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The pressureless damped Euler–Riesz equations

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Abstract. In this paper, we analyze the pressureless damped Euler–Riesz equations posed in either $\mathbb{R}^d$ or $\mathbb{T}^d$. We construct the global-in-time existence and uniqueness of classical solutions for the system around a constant background state. We also establish large-time behaviors of classical solutions showing the solutions towards the equilibrium as time goes to infinity. For the whole space case, we first show an algebraic decay rate of solutions under additional assumptions on the initial data compared to the existence theory. We then refine the argument to have an exponential decay rate of convergence even in the whole space. In the case of the periodic domain, without any further regularity assumptions on the initial data, we provide the exponential convergence of solutions.

1. Introduction

In this paper we are interested in the global well-posedness and large-time behavior for the pressureless Euler–Riesz equations with linear damping posed either in $\Omega = \mathbb{R}^d$ or $\mathbb{T}^d$:

$$\begin{aligned} &\partial_t \rho + \nabla \cdot (\rho u) = 0, \quad x \in \Omega,\ t > 0, \\ &\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) = -\gamma \rho u - \lambda \rho \nabla \Lambda^{\alpha - d} (\rho - c), \end{aligned} \tag{1.1}$$

subject to the initial data

$$(\rho, u)|_{t=0} := (\rho_0, u_0), \quad x \in \Omega, \tag{1.2}$$

where $\rho = \rho(x,t)$ and $u = u(x,t)$ denote the density and velocity of the fluid at time t and position x, respectively. Here, the Riesz operator Λ^s is defined by $(-\Delta)^{s/2}$, and we concentrate on the case $d - 2 < \alpha < d$. The coefficients γ and λ are positive constants, and $c > 0$ is the nonzero background state. For $\Lambda^{\alpha-d}(\rho - c)$ to be well defined, we impose the neutrality condition

$$\int_\Omega (\rho - c)\, dx = 0.$$

Without loss of generality, for simplicity of presentation, we set $\gamma = \lambda = c = 1$.

The pressureless Euler–Riesz system has recently been derived in [27] from the N-interacting particle system governed by Newton's laws. In [27], the interaction between

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particles is given by the force fields ∇K, where $K(x) = |x|^{-\alpha}$ with $d-2 < \alpha < d$, and the modulated kinetic and interaction energies are employed to show the quantitative error estimate between the particle and Euler–Riesz systems. We would like to remark that the case $\alpha = d-2$ with $d \geq 3$ corresponds to the Coulomb potential, and the case $\max\{d-2, 0\} < \alpha < d$ with $d \geq 1$ is called the Riesz potential. The local well-posedness theory for system (1.1) is developed in [12] and the case $(\alpha, d) = (1, 2)$ is discussed in [1]. Strictly speaking, in [12], system (1.1) with zero background state in the undamped case, i.e. $c = 0$ and $\gamma = 0$, is considered; however, a small modification of the strategy used in [12] leads to establishing the local existence and uniqueness of classical solutions to system (1.1); see Theorem 3.1 below for more detailed discussion.

The main purpose of the current work is to establish the global-in-time existence and uniqueness of classical solutions to the pressureless damped Euler–Riesz system (1.1) and its large-time behavior. We would like to emphasize that to the best of our knowledge, even for the multidimensional Euler–Poisson system, in the absence of the pressure, much less is known about the global-in-time regularity of classical solutions or the large-time behavior estimate. For the one-dimensional case, a critical threshold on the initial data distinguishing the global-in-time regularity of solutions and finite-time singularity formation for the pressureless Euler–Poisson system is analyzed in [2, 7, 15, 34]; see also [5, 31] for the case with pressure and other related systems. For higher-dimensional problems, the critical threshold estimate for the two-dimensional *restricted* Euler–Poisson system is studied in [26]; see [25] for more general discussion on the restricted flows. The global existence of smooth solutions for the Euler–Poisson system around a constant background state is discussed in [21, 22, 24]. We also refer to [16, 17, 19] for three-dimensional problems.

In order to state our first main result concerning the global well-posedness theory, we use $\rho > 0$ and let $h = \rho - 1$ to reformulate system (1.1)–(1.2) as

$$\begin{aligned} &\partial_t h + \nabla \cdot (hu) + \nabla \cdot u = 0, \quad x \in \Omega,\ t > 0, \\ &\partial_t u + u \cdot \nabla u = -u - \nabla \Lambda^{\alpha - d} h, \end{aligned} \tag{1.3}$$

with initial data

$$(h, u)|_{t=0} =: (h_0 := \rho_0 - 1, u_0), \quad x \in \Omega. \tag{1.4}$$

For our solution spaces, we consider the following norms:

$$\|(h, u)\|_{X^m}^2 := \|h\|_{H^m}^2 + \|u\|_{H^{m+\frac{d-\alpha}{2}}}^2 + \|h\|_{\dot{H}^{-\frac{d-\alpha}{2}}}^2. \tag{1.5}$$

The notation X^m naturally denotes the space of functions with the finite corresponding norm.

Now we state our first result on the global existence and uniqueness of solutions to (1.1).

Theorem 1.1. *Consider system* (1.3) *on either* $\Omega = \mathbb{R}^d$ *or* $\mathbb{T}^d$ *with* $d \geq 1$ *and* $\max\{d-2, 0\} < \alpha < d$. *For any* $m > \frac{d}{2} + 2$, *suppose that the initial data* (ρ_0, u_0) *satisfy*

$$\inf_{x\in\Omega} h_0(x) > -1, \quad \int_\Omega h_0(x)\,dx = 0, \quad \textit{and} \quad (h_0, u_0) \in X^m.$$

If

$$\|(h_0, u_0)\|_{X^m} < \varepsilon_1$$

for some $\varepsilon_1 > 0$ *sufficiently small, then system* (1.3)–(1.4) *admits a unique solution in* $\mathcal{C}(\mathbb{R}_+; X^m)$.

As mentioned above, the local-in-time existence of solutions for system (1.1) with $\gamma = 0$ is investigated in a recent work [12]. We are currently interested in the linear velocity-damping effect on the global-in-time regularity of solutions, and as stated in Theorem 1.1, the damping can prevent the finite-time breakdown of smoothness of solutions, even in the absence of pressure, when the initial data are sufficiently small and regular. This is reminiscent of the proof of a global Cauchy problem for the compressible Euler equations with damping [29, 33]. However, we only have the Riesz interactions, not the pressure term. On the other hand, for the Euler–Poisson system around a constant background state, i.e. system (1.1) with pressure, $\alpha = d - 2$, and $\gamma = 0$, two- and three-dimensional Cauchy problems are first discussed in [21, 24] and in [16] under the consideration of irrotational flows. Compared to those works, we have linear damping in the velocity instead of the pressure term. The main difficulties lie in the analysis of the highest-order derivative estimates and the dissipation rate on the solutions due to the singularity of Riesz interactions, beyond the Coulomb ones. It is natural to expect that the linear damping gives a good dissipation rate for the velocity u. However, it is not clear how to analyze the stabilizing effect from the Riesz interactions and obtain a proper dissipation rate for the perturbed density h. In order to overcome those difficulties, inspired by [12], we estimate our solutions in the fractional Sobolev space specified in (1.5) and a modified H^m norm for h (see (3.1) below) to have some cancellation of terms with the highest-order derivatives. For the dissipation estimates for h, we clarify the dispersive effect of the Riesz interaction and establish a delicate *hypocoercivity*-type estimate which provides the higher-order dissipation rate. The proof strongly relies on the energy method based on the commutator estimates for the fractional Laplacian and Gagliardo–Nirenberg–Sobolev-type inequalities.

Remark 1.1. In [12], the finite-time singularity formation for the Euler–Riesz system with zero background state in multidimensions $d \geq 1$ is investigated. In the presence of pressure (isentropic or isothermal pressure), either the attractive or repulsive Riesz interaction case is considered, and the finite-time breakdown of smoothness of solutions is observed under suitable assumptions on the initial data. The main idea is based on the lower and upper bound estimates on the internal energy. In the attractive and pressureless case, the estimate of finite-time singularity formation in [12, Theorem 5.4] still holds.

However, it is not clear how to employ those arguments for the repulsive and pressureless case. As far as we know, the global-in-time existence or finite-in-time blowup results have been shown only for the pressureless Euler–Poisson system with zero background state, i.e. (1.1) with $\alpha = d - 2$ and $c = 0$, in a special setting. In [15, 32, 34], the geometrical symmetry was crucially used to deduce such results. To the best of our knowledge, neither the global-in-time existence nor finite-time blowup of solutions to the repulsive and pressureless Euler–Riesz system (1.1) without any symmetry assumptions, has been studied so far in the higher-dimensional case ($d \geq 2$) even for the repulsive and pressureless Euler–Poisson system.

Remark 1.2. In [11], after a suitable scaling, the strong relaxation limit of system (1.1) with the zero background state, i.e. $c = 0$, is investigated, and the following fractional porous medium equation [3, 4] is rigorously and quantitatively derived:

$$\partial_t \rho = \nabla \cdot (\rho \nabla \Lambda^{\alpha - d} \rho).$$

We would like to remark that the argument used in [11] can be extended to the nonzero constant background state case when $\Omega = \mathbb{T}^d$. The local-in-time existence and uniqueness of classical solutions for that limiting equation have been recently established in [10]. Note that as long as there exist classical solutions for those systems, the strong relaxation limit estimate holds. Thus, as a by-product of Theorem 1.1, if one can show the existence of global-in-time classical solutions to the porous medium equation, then the relaxation limit holds for all times. Note that such a global existence was covered as a special case of the singular Euler-alignment system [14, 28] in one dimension. We also refer to [6, 8, 9, 20, 23] for the strong relaxation limits of compressible Euler/Euler–Poisson systems.

Our second result provides the large-time behavior of solutions, obtained in Theorem 1.1, to system (1.1) showing algebraic or exponential decay rates of convergence of solutions in X^m when $\Omega = \mathbb{R}^d$ or $\mathbb{T}^d$.

Theorem 1.2. *Let $d \geq 2$ and the assumptions of Theorem 1.1 be satisfied.*

(i) *(Whole space case): If we additionally assume that*

$$(h_0, u_0) \in \dot{H}^{-s-\frac{d-\alpha}{2}}(\mathbb{R}^d) \times [\dot{H}^{-s}(\mathbb{R}^d)]^d \tag{1.6}$$

for some

$$s \in \left[1 - \frac{d-\alpha}{2}, \frac{\alpha}{2}\right], \tag{1.7}$$

then we have

$$\|(h,u)(\cdot,t)\|_{X^m}^2 + \|h\|_{\dot{H}^{-1+\frac{d-\alpha}{2}}}^2 + \|u\|_{\dot{H}^{d-\alpha-1}}^2 \leq C(1+t)^{-\eta}, \quad t \geq 0,$$

where $\eta > 0$ is given by

$$\eta := \min\left\{\frac{2s}{d-\alpha}, \frac{s+d-\alpha-1}{1-\frac{d-\alpha}{2}}\right\}.$$

Furthermore, if the order $s > 0$ is large enough such that ($\frac{\alpha}{2} \geq$) $s > 2 + d - \alpha$, then we have an exponential decay rate of convergence:

$$\|(h,u)(\cdot,t)\|_{X^m}^2 \leq Ce^{-\zeta t}, \quad t \geq 0$$

for some positive constants C and ζ independent of t.

(ii) *(Periodic case): There exist positive constants C and λ independent of t such that*

$$\|(h,u)(\cdot,t)\|_{X^m} \leq Ce^{-\lambda t}, \quad t \geq 0.$$

Remark 1.3. Condition (1.7) naturally requires $d \geq 2$.

Remark 1.4. Assumption (1.7) and the dimension restriction $d \geq 2$ can be relaxed in the whole space case. Indeed, if we only assume (1.6) for some $s \in (0, \alpha/2]$, then we have

$$\|(h,u)(\cdot,t)\|_{X^m}^2 \leq C(1+t)^{-\frac{s}{1+\frac{d-\alpha}{2}}}, \quad t \geq 0, \tag{1.8}$$

where C is a positive constant independent of t. Note that when $s + \frac{d-\alpha}{2} < 1$, the decay rate of convergence for the whole space case is at most

$$(1+t)^{-\frac{1-\frac{d-\alpha}{2}}{1+\frac{d-\alpha}{2}}} = (1+t)^{-1+\varepsilon}$$

for some constant $\varepsilon \in (0,1)$. In this case, even though the order $s > 0$ is only assumed to be positive, the decay rate does not depend on the dimension d; however this decay estimate provides a good decay estimate for the one-dimensional case. On the other hand, when $s = \frac{\alpha}{2}$, i.e. (1.7) holds, the decay rate becomes

$$(1+t)^{-\min\left\{\frac{\alpha}{d-\alpha}, \frac{d-\frac{\alpha}{2}-1}{1-\frac{d-\alpha}{2}}\right\}}$$

and it becomes

$$(1+t)^{-(d-1)} \quad \text{if } \alpha = d-1.$$

This shows that we can have a better decay rate of convergence in higher dimensions.

Remark 1.5. For the periodic domain case, if we are only interested in the large-time behavior of the lowest-order norm, i.e. $\|u\|_{L^2} + \|h\|_{\dot{H}^{-\frac{d-\alpha}{2}}}$, then the smallness assumption on the solutions in Theorems 1.1–1.2 is not necessarily required. More precisely, if we assume

(i) $\inf_{(x,t)\in\mathbb{T}^d\times\mathbb{R}_+} 1 + h(x,t) \geq h_{\min} > 0$,

(ii) $h \in W^{1,\infty}(\mathbb{T}^d \times \mathbb{R}_+)$, $\nabla \cdot u \in L^\infty(\mathbb{R}_+; [L^\infty(\mathbb{T}^d)]^d)$,

then we have

$$\|u(\cdot,t)\|_{L^2} + \|h(\cdot,t)\|_{\dot{H}^{-\frac{d-\alpha}{2}}} \leq Ce^{-\lambda t}.$$

Here C and λ are positive constants independent of t. In fact, the above estimate plays a crucial role in establishing an exponential decay rate of convergence of $\|(h,u)(\cdot,t)\|_{X^m}^2$; see Proposition 4.2 below.

Remark 1.6. All the results in Theorems 1.1 and 1.2 can be readily extended to the Coulomb interaction case, i.e. system (1.3) with $\alpha = d - 2$. In particular, if $d > 6$ and (1.6) holds with $s \in (4, d-2)$, we have an exponential decay rate of convergence of solutions for system (1.3) with $\alpha = d - 2$, i.e. pressureless damped Euler–Poisson system, even in the whole space.

For the whole space case, as stated in Theorem 1.2, we take into account the negative Sobolev space of solutions. The negative Sobolev norm is first used in [18] for the estimates on the optimal-time decay rates of convergence of solutions to the dissipative equations in the whole space. As mentioned above, we were able to show that the hypocoercivity-type estimate produces the dissipation rate for h; however, it does not give the lower-order norm for h. For this, we find a proper negative order of derivative of solutions that closes the estimates of Sobolev negative norms, and thus an algebraic decay rate of convergence of solutions is established. We would also like to emphasize that an exponential decay rate is found when the negative order is sufficiently large, which subsequently requires $d \geq 1$ large enough, in the whole space. On the other hand, for the periodic domain case, we suitably use the monotonicity of the negative Sobolev norms and construct a modulated energy for system (1.3). More precisely, the modulated energy is equivalent to the lowest-order norm of solutions, $\|u\|_{L^2} + \|h\|_{\dot{H}^{-\frac{d-\alpha}{2}}}$, and this decays to zero exponentially fast as time goes to infinity. This strategy does not require any further integrability in the negative Sobolev space and as stated in Remark 1.5 any smallness assumption on solutions is not needed. This decay estimate on the lower-order norm of solutions, together with the energy estimate established in the proof of Theorem 1.1, yields an exponential decay rate of convergence of solutions in the X^m norm.

Throughout this paper, we use the following notation:

- C denotes a generic positive constant which may differ from line to line;
- $C = C(\alpha, \beta, \dots)$ denotes a positive constant depending on $\alpha, \beta, \dots$;
- $f \lesssim g$ and $f \approx g$ mean that there exists a positive constant $C > 0$ such that $f \leq Cg$ and $C^{-1} f \leq g \leq Cf$, respectively;
- $f \lesssim_{\alpha,\beta,\dots} g$ means that $f \leq C(\alpha, \beta, \dots)g$ for some constant $C(\alpha, \beta, \dots) > 0$;
- ∂^k denotes any partial derivative of order k.

The rest of this paper is organized as follows. In Section 2 we introduce several auxiliary lemmas regarding the commutator estimates and Sobolev embeddings. These estimates will very often be used throughout the paper. Section 3 is devoted to providing the details of the proof of our first main theorem, Theorem 1.1. Since the local well-posedness is by now classical, we mainly discuss the a priori estimates of solutions in the proposed solution space. This yields that the local-in-time solutions can be extended to the global-in-time one. Finally, in Section 4 we study the large-time behavior of classical solutions.

2. Preliminaries

In this section we provide various technical lemmas that will be used frequently throughout the paper.

We first recall from [12] the commutator estimate.

