Ann. Inst. H. Poincaré
Anal. Non Linéaire 40 (2023), 959–1008
DOI 10.4171/AIHPC/75

Dispersive estimates for the Schrödinger equation in a model convex domain and applications

Oana Ivanovici

Abstract. We consider an anisotropic model case for a strictly convex domain $\Omega \subset \mathbb{R}^d$ of dimension $d \geq 2$ with smooth boundary $\partial\Omega \neq \emptyset$ and we describe dispersion for the semiclassical Schrödinger equation with Dirichlet boundary condition. More specifically, we obtain the following fixed time decay rate for the linear semiclassical flow: a loss of $(\frac{h}{t})^{1/4}$ occurs with respect to the boundaryless case due to repeated swallowtail-type singularities, and is proven optimal. Corresponding Strichartz estimates allow us to solve the cubic nonlinear Schrödinger equation on such a three-dimensional model convex domain, hence matching known results on generic compact boundaryless manifolds.

1. Introduction

Let us consider the Schrödinger equation on a manifold (Ω, g), with a strictly convex boundary $\partial\Omega$ (a precise definition of strict convexity will be provided later on in the introduction):

$$-i\partial_t v + \Delta_g v = \kappa |v|^2 v, \quad v|_{t=0} = v_0, \quad v|_{\mathbb{R}\times\partial\Omega} = 0, \tag{1}$$

where Δ_g denotes the Laplace operator with Dirichlet boundary condition, and $\kappa = 0$ (linear equation) or $\kappa = \pm 1$ (defocusing or focusing nonlinear cubic equation, abbreviated to NLS from now on).

For nonlinear partial differential equations on manifolds, understanding the linear flow is a prerequisite to studying nonlinear problems: addressing the Cauchy problem for nonlinear wave equations starts with perturbative techniques and faces the difficulty of controlling solutions to the linear equation in terms of the size of the initial data. Especially at low regularities, mixed norms of Strichartz type ($L^q_t L^r_x$) are particularly useful. For the linear Schrödinger flow $e^{-it\Delta_g} v_0$ ((1) with $\kappa = 0$), local Strichartz estimates (in their most general form) read

$$\|e^{-it\Delta_g} v_0\|_{L^q(0,T)L^r(\Omega)} \leq C_T \|v_0\|_{H^\sigma(\Omega)}, \tag{2}$$

where $2 \leq q, r \leq \infty$ satisfy the Schrödinger admissibility condition, $\frac{2}{q} + \frac{d}{r} \leq \frac{d}{2}$, $(q, r, d) \neq (2, \infty, 2)$ and $\frac{2}{q} + \frac{d}{r} \geq \frac{d}{2} - \sigma$ (scale invariant when equality; otherwise, *loss*

2020 Mathematics Subject Classification. Primary 58J35; Secondary 35J25.
Keywords. Cubic Schrödinger equation, boundary problems, dispersive and Strichartz estimates.

of derivatives in estimate (2) as it deviates from the optimal regularity predicted by scale invariance). In Euclidean space $\mathbb{R}^d$ with $g = (\delta_{ij})$, (2) holds with $\sigma = 0$ and extends globally in time, $T = +\infty$.

The canonical path leading to such Strichartz estimates is to obtain a stronger, fixed time, dispersion estimate, which is then combined with energy conservation, interpolation and a duality argument to obtain (2). Dispersion for the linear Schrödinger flow in $\mathbb{R}^d$ reads

$$\|e^{\pm it\Delta_{\mathbb{R}^d}}\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \le C(d)t^{-d/2} \quad \text{for all } t \neq 0. \tag{3}$$

Indeed, (3) and the unitary property of the flow on $L^2(\mathbb{R}^d)$ are sufficient to obtain all known Strichartz estimates; endpoint cases are more delicate (see [8, 17, 30]).

On any boundaryless Riemann manifold (Ω, g) one may follow the same path, replacing the exact formula by a parametrix, constructed locally within a small ball, thanks to finite speed of propagation for waves or in semiclassical time for Schrödinger – short-time, wavelength-sized intervals (e.g. their size is the inverse of the frequency), allowing for almost finite speed of propagation. By time rescaling, dispersion for the semiclassical Schrödinger equation in Euclidean space reads, with $\psi \in C_0^\infty$ being a smooth cutoff to localize frequencies and $D_t = -i\partial_t$, and t_0 depending on the injectivity radius of (Ω, g),

$$\|\psi(hD_t)e^{\pm ith\Delta_{\mathbb{R}^d}}\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \le \frac{C(d)}{h^d}\min\Big(1, \Big(\frac{h}{t}\Big)^{\frac{d}{2}}\Big) \quad \text{for all } 0 < |t| \le t_0.$$

While for $\Omega = \mathbb{R}^d$, dispersive properties of (1) are well understood, studying dispersive equations of Schrödinger type on manifolds (curved geometry, variable metric) started with Bourgain's work on KdV and Schrödinger on the torus, and then expanded in different directions, all of them with low regularity requirements (e.g. Staffilani–Tataru [27], Burq–Gérard–Tzvetkov [6, 7] for Schrödinger; Smith [22, 23], Tataru [28], Bahouri–Chemin [2, 3], Klainerman–Rodnianski [18] and Smith–Tataru [25, 26] for wave equations). In [7], these linear estimates were used, together with a classical argument due to Yudovitch, to obtain global well-posedness for the defocusing cubic NLS on a generic three-dimensional compact manifold without boundary. We aim to match this result in our context, with a model convex boundary.

For compact manifolds (even without boundary) one cannot expect linear estimates to behave like in the Euclidean case: eventually a loss will occur, due to the volume being finite. No long-time dispersion of wave packets may occur as they have nowhere to disperse. Long-time estimates for the wave equation are unknown, while in the case of the Schrödinger equation, the infinite speed of propagation immediately produces unavoidable losses of derivatives in dispersive estimates. Informally, this may be related to the existence of eigenfunctions, but the complete understanding of the loss mechanism is still a delicate issue, even on the torus. On domains with boundaries, there are additional difficulties related to reflected waves. Partial progress was made in [1] and then in [4, 5], following the general strategy of the low-regularity, boundaryless case: reflect the metric across the boundary and deal with a boundaryless domain whose metric is only Lipschitz at the interface. Such results hold for any (smooth) boundary, regardless of its shape: however, they apply to three-dimensional NLS only for nonlinearities that are weaker than cubic: [5] obtains global well-posedness for smooth nonlinearities $F(v)$ with growth at most $|v|^{2/5}v$.

In the last decade, additional progress has been made for the wave equation on domains with convex boundary. Our first result [13], which deals with the model case of a strictly convex domain, highlights a loss in dispersion for the solution to the linear wave equation that we informally relate to caustics, generated in arbitrarily small time near the boundary. Such caustics appear when optical rays are no longer diverging from each other in the normal direction, where less dispersion occurs as compared to the $\mathbb{R}^d$ case. Our so-called Friedlander model domain is the half-space, for $d \geq 2$, $\Omega_d = \{(x,y) \mid x > 0,\ y \in \mathbb{R}^{d-1}\}$ with the metric g_F inherited from the Laplace operator

$$\Delta_F = \partial_x^2 + \sum_j \partial_{y_j}^2 + x \sum_{j,k} q_{j,k} \partial_{y_j} \partial_{y_k}, \tag{4}$$

where $q_{j,k}$ are constants and $q(\theta) = \sum_{j,k} q_{j,k} \theta_j \theta_k$ is a positive definite quadratic form. Note that q is not, in general, invariant by rotations and we cannot reduce to the radial case in y, unlike [13], where $q(\theta) = |\theta|^2$. One may see Δ_F as the Laplace operator in geodesic normal coordinates near the boundary, but where one would freeze all coefficients $q_{j,k}(x,y)$ to their value on the boundary. Strict convexity of Ω_d with the metric inherited from Δ_F is equivalent to ellipticity of $\sum_{j,k} q_{j,k} \partial_{y_j} \partial_{y_k}$. When $q_{j,j} = 1$ and $q_{j \neq k} = 0$ (i.e. when $q(\theta) = |\theta|^2$) the domain (Ω_d, g_F) is, indeed, a first-order approximation of the unit disk in polar coordinates (r, θ): set $r = 1 - x/2$, $\theta = y$.

Let $h, a \in (0,1)$: if $u_a(t,x,y) = \cos(t\sqrt{|\Delta_F|})(\delta_{x=a,y=0})$ denotes the linear wave flow on $(\Omega, g) = (\Omega_d, g_F)$ with data $\delta_{x=a,y=0}$ and Dirichlet boundary condition, then, for $|t| \geq h$, [13] proves

$$\begin{aligned} &\|\psi(hD_t) u_a(t,\cdot)\|_{L^\infty} \\ &\quad \leq C(d) h^{-d} \min\Big\{1, (h/t)^{\frac{d-2}{2}} \Big(\Big(\frac{h}{t}\Big)^{1/2} + \Big(\frac{h}{t}\Big)^{1/3} + a^{1/4} \Big(\frac{h}{t}\Big)^{1/4}\Big)\Big\}. \end{aligned} \tag{5}$$

Moreover, (5) is sharp, as there exists a sequence $(t_n)_n$ such that equality holds. This optimal $\frac{1}{4}$ loss in the $\frac{h}{t}$ exponent is unavoidable for small a and is due to swallowtail-type singularities in the wave front set of u_a. This first result opened several directions, from the generic convex case [11] to understanding more complicated boundary shapes [19].

