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Simple Proofs of Nakano's Vanishing Theorem and Kazama's Approximation Theorem for Weakly 1-Complete Manifolds

By

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Introduction

Let X be an m-dimensional complex manifold and let E be a vector bundle on X. A hermitian inner product in E is given as usual and is denoted by $H(\xi, \eta)$. In particular, when $\xi=\eta$, we write $H(\xi, \xi)$ as $|\xi|^2$. By $\mathcal{O}(E)$ we denote the sheaf of germs of holomorphic sections of E. X is called a weakly 1-complete manifold when there exists a C^∞-differentiable pseudoconvex function Ψ on X such that $X_c=\{\Psi<c\}$ is relatively compact in X for any real number c. We see that if X is a weakly 1-complete manifold, X_c is also a weakly 1-complete manifold.

Now we consider a weakly 1-complete manifold with a positive vector bundle E (see, Definition (1.4) in § 1). Then the following theorems have been proved by S. Nakano [8] and H. Kazama [4] respectively:

Theorem 1. *For any real number c, we have*

$$H^q(X_c, \mathcal{O}(E\otimes K))=0 \qquad \textit{for} \quad q\geqq 1,$$

where K denotes the canonical line bundle of X.

Theorem 2. *Fix two constants c and d with $c>d$. Then for any holomorphic section $\varphi\in H^0(\overline{X}_d, \mathcal{O}(E\otimes K))$, $\overline{X}_d$ being the closure of X_d in X and for any positive constant ε, there exists a section $\tilde{\varphi}\in H^0(X_c, \mathcal{O}(E\otimes K))$ such that $|\varphi-\tilde{\varphi}|^2<\varepsilon$ everywhere in $\overline{X}_d$.*

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Corollary. $H^q(X, \mathcal{O}(E\otimes K))=0$ *for* $q\geqq 1$.

This follows from Theorems 1 and 2 by a well known technique (see, Gunning and Rossi [2], p. 243, Theorem 14).

In this short note we shall give simple proofs of the above theorems by using the method due to K. Kodaira (see, Theorem 3 in §2) and a key lemma due to A. Andreotti and E. Vesentini (see, [1], p. 93, Proposition 5). The original proof of Theorem 1 is very complicated because of the choices of the metrics of E and X (see, the proof of (iii) in Proposition 1 in p. 172, Nakano [8]). Kazama's proof is very long.

Sections 1 and 2 are devoted to preliminaries and in section 3 our proofs will be done.

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§1. Hermitian Connections of Hermitian Vector Bundles

Let X be an m-dimensional complex manifold and let E be a hermitian vector bundle of rank r on X. We cover X by locally finite coordinate neighborhoods $\{U_\lambda\}$ and denote local coordinates on U_λ by $z_\lambda^1, z_\lambda^2,\dots, z_\lambda^m$. With respect to this covering a hermitian inner product H is expressed by a system of positive definite hermitian matrixes $\{(h_{\lambda,k\bar{j}})\}$ on U_λ: for C^∞-sections $\xi=\{(\xi_\lambda^1, \xi_\lambda^2,\dots, \xi_\lambda^r)\}$ and $\eta=\{(\eta_\lambda^1, \eta_\lambda^2,\dots, \eta_\lambda^r)\}$ of E on X,

$$H(\xi, \eta)=\sum_{kj}^{r} h_{\lambda.k\bar{j}}\xi_\lambda^k\overline{\eta_\lambda^j}. \tag{1.1}$$

By $(h_\lambda^{\bar{k}j})$ we denote the inverse matrix of $(h_{\lambda,k\bar{j}})$. By using H, we can define a hermitian connection in a canonical manner: A system of matrix valued 1-forms $\{\omega_\lambda^*\}$, $\omega_\lambda^*=\{\omega_{\lambda k}^{*i}\}$ on U_λ is called a hermitian connection if

$$\omega_{\lambda k}^{*i}=\sum_{\alpha=1}^{m}\Gamma_{\lambda,\alpha k}^{*\ i}\,dz_\lambda^\alpha \text{ where } \Gamma_{\lambda,\alpha k}^{*\ i}=\sum_{j=1}^{r}h_\lambda^{\bar{j}i}\frac{\partial h_{\lambda,k\bar{j}}}{\partial z_\lambda^\alpha}. \tag{1.2}$$

The curvature tensor of the above connection is defined by

(1.3) $$K_{\lambda,}{}^{i}_{k\bar{\beta}\alpha}=\frac{\partial\Gamma^{*}{}_{\lambda,\alpha k}^{\ i}}{\partial\bar{z}_{\lambda}^{\beta}}.$$

We also define

$$K_{\lambda,ik\bar{\beta}\alpha}=\Sigma h_{\lambda.ji}K_{\lambda,}{}^{j}_{k\bar{\beta}\alpha}.$$

It is easily seen that $K_{\lambda,ik\bar{\beta}\alpha}=\overline{K_{\lambda,\bar{k}\bar{i}\bar{\alpha}\beta}}$. This shows that $(K_{\lambda,ik\bar{\beta}\alpha})$ can be regarded as a hermitian matrix of type (mr, mr).

