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Generation of vortices in the Ginzburg–Landau heat flow

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Abstract. We consider the Ginzburg–Landau heat flow on the flat two-dimensional torus, starting from initial data with a finite number of nondegenerate zeros – but very high initial energy. We show that the initial zeros are conserved, while away from these zeros the modulus quickly grows close to 1, and the flow rapidly enters a logarithmic energy regime, from which the evolution of vortices can be described by the works of Bethuel, Orlandi and Smets.

1. Introduction

In the flat two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ we consider $u(t,x)$, a solution of the Ginzburg–Landau heat flow

$$\begin{aligned} \partial_t u - \varepsilon^2 \Delta u &= (1-|u|^2)u \quad t \geq 0,\ x \in \mathbb{T}^2, \\ u(0,x) &= u_0(x), \end{aligned} \tag{1.1}$$

with $u_0 \in C^1(\mathbb{T}^2)$. The initial condition u_0 may have a finite number of zeros. More precisely, we assume that there exists $\alpha_0 > 0$ such that

$$|u_0(x)| + |\det \nabla u_0(x)| \geq \alpha_0. \tag{1.2}$$

This implies in particular that the zeros of u_0 are nondegenerate and the topological degree of the vector field u_0 at each zero is 1 or -1.

We will denote the energy associated with (1.1) by

$$E_\varepsilon(u) = \int_{\mathbb{T}^2} |\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2.$$

Note that (1.1) is the L^2 gradient flow of E_ε up to a factor ε^2, hence E_ε is decreasing along the flow. The Ginzburg–Landau heat flow has been extensively studied [2, 4, 5, 10, 13, 15–17, 19] in the case of initial data $u_0 = u_{0\varepsilon}$ satisfying a logarithmic energy bound $E_\varepsilon(u_{0\varepsilon}) \leq M \ln(1/\varepsilon)$. This bound enables one to identify vortices, the zeros of $u_{0\varepsilon}$, and to describe their evolution. More precisely, in [10, 13, 15], well-prepared initial data are considered, with a finite number of vortices of degree ± 1 and correspondingly

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quantized energy. These works establish via different methods that, in the accelerated timescale $s = (\varepsilon^2/\ln(1/\varepsilon))t$, vortices move according to the gradient flow of a renormalized energy analyzed in [3], for as long as no collisions happen. This limitation is removed in the works [5, 17, 19], where splittings and collisions of vortices are described rigorously. Specifically, [17] describes the global-in-time motion of vortices, taking collisions into account, in bounded domains with Dirichlet or Neumann boundary conditions. Initial well-preparedness is also relaxed: initial vortices are of degree ± 1, but the energy quantization assumption is less stringent; moreover, splitting of higher-degree vortices into vortices of degree ± 1 is described under specific assumptions. In [5,19], the domain is the whole plane and a global motion law allowing for splittings and collisions is obtained, for initial data satisfying the logarithmic bound $E_\varepsilon(u_{0\varepsilon}) \leq M\ln(1/\varepsilon)$. In the case of $N_\varepsilon \gg 1$ initial vortices, evolution of the vortex density is described by a mean-field equation first obtained rigorously in [12,18].

Here we are interested in initial data that may have much higher energy, and wish to describe the emergence of vortices. This is mentioned as an open problem in [6, Problem 5]. Our methods are strongly inspired by similar results on the emergence of sharp transitions in the Allen–Cahn heat flow [8].

Our first main result concerns the evolution of the zeros of u.

Theorem 1.1. *There exists $C_0 > 0$, depending on u_0, such that, for all $\varepsilon > 0$ sufficiently small (depending on u_0), if $\mathcal{Z}(t)$ denotes the set of zeros of $u(t)$, we have*

$$\#\mathcal{Z}(t) = \#\mathcal{Z}(0) \quad \text{for } 0 \leq t \leq T_\varepsilon := \ln\frac{1}{\varepsilon} - \frac{1}{2}\ln\ln\frac{1}{\varepsilon} - C_0.$$

In other words, no new zeros of the vector field $u(t)$ are generated up to $t = T_\varepsilon$. Additionally, if $z_j(t)$ is the evolution of the jth zero z_j^0 of u_0, then $|z_j(t) - z_j^0| \lesssim \varepsilon\sqrt{\ln(1/\varepsilon)}$, and the topological degree of $u(t)$ at $z_j(t)$ is preserved. Finally, at $t = T_\varepsilon$ we have

$$|u(T_\varepsilon, x)| \geq \frac{1}{2} \quad \text{for } \operatorname{dist}(x, \mathcal{Z}(0)) \gtrsim \varepsilon\sqrt{\ln\frac{1}{\varepsilon}}. \tag{1.3}$$

Above and throughout the paper the symbol $A \lesssim B$ for two nonnegative quantities A, B means that there exists a constant $C > 0$, depending only on u_0, such that $A \leq CB$.

Remark 1.2. In Theorem 1.1 and all our statements, the dependence on u_0 is through the constant $\alpha_0 > 0$ in (1.2), and constants $K_0, r_0 > 0$ such that $\|\nabla u_0\|_{L^\infty} \leq K_0$, the cardinal of $u_0^{-1}(\{0\})$ is bounded by K_0, and u_0 is invertible with $|Du_0^{-1}| \leq K_0$ in $B(z_j^0, r_0)$, for each zero z_j^0. The initial datum u_0 may depend on ε as long as these constants can be chosen ε-independent.

An immediate corollary of Theorem 1.1 is that, if u_0 does not vanish, then $u(t)$ does not vanish for $0 \leq t \leq T_\varepsilon$.

Corollary 1.3. *If $\mathcal{Z}(0) = \emptyset$ then $\mathcal{Z}(t) = \emptyset$ for $t \in [0, T_\varepsilon]$.*

This means that up to time T_ε the Ginzburg–Landau heat flow does not spontaneously create dipoles, as happens in the Berezinsky–Kosterlitz–Thouless phase transition in statistical mechanics. If one allows the initial condition u_0 to depend on ε, one may however observe creation of zeros, as in [14, Proposition 4.1]. More precisely, the construction in [14, Proposition 4.1], adapted to our context (there $\varepsilon = 1$), involves an initial condition $u_0 = u_{0\varepsilon}$ bounded in C^2 and such that the two components of u_0 vanish along two smooth curves passing, with parallel tangents, through two points distant at most ε, so that $|u_0| + |\det(\nabla u_0)| \lesssim \varepsilon$ at these points, and α_0 in (1.2) cannot be taken ε-independent; see Remark 1.2.

Our second main result is a logarithmic energy bound at the time $t = T_\varepsilon$ given by Theorem 1.1.