In the present work, we address the same set of issues for the Schrödinger equation, where parallel developments were expected, at least in the so-called semiclassical setting (recall that "semiclassical" means, in our setting, dealing with time intervals whose size is comparable to the wavelength h, which reduces to almost finite speed of propagation). In the nontrapping case, results for the classical Schrödinger equation may follow when combined with smoothing effects, but we will not address this situation (we model the interior of a convex). In the case of a convex boundary, even the wavelength-sized time behavior is complicated due to the existence of gliding rays. Let $h \in (0,1)$ and consider the semiclassical Schrödinger equation inside the Friedlander domain (Ω_d, g_F), with Δ_F given in (4) and Dirichlet boundary condition

$$ih\partial_t v_h - h^2 \Delta_F v_h = 0, \quad v_{h|t=0} = v_{h,0}, \quad v_{h|\partial\Omega_d} = 0. \tag{6}$$

With this rescaling, we are dealing with (uniformly) bounded intervals rather than h-sized intervals.

Theorem 1. *Let $\psi \in C_0^\infty([\frac{1}{2}, \frac{3}{2}])$, $0 \le \psi \le 1$. There exists $C(d) > 0$, $T_0 > 0$ and $a_0 \le 1$ such that, for all $a \in (0, a_0]$, $h \in (0,1)$, $|t| \in (h, T_0]$, $v_h(t, \cdot)$ solution to (6) with data $v_{h,0}(x,y) = \psi(hD_y)\delta_{x=a,y=0}$,*

$$\|\psi(hD_t)v_h(t,x,y)\|_{L^\infty(\Omega_d)} \le \frac{C(d)}{h^d}\left(\frac{h}{|t|}\right)^{\frac{(d-1)}{2}+\frac{1}{4}}.$$

Moreover, for all $h^{2/3} < a$, for all $|t| \in (\sqrt{a}, \min(T_0, ah^{-1/3})]$, our bound is saturated:

$$\|\psi(hD_t)v_h(t,x,y)\|_{L^\infty(\Omega_d)} \sim \frac{a^{\frac{1}{4}}}{h^d}\left(\frac{h}{|t|}\right)^{\frac{(d-1)}{2}+\frac{1}{4}}.$$

Important additional difficulties appear as compared to the wave equation: for not too small a, the Green function for the wave flow can be explicitly expressed as a sum of “time-almost-orthogonal” waves, which are essentially supported between a finite number of consecutive reflections; in [13], we were therefore reduced to obtaining good dispersion bounds for a *finite* sum of waves well localized in both time and tangential variables. We will establish a suitable subordination formula that yields a similar representation of the Schrödinger flow as a sum of wave packets (see formula (16)); nonetheless, at a given time t, *all* waves in this sum provide important contributions, because they travel with different speeds. To sum all these contributions we need sharp bounds for each of them, similar to those obtained in [15] for waves. Such refined bounds are obtained, for each wave in the sum over consecutive reflections, in Propositions 4, 5 and 6. The dispersive bounds for the Green function are then obtained in Propositions 7, 8, 9 and 10.

For very small a, writing a parametrix as a sum over reflections no longer helps. Using the spectral decomposition of the data in terms of eigenfunctions of the Laplace operator allows us to obtain a parametrix as a sum over the zeros of the Airy function (see formula (11)). With the wave equation, the usual dispersion estimate holds for each term, hence we can sum sufficiently many of them and still get good bounds. However, for the semi-classical Schrödinger flow, even the very first modes – localized at distance $h^{2/3}$ from $\partial\Omega$ (known as gallery modes) yield a *sharp* loss of $\frac{1}{6}$ in both dispersion and Strichartz estimates (see [10]). This regime will be dealt with in Section 3.2.

Theorem 2. *Let $d \ge 2$, (q,r) such that $\frac{1}{q} \le (\frac{d}{2} - \frac{1}{4})(\frac{1}{2} - \frac{1}{r})$ and $s = \frac{d}{2} - \frac{2}{q} - \frac{d}{r}$. There exist $C(d) > 0$, $T_0 > 0$ such that, for v a solution to (6) with data $v_{h,0} \in L^2(\Omega_d)$,*

$$\|\psi(hD_t)v_h\|_{L^q([-T_0,T_0],L^r(\Omega_d))} \le C(d)h^{-s}\|v_{h,0}\|_{L^2(\Omega_d)}.$$

The proof of Theorem 2 follows from Theorem 1 using the classical TT^* argument and the endpoint argument of Keel–Tao [17] for $q = 2$ when $d \geq 3$. The (scale-invariant) loss at the semiclassical level corresponds to $\frac{1}{4}$ derivative in space, as illustrated with $d = 2$, for which the (forbidden) endpoint $(2, \infty)$ with $s = 0$ is replaced by $(8/3, \infty)$ with $s = 1/4$. This improves [4] where for $d = 2$, one has $(3, \infty)$. More generally, [4] obtains $(2, \infty)$ as an endpoint for $d \geq 3$, e.g. $s = d/2 - 1$, whereas we have $(2, 2(2d - 1)/(2d - 5))$ as our endpoint pair, with $s = 1/(2d - 1)$. For $d = 3$, our endpoint pair is $(2, 10)$: that $10 < +\infty$ allows us to adapt the argument from [7] and obtain well-posedness for the cubic equation, as alluded to earlier.

We set Ω to be a compact manifold such that, in a local coordinate chart that intersects its boundary, the metric may be expressed as in our model domain with metric inherited from Δ_F given in (4). One may easily construct such three-dimensional manifolds: for simplicity, we illustrate what can be done in two dimensions, where one can better visualize the corresponding manifold as embedded in $\mathbb{R}^3$. If one periodizes the y variable in Ω_2, we may see it as the surface of an upper cylinder $x > 0$ of radius 1 in $\mathbb{R}^3$, where y is really an angle in the two-dimensional plane $\{x = 0\}$. This surface may be truncated at $x = 1$ and we may extend it smoothly with a (compact) cap to get a Riemannian manifold, say with the induced Euclidean metric within a subset of the cap, the metric from our model in the bottom part of the upper cylinder and a smooth transition in between. One could alternatively connect two copies of our truncated upper cylinder, or connect one with another one where the operator is chosen to be $\partial_x^2 + (1 + x)^{-1}\partial_y^2$ (so as to have a convex boundary on one side and a concave one on the other). These last two examples may also be seen as subdomains of a two-dimensional torus (either sliced circularly in the middle or sliced horizontally). For such a manifold Ω and $d = 3$ we have the following theorem:

Theorem 3. *Let $d = 3$ and $v_0 \in H_0^1(\Omega)$. There exists a unique global-in-time solution $v \in C_t(H_0^1(\Omega))$ to (1) with $\kappa = 1$ (defocusing equation), and its energy is conserved along the flow. For $\kappa = -1$ (focusing equation), the result holds locally in time, and globally provided the mass of v_0 is sufficiently small.*

Moreover, as in the boundaryless case, preservation of regularity holds and one may adapt the argument of [21] to obtain exponential growth for the H^m norm of the solution, where $m \in \mathbb{N}$, $m > 1$. Here we state and prove such growth for $m = 2, 4$ as higher regularity would require dealing with suitable compatibility conditions for the data, which are outside our scope here.

Theorem 4. *Let $d = 3$ and $v_0 \in H_0^1(\Omega) \cap H^m(\Omega)$ with $m = 2, 4$, $\Delta_g v_0 \in H_0^1(\Omega)$ if $m = 4$. Then the solution v from Theorem 3 is $C_t(H^m(\Omega))$, and its norm grows at most exponentially: there exists $C = C(m, \|v_0\|_{H^m(\Omega)})$ such that*

$$\|v(t, \cdot)\|_{H^m(\Omega)} \leq C \exp(Ct).$$

Therefore, well-posedness (and growth of the H^2 norm) for the defocusing cubic equation on such model convex domains is similar to that of generic boundaryless manifolds, and we expect it will hold on any generic three-dimensional compact manifold with strictly convex boundary once Theorem 2 is generalized to such manifolds.