Definition (1.4). E is called positive in the sense of S. Nakano [6] if there exists a hermitian inner product in E such that $(-K_{\lambda,ik\bar{\beta}\alpha})$ is positive definite everywhere.

Set $K_{\lambda,\bar{\beta}\alpha}=\sum_{i=1}^{r}K_{\lambda,}{}^{i}_{i\bar{\beta}\alpha}$. Then $K_{\lambda,\bar{\beta}\alpha}=\partial_{\alpha}\bar{\partial}_{\beta}\log h_{\lambda}$, where $h_{\lambda}=\det(h_{\lambda,k\bar{j}})$. The following is easily proved.

Proposition (1.5). *If E is positive, then*

$$-\Sigma K_{\lambda,\bar{\beta}\alpha}dz_{\lambda}^{\alpha}\wedge d\bar{z}_{\lambda}^{\beta}$$

is positive definite (1.1)*-form on X.*

Then we see that a positive vector bundle induces a kähler metric on X.

Now we shall restrict ourselves to a weakly 1-complete manifold with a positive vector bundle E. The positive metric is denoted by (1.1). Fix a real number c and consider X_c. Then X_c is also a weakly 1-complete manifold with respect to a complete pseudoconvex function

$$\psi=1\Big/\left(1-\frac{\Psi}{c}\right).$$

For a convex increasing function Λ, set

$$a_{\lambda}=h_{\lambda}^{-1}e^{\Lambda(\psi)}.$$

Then we have a kähler metric

(1.6) $$ds^{2}=\sum\frac{\partial^{2}\log a_{\lambda}}{\partial z_{\lambda}^{\alpha}\partial\bar{z}_{\lambda}^{\beta}}\,dz_{\lambda}^{\alpha}\cdot d\bar{z}_{\lambda}^{\beta}.$$

S. Nakano [7] proved

Propositon (1.7). *If* $\int^{\infty}\sqrt{A''(t)}\,dt=\infty$, *then* (1.6) *is a complete kähler metric on* X_c.

In what follows, we fix such a complete metric on X_c, which is denoted by

$$(1.8)\qquad ds^2=\Sigma g_{\lambda,\alpha\bar{\beta}}dz_\lambda^\alpha\cdot d\bar{z}_\lambda^\beta .$$

We define the metric form by

$$\Omega=\sqrt{-1}\,\Sigma g_{\lambda,\alpha\bar{\beta}}dz_\lambda^\alpha\wedge d\bar{z}_\lambda^\beta .$$

From this metric we can define a connection $\{\omega_\lambda\}$, $\omega_\lambda=(\omega_\lambda,{}^\beta_\alpha)$ in a well known manner:

$$(1.9)\qquad \omega_{\lambda,}{}^\beta_\gamma=\sum_{\alpha=1}^m \Gamma_{\lambda,}{}^\beta_{\alpha\gamma}dz_\lambda^\alpha \text{ where } \Gamma_{\lambda,}{}^\beta_{\alpha\gamma}=\sum_{\sigma=1}^m g_\lambda^{\bar{\sigma}\beta}\frac{\partial g_{\lambda,\gamma\bar{\sigma}}}{\partial z_\lambda^\alpha},$$

where $(g_\lambda^{\bar{\alpha}\beta})$ is the inverse of $(g_{\lambda,\alpha\bar{\beta}})$. The Riemann curvature tensor is defined by

$$R_{\lambda,}{}^\alpha_{\beta\bar{\gamma}\delta}=\frac{\partial\Gamma_{\lambda,}{}^\alpha_{\delta\beta}}{\partial\bar{z}_\lambda^\gamma},$$

and also we define

$$R_{\lambda,\bar{\alpha}\beta\bar{\gamma}\delta}=\sum_{\rho=1}^m g_{\lambda,\rho\bar{\alpha}}R_{\lambda,}{}^\rho_{\beta\bar{\gamma}\delta}.$$

As for the conjugates of the above, we define

$$\bar{\Gamma}_{\lambda,}{}^\alpha_{\beta\gamma}=\Gamma_{\lambda,}{}^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}},\ \overline{R_{\lambda,}{}^\rho_{\beta\bar{\gamma}\delta}}=R_{\lambda,}{}^{\bar{\rho}}_{\bar{\beta}\gamma\bar{\delta}} \text{ and } \overline{R_{\lambda,\bar{\alpha}\beta\bar{\gamma}\delta}}=R_{\lambda,\alpha\bar{\beta}\gamma\bar{\delta}}.$$

