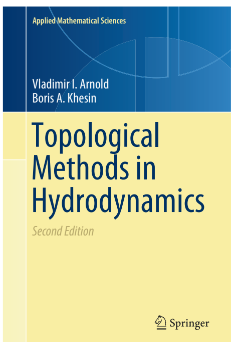
Topological Methods in Hydrodynamics
by Vladimir I. Arnold and Boris A. Khesin

Reviewed by Daniel Peralta-Salas



The theoretical study of fluid flows is a vast area of research that involves many different mathematical disciplines, ranging from the theory of partial differential equations to dynamical systems and differential geometry. More than 250 years after their formulation, the Euler equations (which describe an ideal incompressible fluid) and their viscid counterpart, the Navier–Stokes equations (introduced independently by Claude-Louis Navier and Gabriel Stokes during the first half of the 19th century) still contain a wealth of fundamental open problems. While there are numerous and excellent monographs focusing on the analytic aspects of the equations that govern fluid motions, until recently one can hardly find textbooks on the geometric and topological aspects of fluid flows (which are very rich and significant for understanding hydrodynamics). In 1998 this important gap was filled by the first edition of the book "Topological Methods in Hydrodynamics" by V. I. Arnold and B. Khesin. It is difficult to overestimate the impact this monograph had on those mathematicians who are interested in understanding the equations of fluid mechanics from a geometric viewpoint, as it provides a comprehensive introduction to most of the more remarkable achievements in the area. This includes Arnold's geodesic formulation of the Euler equations, the structure of steady Euler flows, the topological interpretation of helicity and other asymptotic invariants, the effects of the curvature of the group of volume-preserving diffeomorphisms on hydrodynamic instabilities or the fast dynamo problem. More than twenty years later this important book has seen its second edition, the most remarkable novelty being the addition of a very valuable fifty-page appendix that introduces the most significant developments in the area since the publication of the first edition of the book.

Arnold and Khesin's book is the only monograph that presents a thorough introduction to topological fluid mechanics, a young area of research that flourished after the foundational works of Arnold and Moffatt in the 1960's. The interest in the topological and geometric aspects of fluid dynamics probably dates back to Lord Kelvin, who developed an atomic theory in which atoms were understood as knotted vortex tubes in the ether and showed that vorticity is transported by the fluid field in the context of ideal flows, thus implying the preservation of all the vortex structures. In modern times, topological hydrodynamics was considerably developed after the works of Arnold and Moffatt. Arnold realized that the Euler equations of hydrodynamics can be understood as geodesic motions on the infinite-dimensional group of volume-preserving diffeomorphisms and Moffatt unveiled the connection between helicity and the entangledness and knottedness of the fluid. The book under review covers a vast panorama of developments and results in the area, and is an indispensable reference for any researcher interested in fluid mechanics from the geometric, topological or Hamiltonian perspectives. It is impossible to summarize the contents of this book in a few lines, so next I aim to present its chapters, highlighting some landmarks that are introduced in each chapter (paying the price of losing many other interesting results in this short presentation):

Chapter I. Group and Hamiltonian structures of fluid dynamics. This chapter is mainly focused on the study of the Euler equations of ideal fluids from the viewpoints of group theory and Hamiltonian mechanics. A significant part is devoted to developing Arnold's theory relating the Euler equations with the geometry of the infinite-dimensional Lie group of volume-preserving diffeomorphisms of the fluid flow domain. In an important article published in 1966 Arnold showed that the dynamics of an ideal fluid flow can be described by the geodesics on the aforementioned Lie group endowed with the right-invariant metric given by the kinetic energy. This chapter provides a detailed presentation of this result and how it fits within the general framework of the Euler–Poincaré equations for Hamiltonian systems on Lie groups whose action is (right-)invariant, other remarkable examples being the dynamics of the rigid body or the KdV equation. Using this geometric formulation, Ebin and Marsden proved in 1970 the local-in-time existence of solutions to the Euler equations on compact manifolds, both in Sobolev and Hölder classes. The chapter also deals with conserved quantities of the Euler equations (mainly the Casimirs of the adjoint action of the group of volume-preserving diffeomorphisms) and the group setting of ideal magnetohydrodynamics.

Chapter II. Topology of steady fluid flows. This chapter is concerned with the stationary solutions of the Euler equations. It presents in a very detailed way two gems proved by Arnold in the mid 1960s. The first one, nowadays known as Arnold's structure theorem, describes the topological and dynamical structure of analytic 3D fluid steady states in bounded domains whose Bernoulli function is not constant. Under these assumptions, this theorem shows that the Euler flows exhibit the same properties as integrable Hamiltonian systems with two degrees of freedom on an energy hypersurface: presence of subdomains covered by invariant tori or invariant cylinders supporting dynamics that is conjugate to a linear one. In the context of 2D steady states, the second result presented here is Arnold's stability theorem, which provides a sufficient condition for a planar stationary solution to be Lyapunov stable with respect to the L^2-norm of the vorticity. This remarkable result exploits a new variational characterization of steady states discovered by Arnold (in terms of the critical points of the energy functional on the coadjoint orbits of the group of volume-preserving diffeomorphisms) and the Hamiltonian formulation. The topology of the famous Arnold–Beltrami–Childress (ABC) flows, properties of the linearized Euler equations, and Nadirashvili's surprising construction of wandering solutions to the 2D Euler equations on annular regions are also discussed.