Theorem 1.4. *For all sufficiently small $\varepsilon > 0$ (depending on u_0), we have*

$$E_\varepsilon(u(t)) \lesssim \ln\frac{1}{\varepsilon} \quad \forall t \geq T_\varepsilon.$$

Theorem 1.4 shows that the evolution enters an energy regime where the analysis of [4, 5, 19] can be applied. The present context is actually slightly different, because we work on the torus $\mathbb{T}^2$ instead of $\mathbb{R}^2$, but the results of [4, 5, 19] should apply to $\mathbb{T}^2$, with appropriate modifications. Conversely, the results of the present paper could be adapted to $\mathbb{R}^2$, with appropriate conditions at infinity, at the price of minor technical complications.

In particular, the work [19] describes the evolution of the vortices of u as functions of the accelerated time-variable

$$s = \frac{\varepsilon^2}{\ln\frac{1}{\varepsilon}} t.$$

The vortices $a_k(s)$ evolve according to the gradient flow of a renormalized energy $W(a)$, combined with a finite number of collision or branching times. Note that in the torus $\mathbb{T}^2$, the renormalized energy $W(a)$ would be slightly different from the one considered in [19]; see [1, 7, 9]. The initial conditions for the vortices $a_k(s)$ as $s \to 0^+$ are identified via the jacobian $Ju = \det(\nabla u)$ at the initial time [6, Proposition 2]. We therefore complement Theorem 1.4 with our third main result, which characterizes the jacobian at time $t = T_\varepsilon$. Note that, in contrast with the previous theorems, where $\varepsilon > 0$ was small but fixed and therefore we omitted stressing the ε-dependence of $u(t) = u_\varepsilon(t)$ evolving according to (1.1), this only concerns the limit, as $\varepsilon \to 0$, of the map $u(T_\varepsilon) = u_\varepsilon(T_\varepsilon)$.

Theorem 1.5. *We have, as $\varepsilon \to 0$,*

$$Ju(T_\varepsilon) = \det(\nabla u(T_\varepsilon)) \longrightarrow \sum_{j=1}^{N} \hat{d}_j \delta_{z_j^0},$$

in the sense of distributions, where $z_1^0, \dots, z_N^0$ are the zeros of u_0, and $\hat{d}_j \in \{\pm 1\}$ its topological degree at z_j^0.

Now we can be more specific about the initial conditions for the later evolution of the vortices $a_k(s)$, as described in [6, Proposition 2]. Letting d_k denote the topological degree of $u(s)$ at $a_k(s)$ for small $s > 0$, the initial conditions $a_k^0 = \lim_{s\to 0^+} a_k(s)$ must satisfy

$$\sum_{k=1}^{L} d_k \delta_{a_k^0} = \sum_{j=1}^{N} \hat{d}_j \delta_{z_j^0}.$$

This implies in particular that $\{a_k^0\} = \{z_j^0\}$. But the points a_k^0 may not be disjoint: this description does not prevent a priori a single initial zero z_j^0 spontaneously splitting into several vortices $\{a_k\}$, because at $s = 0$ the energy is not yet quantized (in the sense of [19, Theorem 1.5]). In fact, initial splitting into two vortices can easily be ruled out, but it is not clear whether splitting into three or more vortices can occur.

However, note that in the setting of Corollary 1.3, if there are no initial zeros, we can directly conclude that no later vortices appear. A complete proof of this fact would require adapting [6] to our torus-based setting.

The main idea of this paper is that on the timescale considered, the effect of diffusion in the Ginzburg–Landau equation is dominated by the nonlinear effect. This means that the modulus of any initial data instantaneously (on the fast timescale $s = \varepsilon^2 t/(\ln 1/\varepsilon)$) approaches 1, except possibly on small regions where the initial data is close to 0. The methods are elementary and provide explicit pointwise estimates on $u(t,x)$, which directly imply the stated results. To control diffusive effects, the key tool is Lemma 2.2, which is a type of Gronwall inequality (new to the best of our knowledge). The organization of the paper follows that of the presentation of the results, which are proven in the same order in consecutive sections.

2. Zeros of u: Proof of Theorem 1.1

Denote by $\Phi\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ the flow of the ODE $y' = (1 - |y|^2)y$, that is,

$$\partial_t \Phi = (1 - |\Phi|^2)\Phi, \quad \Phi(0, X) = X,$$

given explicitly by

$$\Phi(t, X) = \frac{e^t X}{\sqrt{1 + |X|^2(e^{2t} - 1)}}. \tag{2.1}$$

We want to estimate how far u is from

$$v(t,x) = \Phi(t, g(t,x))$$

for some well-chosen map g with $g(0,x) = u_0(x)$. To this end we define $w = e^{-t}(u - v)$, so that

$$u = v + e^t w.$$

Using the equations satisfied by u and Φ we obtain

$$\begin{aligned} \partial_t w - \varepsilon^2 \Delta w &= -2(v \cdot w)v - |v|^2 w - e^{-t} \mathcal{N}(v, e^t w) - e^{-t} \mathcal{R}, \\ \mathcal{N}(v, X) &= |X|^2 v + 2(v \cdot X)X + |X|^2 X, \\ \mathcal{R} &= \partial_t v - \varepsilon^2 \Delta v - (1 - |v|^2)v \\ &= D\Phi(t, g)(\partial_t g - \varepsilon^2 \Delta g) - \varepsilon^2 D^2 \Phi(t, g) \nabla g \cdot \nabla g. \end{aligned} \tag{2.2}$$

In view of (2.2), it is natural to choose, as in [8], $g(t) = e^{\varepsilon^2 t \Delta} u_0$, that is, g solves

$$\partial_t g - \varepsilon^2 \Delta g = 0, \quad g(0, x) = u_0(x),$$

and therefore

$$\mathcal{R} = -\varepsilon^2 D^2 \Phi(t, g) \nabla g \cdot \nabla g. \tag{2.3}$$

The rest of the article is devoted to obtaining good pointwise estimates on $e^t w = u - v$.

Lemma 2.1. *If w solves*

$$\partial_t w - \varepsilon^2 \Delta w = -2(v \cdot w)v - |v|^2 w + F, \quad t > 0, \ x \in \Omega,$$

with $w(0, x) = 0$, then

$$\|w(t)\|_{L^\infty} \le \int_0^t \|F(s)\|_{L^\infty} \, ds.$$

Proof. Multiplying the equation by $w/|w|$ we obtain

$$\begin{aligned} \partial_t |w| &= \varepsilon^2 \frac{w}{|w|} \cdot \Delta w - |v|^2 |w| - 2 \frac{(v \cdot w)^2}{|w|} + F \cdot \frac{w}{|w|} \\ &\le \varepsilon^2 \frac{w}{|w|} \cdot \Delta w + |F| \\ &\le \varepsilon^2 \Delta |w| + |F|, \end{aligned}$$

so by the comparison principle we have $|w| \le \rho$, where ρ solves $\partial_t \rho - \varepsilon^2 \Delta \rho = |F|$ and $\rho(0, x) = 0$, that is, $\rho(t) = \int_0^t e^{\varepsilon^2 (t-s)\Delta} |F(s)| \, ds$, where $e^{t\Delta}$ denotes the heat semigroup on the torus $\mathbb{T}^2$. Since the L^∞-norm is nonincreasing under the action of that semigroup, we deduce the announced bound. ■