The Ricci form is defined by

$$R_{\lambda,\bar{\beta}\alpha}dz_\lambda^\alpha\wedge d\bar{z}_\lambda^\beta,\quad \text{where}\quad R_{\lambda,\bar{\beta}\alpha}=\sum_{\rho=1}^m R_{\lambda,}{}^\rho_{\rho\bar{\beta}\alpha}.$$

We infer that $\Gamma_{\lambda,}{}^\alpha_{\beta\gamma}=\Gamma_{\lambda,}{}^\alpha_{\gamma\beta}$, since the connection is induced from a kähler metric. The canonical line bundle K of X is defined to be

$$K=\{J_{\lambda\mu}\},\ \text{where } J_{\lambda\mu}=\frac{\partial(z_\mu^1,z_\mu^2,\dots,z_\mu^m)}{\partial(z_\lambda^1,z_\lambda^2,\dots,z_\lambda^m)} \text{ on } U_\lambda\cap U_\mu .$$

We see that

$$|J_{\lambda\mu}|^2=\frac{g_\lambda}{g_\mu} \qquad \text{on } U_\lambda\cap U_\mu \text{ where } g_\lambda=\det(g_{\lambda,\alpha\bar\beta}).$$

Therefore

(1.10)
$$\{g_\lambda^{-1}\}$$

determines a metric of K on X_c. The following is well known:

(1.11)
$$R_{\lambda,\bar\beta\alpha}=\partial_\alpha\bar\partial_\beta\log g_\lambda.$$

In what follows we choose $\{g_\lambda^{-1}\}$ as a metric of K and fix once for all. By using (1.1) and (1.10), we define a hermitian inner product in $E\otimes K$.

(1.12)
$$(\tilde h_{\lambda,k\bar j}) \qquad \text{where} \quad \tilde h_{\lambda,k\bar j}=g_\lambda^{-1}h_{\lambda,k\bar j}.$$

Also for a convex increasing function χ, we take

(1.13)
$$(e^{-\chi(\psi)}\tilde h_{\lambda,k\bar j}).$$

Then we get another inner product in $E\otimes K$. The Riemann curvature tensor induced from (1.13) is denoted by

$$K^{(\chi)i}_{\lambda,\ k\bar\alpha\beta}.$$

We see

(1.14)
$$K^{(\chi)i}_{\lambda,\ j\bar\alpha\beta}=K_{\lambda,\ j\bar\alpha\beta}^{\ i}-\delta^i_j\partial_\alpha\bar\partial_\beta\chi(\psi)-\delta^i_j\partial_\alpha\bar\partial_\beta\log g_\lambda.$$

§2. Differential and Integral Caluculus of $E\otimes K$-valued Forms

We recall differential and integral caluculus of $E\otimes K$-valued forms on X_c. Let $C_{p,q}(X_c, E\otimes K)$ denote the space of C^∞-differentiable $E\otimes K$-valued (p, q)-forms on X_c and let $\mathscr{D}_{p,q}(X_c, E\otimes K)=\{\varphi\in C_{p,q}(X_c, E\otimes K)$: the support of φ is compact$\}$. We express $\varphi=(\varphi^j_\lambda)\in C_{p,q}(X_c, E\otimes K)$ as

$$\varphi^j_\lambda=\frac{1}{p!\,q!}\sum_{\alpha_1,\dots,\alpha_p}\sum_{\beta_1,\dots,\beta_q}(\varphi)^j_{\lambda,\alpha_1\dots\alpha_p,\bar\beta_1\dots\bar\beta_q}dz^{\alpha_1}_\lambda\wedge dz^{\alpha_2}_\lambda\wedge\cdots\wedge d\bar z^{\beta_q}_\lambda.$$

For $\varphi\in C_{p,q}(X_c, E\otimes K)$, we set

$$(\varphi)_{\lambda}^{j,\bar{\alpha}_1\ldots\bar{\alpha}_p,\beta_1\ldots\beta_q}=\Sigma g^{\bar{\alpha}_1\tau_1}\cdot g^{\bar{\alpha}_2\tau_2}\cdot\ldots\cdot g^{\bar{\alpha}_p\tau_p}\cdot g^{\bar{\sigma}_1\beta_1}\cdot g^{\bar{\sigma}_2\beta_2}\cdot\ldots\cdot g^{\bar{\sigma}_q\beta_q}$$

$$\times(\varphi)_{\lambda,\tau_1\tau_2\ldots\tau_p,\bar{\sigma}_1\bar{\sigma}_2\ldots\bar{\sigma}_q}^{j}\,.$$