Chapter III. Topological properties of magnetic and vorticity fields. In this chapter the authors review several results on the topology of solenoidal fields and how it affects energy relaxation in physical processes, such as ideal MHD evolution. This topology is described using functionals on the space of vorticity fields, most of them of "asymptotic type," which means that the functional is defined using a knot invariant, the integral curves of the field and suitable averages. The chapter presents the helicity functional and its topological interpretation in terms of the linking number discovered by Moffatt in 1969, as well as the connection with the asymptotic linking number introduced by Arnold in 1973. Arnold proved a beautiful theorem asserting that these two apparently very different quantities (the former defined using the Riemannian metric and differential forms, and the latter using the flow of the field and a limit process) coincide, thus extending Moffatt's topological interpretation to arbitrary solenoidal fields. Other remarkable theorems covered in this chapter include lower bounds on energy under ideal relaxation using the helicity and Freedman–He's asymptotic crossing number, and Freedman's remarkable proof of the Sakharov–Zeldovich energy minimization conjecture.

Chapter IV. Differential geometry of diffeomorphism groups. This chapter deals with the geometry (from a Riemannian viewpoint) of the infinite-dimensional group of volume-preserving diffeomorphisms, endowed with the right-invariant metric given by the L^2-norm of the velocity field. It pays special attention to the curvature of the group and how it is related to instabilities of the Euler dynamics. Under suitable assumptions, the curvature of the group of volume-preserving diffeomorphisms is negative along many directions, which in view of the geodesic nature of the Euler flow on that group leads to exponential separation of the Lagrangian trajectories of the fluid. An appealing consequence of this claim is that the weather forecast becomes unreliable after a sufficiently long time, a striking consequence of the Riemannian geometry of the diffeomorphism group! Other interesting studies, such as the existence of conjugate points on the aforementioned Lie group, Shnirelman's description of the diameter of the diffeomorphism group, and Brenier's theory of generalized flows are also discussed.

Chapter V. Kinematic fast dynamo problems. This chapter deals with the equation describing the evolution of magnetic fields in magnetohydrodynamics, i.e., the kinematic dynamo equation. When the fluid is a perfect conductor, the magnetic diffusivity is zero and the magnetic field is transported by the velocity-field flow; in the general case the diffusivity appears as a diffusion term of heat type. The chapter describes several results on the existence of fast dynamos (both for the dissipative and non-dissipative models), which are solenoidal fields that give rise to exponential growth in time of the L^2-norm of the magnetic field. This includes a thorough presentation of the connections between the exponential dynamo growth and the Lyapunov exponents, the topological entropy and homoclinic intersections of the velocity field. The authors also present some discrete models (mainly area-preserving diffeomorphisms on surfaces) of fast dynamos, highlighting a very detailed discussion of Arnold's cat map (a paradigmatic model of an Anosov diffeomorphism on the 2-torus). The antidynamo theorem proved by Cowling and some of its generalizations are also discussed.

Chapter VI. Dynamical systems with hydrodynamic background. The final chapter of the first edition of the book is a survey of various partial differential equations that can be studied from the group-theoretic viewpoint presented in Chapter I, i.e., as geodesics of an infinite-dimensional Lie group of symmetries endowed with a right-invariant metric. This includes the KdV equation (related to the Virasoro group), the equations of gas dynamics and compressible fluids, and the filament and nonlinear 1D Schrödinger equations. While the material covered in this chapter is not directly related to the Euler or Navier–Stokes equations, it is very valuable in the sense that it shows the power of the general framework of geodesic motions on Lie groups for studying the evolution of some PDEs.

Appendix. Recent developments in topological hydrodynamics. This chapter is the main new addition to the second edition of Arnold–Khesin's book. It contains an update of the new developments in the area of topological fluid mechanics since 1999 (so we can strictly speak about XXI century mathematics). The material covered by this chapter is huge, so necessarily not very detailed, but with a vast number of references and indications that certainly help the reader to find further results on each subject. The chapter summarizes new remarkable achievements in all the topics of previous chapters, a non-exhaustive list including: the recently obtained classification of Casimirs for 2D and 3D vorticities, the extension of Arnold's geodesic framework to the context of weak solutions of the Euler equations (exhibiting vortex sheets), the realization theorems for knotted vortex lines and tubes in Beltrami flows, a KAM type approach to study ergodicity and mixing properties of the Euler flow, and the connection between problems of optimal mass transport and the evolution of ideal fluids.

Overall Arnold and Khesin's book is a beautiful and extensive introduction to fluid mechanics from a geometric viewpoint. It is a pleasure to read and each chapter contains very valuable material not only for those mathematicians working with the equations of hydrodynamics, but for any researcher interested in the connections between analysis, geometry and topology. I am sure that any professional mathematician can find food for thought in some of the gems that are presented in this monograph. Certainly this was my case as a graduate student in Madrid twenty years ago.

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