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Constrained control of gene-flow models

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Abstract. In ecology and population dynamics, gene flow refers to the transfer of a trait (e.g. genetic material) from one population to another. This phenomenon is of great relevance in studying the spread of diseases or the evolution of social features, such as languages. From the mathematical point of view, gene flow is modeled using bistable reaction–diffusion equations. The unknown is the proportion p of the population that possesses a certain trait, within an overall population N. In such models, gene flow is taken into account by assuming that the population density N depends either on p (if the trait corresponds to fitter individuals) or on the location x (if some zones in the domain can carry more individuals). Recent applications stemming from mosquito-borne-disease control problems or from the study of bilingualism have called for the investigation of the controllability properties of these models. At the mathematical level, this corresponds to boundary control problems and, since we are working with proportions, the control u has to satisfy the constraint $0 \leqslant u \leqslant 1$. In this article, we provide a thorough analysis of the influence of the gene-flow effect on boundary controllability properties. We prove that, when the population density N only depends on the trait proportion p, the geometry of the domain is the only criterion that has to be considered. We then tackle the case of population densities N varying in x. We first prove that, when N varies slowly in x and when the domain is narrow enough, controllability always holds. This result is proved using a robust domain perturbation method. We then consider the case of sharp fluctuations in N: we first give examples that prove that controllability may fail. Conversely, we give examples of heterogeneities N such that controllability will always be guaranteed: in other words the controllability properties of the equation are very strongly influenced by the variations of N. All negative controllability results are proved by showing the existence of nontrivial stationary states, which act as barriers. The existence of such solutions and the methods of proof are of independent interest. Our article is completed by several numerical experiments that confirm our analysis.

1. Introduction

Motivations. In ecology and population dynamics, gene flow refers to the transfer of a trait (e.g. genetic material) from one population to another. This phenomenon strongly depends on the structure of the population density [32] as well as on the proportion of individuals that possess this trait inside the population. Typically, the populations involved in gene-flow phenomena are spatially separated, and the gene-flow effect occurs through

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spatial migrations. Given that it is a key factor in the evolution and differentiation of species, gene flow has received a lot of attention from the biology community [6, 8, 13, 19, 31]. One should note that this effect has also been observed in the evolution of plants [20, 22]. This paper is devoted to the controllability of biological systems involving this effect: Is it possible, acting only on the boundary of the domain, to control the proportion of the trait within the population?

These boundary control problems arise naturally from population dynamics models and have several interpretations. For instance, one might consider the following situation: Given a population of mosquitoes, a proportion of which is carrying a disease, is it possible, acting only on the proportion of sick mosquitoes on the boundary, to drive this population to a state where only sane mosquitoes remain? From the application point of view, some mosquitoes are immune to diseases such as malaria [29], and it is therefore interesting to study the evolution of such proportions over space and time. Governments have recently tackled the issue of controlling diseases transmitted by mosquitoes by releasing genetically modified mosquitoes [3], and such questions have drawn the attention of the mathematical community in recent years [1]. Another example is that of linguistic dynamics: Considering a population of individuals, a part of which is monolingual (speaking only the dominant language), the other part of which is bilingual (speaking the dominant and a minority language), is it possible, acting only on the proportion of bilingual speakers on the boundary of the domain, to drive the population to a state where there remains a nonzero proportion of bilingual speakers, thus ensuring the survival of the minority language? Such models are proposed, for instance, in [35]. In these works, as well as in a variety of other interpretations [2, 7, 12, 23, 33, 35] of bistable models, the gene-flow effect has been acknowledged as crucial in the underlying phenomenon: it states that, when we are interested in the proportion p of a subgroup of a population density N, N may depend on either p (if for instance the subgroup is fitter) or on the space variable x (if some zones in the domain are more favorable and can carry more individuals). The goal of this article is to underline the complexity of the interaction between this gene flow and the controllability properties of the system.

A motivating example: the spatially heterogeneous case. Let us give an example. We consider a population density $N = N(t, x)$, and we are interested in the dynamics of a proportion of the population, which will be denoted by $p = p(t, x)$. The classical gene-flow hierarchical system reads

$$\begin{cases} \dfrac{\partial N}{\partial t} - \Delta N = g(N, x) & \text{in } \mathbb{R}_+ \times \Omega, \\ \dfrac{\partial p}{\partial t} - \Delta p - 2\langle \nabla \ln N, \nabla p \rangle = f(p) & \text{in } \mathbb{R}_+ \times \Omega, \\ \dfrac{\partial N}{\partial \nu} = 0, \quad \dfrac{\partial p}{\partial \nu} = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ N(0, x) > 0, \quad 0 \leqslant p(0, x) \leqslant 1 & \text{in } \Omega. \end{cases} \tag{1}$$

Typically, we can choose a nonlinearity g of monostable type , such as $g(N,x) = N(\kappa(x) - N)$. In that case, $\kappa(x) > 0$ models the resource distribution available for the population inside the domain. Monostable equations have been studied a lot since the seminal [17]. The nonlinearity f is on the other hand assumed to be bistable: a typical example is $f(p) = p(p-\theta)(1-p)$ for some $\theta \in (0;1)$; see Figure 1.

Bistable reaction–diffusion equations are well suited to describe the evolution of a subgroup of a population and are characterized by the so-called *Allee effect*: there exists a threshold for the proportion of this subgroup such that, in the absence of spatial diffusion, above this threshold, this subgroup will invade the whole domain (and drive the other subgroup to extinction) while, under this threshold, it will go extinct.

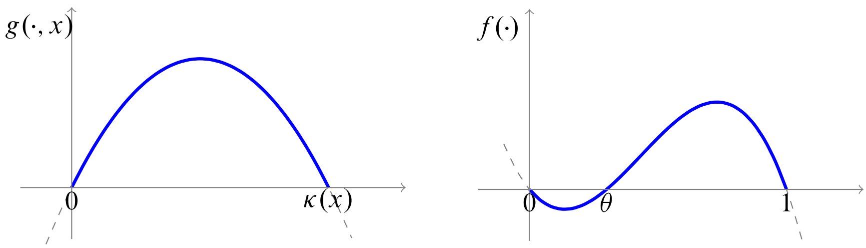

Figure 1. Graph of a typical monostable nonlinearity $g(N,x) = N(\kappa(x) - N)$ (left) and typical bistable nonlinearity $f(p) = p(p-\theta)(1-p)$ (right).

In that context, the goal is to drive the proportion p to a spatially homogeneous equilibrium. Since it is often the case that we cannot control the equation inside the domain (for instance when trying to control a mosquito population), we have to resort to boundary controls. As we will see, the shape of N will have a drastic influence on such controllability properties.

In this article we assume that the first component has reached a steady state, and that the population distribution N is stationary; in other words, we have $N = N(x)$. This reduces the gene-flow system to a scalar equation, in which we place a boundary control action:

$$\begin{cases} \dfrac{\partial p}{\partial t} - \Delta p - 2\langle \nabla \ln N(x), \nabla p\rangle = f(p) & \text{in } \mathbb{R}_+ \times \Omega, \\ p(t,\cdot) = u(t,\cdot) & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ 0 \leqslant p(0,x) \leqslant 1 & \text{in } \Omega. \end{cases}$$

The state p is a proportion, hence our boundary control action has to satisfy the bounds

$$0 \leqslant u(x,t) \leqslant 1.$$

Such bounds on the control are known to lead to fundamental obstructions to the controllability as noticed in [27, 30].

Since our results fit into a growing body of literature, let us recall several results that hold in the absence of the gene-flow effects before stating our contributions.

1.1. Known results regarding the constrained controllability of bistable equations

In [27, 30], the controllability to 0, θ or 1 of the equation

$$\frac{\partial p}{\partial t} - \Delta p = f(p)$$

with a constraint on the boundary control is carried out using the staircase method [10, 26]. The emergence of nontrivial steady states (that is, steady states that are not identically equal to 0, 1 or θ) is the main cause of lack of controllability for large domains, while controllability holds by constructing paths of steady states. In [34], the equation

$$\frac{\partial p}{\partial t} - \Delta p = p(p - \theta(t))(1 - p)$$

is considered, but this time, it is the Allee parameter $\theta = \theta(t)$ that is the control parameter and the target is a traveling wave solution. In [1], an optimal control problem for the equation without diffusion

$$\frac{\partial p}{\partial t} = f(p) + u(t),$$

and with an interior control u (rather than a boundary one), is considered. We underline that, in their study, u only depends on the time, and not on the space variable.

1.2. Main contributions of the paper

Our main contributions can be informally stated as follows:

- **Slowly varying population density.** In that context, we assume that the population density N varies very little from one point to the other. In other words, we assume that there exists a constant N_0, a function $n = n(x)$ and a small $\varepsilon > 0$ such that $N = N(x) = N_0 + \varepsilon n(x)$. If we think about the motivating example (1), this amounts to requiring that the resources distribution κ is itself slowly varying $\kappa = \kappa_0 + \varepsilon \eta(x)$ for some constant κ_0 and some function η. For this reason, we will also refer to this model as *slowly varying spatial heterogeneity*. In that context, the heterogeneity does not qualitatively affect the result from the homogeneous setting; see Theorem 1. The proof is based on a very fine domain perturbation method.
- **Strongly varying population density.** In contrast to Theorem 1, we consider the case when N has rapid variations, which in turn may be interpreted as the effect of a *strongly varying spatial heterogeneity*. In this case, the situation changes dramatically:
  - Rapid variations of population inside the domain lead to lack of controllability due to the emergence of nontrivial steady states which act as barriers. As an example of such a phenomenon, we study the case $\Omega = \mathbb{B}(0; R)$, $N_\sigma(x) = e^{-\frac{\|x\|^2}{\sigma}}$, and we show that, whenever $\sigma > 0$ is small enough, there exist nontrivial stationary solutions to the state equation on p, with boundary values either 0 or 1. In Theorem 2, we give explicit assumptions on the drift to ensure the appearance of such nontrivial steady states. This means, in terms of applications, that if there is a piece of

the domain with high variation in the concentration of individuals, the control will fail.

- On the other side, if the population shows a rapid decay towards the interior, then, surprisingly, there exists a critical threshold in σ for which, independently of the size of the domain, there are no nontrivial solutions acting as barriers, so that controllability is achievable. In Theorem 3 we show that this is the case for

$$N_\sigma(x) = e^{\frac{\|x\|^2}{\sigma}},$$

and prove this result for any σ small enough. This is in sharp contrast with the homogeneous setting (homogeneous being understood in the sense that no drift is present) in which there was a critical size of the domain for which there was always one barrier [27,30]. This result is proved using spectral analysis.

- In Theorem 4 we derive explicit decay rates on the spatial heterogeneity N to ensure controllability, thus obtaining a result analogous to the results set in the homogeneous setting [27,30].

• **Infection-dependent limit.** So far, we have only mentioned the spatially heterogeneous case $N = N(x)$, but another context which is highly relevant for applications is that of infection-dependent models. This model, which also accounts for the gene-flow effect, is for instance obtained in [25, Section 6], in which the authors reduce a system of 2×2 coupled reaction–diffusion equations to a scalar one and prove convergence, under some assumptions, to either spatially heterogeneous models or to infection-dependent models, which are written

$$\frac{\partial p}{\partial t} - \mu\Delta p + 2|\nabla p|^2 \frac{h'(p)}{h(p)} = p(1-p)(p-\theta)$$

for some $\theta \in (0;1)$ and some function h. This equation is most notably studied in [25,33]. This amounts to requiring that the population density N depends on p: $N = N(p)$. In that case we show that the controllability results are exactly analogous to those obtained in the homogeneous setting; see Proposition 1.

Structure of the paper. The structure of the paper is the following:

• In Section 2 we present the mathematical setting and the results.

• Sections 3, 4, 5, 6 and 7 are devoted to the proofs of the theorems in order of presentation.

• Finally, in Section 8 we draw some concluding remarks and state some open problems.

2. Setting and main results

The equations and the control systems. Let us first write down our systems. We say that $f : \mathbb{R} \to \mathbb{R}$ is a *bistable nonlinearity* if

(1) f is $\mathcal{C}^\infty$ on $[0, 1]$;

(2) there exists $\theta \in (0; 1)$ such that 0, θ and 1 are the only three roots of f in $[0, 1]$, where θ is called the Allee parameter;

(3) $f'(0), f'(1) < 0$ and $f'(\theta) > 0$;

(4) without loss of generality, we assume that $\int_0^1 f > 0$. In the typical example $f(p) = p(p-\theta)(1-p)$, this amounts to requiring that θ satisfies $\theta < \frac{1}{2}$.

We gave an example of such a bistable nonlinearity in Figure 1.

We write our two models, the spatially heterogeneous one and the infection-dependent one, in a synthetic way: we consider, in general, a population density of the form $N = N(x, p)$. Infection-dependent models correspond to $N = N(p)$ and spatially heterogeneous models correspond to $N = N(x)$. With a bistable nonlinearity f and such a function N, in a domain $\Omega \subset \mathbb{R}^d$, in its most general form the equation we consider is written

$$\frac{\partial p}{\partial t} - \Delta p - 2\big\langle \nabla_x \big(\ln(N(x, p))\big), \nabla p \big\rangle = f(p). \tag{2}$$

Of particular relevance are the spatially homogeneous steady states of this equation: $p \equiv 0$, $p \equiv \theta$ and $p \equiv 1$. *Our objective in this article is to investigate whether or not it is possible to control any initial datum to these spatially homogeneous steady states.*

Let us formalize this control problem. Given an initial datum $p_0 \in L^2(\Omega)$ such that

$$0 \leqslant p_0 \leqslant 1,$$

we consider the control system

$$\begin{cases} \dfrac{\partial p}{\partial t} - \Delta p - 2\langle \nabla \ln(N), \nabla p\rangle = f(p) & \text{in } \mathbb{R}_+ \times \Omega, \\ p = u(t, x) & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ p(t = 0, \cdot) = p_0, \end{cases} \tag{3}$$

where, for every $t \geqslant 0$, $x \in \partial\Omega$,

$$u(t, x) \in [0, 1] \tag{4}$$

is the control function, and (4) is a natural constraint since we recall that p stands for a proportion of the population. Our goal is to answer the following question: *Given any initial datum* $0 \leqslant p_0 \leqslant 1$*, is it possible to drive* p_0 *to* 0*,* θ *or* 1 *in (in)finite time with a control* u *satisfying* (4)*?* In other words, can we drive any initial datum to one of the spatially homogeneous steady states of the equation? If one thinks about infected mosquitoes, driving any initial population to 0 is relevant for controlling the disease while, if one thinks about mono- or bilingual speakers, driving the initial datum to the intermediate steady state θ ensures the survival of the minority language.

Let us denote the steady states as

$$\forall a \in \{0, \theta, 1\}, \quad z_a \equiv a.$$

By controllability we mean the following, where $a \in \{0, \theta, 1\}$:

- *Controllability in finite time.* We say that p_0 *is controllable to a in finite time* if there exists a finite time $T < \infty$ such that there exists a control u satisfying the constraints (4) and such that the solution $p = p(t, x)$ of (3) satisfies

$$p(T, \cdot) = z_a \text{in } \Omega.$$

- *Controllability in infinite time.* We say that p_0 *is controllable to z_a in infinite time* if there exists a control u satisfying the constraints (4) such that the solution $p = p(t, x)$ of (3) satisfies

$$p(t, \cdot) \xrightarrow[t \to \infty]{\mathcal{C}^0(\Omega)} z_a.$$

Remark 1. Note that, in the definition of controllability in finite time, we do not ask that the controllability time be small; it cannot anyway be arbitrarily small because of the constraint $0 \leqslant u \leqslant 1$ [26]. Such constraints can indeed lead to lack of controllability in a fixed time horizon. The question of the minimal controllability time for such bistable equations is, as far as the authors know, completely open at this point.

Definition 1. We say that (3) is controllable to a in (in)finite time if it is controllable to z_a in (in)finite time for any initial datum $0 \leqslant p_0 \leqslant 1$.

We consider two cases for the flux $N = N(x, p) > 0$ which Are discussed in [2]:

- *The spatially heterogeneous model.* In this case, $N = N(x, p)$ is of the form

$$N = N(x) > 0. \tag{H_1}$$

- *The infection-dependent model.* In this case, the function $N = N(x, p)$ assumes the form

$$N(x, p) = N(p) > 0. \tag{H_2}$$

2.1. Statement of the main controllability results

2.1.1. A brief remark on the statement of the theorems. We are going to present controllability and noncontrollability results for gene-flow models and spatially heterogeneous ones. Regarding obstructions to controllability, the main obstacles are the existence of nontrivial steady states, namely solutions to

$$-\Delta p - 2\left\langle \frac{\nabla N}{N}, \nabla p \right\rangle = f(p) \quad \text{in } \Omega$$

associated with the boundary conditions $p = 0$ or $p = 1$ on $\partial\Omega$. However, given that the existence of nontrivial solutions for the Dirichlet boundary condition $p = 0$ is obtained through sub- and supersolution methods, the natural quantity appearing is the inradius of the domain.