We now briefly discuss linear Strichartz estimates and their optimality. In [9] we proved that there must be a loss of at least $\frac{1}{6}$ derivatives in Strichartz estimates for (6), which is obtained when the data is a gallery mode. Whether this result is sharp, or whether a loss in the semiclassical setting *should* provide losses in classical time in the case of a generic nontrapping domain where concave portions of the boundary could act like mirrors and refocus wave packets (yielding unavoidable losses in dispersion), is unknown at present. In fact, understanding Strichartz estimates in exterior domains seems to be a very delicate task: obstructions from the compact case no longer apply, at least in the case of nontrapping obstacles. Thus, one may ask whether *all Strichartz estimates hold*. The conflict between this questioning and the failure of semiclassical Strichartz (and dispersion) near the boundary is only apparent: for nontrapping domains, a wave packet would spend too short a time in too narrow a region near the boundary to be a contradiction to classical Strichartz.

For the wave equation, Strichartz estimates with losses were obtained in [4] using short-time parametrices constructions from [24]. As already noticed, the main advantage of [4] is also its main weakness: by considering only time intervals that allow for no more than one reflection of a given wave packet, one may handle any boundary but one does not see the full effect of dispersion in the tangential variables. New results in both positive and negative directions were obtained recently, for strictly convex domains: [15] proves Strichartz estimates for the wave equation to hold true on the domain $(\Omega_{d=2}, g_F)$ with at most $\frac{1}{9}$ loss. For $d = 2$, [4] obtained $\frac{1}{6}$ instead of $\frac{1}{9}$ (but for any boundary), while [13] provides $\frac{1}{4}$. Arguments from [15] rely on improving the parametrix construction of [13] and the resulting bounds on the Green function: degenerate stationary phase estimates in [13] may be refined to pinpoint the space-time location of swallowtail singularities (worst-case scenario). It turns out that, for the wave equation, such singularities only happen at an exceptional, discrete set of times. The proof of Theorem 1 will rely on similar refinements of degenerate stationary phase estimates, together with refined estimates on gallery modes from [9], all of which are of independent interest.

Adapting the parametrix construction for the wave flow from [11], one may extend Theorem 1 to a domain Ω whose boundary is everywhere strictly (geodesically) convex: for every point $(0, y_0) \in \partial\Omega$ there exists $(0, y_0, \xi_0, \eta_0) \in T^*\Omega$ where the boundary is micro-locally strictly convex, i.e. such that there exists a bicharacteristic passing through $(0, y_0, \xi_0, \eta_0)$ that intersects $\partial\Omega$ tangentially having exactly second-order contact with the boundary and remaining in the complement of $\partial\bar{\Omega}$. This will be addressed elsewhere.

It should be mentioned that, although Ω_2 is a good approximation of the unit disk in polar coordinates, the proof of Theorem 1 does not (immediately) provide the same results for the disk: in fact, one of the main features of the Friedlander model is that $-\Delta_F$ (given in (4)) with Dirichlet boundary condition has explicit eigenfunctions depending on the zeros of the Airy function (see Lemma 2) which form a Hilbert basis of L^2. Decomposing the initial data in terms of these eigenfunctions allows us to obtain a precise parametrix as a spectral sum (which may be further transformed, via a Poisson-type summation, in a sum of oscillating integrals, depending on the number of reflections on the boundary). Although a complete system of eigenfunctions on the unit disk $\{r \le 1, \ \theta \in [0, 2\pi)\}$ is also explicit and given by $e_{n,m}(r, \theta) := e^{in\theta} J_n(\lambda_{n,m} r)$ (where J_n is the nth Bessel function and $\lambda_{n,m}$ the mth positive zero of J_n), the same approach may turn out to be much more complicated as the expansion of $\lambda_{n,m}$ does not seem to allow us to proceed as in Lemmas 1 and 3 below to obtain a suitable form of the solution in terms of oscillatory integrals. Hence for "not too small a" one would need to solve the eikonal equation (as done in [9, 13] in the case of the wave equation) in order to force a solution as a sum of integrals with Airy phase functions in order to obtain a good parametrix: the main difficulty with this approach is to obtain a "good control" of the symbols as the number of such integrals in the sum defining the parametrix is very large (as, in the Schrödinger case, all the waves overlap at a given time). Combining both methods (the use of Bessel functions for $a \lesssim h^{2/3}$ and of the Airy phase functions for $a \gg h^{2/3}$) may work, but we think that the best approach for the disk is the one in [11] (that gets simplified by the particular form of the operator).

One expects the interior of a strictly convex domain to be a worst-case scenario. At the opposite end, we now have a much better understanding outside a strictly convex obstacle, where the full set of Strichartz estimates are known to hold ([10]) and where dispersion was recently addressed in [12], where diffraction effects related to the Arago–Poisson spot turn out to be significant for $d \ge 4$.

We conclude this introduction with a brief overview of the content in the next sections. In Section 2 we express the (spectrally localized) Green function for (6) first as a spectral sum over the spectrum of (4), then, using the Poisson summation formula, as a sum of oscillatory integrals indexed by the number of reflections on the boundary. In Section 3, both these formulas are used to obtain the dispersive bounds of Theorem 1, depending on the size of the distance to the boundary $a > 0$ of the initial data: firstly, when $a > \max(h^{2/3-\epsilon}, (ht)^{1/2})$, the sum of oscillating integrals (16) is particularly useful as it allows us to apply the stationary phase arguments in order to reduce the Green function to a simpler form (as in Corollary 1). As only the waves that leave at $t = 0$ from $x = a$ within a small cone of directions of aperture $\sqrt{a}$ may provide $\frac{1}{4}$ loss, we separate this "tangential" case, dealt with in Section 3.1.1, from the "transverse" case dealt with in Section 3.1.2. In particular, in the tangential case, we state three main results (Propositions 4, 5, 6) which provide refined estimates for each integral in the sum defining the Green function in this regime, depending on the number of reflections and the position of the spatial variables: these results allow us to achieve the proof of Theorem 1 for "not too small" a. The optimality is shown in Section 3.1.3. The case of small values of a is dealt with in Section 3.2 when we use the spectral formula (11) and provide, as before, sharp dispersive bounds for each term and sum all the contributions.

In Section 4 we prove Propositions 4, 5, 6 in full detail. In Section 5 we deal with nonlinear applications and prove first Theorem 3 (in Section 5.1), then Theorem 4 (in Section 5.2).

In the remainder of the paper, $A \lesssim B$ means that there exists a constant C such that $A \le CB$; this constant may change from line to line and is independent of all parameters except the dimension d. It will be explicit when (very occasionally) needed. Similarly, $A \sim B$ means both $A \lesssim B$ and $B \lesssim A$.

2. The semiclassical Schrödinger propagator: spectral analysis and parametrix construction

We recall some notation, where Ai denotes the standard Airy function (see e.g. [29] for well-known properties of the Airy function), $\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(\frac{\sigma^3}{3}} + \sigma x)\, d\sigma$. Define

$$A_\pm(z) = e^{\mp i\pi/3}\,\mathrm{Ai}(e^{\mp i\pi/3}z) = -e^{\pm 2i\pi/3}\,\mathrm{Ai}(e^{\pm 2i\pi/3}(-z)) \quad \text{for } z \in \mathbb{C}.$$

Then one checks that $\mathrm{Ai}(-z) = A_+(z) + A_-(z)$ (see [29, equation (2.3)]). The next lemma is proved in [14, Lemma 1] and requires the classical notion of asymptotic expansion: a function $f(w)$ admits an asymptotic expansion for $w \to 0$ when there exists a (unique) sequence $(c_n)_n$ such that, for any n, $\lim_{w\to 0} w^{-(n+1)}(f(w) - \sum_0^n c_n w^n) = c_{n+1}$. We denote $f(w) \sim_w \sum_n c_n w^n$.

Lemma 1 (See [14, Lemma 1]). *For $\omega \in \mathbb{R}$, define $L(\omega) = \pi + i\log\frac{A_-(\omega)}{A_+(\omega)}$. Then L is real analytic and strictly increasing. We also have*

$$L(0) = \frac{\pi}{3}, \quad \lim_{\omega\to-\infty} L(\omega) = 0, \quad L(\omega) = \frac{4}{3}\omega^{\frac{3}{2}} + \frac{\pi}{2} - B(\omega^{\frac{3}{2}}) \quad \text{for } \omega \ge 1,$$

with $B(u) \sim_{1/u} \sum_{k=1}^\infty b_k u^{-k}$, $b_k \in \mathbb{R}$, $b_1 > 0$. Finally, $L(\omega_k) = 2\pi k$ and $L'(\omega_k) = 2\pi\int_0^\infty \mathrm{Ai}^2(x-\omega_k)\,dx$, where here and hereafter, $\{-\omega_k\}_{k\ge1}$ denotes the zeros of the Airy function in decreasing order.

2.1. Spectral analysis of the Friedlander model

Our domain is $\Omega_d = \{(x,y) \in \mathbb{R}^d \mid x > 0,\ y \in \mathbb{R}^{d-1}\}$ and the Laplacian Δ_F given by (4). As $-\Delta_F$ has constant coefficients in y, taking the Fourier transform in the y variable, it transforms into $-\partial_x^2 + |\theta|^2 + xq(\theta)$. For $\theta \ne 0$, this operator is a positive self-adjoint operator on $L^2(\mathbb{R}_+)$, with compact resolvent.