We apply Lemma 2.1 to our map w and $F = -e^{-t} \mathcal{N}(v, e^t w) - e^{-t} \mathcal{R}$. We have $|g| \le |u_0| \lesssim 1$, so $|v| \lesssim 1$ and $|\mathcal{N}(v, X)| \lesssim |X|^2$ for $|X| \lesssim 1$ (we will apply this to $X = e^t w = u - v$). Thus we obtain

$$\|e^t w(t)\|_{L^\infty} \lesssim \int_0^t e^{(t-s)} \|e^s w(s)\|_{L^\infty}^2 \, ds + \int_0^t e^{t-s} \|\mathcal{R}(s)\|_{L^\infty} \, ds. \tag{2.4}$$

Recall that

$$\mathcal{R} = -\varepsilon^2 D^2 \Phi(t, g) \nabla g \cdot \nabla g,$$

and $|\nabla g| \le |\nabla u_0| \lesssim 1$, hence

$$\|\mathcal{R}(t)\|_\infty \lesssim \varepsilon^2 \sup_{|X|\lesssim 1} |D^2\Phi(t,X)|.$$

Direct calculation gives

$$\begin{aligned}|D^2\Phi(t,X)| &\lesssim \frac{e^t|X|(e^{2t}-1)}{(1+|X|^2(e^{2t}-1))^{3/2}}\\ &= e^t(e^{2t}-1)^{1/2}\frac{(|X|^2(e^{2t}-1))^{1/2}}{(1+|X|^2(e^{2t}-1))^{3/2}}\\ &\lesssim e^t(e^{2t}-1)^{1/2},\end{aligned}$$

so

$$\begin{aligned}\int_0^t e^{t-s}\|\mathcal{R}(s)\|_{L^\infty}\,ds &\lesssim \varepsilon^2 e^t\int_0^t (e^{2s}-1)^{1/2}\,ds\\ &\lesssim \varepsilon^2 e^t(e^{2t}-1)^{1/2},\end{aligned}$$

where we have used

$$\begin{aligned}\int_0^t (e^{2s}-1)^{1/2}\,ds &= \int_0^{(e^{2t}-1)^{1/2}} \frac{x^2}{1+x^2}\,dx\\ &= (e^{2t}-1)^{1/2} - \arctan((e^{2t}-1)^{1/2})\\ &\le (e^{2t}-1)^{1/2}.\end{aligned}$$

Plugging this into (2.4) we deduce

$$\|e^t w(t)\|_{L^\infty} \lesssim \int_0^t e^{(t-s)}\|e^s w(s)\|_{L^\infty}^2\,ds + \varepsilon^2 e^t(e^{2t}-1)^{1/2}. \tag{2.5}$$

Lemma 2.2. *Assume f, h are continuous positive functions on $(0,\infty)$ satisfying*

$$\limsup_{t\searrow 0}\frac{f(t)}{h(t)} \le 1$$

$$\text{and}\quad f(t) \le c\int_0^t e^{t-s} f(s)^2\,ds + h(t)\quad \forall t>0,$$

for some constant $c>0$. If $T>0$ is such that

$$\sup_{0<t<T}\int_0^t e^{t-s}\frac{h(s)}{h(t)}h(s)\,ds \le \frac{1}{8c},$$

then

$$f(t) \le 2h(t)\quad \forall t\in(0,T).$$

If in addition h is nondecreasing, it suffices to check that

$$\int_0^T e^{T-s}h(s)\,ds \le \frac{1}{8c}.$$

Proof. For all $t > \tilde{t} > 0$ we have

$$\begin{aligned}\frac{f(\tilde{t})}{h(\tilde{t})} &\le c\int_0^{\tilde{t}} e^{\tilde{t}-s}\Big(\frac{f(s)}{h(s)}\Big)^2\frac{h(s)}{h(\tilde{t})}h(s)\,ds+1\\ &\le c\int_0^{\tilde{t}} e^{\tilde{t}-s}\frac{h(s)}{h(\tilde{t})}h(s)\,ds\,\Theta(t)^2+1\\ &\le \frac{1}{8}\Theta(t)^2+1,\end{aligned}$$

where

$$\Theta(t)=\sup_{0<s<t}\frac{f(s)}{h(s)}.$$

Taking the supremum over $0<\tilde{t}<t$ we deduce that

$$\Theta(t)\le\frac{1}{8}\Theta(t)^2+1\quad\forall t\in(0,T),$$

so Θ takes values into

$$\big\{x\in\mathbb{R}\colon \tfrac{x^2}{8}-x+1\ge 0\big\}=(-\infty,4-2\sqrt{2}]\cup[4+2\sqrt{2},+\infty).$$

Since Θ is continuous on $(0,T)$ and $\Theta(0^+)\le 1<4-2\sqrt{2}$, we deduce that $\Theta(t)\le 4-2\sqrt{2}\le 2$ for all $t\in(0,T)$. ∎

We apply Lemma 2.2 to

$$f(t)=\|e^t w(t)\|_{L^\infty},\quad h(t)=A\varepsilon^2e^t(e^{2t}-1)^{1/2},$$

where $A\ge 1$ is the constant hidden in the sign $\lesssim$ in (2.5). By Lemma 2.1, applied to w and $F=-e^{-t}\mathcal{N}(v,e^tw)-e^{-t}\mathcal{R}$ which satisfies $|F|\lesssim 1$ for $0<t<1$, the map w satisfies $\|e^tw(t)\|_{L^\infty}\lesssim t$ for $0<t<1$ so $\limsup_{0^+}(f/h)=0$, and thanks to (2.5) we deduce that

$$\|e^tw(t)\|_{L^\infty}\lesssim\varepsilon^2e^t(e^{2t}-1)^{1/2}\quad\text{for } 0\le t\le T=\ln\tfrac{1}{\varepsilon}-\ln(16A^2),\tag{2.6}$$

since h is nondecreasing, and for this value of T we have

$$8A\int_0^T e^{T-s}h(s)\,ds\le 8A^2\varepsilon^2e^{2T}\le\frac{1}{2}.$$

Estimate (2.6) tells us that u is close to v. Note in particular that (2.6) is valid up to $t=\ln(1/\varepsilon)-\frac{1}{2}\ln\ln(1/\varepsilon)$ if ε is small enough. From (2.6) we also deduce a bound on ∇w, using equation (2.2) again.