Definition 2 (Inradius of a domain). Let $\Omega \subset \mathbb{R}^N$ be bounded:

$$\rho_\Omega = \sup\{r > 0, \exists\, x \in \Omega, \mathbb{B}(x,r) \subset \Omega\},$$

The nonexistence of nontrivial solutions is usually done through the study of the first Laplace–Dirichlet eigenvalue

$$\lambda_1^D(\Omega) := \inf_{\substack{p \in W_0^{1,2}(\Omega) \\ p \neq 0}} \frac{\int_\Omega |\nabla p|^2}{\int_\Omega p^2},$$

which explains why both quantities ρ_Ω and $\lambda_1^D(\Omega)$ appear in our first results. Using Hayman-type inequalities, see [9], we could rewrite $\lambda_1^D(\Omega)$ in terms of the inradius when the set Ω is convex. Indeed, it is proved in [9, Proposition 7.75] that, when Ω is a convex set with $\rho_\Omega < \infty$, then

$$\frac{1}{c\rho_\Omega^2} \leqslant \lambda_1^D(\Omega) \leqslant \frac{C}{\rho_\Omega^2},$$

so that the theorems can be recast in terms of inradius only in the case of convex domains.

2.1.2. Spatially heterogeneous models. In this case we work under assumption (H_1), i.e. with $N = N(x) > 0$ in Ω. As explained in the introduction, we will treat two cases: slowly varying heterogeneities and rapidly varying ones.

Slowly varying heterogeneity. Let us first study the case of a slowly varying total population size: we consider an environment with small spatial changes in the total population size. This amounts to requiring that

$$\left|\frac{\nabla N}{N}\right| \ll 1,$$

where N satisfies (H_1). We can then formally write

$$N \approx N_0 + \frac{\varepsilon}{2} n(x),$$

where N_0 is a constant.[1] We consider, for a function $n \in \mathcal{C}^1(\mathbb{R}^d;\mathbb{R})$ and a parameter $\varepsilon > 0$, the control system

$$\begin{cases} \dfrac{\partial p}{\partial t} - \Delta p - \varepsilon \langle \nabla n, \nabla p \rangle = f(p) & \text{in } \mathbb{R}_+ \times \Omega, \\ p = u(t,x) & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ 0 \leqslant u \leqslant 1, \\ p(t=0,\cdot) = p_0, \;\; 0 \leqslant p_0 \leqslant 1, \end{cases} \tag{5}$$

[1] We can assume, without loss of generality, that $N_0 = 1$. Indeed, equation (3) is invariant under the scaling $N \mapsto \lambda N$ where $\lambda \in \mathbb{R}_+^*$.

For simplicity, we assume that n is defined on $\mathbb{R}^d$ rather than on Ω. Since we have already assumed that N is $\mathcal{C}^1$, this amounts to requiring that n can be extended in a $\mathcal{C}^1$ function outside Ω, which once again would follow from regularity assumptions on Ω.

Theorem 1. *Let, $\Omega \subset \mathbb{R}^d$ be a $\mathcal{C}^2$ domain and let $n \in \mathcal{C}^1(\mathbb{R}^d)$. Then we have the following:*

(1) Lack of controllability for large inradii. *There exists $\rho^* = \rho^*(n, f) > 0$ such that if $\rho_\Omega > \rho^*$, then (5) is not controllable to 0 in (in)finite time in the sense of Definition 1: there exists an initial datum p_0 such that, for any control u satisfying the constraints (4), the solution p of (5) does not converge to 0 as $t \to \infty$.*

(2) Controllability for large Dirichlet eigenvalue and small spatial variations. *If $\lambda_1^D(\Omega) > \|f'\|_{L^\infty}$, there exists $\varepsilon_* = \varepsilon_*(n, f, \Omega)$ such that, when $\varepsilon \leqslant \varepsilon_*$, equation (5) is controllable to 0 and 1 in infinite time f and to θ in finite time in the sense of Definition 1.*

We note that controllability to 0 or 1 cannot hold in finite time, as it would violate the comparison principle; see [30]. To prove this theorem, we have to very finely adapt, using perturbative arguments, the staircase method of [10].

(Lack of) controllability for rapidly varying total population size: (un)blocking phenomena. The previous result, however general, is proved using a very implicit method that does not enable us to give explicit bounds on the perturbation size ε.

As mentioned, the lack of controllability occurs when barriers appear. For instance, if a nontrivial solution to

$$\begin{cases} -\Delta\varphi_0 - 2\left\langle \dfrac{\nabla N}{N}, \nabla\varphi_0 \right\rangle = f(\varphi_0) & \text{in } \Omega, \\ \varphi_0 = 0 & \text{on } \partial\Omega, \end{cases}$$

exists, then it must reach its maximum above θ (this follows from the optimality conditions for maximizers of the function) and thus, from the maximum principle, it is not possible to drive an initial datum $p_0 \geqslant \varphi_0$ to 0 with constrained controls. This kind of counterexample appears when the drift is absent; see [27, 30]. It is usually constructed by means of sub- and supersolutions of the equation. What is more surprising however is that adding a drift may lead to the existence of nontrivial solutions to

$$\begin{cases} -\Delta\varphi_1 - 2\left\langle \dfrac{\nabla N}{N}, \nabla\varphi_1 \right\rangle = f(\varphi_1) & \text{in } \Omega, \\ \varphi_1 = 1 & \text{on } \partial\Omega, \end{cases}$$

which never happens when no drift is present. In that case, driving the population from an initial datum $p_0 \leqslant \varphi_1$ to z_1 is impossible.

In this paragraph we give examples of assumptions on drifts N such that the equation is not controllable to either 0, θ or 1 in a fixed ball $\mathbb{B}(0; R)$, whenever the drift's intensity is too large or, conversely, such that the equation is always controllable regardless of R

when the drift's intensity is large enough. Of course, the assumptions we make are only sufficient to ensure (non)controllability, and not necessary.

To formalize what we mean by "intensity of the drift", let us then consider, for a fixed radius $R > 0$ and a fixed real parameter $\sigma > 0$, the equation

$$\begin{cases} \partial_t p - \Delta p - \frac{2}{\sigma}\left\langle \frac{\nabla N}{N}, \nabla p \right\rangle = f(p) & \text{in } \mathbb{R}_+ \times \mathbb{B}(0;R), \\ p(t,\cdot) = u(t,\cdot) \in \mathbb{R}_+ \times \partial\mathbb{B}(0;R), & \\ 0 \leqslant u \leqslant 1 & \text{in } \mathbb{R}_+ \times \partial\mathbb{B}(0;R), \\ p(0,\cdot) = p_0, \quad 0 \leqslant p_0 \leqslant 1 & \text{in } \mathbb{B}(0;R). \end{cases} \tag{6}$$

The parameter σ quantifies the drift's intensity.

Blocking phenomena for certain classes of drifts. We introduce the following assumptions on the drift N: the first one is

$$\exists C > 0, \quad \frac{\partial_r N}{N} \leqslant -Cr, \tag{$T1$}$$

while the second is

$$\begin{aligned} &N \in \mathcal{C}^1(\Omega), \exists c_0, c_1 > 0, \forall r \in [0;R], \\ &\quad \forall \theta_1, \dots, \theta_{d-1} \in [0;2\pi], \quad e^{-c_0\frac{r^2}{2}} \leqslant N(r,\theta_1,\dots,\theta_{d-1}) \leqslant e^{-c_1\frac{r^2}{2}}. \end{aligned} \tag{$T2$}$$

Our main result is the following theorem:

Theorem 2. *Let $R > 0$ and $\Omega := \mathbb{B}(0;R)$.*

(1) Lack of controllability to 1. *Assume that N is a $\mathcal{C}^1$ function satisfying ($T1$). There exists $\sigma_N > 0$ such that, for any $\sigma \in (0;\sigma_N)$, equation (6) is not controllable to 1 in Ω.*

(2) Lack of controllability to 0. *Assume that N is a $\mathcal{C}^1$ function satisfying ($T2$). There exists $\sigma'_N > 0$ such that, for any $\sigma \in (0;\sigma'_N)$, equation (6) is not controllable to 0 in Ω.*

Corollary. *If N is $\mathcal{C}^1$ and satisfies ($T1$)–($T2$), there exists $\underline{\sigma} > 0$ such that, for any $\sigma \in (0;\underline{\sigma})$, equation (6) is not controllable to either 0 or 1 in Ω.*

Remark 2. (1) It should be noted that, since the lack of controllability is proved using the existence of nontrivial solutions of the equation with homogeneous boundary values 0 or 1, the controllability to θ cannot hold for arbitrary initial conditions.

(2) Sharp changes in the total population size have been known, since [25], to provoke blocking phenomena for the traveling-wave solutions of the bistable equation, and our results seem to lead to the same kind of interpretation: when a sudden change occurs in N, it is hopeless for a population coming from the boundary to settle everywhere in the domain.

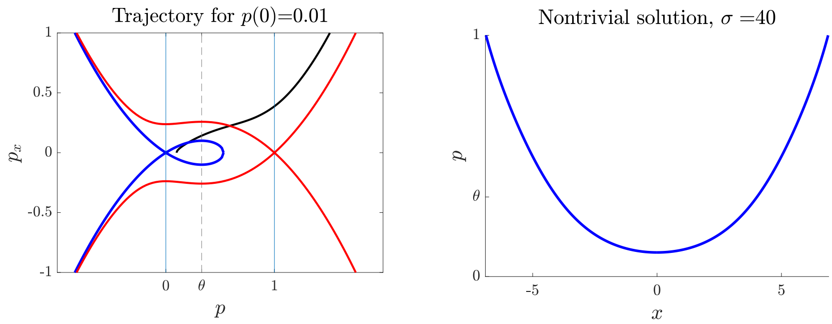

Figure 2. Parameters $\sigma = 40$ and $f(s) = s(1-s)(s-\theta)$, $\theta = 0.33$. (Left) Phase portrait: the trajectory corresponding to the nontrivial solution is in black, the energy set $\{\mathcal{E} = F(1)\}$ in red and the energy set $\{\mathcal{E} = F(0)\}$ in blue. (Right) Nontrivial solution of (8).

(3) We will, for the sake of readability, split the proof of Theorem 2 into two parts, one devoted to the existence of nontrivial steady states with boundary value 1, and one devoted to the existence of such nontrivial solutions with boundary value 0.

(4) The methods used to establish the existence of nontrivial solutions with boundary values 0 or 1 will however be very different: while the existence of a steady state associated with homogeneous boundary value 0 relies on variational arguments, the existence of nontrivial steady states associated with the homogeneous boundary value 1 will rest upon a very fine analysis of the phase plane portrait and will take up most of the sections of the proof. Such complexity in the proof is required by the fact that the steady state $z_1 \equiv 1$ is a global minimizer of the energy functional on the space with homogeneous boundary conditions equal to 1.

(5) This result seems to indicate that the angular component of the drift has very little, if any, influence on controllability.

We illustrate the existence of nontrivial solutions in Figures 2, 3, 4 and 5 for the one-dimensional case, $\Omega = (-L; L)$. In this case, the drift under consideration is $N(r) := e^{-r^2}$, and the equations to which nontrivial solutions must be found are

$$\begin{cases} -\dfrac{\partial^2 p}{\partial x^2} + \dfrac{2x}{\sigma}\dfrac{\partial p}{\partial x} = f(p) \quad \text{in } [-L, L], \\ p(\pm L) = 0, \\ 0 \leqslant p \leqslant 1, \end{cases} \tag{7}$$

and

$$\begin{cases} -\dfrac{\partial^2 p}{\partial x^2} + \dfrac{2x}{\sigma}\dfrac{\partial p}{\partial x} = f(p) \quad \text{in } [-L, L], \\ p(\pm L) = 1, \\ 0 \leqslant p \leqslant 1, \end{cases} \tag{8}$$

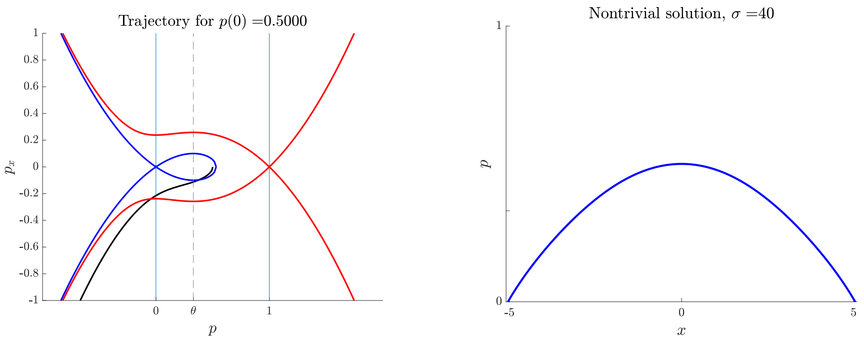

Figure 3. Same class of parameters σ, θ, f. (Left) Phase portrait: the trajectory corresponding to the nontrivial solution is in black, the energy set $\{\mathcal{E} = F(1)\}$ in red and the energy set $\{\mathcal{E} = F(0)\}$ in blue. (Right) Nontrivial solution of (7).

We will study the energy

$$\mathcal{E}: (p, v) \mapsto \frac{1}{2}v^2 + F(p),$$

where $F(p) = \int_0^p f(s)\,ds$.

We also observe this "double-blocking" phenomenon (i.e. the existence of nontrivial solutions to (8) and (7) in the same interval) numerically, when trying to control an initial datum to θ; see Figure 4.

There can also be controllability from 0 to θ, but not from 1 to θ for some drifts, as shown, numerically, in Figure 5.

Unblocking phenomena. Assumptions ($T1$) and ($T2$) essentially state that, when the drift is, loosely speaking, "pushing" towards the boundary intensely enough, barriers will appear and prevent controllability to 0, 1 or θ. We now give, for the sake of completeness, an example of a drift which is pushing "towards" the interior of the domain, and which helps controllability, in the sense that, if it is intense enough, all barriers will disappear. This last situation will be referred to as "unblocking phenomena". For the sake of readability, we once again prove our result in the case of a ball.

Theorem 3. *Let $N(x) := e^{\frac{\|x\|^2}{2}}$. There exists $\sigma_+ > 0$ such that, for every $\Omega := \mathbb{B}(0; R)$ with $R > 0$ and for any $0 < \sigma \leqslant \sigma_+$, (6) is controllable to 0 and 1 in infinite time and to θ in finite time.*

Remark 3. The term "unblocking" is justified by the fact that, as noted already, when the drift is not present, some barriers may appear depending on the value of R; see [27, 30]. The lack of barriers allows us to control to 0, 1 or θ via the static strategy $u \equiv 0$, 1 or θ in infinite time and, to control to θ, we may apply a local exact controllability result (see Proposition 2 below).

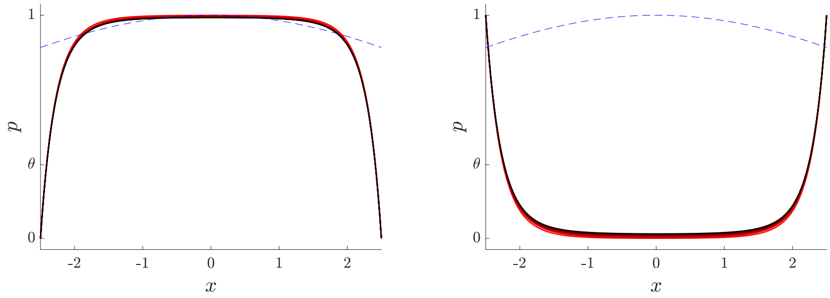

Figure 4. The blue dashed line represents $N^{1/\sigma} = e^{-x^2/\sigma}$ for $\sigma = 40$, $L = 2.5$, initial datum $p_0 = 1$ (left), initial datum $p_0 = 0$ (right). We try to control the initial condition solutions to 0 (left) or 1 (right). Darker red represents a further instance of time and black represents the final time. We clearly observe the lack of controllability due to the presence of a barrier.

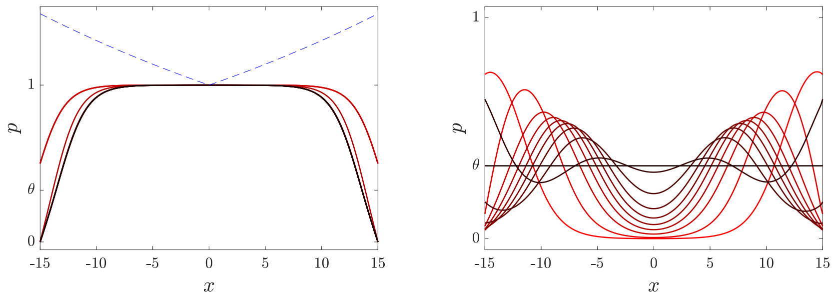

Figure 5. The blue dashed line represents $N^{1/\sigma} = e^{|x|/\sigma}$, $\sigma = 40$, $T = 150$, $L = 15$, initial datum $p_0 = 1$ (left), initial datum $p_0 = 0$ (right). Darker red represents a further instance of time and black is the final time.