Particularly when φ is a $(0, q)$-form, we write $(\varphi)_{\lambda,\bar{\beta}_1\ldots\bar{\beta}_q}^{j}$ and $(\varphi)_{\lambda}^{j\beta_1\ldots\beta_q}$. With respect to (1.8) and (1.12) we define a hermitian inner product in $C_{p,q}(X_c, E\otimes K)$ as follows: For φ and $\psi\in C_{p,q}(X_c, E\otimes K)$

$$H_{\tilde{h}}(\varphi, \psi)=\Sigma\tilde{h}_{\lambda,k\bar{j}}(\varphi)_{\lambda,\alpha_1\ldots\alpha_p,\bar{\beta}_1\ldots\bar{\beta}_q}^{k}\overline{(\psi)_{\lambda}^{j,\bar{\alpha}_1\ldots\bar{\alpha}_p,\beta_1\ldots\beta_q}}\,.$$

Also we define

$$H_{\chi}(\varphi, \psi)=e^{-\chi(\psi)}H_{\tilde{h}}(\varphi, \psi)\,.$$

We define

$$(\varphi, \psi)_{\tilde{h}}=\int_{X_c}H_{\tilde{h}}(\varphi, \psi)dV, \tag{2.1}$$

$$(\varphi, \psi)_{\chi}=\int_{X_c}H_{\chi}(\varphi, \psi)dV \qquad \text{for} \quad \varphi, \psi\in\mathscr{D}_{p,q}(X_c, E\otimes K) \tag{2.2}$$

where $dV=\frac{1}{m!}\,\Omega\wedge\Omega\wedge\cdots\wedge\Omega$($m$-times). Particularly when $\varphi=\psi$, we denote $(\varphi, \varphi)_{\chi}$ (resp. $(\varphi, \varphi)_{\tilde{h}}$) by $\|\varphi\|_{\chi}^{2}$ (resp. $\|\varphi\|_{\tilde{h}}^{2}$). $\bar{\partial}$: $C_{p,q}(X_c, E\otimes K)\to C_{p,q+1}(X_c, E\otimes K)$ is defined as usual. With respect to (2.2) (resp. (2.1)) the formally adjoint operator is defined, which is denoted by ϑ_{χ} (resp. $\vartheta_{\tilde{h}}$). The Laplace-Beltrami operator $\square_{\chi}$ is defined by $\square_{\chi}=\bar{\partial}\vartheta_{\chi}+\vartheta_{\chi}\bar{\partial}$.

Let $\mathscr{T}_{p,q}(X_c, E\otimes K)$ denote $E\otimes K$-valued covariant tensor fields of type (p, q). We write the $(\alpha_1\ldots\alpha_p, \bar{\beta}_1\ldots\bar{\beta}_q)$-component of $\varphi\in\mathscr{T}_{p,q}(X_c, E\otimes K)$, $(\varphi)_{\lambda,\alpha_1\ldots\alpha_p,\bar{\beta}_1\ldots\bar{\beta}_q}^{j}$. The connections (1.2) and (1.9) derive covariant differentiations $\nabla_{\alpha}^{(x)}$ of type (1, 0) and $\overline{\nabla}_{\beta}^{(x)}$ of type (0, 1) in $\mathscr{T}_{p,q}(X_c, E\otimes K)$ respectively:

$$(\nabla_{\alpha}^{(x)}\varphi)_{\lambda,\alpha_1\ldots\alpha_p\alpha,\bar{\beta}_1\ldots\bar{\beta}_q}^{j}=\frac{\partial(\varphi)_{\lambda\alpha_1\ldots\alpha_p,\bar{\beta}_1\ldots\bar{\beta}_q}^{j}}{\partial z_{\lambda}^{\alpha}}$$

$$+\sum_{s=1}^{r}\Gamma_{\lambda,\alpha\ s}^{*(x)j}(\varphi)_{\lambda,\alpha_1\ldots\alpha_p,\bar{\beta}_1\ldots\bar{\beta}_q}^{s}-\sum_{t=1}^{q}\sum_{\tau=1}^{m}\Gamma_{\lambda,\alpha\alpha_t}^{\ \tau}(\varphi)_{\lambda,\alpha_1\ldots\overset{t}{\downarrow}{\tau}\ldots\alpha_p,\bar{\beta}_1\ldots\bar{\beta}_q}^{j}\,,$$

where $\Gamma_{\lambda,\alpha\ s}^{*(x)j}$ denotes the connection coefficients defined from (1.12) as in (1.2), and

$$(\overline{\nabla}_{\beta}^{(\chi)}\varphi)^{j}_{\lambda,\alpha_1\ldots\alpha_p,\bar\beta_1\ldots\bar\beta_q\bar\beta}=\frac{\partial(\varphi)^{j}_{\lambda,\alpha_1\ldots\alpha_p,\bar\beta_1\ldots\bar\beta_q}}{\partial\bar z^{\beta}_{\lambda}}$$

$$-\sum_{t=1}^{q}\sum_{\tau=1}^{m}\Gamma^{\bar\tau}_{\lambda,\bar\beta\bar\beta_t}(\varphi)^{j}_{\lambda,\alpha_1\ldots\alpha_p,\bar\beta_1\ldots\overset{t}{\downarrow}{\bar\tau}\ldots\bar\beta_q}.$$