Lemma 2.3. *If w solves $\partial_t w-\varepsilon^2\Delta w=G$ with $w(0)=0$ in the torus $\mathbb{T}^2$, then we have*

$$\|\nabla w(t)\|_{L^\infty}\lesssim\frac{1}{\varepsilon}\int_0^t\frac{\|G(s)\|_{L^\infty}}{\sqrt{t-s}}\,ds.$$

Proof. We can consider w and G as periodic maps defined on $\mathbb{R}^2$; then w is given by the Duhamel formula

$$w(t) = \int_0^t H_{\varepsilon\sqrt{t-s}} * G(s)\,ds,$$

where the convolution is on $\mathbb{R}^2$ and $H_\delta(x) = \delta^{-2} H(x/\delta)$, $H(x) = (4\pi)^{-1} e^{-|x|^2/4}$. Therefore, we have

$$\|\nabla w(t)\|_{L^\infty} \lesssim \int_0^t \|\nabla H_{\varepsilon\sqrt{t-s}}\|_{L^1} \|G(s)\|_{L^\infty}\,ds,$$

and the estimate follows from

$$\|\nabla H_{\varepsilon\sqrt{t-s}}\|_{L^1} \lesssim \frac{1}{\varepsilon\sqrt{t-s}} \|\nabla H\|_{L^1}. \qquad \blacksquare$$

Applying Lemma 2.3 to equation (2.2) satisfied by w and noting that the choice of T in (2.6) ensures that the right-hand side G of (2.2) satisfies

$$|G| \lesssim \|w\|_{L^\infty} + \varepsilon^2 (e^{2t} - 1)^{1/2} \lesssim \varepsilon^2 (e^{2t} - 1)^{1/2},$$

we obtain

$$\|\nabla w\|_{L^\infty} \lesssim \frac{1}{\varepsilon} \int_0^t \frac{\varepsilon^2 (e^{2s} - 1)^{1/2}}{\sqrt{t-s}}\,ds \lesssim \varepsilon\sqrt{t}(e^{2t} - 1)^{1/2} \tag{2.7}$$

for all $t \leq T = \ln(1/\varepsilon) - \ln(16A^2)$.

All assertions of Theorem 1.1 will follow from the bounds (2.6)–(2.7) on $e^t w = u - v$ and the explicit expression of $v = \Phi(t, g)$. First, we need to gather some information on $g(t) := g(t, \cdot) = e^{\varepsilon^2 t \Delta} u_0$. To that end we use the nondegeneracy assumption (1.2). It implies that u_0 has a finite number of zeros, all of degree ± 1. We denote

$$\{u_0 = 0\} = \{z_1^0, \ldots, z_N^0\}.$$

Since $g(t, x) = \tilde{g}(\varepsilon^2 t, x)$, where $\tilde{g}(\tilde{t}) = e^{\tilde{t}\Delta} u_0$ is C^1 in $[0, \infty) \times \mathbb{T}^2$, we deduce that there exist $t_0, \beta_0, r_0 > 0$ such that, for all $t \leq t_0/\varepsilon^2$,

$$\begin{aligned}
&|g(t)| + |\det(\nabla g(t))| \geq \frac{\alpha_0}{2},\\
&|g(t, x)| \geq \beta_0 \text{ for } \operatorname{dist}(x, \{z_j^0\}) \geq r_0,\\
&g(t) \text{ is invertible and } |\nabla g(t)^{-1}| \lesssim 1 \text{ on } B(z_j^0, r_0).
\end{aligned}$$

In each disk $B(z_j^0, r_0)$, the map $g(t)$ has exactly one zero $\hat{z}_j(t)$, so

$$\{g(t) = 0\} = \{\hat{z}_1(t), \ldots, \hat{z}_N(t)\},$$

and we have

$$\operatorname{dist}(\cdot, \{\hat{z}_j(t)\}) \lesssim |g(t, \cdot)| \lesssim \operatorname{dist}(\cdot, \{\hat{z}_j(t)\}). \tag{2.8}$$

Thanks to the implicit function theorem, the maps $t \mapsto \hat{z}_j(t)$ are C^1, and

$$\frac{d}{dt}\hat{z}_j(t) = -\nabla g(t,\hat{z}_j)^{-1}\partial_t g(t,\hat{z}_j),$$

hence

$$\left|\frac{d}{dt}\hat{z}_j\right| \lesssim \|\partial_t g\|_\infty \lesssim \varepsilon^2\|\Delta g\|_{L^\infty}.$$

Viewing g and u_0 as periodic maps defined on $\mathbb{R}^2$, g is given by the formula

$$g(t) = H_{\varepsilon\sqrt{t}} * u_0,$$

where the convolution is on $\mathbb{R}^2$ and $H_\delta(x) = \delta^{-2}H(x/\delta)$, $H(x) = (4\pi)^{-1}e^{-|x|^2/4}$, so

$$\|\Delta g\|_{L^\infty} \le \|\nabla H_{\varepsilon\sqrt{t}}\|_{L^1}\|\nabla u_0\|_{L^\infty} \lesssim \frac{1}{\varepsilon\sqrt{t}},$$

and we infer

$$\left|\frac{d}{dt}\hat{z}_j\right| \lesssim \frac{\varepsilon}{\sqrt{t}}, \quad |z_j(t) - z_j^0| \lesssim \varepsilon\sqrt{t}. \tag{2.9}$$

Next we combine these properties of $g(t)$ with the explicit expression $v = \Phi(t, g)$ and the bounds (2.6)–(2.7) on $e^t w = u - v$ to obtain the desired properties on u. We denote by $C > 0$ a generic constant depending on u_0 and which may change from line to line. We start by bounding the modulus $|u|$ from below: using (2.1) and (2.6) we obtain, for $0 \le t \le \ln(1/\varepsilon) - C$,

$$\begin{aligned} |u| \ge |v| - e^t|w| &\ge \frac{e^t|g|}{\sqrt{1 + |g|^2(e^{2t}-1)}} - C\varepsilon^2 e^{2t} \\ &\ge \frac{1}{2}\min(e^t|g|, 1) - C\varepsilon^2 e^{2t}. \end{aligned}$$

The last quantity is positive whenever $e^t|g| \ge 1$ and $e^{2t} < 1/(2C\varepsilon^2)$, or $e^t|g| \le 1$ and $|g|^2 > 2C\varepsilon^2$. Hence we deduce that

$$|u| > 0 \text{ in } \{|g| \ge C\varepsilon\} \quad \text{for } 0 \le t \le \ln\tfrac{1}{\varepsilon} - C.$$

In the case without initial zeros, this proves in particular Corollary 1.3. Moreover, combining this with (2.8) we have $|u(t)| > 0$ outside the disks $B(\hat{z}_j(t), C\varepsilon)$. By homotopy invariance of the topological degree, $u(t)$ must have at least one zero $z_j(t) \in B(\hat{z}_j(t), C\varepsilon)$. Next we verify that this zero is unique.