Remark 4. As will be explained in the proof, the key point in Theorem 3 is that the following inequality holds:

$$\lambda_1(\mathbb{R}^d, N) := \inf_{\substack{\psi \in W_0^{1,2}(\mathbb{R}^d) \\ \psi \neq 0}} \frac{\int_{\mathbb{R}^d} N^2 |\nabla \psi|^2}{\int_{\mathbb{R}^d} N^2 \psi^2} > 0.$$

As a consequence of [11, Corollary 1.10], it is thus possible to restate our result as follows: let $\Lambda \in \mathcal{C}^2(\mathbb{R}^d; \mathbb{R}_+)$ be such that

$$\lim_{\|x\| \to \infty} \left(\Delta\Lambda + \frac{1}{2} |\nabla \Lambda|^2 \right) = +\infty$$

and define $N_\sigma := e^{\Lambda(\frac{\cdot}{\sigma})}$. Then there exists $\sigma_\Lambda > 0$ such that for any $0 < \sigma \leqslant \sigma_\Lambda$ and any $R > 0$, if $\Omega = \mathbb{B}(0; R)$, the equation

$$\begin{cases} \partial_t p - \Delta p - 2\left\langle \nabla p, \dfrac{\nabla N_\sigma}{N_\sigma} \right\rangle = f(p) & \text{in } \mathbb{R}_+ \times \Omega, \\ p(t,\cdot) = u(t,\cdot) \in \mathbb{R}_+ \times \partial\Omega, & \\ 0 \leqslant u \leqslant 1 & \text{in } \mathbb{R}_+ \times \partial\Omega, \\ p(0,\cdot) = p_0, 0 \leqslant p_0 \leqslant 1 & \text{in } \Omega, \end{cases} \tag{9}$$

is controllable to 0, θ or 1.

The case of radial drifts. In the case where the total population size $N\colon \Omega \to \mathbb{R}_+^*$ can be extended into a radial function $N\colon \mathbb{R}^d \to \mathbb{R}_+^*$, we can give an explicit bound on the decay rate of N to ensure the controllability of (3). In other words, when the total population size is the restriction to the domain Ω of a radial function, we can obtain controllability results.

Theorem 4. *Let Ω be a bounded smooth domain in $\mathbb{R}^d$. Let $N \in \mathcal{C}^1(\mathbb{R}^d; \mathbb{R}_+^*)$, $\inf N > 0$ and N be radially symmetric. Let*

$$\lambda_1^D(\Omega, N) := \inf_{p \in W_0^{1,2}(\Omega)} \frac{\int_\Omega N^2 |\nabla p|^2}{\int_\Omega N^2 p^2}$$

be the weighted eigenvalue associated with N. If

$$\|f'\|_{L^\infty} < \lambda_1^D(\Omega, N) \tag{10}$$

and if

$$N'(r) \geqslant -\frac{d-1}{2r} N(r), \tag{A_1}$$

then equation (3) is controllable to 0 in infinite time and to θ in finite time.

This theorem is proved using energy methods and adapting the proofs of [30].

2.1.3. High-infection rate models. For the infection-dependent model (H_2), i.e. when N assumes the form

$$N = N(p) > 0,$$

the main equation of (3) reads

$$\frac{\partial p}{\partial t} - \Delta p - 2\frac{N'}{N}(p)|\nabla p|^2 = f(p).$$

Then the controllability properties of the equation are the same as in [30]:

Proposition 1. *Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded set. When $N \in \mathcal{C}^1(\mathbb{R})$ satisfies (H_2), there exists $\rho^* = \rho^*(f)$ such that, for any smooth bounded domain Ω, we have the following:*

(1) Lack of controllability for large inradii. *If* $\rho_\Omega > \rho^*$, *then* (3) *is not controllable to* 0 *in (in)finite time in the sense of Definition* 1*: there exists an initial datum* $0 \leqslant p_0 \leqslant 1$ *such that, for any control* u *satisfying the constraints* (4)*, the solution* p *of* (3) *does not converge to* 0 *as* $t \to \infty$.

(2) Controllability for large Dirichlet eigenvalue. *If* $\lambda_1^D(\Omega) > \|f'\|_{L^\infty}$*, then* (3) *is controllable to* 0, 1 *in infinite time for any initial datum* $0 \leqslant p_0 \leqslant 1$*, and to* θ *in finite time for any initial datum* $0 \leqslant p_0 \leqslant 1$.

A possible interpretation of this result is that even if the domain has a large measure, if it is also very thin, it makes sense that a boundary control should work, while if it has a big bulge, it is intuitive that a lack of boundary controllability should occur.

3. Proof of Theorem 1: slowly varying total population size

3.1. Lack of controllability to 0 for large inradius

We prove here the first point of Theorem 1. Recall that we want to prove that, if the inradius ρ_Ω is bigger than a threshold ρ^* depending only on f, then equation (5) is not controllable to 0 in (in)finite time.
Following [27], we claim that this lack of controllability occurs when the equation

$$\begin{cases} -\Delta\varphi_0 - \varepsilon\langle\nabla n, \nabla\varphi_0\rangle = f(\varphi_0) & \text{in } \Omega, \\ \varphi_0 = 0 & \text{on } \partial\Omega, \\ 0 \leqslant \varphi_0 \leqslant 1, \end{cases} \tag{11}$$

has a nontrivial solution, i.e. a solution such that $\varphi_0 \neq 0$. Indeed, we have the following claim:

Claim 1. *If there exists a nontrivial solution* $\varphi_0 \neq 0$ *to* (11)*, then* (5) *is not controllable to* 0 *in infinite time.*

Proof. This is an easy consequence of the maximum principle. Indeed, let η be a nontrivial solution of (11) and let p_0 be any initial datum satisfying

$$\eta \leqslant p_0 \leqslant 1.$$

Let $u\colon \mathbb{R}_+ \times \partial\Omega \to [0, 1]$ be a boundary control. Let p^u be the associated solution of (5). From the parabolic maximum principle [28, Theorem 12], we have for every $t \in \mathbb{R}_+$,

$$\varphi_0(\cdot) \leqslant p^u(t, \cdot),$$

so that p^u cannot converge to 0 as $t \to \infty$. This concludes the proof. ∎

It thus remains to establish the following lemma:

Lemma 1. *There exists $\rho^* = \rho^*(n, f)$ such that, for any Ω satisfying*

$$\rho_\Omega > \rho^*,$$

there exists a nontrivial solution $\varphi_0 \neq 0$ to equation (11).

Since the proof of this lemma is a straightforward adaptation of [30, Proposition 3.1], we postpone it to Appendix A.

3.2. Controllability to 0 and 1

We now prove the second part of Theorem 1, which we rewrite as the following claim:

Claim 2. *Assume $\lambda_1^D(\Omega) > \|f'\|_{L^\infty}$.*

(1) Controllability to 0. *There exist $\rho_* = \rho_*(n, f)$ and $\varepsilon_0^* > 0$ such that, for any Ω, if $\rho_\Omega \leqslant \rho_*$ and $0 < \varepsilon \leqslant \varepsilon_0^*$, equation* (5) *is controllable to* 0 *in infinite time.*

(2) Controllability to 1. *There exists $\varepsilon_1^* > 0$ such that, for any $0 < \varepsilon \leqslant \varepsilon_1^*$, equation* (5) *is controllable to* 1 *in infinite time.*

Proof. (1) *Controllability to* 0. The key part is to prove the following: *There exists $\rho_* > 0$ such that, if $\rho_\Omega \leqslant \rho_*$, then $y \equiv 0$ is the only solution to*

$$\begin{cases} -\Delta y - \varepsilon\langle \nabla n, \nabla y\rangle = f(y) & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega. \end{cases} \tag{12}$$

Indeed, assuming that the uniqueness of (12) holds, consider the static control $u \equiv 0$ and the solution of

$$\begin{cases} \dfrac{\partial p}{\partial t} - \Delta p - \varepsilon\langle \nabla n, \nabla p\rangle = f(p) & \text{in } \mathbb{R}_+ \times \Omega, \\ p = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ p(t = 0, \cdot) = p_0 & \text{in } \Omega. \end{cases}$$

From standard parabolic regularity and the Arzelà–Ascoli theorem, p converges uniformly in Ω to a solution $\bar{p}$ of (12). However, by the uniqueness of (12), we have $\bar{p} = 0$, whence

$$p(t, \cdot) \xrightarrow[t\to\infty]{\mathscr{C}^0(\bar{\Omega})} 0,$$

which means that the static strategy drives p_0 to 0.

We claim that the uniqueness of solutions to (12) follows from spectral arguments: first of all, uniqueness holds for (12) if the first eigenvalue $\lambda(\varepsilon, n, \Omega)$ of the operator

$$\mathscr{L}_{\varepsilon,n}\colon p \mapsto -\nabla \cdot (e^{\varepsilon n}\nabla p)$$

with Dirichlet boundary conditions satisfies

$$\lambda(\varepsilon, n, \Omega) > \|f'\|_{L^\infty} e^{\varepsilon\|n\|_{L^\infty}},$$

as is standard from classical theory for nonlinear elliptic PDEs [4].

We now notice that, n being bounded, the Rayleigh quotient formulation for the eigenvalue

$$\lambda(\varepsilon, n, \Omega) = \inf_{p \in W_0^{1,2}(\Omega)} \frac{\int_\Omega e^{\varepsilon n} |\nabla p|^2}{\int_\Omega p^2}$$

yields that

$$\lambda(\varepsilon, n, \Omega) \geqslant e^{-\varepsilon \|n\|_{L^\infty}} \lambda_1^D(\Omega),$$

where $\lambda_1^D(\Omega)$ is the first eigenvalue of the Laplace operator with Dirichlet boundary conditions. Thus we are reduced to checking that

$$\lambda_1^D(\Omega) > \|f'\|_{L^\infty} e^{\varepsilon \|n\|_{L^\infty}}$$

for $\varepsilon > 0$ small enough. If the condition $\lambda_1^D(\Omega) > \|f'\|_{L^\infty}$ is fulfilled, taking the limit as $\varepsilon \to 0$ yields the desired result.

(2) *Controllability to* 1. Using the same arguments, we claim that controllability to 1 can be achieved through the static control $u \equiv 1$ provided the only solution to

$$\begin{cases} -\Delta \bar{p} - \varepsilon \langle \nabla n, \nabla \bar{p} \rangle = f(\bar{p}) & \text{in } \Omega, \\ \bar{p} = 1 & \text{on } \partial\Omega, \\ 0 \leqslant \bar{p} \leqslant 1 \end{cases} \tag{13}$$

is $\bar{p} \equiv 1$.

We already know (see [27, 30]) that uniqueness holds for $\varepsilon = 0$. Now this implies that uniqueness holds for ε small enough. Indeed, argue by contradiction and assume that, for every $\varepsilon > 0$ there exists a nontrivial solution $\bar{p}_\varepsilon$ to (13). Since $\bar{p}_\varepsilon \neq 1$, $\bar{p}_\varepsilon$ reaches a minimum at some $\bar{x}_\varepsilon \in \Omega$, and so

$$f(\bar{p}_\varepsilon(\bar{x}_\varepsilon)) < 0,$$

which means that

$$\bar{p}_\varepsilon(\bar{x}_\varepsilon) < \theta.$$

Standard elliptic estimates entail that, as $\varepsilon \to 0$, p_ε converges in $W^{1,2}(\Omega)$ and in $\mathcal{C}^0(\bar{\Omega})$ to $\bar{p}$ satisfying

$$\begin{cases} -\Delta \bar{p} = f(\bar{p}) & \text{in } \Omega, \\ \bar{p} = 1 & \text{on } \partial\Omega, \\ 0 \leqslant \bar{p} \leqslant 1, \end{cases} \tag{14}$$

and such that there exists a point $\bar{x}$ satisfying

$$\bar{p}(\bar{x}) \leqslant \theta,$$

which is a contradiction since we know uniqueness holds for (13). This concludes the proof. ∎

3.3. Proof of the controllability to θ for small inradii

3.3.1. Structure of the proof: the staircase method. We recall that we want to control the semilinear heat equation

$$\begin{cases} \frac{\partial p}{\partial t} - \Delta p - \varepsilon \langle \nabla n, \nabla p \rangle = f(p) & \text{in } \mathbb{R}_+ \times \Omega, \\ p(t,\cdot) = u(t,\cdot) & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ p(t=0,\cdot) = y_0, \end{cases} \tag{15}$$

to $z_\theta \equiv \theta$ with additional constraints on the control u, which we drop for the time being. We first state a local exact controllability result [27, Lemma 1], [26, Lemma 2.1]:

Proposition 2 (Local exact controllability). *Let $T > 0$. There exist $\delta_1(T) > 0$, $C(T) > 0$ such that for all steady state y_f of (15), for all $0 \leqslant y_d \leqslant 1$ satisfying*

$$\|y_d - y_f\|_{\mathcal{C}^0} \leqslant \delta_1(T),$$

(15) is controllable from y_d to y_f in finite time $T < \infty$ through a control u. Furthermore, letting $\bar{u} = y_f|_{\partial\Omega}$, the control function $u = u(t)$ satisfies

$$\|u(t) - \bar{u}\|_{\mathcal{C}^0(\partial\Omega)} \leqslant C(T)\delta_1(T). \tag{16}$$

We now assume that $\rho_\Omega \leqslant \rho^*$, that is, thanks to part (1) of the proof of Claim 2, we assume that we have uniqueness for (12). We then proceed in three steps:

Step 1. Starting from any initial condition $0 \leqslant p_0 \leqslant 1$, we first set the static control

$$u(t,x) = 0.$$

Since n is $\mathcal{C}^1$, standard parabolic estimates and the Arzelà–Ascoli theorem ensure that the solution p^u of (5) converges uniformly, as $t \to \infty$, to a solution $\bar{\eta}$ of (12). However, from Claim 2, $\rho_\Omega \leqslant \rho_*(n, f)$ implies that $z_0 \equiv 0$ is the unique solution of this equation. Thus, this static control guarantees that, for every $\delta > 0$, there exists $T_1 > 0$ such that, for any $t \geqslant T_1$,

$$\|p^u(t,\cdot)\|_{L^\infty} \leqslant \frac{\delta}{2}.$$

Step 2. We prove that there exists a steady state p_0 of (15), that is, a solution of

$$-\Delta p_0 - \varepsilon \langle \nabla n, \nabla p_0 \rangle = f(p_0) \quad \text{in } \Omega,$$

where we do not specify the boundary conditions, but such that

$$0 < \inf_{x \in \Omega} p_0(x) \leqslant \|p_0\|_{L^\infty} \leqslant \frac{\delta}{2},$$

where $\delta > 0$ is chosen to apply Proposition 2; we drive $p^u(T_1,\cdot)$ to p_0 in finite time.

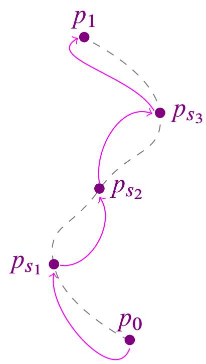

Figure 6. The dashed curve is the path of steady states (for instance in $W^{1,2}(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$), and the points are the close enough steady states. We represent the exact control in finite time $T_1 > 0$ with pink arrows.

Step 3. We drive p_0 to θ using the staircase method.

In this setting, we are thus reduced to the controllability of any initial datum to a small enough p_0 to θ in finite time.

The staircase method. We want to use the staircase method of Coron and Trélat; see [10] for the one-dimensional case and [26] for a full derivation. We briefly recall the most important features of this method: assume that there exists a $\mathcal{C}^0$-continuous path of steady states of (15) $\Gamma = \{p_s\}_{s\in[0,1]}$ such that $p_0 = y_0$ and $p_1 = y_1$. Then (15) is controllable from y_0 to y_1 in finite time. Indeed, as is usually done, we consider a time $T_1 > 0$ and a subdivision

$$0 = s_{i_1} < \cdots < s_{i_K} = 1$$

of $[0, 1]$ such that

$$\forall i \in \{1, \ldots, K-1\}, \quad \|p_{s_i} - p_{s_{i+1}}\|_{\mathcal{C}^0(\Omega)} \leqslant \delta_1,$$

where $\delta_1 = \delta_1(T_1)$ is the controllability parameter given by Proposition 2. We then control each p_{s_i} to $p_{s_{i+1}}$ in finite time by Proposition 2. This result does not necessarily yield constrained controls, but thanks to estimate (16) we can enforce these constraints, by choosing a control parameter δ_1 small enough. Thus, the key part is to find a continuous path of steady states for the perturbed system with slowly varying total population size (5). However, it suffices to have a finite number of steady states that are close enough to each other, starting at y_0 and ending at y_1. We represent the situation in Figure 6.

3.3.2. Perturbation of a path of steady states. We are going to perturb the path of steady states using the implicit function theorem in order to get a sequence of close enough steady states.

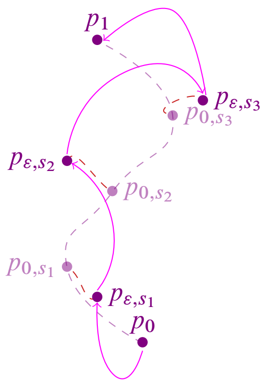

Figure 7. In dark purple, the perturbed steady states, linked to the unperturbed steady states. We do not know whether or not a continuous path of steady states linking these new states exists; however, such points enable us to do exact controllability again and to apply the staircase method.