Lemma 2 (See [14, Lemma 2]). *There exist eigenfunctions $\{e_k(x,\theta)\}_{k\ge0}$ of $-\partial_x^2 + |\theta|^2 + xq(\theta)$ with corresponding eigenvalues $\lambda_k(\theta) = |\theta|^2 + \omega_k q(\theta)^{2/3}$, that form a Hilbert basis for $L^2(\mathbb{R}_+)$. These eigenfunctions are explicit in terms of Airy functions:*

$$e_k(x,\theta) = \frac{\sqrt{2\pi q(\theta)^{1/3}}}{\sqrt{L'(\omega_k)}}\,\mathrm{Ai}(xq(\theta)^{1/3} - \omega_k),$$

and $L'(\omega_k)$ (with L from Lemma 1) is such that $\|e_k(\cdot,\theta)\|_{L^2(\mathbb{R}_+)} = 1$.

For $x_0 > 0$, $\delta_{x=x_0}$ on $\mathbb{R}_+$ may be decomposed as

$$\delta_{x=x_0} = \sum_{k\geq 1} e_k(x,\theta)e_k(x_0,\theta).$$

At fixed t_0, consider $u(t_0,x,y) = \psi(hD_y)\delta_{x=x_0,y=y_0}$, where $h \in (0,1)$ is a small parameter and $\psi \in C_0^\infty([\frac{1}{2},\frac{3}{2}])$. Then the (localized in θ) Green function for (6) on Ω_d reads

$$\begin{aligned} &G_h((t,x,y),(t_0,x_0,y_0)) \\ &\quad = \sum_{k\geq 1}\int_{\mathbb{R}^{d-1}} e^{ih(t-t_0)\lambda_k(\theta)} e^{i\langle y-y_0,\theta\rangle}\psi(h|\theta|)e_k(x,\theta)e_k(x_0,\theta)\,d\theta. \end{aligned} \tag{7}$$

In addition to the cutoff $\psi(h|\theta|)$, we may add a spectral cutoff $\psi_1(h\sqrt{\lambda_k(\theta)})$ under the θ integral without changing its contribution modulo $O(h^\infty)$ terms, where ψ_1 is also such that $\psi_1 \in C_0^\infty([\frac{1}{2},\frac{3}{2}])$. Indeed, as D_y commutes with Δ_F and using Lemma 2,

$$-\Delta_F\big(\psi(h|\theta|)e^{i\langle y,\theta\rangle}e_k(x,\theta)\big) = \lambda_k(\theta)\psi(h|\theta|)e^{i\langle y,\theta\rangle}e_k(x,\theta).$$

On the flow, this is nothing but $\psi_1(hD_t)$ and this smoothes out the Green function that solves (6). As remarked in [13] (see also [15]) for the wave propagator, after adding $\psi_1(h\sqrt{\lambda_k(\theta)})$, the significant part of the sum over k in (7) becomes a finite sum over $k \lesssim \frac{1}{h}$. Indeed, with $\theta = D_y$, $\tau = \frac{h}{i}\partial_t = hD_t$, $\xi = \frac{h}{i}\partial_x = hD_x$, $\eta = \frac{h}{i}\nabla_y = hD_y = h\theta$, the characteristic set of $ih\partial_t - h^2\Delta_F$ is $\tau = \xi^2 + |\eta|^2 + xq(\eta)$. Using $\tau = hD_t = h\lambda_k(D_y)$, one obtains (at the symbolic level) that on the micro-support of any gallery mode associated to ω_k we have

$$\begin{aligned} h^{2/3}\omega_k q^{2/3}(\eta) &= |\xi|^2 + xq(\eta), \\ \theta = D_y &= \eta/h. \end{aligned} \tag{8}$$

We may assume that, on the support of $\psi(\eta)\psi_1(h\sqrt{\lambda_k(\eta/h)})$, one has $h^{2/3}\omega_k \leq \varepsilon_0$ with a small ε_0: this is compatible with (8) as this amounts to $|\xi|^2 \lesssim \varepsilon_0$. Taking into account the asymptotic expansion $\omega_k \sim k^{2/3}$, the condition $h^{2/3}\omega_k \leq \varepsilon_0$ yields $k \lesssim \varepsilon_0/h$, which is the desired finite sum.

As in [13], the remaining part of the Green function (corresponding to larger values of k) will essentially be transverse: at most one reflection for $t \in [0,T_0]$ with T_0 small (depending on the above choice of ε_0). Hence, this regime can be dealt with as in [4] to get the free space decay and we will ignore it in the upcoming analysis.

Reducing the sum to $k \leq \varepsilon_0/h$ is equivalent to adding a spectral cutoff $\phi_{\varepsilon_0}(x + h^2D_x^2/q(\theta))$ in the Green function, where $\phi_{\varepsilon_0} = \phi(\cdot/\varepsilon_0)$ for some smooth cut-off function $\phi \in C_0^\infty([-1,1])$. Using that the eigenfunctions of the operator $-\partial_x^2 + xq(\theta)$ are also $e_k(x,\theta)$ but associated to the eigenvalues $\lambda_k(\theta) - |\theta|^2 = \omega_k q^{2/3}(\theta)$, we can localize with respect to $x + h^2D_x^2/q(\theta)$. Notice that

$$\Big(x + \frac{h^2D_x^2}{q(\theta)}\Big)e_k(x,\theta) = \Big(\frac{\omega_k q^{2/3}(\theta)}{q(\theta)}\Big)e_k(x,\theta),$$

and this new localization operator is exactly associated by symbolic calculus to the cutoff $\phi_{\varepsilon_0}(\omega_k/q(\theta)^{1/3})$. We therefore set, for $(t_0, x_0, y_0) = (0, a, 0)$,

$$G_h^{\varepsilon_0}(t, x, y, 0, a, 0) := \sum_{k\geq 1} \int_{\mathbb{R}^{d-1}} e^{iht\lambda_k(\theta)} e^{i\langle y,\theta\rangle} \psi(h|\theta|)\psi_1(h\sqrt{\lambda_k(\theta)}) \\ \times \phi_{\varepsilon_0}(\omega_k/q(\theta)^{1/3}) e_k(x,\theta) e_k(a,\theta)\, d\theta. \tag{9}$$

Set $a^\natural = \max(a, h^{2/3})$: in the following we introduce a new, small parameter γ satisfying $a^\natural \lesssim \gamma \leq \varepsilon_0$ and then split the (tangential part of the) Green function $G_h^{\varepsilon_0}$ into a dyadic sum $G_{h,\gamma}$ corresponding to a dyadic partition of unity supported for $\omega_k/q(\theta)^{1/3} \sim \gamma \sim 2^j a^\natural \leq \varepsilon_0$. Let $\psi_2(\cdot/\gamma) := \phi_\gamma(\cdot) - \phi_{\gamma/2}(\cdot)$, set $\Gamma_l(a^\natural) = \{\gamma = 2^j a^\natural,\ l \leq j < \log_2(\varepsilon_0/a^\natural)\}$ (we will use $l = 0, 1, 3$) and decompose ϕ_{ε_0} as

$$\phi_{\varepsilon_0}(\cdot) = \phi_{a^\natural}(\cdot) + \sum_{\gamma\in\Gamma_1(a^\natural)} \psi_2(\cdot/\gamma), \tag{10}$$

which allows us to write $G_{h,\varepsilon_0} = \sum_{a^\natural\leq\gamma<1} G_{h,\gamma}$, where the sum is understood as over dyadic γ's, and (rescaling the θ variable for later convenience) $G_{h,\gamma}$ is written

$$G_{h,\gamma}(t, x, a, y) \\ = \sum_{k\geq 1} \frac{1}{h^{d-1}} \int_{\mathbb{R}^{d-1}} e^{iht\lambda_k(\eta/h)} e^{\frac{i}{h}\langle y,\eta\rangle} \psi(|\eta|)\psi_1(h\sqrt{\lambda_k(\eta/h)}) \\ \times \psi_2(h^{2/3}\omega_k/(q(\eta))^{1/3}\gamma)) e_k(x, \eta/h) e_k(a, \eta/h)\, d\eta. \tag{11}$$

Notice that, when $\gamma = a^\natural$, according to (10), we should, in (11), write $\phi_{a^\natural}$ instead of $\psi_2(\cdot/a^\natural)$. However, for values $h^{2/3}\omega_k \lesssim \frac{1}{2}a^\natural$, the corresponding Airy factors are exponentially decreasing and provide an irrelevant contribution: writing $\phi_{a^\natural}$ or $\psi_2(\cdot/a^\natural)$ yields the same contribution in $G_{h,a^\natural}$ modulo $O(h^\infty)$. In fact, when $a < h^{2/3}$ is sufficiently small, there are no ω_k satisfying $h^{2/3}\omega_k/q^{1/3}(\eta) < h^{2/3}/2$ as $\omega_k \geq \omega_1 > 2.33$ and $|\eta| \in [\frac{1}{2}, \frac{3}{2}]$; on the other hand, when $a \gtrsim h^{2/3}$ and $h^{2/3}\omega_k/q^{1/3}(\eta) \leq a/2$ then the Airy factor of $e_k(a, \eta/h)$ is exponentially decreasing (see [29, Section 2.1.4.3] for details). In order to streamline notation, we use the same formula (11) for each $G_{h,\gamma}$. From an operator point of view, with $G_h(\cdot)$ the semiclassical Schrödinger propagator, we are considering (with $iD = \partial$) $G_{h,\gamma} = \psi(hD_y)\psi_1(h\sqrt{-\Delta_F})\psi_2((x + h^2 D_x^2/q(hD_y))/\gamma)G_h$.