Lemma 2.1 ([12]). *Let $s \geq 0$. For a vector field $v \in (H^{\frac{d}{2}+1+s+\varepsilon}(\mathbb{R}^d))^d$ and $f \in H^s(\mathbb{R}^d)$, we have*

$$\|[\Lambda^s, v \cdot \nabla] f\|_{L^2} \lesssim_{s,d,\varepsilon} \|v\|_{H^{\frac{d}{2}+1+s+\varepsilon}} \|f\|_{H^s}.$$

We next present several results on the Gagliardo–Nirenberg interpolation inequalities and Moser-type inequalities.

Proposition 2.1. *We have the following relations:*

(i) *If $f \in H^{s_2}(\Omega)$ and $0 \leq s_1 \leq s_2$,*

$$\|f\|_{H^{s_1}} \leq \|f\|_{L^2}^{\frac{s_2-s_1}{s_2}} \|f\|_{H^{s_2}}^{\frac{s_1}{s_2}}.$$

(ii) *If $s_2, s_3 \geq 0$ and $0 \leq s_1 \leq \min\{s_2, s_3, s_2 + s_3 - d/2\}$,*

$$\|fg\|_{H^{s_1}} \lesssim_{d,s_1,s_2,s_3} \|f\|_{H^{s_2}} \|g\|_{H^{s_3}}.$$

(iii) *If $j, \ell \in \mathbb{N}$ with $0 \leq j \leq \ell$ and $f \in H^\ell(\Omega)$,*

$$\|\nabla^j f\|_{L^2} \lesssim_{d,j,\ell} \|\nabla^\ell f\|_{L^2}^{\frac{j}{\ell}} \|f\|_{L^2}^{1-\frac{j}{\ell}}.$$

(iv) *(Moser-type inequality) If $f, g \in (H^k \cap L^\infty)(\Omega)$,*

$$\|\partial^k(fg)\|_{L^2} \lesssim_{d,k} \|f\|_{L^\infty} \|\partial^k g\|_{L^2} + \|g\|_{L^\infty} \|\partial^k f\|_{L^2}.$$

Moreover, if $\nabla g \in L^\infty(\Omega)$,

$$\|\partial^k(fg) - (\partial^k f)g\|_{L^2} \lesssim_{d,k} \|f\|_{L^\infty} \|\partial^k g\|_{L^2} + \|\nabla g\|_{L^\infty} \|\partial^{k-1} f\|_{L^2}.$$

In addition, if $f, g \in H^k(\Omega)$ with $\nabla f, \nabla g \in L^\infty(\Omega)$,

$$\|\partial^k(fg) - (\partial^k f)g - f(\partial^k g)\|_{L^2} \lesssim_{d,k} \|\nabla f\|_{L^\infty} \|\partial^{k-1} g\|_{L^2} + \|\nabla g\|_{L^\infty} \|\partial^{k-1} f\|_{L^2}.$$

We finally show the total energy estimate of system (1.1) whose proof can be readily obtained.

Proposition 2.2. *For $T > 0$, let (ρ, u) be a classical solution to (1.1) on $[0, T]$. Then we have*

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \rho|u|^2\,dx + \int_\Omega (\rho-1)\Lambda^{\alpha-d}(\rho-1)\,dx\right) + \int_\Omega \rho|u|^2\,dx = 0.$$

3. Global well-posedness for the damped pressureless Euler–Riesz system

In this section we present the proof of Theorem 1.1. Although our proof mostly considers the case $\Omega = \mathbb{R}^d$, similar arguments can be used for the case $\Omega = \mathbb{T}^d$.

3.1. Local well-posedness

Note that the local-in-time existence and uniqueness of strong solutions can be deduced from [12, Theorem 3.1]. Strictly speaking, in [12] the local well-posedness theory is studied in the case that ρ is integrable in Ω; however the proof can be readily extended to our case. Thus, we present the following theorem without providing any details of its proof.

Theorem 3.1. *Let the same assumptions as in Theorem 1.1 be verified. Then for any positive constants $\varepsilon_1 < M_0$, there exists a positive constant T_0 depending only on ε_1 and M_0 such that if $\|(h_0, u_0)\|_{X^m} < \varepsilon_1$, then system (1.3) admits a unique solution $(h, u) \in \mathcal{C}([0, T); X^m)$ satisfying*

$$\sup_{0 \le t \le T_0} \|(h, u)\|_{X^m} \le M_0.$$

We next show the equivalence relation between the reformulated system (1.3) and the original one (1.1). Since its proof is classical, we omit it here.

Proposition 3.1. *Let $m > \frac{d}{2} + 2$. For any fixed $T > 0$, if $(\rho, u) \in \mathcal{C}([0, T); X^m)$ solves system (1.1) with $\rho > 0$, then $(h, u) \in \mathcal{C}([0, T); X^m)$ solves system (1.3) with $1 + h > 0$. Conversely, if $(h, u) \in \mathcal{C}([0, T); X^m)$ solves system (1.3) with $1 + h > 0$, then $(\rho, u) \in \mathcal{C}([0, T); X^m)$ solves system (1.1) with $\rho > 0$.*

3.2. Global well-posedness

In this part we focus on the a priori estimates of solutions (h, u) in the function space $\mathcal{C}(0, T; X^m)$.

Before we move on, we define a modified H^m norm for h as

$$\|h\|_{\tilde{H}^m} := \sum_{0<|k|\le m} \|\rho^{-\frac{1}{2}} \partial^k h\|_{L^2}. \tag{3.1}$$

Note that $\|h\|_{\tilde{H}^m} = \|\rho\|_{\tilde{H}^m}$. Furthermore, if $\|h\|_{L^\infty} < 1$, then

$$\|h\|_{H^m} \approx \|h\|_{L^2} + \|h\|_{\tilde{H}^m},$$

since

$$(1 - \|h\|_{L^\infty})^{1/2} \|h\|_{\tilde{H}^m} \le \sum_{0<|k|\le m} \|\partial^k h\|_{L^2} \le \|\rho\|_{L^\infty}^{\frac{1}{2}} \|h\|_{\tilde{H}^m}.$$

Thus, rather than directly estimating $\|h\|_{H^m}$, we can estimate $\|h\|_{L^2} + \|h\|_{\tilde{H}^m}$.

Next, we investigate higher-order estimates for (h, u). Before proceeding, for notational simplicity we set

$$X(T; m) := \sup_{0 \le t \le T} \|(h, u)(\cdot, t)\|_{X^m}^2 \quad \text{and} \quad X_0(m) := \|(h_0, u_0)\|_{X^m}^2.$$

Since the proofs for the following two lemmas are almost the same as in [12], we omit them here.

Lemma 3.1. *Let $T > 0$, $m > \frac{d}{2} + 2$, and $(h, u) \in \mathcal{C}([0, T); X^m)$ be a solution to system (1.3). Then we have*

$$\frac{1}{2}\frac{d}{dt}\|U_k\|_{L^2}^2 + \|U_k\|_{L^2}^2 \le C\|u\|_{H^{m+\frac{d-\alpha}{2}}}^3 - \int_\Omega \Lambda^{\frac{\alpha-d}{2}} \nabla \partial^k h \cdot U_k \, dx$$

for $0 \le k \le m$, where $U_k := \Lambda^{\frac{d-\alpha}{2}} \partial^k u$ and $C = C(m, k, d, \alpha)$ is a positive constant independent of T.

Remark 3.1. Thanks to the Gagliardo–Nirenberg interpolation inequalities in Proposition 2.1, we have the following equivalence relation: for any $i \in \{0, \dots, m\}$,

$$\|u\|_{L^2} + \sum_{i \le k \le m} \|U_k\|_{L^2} \approx \|u\|_{H^{m+\frac{d-\alpha}{2}}}. \tag{3.2}$$

Lemma 3.2. *Let $T > 0$, $m > \frac{d}{2} + 2$, and $(h, u) \in \mathcal{C}([0, T); X^m)$ be a solution to system (1.3). Then we have*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{1}{\rho}|\partial^k h|^2 \, dx \le C\|u\|_{H^{m+\frac{d-\alpha}{2}}} (1 + \|\nabla \log \rho\|_{L^\infty})^{2(k-1)} \sum_{0<l\le k} \left\| \frac{1}{\sqrt{\rho}} \partial^l h \right\|_{L^2}^2 + \int_\Omega \Lambda^{\frac{\alpha-d}{2}} \nabla R_k \cdot U_k \, dx$$

for $1 \le k \le m$, where $C = C(m, k, d, \alpha)$ is a positive constant independent of T.

Here, we separately consider the L^2-estimate for h.

Lemma 3.3. *Let $T > 0$ and $(h, u) \in \mathcal{C}([0, T); X^m)$ be a solution to system (1.3). Then we have*

$$\frac{1}{2}\frac{d}{dt}\|h\|_{L^2}^2 \le \|u\|_{L^2}\|\nabla h\|_{L^2}\|h\|_{L^\infty} + \int_\Omega \Lambda^{\frac{\alpha-d}{2}} \nabla h \cdot U_0 \, dx.$$

Proof. Direct computation implies

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|h\|_{L^2}^2 &= -\int_\Omega h\nabla \cdot (hu) \, dx - \int_\Omega h\nabla \cdot u \, dx \\
&= \int_\Omega h(\nabla h \cdot u) \, dx + \int_\Omega \nabla h \cdot u \, dx \\
&\le \|h\|_{L^\infty}\|\nabla h\|_{L^2}\|u\|_{L^2} + \int_\Omega \Lambda^{\frac{\alpha-d}{2}} \nabla h \cdot U_0 \, dx,
\end{aligned}$$

and this implies the desired result. ∎

As stated in Lemma 3.1, due to the presence of the linear damping in velocity, we have a dissipation rate for the velocity u. Moreover, the terms with the highest-order derivatives appearing in Lemmas 3.1 and 3.2 cancel each other out. Thus, we now focus on the estimate for the dissipation rate for h. For this, a delicate analysis for the Riesz interaction term based on the hypocoercivity-type estimate is required.

We first begin with the zeroth-order estimate.

Lemma 3.4. *Let $T > 0$, $m > \frac{d}{2} + 2$, and $(h, u) \in \mathcal{C}([0, T); X^m)$ be a solution to system (1.3) satisfying*

$$\sup_{0 \le t \le T} \|h(t)\|_{L^\infty} \le \frac{1}{2}.$$

Then we have

$$\frac{d}{dt} \int_\Omega \frac{1}{\rho} \nabla h \cdot \Lambda^{d-\alpha} u \, dx + \frac{1}{2} \|\nabla h\|_{L^2}^2 \le C \|\nabla h\|_{H^{m-1}} \|u\|_{H^{m+\frac{d-\alpha}{2}}}^2 + \|\Lambda^{\frac{d-\alpha}{2}} (\nabla \cdot u)\|_{L^2}^2 + 6 \|\Lambda^{d-\alpha} u\|_{L^2}^2,$$

where $C = C(m, d, \alpha)$ is a positive constant independent of T.

Proof. We first find

$$\begin{aligned}
&\frac{d}{dt} \int_\Omega \frac{1}{\rho} \nabla h \cdot \Lambda^{d-\alpha} u \, dx \\
&\quad = -\int_\Omega \frac{\partial_t \rho}{\rho^2} \nabla h \cdot \Lambda^{d-\alpha} u \, dx + \int_\Omega \frac{1}{\rho} \nabla (\partial_t h) \cdot \Lambda^{d-\alpha} u \, dx \\
&\qquad + \int_\Omega \frac{1}{\rho} \nabla h \cdot \Lambda^{d-\alpha} (\partial_t u) \, dx \\
&\quad =: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,
\end{aligned}$$

where we use the equation of h in (1.3) to estimate

$$\begin{aligned}
\mathcal{I}_2 &= \int_\Omega \frac{\nabla h}{\rho^2} \partial_t h \cdot \Lambda^{d-\alpha} u \, dx - \int_\Omega \frac{1}{\rho} \partial_t h \Lambda^{d-\alpha} (\nabla \cdot u) \, dx \\
&= -\mathcal{I}_1 + \int_\Omega \frac{1}{\rho} \nabla \cdot (\rho u) \Lambda^{d-\alpha} (\nabla \cdot u) \, dx \\
&= -\mathcal{I}_1 + \int_\Omega (\nabla \cdot u) \Lambda^{d-\alpha} (\nabla \cdot u) \, dx + \int_\Omega \Big(\frac{\nabla \rho}{\rho} \cdot u \Big) \Lambda^{d-\alpha} (\nabla \cdot u) \, dx \\
&\le -\mathcal{I}_1 + \|\Lambda^{\frac{d-\alpha}{2}} (\nabla \cdot u)\|_{L^2}^2 + \frac{\|u\|_{L^\infty}}{1 - \|h\|_{L^\infty}} \|\nabla h\|_{L^2} \|\Lambda^{d-\alpha} (\nabla \cdot u)\|_{L^2} \\
&\le -\mathcal{I}_1 + \|\Lambda^{\frac{d-\alpha}{2}} (\nabla \cdot u)\|_{L^2}^2 + C \|\nabla h\|_{L^2} \|u\|_{H^{m+\frac{d-\alpha}{2}}}^2 .
\end{aligned}$$

Here $C = C(m, d, \alpha)$ is a positive constant independent of T.

On the other hand, $\mathcal{I}_3$ can be estimated as

$$\begin{aligned}
\mathcal{I}_3 &= -\int_\Omega \frac{1}{\rho}\nabla h\cdot\Lambda^{d-\alpha}(u\cdot\nabla u + u + \Lambda^{\alpha-d}\nabla h)\,dx \\
&\le C\frac{1}{1-\|h\|_{L^\infty}}\|\nabla h\|_{L^2}\|u\|^2_{H^{m+\frac{d-\alpha}{2}}} + \frac{1}{1-\|h\|_{L^\infty}}\|\nabla h\|_{L^2}\|\Lambda^{d-\alpha}u\|_{L^2} \\
&\quad - \frac{1}{1+\|h\|_{L^\infty}}\|\nabla h\|^2_{L^2} \\
&\le C\|\nabla h\|_{L^2}\|u\|^2_{H^{m+\frac{d-\alpha}{2}}} + 2\|\nabla h\|_{L^2}\|\Lambda^{d-\alpha}u\|_{L^2} - \frac{2}{3}\|\nabla h\|^2_{L^2} \\
&\le -\frac{1}{2}\|\nabla h\|^2_{L^2} + C\|\nabla h\|_{L^2}\|u\|^2_{H^{m+\frac{d-\alpha}{2}}} + 6\|\Lambda^{d-\alpha}u\|^2_{L^2}.
\end{aligned}$$

Thus, we combine the estimates for $\mathcal{I}_i$, $i = 1, 2, 3$, to conclude the desired result. ■

Before proceeding to the higher-order estimates, we provide some technical estimates based on the Moser-type inequality below. For smoothness of reading, we postpone its proof to Appendix A.

Lemma 3.5. *Let $T > 0$, $m > \frac{d}{2} + 2$, and $(h, u) \in \mathcal{C}([0, T); X^m)$ be a solution to system (1.3) satisfying*

$$\sup_{0\le t\le T}\|h(t)\|_{L^\infty} \le \frac{1}{2}.$$

Then for $1 \le k \le m-1$ we have

(i) $\left\|\nabla\left(\frac{\nabla h}{\rho^2}\cdot\partial^k(\rho u)\right)\right\|_{L^2} \le C(1+\|\nabla h\|_{H^{m-1}})^2\|\nabla h\|_{H^{m-1}}\|u\|_{H^{m+\frac{d-\alpha}{2}}}$,

(ii) $\left\|\nabla\left[\nabla\cdot\left(\frac{1}{\rho}\left(\partial^k(\rho u) - (\partial^k\rho)u - \rho(\partial^k u)\right)\right)\right]\right\|_{L^2}$
$\le C(1+\|\nabla h\|_{H^{m-1}})^2\|\nabla h\|_{H^{m-1}}\|u\|_{H^{m+\frac{d-\alpha}{2}}}$,

(iii) $\left\|\partial\left(\frac{1}{\rho^2}(\partial^k h)\nabla h\right)\right\|_{L^2} \le C(1+\|\nabla h\|_{H^{m-1}})\|\nabla h\|^2_{H^{m-1}}$,

(iv) $\|\Lambda^{d-\alpha}\partial^{k-1}\partial_t u\|_{L^2} \le C\left(\|u\|^2_{H^{m+\frac{d-\alpha}{2}}} + \|u\|_{H^{m+\frac{d-\alpha}{2}}} + \|\nabla h\|_{H^{m-1}}\right)$, *and*

(v) $\|\Lambda^{d-\alpha}\partial^{k-1}(\nabla u : (\nabla u)^T)\|_{L^2} \le C\|u\|^2_{H^{m+\frac{d-\alpha}{2}}}$.

Here $\rho = h + 1$ and $C = C(m, k, d, \alpha)$ is a positive constant independent of T.