Recall that $g(t)$ is invertible on $B(z_0, r)$, and maps $B(z_j(t), C\varepsilon)$ into $B(0, K\varepsilon)$, for some constant K depending on u_0, thanks to (2.8). The flow map $\Phi(t) = \Phi(t,\cdot)$ is invertible from $B(0, K\varepsilon)$ onto $B(0, R)$ given by $R = \widehat{\Phi}(t, K\varepsilon)$, with inverse $\Phi(t)^{-1} = \Phi(-t)$. Here, $\widehat{\Phi}(t, r) = |\Phi(t, re^{i\theta})|$ for any $r > 0$ and $\theta \in \mathbb{R}$.

Therefore, $v(t) = \Phi(t) \circ g(t)$ is invertible on $B(z_j(t), C\varepsilon)$, and

$$\sup_{v(B(\hat{z}_j(t),C\varepsilon))} |\nabla v(t)^{-1}| \lesssim \sup_{|X|\leq \hat{\Phi}(t,K\varepsilon)} |\nabla \Phi(-t,X)|.$$

For $t \leq \ln(1/\varepsilon) - C$ we have

$$\Phi(t, K\varepsilon) = \frac{e^t K\varepsilon}{\sqrt{1 + K^2\varepsilon^2(e^{2t}-1)}} \leq e^t K\varepsilon \leq \frac{1}{2},$$

provided C is large enough, and, for $|X| \leq 1/2$,

$$\nabla\Phi(-t,X) = \frac{e^{-t}}{\sqrt{1-|X|^2(1-e^{-2t})}}\Big(I + \frac{1-e^{-2t}}{1-|X|^2(1-e^{-2t})} X \otimes X\Big),$$

so we infer

$$\sup_{v(B(\hat{z}_j(t),C\varepsilon))} |\nabla v(t)^{-1}| \lesssim e^{-t}.$$

We use this to show that $u(t)$ is injective on $B(\hat{z}_j(t), C\varepsilon)$. Since the equation $y = u(x) = v(x) + e^t w(x)$ is equivalent to $x = v^{-1}(y - e^t w(x))$, it suffices to check that the map $F\colon x \mapsto v^{-1}(y - e^t w(x))$ is a contraction on $B(\hat{z}_j(t), C\varepsilon)$, for $|y| < \delta$. Here, $\delta > 0$ is a small constant such that v^{-1} is well defined on $B(0, 2\delta)$. Thanks to (2.6) we have $|e^t w| \leq \delta$ provided C is large enough, and

$$\sup_{B(\hat{z}_j(t),C\varepsilon)} |\nabla F| \lesssim e^{-t}\|e^t \nabla w\|_\infty \lesssim \|\nabla w\|_\infty.$$

Since $\|\nabla w\|_\infty \lesssim \varepsilon\sqrt{t}e^t$ thanks to (2.7), we deduce that F is a contraction for

$$0 \leq t \leq T_\varepsilon = \ln\tfrac{1}{\varepsilon} - \tfrac{1}{2}\ln\ln\tfrac{1}{\varepsilon} - C_0,$$

if the constant C_0 is large enough, depending on u_0. By the above discussion this shows that $u(t)$ is injective on $B(\hat{z}_j(t), C\varepsilon)$, and

$$\{u(t) = 0\} = \{z_1(t), \ldots, z_N(t)\},$$

for some $z_j(t) \in B(\hat{z}_j(t), C\varepsilon)$. This proves Theorem 1.1, except for its last assertion (1.3). To verify (1.3), we note that (2.6) ensures $|u - v| = |e^t w| \leq 1/4$ for $t = T_\varepsilon$, so it suffices to check that $|v(T_\varepsilon, x)| \geq 3/4$ for $\mathrm{dist}(x, \{z_j^0\}) \gtrsim \varepsilon\sqrt{\ln(1/\varepsilon)}$. We have

$$\begin{aligned}|v(T_\varepsilon)| = \frac{e^{T_\varepsilon}|g|}{\sqrt{1+|g|^2(e^{2T_\varepsilon}-1)}} &= \frac{1}{\sqrt{1+(1-|g|^2)e^{-2T_\varepsilon}|g|^{-2}}} \\ &= \frac{1}{\sqrt{1+(1-|g|^2)e^{C_0}\big(\frac{\varepsilon\sqrt{\ln(1/\varepsilon)}}{g}\big)^2}}.\end{aligned}$$

If $|g| \geq M\varepsilon\sqrt{\ln(1/\varepsilon)}$ for some large enough $M > 0$, we deduce $|v(T_\varepsilon)| \geq 3/4$. Thanks to (2.8) this implies that $|v(T_\varepsilon, x)| \geq 3/4$ for $\mathrm{dist}(x, \{z_j^0\}) \gtrsim \varepsilon\sqrt{\ln(1/\varepsilon)}$ and concludes the proof of Theorem 1.1.

3. Energy of u: Proof of Theorem 1.4

First, we obtain more precise estimates for w away from the bad disks $B(z_j^0, C\varepsilon\sqrt{\ln(1/\varepsilon)})$. To this end we localize the equation by setting

$$\widetilde{w} = \chi^2 w,$$

for some appropriate smooth cutoff function $0 \le \chi(x) \le 1$, to be chosen later. From equation (2.2) satisfied by w we deduce

$$\begin{aligned}\partial_t \widetilde{w} - \varepsilon^2 \Delta \widetilde{w} = -2(v\cdot\widetilde{w})v - |v|^2\widetilde{w} - e^{-t}\chi^2\mathcal{N}(v, e^t w) - e^{-t}\chi^2\mathcal{R} \\ - \varepsilon^2(\Delta\chi^2)w - 2\varepsilon^2\nabla\chi^2\cdot\nabla w.\end{aligned} \tag{3.1}$$

Applying Lemma 2.1 to equation (3.1) satisfied by $\widetilde{w}$, and using (2.6)–(2.7) to estimate the two last terms, we deduce

$$\begin{aligned}\|e^t\widetilde{w}\|_{L^\infty} \lesssim \int_0^t e^{t-s}\|e^s\widetilde{w}\|_{L^\infty}^2\,ds + \int_0^t e^{t-s}\|\chi^2\mathcal{R}(s)\|_{L^\infty}\,ds \\ + (\varepsilon\sqrt{t}\|\nabla\chi\|_{L^\infty} + \varepsilon^2\|\nabla^2\chi\|_{L^\infty})\varepsilon^2 e^t(e^{2t}-1)^{1/2}\end{aligned}$$

for all $t \le \ln(1/\varepsilon) - C$. Applying Lemma 2.2 we therefore have

$$\begin{aligned}\|e^t\chi^2 w\|_{L^\infty} \lesssim \int_0^t e^{t-s}\|\chi^2\mathcal{R}(s)\|_{L^\infty}\,ds \\ + (\varepsilon\sqrt{t}\|\nabla\chi\|_{L^\infty} + \varepsilon^2\|\nabla^2\chi\|_{L^\infty})\varepsilon^2 e^t(e^{2t}-1)^{1/2},\end{aligned} \tag{3.2}$$