Remark 1. For $a \lesssim h^{2/3}$, [9] proved $\|G_{h,h^{2/3}}(t,\cdot,a,\cdot)\|_{L^\infty} \lesssim \frac{1}{h^d}(\frac{h}{t})^{(d-1)/2}h^{1/3}$. The proof in [9] has $q(\eta) = |\eta|^2$ but easily extends to a positive definite quadratic form q. The subsequent $\frac{1}{6}$ loss in homogeneous Strichartz estimates is optimal for $a \lesssim h^{2/3}$: in [9, Theorem 1.8] we suitably chose Gaussian data whose associated semiclassical Schrödinger flow saturates the above bound (the so-called gallery modes).

We briefly recall a variant of the Poisson summation formula that will be crucial to analyze the spectral sum defining $G_{h,\gamma}$ (see [14, Lemma 3] for the proof).

Lemma 3. *In $\mathcal{D}'(\mathbb{R}_\omega)$, one has $\sum_{N\in\mathbb{Z}} e^{-iNL(\omega)} = 2\pi \sum_{k\in\mathbb{N}^*} \frac{1}{L'(\omega_k)}\delta(\omega-\omega_k)$, e.g. for all $\phi \in C_0^\infty$,*

$$\sum_{N\in\mathbb{Z}} \int e^{-iNL(\omega)}\phi(\omega)\,d\omega = 2\pi \sum_{k\in\mathbb{N}^*} \frac{1}{L'(\omega_k)}\phi(\omega_k). \tag{12}$$

Using (12) on $G_{h,\gamma}$, we transform the sum over k into a sum over $N \in \mathbb{Z}$, as

$$\begin{aligned}
&\widehat{G}_{h,\gamma}(t,x,a,\eta/h)\\
&\quad= \frac{1}{2\pi}\sum_{N\in\mathbb{Z}} \int_{\mathbb{R}} e^{-iNL(\omega)}(|\eta|/h)^{2/3}q^{1/3}(\eta/|\eta|)e^{\frac{i}{h}t|\eta|^2(1+h^{2/3}\omega q^{2/3}(\eta/|\eta|)/|\eta|^{2/3})}\\
&\qquad\qquad\times \psi_1\big(|\eta|\sqrt{1+h^{2/3}\omega q^{2/3}(\eta/|\eta|)/|\eta|^{2/3}}\big)\psi_2\big(h^{2/3}\omega/(q^{1/3}(\eta)\gamma)\big)\\
&\qquad\qquad\times \mathrm{Ai}(xq^{1/3}(\eta)/h^{2/3}-\omega)\,\mathrm{Ai}(aq^{1/3}(\eta)/h^{2/3}-\omega)\,d\omega,
\end{aligned} \tag{13}$$

where $\widehat{G}_{h,\gamma}$ is the Fourier transform in y. For $\sup(a,h^{2/3}) \le \gamma < 1$, we let $\lambda_\gamma = \frac{\gamma^{3/2}}{h}$; when $h^{2/3} \lesssim a$ and $\gamma \sim a$ we write $\lambda := \frac{a^{3/2}}{h}$. The Airy factors are (after rescaling)

$$\mathrm{Ai}(xq^{1/3}(\eta)/h^{2/3}-\omega) = \frac{q^{1/6}(\eta)\lambda_\gamma^{1/3}}{2\pi}\int e^{iq^{1/2}(\eta)\lambda_\gamma\left(\frac{\sigma^3}{3}+\sigma\left(\frac{x}{\gamma}-\omega/(q^{1/3}(\eta)\lambda_\gamma^{2/3})\right)\right)}\,d\sigma.$$

Rescaling $\omega = q^{1/3}(\eta)\lambda_\gamma^{2/3}\alpha = q^{1/3}(\eta)\gamma\alpha/h^{2/3}$ in (13) yields

$$\begin{aligned}
\widehat{G}_{h,\gamma}(t,x,a,\eta/h) = \frac{\lambda_\gamma^{4/3}}{(2\pi)^3h^{2/3}}\sum_{N\in\mathbb{Z}}\int_{\mathbb{R}}\int_{\mathbb{R}^2} &e^{\frac{i}{h}\widetilde{\Phi}_{N,a,\gamma}(\eta,\alpha,s,\sigma,t,x)}q(\eta)\\
&\times \psi_1\big(|\eta|\sqrt{1+\gamma\alpha q(\eta/|\eta|)}\big)\\
&\times \psi_2(\alpha)\,ds\,d\sigma\,d\alpha,
\end{aligned} \tag{14}$$

$$\begin{aligned}
\widetilde{\Phi}_{N,a,\gamma}(\eta,\alpha,s,\sigma,t,x) &= t|\eta|^2(1+\gamma\alpha q(\eta/|\eta|)) - NhL(q^{1/3}(\eta)\lambda_\gamma^{2/3}\alpha)\\
&\quad+\gamma^{3/2}q^{1/2}(\eta)\Big(\frac{\sigma^3}{3}+\sigma\Big(\frac{x}{\gamma}-\alpha\Big)+\frac{s^3}{3}+s\Big(\frac{a}{\gamma}-\alpha\Big)\Big).
\end{aligned} \tag{15}$$

Here,

$$NhL(q^{1/3}(\eta)\lambda_\gamma^{2/3}\alpha) = \frac{4}{3}Nq^{1/2}(\eta)(\gamma\alpha)^{3/2} - NhB(q^{1/2}(\eta)\lambda_\gamma\alpha^{3/2}) + Nh\pi/2,$$

and we recall that, asymptotically,

$$B(q^{1/2}(\eta)\lambda_\gamma\alpha^{3/2}) \sim_{1/(\lambda_\lambda\alpha^{3/2})} \sum_{k\ge1}\frac{b_k}{(q^{1/2}(\eta)\lambda_\gamma\alpha^{3/2})^k},$$

where on the support of $\psi_2(\alpha)$ we have $\alpha \sim 1$. At this point, as $|\eta| \in [\frac{1}{2},\frac{3}{2}]$, we may drop the ψ_1 localization in (14) by support considerations (slightly changing any cut-off support if necessary). Therefore,

$$G_{h,\gamma}(t,x,a,y) = \frac{1}{(2\pi)^3}\frac{\gamma^2}{h^{d+1}}\sum_{N\in\mathbb{Z}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^2} e^{\frac{i}{h}(\langle y,\eta\rangle+\tilde{\Phi}_{N,a,\gamma})}q(\eta)\psi(|\eta|)$$
$$\times \psi_2(\alpha)\, ds\, d\sigma\, d\alpha\, d\eta. \qquad (16)$$

Both formulas (16) and (11) define exactly the same object and both will be necessary to prove the dispersive estimates. The sum over the eigenmodes e_k will be particularly useful for small values of $a \lesssim (ht)^{1/2}$, while for large values of the initial distance to the boundary the sum over N will take over. While both formulas coincide, they are in some sense the dual of each other: for small a, there are fewer terms in the sum over k in (11), while for $a > (ht)^{1/2}$ there are fewer terms in the sum over the reflections N.

As noticed in [13], the symmetry of the Green function (or its suitable spectral truncations) with respect to x and a allows us to restrict the computations of the L^∞ norm to the region $0 \le x \le a$. In other words, instead of evaluating $\|G_h^{\varepsilon_0}\|_{L^\infty(0\le x,y)}(t,\cdot)$ it will be enough to bound $\|G_h^{\varepsilon_0}\|_{L^\infty(0\le x\le a,y)}(t,\cdot)$.

Remark 2. In order to generalize Theorem 1 to a convex domain as outlined in the introduction, our construction of "quasi-modes" from [11] will turn out to be crucial. In the general situation, the regime $a \le h$ turns out to have its own difficulties: even deciding how the initial data should be chosen in order for the Dirichlet condition to be satisfied on the boundary becomes a nontrivial issue. In [11], we bypass our lack of understanding of the eigenfunctions for the Laplace operator and use spectral theory for the model Laplace operator (4) in order to construct suitable initial data for very small a. Thus, constructing a parametrix in the model case (in terms of both eigenmodes and the sum over reflections) and obtaining its best possible decay properties is important in order to further generalize Theorem 1.