Lemma 3.6. *Let $T > 0$, $m > \frac{d}{2} + 2$, and $(h, u) \in \mathcal{C}([0, T); X^m)$ be a solution to system (1.3) satisfying*

$$\sup_{0\le t\le T}\|h(t)\|_{L^\infty} \le \frac{1}{2}.$$

Then we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{1}{\rho}\partial^k\nabla h\cdot\Lambda^{d-\alpha}\partial^k u\,dx+\frac{1}{2}\|\partial^k\nabla h\|_{L^2}^2\\ \le C(1+\|\nabla h\|_{H^{m-1}})^2\big(\|\nabla h\|_{H^{m-1}}^3+\|u\|_{H^{m+\frac{d-\alpha}{2}}}^3\big)+\|\Lambda^{\frac{d-\alpha}{2}}\partial^k(\nabla\cdot u)\|_{L^2}^2\\ +6\|\Lambda^{d-\alpha}\partial^k u\|_{L^2}^2\end{aligned}$$

for $1\le k\le m-1$*, where* $C=C(m,k,d,\alpha)$ *is a positive constant independent of* T*.*

Proof. Throughout this proof, $C>0$ denotes the generic constant depending only on m,k,d, and α, independent of T.

Direct computation yields

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{1}{\rho}\partial^k\nabla h\cdot\Lambda^{d-\alpha}\partial^k u\,dx\\ =-\int_\Omega\frac{\partial_t\rho}{\rho^2}\partial^k\nabla h\cdot\Lambda^{d-\alpha}\partial^k u\,dx+\int_\Omega\frac{1}{\rho}\partial^k\nabla\partial_t h\cdot\Lambda^{d-\alpha}\partial^k u\,dx\\ +\int_\Omega\frac{1}{\rho}\partial^k\nabla h\cdot\Lambda^{d-\alpha}\partial^k\partial_t u\,dx\\ =:\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3.\end{aligned}$$

We estimate $\mathcal{J}_1$, $\mathcal{J}_2$, and $\mathcal{J}_3$ one by one as follows.

◇ *Estimates for* $\mathcal{J}_1$*.* One obtains

$$\begin{aligned}\mathcal{J}_1&=\int_\Omega\frac{\nabla\cdot(\rho u)}{\rho^2}\partial^k\nabla h\cdot\Lambda^{d-\alpha}\partial^k u\,dx\\ &=\int_\Omega\frac{\nabla\cdot u}{\rho}\partial^k\nabla h\cdot\Lambda^{d-\alpha}\partial^k u\,dx+\int_\Omega\frac{\nabla h\cdot u}{\rho^2}\partial^k\nabla h\cdot\Lambda^{d-\alpha}\partial^k u\,dx\\ &\le\frac{\|\nabla u\|_{L^\infty}}{1-\|h\|_{L^\infty}}\|\partial^k\nabla h\|_{L^2}\|\Lambda^{d-\alpha}\partial^k u\|_{L^2}+\frac{\|\nabla h\|_{L^\infty}\|u\|_{L^\infty}}{(1-\|h\|_{L^\infty})^2}\|\partial^k\nabla h\|_{L^2}\|\Lambda^{d-\alpha}\partial^k u\|_{L^2}\\ &\le C(1+\|\nabla h\|_{H^{m-1}})\|\nabla h\|_{H^{m-1}}\|u\|_{H^{m+\frac{d-\alpha}{2}}}^2.\end{aligned}$$

◇ *Estimates for* $\mathcal{J}_2$*.* We get

$$\begin{aligned}\mathcal{J}_2&=\int_\Omega\frac{1}{\rho^2}\partial^k\partial_t h\nabla h\cdot\Lambda^{d-\alpha}\partial^k u\,dx-\int_\Omega\frac{1}{\rho}\partial^k\partial_t h\Lambda^{d-\alpha}\partial^k(\nabla\cdot u)\,dx\\ &=-\int_\Omega\frac{1}{\rho^2}\partial^k(\nabla\cdot(hu))\nabla h\cdot\Lambda^{d-\alpha}\partial^k u\,dx-\int_\Omega\frac{1}{\rho^2}\partial^k(\nabla\cdot u)\nabla h\cdot\Lambda^{d-\alpha}\partial^k u\,dx\\ &\quad+\int_\Omega\frac{1}{\rho}\partial^k(\nabla\cdot(\rho u))\Lambda^{d-\alpha}\partial^k(\nabla\cdot u)\,dx\\ &=-\int_\Omega\frac{1}{\rho^2}\partial^k(\nabla\cdot(hu))\nabla h\cdot\Lambda^{d-\alpha}\partial^k u\,dx-\int_\Omega\frac{1}{\rho^2}\partial^k(\nabla\cdot u)\nabla h\cdot\Lambda^{d-\alpha}\partial^k u\,dx\\ &\quad+\int_\Omega\frac{\nabla h}{\rho^2}\cdot\partial^k(\rho u)\Lambda^{d-\alpha}\partial^k(\nabla\cdot u)\,dx-\int_\Omega\frac{1}{\rho}\partial^k(\rho u)\cdot\nabla\Lambda^{d-\alpha}\partial^k(\nabla\cdot u)\,dx\end{aligned}$$

$$
\begin{aligned}
&= -\int_\Omega \frac{1}{\rho^2}\partial^k(\nabla\cdot(hu))\nabla h\cdot\Lambda^{d-\alpha}\partial^k u\,dx - \int_\Omega \frac{1}{\rho^2}\partial^k(\nabla\cdot u)\nabla h\cdot\Lambda^{d-\alpha}\partial^k u\,dx \\
&\quad - \int_\Omega \nabla\Big(\frac{\nabla h}{\rho^2}\cdot\partial^k(\rho u)\Big)\cdot\Lambda^{d-\alpha}\partial^k u\,dx - \int_\Omega \frac{1}{\rho}(\partial^k h)u\cdot\nabla\Lambda^{d-\alpha}\partial^k(\nabla\cdot u)\,dx \\
&\quad - \int_\Omega \partial^k u\cdot\nabla\Lambda^{d-\alpha}\partial^k(\nabla\cdot u)\,dx \\
&\quad - \int_\Omega \frac{1}{\rho}\big(\partial^k(\rho u) - (\partial^k\rho)u - \rho(\partial^k u)\big)\cdot\nabla\Lambda^{d-\alpha}\partial^k(\nabla\cdot u)\,dx \\
&=: \sum_{i=1}^{6}\mathcal{J}_{2i}.
\end{aligned}
$$

For $\mathcal{J}_{21}$, we use the Moser-type inequality in Proposition 2.1 to get

$$
\begin{aligned}
\mathcal{J}_{21} &= -\int_\Omega \frac{1}{\rho^2}\partial^k(\nabla h\cdot u)\nabla h\cdot\Lambda^{d-\alpha}\partial^k u\,dx \\
&\quad - \int_\Omega \frac{1}{\rho^2}\big(\partial^k(h\nabla\cdot u) - h\partial^k(\nabla\cdot u)\big)\nabla h\cdot\Lambda^{d-\alpha}\partial^k u\,dx \\
&\quad - \int_\Omega \frac{1}{\rho^2}h\partial^k(\nabla\cdot u)\nabla h\cdot\Lambda^{d-\alpha}\partial^k u\,dx \\
&\le \frac{\|\nabla h\|_{L^\infty}}{(1-\|h\|_{L^\infty})^2}\big(\|\partial^k(\nabla h\cdot u)\|_{L^2} + \|\partial^k(h\nabla\cdot u) - h\partial^k(\nabla\cdot u)\|_{L^2}\big)\|\Lambda^{d-\alpha}\partial^k u\|_{L^2} \\
&\quad - \int_\Omega \frac{1}{\rho^2}h\partial^k(\nabla\cdot u)\nabla h\cdot\Lambda^{d-\alpha}\partial^k u\,dx \\
&\le C\|\nabla h\|_{L^\infty}\big(\|\nabla h\|_{L^\infty}\|\partial^k u\|_{L^2} + \|u\|_{W^{1,\infty}}\|\partial^k h\|_{H^1}\big)\|\Lambda^{d-\alpha}\partial^k u\|_{L^2} \\
&\quad - \int_\Omega \frac{1}{\rho^2}h\partial^k(\nabla\cdot u)\nabla h\cdot\Lambda^{d-\alpha}\partial^k u\,dx \\
&\le C\|\nabla h\|^2_{H^{m-1}}\|u\|^2_{H^{m+\frac{d-\alpha}{2}}} - \int_\Omega \frac{1}{\rho^2}h\partial^k(\nabla\cdot u)\nabla h\cdot\Lambda^{d-\alpha}\partial^k u\,dx.
\end{aligned}
$$

This implies

$$
\begin{aligned}
\mathcal{J}_{21} + \mathcal{J}_{22} &\le C\|\nabla h\|^2_{H^{m-1}}\|u\|^2_{H^{m+\frac{d-\alpha}{2}}} - \int_\Omega \frac{1}{\rho}\partial^k(\nabla\cdot u)\nabla h\cdot\Lambda^{d-\alpha}\partial^k u\,dx \\
&\le C(1+\|\nabla h\|_{H^{m-1}})\|\nabla h\|_{H^{m-1}}\|u\|^2_{H^{m+\frac{d-\alpha}{2}}}.
\end{aligned}
$$

For $\mathcal{J}_{23}$, we use Hölder's inequality and Lemma 3.5(i) to obtain

$$
\begin{aligned}
\mathcal{J}_{23} &\le \Big\|\nabla\Big(\frac{\nabla h}{\rho^2}\cdot\partial^k(\rho u)\Big)\Big\|_{L^2}\|\Lambda^{d-\alpha}\partial^k u\|_{L^2} \\
&\le C(1+\|\nabla h\|_{H^{m-1}})^2\|\nabla h\|_{H^{m-1}}\|u\|^2_{H^{m+\frac{d-\alpha}{2}}}.
\end{aligned}
$$

We next use integration by parts to deduce

$$
\mathcal{J}_{25} = \|\Lambda^{\frac{d-\alpha}{2}}\partial^k(\nabla\cdot u)\|^2_{L^2}.
$$

For the estimate of $\mathcal{J}_{26}$ we use Lemma 3.5(ii) to deduce

$$\begin{aligned}\mathcal{J}_{26} &= -\int_\Omega \nabla\Big[\nabla\cdot\Big(\frac{1}{\rho}\big(\partial^k(\rho u)-(\partial^k\rho)u-\rho(\partial^k u)\big)\Big)\Big]\cdot\Lambda^{d-\alpha}\partial^k u\,dx\\ &\le \Big\|\nabla\Big[\nabla\cdot\Big(\frac{1}{\rho}\big(\partial^k(\rho u)-(\partial^k\rho)u-\rho(\partial^k u)\big)\Big)\Big]\Big\|_{L^2}\|\Lambda^{d-\alpha}\partial^k u\|_{L^2}\\ &\le C\big(1+\|\nabla h\|_{H^{m-1}}\big)^2\|\nabla h\|_{H^{m-1}}\|u\|^2_{H^{m+\frac{d-\alpha}{2}}}.\end{aligned}$$

Hence, we gather the estimates for the $\mathcal{J}_{2i}$ to yield

$$\begin{aligned}\mathcal{J}_2 \le &-\int_\Omega \frac{1}{\rho}(\partial^k h)u\cdot\nabla\Lambda^{d-\alpha}\partial^k(\nabla\cdot u)\,dx+\|\Lambda^{\frac{d-\alpha}{2}}\partial^k(\nabla\cdot u)\|^2_{L^2}\\ &+C(1+\|\nabla h\|_{H^{m-1}})^2\|\nabla h\|_{H^{m-1}}\|u\|^2_{H^{m+\frac{d-\alpha}{2}}}.\end{aligned}$$

$\diamond$ *Estimates for* $\mathcal{J}_3$. In this case,

$$\begin{aligned}\mathcal{J}_3 &= \int_\Omega \frac{1}{\rho^2}(\partial^k h)\nabla h\cdot\Lambda^{d-\alpha}\partial^k\partial_t u\,dx-\int_\Omega\frac{1}{\rho}\partial^k h\Lambda^{d-\alpha}\partial^k\partial_t(\nabla\cdot u)\,dx\\ &= \int_\Omega \frac{1}{\rho^2}(\partial^k h)\nabla h\cdot\Lambda^{d-\alpha}\partial^k\partial_t u\,dx\\ &\quad+\int_\Omega\frac{1}{\rho}(\partial^k h)\Lambda^{d-\alpha}\partial^k(\nabla\cdot(u\cdot\nabla u)+\nabla\cdot u+\Lambda^{\alpha-d}\Delta h)\,dx\\ &= \int_\Omega \frac{1}{\rho^2}(\partial^k h)\nabla h\cdot\Lambda^{d-\alpha}\partial^k\partial_t u\,dx+\int_\Omega\frac{1}{\rho}(\partial^k h)\Lambda^{d-\alpha}\partial^k(\nabla\cdot(u\cdot\nabla u))\,dx\\ &\quad+\int_\Omega\frac{1}{\rho}(\partial^k h)\Lambda^{d-\alpha}\partial^k(\nabla\cdot u)\,dx+\int_\Omega\frac{1}{\rho}(\partial^k h)\Delta(\partial^k h)\,dx\\ &=: \sum_{i=1}^4\mathcal{J}_{3i}.\end{aligned}$$

For the term $\mathcal{J}_{31}$, we use Lemma 3.5(iii) & (iv) to obtain

$$\begin{aligned}\mathcal{J}_{31} &= \int_\Omega \partial\Big(\frac{1}{\rho^2}(\partial^k h)\nabla h\Big)\cdot\Lambda^{d-\alpha}\partial^{k-1}\partial_t u\,dx\\ &\le \Big\|\partial\Big(\frac{1}{\rho^2}(\partial^k h)\nabla h\Big)\Big\|_{L^2}\|\Lambda^{d-\alpha}\partial^{k-1}\partial_t u\|_{L^2}\\ &\le C(1+\|\nabla h\|_{H^{m-1}})\|\nabla h\|^2_{H^{m-1}}\big(\|u\|^2_{H^{m+\frac{d-\alpha}{2}}}+\|u\|_{H^{m+\frac{d-\alpha}{2}}}+\|\nabla h\|_{H^{m-1}}\big)\\ &\le C\big(1+\|\nabla h\|_{H^{m-1}}+\|u\|_{H^{m+\frac{d-\alpha}{2}}}\big)\|\nabla h\|^2_{H^{m-1}}\big(\|u\|_{H^{m+\frac{d-\alpha}{2}}}+\|\nabla h\|_{H^{m-1}}\big).\end{aligned}$$

We then estimate $\mathcal{J}_{32}$ as

$$\mathcal{J}_{32} = -\int_\Omega \partial\Big(\frac{1}{\rho}(\partial^k h)\Big)\Lambda^{d-\alpha}\partial^{k-1}(\nabla\cdot(u\cdot\nabla u))\,dx$$

$$
\begin{aligned}
&= -\int_\Omega \partial\Big(\frac{1}{\rho}(\partial^k h)\Big)\Lambda^{d-\alpha}\partial^{k-1}((u\cdot\nabla)(\nabla\cdot u))\,dx \\
&\quad -\int_\Omega \partial\Big(\frac{1}{\rho}(\partial^k h)\Big)\Lambda^{d-\alpha}\partial^{k-1}(\nabla\cdot(u\cdot\nabla u)-u\cdot\nabla(\nabla\cdot u))\,dx \\
&= -\int_\Omega \partial\Big(\frac{1}{\rho}(\partial^k h)\Big)u\cdot\nabla\Lambda^{d-\alpha}\partial^{k-1}(\nabla\cdot u)\,dx \\
&\quad -\int_\Omega \partial\Big(\frac{1}{\rho}(\partial^k h)\Big)[\Lambda^{d-\alpha},u\cdot\nabla]\partial^{k-1}(\nabla\cdot u)\,dx \\
&\quad -\int_\Omega \partial\Big(\frac{1}{\rho}(\partial^k h)\Big)\Lambda^{d-\alpha}[\partial^{k-1}(u\cdot\nabla(\nabla\cdot u))-u\cdot\nabla\partial^{k-1}(\nabla\cdot u)]\,dx \\
&\quad -\int_\Omega \partial\Big(\frac{1}{\rho}(\partial^k h)\Big)\Lambda^{d-\alpha}\partial^{k-1}(\nabla\cdot(u\cdot\nabla u)-u\cdot\nabla(\nabla\cdot u))\,dx \\
&=: \sum_{i=1}^{4}\mathcal{J}_{32i}.
\end{aligned}
$$

First, we estimate $\mathcal{J}_{321}$ as

$$
\begin{aligned}
\mathcal{J}_{321} &= \int_\Omega \frac{1}{\rho}(\partial^k h)\partial u\cdot\nabla\Lambda^{d-\alpha}\partial^{k-1}(\nabla\cdot u)\,dx + \int_\Omega \frac{1}{\rho}(\partial^k h)u\cdot\nabla\Lambda^{d-\alpha}\partial^{k}(\nabla\cdot u)\,dx \\
&= -\int_\Omega \nabla\cdot\Big(\frac{1}{\rho}(\partial^k h)\partial u\Big)\Lambda^{d-\alpha}\partial^{k-1}(\nabla\cdot u)\,dx + \int_\Omega \frac{1}{\rho}(\partial^k h)u\cdot\nabla\Lambda^{d-\alpha}\partial^{k}(\nabla\cdot u)\,dx \\
&\le C\Big(\frac{\|\nabla h\|_{L^\infty}\|\partial^k h\|_{L^2}\|\partial u\|_{L^\infty}}{(1-\|h\|_{L^\infty})^2} + \frac{\|\partial^k\nabla h\|_{L^2}\|\partial u\|_{L^\infty}+\|\partial^k h\|_{L^2}\|\nabla^2 u\|_{L^\infty}}{1-\|h\|_{L^\infty}}\Big) \\
&\quad \times\|\Lambda^{d-\alpha}\partial^{k-1}(\nabla\cdot u)\|_{L^2} + \int_\Omega \frac{1}{\rho}(\partial^k h)u\cdot\nabla\Lambda^{d-\alpha}\partial^{k}(\nabla\cdot u)\,dx \\
&\le C(1+\|\nabla h\|_{H^{m-1}})\|\nabla h\|_{H^{m-1}}\|u\|^2_{H^{m+\frac{d-\alpha}{2}}} + \int_\Omega \frac{1}{\rho}(\partial^k h)u\cdot\nabla\Lambda^{d-\alpha}\partial^{k}(\nabla\cdot u)\,dx.
\end{aligned}
$$

Next, Lemma 2.1 yields

$$
\begin{aligned}
\mathcal{J}_{322} &\le C\Big(\frac{\|\nabla h\|_{L^\infty}\|\partial^k h\|_{L^2}}{(1-\|h\|_{L^\infty})^2} + \frac{\|\partial^{k+1}h\|_{L^2}}{1-\|h\|_{L^\infty}}\Big)\|u\|_{H^{\frac{d}{2}+1+(d-\alpha)+\varepsilon}}\|\partial^{k-1}(\nabla\cdot u)\|_{H^{d-\alpha}} \\
&\le C(1+\|\nabla h\|_{H^{m-1}})\|\nabla h\|_{H^{m-1}}\|u\|^2_{H^{m+\frac{d-\alpha}{2}}},
\end{aligned}
$$

where ε satisfies $(d-\alpha)/2+\varepsilon\le 1$ so that $\frac{d}{2}+1+(d-\alpha)+\varepsilon\le m+\frac{d-\alpha}{2}$.