Then we obtain the commutation formula: For $\varphi\in\mathscr{T}_{p,q}(X_c, E\otimes K)$,

$$(2.3)\quad ([\nabla_{\alpha}^{(\chi)}\overline{\nabla}_{\beta}^{(\chi)}-\overline{\nabla}_{\beta}^{(\chi)}\nabla_{\alpha}^{(\chi)}]\varphi)^{j}_{\lambda,\alpha_1\ldots\alpha_p\alpha,\bar\beta_1\ldots\bar\beta_q\bar\beta}$$

$$=-\sum_{t=1}^{q}\sum_{\sigma=1}^{m}R_{\lambda,\bar\beta_t\alpha\bar\beta}^{\ \ \bar\sigma}(\varphi)^{j}_{\lambda,\alpha_1\ldots\alpha_p,\bar\beta_1\ldots\overset{t}{\downarrow}{\bar\sigma}\ldots\bar\beta_q}$$

$$+\sum_{t=1}^{p}\sum_{\tau=1}^{m}R_{\lambda,\alpha_t\bar\beta\alpha}^{\ \ \tau}(\varphi)^{s}_{\lambda,\alpha_1\ldots\overset{t}{\downarrow}{\tau}\ldots\alpha_p,\bar\beta_1\ldots\bar\beta_q}+\sum_{s=1}^{r}K^{(\chi)j}_{\lambda,\ s\bar\beta\alpha}(\varphi)^{s}_{\lambda,\alpha_1\ldots\alpha_p,\bar\beta_1\ldots\bar\beta_q}.$$

In the same manner as in Kodaira and Morrow [5] (see, p. 110, Proposition (5.3) and Theorem (5.2), and p. 112, Proposition (6.7)), we get for $\varphi\in\mathscr{D}_{p,q}(X_c, E\otimes K)$,

$$(2.4)\quad\begin{cases}(\bar\partial\varphi)^{j}_{\lambda,\alpha_1\ldots\alpha_p,\bar\beta_0\ldots\bar\beta_q}=\sum_{\mu=0}^{q}(-1)^{\mu+p}\overline{\nabla}_{\beta_\mu}^{(\chi)}(\varphi)^{j}_{\lambda,\alpha_1\ldots\alpha_p,\bar\beta_0\ldots\hat{\bar\beta}_\mu\ldots\bar\beta_q},\\ (\vartheta_\chi\varphi)^{j}_{\lambda,\alpha_1\ldots\alpha_p,\bar\beta_1\ldots\bar\beta_{q-1}}=-\sum_{\alpha,\beta=1}^{m}g^{\bar\beta\alpha}\nabla_{\alpha}^{(\chi)}(\varphi)^{j}_{\lambda,\bar\beta\alpha_1\ldots\alpha_p,\bar\beta_1\ldots\bar\beta_{q-1}}.\end{cases}$$

Therefore

$$(\Box_\chi\varphi)^{j}_{\lambda,\alpha_1\ldots\alpha_p,\bar\beta_1\ldots\bar\beta_q}=-\sum_{\alpha_0\beta_0}g^{\bar\beta_0\alpha}\nabla_{\alpha}^{(\chi)}\overline{\nabla}_{\beta_0}^{(\chi)}(\varphi)^{j}_{\lambda,\alpha_1\ldots\alpha_p,\bar\beta_1\ldots\bar\beta_q}$$

$$-\sum_{\alpha,\beta}\sum_{\mu=1}^{q}(-1)^{\mu}g^{\bar\beta\alpha}(\nabla_{\alpha}^{(\chi)}\overline{\nabla}_{\beta_\mu}^{(\chi)}-\nabla_{\beta_\mu}^{(\chi)}\nabla_{\alpha}^{(\chi)})(\varphi)^{j}_{\lambda,\alpha_1\ldots\alpha_p,\bar\beta\bar\beta_1\ldots\hat{\bar\beta}_\mu\ldots\bar\beta_q}.$$

