|^2_{L^2(\mathbb R^d)}\\&\quad\le\Big|1-\frac1{c_\gamma}\Big|^2\|\nabla\varphi_\gamma(t)\|^2_{L^2(\mathbb R^d)}+\Big|1-\frac1{c_{\gamma'}}\Big|^2\|\nabla\varphi_{\gamma'}(t)\|^2_{L^2(\mathbb R^d)}\\&\qquad+Ct\Big(\frac1\gamma+\frac1{\gamma'}\Big)+\|\nabla(\tilde\varphi_\gamma-\tilde\varphi_{\gamma'})(0)\|^2_{L^2(\mathbb R^d)}\\&\quad\le\frac1\gamma\Big(Ct+\gamma\Big|1-\frac1{c_\gamma}\Big|^2\sup_\gamma\|n_\gamma(t)\|^2_{\dot H^{-1}(\mathbb R^d)}\Big)\\&\qquad+\frac1{\gamma'}\Big(Ct+\gamma'\Big|1-\frac1{c_{\gamma'}}\Big|^2\sup_{\gamma'}\|n_{\gamma'}(t)\|^2_{\dot H^{-1}(\mathbb R^d)}\Big)+\|\nabla(\tilde\varphi_\gamma-\tilde\varphi_{\gamma'})(0)\|^2_{L^2(\mathbb R^d)}.\end{aligned}$$

By the definition of c_γ, $\gamma|1-\frac1{c_\gamma}|^2\to0$ as $\gamma\to\infty$. Thus, we have

$$\|\nabla(\varphi_\gamma-\varphi_{\gamma'})(t)\|^2_{L^2(\mathbb R^d)}\le(Ct+C)\Big(\frac1\gamma+\frac1{\gamma'}\Big)+3\|\nabla(\tilde\varphi_\gamma-\tilde\varphi_{\gamma'})(0)\|^2_{L^2(\mathbb R^d)}.$$

By the same argument, we find

$$\|\nabla(\tilde{\varphi}_\gamma - \tilde{\varphi}_{\gamma'})(0)\|^2_{L^2(\mathbb{R}^d)} \le C\Big(\frac{1}{\gamma} + \frac{1}{\gamma'}\Big) + 3\|\nabla(\varphi_\gamma - \varphi_{\gamma'})(0)\|^2_{L^2(\mathbb{R}^d)}.$$

Finally, we conclude

$$\|\nabla(\varphi_\gamma - \varphi_{\gamma'})(t)\|^2_{L^2(\mathbb{R}^d)} \le (Ct + C)\Big(\frac{1}{\gamma} + \frac{1}{\gamma'}\Big) + 9\|\nabla(\varphi_\gamma - \varphi_{\gamma'})(0)\|^2_{L^2(\mathbb{R}^d)}. \tag{14}$$

Consequently, arguing as before and letting $\gamma' \to \infty$, we find

$$\|n_\gamma(t) - n_\infty(t)\|_{\dot{H}^{-1}(\mathbb{R}^d)} \le \frac{C\sqrt{t} + C}{\sqrt{\gamma}} + 9\|n^0_\gamma - n^0_\infty\|_{\dot{H}^{-1}(\mathbb{R}^d)}.$$

Again, under the additional BV assumptions we obtain (4) thanks to the interpolation inequality in Lemma 1.6.

4. Including drift and reaction terms

Having obtained the announced rate of convergence due to the nonlinear diffusion term, we now show that we can include the drift and reaction terms. In fact, due to our assumptions on the proliferation rate and the chemical potential, all the additional terms will either have an appropriate sign, or be absorbed into the L^2-norm of the potential φ. We now write (1) as

$$\frac{\partial n_\gamma}{\partial t} - \Delta A_\gamma(n_\gamma) = \nabla \cdot (n_\gamma \nabla V) + n_\gamma g,$$