For $\mathcal{J}_{323}$ we need to estimate

$$
\|\Lambda^{d-\alpha}(\partial^\ell u\cdot\nabla\partial^{k-1-\ell}(\nabla\cdot u))\|_{L^2}
$$

for $1\le\ell\le k-1$. For $\ell=1$,

$$
\|\Lambda^{d-\alpha}\partial u\cdot\nabla\partial^{k-2}(\nabla\cdot u)\|_{L^2}\le\|\partial u\cdot\nabla\partial^{k-2}(\nabla\cdot u)\|_{H^{d-\alpha}}\le C\|u\|^2_{H^{m+\frac{d-\alpha}{2}}},
$$

where we used Proposition 2.1(ii) with $s_1=s_3=d-\alpha$ and $s_2=\frac{d}{2}+2+\frac{d-\alpha}{2}$.

For $2 \le \ell \le k-1$, we use Proposition 2.1 to get

$$\begin{aligned}\|\Lambda^{d-\alpha}\partial^\ell u \cdot \nabla \partial^{k-1-\ell}(\nabla \cdot u)\|_{L^2} &\le \|\partial^\ell u \cdot \nabla \partial^{k-1-\ell}(\nabla \cdot u)\|_{H^2} \\ &\le C\|\partial^\ell u\|_{H^{m-\ell}}\|\partial^{k-\ell+1}u\|_{H^{m-(k-\ell+1)}} \\ &\le C\|u\|^2_{H^{m+\frac{d-\alpha}{2}}},\end{aligned}$$

and this implies

$$\begin{aligned}\mathcal{J}_{323} &\le C\Big(\frac{\|\nabla h\|_{L^\infty}\|\partial^k h\|_{L^2}}{(1-\|h\|_{L^\infty})^2} + \frac{\|\partial^{k+1}h\|_{L^2}}{1-\|h\|_{L^\infty}}\Big)\|u\|^2_{H^{m+\frac{d-\alpha}{2}}} \\ &\le C(1+\|\nabla h\|_{H^{m-1}})\|\nabla h\|_{H^{m-1}}\|u\|^2_{H^{m+\frac{d-\alpha}{2}}}.\end{aligned}$$

Now, for $\mathcal{J}_{324}$ we use the estimate

$$\begin{aligned}\Big\|\partial\Big(\frac{1}{\rho}(\partial^k h)\Big)\Big\|_{L^2} &\le C\Big(\frac{\|\nabla h\|_{L^\infty}\|\partial^k h\|_{L^2}}{(1-\|h\|_{L^\infty})^2} + \frac{\|\partial^{k+1}h\|_{L^2}}{1-\|h\|_{L^\infty}}\Big) \\ &\le C(1+\|\nabla h\|_{H^{m-1}})\|\nabla h\|_{H^{m-1}},\end{aligned}$$

together with Lemma 3.5(v), to obtain

$$\begin{aligned}\mathcal{J}_{324} &= -\sum_{i,j=1}^{d}\int_\Omega \partial\Big(\frac{1}{\rho}(\partial^k h)\Big)\Lambda^{d-\alpha}\partial^{k-1}(\partial_{x_i}u_j\partial_{x_j}u_i)\,dx \\ &= -\int_\Omega \partial\Big(\frac{1}{\rho}(\partial^k h)\Big)\Lambda^{d-\alpha}\partial^{k-1}(\nabla u : (\nabla u)^T)\,dx \\ &\le \Big\|\partial\Big(\frac{1}{\rho}(\partial^k h)\Big)\Big\|_{L^2}\|\Lambda^{d-\alpha}\partial^{k-1}(\nabla u : (\nabla u)^T)\|_{L^2} \\ &\le C(1+\|\nabla h\|_{H^{m-1}})\|\nabla h\|_{H^{m-1}}\|u\|^2_{H^{m+\frac{d-\alpha}{2}}}.\end{aligned}$$

Then we collect the estimates for the $\mathcal{J}_{32i}$ to yield

$$\mathcal{J}_{32} \le C(1+\|\nabla h\|_{H^{m-1}})\|\nabla h\|_{H^{m-1}}\|u\|^2_{H^{m+\frac{d-\alpha}{2}}} + \int_\Omega \frac{1}{\rho}(\partial^k h)u \cdot \nabla\Lambda^{d-\alpha}\partial^k(\nabla \cdot u)\,dx.$$

For $\mathcal{J}_{33}$ and $\mathcal{J}_{34}$,

$$\begin{aligned}\mathcal{J}_{33} &= \int_\Omega \frac{1}{\rho^2}(\partial^k h)\nabla h \cdot \Lambda^{d-\alpha}\partial^k u\,dx - \int_\Omega \frac{1}{\rho}\partial^k \nabla h \cdot \Lambda^{d-\alpha}\partial^k u\,dx \\ &\le \frac{\|\nabla h\|_{L^\infty}}{(1-\|h\|_{L^\infty})^2}\|\partial^k h\|_{L^2}\|\Lambda^{d-\alpha}\partial^k u\|_{L^2} + \frac{1}{1-\|h\|_{L^\infty}}\|\partial^k \nabla h\|_{L^2}\|\Lambda^{d-\alpha}\partial^k u\|_{L^2} \\ &\le C\|\nabla h\|^2_{H^{m-1}}\|u\|_{H^{m+\frac{d-\alpha}{2}}} + \frac{1}{6}\|\partial^k \nabla h\|^2_{L^2} + 6\|\Lambda^{d-\alpha}\partial^k u\|^2_{L^2},\end{aligned}$$

$$
\begin{aligned}
\mathcal{J}_{34} &= \int_\Omega \frac{1}{\rho^2}(\partial^k h)\nabla h \cdot \partial^k \nabla h \, dx - \int_\Omega \frac{1}{\rho}|\partial^k \nabla h|^2 \, dx \\
&\le \frac{\|\nabla h\|_{L^\infty}}{(1-\|h\|_{L^\infty})^2}\|\partial^k h\|_{L^2}\|\partial^k \nabla h\|_{L^2} - \frac{1}{1+\|h\|_{L^\infty}}\|\partial^k \nabla h\|_{L^2}^2 \\
&\le C\|\nabla h\|_{H^{m-1}}^3 - \frac{2}{3}\|\partial^k \nabla h\|_{L^2}^2.
\end{aligned}
$$

Thus, we combine the estimates for the $\mathcal{J}_{3i}$ to obtain

$$
\begin{aligned}
\mathcal{J}_3 \le{}& C(1+\|\nabla h\|_{H^{m-1}})(\|\nabla h\|_{H^{m-1}}^3 + \|u\|_{H^{m+\frac{d-\alpha}{2}}}^3) - \frac{1}{2}\|\partial^k \nabla h\|_{L^2}^2 + 6\|\Lambda^{d-\alpha}\partial^k u\|_{L^2}^2 \\
&+ \int_\Omega \frac{1}{\rho}(\partial^k h)u \cdot \nabla \Lambda^{d-\alpha}\partial^k(\nabla \cdot u)\, dx.
\end{aligned}
$$

Therefore, we gather all the results for the $\mathcal{J}_i$ to obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{1}{\rho}\partial^k \nabla h \cdot \Lambda^{d-\alpha}\partial^k u \, dx + \frac{1}{2}\|\partial^k \nabla h\|_{L^2}^2 \\
&\quad \le C(1+\|\nabla h\|_{H^{m-1}})^2(\|\nabla h\|_{H^{m-1}}^3 + \|u\|_{H^{m+\frac{d-\alpha}{2}}}^3) \\
&\qquad + \|\Lambda^{\frac{d-\alpha}{2}}\partial^k(\nabla \cdot u)\|_{L^2}^2 + 6\|\Lambda^{d-\alpha}\partial^k u\|_{L^2}^2.
\end{aligned}
$$

■

Based on the results so far, below we provide a uniform-in-time bound estimate of solutions.

Proposition 3.2. *Let $T > 0$, $m > \frac{d}{2} + 2$, and $(h, u) \in \mathcal{C}([0, T); X^m)$ be a solution to system (1.3). Suppose that $X(T; m) \le \varepsilon_0^2 \ll 1$, so that*

$$
\sup_{0\le t\le T} \|h(t)\|_{L^\infty} \le \frac{1}{2}.
$$

Then there exists a positive constant C^* independent of T such that

$$
X(T; m) \le C^* X_0(m).
$$

Proof. Applying Lemma 3.6 and (3.2) implies that we can find $C_1 > 0$ independent of T such that

$$
\begin{aligned}
&\frac{d}{dt}\Big(\sum_{0\le k\le m-1}\int_\Omega \frac{1}{\rho}\partial^k \nabla h \cdot \Lambda^{d-\alpha}\partial^k u \, dx\Big) + \frac{1}{2}\|\nabla h\|_{H^{m-1}}^2 \\
&\quad \le C(1+\|\nabla h\|_{H^{m-1}})^2(\|\nabla h\|_{H^{m-1}}^3 + \|u\|_{H^{m+\frac{d-\alpha}{2}}}^3) \\
&\qquad + 2\sum_{0\le k\le m-1}\|(\nabla \cdot U_k)\|_{L^2}^2 + 6\sum_{0\le k\le m-1}\|\Lambda^{d-\alpha}\partial^k u\|_{L^2}^2 \\
&\quad \le C(1+\|\nabla h\|_{H^{m-1}})^2(\|\nabla h\|_{H^{m-1}}^3 + \|u\|_{H^{m+\frac{d-\alpha}{2}}}^3) + C_1\sum_{0\le k\le m}\|U_k\|_{L^2}^2, \qquad (3.3)
\end{aligned}
$$

where we used

$$\|\Lambda^{d-\alpha}\partial^k u\|_{L^2} = \|\Lambda^{\frac{d-\alpha}{2}} U_k\|_{L^2} \le \|U_k\|_{H^1}.$$

On the other hand, it follows from Lemmas 3.1 and 3.2 that

$$\begin{aligned}\frac{d}{dt}\Big[\sum_{1\le k\le m}\Big(\|U_k\|_{L^2}^2 + \Big\|\frac{1}{\sqrt{\rho}}\partial^k h\Big\|_{L^2}^2\Big)\Big] + 2\sum_{1\le k\le m}\|U_k\|_{L^2}^2 \\ \le C(1+\|\nabla h\|_{H^{m-1}})^{2(m-1)}(\|\nabla h\|_{H^{m-1}}^3 + \|u\|_{H^{m+\frac{d-\alpha}{2}}}^3).\end{aligned} \tag{3.4}$$

Here we used

$$(1+\|\nabla\log\rho\|_{L^\infty})^{2(k-1)}\sum_{0<l\le k}\Big\|\frac{1}{\sqrt{\rho}}\partial^l h\Big\|_{L^2}^2 \le C(1+\|\nabla h\|_{H^{m-1}})^{2(m-1)}\|\nabla h\|_{H^{k-1}}^2$$

for $k = 1,\dots,m-1$, where $C > 0$ is independent of t.

Now we use Lemma 3.1 for $k = 0$ and Lemma 3.3 to get

$$\frac{d}{dt}(\|h\|_{L^2}^2 + \|U_0\|_{L^2}^2) + 2\|U_0\|_{L^2}^2 \le C\|u\|_{H^{m+\frac{d-\alpha}{2}}}^3 + 2\|u\|_{L^2}\|\nabla h\|_{L^2}\|h\|_{L^\infty},$$

and combine this with Proposition 2.2 to obtain

$$\begin{aligned}\frac{d}{dt}\Big(\int_\Omega \rho|u|^2\,dx + \|h\|_{\dot H^{-\frac{d-\alpha}{2}}}^2 + \|h\|_{L^2}^2 + \|U_0\|_{L^2}^2\Big) + \|u\|_{L^2}^2 + 2\|U_0\|_{L^2}^2 \\ \le C\|u\|_{H^{m+\frac{d-\alpha}{2}}}^3 + 2\|u\|_{L^2}\|\nabla h\|_{L^2}\|h\|_{L^\infty}.\end{aligned}$$

We next choose a positive constant $\eta_1 \ll 1$ satisfying $C_1\eta_1 < 1$ and combine (3.3) and (3.4) to find

$$\begin{aligned}&\frac{d}{dt}\Big[\int_\Omega \rho|u|^2\,dx + \|h\|_{\dot H^{-\frac{d-\alpha}{2}}}^2 + \|h\|_{L^2}^2 + \|U_0\|_{L^2}^2 \\ &\quad + \sum_{1\le k\le m}\Big(\|U_k\|_{L^2}^2 + \Big\|\frac{1}{\sqrt{\rho}}\partial^k h\Big\|_{L^2}^2\Big) + \eta_1\sum_{0\le k\le m-1}\int_\Omega \frac{1}{\rho}\partial^k\nabla h\cdot\Lambda^{d-\alpha}\partial^k u\,dx\Big] \\ &\quad + \frac{\eta_1}{2}\|\nabla h\|_{H^{m-1}}^2 + \Big(\|u\|_{L^2}^2 + \sum_{1\le k\le m}\|U_k\|_{L^2}^2\Big) \\ &\le C(1+\|\nabla h\|_{H^{m-1}})^2(\|\nabla h\|_{H^{m-1}}^3 + \|u\|_{H^{m+\frac{d-\alpha}{2}}}^3) \\ &\quad + C\|h\|_{H^m}(\|u\|_{L^2}^2 + \|\nabla h\|_{L^2}^2).\end{aligned} \tag{3.5}$$

Since we have the equivalence relations

$$\begin{aligned}\int_\Omega \rho|u|^2\,dx \approx \|u\|_{L^2}^2, \quad \|\nabla h\|_{H^{m-1}} \approx \sum_{1\le k\le m}\Big\|\frac{1}{\sqrt{\rho}}\partial^k h\Big\|_{L^2}, \\ \|u\|_{H^{m+\frac{d-\alpha}{2}}} \approx \|u\|_{L^2} + \sum_{0\le k\le m}\|U_k\|_{L^2},\end{aligned} \tag{3.6}$$

and

$$\sum_{0\le k\le m}\|U_k\|_{L^2}^2+\sum_{1\le k\le m}\left\|\frac{1}{\sqrt{\rho}}\partial^k h\right\|_{L^2}^2+\eta_1\sum_{0\le k\le m-1}\int_\Omega\frac{1}{\rho}\partial^k\nabla h\cdot\Lambda^{d-\alpha}\partial^k u\,dx$$
$$\approx\sum_{0\le k\le m}\|U_k\|_{L^2}^2+\sum_{1\le k\le m}\left\|\frac{1}{\sqrt{\rho}}\partial^k h\right\|_{L^2}^2 \tag{3.7}$$

for $\eta_1>0$ sufficiently small, we can find positive constants η_2 and C_2 independent of T such that

$$\frac{d}{dt}(\mathcal{Y}^m+\|h\|_{L^2}^2+\|h\|_{\dot H^{-\frac{d-\alpha}{2}}}^2)+\eta_2\mathcal{Y}^m\le C_2\varepsilon_0\mathcal{Y}^m, \tag{3.8}$$

where $\mathcal{Y}^m=\mathcal{Y}^m(t)$ is given by

$$\mathcal{Y}^m:=\int_\Omega\rho|u|^2\,dx+\sum_{0\le k\le m}\|U_k\|_{L^2}^2+\sum_{1\le k\le m}\left\|\frac{1}{\sqrt{\rho}}\partial^k h\right\|_{L^2}^2$$
$$+\eta_1\sum_{0\le k\le m-1}\int_\Omega\frac{1}{\rho}\partial^k\nabla h\cdot\Lambda^{d-\alpha}\partial^k u\,dx.$$

Thus, once ε_0 is chosen sufficiently small so that $\eta_2-C_2\varepsilon_0>0$, we set $\lambda:=\eta_2-C_2\varepsilon_0$ and use Grönwall's lemma to get

$$\mathcal{Y}^m(t)+\|h(\cdot,t)\|_{L^2}^2+\|h(\cdot,t)\|_{\dot H^{-\frac{d-\alpha}{2}}}^2+\lambda\int_0^t\mathcal{Y}^m(\tau)\,d\tau$$
$$\le\mathcal{Y}^m(0)+\|h_0\|_{L^2}^2+\|h_0\|_{\dot H^{-\frac{d-\alpha}{2}}}^2.$$

Since

$$\mathcal{Y}^m(t)+\|h(\cdot,t)\|_{L^2}^2+\|h(\cdot,t)\|_{\dot H^{-\frac{d-\alpha}{2}}}^2\approx\|(h,u)(\cdot,t)\|_{X^m}^2$$

for $\eta_1>0$ small enough, we conclude the desired result. ∎

3.3. Proof of Theorem 1.1

We are now ready to provide the details of the proof of Theorem 1.1.