In what follows we consider only $(0, q)$-forms. Then by (2.3)

$$(\Box_\chi\varphi)^{j}_{\lambda,\bar\beta_1\ldots\bar\beta_q}=-\sum_{\alpha\beta}g^{\bar\beta\alpha}\nabla_{\alpha}^{(\chi)}\overline{\nabla}_{\beta}^{(\chi)}(\varphi)^{j}_{\lambda\ ,\bar\beta_1\ldots\bar\beta_q}$$

$$-\sum R^{\bar\tau}_{\bar\beta_\mu}(\varphi)^{j}_{\lambda,\ \bar\beta_1\ldots\underset{\mu}{\uparrow}{\bar\tau}\ldots\bar\beta_q}-\sum g^{\bar\tau\alpha}K^{(\chi)j}_{\lambda,\ s\bar\beta_\mu\alpha}(\varphi)^{s}_{\lambda,\ \bar\beta_1\ldots\overset{\mu}{\downarrow}{\bar\tau}\ldots\bar\beta_q}$$

where $$R^{\bar\tau}_{\bar\beta_\mu}=\sum g^{\bar\tau\alpha}R_{\lambda,\bar\beta_\mu\alpha}.$$

Then

$$H_\chi(\square_\chi\varphi,\ \varphi)=-\sum e^{-\chi(\psi)}\tilde{h}_{\lambda,k\bar{j}}g^{\bar{\beta}\alpha}\nabla^{(\chi)}_\alpha\overline{\nabla}^{(\chi)}_\beta(\varphi)^k_{\lambda,\bar{\beta}_1\dots\bar{\beta}_q}\overline{(\varphi)_\lambda{}^{j\beta_1\dots\beta_q}}$$
$$-\sum e^{-\chi(\psi)}\tilde{h}_{\lambda,k\bar{j}}R^{\bar{\tau}}_{\bar{\beta}_\mu}(\varphi)^k_{\lambda,\bar{\beta}_1\dots\underset{\mu}{\bar{\tau}}\dots\bar{\beta}_q}\overline{(\varphi)^{j\beta_1\dots\beta_q}_\lambda}$$
$$-\sum e^{-\chi(\psi)}\tilde{h}_{\lambda,k\bar{j}}g^{\bar{\tau}\alpha}K^{(\chi)k}_{\lambda,\ s\bar{\beta}_\mu\alpha}(\varphi)^k_{\lambda,\bar{\beta}_1\dots\underset{\mu}{\bar{\tau}}\dots\bar{\beta}_q}\overline{(\varphi)^{j\beta_1\dots\beta_q}_\lambda}\,.$$

As in Kodaira and Morrow [5] (see, p. 126), we can prove

$$-\sum\int_{X_c}e^{-\chi(\psi)}g^{\bar{\beta}\alpha}_\lambda\tilde{h}_{\lambda,k\bar{j}}\nabla^{(\chi)}_\alpha\overline{\nabla}^{(\chi)}_\beta(\varphi)^k_{\lambda,\bar{\beta}_1\dots\bar{\beta}_q}\overline{(\varphi)^{k,\beta_1\dots\beta_q}_\lambda}dV\geqq 0\,.$$

Thus we obtain

$$(2.5)\qquad (\square_\lambda\varphi,\ \varphi)_\chi\geqq-\int_{X_c}\sum e^{-\chi(\psi)}\tilde{h}_{\lambda,k\bar{j}}R^{\bar{\tau}}_{\bar{\beta}_\mu}(\varphi)^k_{\lambda,\bar{\beta}_1\dots\overset{\mu}{\bar{\tau}}\dots\bar{\beta}_q}\overline{(\varphi)^{j\beta_1\dots\beta_q}_\lambda}dV$$
$$-\int_{X_c}\sum e^{-\chi(\psi)}g^{\bar{\tau}\alpha}\tilde{h}_{\lambda,k\bar{j}}K^{(\chi)k}_{\lambda,\ s\bar{\beta}_\mu\alpha}(\varphi)^s_{\lambda,\beta_1\dots\underset{\mu}{\bar{\tau}}\dots\beta_q}\overline{(\varphi)^{j\beta_1\dots\beta_q}_\lambda}dV.$$