where $g = g(t,x)$ and A_γ is chosen appropriately depending on the state law for the pressure. As seen before, there is no harm in assuming the uniform bound $n \le 1$. Then, arguing in the same way as previously, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |\nabla(\varphi_\gamma - \varphi_{\gamma'})|^2 + \int_{\mathbb{R}^d} (n_\gamma - n_{\gamma'})(A_\gamma(n_\gamma) - A_{\gamma'}(n_{\gamma'}))&\\
= -\int_{\mathbb{R}^d} (n_\gamma - n_{\gamma'})\nabla(\varphi_\gamma - \varphi_{\gamma'}) \cdot \nabla V + \int_{\mathbb{R}^d} g(t,x)(n_\gamma - n_{\gamma'})(\varphi_\gamma - \varphi_{\gamma'})&\\
= \int_{\mathbb{R}^d} \Delta(\varphi_\gamma - \varphi_{\gamma'})\nabla(\varphi_\gamma - \varphi_{\gamma'}) \cdot \nabla V - \int_{\mathbb{R}^d} g(t,x)\Delta(\varphi_\gamma - \varphi_{\gamma'})(\varphi_\gamma - \varphi_{\gamma'}).&
\end{aligned}$$

It only remains to consider the two new terms on the right-hand side. For the first one we can write

$$\begin{aligned}
&\int_{\mathbb{R}^d} \Delta(\varphi_\gamma - \varphi_{\gamma'})\nabla(\varphi_\gamma - \varphi_{\gamma'}) \cdot \nabla V\\
&\quad = -\int_{\mathbb{R}^d} \nabla(\varphi_\gamma - \varphi_{\gamma'})^T D^2(\varphi_\gamma - \varphi_{\gamma'})\nabla V - \int_{\mathbb{R}^d} \nabla(\varphi_\gamma - \varphi_{\gamma'})^T D^2 V \nabla(\varphi_\gamma - \varphi_{\gamma'})
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\int_{\mathbb{R}^d} \nabla|\nabla(\varphi_\gamma - \varphi_{\gamma'})|^2 \cdot \nabla V - \int_{\mathbb{R}^d} \nabla(\varphi_\gamma - \varphi_{\gamma'})^T D^2 V \nabla(\varphi_\gamma - \varphi_{\gamma'}) \\
&= \frac{1}{2}\int_{\mathbb{R}^d} |\nabla(\varphi_\gamma - \varphi_{\gamma'})|^2 \Delta V - \int_{\mathbb{R}^d} \nabla(\varphi_\gamma - \varphi_{\gamma'})^T D^2 V \nabla(\varphi_\gamma - \varphi_{\gamma'}) \\
&\leq -\lambda \int_{\mathbb{R}^d} |\nabla(\varphi_\gamma - \varphi_{\gamma'})|^2,
\end{aligned}
$$

where we have integrated by parts and used assumptions (A-drift). For the remaining term we integrate by parts to obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^d} g|\nabla(\varphi_\gamma - \varphi_{\gamma'})|^2 + \int_{\mathbb{R}^d} (\varphi_\gamma - \varphi_{\gamma'})\nabla(\varphi_\gamma - \varphi_{\gamma'}) \cdot \nabla g \\
&\qquad \leq \|g_+\|_{L^\infty((0,T)\times\mathbb{R}^d)} \int_{\mathbb{R}^d} |\nabla(\varphi_\gamma - \varphi_{\gamma'})|^2 + \underbrace{\int_{\mathbb{R}^d} (\varphi_\gamma - \varphi_{\gamma'})\nabla(\varphi_\gamma - \varphi_{\gamma'}) \cdot \nabla g}_{\mathcal{A}}.
\end{aligned}
$$

In case of $d = 2$, we suppose that g satisfies assumption (A-reaction). Then we can integrate by parts in the last term to obtain

$$
\mathcal{A} = -\frac{1}{2}\int_{\mathbb{R}^d} |\varphi_\gamma - \varphi_{\gamma'}|^2 \Delta g \leq 0.
$$