Remark 5. Here, if we were to try and prove, for ε small enough, the existence of a continuous path of steady states, the idea would be to start from a path $(p_{0,s})_{s\in[0,1]}$ for $\varepsilon = 0$ (which we know exists from [27, 30]) and to try to perturb it into a path for $\varepsilon > 0$ small enough, thus giving us a path $\{p_{\varepsilon,s}\}_{s\in[0,1],\varepsilon>0}$. However, doing it for the whole path requires some kind of implicit function theorem or, at least, some bifurcation argument. Namely, to construct the path, we would need to ensure that either

$$\mathcal{L}^{s,\varepsilon} := -\nabla\cdot(e^{\varepsilon n}\nabla) - e^{\varepsilon n} f'(p_{0,s})$$

has no zero eigenvalue for $\varepsilon = 0$ or that it has a nonzero crossing number (namely, a nonzero number of eigenvalues enter or leave $\mathbb{R}_+^*$ as ε increases from $-\delta$ to δ). In the first case, the implicit function theorem would apply; in the second case, bifurcation theory (see [16, Theorem II.7.3]) would ensure the existence of a branch $p_{\varepsilon,s}$ for ε small enough. These conditions seem too hard to check for an arbitrary path of continuous steady states. Hence, we focus on perturbing a finite number of points close enough on the path since, as we noted, this is enough to ensure exact controllability. We illustrate this construction in Figure 7.

Henceforth, our goal is the following proposition:

Proposition 3. *Let $\delta > 0$. There exist $K \in \mathbb{N}$ and $\varepsilon_\theta^* > 0$ such that, for any $0 < \varepsilon \leqslant \varepsilon_\theta^*$, there exists a sequence $\{p_{\varepsilon,i}\}_{i=1,\dots,K}$ satisfying the following:*

- *For every $i = 1,\dots,K$, $p_{\varepsilon,i}$ is a steady state of (5):*

$$-\Delta p_{\varepsilon,i} - \varepsilon\langle\nabla n, \nabla p_{\varepsilon,i}\rangle = f(p_{\varepsilon,i}).$$

- $p_{\varepsilon,K} = z_\theta \equiv \theta$, $0 < \inf p_{\varepsilon,1} \leqslant \|p_{\varepsilon,1}\|_{L^\infty} \leqslant \delta.$

- *For every* $i = 1, \dots, K$,

$$\frac{\delta}{2} \leqslant p_{\varepsilon,i} \leqslant \|p_{\varepsilon,i}\|_{L^\infty} \leqslant 1 - \frac{\delta}{2}.$$

- *For every* $i = 1, \dots, K-1$,

$$\|p_{\varepsilon,i+1} - p_{\varepsilon,i}\|_{L^\infty} \leqslant \delta.$$

As explained, this proposition gives us the desired conclusion:

Claim 3. *Proposition 3 implies the controllability to* θ *for any initial datum* p_0 *in equation* (3).

We strongly rely on the explicit construction of the path of steady states for $\varepsilon = 0$ in [27, 30].

Known constructions of a path of steady states ($\varepsilon = 0$). For the multidimensional case, it has been shown in [30] that one can construct a path of steady states linking $z_0 \equiv 0$ to $z_\theta \equiv \theta$ in the following way. Let Ω be the domain where the equation is set and let $R_\Omega > 0$ be such that

$$\Omega \subseteq \mathbb{B}(0; R_\Omega).$$

The path of steady states is defined as follows in [30]: first of all, if uniqueness holds for

$$\begin{cases} -\Delta p = f(p) & \text{in } \mathbb{B}(0; R_\Omega), \\ p = 0 & \text{on } \partial\mathbb{B}(0; R_\Omega), \end{cases}$$

then, for $\eta > 0$ small enough, there exists a unique solution to

$$\begin{cases} -\Delta p_\eta = f(p_\eta) & \text{in } \mathbb{B}(0; R_\Omega), \\ p_\eta = \eta & \text{on } \partial\mathbb{B}(0; R_\Omega). \end{cases}$$

Define, for any $s \in [0, 1]$, $p^{0,s}$ as the unique solution to the problem

$$\begin{cases} -\Delta p^{0,s} = f(p^{0,s}) & \text{in } \mathbb{B}(0; R_\Omega), \\ p^{0,s}(0) = s\theta + (1-s)p_\eta(0), \\ p^{0,s} \text{ is radial.} \end{cases} \tag{17}$$

Using polar coordinates, the authors prove that the equation above has a unique solution, and that the map $s \mapsto p^{0,s}$ is continuous in the $\mathscr{C}^0$ topology. Using energy-type methods, they prove that this solution is admissible, i.e. that for any $0 < t_0 < 1$,

$$0 < \inf_{\substack{s \in [t_0; 1] \\ x \in \mathbb{B}(0; R)}} p^{0,s}(x) \leqslant \sup_{\substack{s \in [0,1] \\ x \in \mathbb{B}(0; R)}} p^{0,s}(x) < 1.$$

This gives a path on $\mathbb{B}(0; R_\Omega)$. To construct the path on Ω, it suffices to set

$$\tilde{p}^{0,s} := p^{0,s}|_\Omega.$$

Furthermore, by elliptic regularity or by studying the equation in polar coordinates, we see that, for every $s \in [0,1]$,

$$p^{0,s} \in \mathscr{C}^{2,\alpha}(\mathbb{B}(0; R_\Omega))$$

for any $0 < \alpha < 1$. Instead of perturbing the functions $\tilde{p}^{0,s} \in \mathscr{C}^{2,\alpha}(\Omega)$, we will perturb the functions $p^{0,s} \in \mathscr{C}^2(\mathbb{B}(0; R_\Omega))$.

Notation 1. Henceforth, the parameter $R_\Omega > 0$ is fixed and, for any $s \in [0,1]$, $p^{0,s}$ is the unique solution to (17).

Proof of Proposition 3. Let $\delta > 0$. Let $\{s_i\}_{i=0,\dots,K}$ be a sequence of points such that

$$0 < p^{0,s_0} \leqslant \|p^{0,s_0}\|_{L^\infty} \leqslant \frac{\delta}{2}, \tag{18}$$

and

$$\forall i \in \{0,\dots,K-1\}, \quad \|p^{0,s_i} - p^{0,s_{i+1}}\|_{L^\infty} \leqslant \frac{\delta}{4}. \tag{19}$$

We define, for any $i = 1,\dots,K$,

$$p_{0,i} = p^{0,s_i}.$$

Fix a parameter $\alpha \in (0;1)$. We define a one-parameter family of mappings as follows: for any $i = 1,\dots,K$, let

$$\mathscr{F}_i: \begin{cases} \mathscr{C}^{2,\alpha}(\mathbb{B}(0;R_\Omega))\times[-1;1] \to \mathscr{C}^{0,\alpha}(\mathbb{B}(0;R_\Omega)) \times \mathscr{C}^0(\partial\mathbb{B}(0;R_\Omega)), \\ \qquad (p,\varepsilon) \mapsto (-\nabla\cdot(e^{\varepsilon n}\nabla p) - f(p)e^{\varepsilon n},\ p|_{\partial\mathbb{B}(0;R_\Omega)} - p_{0,i}|_{\partial\mathbb{B}(0;R_\Omega)}). \end{cases}$$

We note that

$$\forall i \in \{1,\dots,K\}, \quad \mathscr{F}_i(p_{0,i},0) = 0.$$

We wish to apply the implicit function theorem, which is permitted provided the operator

$$\mathcal{L}_i : \xi \mapsto -\Delta\xi - f'(p_{0,i})\xi$$

with Dirichlet boundary conditions is invertible. If this is the case we know that there exists a continuous path $\varepsilon \mapsto p_{\varepsilon,i}$ (for $\varepsilon \in [0;\varepsilon_0)$, where $\varepsilon_0 > 0$ is small enough) starting from $p_{0,i}$ such that

$$\mathscr{F}_i(p_{\varepsilon,i},\varepsilon) = 0 \quad \text{for any } \varepsilon \in [0;\varepsilon_0).$$

Denoting, for any differential operator $\mathcal{A}$ its spectrum by $\Sigma(\mathcal{A})$, this invertibility property amounts, thanks to elliptic regularity (see [14]) to requiring that

$$\forall i \in \{1,\dots,K\}, \quad 0 \notin \Sigma(\mathcal{L}_i). \tag{20}$$

If condition (20) is satisfied, then $p_{0,i}$ perturbs into $p_{\varepsilon,i}$ and we can define

$$\tilde{p}_{\varepsilon,i} := p_{\varepsilon,i}|_{\Omega}$$

as a suitable sequence of steady states in Ω. Since we are working with a finite number of points, taking ε small enough guarantees

$$\forall i = 1, \dots, K, \quad \|p_{\varepsilon,i} - p_{0,i}\|_{L^\infty} \leqslant \frac{\delta}{4}$$

and we would then have, for any $i = 1, \dots, K-1$,

$$\begin{aligned}\|p_{\varepsilon,i+1} - p_{\varepsilon,i}\|_{L^\infty} &\leqslant \|p_{\varepsilon,i+1} - p_{0,i+1}\|_{L^\infty} + \|p_{0,i} - p_{\varepsilon,i}\|_{L^\infty} + \|p_{0,i+1} - p_{0,i}\|_{L^\infty} \\ &\leqslant \frac{\delta}{4} + \frac{\delta}{4} + \frac{\delta}{2} = \delta,\end{aligned}$$

which is what we require of the sequence.

Let us define the set of resonant points (i.e. the points where (20) does not hold) as

$$\Gamma := \big\{ j \in \{1, \dots, K\},\ 0 \in \Sigma(\mathcal{L}_j) \big\}.$$

We note that $1 \notin \Gamma$ because the first eigenvalue of

$$\mathcal{L}_1 = -\Delta - f'(0)$$

is positive: indeed, since $f'(0) < 0$ and $\|p_{0,1}\|_{L^\infty}$ is small, this first eigenvalue is bounded from below by the first Dirichlet eigenvalue of the ball $\mathbb{B}(0; R_\Omega)$. Hence $1 \notin \Gamma$. We proceed as follows:

(1) Whenever $i \notin \Gamma$, we can apply the implicit function theorem to obtain the existence of a continuous path $p_{\varepsilon,i}$ starting from $p_{0,i}$ such that

$$p_{\varepsilon,i}|_{\partial\mathbb{B}(0;R_\Omega)} = p_{0,i}|_{\partial\mathbb{B}(0;R_\Omega)}, \quad \mathcal{F}(p_{\varepsilon,i}, \varepsilon) = 0,$$

so that, taking ε small enough, we can ensure that, for any $i \notin \Gamma$,

$$\|p_{\varepsilon,i} - p_{0,i}\|_{L^\infty} \leqslant \frac{\delta}{4}.$$

(2) Whenever $i \in \Gamma$, we apply the implicit function theorem on a larger domain $\mathbb{B}(0; R_\Omega + \tilde{\delta})$, $\tilde{\delta} > 0$; this construction is illustrated in Figure 8.

Let, for any $i \in \Gamma$, $\lambda_i(k, R_\Omega)$ be the kth eigenvalue of $\mathcal{L}_i$ with Dirichlet boundary conditions on $\mathbb{B}(0; R_\Omega)$. Let, for any $i \in \Gamma$,

$$k_i := \sup\big\{k, \lambda_i(k, R_\Omega) = 0\big\}.$$

Obviously, there exists $M > 0$ such that $k_i \leqslant M$ uniformly in i, since $\lambda_i(k, R_\Omega) \to \infty$ as $k \to \infty$. We then invoke the monotonicity of the eigenvalues with respect to the domain.

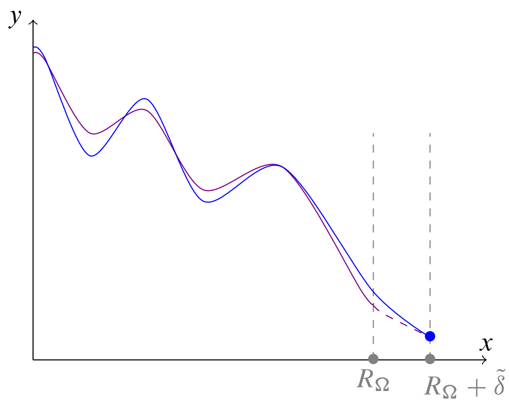

Figure 8. The initial solution $p_{0,i}$ on $\mathbb{B}(0; R_\Omega)$ is continued into a solution on $\mathbb{B}(0; R_\Omega + \tilde{\delta})$, and we apply the implicit function theorem on this domain to obtain the blue curve.

Let, for any $\tilde{\delta}$, $p_{0,i}^{\tilde{\delta}}$ be the extension of $p_{0,i}$ to $\mathbb{B}(0; R_\Omega + \tilde{\delta})$; this is possible given that $p_{0,i}$ is given by the radial equation (17).

Let $\tilde{\mathcal{L}}_i : y \mapsto -\Delta y - f'(p_{0,i}^{\tilde{\delta}})y$ and $\tilde{\lambda}_i(\cdot, R_\Omega + \tilde{\delta})$ be its eigenvalues. By the min-max principle of Courant (see [15]) we have, for any $k \in \mathbb{N}$ and any $\tilde{\delta} > 0$,

$$\tilde{\lambda}_i(k, R_\Omega + \tilde{\delta}) < \lambda_i(k, R_\Omega).$$

Hence, for every $i \in \Gamma$, there exists $\tilde{\delta}_i > 0$ small enough so that, for any $0 < \tilde{\delta} < \tilde{\delta}_i$,

$$0 \notin \Sigma(\tilde{\mathcal{L}}_i).$$

We then choose $\approx{\delta} = \min_{i\in\Gamma} \tilde{\delta}_i$ and apply the implicit function theorem on $\mathbb{B}(0; R_\Omega + \frac{\approx{\delta}}{2})$. This gives the existence of $\tilde{\varepsilon} > 0$ such that, for any $\varepsilon < \tilde{\varepsilon}$ and any $i \in \Gamma$, there exists a solution $p_{\varepsilon,i}^{\approx{\delta}}$ of

$$\begin{cases} -\Delta p_{\varepsilon,i}^{\approx{\delta}} - \varepsilon\langle \nabla n, \nabla p_{\varepsilon,i}^{\approx{\delta}}\rangle = f(p_{\varepsilon,i}^{\approx{\delta}}) & \text{in } \mathbb{B}(0; R_\Omega + \frac{\approx{\delta}}{2}), \\ p_{\varepsilon,i}^{\approx{\delta}} = p_{0,i}\big|_{\partial\mathbb{B}(0;R_\Omega+\frac{\approx{\delta}}{2})}, \\ p_{\varepsilon,i} \xrightarrow[\varepsilon\to 0]{\mathscr{C}^0(\mathbb{B}(0;R_\Omega+\frac{\approx{\delta}}{2}))} p_{0,i}^{\frac{\approx{\delta}}{2}}. \end{cases} \tag{21}$$

Furthermore,

$$p_{0,i}^{\delta}|_{\partial\mathbb{B}(0;R_\Omega)} \xrightarrow[\delta\to 0]{} p_{0,i}|_{\partial\mathbb{B}(0;R_\Omega)}.$$

Thus, by choosing $\approx{\delta}$ small enough, we can guarantee that, by defining

$$\tilde{p}_{\varepsilon,i} := p_{\varepsilon,i}^{\approx{\delta}}|_{\mathbb{B}(0;R_\Omega)}$$

we have for every ε small enough,

$$\|\tilde{p}_{\varepsilon,i} - p_{0,i}\|_{L^\infty} \leqslant \frac{\delta}{4}.$$

We note that on $\partial\mathbb{B}(0; R_\Omega)$, $\tilde{p}_{\varepsilon,i}$ does not satisfy, the same boundary condition as $p_{0,i}$, but this would be too strong a requirement.

This concludes the proof of Proposition 3 and, thus, of Theorem 1. ■

4. Proof of Theorem 2: blocking phenomenon

We split the proof of this theorem in two parts: the first one is devoted to the blocking phenomenon towards 1, the second to the blocking phenomenon towards 0.

4.1. Proof of Theorem 2: blocking phenomenon towards 1

We fix our drift N, as well as the constant C given by assumption ($T1$). We define the first relevant equation:

$$\begin{cases} -\Delta p - \dfrac{2}{\sigma}\left\langle \nabla p, \dfrac{\nabla N}{N}\right\rangle = f(p) & \text{in } \Omega = \mathbb{B}(0; R), \\ p = 1 & \text{on } \partial\Omega. \end{cases} \tag{22}$$

Noncontrollability to 1 is implied by the existence of nontrivial solutions p satisfying $0 \leqslant p \leqslant 1$ to (22); such solutions are called admissible. Thus the blocking phenomenon towards 1 of Theorem 2 is an immediate consequence of the following lemma:

Lemma 2. *Assume N satisfies ($T1$). There exists $\sigma_{N,1} > 0$ such that, for any $\sigma \in (0; \sigma_{N,1})$, there exists a nontrivial admissible solution of* (22).