3. Dispersive estimates for the semiclassical Schrödinger flow

We now prove dispersive bounds for $G_h^{\varepsilon_0}(t,x,a,y)$ on Ω_d for fixed $|t|\in[h,T_0]$, with small $T_0>0$. We will separately estimate $\|G_{h,\gamma}(t,\cdot)\|_{L^\infty(\Omega_d)}$ for every γ such that $a^\natural \lesssim \gamma \le \varepsilon_0$. Henceforth we assume $t>0$. We sort out several situations, with a fixed (small) $\epsilon>0$. First, $\max(h^{2/3-\epsilon},(ht)^{1/2}) \le a \le \varepsilon_0$: in this case, for all γ such that $a^\natural \lesssim \gamma \le \varepsilon_0$ we have $\max(h^{2/3-\epsilon},(ht)^{1/2}) \le a \lesssim \gamma \le \varepsilon_0$. This is our main case, where only formula (16) is useful; integrals with respect to σ, s have up to third-order degenerate critical points and we need to perform a very detailed analysis of these integrals. In particular, the "tangential" case $\gamma \sim a$ provides the worst decay estimates. When $8a \le \gamma$, integrals in (16) have degenerate critical points of order at most 2. We call this regime "transverse": summing up $\sum_{8a\le\gamma}\|G_{h,\gamma}(t,\cdot)\|_{L^\infty}$ still provides a better contribution than $\|G_{h,a}(t,\cdot)\|_{L^\infty}$. Second, for $a \lesssim \max(h^{2/3-\epsilon},(ht)^{1/2})$, we further subdivide: either $\max(h^{2/3-\epsilon},(ht)^{1/2}) \le \gamma \le \varepsilon_0$, which is similar to the previous "transverse" regime, and estimates will follow using (16);

or $a^\natural \lesssim \gamma \lesssim \max(h^{2/3-\epsilon}, (ht)^{1/2})$, and we will use (11) to evaluate the L^∞ norm of $G_{h,\gamma}$ and its sum over relevant γ's.

3.1. Case $\max(h^{2/3-\epsilon}, (ht)^{1/2}) \le a \le \varepsilon_0$, with (small) $\epsilon > 0$

Here we use (16). As $a^\natural = a$, we consider γ such that $a \lesssim \gamma \le \varepsilon_0$. Let $\lambda_\gamma := \gamma^{3/2}/h$, then $\lambda_\gamma \ge h^{-3\epsilon/2}$. It is worth mentioning that the approach in this section applies for all $h^{2/3-\epsilon} \lesssim a \le \varepsilon_0$, providing sharp estimates for each $G_{h,\gamma}$ for all $h^{2/3-\epsilon} \lesssim a \lesssim \gamma \le \varepsilon_0$; however, when summing over $a \lesssim \gamma \le (ht)^{1/2}$, bounds for $G_h^{\varepsilon_0}$ get worse than those from Theorem 1. Hence we restrict to values $\max(h^{2/3-\epsilon}, (ht)^{1/2}) \le a \le \varepsilon_0$, while lesser values will be dealt with differently later.

First, we prove that the sum defining $G_{h,\gamma}$ in (16) over N is essentially finite and we estimate the number of terms in the relevant sum.

Proposition 1. *For a fixed $t \in (h, T_0]$ the sum (16) over N is essentially finite and $0 \le N \lesssim \frac{t}{\sqrt{\gamma}}$. In other words, if M is a sufficiently large constant (depending only on q), then*

$$\frac{1}{(2\pi)^3}\frac{\gamma^2}{h^{d+1}} \sum_{N\in\mathbb{Z}, N \ge \frac{Mt}{\sqrt{\gamma}}} \int_{\mathbb{R}\times\mathbb{R}^{d-1}} \int_{\mathbb{R}^2} e^{\frac{i}{h}(\langle y,\eta\rangle + \tilde{\Phi}_{N,a,\gamma})} q(\eta)\psi(|\eta|)\psi_2(\alpha)\, ds\, d\sigma\, d\alpha\, d\eta$$
$$= O(h^\infty).$$

Proof. The proof follows easily using nonstationary phase arguments for $N \ge M\frac{t}{\sqrt{\gamma}}$ for some M sufficiently large. Critical points with respect to σ, s are such that

$$\sigma^2 = \alpha - x/\gamma, \quad s^2 = \alpha - a/\gamma, \tag{17}$$

and as $x \ge 0$, $\tilde{\Phi}_{N,a,\gamma}$ may be stationary in σ, s only if $|(\sigma, s)| \le \sqrt{\alpha}$. As $\psi_2(\alpha)$ is supported near 1, it follows that we must also have $x \le 2\gamma$, otherwise $\tilde{\Phi}_{N,a,\gamma}$ is nonstationary with respect to σ. If $|(\sigma, s)| \ge (1 + N^\epsilon)\sqrt{\alpha}$ for some $\epsilon > 0$ we can perform repeated integrations by parts in σ, s to obtain $O(((1 + N^\epsilon)\lambda_\gamma)^{-n})$ for all $n \ge 1$. Let χ be a smooth cutoff supported in $[-1, 1]$ and write $1 = \chi(\sigma/(N^\epsilon\sqrt{\alpha})) + (1 - \chi)(\sigma/(N^\epsilon\sqrt{\alpha}))$. Then

$$\begin{aligned}\psi(|\eta|) \sum_{N\in\mathbb{Z}} \int_{\mathbb{R}} \int_{\mathbb{R}^2} & e^{\frac{i}{h}\tilde{\Phi}_{N,a,\gamma}} \psi_2(\alpha)\chi(s/(N^\epsilon\sqrt{\alpha}))(1 - \chi)(\sigma/(N^\epsilon\sqrt{\alpha}))\, ds\, d\sigma\, d\alpha \\ &\lesssim \lambda_\gamma^{-1/3} \sup_{\alpha, |\eta| \in [\frac{1}{2}, \frac{3}{2}]} \left|\mathrm{Ai}\big((a - \gamma\alpha)q^{1/3}(\eta)/h^{2/3}\big)\right| \sum_{N\in\mathbb{Z}} ((1 + N^\epsilon)\lambda_\gamma)^{-n}) \\ &= O(h^\infty),\end{aligned}$$

where in the last line we used $\lambda_\gamma \ge h^{-3\epsilon/2}$, $\epsilon > 0$. In the same way, we can sum on the support of $(1 - \chi)(s/(N^\epsilon\sqrt{\alpha}))$ and obtain an $O(h^\infty)$ contribution. Therefore, we may add cutoffs $\chi(\sigma/(N^\epsilon\sqrt{\alpha}))$ and $\chi(s/(N^\epsilon\sqrt{\alpha}))$ in $G_{h,\gamma}$ without changing its contribution modulo $O(h^\infty)$. Using (15) again, we have, at the critical point of $\tilde{\Phi}_{N,a,\gamma}$ with respect to α,

$$\frac{t}{\gamma^{1/2}}q(\eta) - q^{1/2}(\eta)(s + \sigma) = 2Nq^{1/2}(\eta)\sqrt{\alpha}\Big(1 - \frac{3}{4}B'(\eta\lambda\alpha^{3/2})\Big), \tag{18}$$

and as $|(\sigma,s)|/\sqrt{\alpha}\lesssim 1+N^{\epsilon}$ on the support of $\chi(\sigma/(N^{\epsilon}\sqrt{\alpha}))\chi(s/(N^{\epsilon}\sqrt{\alpha}))$, $\widetilde{\Phi}_{N,a,\gamma}$ may be stationary with respect to α only when $\frac{t}{\sqrt{\gamma}}\sim 2N$. As $B'(\eta\lambda\alpha^{3/2})=O(\lambda_{\gamma}^{-3})=O(h^{9\epsilon/2})$, its contribution is irrelevant. From (17) and (18), if

$$\frac{t}{\gamma^{1/2}}\frac{q^{1/2}(\eta)}{\sqrt{\alpha}}\notin[2(N-1),2(N+1)],\tag{19}$$

then the phase is nonstationary in α. Recall that q is positive definite and let

$$m_0:=\inf_{\Theta\in\mathbb{S}^{d-2}}q^{1/2}(\Theta),\quad M_0=\sup_{\Theta\in\mathbb{S}^{d-2}}q^{1/2}(\Theta).\tag{20}$$

As $|\eta|,\alpha\in[\frac{1}{2},\frac{3}{2}]$ on the support of the symbol and $q^{1/2}(\eta)=|\eta|q^{1/2}(\eta/|\eta|)$, if $2(N-1)>\frac{t}{\sqrt{\gamma}}\times M_0\frac{3/2}{\sqrt{1/2}}$ or if $2(N+1)<\frac{t}{\sqrt{\gamma}}\times m_0\frac{1/2}{\sqrt{3/2}}$, then the phase is nonstationary in α as its first-order derivative behaves like N. Repeated integrations by parts allow us to sum in N as above, and conclude. ∎

Remark 3. We can in fact add an even better localization with respect to σ and s: on the support of $(1-\chi)(\sigma/(2\sqrt{\alpha}))$ and $(1-\chi)(s/(2\sqrt{\alpha}))$ the phase is nonstationary in σ or s, and integrations by parts yield an $O(\lambda_{\gamma}^{-\infty})$ contribution. According to Proposition 1, the sum over N has finitely many terms, and therefore summing yields an $O(h^{\infty})$ contribution.