If instead $d \geq 3$, we may alternatively assume that g satisfies assumption (A-reaction$'$) or assumption (A-reaction$''$). In the first case, successively using the Hölder and Sobolev inequalities, we obtain

$$
\mathcal{A} \leq \frac{1}{2}\|\varphi_\gamma - \varphi_{\gamma'}\|^2_{L^{2^*}(\mathbb{R}^d)} \|(\Delta g)_-\|_{L^{d/2}(\mathbb{R}^d)} \leq C_S \|(\Delta g)_-\|_{L^{d/2}(\mathbb{R}^d)} \int_{\mathbb{R}^d} |\nabla(\varphi_\gamma - \varphi_{\gamma'})|^2,
$$

where C_S denotes the constant from the Sobolev inequality, and $2^* = \frac{2d}{d-2}$ is the Sobolev conjugate exponent. Otherwise, if g satisfies (A-reaction$''$), to estimate the term $\mathcal{A}$ we do not integrate it by parts, but we use in turn the Young, Hölder, and Sobolev inequalities, to obtain

$$
\begin{aligned}
2\mathcal{A} &\leq \int_{\mathbb{R}^d} |\nabla(\varphi_\gamma - \varphi_{\gamma'})|^2 + \int_{\mathbb{R}^d} |(\varphi_\gamma - \varphi_{\gamma'})|^2 |\nabla g|^2 \\
&\leq \int_{\mathbb{R}^d} |\nabla(\varphi_\gamma - \varphi_{\gamma'})|^2 + \|\varphi_\gamma - \varphi_{\gamma'}\|^2_{L^{2^*}(\mathbb{R}^d)} \|\nabla g\|^2_{L^d(\mathbb{R}^d)} \\
&\leq (1 + C_S \|\nabla g\|^2_{L^d(\mathbb{R}^d)}) \int_{\mathbb{R}^d} |\nabla(\varphi_\gamma - \varphi_{\gamma'})|^2.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |\nabla(\varphi_\gamma - \varphi_{\gamma'})|^2 + \int_{\mathbb{R}^d} (n_\gamma - n_{\gamma'})(A_\gamma(n_\gamma) - A_{\gamma'}(n_{\gamma'})) \\
&\qquad \leq C \int_{\mathbb{R}^d} |\nabla(\varphi_\gamma - \varphi_{\gamma'})|^2.
\end{aligned}
$$

Assuming for concreteness the power law pressure, using inequality (14) and a Grönwall inequality, we deduce

$$\sup_{t\in[0,T]} \|\nabla(\varphi_\gamma - \varphi_{\gamma'})(t)\|_{L^2(\mathbb{R}^d)} \le C\Big(\frac{1}{\sqrt{\gamma}} + \frac{1}{\sqrt{\gamma'}}\Big) + \|\nabla(\varphi_\gamma - \varphi_{\gamma'})(0)\|_{L^2(\mathbb{R}^d)}.$$

Finally, passing to the limit $\gamma' \to \infty$, we conclude the proof of Theorem 1.1. Using the uniform BV-bound and (9) we obtain Theorem 1.2.

4.1. Limit relation between n_∞ and p_∞

Here we prove relation (5) between the limit density and pressure, where p_∞ is defined as the weak* limit (up to a subsequence) of p_γ in $L^\infty((0,T)\times\mathbb{R}^d)$.

Proof of Theorem 1.3. The relation is a straightforward consequence of the main estimate obtained in Section 3. We inspect (13), this time not ignoring the non-positive terms. After integration in time, using (14) these terms can be bounded as

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} \tilde{n}_{\gamma'}^{\gamma'}(1-\tilde{n}_\gamma)c_{\gamma'}^{\gamma'-1} + \int_0^T \int_{\mathbb{R}^d} \tilde{n}_\gamma^{\gamma}(1-\tilde{n}_{\gamma'})c_\gamma^{\gamma-1} \\ \le C(T)\Big(\frac{1}{\gamma} + \frac{1}{\gamma'}\Big) + \int_{\mathbb{R}^d} |\nabla(\varphi_\gamma - \varphi_{\gamma'})(0)|^2.\end{aligned}$$