First, choose ε_0 as required in Proposition 3.2. Then we set ε_1 to

$$\varepsilon_1^2:=\frac{\varepsilon_0^2}{2(1+C^*)}.$$

By the local existence theory, Theorem 3.1, we can find $T_0>0$ such that if the initial data (h_0,u_0) satisfies $X_0(m)<\varepsilon_1^2$, a solution (h,u) to (1.3) exists in $\mathcal{C}([0,T_0);X^m)$. Assume for a contradiction that

$$T^*:=\sup\{T>0\mid X(T;m)\le\varepsilon_0^2\}<\infty.$$

Then by definition,

$$\varepsilon_0^2=X(T^*;m)\le C^*X_0(m)<C^*\varepsilon_1^2=\frac{C^*}{2(1+C^*)}\varepsilon_0^2<\varepsilon_0^2,$$

and this contradicts the assumption. Thus, the solution exists in $\mathcal{C}(\mathbb{R}_+;X^m)$.

4. Large-time behavior of solutions

4.1. Whole space case: Algebraic decay rate of convergence

To get the large-time behavior estimates for the whole space case, we investigate negative Sobolev norms. First, we present an auxiliary lemma below.

Lemma 4.1. (i) *Let $-d < s_1 < s < s_2 < d$ and $f \in (\dot{H}^{s_1} \cap \dot{H}^{s_2})(\mathbb{R}^d)$. Then we have*

$$\|f\|_{\dot{H}^s} \le \|f\|_{\dot{H}^{s_1}}^{\frac{s_2-s}{s_2-s_1}} \|f\|_{\dot{H}^{s_2}}^{\frac{s-s_1}{s_2-s_1}}.$$

(ii) *If $s \in (0,d)$, $1 < p < q < \infty$, and $1/q + s/d = 1/p$, then we have*

$$\|\Lambda^{-s} f\|_{L^q} \lesssim_p \|f\|_{L^p}.$$

Proof. For (i), since we have

$$s = \frac{s_2 - s}{s_2 - s_1} s_1 + \frac{s - s_1}{s_2 - s_1} s_2,$$

we use Hölder's inequality to obtain

$$\begin{aligned}\int_{\mathbb{R}^d} |\xi|^{2s} |\hat{f}(\xi)|^2 \, d\xi &= \int_{\mathbb{R}^d} |\xi|^{2s_1(\frac{s_2-s}{s_2-s_1})} |\xi|^{2s_2(\frac{s-s_1}{s_2-s_1})} |\hat{f}(\xi)|^{2(\frac{s_2-s}{s_2-s_1})} |\hat{f}(\xi)|^{2(\frac{s-s_1}{s_2-s_1})} \, d\xi \\ &\le \left(\int_{\mathbb{R}^d} |\xi|^{2s_1} |\hat{f}(\xi)|^2 \, d\xi \right)^{\frac{s_2-s}{s_2-s_1}} \left(\int_{\mathbb{R}^d} |\xi|^{2s_2} |\hat{f}(\xi)|^2 \, d\xi \right)^{\frac{s-s_1}{s_2-s_1}},\end{aligned}$$

and this implies the desired result.

The inequality in (ii) is the well-known Hardy–Littlewood–Sobolev inequality, and for the proof, we refer to [30, p. 119, Theorem 1]. ∎

Lemma 4.2. *Let $T > 0$, $m > \frac{d}{2} + 2$, and $0 < s \le \frac{\alpha}{2}$. Let $(h, u) \in \mathcal{C}([0,T); X^m)$ be a solution to system (1.3) satisfying*

$$\sup_{0 \le t \le T} \|h(t)\|_{L^\infty} \le \frac{1}{2}.$$

Then we have

$$\begin{aligned}&\frac{d}{dt}\left(\|\Lambda^{-s} u\|_{L^2}^2 + \|\Lambda^{-s-\frac{d-\alpha}{2}} h\|_{L^2}^2\right) + 2\|\Lambda^{-s} u\|_{L^2}^2 \\ &\qquad \le C\|u\|_{H^{m+\frac{d-\alpha}{2}}}^2 \|\Lambda^{-s} u\|_{L^2} + C\|h\|_{L^2} \|u\|_{H^{m+\frac{d-\alpha}{2}}} \|\Lambda^{-s-\frac{d-\alpha}{2}} \nabla h\|_{L^2},\end{aligned}$$

where $C = C(s, \alpha, d, m)$ is a positive constant independent of T.

Proof. Direct estimation gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\Lambda^{-s} u\|_{L^2}^2 + \|\Lambda^{-s-\frac{d-\alpha}{2}} h\|_{L^2}^2\right) + \|\Lambda^{-s} u\|_{L^2}^2 \\ &\qquad = -\int_{\mathbb{R}^d} \Lambda^{-s}(u \cdot \nabla u) \cdot \Lambda^{-s} u \, dx - \int_{\mathbb{R}^d} \Lambda^{-s-\frac{d-\alpha}{2}} (\nabla \cdot (hu)) \Lambda^{-s-\frac{d-\alpha}{2}} h \, dx\end{aligned}$$

$$
\begin{aligned}
&= -\int_{\mathbb{R}^d} \Lambda^{-s}(u\cdot\nabla u)\cdot\Lambda^{-s}u\,dx + \int_{\mathbb{R}^d} \Lambda^{-s-\frac{d-\alpha}{2}}(hu)\cdot\Lambda^{-s-\frac{d-\alpha}{2}}\nabla h\,dx \\
&=: \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned}
$$

For $\mathcal{I}_1$ one gets

$$
\begin{aligned}
\mathcal{I}_1 &\le \|\Lambda^{-s}(u\cdot\nabla u)\|_{L^2}\|\Lambda^{-s}u\|_{L^2} \\
&\le \|u\cdot\nabla u\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{d}}}}\|\Lambda^{-s}u\|_{L^2} \\
&\le \|\nabla u\|_{L^2}\|u\|_{L^{\frac{d}{s}}}\|\Lambda^{-s}u\|_{L^2} \\
&\le C\|\nabla u\|_{L^2}\|\nabla^{[\frac{d}{2}]+1-s}u\|_{L^2}^{\theta}\|u\|_{L^2}^{1-\theta}\|\Lambda^{-s}u\|_{L^2} \\
&\le C\|u\|^2_{H^{m+\frac{d-\alpha}{2}}}\|\Lambda^{-s}u\|_{L^2},
\end{aligned}
$$

where we used Lemma 4.1(ii) and the Gagliardo–Nirenberg interpolation inequality with

$$
\frac{s}{d} = \Big(\frac{1}{2}-\frac{k}{d}\Big)\theta + \frac{1}{2}(1-\theta), \quad k = \Big[\frac{d}{2}\Big] + 1 - s, \quad \theta = \frac{\frac{d}{2}-s}{\big[\frac{d}{2}\big]+1-s}.
$$

For $\mathcal{I}_2$ we obtain

$$
\begin{aligned}
\mathcal{I}_2 &\le \|\Lambda^{-s-\frac{d-\alpha}{2}}(hu)\|_{L^2}\|\Lambda^{-s-\frac{d-\alpha}{2}}\nabla h\|_{L^2} \\
&\le \|hu\|_{L^{\frac{1}{\frac{1}{2}+\frac{s+\frac{d-\alpha}{2}}{d}}}}\|\Lambda^{-s-\frac{d-\alpha}{2}}\nabla h\|_{L^2} \\
&\le \|h\|_{L^2}\|u\|_{L^{\frac{d}{s+\frac{d-\alpha}{2}}}}\|\Lambda^{-s-\frac{d-\alpha}{2}}\nabla h\|_{L^2} \\
&\le C\|h\|_{L^2}\|\nabla^{[\frac{\alpha}{2}]+1-s}u\|_{L^2}^{\theta}\|u\|_{L^2}^{1-\theta}\|\Lambda^{-s-\frac{d-\alpha}{2}}\nabla h\|_{L^2} \\
&\le C\|h\|_{L^2}\|u\|_{H^{m+\frac{d-\alpha}{2}}}\|\Lambda^{-s-\frac{d-\alpha}{2}}\nabla h\|_{L^2}.
\end{aligned}
$$

Here we also used Lemma 4.1(ii) and the Gagliardo–Nirenberg interpolation inequality with

$$
\frac{s+\frac{d-\alpha}{2}}{d} = \Big(\frac{1}{2}-\frac{k}{d}\Big)\theta + \frac{1}{2}(1-\theta), \quad k = \Big[\frac{\alpha}{2}\Big] + 1 - s, \quad \theta = \frac{\frac{\alpha}{2}-s}{\big[\frac{\alpha}{2}\big]+1-s}.
$$

Now we combine all the estimates for the $\mathcal{I}_i$ to deduce the desired result. ■

Lemma 4.3. *Let $T > 0$, $m > \frac{d}{2} + 2$, and $0 < s \le \frac{\alpha}{2}$. Let $(h, u) \in \mathcal{C}([0, T); X^m)$ be a solution to system (1.3) satisfying*

$$
\sup_{0\le t\le T}\|h(t)\|_{L^\infty} \le \frac{1}{2}.
$$

Then we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d} \Lambda^{-s}\nabla h\cdot\Lambda^{-s}u\,dx + \frac{1}{2}\|\Lambda^{-s-\frac{d-\alpha}{2}}\nabla h\|_{L^2}^2 \\
&\quad\le C\left(\|h\|_{H^m} + \|\Lambda^{-s-\frac{d-\alpha}{2}}\nabla h\|_{L^2}\right)\|u\|_{H^{m+\frac{d-\alpha}{2}}}^2 + C\|h\|_{L^2}\|\Lambda^{-s}u\|_{L^2}^2 \\
&\qquad + \|\Lambda^{-s}(\nabla\cdot u)\|_{L^2}^2 + \frac{1}{2}\|\Lambda^{-s+\frac{d-\alpha}{2}}u\|_{L^2}^2,
\end{aligned}$$

where $C > 0$ is independent of T.

Proof. Straightforward calculation yields

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^d} \Lambda^{-s}\nabla h\cdot\Lambda^{-s}u\,dx &= -\int_{\mathbb{R}^d} \Lambda^{-s}(\partial_t h)\Lambda^{-s}\nabla\cdot u\,dx \\
&\quad + \int_{\mathbb{R}^d} \Lambda^{-s}\nabla h\cdot\Lambda^{-s}(\partial_t u)\,dx \\
&=: \mathcal{J}_1 + \mathcal{J}_2.
\end{aligned}$$

For $\mathcal{J}_1$ we use Lemma 4.1 to get

$$\begin{aligned}
\mathcal{J}_1 &= \int_{\mathbb{R}^d} \Lambda^{-s}(\nabla\cdot(hu))\Lambda^{-s}\nabla\cdot u\,dx + \|\Lambda^{-s}(\nabla\cdot u)\|_{L^2}^2 \\
&= -\int_{\mathbb{R}^d} \Lambda^{-s}(hu)\cdot\nabla\Lambda^{-s}\nabla\cdot u\,dx + \|\Lambda^{-s}(\nabla\cdot u)\|_{L^2}^2 \\
&\le \|\Lambda^{-s}(hu)\|_{L^2}\|\Lambda^{-s}\nabla(\nabla\cdot u)\|_{L^2} + \|\Lambda^{-s}(\nabla\cdot u)\|_{L^2}^2 \\
&\le C\|hu\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{d}}}}\|\Lambda^{-s}\nabla(\nabla\cdot u)\|_{L^2} + \|\Lambda^{-s}(\nabla\cdot u)\|_{L^2}^2 \\
&\le C\|h\|_{L^2}\|u\|_{H^{m+\frac{d-\alpha}{2}}}\|\Lambda^{-s}\nabla(\nabla\cdot u)\|_{L^2} + \|\Lambda^{-s}(\nabla\cdot u)\|_{L^2}^2 \\
&\le C\|h\|_{L^2}\|u\|_{H^{m+\frac{d-\alpha}{2}}}\left(\|u\|_{H^{m+\frac{d-\alpha}{2}}} + \|\Lambda^{-s}u\|_{L^2}\right) + \|\Lambda^{-s}(\nabla\cdot u)\|_{L^2}^2.
\end{aligned}$$

Here we used

$$\|\Lambda^{-s}\nabla(\nabla\cdot u)\|_{L^2} \le \|\Lambda^{-s}u\|_{L^2}^{\frac{s}{s+2}}\|\nabla^2 u\|_{L^2}^{\frac{2}{s+2}} \le \frac{s}{s+2}\|\Lambda^{-s}u\|_{L^2} + \frac{2}{s+2}\|\nabla^2 u\|_{L^2}.$$

For $\mathcal{J}_2$ we also have

$$\begin{aligned}
\mathcal{J}_2 &= -\int_{\mathbb{R}^d} \Lambda^{-s}\nabla h\cdot\Lambda^{-s}(u\cdot\nabla u)\,dx - \int_{\mathbb{R}^d} \Lambda^{-s}\nabla h\cdot\Lambda^{-s}u\,dx - \|\Lambda^{-s-\frac{d-\alpha}{2}}\nabla h\|_{L^2}^2 \\
&\le \|\Lambda^{-s}\nabla h\|_{L^2}\|\Lambda^{-s}(u\cdot\nabla u)\|_{L^2} - \frac{1}{2}\|\Lambda^{-s-\frac{d-\alpha}{2}}\nabla h\|_{L^2}^2 + \frac{1}{2}\|\Lambda^{-s+\frac{d-\alpha}{2}}u\|_{L^2}^2 \\
&\le C\left(\|\Lambda^{-s-\frac{d-\alpha}{2}}\nabla h\|_{L^2} + \|\nabla h\|_{L^2}\right)\|u\|_{H^{m+\frac{d-\alpha}{2}}}^2 \\
&\quad - \frac{1}{2}\|\Lambda^{-s-\frac{d-\alpha}{2}}\nabla h\|_{L^2}^2 + \frac{1}{2}\|\Lambda^{-s+\frac{d-\alpha}{2}}u\|_{L^2}^2,
\end{aligned}$$

where we used Lemma 4.1 and Young's inequality to get

$$\begin{aligned}\|\Lambda^{-s}\nabla h\|_{L^2} &\le \|\Lambda^{-s-\frac{d-\alpha}{2}}\nabla h\|_{L^2}^{\theta}\|\nabla h\|_{L^2}^{1-\theta}\\ &\le \theta\|\Lambda^{-s-\frac{d-\alpha}{2}}\nabla h\|_{L^2}+(1-\theta)\|\nabla h\|_{L^2}, \quad \theta := \frac{s}{s+\frac{d-\alpha}{2}}.\end{aligned}$$

Thus, we gather the estimates for $\mathcal{J}_1$ and $\mathcal{J}_2$ to conclude the desired result. ∎

Proposition 4.1. *Let $T>0$, $m>\frac{d}{2}+2$, and $0<s\le\frac{\alpha}{2}$. Let $(h,u)\in\mathcal{C}([0,T);X^m)$ be a solution to system (1.3). Suppose that $X(T;m)\le\varepsilon_0^2\ll 1$ so that*

$$\sup_{0\le t\le T}\|h(t)\|_{L^\infty}\le\frac{1}{2}.$$

Then we have

$$\begin{aligned}&\|u(\cdot,t)\|_{H^{m+\frac{d-\alpha}{2}}}^2+\|u(\cdot,t)\|_{\dot H^{-s}}^2+\|h(\cdot,t)\|_{H^m}^2+\|h(\cdot,t)\|_{\dot H^{-s-\frac{d-\alpha}{2}}}^2\\ &\quad+\int_0^t\Big(\|u(\cdot,\tau)\|_{H^{m+\frac{d-\alpha}{2}}}^2+\|u(\cdot,\tau)\|_{\dot H^{-s}}^2+\|\nabla h(\cdot,\tau)\|_{H^{m-1}}^2+\|h(\cdot,\tau)\|_{\dot H^{1-s-\frac{d-\alpha}{2}}}^2\Big)\,d\tau\\ &\le C\Big(\|u_0\|_{H^{m+\frac{d-\alpha}{2}}}^2+\|u_0\|_{\dot H^{-s}}^2+\|h_0\|_{H^m}^2+\|h_0\|_{\dot H^{-s-\frac{d-\alpha}{2}}}^2\Big),\end{aligned}$$

where $C>0$ is independent of T.