Lemma 4. *For $\gamma\gtrsim a\geq(ht)^{1/2}$, the factor $e^{iNB(q^{1/2}(\eta)\lambda_{\gamma}\alpha^{3/2})}$ can be moved into the symbol.*

Proof. As $\alpha,q(\eta)\in[\frac{1}{2},\frac{3}{2}]$ on the support of ψ_2, ψ and $N\sim\frac{t}{\sqrt{\gamma}}$, we obtain, using Lemma 1,

$$NB(q^{1/2}(\eta)\lambda_{\gamma}\alpha^{3/2})\sim N\sum_{k\geq 1}\frac{b_k}{(q^{1/2}(\eta)\lambda_{\gamma}\alpha^{3/2})^k}\sim\frac{Nb_1}{q^{1/2}(\eta)\lambda_{\gamma}}\sim\frac{ht}{\gamma^2}.$$

As here we consider only values $(ht)^{1/2}\lesssim\gamma$, this term remains bounded (so it does not oscillate). ∎

We set $\Phi_{N,a,\gamma}=\langle y,\eta\rangle+\widetilde{\Phi}_{N,a,\gamma}-NhB(q^{1/2}(\eta)\lambda_{\gamma}\alpha^{3/2})$: from Lemma 4, in this regime, $\Phi_{N,a,\gamma}$ are the phase functions in the sum of $G_{h,\gamma}$ defined by (16). We have

$$\begin{aligned}\Phi_{N,a,\gamma}&(\eta,\alpha,s,\sigma,t,x,y)\\&=\langle y,\eta\rangle+t|\eta|^2(1+\gamma\alpha q(\eta/|\eta|))\\&\quad+\gamma^{3/2}q^{1/2}(\eta)\Big(\frac{\sigma^3}{3}+\sigma\Big(\frac{x}{\gamma}-\alpha\Big)+\frac{s^3}{3}+s\Big(\frac{a}{\gamma}-\alpha\Big)-\frac{4}{3}N\alpha^{3/2}\Big).\end{aligned}$$

In the following we study, at fixed $|N|\lesssim\frac{1}{\sqrt{\gamma}}$, the integral appearing in the sum (16), which we denote by $V_{N,h,\gamma}(t,x,y)$. Notice that when $N=0$ we deal with the free semiclassical Schrödinger flow.

Proposition 2. *For all $a \in (0, a_0]$, $h \in (0,1)$ and $t \in (h, T_0]$,*

$$\Big| \sum_{\gamma \in \Gamma_0(a)} V_{0,h,\gamma}(t,x,y) \Big| \lesssim \frac{1}{h^d} \Big(\frac{h}{t}\Big)^{d/2}.$$

Proof. In this case ($N = 0$) we use (9), (10) and (16) to write the sum over γ as

$$\sum_{\gamma \in \Gamma_0(a)} V_{0,h,\gamma}(t,x,y)$$
$$= \frac{1}{(2\pi)^3} \frac{1}{h^{d+1}} \int \psi(|\eta|) q(\eta) \phi_{\varepsilon_0}(\alpha)$$
$$\times e^{\frac{i}{h}(\langle y,\eta\rangle + t|\eta|^2(1+\alpha q(\eta/|\eta|)) + q^{1/2}(\eta)(\frac{\sigma^3}{3} + \sigma(x-\alpha) + \frac{s^3}{3} + s(a-\alpha)))} \, d\sigma \, ds \, d\alpha \, d\eta.$$

Set $\xi_1 = \frac{s+\sigma}{2}$ and $\xi_2 = \frac{\sigma - s}{2}$. Then $\sigma = \xi_1 + \xi_2$ and $s = \xi_1 - \xi_2$. The phase in the above integral becomes

$$\langle y, \eta\rangle + t|\eta|^2(1 + \alpha q(\eta/|\eta|)) + q^{1/2}(\eta)\Big(\frac{2}{3}\xi_1^3 + 2\xi_1\xi_2^2 + \xi_1(x + a - 2\alpha) + \xi_2(x-a)\Big)$$
$$= \Phi_{0,a,1}.$$

As $\partial^2_\alpha \Phi_{0,a,1} = 0$ and $\partial^2_{\xi_1,\alpha}\Phi_{0,a,1} = -2q^{1/2}(\eta)$, the usual stationary phase applies in both ξ_1, α and yields a factor h. The critical points are $\xi_{1,c} = \frac{tq^{1/2}(\eta)}{2}$, $\alpha_c = \xi_{1,c}^2 + \xi_2^2 + \frac{x+a}{2}$. The critical point with respect to ξ_2 satisfies $\partial_{\xi_2}\Phi_{0,a,1}|_{\xi_{1,c},\alpha_c} = q^{1/2}(\eta)(4\xi_{1,c}\xi_2 + x - a)$ and the second derivative equals $\partial^2_{\xi_2}\Phi_{0,a,1}|_{\xi_{1,c},\alpha_c} = q^{1/2}(\eta) \times 4\xi_{1,c} = 2tq(\eta)$. For $t/h \gg 1$, the stationary phase applies and yields a factor $(h/t)^{1/2}$. We are left with the integration with respect to η. Using $\alpha \le \varepsilon_0$ on the support of $\phi_{\varepsilon_0}(\alpha)$ and $x \ge 0$, it follows that $\xi_{1,c}^2 + \xi_{2,c}^2 \le \varepsilon_0$. Writing $t|\eta|^2 q(\eta/|\eta|) = tq(\eta) = 2q^{1/2}(\eta)\xi_{1,c}$, the critical value equals

$$t|\eta|^2(1 + \alpha_c q(\eta/|\eta|)) - q^{1/2}(\eta)\Big(\frac{4}{3}\xi_{1,c}^3 + 4\xi_{1,c}\xi_{2,c}^2\Big)$$
$$= t|\eta|^2 + 2q^{1/2}(\eta)\xi_{1,c}\Big(\alpha_c - \frac{2}{3}\xi_{1,c}^2 - 2\xi_{2,c}^2\Big),$$

and a derivative with respect to η_j equals $y_j + 2t\eta_j + \partial_{\eta_j}(q^{1/2}(\eta))\xi_{1,c}(\frac{4}{3}\xi_{1,c}^2 + x + a)$. We conclude by the stationary phase as this yields $\nabla^2_\eta \Phi_{0,a,1}|_{\xi_{1,c},\xi_{2,c},\alpha_c} = 2t\mathbb{I}_{d-1}(1 + O(\varepsilon_0))$. The proof above applies also separately yielding dispersive bounds without loss for each $V_{0,h,\gamma}$. ■

As we set $t > 0$, from now on we only consider $N \ge 1$.

Proposition 3. *Let $N \ge 1$. The phase function $\Phi_{N,a,\gamma}$ can have at most one critical point (α_c, η_c) on the support $[\frac{1}{2}, \frac{3}{2}]$ of the symbol. At critical points in (α, η), the determinant of the Hessian matrix is comparable to $t^{d-1} \times \gamma^{3/2} N$. The stationary phase applies in both $\alpha \in [\frac{1}{2}, \frac{3}{2}]$ and $\eta \in \mathbb{R}^{d-1}$ and yields a decay factor $(h/t)^{(d-1)/2} \times (\lambda_\gamma N)^{-1/2}$.*

Proof. The derivatives of the phase $\Phi_{N,a,\gamma}$ with respect to α, η are

$$\partial_\alpha \Phi_{N,a,\gamma} = \gamma^{3/2} q^{1/2}(\eta)\Big(\frac{t}{\sqrt{\gamma}} q^{1/2}(\eta) - (\sigma+s) - 2N\sqrt{\alpha}\Big),$$

$$\nabla_\eta \Phi_{N,a,\gamma} = y + 2\eta t + \frac{\gamma^{3/2}\nabla q(\eta)}{2q^{1/2}(\eta)}\Big(\frac{\sigma^3}{3} + \sigma\Big(\frac{x}{\gamma} - \alpha\Big) + \frac{s^3}{3} + s\Big(\frac{a}{\gamma} - \alpha\Big) - \frac{4}{3}N\alpha^{3/2} + \frac{2\alpha t}{\sqrt{\gamma}} q^{1/2}(\eta)\Big).$$