Referring to (1.11) and (1.14), the second term of the right side of (2.5) becomes

$$-\int_{X_c}\sum e^{-\chi(\psi)}\tilde{h}_{\lambda,k\bar{j}}g^{\bar{\tau}\alpha}K_{\lambda,\ s\bar{\beta}_\mu\alpha}^{\ k}(\varphi)^s_{\lambda,\bar{\beta}_1\dots\overset{\mu}{\bar{\tau}}\dots\bar{\beta}_q}\overline{(\varphi)^{j\beta_1\dots\beta_q}_\lambda}\,dV$$
$$(2.6)\qquad +\int_{X_c}\sum e^{-\chi(\psi)}\tilde{h}_{\lambda,k\bar{j}}R^{\bar{\tau}}_{\bar{\beta}_\mu}(\varphi)^k_{\lambda,\bar{\beta}_1\dots\overset{\mu}{\bar{\tau}}\dots\bar{\beta}_q}\overline{(\varphi)^{j\beta_1\dots\beta_q}_\lambda}\,dv$$
$$+\int_{X_c}\sum e^{-\chi(\psi)}\tilde{h}_{\lambda,k\bar{j}}g^{\bar{\tau}\alpha}\partial_\alpha\bar{\partial}_\beta\chi(\psi)(\varphi)^k_{\lambda,\bar{\beta}_1\dots\underset{\mu}{\bar{\tau}}\dots\bar{\beta}_q}\overline{(\varphi)^{j\beta_1\dots\beta_q}_\lambda}\,dV.$$

Here note that since $\partial\bar{\partial}(\psi)\geqq 0$, the last term in (2.6) is non-negative and that the first term in (2.5) and the second term in (2.6) cancel each other. Finally we obtain

Theorem 3. *For* $\varphi\in\mathscr{D}_{0,q}(X_c,\ E\otimes K)$, *we have*

$$(\square_\chi\varphi,\varphi)_\chi\geqq-\int_{X_c}\sum e^{-\chi(\psi)}\tilde{h}_{\lambda,k\bar{j}}g^{\bar{\tau}\alpha}K_{\lambda,\ s\bar{\beta}_\mu\alpha}^{\ k}\,(\varphi)^s_{\lambda,\bar{\beta}_1\dots\overset{\mu}{\bar{\tau}}\dots\bar{\beta}_q}\overline{(\varphi)^{j\beta_1\dots\beta_q}_\lambda}\,dV.$$

§3. Proofs of Theorems 1 and 2

First we prove Theorem 1. Making completion of $\mathscr{D}_{0,q}(X_c,\ E\otimes K)$

with respect to $\|\varphi\|_{\chi}^{2}$ (resp. $\|\varphi\|_{\tilde{h}}^{2}$) we obtain a Hilbert space $\mathscr{L}_{0,q}^{2}(X_c, E\otimes K, \chi)$ (resp. $\mathscr{L}_{0,q}^{2}(X_c, E\otimes K, \tilde{h})$). We extend $\bar{\partial}: \mathscr{D}_{0,q}(X_c, E\otimes K)\to\mathscr{D}_{0,q+1}(X_c, E\otimes K)$ (resp. $\bar{\partial}: \mathscr{D}_{0,q+1}(X_c, E\otimes K)\to\mathscr{D}_{0,q+2}(X_c, E\otimes K)$) to the differential operator in the sense of distribution, which is denoted by $T: \mathscr{L}_{0,q}^{2}(X_c, E\otimes K, \chi)\to\mathscr{L}_{0,q+1}^{2}(X_c, E\otimes K, \chi)$ (resp. $S: \mathscr{L}_{0,q+1}^{2}(X_c, E\otimes K, \chi)\to\mathscr{L}_{0,q+2}^{2}(X_c, E\otimes K, \chi)$). Then T (resp. S) is a densely defined closed operator, so the adjoint operator T^* (resp. S^*) can be defined.
Consider

$$\mathscr{L}_{0,q}^{2}(X_c, E\otimes K, \chi) \underset{T^*}{\overset{T}{\rightleftarrows}} \mathscr{L}_{0,q+1}^{2}(X_c, E\otimes K, \chi) \underset{S^*}{\overset{S}{\rightleftarrows}} \mathscr{L}_{0,q+2}^{2}(X_c, E\otimes K, \chi).$$

First we infer that by the completeness of ψ, there exists a convex increasing function χ such that $\|\varphi\|_{\chi}^{2}<+\infty$ for any $\varphi\in C_{0,q}(X_c, E\otimes K)$. Then in view of Dolbault isomorphism and a lemma on L^2-estimate (see, Hörmander [3], p. 78, Lemma 4.1.1), it is sufficient for the proof of Theorem 1 to prove the following

Theorem 4. *There exists a positive constant C_0 which does not depend on the choice of χ such that*

$$(*)\qquad \|\varphi\|_{\chi}^{2}\leqq C_0(\|T^*\varphi\|_{\chi}^{2}+\|S\varphi\|_{\chi}^{2})\qquad \textit{for}\quad \varphi\in D(T^*)\cap D(S),$$

where

$$D(T^*)=\{\varphi\in\mathscr{L}_{0,q+1}^{2}(X_c, E\otimes K, \chi):\ T^*\varphi\in\mathscr{L}_{0,q}^{2}(X_c, E\otimes K, \chi)\},$$

$$D(S)=\{\varphi\in\mathscr{L}_{0,q+1}^{2}(X_c, E\otimes K, \chi):\ S\varphi\in\mathscr{L}_{0,q+2}^{2}(X_c, E\otimes K, \chi)\}.$$