provided $t \le \ln(1/\varepsilon) - C$ and $\varepsilon\sqrt{t}\|\nabla\chi\|_{L^\infty} + \varepsilon^2\|\nabla^2\chi\|_{L^\infty} \le 1$. Using properties (2.8) of g, the fact that $|\hat{z}_j(t) - z_j^0| \lesssim \varepsilon\sqrt{\ln(1/\varepsilon)}$ for $t \le \ln(1/\varepsilon)$ thanks to (2.9), and letting

$$D(x) = \operatorname{dist}(x, \{z_j^0\}),$$

we have

$$D \lesssim |g| \lesssim D \quad \text{in } \left\{D \ge M\varepsilon\sqrt{\ln(1/\varepsilon)}\right\},$$

for $t \le \ln(1/\varepsilon)$. Here, $M > 0$ is a large constant that depends only on u_0. More precisely, it suffices to choose M such that $2|\hat{z}_j(t) - z_j^0| \le M\varepsilon\sqrt{\ln(1/\varepsilon)}$, which implies that $D/2 \le \operatorname{dist}(\cdot, \hat{z}_j(t)) \le 3D/2$ in $\{D \ge M\varepsilon\sqrt{\ln(1/\varepsilon)}\}$. Since $|\nabla g| \lesssim 1$, recalling the explicit formulas (2.3) and (2.1) we deduce

$$\begin{aligned}|\mathcal{R}| &\lesssim \varepsilon^2 \frac{e^t|g|(e^{2t}-1)}{(1+|g|^2(e^{2t}-1))^{3/2}} \\ &\lesssim \varepsilon^2 e^t(e^{2t}-1)\frac{D}{(1+C^{-2}D^2(e^{2t}-1))^{3/2}},\end{aligned}$$

in $\{D \geq M\varepsilon\sqrt{\ln(1/\varepsilon)}\}$ for $t \leq \ln(1/\varepsilon) - C$. Therefore, choosing cutoff functions χ satisfying

$$\mathbf{1}_{2\lambda \leq D \leq 3\lambda} \leq \chi \leq \mathbf{1}_{\lambda \leq D \leq 4\lambda}, \quad |\nabla\chi| \lesssim \frac{1}{\lambda}, \ |\nabla^2\chi| \lesssim \frac{1}{\lambda^2},$$

for some $\lambda \gtrsim \varepsilon\sqrt{\ln(1/\varepsilon)}$, from (3.2) we infer

$$\begin{aligned} |e^t w| \lesssim \varepsilon^2 e^t D \int_0^t \frac{e^{2s}-1}{(1+C^{-2}D^2(e^{2s}-1))^{3/2}}\,ds \\ + \frac{\varepsilon}{D}\sqrt{1+t}\,\varepsilon^2 e^t (e^{2t}-1)^{1/2} \end{aligned} \tag{3.3}$$

in $\{D \geq M\varepsilon\sqrt{\ln(1/\varepsilon)}\}$ for $t \leq \ln(1/\varepsilon) - C$ and ε small enough.

For any $\alpha \in (0, 1/2)$ we have

$$\begin{aligned} &\int_0^t \frac{e^{2s}-1}{(1+\alpha^2(e^{2s}-1))^{3/2}}\,ds \\ &\quad = \frac{1}{\alpha^2}\int_1^{(1+\alpha^2(e^{2t}-1))^{1/2}} \frac{(x^2-1)}{x^2(x^2-1+\alpha^2)}\,dx \leq \frac{1}{\alpha^2}\int_1^\infty \frac{dx}{x^2}, \end{aligned}$$

thanks to the change of variable $x = (1+\alpha^2(e^{2s}-1))^{1/2}$. Hence, taking C large enough that $D/C < 1/2$, from (3.3) we deduce

$$\frac{1}{\varepsilon}|e^t w| \lesssim \varepsilon e^t \frac{\sqrt{1+t}}{D} \tag{3.4}$$

in $\{D \geq M\varepsilon\sqrt{\ln(1/\varepsilon)}\}$ for $t \leq \ln(1/\varepsilon) - C$ and ε small enough. Using Lemma 2.3 and (3.4) to estimate the right-hand side of (3.1), we also obtain gradient bounds

$$e^t|\nabla w| \lesssim \varepsilon e^t \sqrt{t}\,\frac{\sqrt{1+t}}{D} \tag{3.5}$$

in $\{D \geq M\varepsilon\sqrt{\ln(1/\varepsilon)}\}$ for $t \leq \ln(1/\varepsilon) - C$ and ε small enough.

Next we refine these estimates by including the effect of the second term $-|v|^2\widetilde{w}$ in the right-hand side of (3.1). We choose as above a cutoff function χ supported in $\{\lambda \leq D \leq 4\lambda\}$, with $|\nabla\chi| \lesssim \lambda^{-1}$ and $|\nabla^2\chi| \lesssim \lambda^{-2}$, for some $\lambda \geq M\varepsilon\sqrt{\ln(1/\varepsilon)}$. Combining (2.6)–(2.7) and (3.4)–(3.5) to bound the two last terms in (3.1), we have

$$\begin{aligned} &\partial_t\widetilde{w} - \varepsilon^2\Delta\widetilde{w} = -2(v\cdot\widetilde{w})v - |v|^2\widetilde{w} + \widetilde{F}, \\ &|\widetilde{F}| \lesssim e^{-t}\|e^t\widetilde{w}\|^2_{L^\infty} + e^{-t}\|\chi^2\mathcal{R}\|_{L^\infty} \\ &\qquad + \frac{\varepsilon^3}{\lambda}\sqrt{1+t}\,\min\Big(\frac{\sqrt{1+t}}{\lambda}, (e^{2t}-1)^{1/2}\Big). \end{aligned} \tag{3.6}$$

Arguing as in the proof of Lemma 2.1 but retaining the second term in the right-hand side of (3.6), we have

$$\partial_t|\widetilde{w}| + |v|^2|\widetilde{w}| - \varepsilon^2\Delta|\widetilde{w}| \leq |\widetilde{F}|.$$

In the support of χ we have

$$|v|^2 = \frac{e^{2t}|g|^2}{1+|g|^2(e^{2t}-1)} = 1 - \frac{1-|g|^2}{1+|g|^2(e^{2t}-1)} \geq \max\Big(1 - \frac{e^{-2t}}{C^2\lambda^2}, 0\Big);$$

hence

$$\partial_t|\widetilde{w}| + \max\Big(1 - \frac{e^{-2t}}{C^2\lambda^2}, 0\Big)|\widetilde{w}| - \varepsilon^2\Delta|\widetilde{w}| \leq |\widetilde{F}|.$$