Proof. We collect the estimates in Lemmas 4.2 and 4.3, combine these with (3.8), and use Young's inequality to find positive constants λ_2, η_3, and C_3 satisfying

$$\begin{aligned}&\frac{d}{dt}\Big(\mathcal{Y}^m+\|\Lambda^{-s}u\|_{L^2}^2+\|h\|_{L^2}^2+\|h\|_{\dot H^{-\frac{d-\alpha}{2}}}^2+\|\Lambda^{-s-\frac{d-\alpha}{2}}h\|_{L^2}^2\\ &\qquad+\eta_3\int_{\mathbb{R}^d}\Lambda^{-s}\nabla h\cdot\Lambda^{-s}u\,dx\Big)+\lambda_2\big(\mathcal{Y}^m+\|\Lambda^{-s}u\|_{L^2}^2+\|\Lambda^{-s-\frac{d-\alpha}{2}}\nabla h\|_{L^2}^2\big)\\ &\le C_3\varepsilon_0\big(\mathcal{Y}^m+\|\Lambda^{-s}u\|_{L^2}^2+\|\Lambda^{-s-\frac{d-\alpha}{2}}\nabla h\|_{L^2}^2\big),\end{aligned}$$

where we used

$$\|\Lambda^{-s}(\nabla\cdot u)\|_{L^2}\lesssim_s\|\Lambda^{-s}u\|_{L^2}^{\frac{s}{s+1}}\|\nabla u\|_{L^2}^{\frac{1}{s+1}}$$

and

$$\|\Lambda^{-s+\frac{d-\alpha}{2}}u\|_{L^2}\lesssim_{s,d,\alpha}\|\Lambda^{-s}u\|_{L^2}^{\frac{s-\frac{d-\alpha}{2}}{s}}\|u\|_{L^2}^{\frac{\frac{d-\alpha}{2}}{s}}.$$

Thus, we use the smallness of ε_0 to get a constant $\lambda_3>0$ and the relations

$$\begin{aligned}\|\Lambda^{-\frac{d-\alpha}{2}}h\|_{L^2} &\le \|\Lambda^{-s-\frac{d-\alpha}{2}}h\|_{L^2}^{\theta}\|h\|_{L^2}^{1-\theta}\\ &\le \theta\|\Lambda^{-s-\frac{d-\alpha}{2}}h\|_{L^2}+(1-\theta)\|h\|_{L^2}, \quad \theta := \frac{\frac{d-\alpha}{2}}{s+\frac{d-\alpha}{2}}\end{aligned}$$

and

$$\begin{aligned}\|\Lambda^{-s}\nabla h\|_{L^2} &\le \|\Lambda^{-s-\frac{d-\alpha}{2}}h\|_{L^2}^{\bar\theta}\|\nabla h\|_{L^2}^{1-\bar\theta}\\ &\le \bar\theta\|\Lambda^{-s-\frac{d-\alpha}{2}}h\|_{L^2}+(1-\bar\theta)\|h\|_{L^2}, \quad \bar\theta := \frac{s}{1+s+\frac{d-\alpha}{2}}\end{aligned}$$

to yield

$$\begin{aligned}&\mathcal{Y}^m(t)+\|u(\cdot,t)\|_{\dot H^{-s}}^2+\|h(\cdot,t)\|_{L^2}^2+\|h(\cdot,t)\|_{\dot H^{-s-\frac{d-\alpha}{2}}}^2\\ &\quad+\lambda_3\int_0^t\big(\mathcal{Y}^m(\tau)+\|u(\cdot,\tau)\|_{\dot H^{-s}}^2+\|h(\cdot,\tau)\|_{\dot H^{1-s-\frac{d-\alpha}{2}}}^2\big)\,d\tau\\ &\le C\big(\mathcal{Y}^m(0)+\|u_0\|_{\dot H^{-s}}^2+\|h_0\|_{L^2}^2+\|h_0\|_{\dot H^{-s-\frac{d-\alpha}{2}}}^2\big).\end{aligned}$$

This asserts the desired result. ■

4.1.1. Proof of Theorem 1.2. In this part, we provide the details of the proof of Theorem 1.2 on the large-time behavior of solutions in the whole space.

Before getting into the main estimates, we first deal with the decay estimate (1.8) in Remark 1.4. That introduces the main ideas behind our arguments for the better decay estimates of solutions.

From Lemma 4.1, we have that for $s \ge 0$,

$$\|h\|_{L^2} \le \|\nabla h\|_{L^2}^{\frac{s+\frac{d-\alpha}{2}}{1+s+\frac{d-\alpha}{2}}}\|h\|_{\dot H^{-s-\frac{d-\alpha}{2}}}^{\frac{1}{1+s+\frac{d-\alpha}{2}}} \quad\text{and}\quad \|h\|_{\dot H^{-\frac{d-\alpha}{2}}} \le \|\nabla h\|_{L^2}^{\frac{s}{1+s+\frac{d-\alpha}{2}}}\|h\|_{\dot H^{-s-\frac{d-\alpha}{2}}}^{\frac{1+\frac{d-\alpha}{2}}{1+s+\frac{d-\alpha}{2}}}.$$

We then use the uniform bound in Proposition 4.1 and the smallness assumptions on the solutions to get

$$\big(\|h\|_{L^2}^2+\|h\|_{\dot H^{-\frac{d-\alpha}{2}}}^2\big)^{\frac{1+s+\frac{d-\alpha}{2}}{s}} \lesssim \|\nabla h\|_{L^2}^2.$$

Thus, we now set

$$\mathcal{F}^m(t) := \mathcal{Y}^m(t)+\|h(\cdot,t)\|_{L^2}^2+\eta_2\|h(\cdot,t)\|_{\dot H^{-\frac{d-\alpha}{2}}}^2, \tag{4.1}$$

where $\mathcal{Y}^m$ is from Proposition 4.1. From the smallness of solutions and estimates in (3.8), we can find a constant $\lambda_4 > 0$ satisfying

$$\frac{d}{dt}\mathcal{F}^m+\lambda_4(\mathcal{F}^m)^{\frac{1+s+\frac{d-\alpha}{2}}{s}} \le 0.$$

This gives

$$-\frac{1}{\frac{1+\frac{d-\alpha}{2}}{s}}\frac{d}{dt}\big((\mathcal{F}^m)^{-\frac{1+\frac{d-\alpha}{2}}{s}}\big) \le -\lambda_4.$$

We integrate the above with respect to t and get

$$(\mathcal{F}^m(t))^{-\frac{1+\frac{d-\alpha}{2}}{s}} \geq (\mathcal{F}^m(0))^{-\frac{1+\frac{d-\alpha}{2}}{s}} + \lambda_4 \frac{1+\frac{d-\alpha}{2}}{s} t,$$

or equivalently,

$$\mathcal{F}^m(t) \leq \Big((\mathcal{F}^m(0))^{-\frac{1+\frac{d-\alpha}{2}}{s}} + \lambda_4 \frac{1+\frac{d-\alpha}{2}}{s} t \Big)^{-\frac{s}{1+\frac{d-\alpha}{2}}}.$$

Thus, we obtain

$$\mathcal{F}^m(t) \lesssim (1+t)^{-\frac{s}{1+\frac{d-\alpha}{2}}}.$$

Since $\mathcal{F}^m(t) \approx \|(h,u)(\cdot,t)\|_{X^m}^2$, this concludes the desired result.

As mentioned in Remark 1.4, the above estimates do not allow us to have a better decay rate of convergence even in higher dimensions. For that reason, we refine the above arguments by taking the negative order $(\frac{\alpha}{2} \geq)\ s > 0$ large enough.

We present the proof by dividing into two cases: $s \geq 1 - \frac{d-\alpha}{2}$ and $s > 2 + d - \alpha$. In the former case, we obtain an algebraic decay rate of convergence of solutions. On the other hand, in the latter case, an exponential decay rate is found.

$\diamond$ *Case A:* $s \geq 1 - \frac{d-\alpha}{2}$. In this case, we first note that

$$-s - \frac{d-\alpha}{2} \leq -1 + \frac{d-\alpha}{2} < 0 \quad \text{and} \quad -s < d - \alpha - 1 < 1$$

with our assumption on s and α, and thus the interpolation inequality implies

$$(h_0, u_0) \in \dot{H}^{-1+\frac{d-\alpha}{2}}(\mathbb{R}^d) \times [\dot{H}^{d-\alpha-1}(\mathbb{R}^d)]^d.$$

Then, similarly to Lemma 4.2, we estimate

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|\Lambda^{-1+\frac{d-\alpha}{2}} h\|_{L^2}^2 + \|\Lambda^{d-\alpha-1} u\|_{L^2}^2\big) + \|\Lambda^{d-\alpha-1} u\|_{L^2}^2 \\
&\quad = -\int_{\mathbb{R}^d} (\Lambda^{-1+\frac{d-\alpha}{2}} h)\Lambda^{-1+\frac{d-\alpha}{2}}(\nabla\cdot(hu))\,dx - \int_{\mathbb{R}^d} (\Lambda^{-1+\frac{d-\alpha}{2}} h)\Lambda^{-1+\frac{d-\alpha}{2}}(\nabla\cdot u)\,dx \\
&\qquad - \frac{1}{2}\int_{\mathbb{R}^d} \Lambda^{d-\alpha-1}(u\cdot\nabla u)\Lambda^{d-\alpha-1} u\,dx - \int_{\mathbb{R}^d} \nabla\Lambda^{-1} h \Lambda^{d-\alpha-1} u\,dx \\
&\quad = \int_{\mathbb{R}^d} (\Lambda^{-1+\frac{d-\alpha}{2}}\nabla h)\cdot\Lambda^{-1+\frac{d-\alpha}{2}}(hu)\,dx - \frac{1}{2}\int_{\mathbb{R}^d} \Lambda^{d-\alpha-1}(u\cdot\nabla u)\Lambda^{d-\alpha-1} u\,dx \\
&\quad \leq \|\Lambda^{-1+\frac{d-\alpha}{2}}\nabla h\|_{L^2}\|\Lambda^{-1+\frac{d-\alpha}{2}}(hu)\|_{L^2} + \frac{1}{2}\|\Lambda^{d-\alpha-1}(u\cdot\nabla u)\|_{L^2}\|\Lambda^{d-\alpha-1} u\|_{L^2} \\
&\quad \leq \|\Lambda^{-1+\frac{d-\alpha}{2}}\nabla h\|_{L^2}\|hu\|_{L^{\frac{1}{\frac{1}{2}+\frac{1-\frac{d-\alpha}{2}}{d}}}} + \frac{1}{2}\|\Lambda^{d-\alpha-1}(u\cdot\nabla u)\|_{L^2}\|\Lambda^{d-\alpha-1} u\|_{L^2}.
\end{aligned}$$

Here, if $\alpha \leq d-1$, we get

$$\|\Lambda^{d-\alpha-1}(u\cdot\nabla u)\|_{L^2} \leq C\|u\cdot\nabla u\|_{H^1} \leq C\|u\|_{H^{m+\frac{d-\alpha}{2}}}^2,$$

and if $\alpha \in (d-1, d)$, we deduce

$$\|\Lambda^{-(d-\alpha-1)}(u\cdot\nabla u)\|_{L^2} \le \|u\cdot\nabla u\|_{L^{\frac{1}{\frac{1}{2}+\frac{d-\alpha-1}{d}}}} \le \|\nabla u\|_{L^2}\|u\|_{L^{\frac{d}{d-\alpha-1}}} \le C\|u\|^2_{H^{m+\frac{d-\alpha}{2}}}.$$

In either case, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|\Lambda^{-1+\frac{d-\alpha}{2}}h\|^2_{L^2} + \|\Lambda^{d-\alpha-1}u\|^2_{L^2}\big) + \|\Lambda^{d-\alpha-1}u\|^2_{L^2}\\
&\quad\le \|\Lambda^{-1+\frac{d-\alpha}{2}}\nabla h\|_{L^2}\|hu\|_{L^{\frac{1}{\frac{1}{2}+\frac{1-\frac{d-\alpha}{2}}{d}}}} + C\|u\|^2_{H^{m+\frac{d-\alpha}{2}}}\|\Lambda^{d-\alpha-1}u\|_{L^2}\\
&\quad\le \|\Lambda^{-1+\frac{d-\alpha}{2}}\nabla h\|_{L^2}\|h\|_{L^2}\|u\|_{L^{\frac{d}{1-\frac{d-\alpha}{2}}}} + C\|u\|^2_{H^{m+\frac{d-\alpha}{2}}}\|\Lambda^{d-\alpha-1}u\|_{L^2}\\
&\quad\le C\|h\|^2_{H^m}\|u\|_{H^{m+\frac{d-\alpha}{2}}} + C\|u\|^2_{H^{m+\frac{d-\alpha}{2}}}\|\Lambda^{d-\alpha-1}u\|_{L^2},
\end{aligned}$$

where we used $1-\frac{d-\alpha}{2}\in(0,1)$ to have $\|\Lambda^{-1+\frac{d-\alpha}{2}}\nabla h\|_{L^2} \lesssim \|h\|_{H^m}$. On the other hand, similarly to Lemma 4.3, we can get a constant $\gamma > 0$ satisfying

$$\begin{aligned}
&-\frac{d}{dt}\int_{\mathbb{R}^d} h\Lambda^{d-\alpha-2}\nabla\cdot u\,dx\\
&\quad= -\int_{\mathbb{R}^d}(\partial_t h)\Lambda^{d-\alpha-2}\nabla\cdot u\,dx - \int_{\mathbb{R}^d} h\Lambda^{-\alpha}(\partial_t(\nabla\cdot u))\,dx\\
&\quad= -\int_{\mathbb{R}^d} hu\cdot\Lambda^{d-\alpha-2}\nabla(\nabla\cdot u)\,dx + \|\Lambda^{-1+\frac{d-\alpha}{2}}\nabla\cdot u\|^2_{L^2}\\
&\qquad+\frac{1}{2}\int_{\mathbb{R}^d} h\Lambda^{d-\alpha-2}(\nabla\cdot(u\cdot\nabla u))\,dx - \int_{\mathbb{R}^d} h\Lambda^{d-\alpha-2}\nabla\cdot u\,dx - \|h\|^2_{L^2}\\
&\quad\le \|u\|_{L^\infty}\|h\|_{L^2}\|\Lambda^{d-\alpha-2}\nabla(\nabla\cdot u)\|_{L^2} + \|\Lambda^{-1+\frac{d-\alpha}{2}}\nabla\cdot u\|^2_{L^2}\\
&\qquad+C\|h\|_{L^2}\|\Lambda^{d-\alpha-2}\nabla\cdot(u\cdot\nabla u)\|_{L^2} - \frac{1}{2}\|h\|^2_{L^2} + \frac{1}{2}\|\Lambda^{d-\alpha-2}\nabla\cdot u\|^2_{L^2}\\
&\quad\le C\|h\|_{L^2}\|u\|^2_{H^{m+\frac{d-\alpha}{2}}} + \gamma\big(\|u\|^2_{H^1} + \|\Lambda^{d-\alpha-1}u\|^2_{L^2}\big) - \frac{1}{2}\|h\|^2_{L^2}.
\end{aligned}$$

Then we use the estimates in Proposition 4.1 to get

$$\begin{aligned}
&\frac{d}{dt}\left(\mathcal{F}^m + \|\Lambda^{-1+\frac{d-\alpha}{2}}h\|^2_{L^2} + \|\Lambda^{d-\alpha-1}u\|^2_{L^2} - \eta_4\int_{\mathbb{R}^d} h\Lambda^{d-\alpha-2}\nabla\cdot u\,dx\right)\\
&\qquad+\eta_5\big(\mathcal{Y}^m + \|\Lambda^{d-\alpha-1}u\|^2_{L^2} + \|h\|^2_{L^2}\big)\\
&\quad\le C\varepsilon_0\big(\mathcal{Y}^m + \|\Lambda^{d-\alpha-1}u\|^2_{L^2} + \|h\|^2_{L^2}\big)
\end{aligned}\tag{4.2}$$

for some positive constants η_4 and η_5, where $\mathcal{F}^m$ appears in (4.1). Noting that

$$s + \frac{d-\alpha}{2} \ge 1 > \max\Big\{1-\frac{d-\alpha}{2}, \frac{d-\alpha}{2}\Big\},$$

we find

$$\|h\|_{\dot{H}^{-\frac{d-\alpha}{2}}} \le \|h\|_{L^2}^{\frac{s}{s+\frac{d-\alpha}{2}}} \|h\|_{\dot{H}^{-s-\frac{d-\alpha}{2}}}^{\frac{\frac{d-\alpha}{2}}{s+\frac{d-\alpha}{2}}} \quad \text{and} \quad \|h\|_{\dot{H}^{-1+\frac{d-\alpha}{2}}} \le \|h\|_{L^2}^{\frac{s+d-\alpha-1}{s+\frac{d-\alpha}{2}}} \|h\|_{\dot{H}^{-s-\frac{d-\alpha}{2}}}^{\frac{1-\frac{d-\alpha}{2}}{s+\frac{d-\alpha}{2}}},$$

which subsequently implies

$$\|h\|_{\dot{H}^{-\frac{d-\alpha}{2}}}^{\frac{s+\frac{d-\alpha}{2}}{s}} \lesssim \|h\|_{L^2} \quad \text{and} \quad \|h\|_{\dot{H}^{-1+\frac{d-\alpha}{2}}}^{\frac{s+\frac{d-\alpha}{2}}{s+d-\alpha-1}} \lesssim \|h\|_{L^2}.$$

From the smallness condition on $\|h\|_{L^2}$ we have

$$\left(\|h\|_{\dot{H}^{-1+\frac{d-\alpha}{2}}}^2 + \|h\|_{\dot{H}^{-\frac{d-\alpha}{2}}}^2\right)^{1+\zeta} \lesssim \|h\|_{L^2}^2, \quad \zeta := \max\left\{\frac{d-\alpha}{2s}, \frac{1-\frac{d-\alpha}{2}}{s+d-\alpha-1}\right\}. \tag{4.3}$$

We now define

$$\mathcal{Z}^m := \mathcal{F}^m + \|\Lambda^{-1+\frac{d-\alpha}{2}} h\|_{L^2}^2 + \|\Lambda^{d-\alpha-1} u\|_{L^2}^2 - \eta_4 \int_{\mathbb{R}^d} h \Lambda^{d-\alpha-2} \nabla \cdot u \, dx;$$

then a simple combination (4.2) and (4.3) leads to

$$\frac{d}{dt}\mathcal{Z}^m + \lambda_3 (\mathcal{Z}^m)^{1+\zeta} \le 0.$$

Solving the above differential inequality gives

$$(\mathcal{Z}^m)(t) \le \left((\mathcal{Z}^m(0))^{-\zeta} + \lambda_3 \zeta t\right)^{-\frac{1}{\zeta}},$$

and this proves the first assertion in Theorem 1.2.