4.1.1. Reduction to the Gaussian case. The key argument in this proof is the use of a comparison principle, which will enable us to work only with Gaussian drifts, that is, with drifts of the form

$$\mathcal{N}_C(x) := e^{-\frac{C}{2}\|x\|^2}.$$

Here, C is the constant given by assumption ($T1$). Let us then fix this drift.

Blocking phenomenon towards 1 in the Gaussian case. Let us first consider the equation

$$\begin{cases} -\Delta \eta - \dfrac{2}{\sigma}\left\langle \nabla \eta, \dfrac{\nabla \mathcal{N}_C}{\mathcal{N}_C}\right\rangle = f(\eta) & \text{in } \Omega = \mathbb{B}(0; R), \\ \eta = 1 & \text{on } \partial\Omega. \end{cases} \tag{23}$$

The first result to be established is the following:

Lemma 3. *There exists $\bar{\sigma}_C > 0$ such that, for any $\sigma \in (0; \bar{\sigma}_C)$, there exists a nontrivial radially symmetric solution $\eta_{C,1,\sigma}$ of (23). This solution is radially symmetric and nondecreasing, and satisfies $0 \leqslant \eta_{C,1,\sigma} \leqslant 1$.*

We prove it in the next paragraph. Throughout the rest of this section, such a $\bar{\sigma}_C > 0$ *is fixed.*

The link with Lemma 2 is addressed in the following lemma:

Lemma 4. *Lemma 3 implies Lemma 2.*

Proof. Let $\sigma \in (0; \bar{\sigma}_C)$, and let $\eta_{C,1,\sigma}$ be the nontrivial radially symmetric solution given by Lemma 3, which we abbreviate as η_1. Since $\partial_r \eta_1 \geqslant 0$ and η_1 is radially symmetric, it follows that

$$-\left\langle \frac{\nabla N}{N}, \nabla \eta_1 \right\rangle = -\frac{\partial_r N}{N} \partial_r \eta_1 \geqslant C r \partial_r \eta_1 = -\left\langle \frac{\nabla \mathcal{N}_C}{\mathcal{N}_C}, \nabla \eta_1 \right\rangle.$$

The last inequality is a consequence of assumption ($T1$). Hence,

$$-\Delta \eta_1 - \frac{2}{\sigma}\left\langle \frac{\nabla N}{N}, \nabla \eta_1 \right\rangle - f(\eta_1) \geqslant -\Delta \eta_1 - \frac{2}{\sigma}\left\langle \frac{\nabla \mathcal{N}_C}{\mathcal{N}_C}, \nabla \eta_1 \right\rangle - f(\eta_1) = 0.$$

In other words, η_1 is a supersolution of (22). Since $z \equiv 0$ is always a subsolution of (22), the classical method of sub- and supersolutions [18, Theorem 5.17] ensures the existence of a nontrivial solution of (22). ∎

We now prove Lemma 3.

4.1.2. Proof of Lemma 3. We first simplify the proof by noticing the following claim:

Claim 4. *Let $\sigma > 0$ be arbitrary. Assume there exists a nontrivial radially symmetric solution $\eta_{C,1,\sigma}$ of (23) that is radially symmetric and nondecreasing, and satisfies $0 \leqslant \eta_{C,1,\sigma} \leqslant 1$. Then, for any $\tilde{\sigma} \in (0; \sigma)$, there exists a solution $\eta_{C,1,\tilde{\sigma}}$ of (23)*

Proof. We once again use the method of sub- and supersolutions. Indeed, it suffices to notice that

$$-\frac{1}{\tilde{\sigma}}\left\langle \frac{\nabla \mathcal{N}_C}{\mathcal{N}_C}, \nabla \eta_{C,1,\sigma} \right\rangle = \frac{C r}{\tilde{\sigma}} \partial_r \eta_{C,1,\sigma} \geqslant -\frac{1}{\sigma}\left\langle \frac{\nabla \mathcal{N}_C}{\mathcal{N}_C}, \nabla \eta_{C,1,\sigma} \right\rangle.$$

Hence

$$\begin{aligned} &-\Delta \eta_{C,\sigma,1} - \frac{2}{\tilde{\sigma}}\left\langle \frac{\nabla N}{N}, \nabla \eta_{C,\sigma,1} \right\rangle - f(u_{C,\sigma,1}) \\ &\quad \geqslant -\Delta \eta_{C,\sigma,1} - \frac{2}{\sigma}\left\langle \frac{\nabla \mathcal{N}_C}{\mathcal{N}_C}, \nabla \eta_{C,\sigma,1} \right\rangle - f(u_{C,\sigma,1}) = 0. \end{aligned}$$

This hence gives us a supersolution for the equation with $\tilde{\sigma}$, and the conclusion follows in the same way as Lemma 4. ∎

Proof of Lemma 3. Given Claim 4, it suffices to prove that a solution exists for at least one $\sigma > 0$. To prove that this is the case, we use a phase plane analysis and a shooting method. Let us briefly outline the main steps.

For $\alpha \in (0;\theta)$, we consider the solution $p_{\alpha,\sigma} = p_{\alpha,\sigma}(r)$ of the differential equation[2]

$$\begin{cases} -p''_{\alpha,\sigma} + \frac{2r}{\sigma} p'_{\alpha,\sigma} - \frac{d-1}{r} p'_{\alpha,\sigma} = f(p_{\alpha,\sigma}), \\ p_{\alpha,\sigma}(0) = \alpha, \quad p'_{\alpha,\sigma}(0) = 0. \end{cases} \tag{24}$$

We prove the following successively:

Step 1 (Claim 5) For any $\alpha \in (0;\theta)$, there exists $r_{\alpha,\sigma,\theta} > 0$ such that

$$p_{\alpha,\sigma}(r_{\alpha,\sigma,\theta}) = \theta, \quad \alpha < p_{\alpha,\sigma} < \theta \text{ in } (0; r_{\alpha,\sigma,\theta}), \quad p'_{\alpha,\sigma} > 0 \text{ in } (0; r_{\alpha,\sigma,\theta}).$$

Step 2 (Claim 6) There holds

$$r_{\alpha,\sigma,\theta} \xrightarrow[\alpha\to 0^+]{} +\infty.$$

Step 3 (Claims 7, 8) For any $\sigma > 0$, there exists $\alpha \in (0;\theta)$ such that

$$p'_{\alpha,\sigma}(r) \xrightarrow[r\to\infty]{} +\infty, \quad p'_{\alpha,\sigma}(r) > 0 \text{ on } [r_{\alpha,\sigma,\theta}; +\infty).$$

This will enable us to show that, when σ is fixed, there exists $R(\sigma, 1)$ such that there exists $\alpha \in (0;\theta)$ satisfying

$$p_{\alpha,\sigma}(R(\sigma, 1)) = 1, \quad p_{\alpha,\sigma} \text{ is increasing in } (0; R(\sigma, 1)).$$

Let R^*_σ be the smallest value such that there exists such a nontrivial solution. We will prove that, for any $R > R_{\sigma^*}$, there exists a nontrivial solution in $\mathbb{B}(0; R)$ with boundary value 1.

Step 4 (Claim 9) We prove that

$$R^*_\sigma \xrightarrow[\sigma\to 0]{} 0,$$

hence concluding the proof by choosing $\sigma_C > 0$ such that $R_{\sigma_C} < R$.

Step 1. The goal of this paragraph is to prove the following claim:

Claim 5. *For any $\alpha \in (0;\theta)$, there exists $r_{\alpha,\sigma,\theta} > 0$ such that*

$$p_{\alpha,\sigma}(r_{\alpha,\sigma,\theta}) = \theta, \quad \alpha < p_{\alpha,\sigma} < \theta \textit{ in } (0; r_{\alpha,\sigma,\theta}), \quad p'_{\alpha,\sigma} > 0 \textit{ in } (0; r_{\alpha,\sigma,\theta}).$$

Proof. Since $p_{\alpha,\sigma}$ is continuous and since $p_{\alpha,\sigma}(0) = \alpha < \theta$, there exists $\delta > 0$ such that

$$p_{\alpha,\sigma}([0;\delta]) \subset [0;\theta].$$

[2]The existence and uniqueness of such an equation is discussed in Claim 10.

Let $r_{\alpha,\sigma,\theta}$ be defined as

$$r_{\alpha,\sigma,\theta} := \sup\left\{\delta > 0,\ p_{\alpha,\sigma}([0;\delta]) \subset [0;\theta]\right\} > 0.$$

We note that we do not yet rule out the case $r_{\alpha,\sigma,\theta} = \infty$.

Let us first show that $p_{\alpha,\sigma}$ is increasing on $[0; r_{\alpha,\sigma,\theta})$. On $[0; r_{\alpha,\sigma,\theta})$, we have $f(p_{\alpha,\sigma}(r)) < 0$, so that

$$\begin{cases} p''_{\alpha,\sigma} \geqslant \left(\frac{2r}{\sigma} - \frac{d-1}{r}\right) p'_{\alpha,\sigma} & \text{in } (0; r_{\alpha,\sigma,\theta}), \\ p'_{\alpha,\sigma}(0) = 0. \end{cases}$$

Integrating this inequality gives

$$r \mapsto e^{-\frac{r^2}{\sigma}} r^{d-1} p'_{\alpha,\sigma}(r) \text{ is nondecreasing.} \tag{25}$$

Furthermore, since $\alpha \in (0;\theta)$, $p_{\alpha,\sigma}$ is not constant in $(0; r_{\sigma,\alpha,\theta})$.This immediately gives

$$p'_{\alpha,\sigma} > 0 \quad \text{in } (0; r_{\alpha,\sigma,\theta}).$$

This also proves that $r_{\alpha,\sigma,\theta} < +\infty$: we argue by contradiction. If $r_{\alpha,\sigma,\theta} = +\infty$ then (25) guarantees that $p'_{\alpha,\sigma}(r) \xrightarrow[r\to\infty]{} +\infty$, leading to an immediate contradiction. The same argument gives

$$p'_{\alpha,\sigma}(r_{\alpha,\sigma,\theta}) > 0. \qquad \blacksquare$$

This claim allows us to define

$$r_{\alpha,\sigma,\theta} := \inf\left\{r > 0,\, p_{\alpha,\sigma}(r) = \theta\right\} \in (0; +\infty).$$

Step 2. The goal of this paragraph is the following claim:

Claim 6. *Let* $\sigma > 0$ *be fixed. There holds*

$$r_{\alpha,\sigma,\theta} \xrightarrow[\alpha\to 0^+]{} +\infty.$$

Proof. The proof relies on the study of the function

$$\xi(r) := \frac{1}{2}(p_{\alpha,\sigma}^2 + p_{\alpha,\sigma}'^2).$$

We introduce

$$M := \sup_{s\in(0;1)} \frac{-f(s)}{s} > 0.$$

This quantity is finite due to the assumptions on f. Differentiating ξ in $(0; r_{\alpha,\sigma,\theta})$ gives

$$\begin{aligned} \xi'(r) &= p'_{\alpha,\sigma}(r)(p_{\alpha,\sigma}(r) + p''_{\alpha,\sigma}(r)) \\ &= p'_{\alpha,\sigma}(r)\Big(p_{\alpha,\sigma}(r) - f(p_{\alpha,\sigma}) + 2\frac{r}{\sigma} p'_{\alpha,\sigma}(r) - \frac{d-1}{r} p'_{\alpha,\sigma}(r)\Big) \end{aligned}$$

$$\leqslant p'_{\alpha,\sigma}(r)\Big(p_{\alpha,\sigma}(r) + Mp_{\alpha,\sigma}(r) + 2\frac{r}{\sigma}p'_{\alpha,\sigma}(r)\Big) \quad \text{since } p'_{\alpha,\sigma} > 0 \text{ in } (0; r_{\alpha,\sigma,\theta})$$
$$\leqslant p'_{\alpha,\sigma}(r)p_{\alpha,\sigma}(r)(M+1) + 2\frac{r}{\sigma}p'_{\alpha,\sigma}(r)^2$$
$$\leqslant \frac{M+1}{2}(p'_{\alpha,\sigma}(r)^2 + p_{\alpha,\sigma}(r)^2) + 2\frac{r}{\sigma}(p'_{\alpha,\sigma}(r)^2 + p_{\alpha,\sigma}(r)^2)$$
$$\leqslant \xi(r)\Big(M+1+4\frac{r}{\sigma}\Big).$$

Since $\xi(0) = \frac{1}{2}\alpha^2$ we conclude from Grönwall's lemma that

$$\xi(r) \leqslant \frac{\alpha^2}{2}e^{(M+1)r+2\frac{r^2}{\sigma}}.$$

Finally,

$$\xi(r_{\alpha,\sigma,\theta}) \geqslant \frac{1}{2}p_{\alpha,\sigma}(r_{\alpha,\sigma,\theta})^2 = \frac{1}{2}\theta^2,$$

so that

$$\frac{1}{2}\theta^2 \leqslant \frac{\alpha^2}{2}e^{(M+1)r_{\alpha,\sigma,\theta}+2\frac{r^2_{\alpha,\sigma,\theta}}{\sigma}}.$$

The conclusion follows. ■

Remark 6. If we define $r_{\alpha,\sigma,\frac{\theta}{2}}$ as the first root of $p_{\alpha,\sigma}(r_{\alpha,\sigma,\frac{\theta}{2}}) = \frac{\theta}{2}$, the same proof shows that

$$r_{\alpha,\sigma,\frac{\theta}{2}} \xrightarrow[\alpha\to 0]{} +\infty.$$

Step 3. In this paragraph, we prove the two following claims:

Claim 7. *For any $\sigma > 0$, there exists $\alpha \in (0;\theta)$ such that*

$$p'_{\alpha,\sigma}(r) \xrightarrow[r\to\infty]{} +\infty, \quad p'_{\alpha,\sigma}(r) > 0 \quad \text{on } [r_{\alpha,\sigma,\theta}; +\infty).$$

Claim 8. *Let $\sigma > 0$ be fixed. There exists $R(\sigma,1)$ such that there exists $\alpha \in (0;\theta)$ satisfying*

$$p_{\alpha,\sigma}(R(\sigma,1)) = 1, \quad p_{\alpha,\sigma} \text{ is increasing in } (0; R(\sigma,1)).$$

As a consequence, a nontrivial solution of (22) *exists in* $\mathbb{B}(0; R(\sigma,1))$*, and this solution is radially symmetric and nondecreasing. Furthermore, for any* $R \geqslant R(\sigma,1)$*, a nontrivial solution of* (22) *exists in* $\mathbb{B}(0; R)$.

Proof of Claim 7. To prove this claim, the first essential step is to prove that, when $\sigma > 0$ is fixed, there exists $\underline{m} > 0$ such that, for any $\alpha > 0$ small enough,

$$p'_{\alpha,\sigma}(r_{\sigma,\alpha,\theta}) \geqslant \underline{m}.$$

This is done through energy arguments.

We first observe that we can choose $\alpha > 0$ small enough so that the energy

$$\mathcal{E}_{\alpha,\sigma}: \mathbb{R}_+ \ni r \mapsto \frac{1}{2}(p'_{\alpha,\sigma}(r))^2 + F(p_{\alpha,\sigma}(r))$$

is increasing on $(r_{\sigma,\alpha,\frac{\theta}{2}}; +\infty)$. Here, we recall that $r_{\sigma,\alpha,\frac{\theta}{2}}$ was defined in Remark 6 as the first solution of $p(r_{\sigma,\alpha,\frac{\theta}{2}}) = \frac{\theta}{2}$ in $(0; r_{\alpha,\sigma,\theta})$.

Indeed, this energy satisfies

$$\frac{d\mathcal{E}_{\alpha,\sigma}}{dr} = \Big(\frac{2r}{\sigma} - \frac{d-1}{r}\Big) p'_{\sigma,\alpha}(r)^2.$$

This last term is positive whenever

$$r \geqslant \sqrt{\frac{\sigma(d-1)}{2}}.$$

As a consequence, to obtain the monotonicity of the energy on $(r_{\alpha,\sigma,\theta}; +\infty)$, it is sufficient to ensure that

$$r_{\alpha,\sigma,\theta} > \sqrt{\frac{\sigma(d-1)}{2}}.$$

Claim 6 guarantees that this is possible provided $\alpha > 0$ is small enough. We will even require something stronger than $r_{\alpha,\sigma,\theta} > \sqrt{\sigma(d-1)/2}$, that is, we fix (thanks to Remark 6) $\alpha > 0$ small enough so that

$$r_{\alpha,\sigma,\frac{\theta}{2}} > \sqrt{\frac{\sigma(d-1)}{2}}.$$

Further, $p_{\alpha,\sigma}$ satisfies

$$p''_{\alpha,\sigma}(r) = -f(p_{\alpha,\sigma}) + p'_{\alpha,\sigma}\Big(\frac{2r}{\sigma} - \frac{d-1}{r}\Big).$$

The previous computation then shows that $\mathcal{E}_{\alpha,\sigma}$ is increasing on $(r_{\sigma,\alpha,\frac{\theta}{2}}; +\infty)$, whence it follows that

$$\mathcal{E}_{\alpha,\sigma}(r_{\alpha,\sigma,\theta}) > \mathcal{E}_{\alpha,\sigma}(r_{\sigma,\alpha,\frac{\theta}{2}}) \geqslant F\Big(\frac{\theta}{2}\Big).$$

In particular, we obtain

$$p'_{\alpha,\sigma}(r_{\alpha,\sigma,\theta}) > \sqrt{2\Big(F\Big(\frac{\theta}{2}\Big) - F(\theta)\Big)} =: \underline{m}.$$

We quickly remark that, since f is negative on $(0; \theta)$, $F(\theta) < F(\frac{\theta}{2})$, so that the right-hand side of the previous inequality is indeed positive.