Proof. By the choice of the base metric, it is a complete metric. So referring to a key lemma which is due to A. Andreotti and E. Vesentini [1] (see, p. 93, Proposition 5), we have only to prove $(*)$ for $\varphi\in\mathscr{D}_{0,q+1}(X_c, E\otimes K)$. Let C denote the minimum of the eigen values of $(-K_{\lambda,\bar{k}j\beta\bar{\alpha}})$ on X_c, then we see that $C>0$. Thus by Theorem 3 we have

$$(\square_{\chi}\varphi, \varphi)_{\chi}\geqq C_0\|\varphi\|_{\chi}^{2},\qquad \text{where}\quad C_0=(q+1)C,$$

which proves $(*)$.

Next we prove Theorem 2. We follow the proof given in the approximation theorem on Stein manifolds (see, L. Hörmander [3], p. 89–90). For $E\otimes K$-valued forms φ and ψ, we set

$$(\varphi, \psi)_{\tilde{h}|d}=\int_{X_d} H_{\tilde{h}}(\varphi, \psi)dV.$$

To prove Theorem 2 it is sufficient to show that if $u\in\mathscr{L}^2_{0,0}(X_d, E\otimes K, \chi)$ satisfies $(u, \varphi)_{\tilde{h}|d}=0$ for any $\varphi\in H^0(X_c, \mathcal{O}(E\otimes K))$, then $(u, \tilde{\varphi})_{\tilde{h}|d}=0$ for any $\tilde{\varphi}\in H^0(\bar{X}_d, \mathcal{O}(E\otimes K))$. Take such a u. We extend u by setting 0 outside of $\bar{X}_d$ and denote it by the same latter u. Let N_T be the null space of T: $\mathscr{L}^2_{0,0}(X_c, E\otimes K, \chi)\to\mathscr{L}^2_{0,1}(X_c, E\otimes K, \chi)$, then we see that

$$N_T^{\perp}=H^0(X_c, \mathcal{O}(E\otimes K))\cap\mathscr{L}^2_{0,0}(X_c, E\otimes K, \chi),$$

where $N_T^{\perp}$ denotes the orthogonal complement of N_T. So $ue^{\chi(\psi)}$ is contained in $N_T^{\perp}$. By a lemma due to L. Hörmander [3] (see, p. 79, Lemma 4.1.2) we see that there exists an $f\in\mathscr{L}^2_{0,1}(X_c, E\otimes K, \chi)$ such that

$$\text{(3.1)} \qquad ue^{\chi(\psi)}=T^*f \quad \text{and} \quad \|f\|^2_{\chi}\leqq C_0\|u\|^2_{\chi}.$$

Set $g=e^{-\chi(\psi)}f$. Then we have by (3.1)

$$u=\vartheta_{\tilde{h}}g.$$

Now we choose a sequence of functions $\{\chi_\nu\}$ such that (1) χ_ν is a convex increasing function, (2) $\chi_\nu\geqq\chi_1$ for each ν, (3) $\chi_\nu(t)=1$ if $t\leqq d$ and (4) for any $t\in(d, c)$ $\chi_\nu(t)\to\infty$ $(\nu\to\infty)$.

For each χ_ν we get $g^{(\nu)}$. By (3.1) and (3) there exists a positive constant M which does not depend on ν such that

$$\int_{X_c} e^{\chi_\nu(\psi)}H_{\tilde{h}}(g^{(\nu)}, g^{(\nu)})dV\leqq M.$$

Then $g^{(\nu)}\in\mathscr{L}^2_{0,0}(X_c, E\otimes K, -\chi_1)$ and $g^{(\nu)}$ is bounded. Therefore there exists a subsequence which converges weakly to a limit g_0. By (4) we see that $g_0=0$ on X_c-X_d. Also by the continuity of differentiation in the sense of distribution, we have $u=\vartheta_{\tilde{h}}g_0$. Therefore, $(u, \alpha)_{\tilde{h}}=(g_0, \bar{\partial}\alpha)_{\tilde{h}}$ for $\alpha\in\mathscr{D}_{0,0}(X_c, E\otimes K)$. Take $\tilde{\varphi}\in H^0(X_d, \mathcal{O}(E\otimes K))$ and extend $\tilde{\varphi}$ to $\tilde{\varphi}^*$ such that $\tilde{\varphi}^*\in\mathscr{D}_{0,0}(X_c, E\otimes K)$. Then we see that $(u, \tilde{\varphi})_{\tilde{h}|d}=0$, which proves Theorem 2.

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