$\diamond$ *Case B: $s > 2 + d - \alpha$.* Note that

$$\|u\|_{L^2} \le \|u\|_{\dot{H}^{1+\frac{d-\alpha}{2}}}^{\frac{s}{1+\frac{d-\alpha}{2}+s}} \|u\|_{\dot{H}^{-s}}^{\frac{1+\frac{d-\alpha}{2}}{1+\frac{d-\alpha}{2}+s}}$$

and $s > 2 + d - \alpha$ is equivalent to $\frac{s}{1+\frac{d-\alpha}{2}+s} > \frac{2}{3}$. Since we have a uniform bound for $\|u\|_{\dot{H}^{-s}}$, we can get

$$\|u\|_{L^2}^3 \lesssim \varepsilon_0^{\frac{3s}{1+\frac{d-\alpha}{2}+s}-2} \|u\|_{\dot{H}^{1+\frac{d-\alpha}{2}}}^2 = \varepsilon_0^{\frac{3s}{1+\frac{d-\alpha}{2}+s}-2} \|U_1\|_{L^2}^2,$$

where U_1 was defined as $U_1 := \nabla \Lambda^{\frac{d-\alpha}{2}} u$.

We now define a function

$$\begin{aligned} \bar{\mathcal{Y}}^m := & \sum_{1 \le k \le m} \left(\|U_k\|_{L^2}^2 + \left\|\frac{1}{\sqrt{\rho}} \partial^k h\right\|_{L^2}^2\right) \\ & + \eta_1 \sum_{0 \le k \le m-1} \int_{\mathbb{R}^d} \frac{1}{\rho} \partial^k \nabla h \cdot \Lambda^{d-\alpha} \partial^k u \, dx. \end{aligned} \tag{4.4}$$

Then it follows from (3.3) and (3.4) that there exist positive constants η_5 and C, independent of ε_0 and T, such that

$$\frac{d}{dt}\bar{y}^m + \eta_5 \bar{y}^m \le C\varepsilon_0 \bar{y}^m + C\|u\|_{L^2}^3 \le C\varepsilon_0 \bar{y}^m + C\varepsilon_0^{\frac{3s}{1+\frac{d-\alpha}{2}+s}-2}\|U_1\|_{L^2}^2.$$

Then the smallness condition on ε_0 implies

$$\frac{d}{dt}\bar{y}^m + \eta_7 \bar{y}^m \le 0$$

for some constant $\eta_7 > 0$, and Grönwall's lemma implies an exponential decay rate of convergence of solutions. This completes the proof.

4.2. Periodic case: Exponential decay rate of convergence

In this part we consider the periodic domain, i.e. $\Omega = \mathbb{T}^d$, and study the large-time behavior estimates of system (1.3). Instead of dealing with the negative Sobolev norm of (h, u), we take advantage of the boundedness of the domain and show an exponential decay estimate of its L^2 norm.

Let us define a modulated energy:

$$\mathcal{E}(t) := \frac{1}{2}\int_{\mathbb{T}^d}(1+h)|u-m_c|^2\,dx + \frac{1}{2}\int_{\mathbb{T}^d} h\Lambda^{\alpha-d}h\,dx,$$

where m_c denotes the average of the momentum:

$$m_c := \int_{\mathbb{T}^d}(1+h)u\,dx.$$

Here we remind the reader that the mass is assumed to be zero, i.e. $\int_{\mathbb{T}^d} h\,dx = 0$ for all $t \ge 0$.

Note that if there exists a positive constant lower bound on $1+h$, i.e. $h(x,t)+1 > h_{\min} > 0$ for all $(x,t) \in \mathbb{T}^d \times \mathbb{R}_+$, then the modulated energy satisfies

$$h_{\min}\|(u-m_c)(\cdot,t)\|_{L^2}^2 + \|h(\cdot,t)\|_{\dot{H}^{-\frac{d-\alpha}{2}}}^2 \le \mathcal{E}(t) \tag{4.5}$$

for all $t \ge 0$. This implies that the exponential decay of $\mathcal{E}(t)$ also gives an estimate of the lowest-order norm of solutions. For that reason, our first goal is to prove the following proposition.

Proposition 4.2. *Let (h,u) be a global classical solution to (1.3) with sufficient regularity. Suppose that*

(i) $\inf_{(x,t)\in\mathbb{T}^d\times\mathbb{R}_+} 1 + h(x,t) \ge h_{\min} > 0$ *and*

(ii) $h \in W^{1,\infty}(\mathbb{T}^d \times \mathbb{R}_+)$, $\nabla\cdot u \in L^\infty(\mathbb{R}_+; [L^\infty(\mathbb{T}^d)]^d)$.

Then we have

$$\|(u-m_c)(\cdot,t)\|_{L^2}^2 + \|h(\cdot,t)\|_{\dot{H}^{-\frac{d-\alpha}{2}}} \le Ce^{-\lambda t}.$$

Here C and λ are positive constants independent of t.

Remark 4.1. We notice that the required regularity and assumptions for solutions (h,u) are guaranteed by Theorem 1.1.

In the lemma below, we first show that the modulated energy $\mathcal{E}(t)$ is not increasing in time.

Lemma 4.4. *Let (h,u) be a global classical solution to (1.3) with sufficient regularity. Then we have*

$$\frac{d}{dt}\mathcal{E}(t) + \mathcal{D}(t) = 0,$$

where the dissipation rate function $\mathcal{D}$ is given by

$$\mathcal{D}(t) := \int_{\mathbb{T}^d} (h+1)|u-m_c|^2\,dx.$$

Proof. Direct computation gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d}(h+1)|u-m_c|^2\,dx \\
&\quad= \frac{1}{2}\int_{\mathbb{T}^d}\partial_t h|u-m_c|^2\,dx + \int_{\mathbb{T}^d}(h+1)(u-m_c)(\partial_t u - m_c')\,dx \\
&\quad= \int_{\mathbb{T}^d}((h+1)u\cdot\nabla u)\cdot(u-m_c)\,dx \\
&\qquad - \int_{\mathbb{T}^d}(h+1)(u-m_c)\cdot\big(u\cdot\nabla u + u + \nabla\Lambda^{\alpha-d}h\big)\,dx \\
&\quad= -\int_{\mathbb{T}^d}(h+1)(u-m_c)\cdot u\,dx - \int_{\mathbb{T}^d}(h+1)(u-m_c)\cdot\nabla\Lambda^{\alpha-d}h\,dx \\
&\quad= -\int_{\mathbb{T}^d}(h+1)|u-m_c|^2\,dx - \int_{\mathbb{T}^d}(h+1)u\cdot\nabla\Lambda^{\alpha-d}h\,dx.
\end{aligned}$$

Here we used the symmetry of the operator $\Lambda^{\alpha-d}$:

$$\int_{\mathbb{T}^d}(h+1)\nabla\Lambda^{\alpha-d}h\,dx = \int_{\mathbb{T}^d}h\nabla\Lambda^{\alpha-d}h\,dx = 0.$$

Since we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d}h\Lambda^{\alpha-d}h\,dx = \int_{\mathbb{T}^d}(h+1)u\cdot\nabla\Lambda^{\alpha-d}h\,dx,$$

we combine the above results to conclude the desired result. ∎

Since the dissipation rate function $\mathcal{D}$ does not have a dissipation with respect to h, motivated by [13], we introduce a perturbed modulated energy $\mathcal{E}^\sigma$:

$$\mathcal{E}^\sigma := \mathcal{E} + \sigma \int_{\mathbb{T}^d} (u - m_c) \cdot \nabla W \star h \, dx,$$

where $\sigma > 0$ will be chosen appropriately later and W satisfies the following relations:

(i) The potential W is an even function explicitly written as

$$W(x) = \begin{cases} -c_0 \log |x| + G_0(x) & \text{if } d = 2, \\ c_1 |x|^{2-d} + G_1(x) & \text{if } d \geq 3, \end{cases}$$

where $c_0 > 0$ and $c_1 > 0$ are normalization constants and G_0 and G_1 are smooth functions over $\mathbb{T}^2$ and $\mathbb{T}^d$ $(d \geq 3)$, respectively.

(ii) For any $h \in L^2(\mathbb{T}^d)$ with $\int_{\mathbb{T}^d} h \, dx = 0$, $U := W \star h \in H^1(\mathbb{T}^d)$ is the unique function that satisfies the condition

$$\int_{\mathbb{T}^d} U \, dx = 0 \quad \text{and} \quad \int_{\mathbb{T}^d} \nabla U \cdot \nabla \psi \, dx = \int_{\mathbb{T}^d} h \, \psi \, dx \quad \forall \psi \in H^1(\mathbb{T}^d), \tag{4.6}$$

i.e. U is the unique weak solution to $-\Delta U = h$.

Remark 4.2. For $h \in L^2(\mathbb{T}^d)$ with $\int_{\mathbb{T}^d} h \, dx = 0$, the following hold:

(i) For W defined above we have

$$\|h\|_{\dot{H}^{-1}} \approx \|\nabla W \star h\|_{L^2}.$$

(ii) We can define the $\dot{H}^{-s}(\mathbb{T}^d)$-norm as

$$\|h\|_{\dot{H}^{-s}} := \left(\sum_{n \in \mathbb{Z}^d} |n|^{-2s} |\hat{h}(n)|^2 \right)^{1/2}.$$

Then, by definition, it is clear that for $s_1 \geq s_2 \geq 0$,

$$\|h\|_{\dot{H}^{-s_1}} \leq \|h\|_{\dot{H}^{-s_2}}.$$

In particular, we have

$$\|h\|_{\dot{H}^{-1}} \leq \|h\|_{\dot{H}^{-\frac{d-\alpha}{2}}},$$

due to $(d - \alpha)/2 \in (0, 1)$.

Remark 4.3. Due to Remark 4.2 and the assumptions in Proposition 4.2, we find

$$\left| \int_{\mathbb{T}^d} (u - m_c) \cdot \nabla W \star h \, dx \right| \lesssim \int_{\mathbb{T}^d} (h + 1) |u - m_c|^2 \, dx + \|h\|_{\dot{H}^{-1}}.$$

Since $\|h\|_{\dot{H}^{-1}} \leq \|h\|_{\dot{H}^{-\frac{d-\alpha}{2}}}$, for sufficiently small σ, we get

$$\mathcal{E} \approx \mathcal{E}^\sigma. \tag{4.7}$$

We are now in a position to provide the details of the proof for Proposition 4.2.

Proof of Proposition 4.2. It is obvious that $\mathcal{E}^\sigma$ satisfies

$$\frac{d}{dt}\mathcal{E}^\sigma + \mathcal{D}^\sigma = 0, \tag{4.8}$$

where $\mathcal{D}^\sigma = \mathcal{D}^\sigma(t)$ is given as

$$\mathcal{D}^\sigma := \mathcal{D} - \sigma \frac{d}{dt}\int_{\mathbb{T}^d} (u - m_c)\cdot \nabla W \star h\, dx.$$

Now we claim that

$$\mathcal{D}^\sigma(t) \geq c\mathcal{E}^\sigma(t)$$

for some positive constant c independent of t. First, we estimate

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^d} (u - m_c)\cdot \nabla W \star h\, dx \\
&\quad = -\int_{\mathbb{T}^d} \big(u\cdot\nabla u + u + \nabla\Lambda^{\alpha-d}h\big)\cdot \nabla W \star h\, dx - m_c' \int_{\mathbb{T}^d} \nabla W \star h\, dx \\
&\qquad + \int_{\mathbb{T}^d} (u - m_c)\cdot \nabla W \star (\partial_t h)\, dx \\
&\quad = -\int_{\mathbb{T}^d} (u\cdot\nabla u)\cdot \nabla W \star h\, dx - \int_{\mathbb{T}^d} (u - m_c)\cdot \nabla W \star h\, dx \\
&\qquad - \int_{\mathbb{T}^d} \nabla\Lambda^{\alpha-d}h\cdot \nabla W \star h\, dx - \int_{\mathbb{T}^d} (u - m_c)\cdot \nabla W \star (\nabla\cdot((h+1)u))\, dx \\
&\quad =: \sum_{i=1}^{4} \mathcal{J}_i.
\end{aligned}$$

For $\mathcal{J}_1$ we recall that for $a = (a_1, \dots, a_d) \in \mathbb{R}^d$ and $b = (b_1, \dots, b_d) \in \mathbb{R}^d$,

$$\nabla\cdot(a\otimes b) = \sum_{j=1}^{d} \partial_{x_j}(a_i b_j) = a(\nabla\cdot b) + (b\cdot\nabla)a.$$

This gives

$$\begin{aligned}
\mathcal{J}_1 &= -\int_{\mathbb{T}^d} (u\cdot\nabla(u - m_c))\cdot \nabla W \star h\, dx \\
&= -\int_{\mathbb{T}^d} \nabla\cdot((u - m_c)\otimes u)\cdot \nabla W \star h\, dx + \int_{\mathbb{T}^d} (u - m_c)(\nabla\cdot u)\cdot \nabla W \star h\, dx \\
&= \int_{\mathbb{T}^d} ((u - m_c)\otimes u) : \nabla^2 W \star h\, dx + \int_{\mathbb{T}^d} (u - m_c)(\nabla\cdot u)\cdot \nabla W \star h\, dx.
\end{aligned}$$

For $\mathcal{J}_3$ we use (4.6) to get

$$\mathcal{J}_3 = \int_{\mathbb{T}^d} (\Lambda^{\alpha-d}h)\Delta W \star h\, dx = -\int_{\mathbb{T}^d} h\Lambda^{\alpha-d}h\, dx.$$

For $\mathcal{J}_4$ we find

$$\begin{aligned}\mathcal{J}_4 = &-\int_{\mathbb{T}^d} (u-m_c)\cdot\nabla W \star \big(\nabla\cdot((h+1)(u-m_c))\big)\,dx\\ &-\int_{\mathbb{T}^d} (u-m_c)\cdot\nabla W \star \big(\nabla\cdot((h+1)m_c)\big)\,dx.\end{aligned}$$

Here we rewrite the second term on the right-hand side of the above as

$$\begin{aligned}&-\int_{\mathbb{T}^d} (u-m_c)\cdot\nabla W \star \big(\nabla\cdot((h+1)m_c)\big)\,dx\\ &\quad=\int_{\mathbb{T}^d} (\nabla\cdot(u-m_c))W \star \big(\nabla\cdot((h+1)m_c)\big)\,dx\\ &\quad=\iint_{\mathbb{T}^d\times\mathbb{T}^d} (\nabla\cdot(u-m_c))W(x-y)\nabla_y\cdot(h(y)m_c)\,dx\,dy\\ &\quad=\iint_{\mathbb{T}^d\times\mathbb{T}^d} (\nabla\cdot(u-m_c))m_c\cdot\nabla W(x-y)h(y)\,dx\,dy\\ &\quad=\int_{\mathbb{T}^d} m_c(\nabla\cdot(u-m_c))\cdot\nabla W\star h\,dx\\ &\quad=\int_{\mathbb{T}^d} \nabla\cdot(m_c\otimes(u-m_c))\cdot\nabla W\star h\,dx\\ &\quad=-\int_{\mathbb{T}^d} (m_c\otimes(u-m_c)):\nabla^2 W\star h\,dx\\ &\quad=-\int_{\mathbb{T}^d} ((u-m_c)\otimes m_c):\nabla^2 W\star h\,dx,\end{aligned}$$

where we used the symmetry of $\nabla^2 W\star h$ to get the last equality. Thus, $\mathcal{J}_4$ can be estimated as

$$\begin{aligned}\mathcal{J}_4 = &-\int_{\mathbb{T}^d} (u-m_c)\cdot\nabla W \star \big(\nabla\cdot((h+1)(u-m_c))\big)\,dx\\ &-\int_{\mathbb{T}^d} ((u-m_c)\otimes m_c):\nabla^2 W\star h\,dx.\end{aligned}$$