We rewrite this as

$$\partial_t e^{h(t)}|\widetilde{w}| - \varepsilon^2\Delta e^{h(t)}|\widetilde{w}| \leq e^{h(t)}|\widetilde{F}|,$$

where

$$\begin{aligned} h(t) &= \int_0^t \max\Big(1 - \frac{e^{-2t}}{C^2\lambda^2}, 0\Big)\,ds \\ &= \begin{cases} 0 & \text{for } 0 < t < t_\lambda, \\ t - t_\lambda - \dfrac{1}{2C^2\lambda^2}(e^{-2t} - e^{-2t_\lambda}) & \text{for } t > t_\lambda, \end{cases} \end{aligned}$$

where $t_\lambda = \ln(1/(C\lambda))$ is such that $1 - e^{-2t_\lambda}/(C^2\lambda^2) = 0$. Arguing again as in Lemma 2.1 we deduce

$$\begin{aligned} \|\widetilde{w}(t)\|_{L^\infty} &\leq \int_0^t e^{h(s)-h(t)}\|\widetilde{F}(s)\|_{L^\infty}\,ds \\ &= \int_0^{t_\lambda} \|\widetilde{F}(s)\|_{L^\infty}\,ds \\ &\quad + \int_{t_\lambda}^t e^{s-t}e^{\frac{1}{2C^2\lambda^2}(e^{-2t}-e^{-2s})}\|\widetilde{F}(s)\|_{L^\infty}\,ds \\ &\leq \int_0^{t_\lambda} \|\widetilde{F}(s)\|_{L^\infty}\,ds + C\int_{t_\lambda}^t e^{s-t}\|\widetilde{F}(s)\|_{L^\infty}\,ds; \end{aligned}$$

hence, from the bound on $\widetilde{F}$ in (3.6), and estimating the term $\chi^2\mathcal{R}$ exactly as before (because the worst term in (3.6) is the last one anyway),

$$\begin{aligned} \|e^t\widetilde{w}(t)\|_{L^\infty} &\lesssim \int_0^t e^{t-s}\|e^s\widetilde{w}(s)\|_{L^\infty}^2\,ds + \int_0^t e^{t-s}\|\chi^2\mathcal{R}(s)\|_{L^\infty}\,ds \\ &\quad + \frac{\varepsilon^3}{\lambda}e^t\int_0^{t_\lambda}\sqrt{1+s}(e^{2s}-1)^{1/2}\,ds + \frac{\varepsilon^3}{\lambda^2}\int_{t_\lambda}^t (1+s)e^s\,ds \\ &\lesssim \int_0^t e^{t-s}\|e^s\widetilde{w}(s)\|_{L^\infty}^2\,ds + \frac{\varepsilon^2}{\lambda}e^t + \frac{\varepsilon^3}{\lambda^2}e^t(1+t). \end{aligned}$$

Applying Lemma 2.2 we obtain

$$\frac{1}{\varepsilon}|e^t w| \lesssim \varepsilon e^t\frac{1+(\varepsilon/D)(1+t)}{D}$$

and, with the help of Lemma 2.3, the gradient bound

$$|e^t \nabla w| \lesssim \varepsilon e^t \sqrt{t}\,\frac{1+(\varepsilon/D)(1+t)}{D}.$$

These bounds are valid in $\{D \geq M\varepsilon\sqrt{\ln(1/\varepsilon)}\}$ for $t \leq \ln(1/\varepsilon) - C$ and ε small enough. Using

$$\begin{aligned}\int_{\{D\geq M\varepsilon\sqrt{\ln(1/\varepsilon)}\}} &\frac{1+(\varepsilon^2/D^2)(1+t)^2}{D^2}\,dx \\ &\lesssim \int_{\varepsilon\sqrt{\ln(1/\varepsilon)}}^1 \frac{dr}{r} + \varepsilon^2(1+t)^2\int_{\varepsilon\sqrt{\ln(1/\varepsilon)}}^1 \frac{dr}{r^3} \lesssim \ln\frac{1}{\varepsilon},\end{aligned}$$

and $u - v = e^t w$, we deduce the energy bounds

$$\int_{\{D\geq M\varepsilon\sqrt{\ln(1/\varepsilon)}\}} \frac{|u-v|^2}{\varepsilon^2}\,dx \lesssim \varepsilon^2 e^{2t}\ln\frac{1}{\varepsilon},$$

$$\int_{\{D\geq M\varepsilon\sqrt{\ln(1/\varepsilon)}\}} |\nabla u - \nabla v|^2\,dx \lesssim \varepsilon^2 e^{2t} t\ln\frac{1}{\varepsilon},$$

and, using (2.6)–(2.7) to estimate the contributions from $\{D \lesssim \varepsilon\sqrt{\ln(1/\varepsilon)}\}$,

$$\int_\Omega \left(|\nabla u - \nabla v|^2 + \frac{|u-v|^2}{\varepsilon^2}\right) dx \lesssim \varepsilon^2 e^{2t} t \ln\frac{1}{\varepsilon}. \tag{3.7}$$

Note that this upper bound is $\lesssim \ln(1/\varepsilon)$ at $t = T_\varepsilon \leq \ln(1/\varepsilon\sqrt{\ln(1/\varepsilon)})$. Next we derive energy bounds for v. We have

$$\begin{aligned}1-|v|^2 = 1 - \frac{e^{2t}|g|^2}{1+|g|^2(e^{2t}-1)} &= \frac{1-|g|^2}{1+|g|^2(e^{2t}-1)} \\ &\leq \frac{1}{1+|g|^2(e^{2t}-1)},\end{aligned}$$

and since $|g|$ is of the same order as $\mathrm{dist}(\cdot, \{\hat{z}_j(t)\})$ thanks to (2.8) we deduce

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_\Omega (1-|v|^2)^2\,dx &\leq \frac{1}{\varepsilon^2}\int_\Omega \frac{1}{(1+|g|^2(e^{2t}-1))^2}\,dx \\ &\lesssim \frac{1}{\varepsilon^2}\int_0^1 \frac{1}{(1+C^{-2}r^2(e^{2t}-1))^2}\,r\,dr \\ &\lesssim \frac{1}{\varepsilon^2}\frac{1}{e^{2t}-1}.\end{aligned}$$

We also have

$$|\nabla v| = |D_X\Phi(t,g)\nabla g| \lesssim \frac{e^t}{(1+|g|^2(e^{2t}-1))^{1/2}};$$

hence

$$\int_\Omega |\nabla v|^2 \lesssim e^{2t} \int_0^1 \frac{1}{1 + C^{-2} r^2 (e^{2t} - 1)} r\, dr$$
$$\lesssim \frac{e^{2t}}{e^{2t} - 1} \ln(1 + C^{-2}(e^{2t} - 1)).$$

Gathering the above and recalling $T_\varepsilon = \ln(1/\varepsilon) - \frac{1}{2} \ln\ln(1/\varepsilon) - C_0$, we obtain

$$\int_\Omega \Big(|\nabla v|^2 + \frac{1}{\varepsilon^2}(1 - |v|^2)^2 \Big)\, dx \lesssim \ln \frac{1}{\varepsilon} \quad \text{at } t = T_\varepsilon.$$

Combining this with the bounds (3.7) concludes the proof of Theorem 1.4.