Now let ψ be a compactly supported test function and consider the quantity

$$\begin{aligned}\Big|\int_0^T \int_{\mathbb{R}^d} \psi \tilde{n}_\gamma^\gamma (1-\tilde{n}_{\gamma'})\Big| &\le \|\psi\|_\infty \int_0^T \int_{\operatorname{supp}\psi} \tilde{n}_\gamma^\gamma(1-\tilde{n}_{\gamma'}) \\ &= \|\psi\|_\infty \int_0^T \int_{\operatorname{supp}\psi} \tilde{p}_\gamma^{\frac{\gamma}{\gamma-1}}(1-\tilde{n}_{\gamma'}).\end{aligned}$$

Using weak lower semicontinuity of convex functionals and weak* convergence of the pressure and the density, we can pass to the limit with γ' and γ in turn to obtain

$$\int_0^T \int_{\mathbb{R}^d} \psi p_\infty (1-n_\infty) = 0,$$

which concludes the proof. ∎

5. Conclusions and open problems

We computed the rate of convergence of the solutions of a reaction–advection–diffusion equation of porous medium type in the incompressible limit. Our result in a negative Sobolev norm can be interpolated with uniform BV-estimates in order to find a rate in any L^p-space for $1 < p < \infty$.

How to assess the accuracy of our estimate remains an open problem. For the pure porous medium equation it might seem tempting to attempt a calculation for the illustrious example of the Barenblatt solution (taking as initial data the solution at some time $t > 0$). However, a direct calculation shows that in this case the data is “ill prepared” in the sense that it converges (in L^1) to its limit profile with too slow a rate of $\sim \ln \gamma/\gamma$. It is unclear how to approach the question of optimality in general. We expect that the “worst” rate would be exhibited by a *focusing solution*, whose support is initially contained outside a compact set and closes up in finite time, thus generating a singularity.

Another challenging problem is to find an estimate for the convergence rate of the pressure, for which the method used above seems inapplicable as it is not clear how to relate the quantities $p_\gamma - p_{\gamma'}$ and $\varphi_\gamma - \varphi_{\gamma'}$. Consequently, we are also currently unable to treat more general, pressure-dependent, reaction terms. Finally, it would be of interest to investigate whether it is possible to strengthen the estimate of Theorem 1.1 to Lebesgue norms without interpolation with BV. One advantage of any such alternative approach could be to allow for passing to the incompressible limit when BV bounds are not available, as is the case for systems of equations like (1). Additionally, it could allow for estimating the rate of convergence in the L^1-norm rather than the seemingly arbitrary $L^{4/3}$-norm.

A. Bounding the W_2-norm by the $\dot{H}^{-1}$-norm

We consider here the conservative case of (1), assuming $\int n_\gamma(t) = \int n_\infty(t) = 1$. Moreover, rather than the Cauchy problem set in the whole space, we consider the boundary value problem set in a bounded domain $\Omega \subset \mathbb{R}^d$ with homogeneous Neumann boundary conditions.

We put $\mathrm{d}\mu_\gamma = n_\gamma(x)\,\mathrm{d}x$, $\mathrm{d}\mu_\infty = n_\infty(x)\,\mathrm{d}x$, ignoring time dependence for the sake of brevity. Furthermore, we make the additional assumption that $n_\infty \geq \underline{n} > 0$ for some constant $\underline{n}$.

Consider the curve $\rho\colon [0,1] \to \mathcal{P}_2(\mathbb{R}^d)$ given by $\tau \mapsto \rho_\tau := (1-\tau)\mu_\gamma + \tau\mu_\infty$, together with the vector field

$$V_\tau(x) = \frac{1}{(1-\tau)n_\gamma(x) + \tau n_\infty(x)} \nabla(\varphi_\gamma - \varphi_\infty).$$