The key part is that this lower estimate on $p'_{\alpha,\sigma}(r_{\alpha,\sigma,\theta})$ is uniform in α.

We now turn back to the equation for $p_{\alpha,\sigma}$:

$$p''_{\alpha,\sigma}(r) = -f(p_{\alpha,\sigma}(r)) + p'_{\alpha,\sigma}(r)\Big(\frac{2r}{\sigma} - \frac{d-1}{r}\Big) =: g(r).$$

We will obtain that $p'_{\alpha,\sigma}$ is increasing and goes to $+\infty$ by studying the growth of g. First, notice that

$$g(r_{\alpha,\sigma,\theta}) \geqslant \Big(2\frac{r_{\alpha,\sigma,\theta}}{\sigma} - \frac{d-1}{r_{\alpha,\sigma,\theta}}\Big)\underline{m} > 0$$

because of the uniform lower bound on $p'_{\alpha,\sigma}(r_{\alpha,\sigma,\theta}) \geqslant \underline{m}$ and because α was chosen small enough to ensure $r_{\alpha,\sigma,\theta} > \sqrt{\sigma(d-1)/2}$.

As a consequence, $p'_{\alpha,\sigma}$ is locally increasing around $r_{\alpha,\sigma,\theta}$, which allows us to define

$$A_1 := \sup\big\{A \in \mathbb{R}_+^*,\ p'_{\alpha,\sigma} \geqslant p'_{\alpha,\sigma}(r_{\alpha,\sigma,\theta}) \text{ in } [r_{\alpha,\sigma,\theta}; r_{\alpha,\sigma,\theta} + A]\big\} > 0.$$

We are going to prove that $A_1 = +\infty$. Let us first compute $g'(r)$:

$$\begin{aligned} g'(r) &= -f'(p_{\alpha,\sigma})p'_{\alpha,\sigma} + \Big(\frac{2r}{\sigma} - \frac{d-1}{r}\Big)p''_{\alpha,\sigma} + p'_{\alpha,\sigma}\Big(\frac{2}{\sigma} + \frac{d-1}{r^2}\Big) \\ &= p'_{\alpha,\sigma}\Big(-f'(p_{\alpha,\sigma}) + \frac{2}{\sigma} + \Big(\frac{2r}{\sigma} - \frac{d-1}{r}\Big)\Big(\frac{2r}{\sigma} - \frac{f(p_{\alpha,\sigma})}{p'_{\alpha,\sigma}} - \frac{d-1}{r}\Big) + \frac{d-1}{r^2}\Big) \\ &= p'_{\alpha,\sigma}\Big(-f'(p_{\alpha,\sigma}) + \frac{2}{\sigma} + \Big(\frac{2r}{\sigma} - \frac{d-1}{r}\Big)^2 - \Big(\frac{2r}{\sigma} - \frac{d-1}{r}\Big)\frac{f(p_{\alpha,\sigma})}{p'_{\alpha,\sigma}} + \frac{d-1}{r^2}\Big) \\ &= p'_{\alpha,\sigma}G(r, p_{\alpha,\sigma}, p'_{\alpha,\sigma}), \end{aligned}$$

with

$$G(r,p,v) := \Big(-f'(p) + \frac{2}{\sigma} + \Big(\frac{2r}{\sigma} - \frac{d-1}{r}\Big)^2 - \Big(\frac{2r}{\sigma} - \frac{d-1}{r}\Big)\frac{f(p)}{v} + \frac{d-1}{r^2}\Big).$$

If we can guarantee that

$$\forall v \geqslant \underline{m},\ \forall r \geqslant r_{\alpha,\sigma,\theta}, \quad G(r,p,v) \geqslant 0, \tag{26}$$

then we are done by considering the system

$$p''_{\alpha,\sigma} = g, g' = p'_{\alpha,\sigma}G,$$

and we will have established that $A_1 = +\infty$. Let us now prove that (26) holds for $\alpha > 0$ small enough: extending if needed f into a $W^{1,\infty}$ function outside [0, 1], we see that this condition is guaranteed if, for any $r \geqslant r_{\alpha,\sigma,\theta}$, we have

$$\|f'\|_{L^\infty} + \Big(\frac{2r}{\sigma} - \frac{d-1}{r}\Big)\frac{\|f\|_{L^\infty}}{\underline{m}} \leqslant \frac{2}{\sigma} + \Big(\frac{2r}{\sigma} - \frac{d-1}{r}\Big)^2 + \frac{d-1}{r^2}. \tag{27}$$

However, this inequality always holds for any $r \geqslant r_{\alpha,\sigma,\theta}$, provided $r_{\alpha,\sigma,\theta}$ is large enough, which is in turn guaranteed provided $\alpha > 0$ is small enough. With such an α fixed, we have $A_1 = +\infty$, and so we have

$$\forall r \geqslant r_{\alpha,\sigma,\theta}, \quad p'_{\alpha,\sigma}(r) \geqslant p'_{\alpha,\sigma}(r_{\alpha,\sigma,\theta}) \geqslant \underline{m} > 0.$$

As a by-product, we get

$$\forall r \geqslant r_{\alpha,\sigma,\theta}, \quad g(r) \geqslant g(r_{\alpha,\sigma,\theta}) > 0,$$

so that, integrating the inequality

$$(p'_{\alpha,\sigma})'(r) = g(r) \geqslant g(r_{\alpha,\sigma,\theta}) > 0,$$

we obtain

$$p'_{\alpha,\sigma}(r) \xrightarrow[r\to\infty]{} +\infty. \qquad \blacksquare$$

Proof of Claim 8. The existence of such an $R(\sigma, 1)$ is an immediate consequence of Claim 7. Indeed, choosing $\alpha > 0$ small enough so that the conclusions of Claim 7 are satisfied and keeping in mind that $p_{\alpha,\sigma}$ is increasing on $[0; r_{\alpha,\sigma,\theta}]$, it suffices to define $R(\sigma, 1, \alpha)$ as the first solution of

$$p_{\alpha,\sigma}(R(\sigma, 1, \alpha)) = 1$$

to obtain the desired conclusion. Let us now fix such an $\bar{\alpha} > 0$ and define $R(\sigma, 1) := R(\sigma, 1, \bar{\alpha})$.

To obtain the same conclusion for any $R > R(\sigma, 1)$, it suffices to observe that, first, if $0 < \alpha < \bar{\alpha}$, the solution $p_{\alpha,\sigma}$ satisfies the conclusion of Claim 7 and that $p_{\alpha,\sigma} < p_{\bar{\alpha},\sigma}$ by a standard comparison argument, so that $\alpha \mapsto R(\sigma, 1, \alpha)$ is nonincreasing, and, second, that $R(\sigma, 1, \alpha) \xrightarrow[\alpha\to 0]{} +\infty$. This behavior as $\alpha \to 0^+$ is a simple consequence of the fact that

$$R(\sigma, 1, \alpha) > r_{\alpha,\sigma,\theta} \xrightarrow[\alpha\to 0^+]{} +\infty. \qquad \blacksquare$$

We now define

$$R^*_\sigma := \inf\big\{R_1 > 0,\ \forall R' \geqslant R_1, \text{there exists a nontrivial radially symmetric nondecreasing solution of (22) in } \mathbb{B}(0; R')\big\}. \tag{28}$$

Step 4. In this final step, we prove the following claim:

Claim 9.

$$R^*_\sigma \xrightarrow[\sigma\to 0]{} 0.$$

Proof. We argue by contradiction. Assume that there exists a sequence $\{\sigma_k\}_{k\in\mathbb{N}}$ such that

$$R^*_{\sigma_k} \underset{k\to\infty}{\not\longrightarrow} 0, \quad \sigma_k \xrightarrow[k\to\infty]{} 0.$$

With a slight abuse of notation, we assume that

$$\underline{R} := \varliminf_{k\to\infty} R^*_{\sigma_k} > 0.$$

Let $\alpha > 0$ be fixed. From Claim 5 we know that, for every $\sigma > 0$, there exists $r_{\alpha,\sigma,\theta} > 0$ such that

$$p_{\alpha,\sigma}(r_{\alpha,\sigma,\theta}) = \theta, \quad p_{\alpha,\sigma} \text{ is increasing on } [0; r_{\alpha,\sigma,\theta}].$$

Let

$$p_k := p_{\alpha,\sigma_k}, \quad r_k := r_{\alpha,\sigma_k,\theta}.$$

We reach a contradiction by distinguishing two cases:

(1) 0 *is an accumulation point of* $\{r_k\}$. Assume that, up to a subsequence, we have

$$r_k \xrightarrow[k\to\infty]{} 0.$$

The mean value theorem ensures, for any $k \in \mathbb{N}$, the existence of $y_k \in (0; r_k)$ such that

$$p_k'(y_k) = \frac{\theta - \alpha}{r_k} \xrightarrow[k\to\infty]{} +\infty.$$

We first note that we can obtain a crude estimate on y_k, namely, that, for k large enough, we have

$$y_k \geqslant \sqrt{\frac{\sigma_k(d-1)}{2}} =: r_k^*. \tag{29}$$

To get this estimate, we note that, on $(0; r_k^*)$, we have

$$\left(\frac{2r}{\sigma} - \frac{d-1}{r}\right) < 0,$$

and so

$$p_k'' \leqslant f(p_k).$$

It thus follows that

$$\forall r \in [0; r_k^*], \quad p_k'(r) \leqslant r\|f\|_{L^\infty} \leqslant r_k^*\|f\|_{L^\infty}.$$

Since

$$p_k'(y_k) \xrightarrow[k\to\infty]{} +\infty,$$

it follows that, for k large enough, $y_k \geqslant r_k^*$. We claim that this implies that $p_k' \to +\infty$ uniformly in $[y_k; \frac{R}{2}]$ as made precise in the following statement:

$$\forall M \in \mathbb{R}_+^*, \ \exists k_M \in \mathbb{N}, \ \forall k \geqslant k_M, \quad p_k' \geqslant M \text{ in } \left[y_k; \frac{R}{2}\right]. \tag{30}$$

To prove (30), we first note that $p_k'(y_k) \xrightarrow[k\to\infty]{} +\infty$ implies

$$p_k'(r_k) \xrightarrow[k\to\infty]{} +\infty.$$

Indeed, this follows from the fact that, since $(\frac{2r}{\sigma_k} - \frac{d-1}{r}) > 0$ in $(y_k; +\infty)$ (because $y_k > r_k^*$), since $p_k' \geqslant 0$ in $(0; r_k)$ (because of Claim 5) and since $f(p_k) \leqslant 0$ (because $p_k \leqslant \theta$ in $(y_k; r_k)$), we have

$$p_k'' = \Big(\frac{2r}{\sigma_k} - \frac{d-1}{r}\Big)p_k' - f(p_k) \geqslant 0 \quad \text{in } (y_k; r_k)$$

and so

$$p_k'(r_k) \geqslant p_k'(y_k).$$

To prove that this implies (30), we use a comparison principle on $(r_k; \frac{R}{2})$: define q_k as the solution of

$$\begin{cases} q_k' = -f(p_k) & \text{in } (r_k; \frac{R}{2}), \\ q_k(r_k) = p_k'(r_k). \end{cases}$$

A crude bound on q_k is

$$\forall t \in \Big(r_k; \frac{R}{2}\Big), \quad q_k(t) \geqslant q_k(r_k) - (t - r_k)\|f\|_{L^\infty}.$$

Since $r_k \xrightarrow[k\to\infty]{} 0$ and since $q_k(r_k) \xrightarrow[k\to\infty]{} +\infty$, q_k diverges to ∞ uniformly on $(r_k; \frac{R}{2})$. A simple consequence is that $q_k > 0$ for k large enough.

We now define $z_k := q_k - p_k'$. We immediately obtain that

$$z_k' - \Big(\frac{2r}{\sigma_k} - \frac{d-1}{r}\Big)z_k = -\Big(\frac{2r}{\sigma_k} - \frac{d-1}{r}\Big)q_k.$$

Then again, since $r_k \geqslant y_k$ it follows that $(\frac{2r}{\sigma_k} - \frac{d-1}{r}) > 0$ in $(r_k; \frac{R}{2})$, hence

$$z_k' - \Big(\frac{2r}{\sigma_k} - \frac{d-1}{r}\Big)z_k < 0 \quad \text{in } \Big(r_k; \frac{R}{2}\Big).$$

Integrating this differential inequality yields that

$$r \mapsto e^{-\frac{r^2}{\sigma}} r^{d-1} z_k \text{ is nonincreasing on } \Big[r_k; \frac{R}{2}\Big],$$

hence, for any $r \in [r_k; \frac{R}{2}]$,

$$z_k(r) \leqslant e^{\frac{r^2}{\sigma}} r^{1-d} e^{-\frac{r_k^2}{\sigma}} r_k^{d-1} z_k(r_k) = 0 \quad \text{because } z_k(r_k) = 0.$$

As a consequence $z_k < 0$.

It follows that

$$p_k' \geqslant q_k \quad \text{in } \Big[r_k; \frac{R}{2}\Big]$$

and thus converges uniformly to $+\infty$ in that interval. As a consequence, the equation $p_k(x) = 1$ has a unique root $x_k \in [r_k; \frac{R}{2}]$ for any k large enough, and is increasing in $(0; x_k)$, which is in contradiction with the definition of $\underline{R}$.

(2) 0 *is not an accumulation point of* $\{r_k\}$. Assuming 0 is not an accumulation point of $\{r_k\}_{k\in\mathbb{N}}$, a contradiction ensues in the following manner. We know that there thus exists a point $y > 0$ such that

$$y \leqslant \underline{\lim}_{k\to\infty} r_k, \qquad \lim_{k\to\infty} p_k(y) \leqslant \theta - \delta$$

for some $\delta > 0$. Then we note that, by explicit integration of

$$-p_k'' + \Big(\frac{2r}{\sigma_k} - \frac{d-1}{r}\Big) p_k' = f(p_k),$$

we get

$$r^{d-1} p_k'(r) = e^{\frac{r^2}{\sigma}} \int_0^r e^{\frac{-t^2}{\sigma}} (-f(p_k(t))) t^{d-1}\, dt. \tag{31}$$

Since $y \in (0; r_k)$ for every k large enough and since p_k is increasing in $(0; r_k)$, we have $\alpha \leqslant p_k \leqslant \theta - \delta$ for every $t \in [0; y]$, so that

$$\exists\, \delta' > 0, \quad f(p_k) \leqslant -\delta' \text{ on } [0; y].$$

Plugging this into the integral formulation (31) gives the lower bound

$$r^{d-1} p_k'(r) \geqslant \delta' e^{\frac{r^2}{\sigma}} \int_0^r e^{-\frac{t^2}{\sigma}} t^{d-1}\, dt, \quad r \in [0, y].$$

Let us now study the interval $[\frac{y}{2}; y]$ and prove that p_k' converges uniformly to $+\infty$ in $[\frac{y}{2}; y]$, which would immediately yield the desired contradiction.

We note that, for any $r \in [\frac{y}{2}; y]$, we have

$$\int_0^r e^{-\frac{t^2}{\sigma}} t^{d-1}\, dt \geqslant \int_0^{y/2} e^{-\frac{t^2}{\sigma}} t^{d-1}\, dt \underset{\sigma\to 0}{\sim} C\sqrt{\sigma^d}$$

for some $C > 0$ by the Laplace method (recalled in detailed below (34)), which immediately gives

$$p_k'(r) \xrightarrow[k\to\infty]{} +\infty \quad \text{uniformly in } \Big[\frac{y}{2}; y\Big].$$

The conclusion follows. ■

Proof of Lemma 3. Since $R > 0$, there exists $\underline{\sigma} > 0$ such that, for any $\sigma \leqslant \underline{\sigma}$, $R_\sigma^* < R$. As a consequence of the definition of R_σ^*, a nontrivial solution of (22) exists in $\mathbb{B}(0; R)$ for any $\sigma \leqslant \underline{\sigma}$. ■

4.2. Proof of Theorem 2: blocking phenomenon towards 0

The relevant equation is, in this case,

$$\begin{cases} -\Delta p - \dfrac{2}{\sigma}\Big\langle \nabla p, \dfrac{\nabla N}{N}\Big\rangle = f(p) & \text{in } \Omega, \\ p = 0 & \text{on } \partial\Omega. \end{cases} \tag{32}$$

A nontrivial solution p to this equation is called admissible if $0 \leqslant p \leqslant 1$.