Hence, we combine the estimates for the J_i to yield

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{T}^d} (u-m_c)\cdot\nabla W\star h\,dx\\ &\quad=\int_{\mathbb{T}^d} ((u-m_c)\otimes(u-m_c)):\nabla^2 W\star h\,dx\\ &\qquad+\int_{\mathbb{T}^d} (\nabla\cdot u-1)(u-m_c)\cdot\nabla W\star h\,dx\\ &\qquad-\int_{\mathbb{T}^d} h\Lambda^{\alpha-d}h\,dx-\int_{\mathbb{T}^d} (u-m_c)\cdot\nabla W\star\big(\nabla\cdot((h+1)(u-m_c))\big)\,dx.\end{aligned}$$

Therefore, we choose sufficiently small $\sigma > 0$ to obtain

$$\begin{aligned}
\mathcal{D}^\sigma &= \mathcal{D} - \sigma \int_{\mathbb{T}^d} ((u-m_c)\otimes(u-m_c)) : \nabla^2 W \star h \, dx \\
&\quad - \sigma \int_{\mathbb{T}^d} (\nabla\cdot u - 1)(u-m_c)\cdot \nabla W \star h \, dx \\
&\quad + \sigma \int_{\mathbb{T}^d} h\Lambda^{\alpha-d} h \, dx + \sigma \int_{\mathbb{T}^d} (u-m_c)\cdot\nabla W \star \big(\nabla\cdot((h+1)(u-m_c))\big)\, dx \\
&\geq \mathcal{D} - \sigma \|\nabla^2 W \star h\|_{L^\infty} \|(u-m_c)\|_{L^2}^2 \\
&\quad - \sigma(1+\|\nabla\cdot u\|_{L^\infty})\|u-m_c\|_{L^2}\|\nabla W \star h\|_{L^2} \\
&\quad + \sigma \|h\|^2_{\dot{H}^{-\frac{d-\alpha}{2}}} - \sigma\|u-m_c\|_{L^2}\big\|\nabla W \star \big(\nabla\cdot((h+1)(u-m_c))\big)\big\|_{L^2} \\
&\geq \mathcal{D} - C\sigma\|\nabla W\|_{L^1}\|\nabla h\|_{L^\infty}\left(\int_{\mathbb{T}^d}(h+1)|u-m_c|^2\, dx\right) \\
&\quad - C\sigma\left(\int_{\mathbb{T}^d}(h+1)|u-m_c|^2\, dx\right)^{1/2}\|h\|_{\dot{H}^{-1}} \\
&\quad + \sigma\|h\|^2_{\dot{H}^{-\frac{d-\alpha}{2}}} - C\sigma\|u-m_c\|_{L^2}\|\nabla\cdot((h+1)(u-m_c))\|_{\dot{H}^{-1}} \\
&\geq (1 - C(\sigma^{1/2}+\sigma))\mathcal{D} + (\sigma - C\sigma^{3/2})\|h\|^2_{\dot{H}^{-\frac{d-\alpha}{2}}} \\
&\geq c\mathcal{E}^\sigma,
\end{aligned}$$

where $c = c(h_{\min}, \|\nabla W\|_{L^1}, \|h\|_{W^{1,\infty}}, \|\nabla\cdot u\|_{L^\infty})$ is a positive constant independent of t. Thus, the claim is proved, and we have from (4.8) that

$$\frac{d}{dt}\mathcal{E}^\sigma + c\mathcal{E}^\sigma \leq 0.$$

Applying Grönwall's lemma to the above gives exponential decay of $\mathcal{E}^\sigma$, and this, combined with (4.5) and (4.7), concludes the desired result. ∎

4.2.1. Proof of Theorem 1.2: Periodic case. We first notice that the average of momentum satisfies

$$m_c'(t) = -m_c(t), \quad \text{i.e. } m_c(t) = m_c(0)e^{-t},$$

due to the symmetry of the operator $\Lambda^{\alpha-d}$. This together with Proposition 4.2 gives

$$\int_{\mathbb{T}^d} |u|^2\, dx \leq 2\int_{\mathbb{T}^d}|u-m_c|^2\, dx + 2|m_c|^2 \leq C_1 e^{-C_2 t}$$

for some $C_i > 0$, $i = 1, 2$, independent of t. On the other hand, it follows from (3.5) and the equivalence relations (3.6)–(3.7) that

$$\frac{d}{dt}\bar{y}^m + \lambda_5 \bar{y}^m \leq C_1 e^{-C_2 t}$$

for some $\lambda_5 > 0$ which is independent of t, where $\bar{\mathcal{Y}}^m$ appears in (4.4). Applying Grönwall's lemma to the above yields exponential decay of $\bar{\mathcal{Y}}^m$ towards zero as $t \to \infty$. Note that the L^2-norm of h is not included in $\bar{\mathcal{Y}}^m$. For the decay estimate of $\|h(\cdot,t)\|_{L^2}$, we use Lemma 4.1 to get

$$\|h\|_{L^2} \le \|h\|^{\theta}_{\dot{H}^{-\frac{d-\alpha}{2}}} \|\nabla h\|^{1-\theta}_{L^2} \quad \text{with } \theta = \frac{1}{1+\frac{d-\alpha}{2}}.$$

Since the right-hand side of the above converges to zero exponentially fast, we also have the same exponential decay rate of convergence of $\|h\|_{L^2}$. This completes the proof.

A. Proof of Lemma 3.5

In this appendix we provide the details of the proof of Lemma 3.5.

(i) We apply Proposition 2.1 to obtain

$$\begin{aligned}
&\Big\|\nabla\Big(\frac{\nabla h}{\rho^2}\cdot\partial^k(\rho u)\Big)\Big\|_{L^2} \\
&\quad\le \Big\|\nabla\Big(\frac{\nabla h}{\rho^2}\cdot(\partial^k(hu)-h\partial^k u)\Big)\Big\|_{L^2} + \Big\|\nabla\Big(\frac{\nabla h}{\rho}\cdot\partial^k u\Big)\Big\|_{L^2} \\
&\quad\le C\Big(\Big\|\frac{\nabla h}{\rho^2}\Big\|_{W^{1,\infty}} + \Big\|\frac{\nabla h}{\rho}\Big\|_{W^{1,\infty}}\Big)\big(\|\partial^k(hu)-h\partial^k u\|_{H^1} + \|\partial^k u\|_{H^1}\big) \\
&\quad\le C\Big(\frac{\|\nabla^2 h\|_{L^\infty}}{(1-\|h\|_{L^\infty})^2} + \frac{\|\nabla h\|^2_{L^\infty}}{(1-\|h\|_{L^\infty})^3}\Big) \\
&\qquad\times\big(\|\nabla h\|_{W^{1,\infty}}\|\partial^{k-1}u\|_{H^1} + \|\partial^k h\|_{H^1}\|u\|_{L^\infty} + \|\partial^k u\|_{H^1}\big) \\
&\quad\le C(1+\|\nabla h\|_{H^{m-1}})^2\|\nabla h\|_{H^{m-1}}\|u\|_{H^{m+\frac{d-\alpha}{2}}},
\end{aligned}$$

where $C = C(m,k,d,\alpha)$ is a positive constant independent of T.

(ii) Note that the left-hand side equals zero when $k = 1$. For $k \ge 2$, we again use Proposition 2.1 to estimate

$$\begin{aligned}
&\Big\|\nabla\Big[\nabla\cdot\Big(\frac{1}{\rho}\big(\partial^k(\rho u)-(\partial^k\rho)u-\rho(\partial^k u)\big)\Big)\Big]\Big\|_{L^2} \\
&\quad\le C\Big\|\nabla^2\Big(\frac{1}{\rho}\Big)\Big\|_{L^\infty}\big\|\partial^k(\rho u)-(\partial^k\rho)u-\rho(\partial^k u)\big\|_{L^2} \\
&\qquad + C\Big\|\nabla\Big(\frac{1}{\rho}\Big)\Big\|_{L^\infty}\big\|\nabla\big(\partial^k(\rho u)-(\partial^k\rho)u-\rho(\partial^k u)\big)\big\|_{L^2} \\
&\qquad + \frac{1}{1-\|h\|_{L^\infty}}\big\|\nabla^2\big(\partial^k(\rho u)-(\partial^k\rho)u-\rho(\partial^k u)\big)\big\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&\le C\Big(\frac{\|\nabla^2 h\|_{L^\infty}}{(1-\|h\|_{L^\infty})^2}+\frac{\|\nabla h\|_{L^\infty}^2}{(1-\|h\|_{L^\infty})^3}\Big)\big(\|\nabla h\|_{L^\infty}\|\partial^{k-1}u\|_{L^2}+\|\partial^{k-1}h\|_{L^2}\|\nabla u\|_{L^\infty}\big)\\
&\quad + C\frac{\|\nabla h\|_{L^\infty}}{(1-\|h\|_{L^\infty})^2}\big[\big(\|\nabla^2 h\|_{L^\infty}\|\partial^{k-1}u\|_{L^2}+\|\nabla\partial^{k-1}h\|_{L^2}\|\nabla u\|_{L^\infty}\big)\\
&\qquad\qquad + \big(\|\nabla h\|_{L^\infty}\|\partial^{k-1}\nabla u\|_{L^2}+\|\partial^{k-1}h\|_{L^2}\|\nabla^2 u\|_{L^\infty}\big)\big]\\
&\quad + \frac{C}{1-\|h\|_{L^\infty}}\big[\big(\|\nabla^3 h\|_{L^\infty}\|\partial^{k-1}u\|_{L^2}+\|\partial^{k-1}\nabla^2 h\|_{L^2}\|\nabla u\|_{L^\infty}\big)\\
&\qquad\qquad + \big(\|\nabla^2 h\|_{L^\infty}\|\partial^{k-1}\nabla u\|_{L^2}+\|\partial^{k-1}\nabla h\|_{L^2}\|\nabla^2 u\|_{L^\infty}\big)\\
&\qquad\qquad + \big(\|\nabla h\|_{L^\infty}\|\partial^{k-1}\nabla^2 u\|_{L^2}+\|\partial^{k-1}h\|_{L^2}\|\nabla^3 u\|_{L^\infty}\big)\big]\\
&\le C(1+\|\nabla h\|_{W^{1,\infty}})^2\|\nabla h\|_{H^{m-1}}\|u\|_{H^{m+\frac{d-\alpha}{2}}}\\
&\le C(1+\|\nabla h\|_{H^{m-1}})^2\|\nabla h\|_{H^{m-1}}\|u\|_{H^{m+\frac{d-\alpha}{2}}}.
\end{aligned}
$$

(iii) Note that

$$
\partial\Big(\frac{1}{\rho^2}(\partial^k h)\nabla h\Big)=\frac{1}{\rho^2}\Big(-\frac{\partial h}{2\rho}(\partial^k h)\nabla h+\partial^{k+1}h\nabla h+(\partial^k h)\nabla\partial h\Big),
$$

and thus taking the L^2 norm on both sides of the above and using Proposition 2.1 gives

$$
\begin{aligned}
&\Big\|\partial\Big(\frac{1}{\rho^2}(\partial^k h)\nabla h\Big)\Big\|_{L^2}\\
&\quad\le \frac{C}{(1-\|h\|_{L^\infty})^2}\Big(\frac{\|\nabla h\|_{L^\infty}^2}{2(1-\|h\|_{L^\infty})}\|\partial^k h\|_{L^2}+\|\partial^{k+1}h\|_{L^2}\|\nabla h\|_{L^\infty}\\
&\qquad\qquad + \|\partial^k h\|_{L^2}\|\nabla^2 h\|_{L^\infty}\Big)\\
&\quad\le C(1+\|\nabla h\|_{H^{m-1}})\|\nabla h\|_{H^{m-1}}^2.
\end{aligned}
$$

(iv) It follows from the equation for u in (1.3) that

$$
\begin{aligned}
-\Lambda^{d-\alpha}\partial^{k-1}\partial_t u &= \Lambda^{d-\alpha}\partial^{k-1}(u\cdot\nabla u+u+\Lambda^{\alpha-d}\nabla h)\\
&= u\cdot\nabla\Lambda^{d-\alpha}\partial^{k-1}u+[\Lambda^{d-\alpha},u\cdot\nabla]\partial^{k-1}u\\
&\quad + \Lambda^{d-\alpha}[\partial^{k-1},u\cdot\nabla]u+\Lambda^{d-\alpha}\partial^{k-1}u+\partial^{k-1}\nabla h.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\|\Lambda^{d-\alpha}\partial^{k-1}\partial_t u\|_{L^2}\\
&\quad\le C\Big(\|u\|_{L^\infty}\|\nabla\Lambda^{d-\alpha}\partial^{k-1}u\|_{L^2}+\|u\|_{H^{\frac{d}{2}+1+(d-\alpha)+\varepsilon}}\|\partial^{k-1}u\|_{H^{d-\alpha}}\\
&\qquad + \sum_{j=1}^{k-1}\|\Lambda^{d-\alpha}(\partial^j u\cdot\nabla\partial^{k-1-j}u)\|_{L^2}+\|\Lambda^{d-\alpha}\partial^{k-1}u\|_{L^2}+\|\partial^{k-1}\nabla h\|_{L^2}\Big)\\
&\quad\le C\big(\|u\|_{H^{m+\frac{d-\alpha}{2}}}^2+\|u\|_{H^{m+\frac{d-\alpha}{2}}}+\|\nabla h\|_{H^{m-1}}\big),
\end{aligned}
$$

where we used Lemma 2.1 with $\varepsilon > 0$ satisfying $(d-\alpha)/2 + \varepsilon < 1$, together with Proposition 2.1, to get

$$\begin{aligned}\|\Lambda^{d-\alpha}(\partial^j u \cdot \nabla \partial^{k-1-j} u)\|_{L^2} &\lesssim_{\alpha,d} \|\partial^j u \cdot \nabla \partial^{k-1-j} u\|_{H^2} \\ &\lesssim_{\alpha,d,k,j} \|\partial^j u\|_{H^{m-j}} \|\partial^{k-j} u\|_{H^{m-(k-j)}} \\ &\lesssim_{\alpha,d,k,j} \|u\|_{H^m}^2\end{aligned}$$

for $1 \le j \le k-1$.

(v) By adding and subtracting, we obtain

$$\begin{aligned}&\Lambda^{d-\alpha}\partial^{k-1}(\nabla u : (\nabla u)^T) \\ &\quad = \sum_{i=1}^d \Lambda^{d-\alpha}\partial^{k-1}(\partial_{x_i} u \cdot \nabla u_i) \\ &\quad = \sum_{i=1}^d \Lambda^{d-\alpha}\big((\partial^{k-1}\partial_{x_i} u) \cdot \nabla u_i + \partial_{x_i} u \cdot \nabla \partial^{k-1} u_i\big) \\ &\qquad + \sum_{i=1}^d \Lambda^{d-\alpha}\big(\partial^{k-1}(\partial_{x_i} u \cdot \nabla u_i) - \partial_{x_i} u \cdot \nabla \partial^{k-1} u_i - \partial_{x_i}(\partial^{k-1} u) \cdot \nabla u_i\big) \\ &\quad = 2\sum_{i=1}^d \partial_{x_i} u \cdot \nabla \Lambda^{d-\alpha}\partial^{k-1} u_i + 2[\Lambda^{d-\alpha}, \partial_{x_i} u \cdot \nabla]\partial^{k-1} u_i \\ &\qquad + \sum_{i=1}^d \Lambda^{d-\alpha}\big(\partial^{k-1}(\partial_{x_i} u \cdot \nabla u_i) - \partial_{x_i} u \cdot \nabla \partial^{k-1} u_i - \partial_{x_i}(\partial^{k-1} u) \cdot \nabla u_i\big).\end{aligned}$$

We then apply Lemma 2.1 and Proposition 2.1 to deduce

$$\begin{aligned}&\|\Lambda^{d-\alpha}\partial^{k-1}(\nabla u : (\nabla u)^T)\|_{L^2} \\ &\quad \le C\Bigg(\|\nabla u\|_{L^\infty}\|u\|_{H^{k+(d-\alpha)}} + \|\nabla u\|_{H^{\frac{d}{2}+1+(d-\alpha)+\varepsilon}}\|\partial^{k-1} u\|_{H^{d-\alpha}} \\ &\qquad\qquad + \sum_{\ell=1}^{k-2}\|\Lambda^{d-\alpha}(\partial^{\ell+1} u \cdot \nabla \partial^{k-1-\ell} u)\|_{L^2}\Bigg) \\ &\quad \le C\|u\|_{H^{m+\frac{d-\alpha}{2}}}^2,\end{aligned}$$

where ε satisfies $(d-\alpha)/2 + \varepsilon < 1$.

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