For any test function $\psi \in C_c^\infty((0,1)\times\Omega)$ we have

$$\begin{aligned}
\int_0^1 \int_\Omega \frac{\partial \psi}{\partial \tau}\,\mathrm{d}\rho_\tau(x)\,\mathrm{d}\tau &= \int_0^1 \int_\Omega \frac{\partial \psi}{\partial \tau}((1-\tau)n_\gamma(x) + \tau n_\infty(x))\,\mathrm{d}x\,\mathrm{d}\tau \\
&= \int_0^1 \int_\Omega \psi(n_\gamma(x) - n_\infty(x))\,\mathrm{d}x\,\mathrm{d}\tau \\
&= \int_0^1 \int_\Omega \nabla\psi \cdot \nabla(\varphi_\gamma - \varphi_\infty)\,\mathrm{d}x\,\mathrm{d}\tau \\
&= \int_0^1 \int_\Omega \nabla\psi \cdot V_\tau\,\mathrm{d}\rho_\tau(x)\,\mathrm{d}\tau.
\end{aligned}$$

Therefore, the pair (ρ, V) solves the continuity equation

$$\frac{\partial \rho_\tau}{\partial \tau} + \nabla \cdot (V_\tau(x)\rho_\tau) = 0,$$

posed on $(0,1) \times \mathbb{R}^d$ with the marginal constraints

$$\rho_0 = \mu_\gamma, \quad \rho_1 = \mu.$$

Consequently, from [27, Theorem 5.15], we deduce that ρ is absolutely continuous and the inequality

$$|\rho'|(\tau) \le \|V_\tau\|_{L^2(\mathbb{R}^d, \mathrm{d}\rho_\tau)}$$

holds, where $|\rho'|$ denotes the metric derivative of the curve ρ with respect to the Wasserstein distance. Furthermore, since $(\mathcal{P}_2(\mathbb{R}^d), W_2)$ is a length space, we have

$$W_2(\mu_\gamma, \mu_\infty) \le \int_0^1 |\rho'|(\tau)\, \mathrm{d}\tau.$$

Combining these last two inequalities, we obtain the bound

$$\begin{aligned} W_2(\mu_\gamma, \mu_\infty) &\le \int_0^1 \|V_\tau(x)\|_{L^2(\mathbb{R}^d, \mathrm{d}\rho_\tau)}\, \mathrm{d}\tau \\ &\le \frac{1}{\sqrt{\underline{n}}} \|\nabla(\varphi_\gamma - \varphi_\infty)\|_{L^2(\mathbb{R}^d)} \int_0^1 \frac{1}{\sqrt{\tau}}\, \mathrm{d}\tau \\ &= \frac{2}{\sqrt{\underline{n}}} \|n_\gamma - n_\infty\|_{\dot{H}^{-1}(\mathbb{R}^d)}. \end{aligned}$$

Interestingly, a reverse bound can also be shown. Rather than a positive lower bound, a common upper bound is now required of all the densities (which is of course the case here). Now let $\sigma\colon [0,1] \to \mathcal{P}_2(\mathbb{R}^d)$ be a constant-speed geodesic from μ_γ to μ_∞ and E be a vector field such that (σ, E) satisfy the continuity equation, and $\|E_\tau\|_{L^2(\mathbb{R}^d;\sigma_\tau)} = W_2(\mu_\gamma, \mu_\infty)$. Then

$$\begin{aligned} \|\nabla\varphi_\gamma - \nabla\varphi_\infty\|_{L^2}^2 &= \int_\Omega (\varphi_\gamma - \varphi_\infty)(n_\gamma - n_\infty) \\ &= \int_0^1 \int_\Omega \nabla(\varphi_\gamma - \varphi_\infty) \cdot E_\tau\, \mathrm{d}\rho_\tau\, \mathrm{d}\tau \\ &\le \frac{1}{2}\|\nabla\varphi_\gamma - \nabla\varphi_\infty\|_{L^2}^2 + \frac{1}{2} W_2(\mu_\gamma, \mu_\infty)^2. \end{aligned}$$

We refer the reader to [27, Section 5.5.2], and references therein, for further discussion of the equivalence of the two distances.

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