Lemma 5. *Assume N satisfies ($T2$). There exists $\sigma_{N,0} > 0$ such that, for any $\sigma \in (0;\sigma_{N,0})$, there exists a nontrivial admissible solution of* (32).

Theorem 2 (2) is an immediate consequence of this lemma.

We will use the Laplace method to prove that, for any $R > 0$, there exists $\sigma_N > 0$ such that, for any $\sigma \in (0;\sigma_N)$, the equation

$$\begin{cases} -\Delta p - \dfrac{2}{\sigma}\left\langle \dfrac{\nabla N}{N}, \nabla p \right\rangle = f(p) & \text{in } \Omega, \\ p = 0 & \text{on } \partial\Omega \end{cases} \tag{33}$$

has a nontrivial solution. In order to do so, we use the classical method of [4]. First of all, let us note that (33) admits a variational formulation. Indeed, multiplying (33) by $N^{\frac{2}{\sigma}}$ we obtain that any solution p of (33) satisfies

$$-N^{\frac{2}{\sigma}}\Delta p - \frac{2}{\sigma}N^{\frac{2}{\sigma}-1}\langle \nabla N, \nabla p\rangle = N^{\frac{2}{\sigma}} f(p);$$

that is, in other words,

$$-\nabla \cdot (N^{\frac{2}{\sigma}}\nabla p) = N^{\frac{2}{\sigma}} f(p).$$

This leads to introducing the natural energy functional

$$\mathcal{E}_{N,\sigma} : W_0^{1,2}(\Omega) \ni p \mapsto \frac{1}{2}\int_\Omega N^{\frac{2}{\sigma}}|\nabla p|^2 - \int_\Omega N^{\frac{2}{\sigma}} F(p).$$

However, depending on the nonlinearity f, this functional may not be coercive or admit minimizers. To overcome this difficulty, we first follow the strategy of [4] and assume that f was extended by 0 outside [0; 1] (as we are looking for solutions between 0 and 1, if we can prove that, for such an extension, we have a minimizer between 0 and 1, then the only thing that matters is the definition of f on [0; 1], so that the chosen extension is unimportant). We can now prove that $\mathcal{E}_{N,\sigma}$ admits a minimizer p^* in $W_0^{1,2}(\Omega)$ that further satisfies $0 \leqslant p^* \leqslant 1$: consider a minimizing sequence $\{p_k\}_{k\in\mathbb{N}} \in W_0^{1,2}(\Omega)$. Define, for any $k \in \mathbb{N}$,

$$\tilde{p}_k := p_k(\mathbb{1}_{\{p_k \geqslant 0\}} + \mathbb{1}_{\{p_k \leqslant 1\}}) \in W_0^{1,2}(\Omega).$$

As for any $x \leqslant 0$ we have $F(x) = 0$ and, for any $x \geqslant 1$, $F(x) = F(1)$ we obtain

$$\forall k \in \mathbb{N}, \quad \int_\Omega N^{\frac{2}{\sigma}} F(p_k) = \int_\Omega N^{\frac{2}{\sigma}} F(\tilde{p}_k).$$

Similarly,

$$\forall k \in \mathbb{N}, \quad \int_\Omega N^{\frac{2}{\sigma}}|\nabla p_k|^2 \geqslant \int_\Omega N^{\frac{2}{\sigma}}|\nabla \tilde{p}_k|^2,$$

so that, for any $k \in \mathbb{N}$, $\mathcal{E}_{N,\sigma}(p_k) \geqslant \mathcal{E}_{N,\sigma}(\tilde{p}_k)$. Hence, $\{\tilde{p}_k\}_{k\in\mathbb{N}}$ is a minimizing sequence, which is moreover bounded in $L^\infty(\Omega)$. Consequently, it follows from the definition of $\mathcal{E}_{N,\sigma}$ that $\{\tilde{p}_k\}_{k\in\mathbb{N}}$ is bounded in $W_0^{1,2}(\Omega)$, and hence, up to a subsequence, converges (strongly in $L^2(\Omega)$, weakly in $W_0^{1,2}(\Omega)$) to a minimizer p^* that further satisfies $0 \leqslant p^* \leqslant 1$ almost everywhere.

The fact that the minimum is nonzero is a consequence of the next lemma:

Lemma 6. *Assume N satisfies ($T2$). There exists $\sigma_{N,0} > 0$ such that, for any $\sigma \in (0; \sigma_{N,0})$, there holds*

$$\min_{p \in W_0^{1,2}(\Omega)} \mathcal{E}_{N,\sigma}(p) < 0.$$

Then, since $\mathcal{E}_{N,\sigma}$ admits a minimum at a minimizer which is admissible (i.e. between 0 and 1 almost everywhere) , and since Lemma 6 ensures this minimum is not identically 0, the existence of a nontrivial solution follows.

Lemma 6 relies on the Laplace method and, more precisely, on Watson's lemma; this method is presented in [5, 37]. We briefly recall the following conclusion of this method (see for instance [37, Theorem 1]): let $r_1 > 0$. If $\phi : [0; R] \to \mathbb{R}$ is a $\mathcal{C}^1$ function such that $\phi(0) \neq 0$, if $\alpha > 0$ is a positive parameter, then

$$\int_0^{r_1} t^{\alpha-1} \phi(t) e^{-c_0 \frac{t^2}{\varepsilon}} \, dt \underset{\varepsilon \to 0^+}{\sim} M(c_0, d) \phi(0) \varepsilon^{\frac{\alpha}{2}}, \tag{34}$$

where $M(c_0, d)$ is a constant that only depends on c_0 and α.

Proof of Lemma 6. We fix $c_0, c_1 > 0$ as given by assumption ($T2$).

We construct a function $\eta > 0$ such that, whenever σ is small enough,

$$\mathcal{E}_{N,\sigma}(\eta) < 0.$$

To do so, we define η as follows: let $\delta \in (0; \frac{R}{2})$. Let $\eta \equiv 1$ in $\mathbb{B}(0; \delta)$ and $\eta \equiv 0 \in \mathbb{B}(0; R) \backslash \mathbb{B}(0; 2\delta)$. We extend this function to a radially symmetric nonincreasing function $\eta \in \mathcal{C}^1(\mathbb{B}(0; R))$.

Let us split the energy $\mathcal{E}_{N,\sigma}$ into two parts:

(1) The first part corresponds to the gradient: we note that

$$\begin{aligned} 0 \leqslant \int_\Omega N^{\frac{2}{\sigma}} |\nabla \eta|^2 &\leqslant \int_\Omega e^{-c_1 \frac{\|x\|^2}{\sigma}} |\nabla \eta(x)|^2 \, dx \\ &= \int_{\mathbb{B}(0;R) \backslash \mathbb{B}(0;\delta)} e^{-c_1 \frac{\|x\|^2}{\sigma}} |\nabla \eta(x)|^2 \, dx \qquad \text{because } \nabla \eta \equiv 0 \text{ in } \mathbb{B}(0; \delta) \\ &\leqslant |\mathbb{B}(0; R)| e^{-c_1 \frac{\delta^2}{\sigma}} \|\nabla \eta\|_{L^\infty} = M_{\mathbf{I}} e^{-c_1 \frac{\delta^2}{\sigma}}. \end{aligned}$$

Here, $M_{\mathbf{I}} > 0$.

(2) The second part is trickier. Let us consider

$$\int_\Omega F(\eta(x)) N^{\frac{2}{\sigma}}(x) \, dx.$$

Since η is radially nonincreasing and since $F(\eta(0)) = F(1) > 0$, let $r_1 > 0$ be the first real number such that

$$\forall x \in \Omega, \quad \|x\| = r_1 \Rightarrow F(\eta(x)) = 0.$$

We then have

$$\int_\Omega F(\eta(x)) N^{\frac{2}{\sigma}}(x)\, dx = \int_{\mathbb{B}(0;r_1)} F(\eta(x)) N^{\frac{2}{\sigma}}(x)\, dx + \int_{\mathbb{B}(0;R)\setminus\mathbb{B}(0;r_1)} F(\eta(x)) N^{\frac{2}{\sigma}}(x)\, dx.$$

We note that

$$\left| \int_{\mathbb{B}(0;R)\setminus\mathbb{B}(0;r_1)} F(\eta(x)) N^{\frac{2}{\sigma}}(x)\, dx \right| \leqslant \|F(\eta)\|_{L^\infty} M' e^{-c_1 \frac{r_1^2}{\sigma}} \tag{35}$$

for some constant M' by the same arguments that gave us $M_{\mathbf{I}}$, so that this part decays exponentially as $\sigma \to 0^+$. For the first part, since $F(\eta) \geqslant 0$ in $\mathbb{B}(0; r_1)$ by definition of r_1, we have

$$\int_{\mathbb{B}(0;r_1)} F(\eta(x)) N^{\frac{2}{\sigma}}(x)\, dx \geqslant \int_{\mathbb{B}(0;r_1)} F(\eta(x)) e^{-c_0 \frac{\|x\|^2}{\sigma}}\, dx.$$

Since all the functions involved are now radially symmetric, passing to polar coordinates gives

$$\int_{\mathbb{B}(0;r_1)} F(\eta(x)) e^{-c_0 \frac{\|x\|^2}{\sigma}}\, dx = S_d \int_0^{r_1} F(\eta(r)) r^{d-1} e^{-c_0 \frac{r^2}{\sigma}}\, dr,$$

where, with a slight abuse of notation and since η is radially symmetric, we keep the notation $F(\eta)$ for its one-dimensional counterpart. In the formula above, S_d only depends on the dimension. From the Laplace method, it follows that

$$\int_{\mathbb{B}(0;r_1)} F(\eta(x)) e^{-c_0 \frac{\|x\|^2}{\sigma}}\, dx \underset{\sigma \to 0^+}{\sim} M'' F(1) \sigma^{\frac{d}{2}}$$

for some constant $M'' > 0$. Combining this with (35), we obtain, for some constant $M_{\mathbf{II}} > 0$,

$$\int_\Omega F(\eta(x)) N^{\frac{2}{\sigma}}(x)\, dx \geqslant M_{\mathbf{II}} F(1) \sigma^{\frac{d}{2}}.$$

Combining these two steps, we obtain

$$\begin{aligned} \mathcal{E}_{N,\sigma}(\eta) &\leqslant M_{\mathbf{I}} e^{-c_1 \frac{\delta^2}{\sigma}} - M_{\mathbf{II}} F(1) \sigma^{\frac{d}{2}} \\ &< 0 \end{aligned}$$

whenever σ is small enough. The conclusion follows. ■

As a consequence, whenever N satisfies assumption ($T2$), a nontrivial solution to (33) exists.

This concludes the proof of the theorem.

5. Proof of Theorem 3: unblocking phenomenon

Proof of Theorem 3. The key point is that, when $\sigma > 0$ is small enough, we have uniqueness of solutions to

$$\begin{cases} -\Delta p - \dfrac{2}{\sigma}\Big\langle \dfrac{\nabla N}{N}, \nabla p\Big\rangle = f(p) & \text{in } \Omega = \mathbb{B}(0;R), \\ p \equiv a & \text{on } \partial\Omega, \end{cases} \tag{36}$$

where $a = 0$, 1 or θ. Indeed, should this uniqueness hold, a static control $u \equiv a$ will drive any initial condition to z_a (in finite time for θ, and in infinite time for 0 and 1). We first note that the main equation of (36) is equivalent to

$$-\nabla \cdot (N^{\frac{2}{\sigma}} \nabla p) = N^{\frac{2}{\sigma}} f(p).$$

However, defining

$$M := \sup_{x,y} \Big| \frac{f(x) - f(y)}{x - y} \Big|$$

this new form allows us to state that uniqueness for (36) holds provided

$$\lambda_\sigma(\Omega, N) := \inf_{\substack{\psi \in W_0^{1,2}(\Omega) \\ \psi \neq 0}} \frac{\int_\Omega N^{\frac{2}{\sigma}} |\nabla \psi|^2}{\int_\Omega N^{\frac{2}{\sigma}} \psi^2} > M.$$

This is readily seen by taking the difference of two different solutions of (36). We are now going to prove that, with $N(x) = e^{\|x\|^2}$, we have

$$\lambda_\sigma(\Omega, N) \underset{\sigma \to 0}{\longrightarrow} +\infty. \tag{37}$$

Note that (37) follows from an elementary observation: we obviously have

$$\lambda_\sigma(\Omega, N) \geqslant \lambda_\sigma(\mathbb{R}^d, N) = \inf_{\substack{\psi \in W_0^{1,2}(\mathbb{R}^d) \\ \psi \neq 0}} \frac{\int_{\mathbb{R}^d} N^{\frac{2}{\sigma}} |\nabla \psi|^2}{\int_{\mathbb{R}^d} N^{\frac{2}{\sigma}} \psi^2}.$$

By a simple change of variables (since $N : x \mapsto e^{\frac{\|x\|^2}{2}}$), we have

$$\lambda_\sigma(\mathbb{R}^d, N) = \frac{1}{\sigma} \lambda_1(\mathbb{R}^d, N) = \frac{1}{\sigma} \inf_{\substack{\psi \in W_0^{1,2}(\mathbb{R}^d) \\ \psi \neq 0}} \frac{\int_{\mathbb{R}^d} N^2 |\nabla \psi|^2}{\int_{\mathbb{R}^d} N^2 \psi^2}$$

and, from [11, Corollary 1.10],

$$\lambda_1(\mathbb{R}^d, N) > 0.$$

Now (37) follows immediately. ∎

6. Proof of Theorem 4: radial drifts

Proof of Theorem 4. Proceeding along the same lines as in Theorem 1, we prove that for any drift $N \in \mathcal{C}^\infty(\Omega;\mathbb{R})$ (regardless of whether or not it is the restriction of a radial drift N to the domain Ω), if condition (10) holds, then $z_0 \equiv 0$ is the only solution to

$$\begin{cases} -\Delta p - 2\Big\langle \frac{\nabla N}{N}, \nabla p\Big\rangle = f(p) & \text{in } \Omega, \\ p = 0 & \text{on } \partial\Omega, \\ 0 \leqslant p \leqslant 1, \end{cases} \tag{38}$$

and note that the main equation is equivalent to

$$-\nabla \cdot (N^2 \nabla p) = f(p) N^2.$$

Indeed, assuming there exists a nontrivial solution p to (38), then from the mean value theorem, we can write

$$f(p) = f'(y)p$$

for some function y. Multiplying the equation by p and integrating by parts gives, using the Rayleigh quotient formulation of $\lambda_1^D(\Omega, N)$,

$$\lambda_1^D(\Omega, N)\int_\Omega N^2 p^2 \leqslant \int N^2 |\nabla p|^2 = \int_\Omega N^2 f'(y) p^2 \leqslant \|f'\|_{L^\infty}\int_\Omega N^2 p^2,$$

which is a contradiction unless $p \equiv 0$.

Once we have uniqueness for (38) we follow, for any initial datum p_0, the staircase method explained in the proof of Theorem 1: we first set the static control $u = 0$, we drive the solution to a $\mathcal{C}^0$ neighborhood of z_0, then to a steady-state solution of (3) in this neighborhood. Thus, we only need to prove the existence of a path of steady states linking z_0 to z_θ. In order to prove that such a path of steady states exists under assumption (A_1), we use an energy method.

Let $R > 0$ be such that $\Omega \subset \mathbb{B}(0; R)$. As in [30], we define, for any $s \in [0, 1]$, p_s as the unique solution of

$$\begin{cases} -\Delta p_s - 2\Big\langle \frac{\nabla N}{N}, \nabla p_s\Big\rangle = f(p_s) & \text{in } \mathbb{B}(0; R), \\ p_s \text{ is radial in } \mathbb{B}(0; R), \\ p_s(0) = s\theta. \end{cases} \tag{39}$$

We notice that the first equation in (39) can be rewritten as

$$-\nabla \cdot (N^2 \nabla p_s) = f(p_s) N^2.$$

Since N is radially symmetric, this amounts to solving, in radial coordinates

$$\begin{cases} -\frac{1}{r^{d-1}}(r^{d-1} N^2 p_s')' = f(p_s) N^2 & \text{in } [0; R], \\ p_s(0) = s\theta, \quad p_s'(0) = 0. \end{cases} \tag{40}$$

We prove the existence and uniqueness of solutions to (40) below but underline that the core difficulty here is ensuring that

$$0 \leqslant p_s \leqslant 1.$$

Claim 10. *For any $s \in [0,1]$, there exists a unique solution to* (40)*.*

Proof. This follows from a standard contraction argument. On $L^\infty(0; r_1)$, where $r_1 < R$ will be fixed later on, define the map

$$T: \varphi \mapsto s\theta + \int_0^r \frac{1}{l^{d-1}N^2} \int_0^l -t^{d-1} f(\varphi) N^2 \, dt \, dl.$$

We have the following estimate:

$$\begin{aligned} \|T\varphi - T\phi\|_{L^\infty} &\leqslant \int_0^r \frac{1}{l^{d-1}N^2} \int_0^l t^{d-1} M \|\varphi - \phi\|_{L^\infty} N^2 \, dt \, dl \\ &\leqslant M \|\varphi - \phi\|_{L^\infty} \|N^2\|_{L^\infty} \left\| \frac{1}{N^2} \right\|_{L^\infty} \frac{r^2}{d}, \end{aligned}$$

where M is the Lipschitz constant of f. If r_1 is small enough, T is a contraction in $L^\infty(0; r_1)$ and so the existence and uniqueness of a solution follow in $(0; r_1)$. In $(r_1; R)$, the standard Cauchy–Lipschitz theory applies. ■

Claim 11. *Under assumption* (A_1) *the path is admissible: we have, for any $s \in [0,1]$,*

$$0 \leqslant p_s \leqslant 1. \tag{41}$$

Furthermore, the path $\{p_s\}_{s \in [0,1]}$ is continuous in the $\mathcal{C}^0$ topology.