4. Jacobian of u: Proof of Theorem 1.5

Define $\mathfrak{u}_\varepsilon(x) = u(T_\varepsilon, x)$, where $T_\varepsilon = \ln(1/\varepsilon) - (1/2) \ln\ln(1/\varepsilon) - C_0$ for a large enough constant C_0. We consider the jacobian

$$J\mathfrak{u}_\varepsilon = \det(\nabla \mathfrak{u}_\varepsilon),$$

and show, as $\varepsilon \to 0$, the convergence

$$J\mathfrak{u}_\varepsilon \to \pi \sum_{j=1}^{N} \hat{d}_j \delta_{z_j^0}, \tag{4.1}$$

in the sense of distributions, where $\hat{d}_j \in \{\pm 1\}$ is the topological degree of u_0 at z_j^0.

Note that one can check, by direct calculation, that $Jv(T_\varepsilon)$ converges to this sum of Dirac masses. But the bounds we have obtained on $e^t w = u - v$ are not enough to directly infer (4.1). Instead, we invoke the compactness result of [11, Theorem 3.1]: thanks to the energy bound

$$E_\varepsilon(\mathfrak{u}_\varepsilon) \lesssim \ln \frac{1}{\varepsilon},$$

there exists a sequence $\varepsilon_n \to 0$, integers $\tilde{d}_k \in \mathbb{Z} \setminus \{0\}$, and distinct points $a_k \in \mathbb{T}^2$ such that

$$J\mathfrak{u}_{\varepsilon_n} \to \pi \sum_{k=1}^{M} \tilde{d}_k \delta_{a_k}.$$

We show next that we must have $M = N$, $\{a_k\} = \{z_j^0\}$, without loss of generality $a_j = z_j^0$ for $j = 1, \dots, N$, and $\tilde{d}_j = \hat{d}_j$. Therefore, the limit is unique and this proves (4.1).

First, we prove that $\{a_k\} \subset \{z_j^0\}$. This is a consequence of the bounds obtained above on the map u, and the fact that the limit of $J\mathfrak{u}_\varepsilon$ provides a lower bound for $E_\varepsilon(\mathfrak{u}_\varepsilon)/\ln(1/\varepsilon)$ [11, Theorem 4.1]. By that lower bound, for any $\delta > 0$ we must have

$$E_{\varepsilon_n}(\mathfrak{u}_{\varepsilon_n}; B(a_k, \delta)) \geq \pi |d_k| \ln \frac{1}{\varepsilon_n} + o\Big(\ln \frac{1}{\varepsilon_n}\Big)$$

as $n \to \infty$. Note that $|d_k| \geq 1$. Therefore, to show that $\{a_k\} \subset \{z_j^0\}$ it suffices to obtain an upper bound of the form

$$E_\varepsilon(\mathfrak{u}_\varepsilon; B(a,\delta)) \leq \frac{\pi}{2} \ln \frac{1}{\varepsilon} \quad \text{for } \varepsilon \ll 1,$$

for any $a \notin \{z_j^0\}$ and some $\delta > 0$. Recall that we have $u = v + e^t w$, and the pointwise bounds from Section 3,

$$\begin{aligned} |\nabla v|^2 + \frac{1}{\varepsilon^2}(1-|v|^2)^2 &\lesssim \frac{1}{1+D^2e^{2t}}\Big(e^{2t} + \frac{1}{\varepsilon^2}\frac{1}{1+D^2e^{2t}}\Big), \\ |\nabla e^t w|^2 + \frac{1}{\varepsilon^2}|e^t w|^2 &\lesssim \varepsilon^2 e^{2t}\frac{t^2}{D^2}, \end{aligned}$$

in $\{D \geq C\varepsilon \ln^{1/2}(1/\varepsilon)\}$ and for $1 \leq t \leq \ln(1/\varepsilon) - C_0$.

For $t = T_\varepsilon = \ln(1/\varepsilon) - (1/2)\ln\ln(1/\varepsilon) - C_0$, we deduce

$$|\nabla u|^2 + \frac{1}{\varepsilon^2}(1-|u|^2)^2 \lesssim \frac{1 + (\varepsilon/D)^2 e^{4C_0}\ln^2(1/\varepsilon) + e^{-2C_0}\ln(1/\varepsilon)}{D^2},$$

in $\{D \geq C\varepsilon \ln^{1/2}(1/\varepsilon)\}$. Hence at time $t = T_\varepsilon$, for any $\delta \geq \varepsilon \ln(1/\varepsilon)$ and for $\operatorname{dist}(a, \{z_j^0\}) \geq 4\delta$, we have

$$E_\varepsilon(\mathfrak{u}_\varepsilon; B(a,\delta)) \lesssim 1 + \Big(\frac{\varepsilon}{\delta}\Big)^2 e^{4C_0}\ln^2\frac{1}{\varepsilon} + e^{-2C_0}\ln\frac{1}{\varepsilon} \leq \frac{\pi}{2}\ln\frac{1}{\varepsilon},$$

for $\varepsilon \ll 1$, provided C_0 is chosen large enough. By the above discussion, this implies that $\{a_k\} \subset \{z_j^0\}$.

Therefore, we may write

$$J\mathfrak{u}_{\varepsilon_n} \to \pi \sum_{j=1}^{N} \tilde{d}_j \delta_{z_j^0}, \tag{4.2}$$

where $\tilde{d}_j \in \mathbb{Z}$. Here we allow the possibility that $\tilde{d}_j = 0$ because we have not yet proven that $\{z_j^0\} \subset \{a_k\}$. To prove (4.1), it suffices to show that $\tilde{d}_j = \hat{d}_j$. To that end, note that for all small $\varepsilon > 0$ we have

$$\frac{1}{\pi}\int_{B(z_j^0,r)} \det(\nabla \mathfrak{u}_\varepsilon) = \deg(\mathfrak{u}_\varepsilon, \partial B(z_j^0, r)) = \hat{d}_j$$

for any small $r > 0$ and $j = 1, \ldots, N$. This is because $t \mapsto u(t)$ is smooth and $u(t)$ does not vanish on $\partial B(z_j^0, r)$ for small $\varepsilon > 0$ and all $t \in [0, T_\varepsilon]$, so the degree of $\mathfrak{u}_\varepsilon = u(T_\varepsilon)$ is equal to the degree of $u_0 = u(0)$, which is $\hat{d}_j$ by definition. Therefore, testing (4.2) with a test function $\varphi \approx \mathbf{1}_{B(z_j^0,r)}$, we obtain $\tilde{d}_j \approx \hat{d}_j$, hence $\tilde{d}_j = \hat{d}_j$ because these are integers. This concludes the proof of (4.1).

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