Proof. (1) *Admissibility of the path under assumption* (A_1). We now prove estimate (41). To do so, we introduce the energy functional

$$\mathcal{E}_1: x \mapsto \frac{1}{2}(p_s'(x))^2 + F(p_s(x)),$$

where $F: x \mapsto \int_0^x f$ is the antiderivative of f. Differentiating $\mathcal{E}_1$ with respect to x, we get

$$\begin{aligned} \mathcal{E}_1'(x) &= (p_s''(x) + f(p_s))p_s'(x) \\ &= \left(-\frac{d-1}{r} - 2\frac{N'(r)}{N(r)}\right)(p_s'(r))^2 && \text{from equation (40)} \\ &\leqslant 0 && \text{from hypothesis } (A_1). \end{aligned}$$

In particular, for any $s \neq 0$, $p_s \neq 0$ in $(0; R)$, arguing by contradiction, we have that if, for $\underline{x} \in (0; R)$ we had $p_s(\underline{x}) = 0$, then

$$\mathcal{E}_1(\underline{x}) = \frac{1}{2}(p_s'(\underline{x}))^2 \geqslant 0.$$

However, $\mathcal{E}_1(0) = F(s\theta) < 0$, so that a contradiction follows. For the same reason, $p_s \neq 1$ in $[0; R]$, for otherwise, if $p_s(\bar{x}) = 1$ at some $\bar{x} \in [0, 1]$, we would have

$$\mathcal{E}_1(\bar{x}) \geqslant F(1) > 0,$$

which is once again a contradiction. It follows that, for any $s \in (0; 1]$,

$$0 \leqslant p_s \leqslant 1,$$

as claimed.

(2) *Continuity of the path.* We want to prove the $\mathcal{C}^0$ continuity of the path. Let $s \in [0, 1]$ and let $\{s_k\}_{k\in\mathbb{N}} \in [0, 1]^{\mathbb{N}}$ be a sequence such that

$$s_k \xrightarrow[k\to\infty]{} s.$$

Let $p_k := p_{s_k}$. Our goal is to show that

$$p_k \xrightarrow[k\to\infty]{\mathcal{C}^0(\mathbb{B}(0;R))} p_s. \tag{42}$$

We will use elliptic regularity to ensure it. We first derive a $W^{1,\infty}$ estimate from the one-dimensional equation, and use it to obtain, for any $\alpha \in (0; 1)$, a $\mathcal{C}^{2,\alpha}$ estimate for the equation set in $\mathbb{B}(0; R)$. By the admissibility of the path we have, for every $k \in \mathbb{N}$,

$$0 \leqslant p_k \leqslant 1.$$

Passing into radial coordinates and integrating equation (40) between 0 and x gives

$$-p_k'(x) = \frac{1}{N^2(x)x^{d-1}} \int_0^x f(p_k(s))N^2(s)s^{d-1}\,ds. \tag{43}$$

Thus the sequence $\{p_k\}_{k\in\mathbb{N}}$ is uniformly bounded in $W^{1,\infty}((0; 1))$. We now consider equation (39). Since, by the first step, $\{p_k\}_{k\in\mathbb{N}}$ is uniformly bounded in $\mathcal{C}^{0,\alpha}(\mathbb{B}(0; R))$ for any $\alpha \in (0; 1)$, and since $N \in \mathcal{C}^\infty(\mathbb{B}(0; R))$, it follows from Hölder elliptic regularity (see [14]) that, for any $\alpha \in (0; 1)$, there exists $M_\alpha \in \mathbb{R}$ such that, for every $k \in \mathbb{N}$,

$$\|p_k\|_{\mathcal{C}^{2,\alpha}(\mathbb{B}(0;R))} \leqslant M_\alpha,$$

hence $\{p_k\}_{k\in\mathbb{N}}$ converges in $\mathcal{C}^1(\mathbb{B}(0; R))$, up to a subsequence, to p_∞. Passing to the limit in the weak formulation of the equation, we see that p_∞ satisfies

$$-\nabla \cdot (N^2 \nabla p_\infty) = f(p_\infty)N^2.$$

Passing to the limit in

$$\forall k \in \mathbb{N}, \quad p_k(0) = s_k\theta,$$

we get $p_\infty(0) = s\theta$ and, finally, since for every $k \in \mathbb{N}$, p_k is radial, i.e.

$$\forall k \in \mathbb{N}, \forall i, j \in \{1, \ldots, d\}, \quad x_j \frac{\partial p_k}{\partial x_i} - x_i \frac{\partial p_k}{\partial x_j} = 0,$$

we can pass to the limit in this identity to obtain that p_∞ is radial. In particular,

$$p_\infty = p_s$$

and so the continuity of the path holds. ■

To conclude the proof of Theorem 4, it suffices to apply the staircase method. ■

7. Proof of Proposition 1: high-infection rate models

Proof of Proposition 1. The proof consists in transforming the equation

$$\frac{\partial p}{\partial t} - \Delta p - 2\frac{N'}{N}(p)|\nabla p|^2 = f(p) \tag{44}$$

into a simpler one. This is done by following the idea of [25, Proof of Theorem 1]. Let us introduce the antiderivative of N^2 as

$$\mathcal{N}\colon x \mapsto \int_0^x N^2(\xi)\, d\xi.$$

We first note that multiplying N by any factor λ leaves equation (44) invariant. We thus fix

$$\int_0^1 N^2(\xi)\, d\xi = 1.$$

Multiplying (44) by N^2 we get

$$\begin{aligned} N^2(p)\frac{\partial p}{\partial t} - N^2(p)\Delta p - 2N(p)N'(p)|\nabla p|^2 &= (\mathcal{N}(p))_t - \nabla \cdot (N^2(p)\nabla p) \\ &= (\mathcal{N}(p))_t - \Delta(\mathcal{N}(p)). \end{aligned}$$

Hence, as $\mathcal{N}$ is a diffeomorphism, the function $\tilde{p} := \mathcal{N}(p)$ satisfies

$$\frac{\partial \tilde{p}}{\partial t} - \Delta \tilde{p} = f(\mathcal{N}^{-1}(\tilde{p}))N^2(\mathcal{N}^{-1}(\tilde{p})) =: \tilde{f}(\tilde{p}).$$

However, it is easy to see that, f being bistable, so is $\tilde{f}$. Furthermore, $\mathcal{N}$ is a $\mathcal{C}^1$ diffeomorphism of $[0, 1]$, and it is easy to see that p is controllable to 0, $\mathcal{N}(\theta)$ or 1 if and only if $\tilde{p}$ is controllable to 0, θ or 1, and we are thus reduced to the statement of [30, Theorem 1.2], from which the conclusion follows. ■

8. Conclusion

8.1. Obtaining the results for general coupled systems

As explained in the introduction, the equations considered in this article correspond to some scaling limits for more general coupled systems of reaction–diffusion equations, and it seems interesting to investigate whether or not the results we obtained in this article might be generalized to encompass the case of such general systems. As was explained in the introduction, these models can be used to control populations of infected mosquitoes and arise in evolutionary dynamics. Obtaining a finer understanding of the real underlying dynamics rather than the simplified version under scrutiny here seems, however, challenging. Indeed, although controllability results for linear systems of equations exist (see for instance [21]), the nonlinear case has not yet been completely studied.

From the application point of view, we observe that a qualitative understanding of the heterogeneity is a must. Indeed, the mildness of the assumptions ($T1$)–($T2$) prove that, whenever a localized sharp transition in this heterogeneity occurs, controllability to steady states may fail.

However, given that, as explained in the introduction, gene-flow models and spatially heterogeneous models are limits in a certain scaling of such systems, it would be interesting to see whether or not our perturbation arguments, which were introduced to pass from the spatially homogeneous model to the slowly varying one, could work to pass from this scaling limit to the whole system in a certain regime.

In the homogeneous case, when $\int_0^1 f = 0$, there does not exist any nontrivial solution with boundary values 0 or 1 [30]. However, in the heterogeneous setting, there can exist such nontrivial solutions. Note that in the proof of the first point of Theorem 2, that is, for the blocking phenomenon towards 1, we have not used the fact that the primitive at 1 has a particular sign.

8.2. Open problem

Let us now list a few questions which, to the best of our knowledge, are still open and seem worth investigating.

- The qualitative properties of time-optimal controls.
  As suggested in [27] one might try to optimize the control with respect to the controllability time. Indeed, it is known that, under constraints on the control, parabolic equations have a minimal controllability time; see for instance [26, 36].
  For constrained controllability it is known that there exists a minimal controllability time to control, for instance, from 0 to θ (see [27]). We may try to optimize the control strategies so as to minimize the controllability time. In our case, that is, the spatially heterogeneous case, are these controls of bang-bang type? Another qualitative question that is relevant in this context is that of symmetry: In the one-dimensional case,

when working on an interval $[-L, L]$, are time-optimal controls symmetric? In the multidimensional case, when the domain Ω is a ball, is it possible to prove radial symmetry of time-optimal controls?

- The influence of spatial heterogeneity on controllability time.
  Adding a drift (which corresponds to the spatially heterogeneous model) modifies the controllability time. As we have seen, such heterogeneities might lead to a lack of controllability. However, it is also suggested in the numerical experiments shown below that adding a drift might be beneficial for the controllability time. It might be interesting to consider the following question: Given L^∞ and L^1 bounds on the spatial heterogeneity N, which is the drift yielding the minimal controllability time? In other terms, how can we design the domain so as to minimize the controllability time? In the simulation below, we thus considered the following optimization problem: Letting, for any drift $m = N_x/N$, $T(m)$ be the minimal controllability time from 0 to θ of the spatially heterogeneous equation (3) (with $T(m) \in (0; +\infty]$), solve

$$\inf_{-M \leqslant m \leqslant M} T(m).$$

We obtain the graph shown in Figure 9 with $M = 250$ and $L = 2.5$.

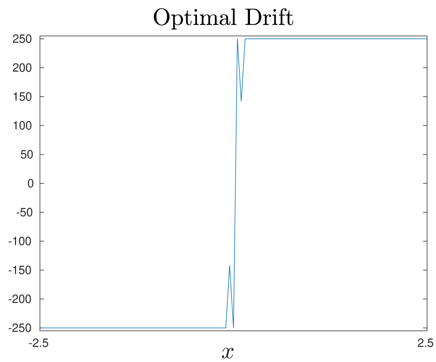

Figure 9. Time-optimal spatial heterogeneity.

In Figure 10, we numerically observe that the minimal controllability time goes to 0 for the case in which the radial derivative goes inwards, while it blows up in the other two cases. For the case of the Gaussian, we observe the emergence of an upper barrier as the drift becomes stronger, and the same is happening for the case of the sinusoidal drift. Even if these simulations may fit the intuition, a proper analysis of the minimal controllability time should be carried out.

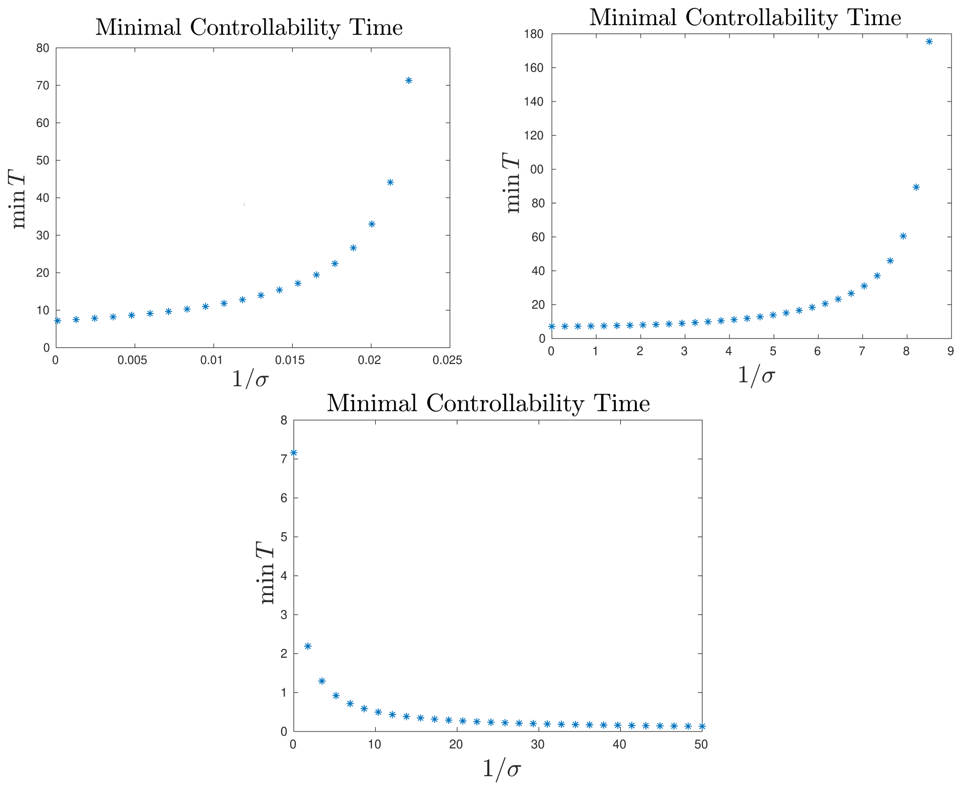

Figure 10. Minimal controllability time depending on the strength of the drift for $N = e^{-\frac{x^2}{\sigma}}$ (top left), $N' = \frac{\sin(x)}{\sigma} N$ (top right), $N = e^{\frac{x^2}{\sigma}}$ (bottom).

A. Proof of Lemma 1

Proof of Lemma 1. Let us first remark that (11) has a variational structure. Indeed, p is a solution of

$$-\Delta p + \varepsilon \langle \nabla n, \nabla p \rangle = f(p), \quad p \in W_0^{1,2}(\Omega)$$

if and only if

$$-\nabla \cdot (e^{\varepsilon n} \nabla p) = f(p) e^{\varepsilon n}, \quad p \in W_0^{1,2}(\Omega). \tag{45}$$

Following the arguments of [4, Remark II.2], we introduce the energy functional associated with (45): let

$$\mathcal{E}_1 \colon W_0^{1,2}(\Omega) \ni p \mapsto \frac{1}{2} \int_\Omega e^{\varepsilon n} |\nabla p|^2 - \int_\Omega e^{\varepsilon n} F(p).$$

From standard arguments in the theory of sub- and supersolutions [4], if there exists $v \in W_0^{1,2}(\Omega)$ such that

$$\mathcal{E}_1(v) < 0 \tag{46}$$

then there exists a nontrivial solution to (11). We now prove that there exists $v \in W_0^{1,2}(\Omega)$ such that (46) holds, by adapting the construction and computations of [30] (we only sketch the $d \geqslant 2$ case). Let $\mathbb{B}(\bar{x}; \rho_\Omega)$ be one of the balls of maximum radius inscribed in Ω. Up to a translation, we assume that $\bar{x} = 0$.

Let $\delta > 0$. We define v_δ as

$$v_\delta \colon \begin{cases} x \in \mathbb{B}(0; \rho_\Omega - \delta) \mapsto 1, \\ x \in \mathbb{B}(0; \rho_\Omega) \backslash \mathbb{B}(0; \rho_\Omega - \delta) \mapsto \dfrac{\rho_\Omega^2 - \|x\|^2}{\rho_\Omega^2 - (\rho_\Omega - \delta)^2}, \\ x \in \Omega \backslash \mathbb{B}(0; \rho_\Omega) \mapsto 0. \end{cases}$$

An explicit computation yields

$$\int_\Omega e^{\varepsilon n} |\nabla v_\delta|^2 \sim_{\delta \to 0} C_1 \delta \rho_\Omega^{d-1} e^{\varepsilon n(\rho_\Omega)}$$

for some constant $C_1 > 0$, and

$$\int_\Omega e^{\varepsilon n} F(v_\delta) \geqslant C_2 F(1)(\rho_\Omega - \delta)^d.$$

Hence, since n is bounded, as ρ_Ω grows the second term in the energy functional will dominate and the conclusion follows: as $\rho_\Omega \to \infty$ and $\delta \to 0$ the energy of v_1 is negative. ∎

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