At $\partial_\alpha \Phi_{N,a,\gamma} = 0$ and $\nabla_\eta \Phi_{N,a,\gamma} = 0$, the critical points are such that

$$\sqrt{\alpha} = \frac{tq^{1/2}(\eta)}{2N\sqrt{\gamma}} - \frac{s+\sigma}{2N} \tag{21}$$

and also (replacing $2N\sqrt{\alpha}$ by $\frac{t}{\sqrt{\gamma}} q^{1/2}(\eta) - (\sigma+s)$ in the expression of $\nabla_\eta \Phi_{N,a,\gamma}$)

$$2t\Big(\eta + \frac{1}{2}\gamma\alpha\nabla q(\eta)\Big) = -y - \gamma^{3/2}\frac{\nabla q(\eta)}{2q^{1/2}(\eta)}\Big[\frac{\sigma^3}{3} + \sigma\frac{x}{\gamma} + \frac{s^3}{s} + s\frac{a}{\gamma} - \frac{(s+\sigma)\alpha}{3}\Big]. \tag{22}$$

From (19) (and the support condition on η, α), the critical points $\eta_c, \alpha_c \in [\frac{1}{2}, \frac{3}{2}]$ do exist only if

$$(1 - 1/N)\frac{\sqrt{1/2}}{3M_0/2} \le \frac{t}{2N\sqrt{\gamma}} \le (1 + 1/N)\frac{\sqrt{3/2}}{m_0/2}. \tag{23}$$

For $N \ge 2$, fix M sufficiently large that $[(1-\frac{1}{2})\frac{\sqrt{1/2}}{3M_0/2}, (1+\frac{1}{2})\frac{\sqrt{3/2}}{m_0/2}] \subset [\frac{1}{M}, M]$. Then (21) may have a solution on the support of ψ_2 only when $\frac{t}{2N\sqrt{\gamma}} \in [\frac{1}{M}, M]$. For $N = 1$, we obtain the upper bound $\frac{t}{2\sqrt{\gamma}} \le \frac{4}{m_0}\sqrt{3/2}$ but also, using (17), the following lower bounds: either $s + \sigma \ge -\frac{3}{2}\sqrt{\alpha}$, in which case $\frac{t}{2\sqrt{\gamma}} \ge \frac{\sqrt{\alpha}}{4|\eta| M_0}$, or $(s+\sigma) \le -\frac{3}{2}\sqrt{\alpha}$ in which case both s and σ must take nonpositive values and in this case,

$$\frac{t}{2\sqrt{\gamma}} q^{1/2}(\eta) \ge \sqrt{\alpha} + \frac{s+\sigma}{2} \ge \frac{a/\gamma}{2(\sqrt{\alpha} - s)} + \frac{x/\gamma}{2(\sqrt{\alpha} - \sigma)} \ge \frac{a/\gamma}{4\sqrt{\alpha}}.$$

Hence, for $\frac{t}{\sqrt{\gamma}} \le \frac{a/\gamma}{3\sqrt{3/2}M_0}$ the flow does not reach the boundary (no reflections).

Let $N \ge 1$ and $t/\sqrt{\gamma} \ge \frac{a/\gamma}{3\sqrt{3/2}M_0}$ (as otherwise the phase is nonstationary). As $\alpha \in [\frac{1}{2}, \frac{3}{2}]$ and $\gamma \le \varepsilon_0$, (22) may have a critical point η_c only when $|y|/2t \in [\frac{1}{2} + O(\varepsilon_0), \frac{3}{2} + O(\varepsilon_0)]$. Using $\partial_{\eta_j} q(\eta) = 2q_{j,j}\eta_j + \sum_{k\ne j} q_{j,k}\eta_k$, $q_{j,k} = q_{k,j}$ the second-order derivatives become

$$\partial^2_{\alpha,\alpha}\Phi_{N,a,\gamma} = -\gamma^{3/2} q^{1/2}(\eta)\frac{N}{\sqrt{\alpha}},$$

$$\partial_{\eta_j}\partial_\alpha \Phi_{N,a,\gamma} = \frac{\partial_{\eta_j} q(\eta)}{2q(\eta)}\partial_\alpha \Phi_{N,a,\gamma} + \gamma^{3/2}\frac{t}{2\sqrt{\gamma}}\partial_{\eta_j} q(\eta),$$

$$\partial^2_{\eta_j,\eta_j}\Phi_{N,a,\gamma} = 2t\Big(1+\gamma\alpha\frac{(\partial_{\eta_j}q(\eta))^2}{4q(\eta)}\Big)+\frac{\gamma^{3/2}}{q^{1/2}(\eta)}\Big(q_{j,j}-\frac{(\partial_{\eta_j}q(\eta))^2}{4q(\eta)}\Big)$$
$$\times\Big(\frac{\sigma^3}{3}+\sigma\Big(\frac{x}{\gamma}-\alpha\Big)+\frac{s^3}{3}+s\Big(\frac{a}{\gamma}-\alpha\Big)-\frac{4}{3}N\alpha^{3/2}+2\alpha\frac{t}{\sqrt{\gamma}}q^{1/2}(\eta)\Big),$$
$$\partial^2_{\eta_j,\eta_k}\Phi_{N,a,\gamma} = 2t\gamma\alpha\frac{\partial_{\eta_j}q(\eta)}{2q^{1/2}(\eta)}\frac{\partial_{\eta_k}q(\eta)}{2q^{1/2}(\eta)}+\frac{\gamma^{3/2}}{q^{1/2}(\eta)}\Big(q_{j,k}-\frac{\partial_{\eta_j}q(\eta)\partial_{\eta_k}q(\eta)}{4q(\eta)}\Big)$$
$$\times\Big(\frac{\sigma^3}{3}+\sigma\Big(\frac{x}{\gamma}-\alpha\Big)+\frac{s^3}{3}+s\Big(\frac{a}{\gamma}-\alpha\Big)-\frac{4}{3}N\alpha^{3/2}+2\alpha\frac{t}{\sqrt{\gamma}}q^{1/2}(\eta)\Big).$$

At the stationary points, $\nabla^2_{\eta,\eta}\Phi_{N,a,\gamma}\sim 2t(1+O(\gamma))\mathbb{I}_{d-1}+O(\gamma^{3/2})$, where $\mathbb{I}_{d-1}$ denotes the identity matrix in dimension $d-1$. As $\varepsilon_0<1$ is small, we deduce that $\nabla^2_{\eta,\eta}\Phi_{N,a,\gamma}\sim 2t\mathbb{I}_{d-1}$. Hence, the stationary phase with respect to η yields a factor of $(h/t)^{\frac{d-1}{2}}$, while the stationary phase in α yields a factor $(\lambda_\gamma N)^{-1/2}$ for $N\geq 1$. ■

Lemma 5. *Let $N\geq 1$ and $a\lesssim\gamma\leq\varepsilon_0$. The critical point η_c of $\Phi_{N,a,\gamma}$ is a function of $s+\sigma$, $(\sigma-s)^2$, $(\sigma-s)\frac{(x-a)}{\gamma}$, $\frac{y}{2t}$ and $\frac{t}{2N\sqrt{\gamma}}$. There exist smooth, uniformly bounded (vector-valued) functions $\Theta,\tilde{\Theta}$ depending on the small parameter γ, such that*

$$\eta_c^0 := \eta_c|_{\sigma=s=0} = -\frac{y}{2t}+\gamma\Theta\Big(\frac{y}{2t},\frac{t}{2N\sqrt{\gamma}},\gamma\Big),$$
$$\Theta\Big(\frac{y}{2t},\frac{t}{2N\sqrt{\gamma}},\gamma\Big) = -\frac{1}{2}\Big(\frac{t}{2N\sqrt{\gamma}}\Big)^2(q\nabla q)\Big(-\frac{y}{2t}\Big)+\gamma\tilde{\Theta}\Big(\frac{y}{2t},\frac{t}{2N\sqrt{\gamma}},\gamma\Big).$$

Moreover, $\Theta_1 := \frac{t}{\gamma^{3/2}}\partial_\sigma\eta_c$ and $\Theta_2 := \frac{t}{\gamma^{3/2}}\partial_s\eta_c$ are smooth, uniformly bounded functions.

Proof. We start with the second statement. First, let $N\geq 2$ and define M as

$$M := 3\max\Big\{\frac{\sqrt{3/2}}{m_0},\frac{M_0}{\sqrt{1/2}}\Big\},\quad \text{with } m_0, M_0 \text{ introduced in (20).} \tag{24}$$

Then M is large enough that $[(1-\frac{1}{2})\frac{\sqrt{1/2}}{3M_0/2},(1+\frac{1}{2})\frac{\sqrt{3/2}}{m_0/2}]\subset[\frac{1}{M},M]$ and for $\frac{t}{2N\sqrt{\gamma}}\in[\frac{1}{M},M]$ and $\frac{|y|}{2t}\in[\frac{1}{4},2]$, the critical points α_c and η_c of $\Phi_{N,a,\gamma}$ solve (21) and (22). Let $\eta_c^0 := \eta_c|_{\sigma=s=0}$ denote the value of η_c at $\sigma=s=0$, then, using (22), η_c^0 solves the equation

$$\eta_c^0+\frac{1}{2}\gamma\Big(\frac{t}{2N\sqrt{\gamma}}\Big)^2q(\eta_c^0)\nabla q(\eta_c^0) = -\frac{y}{2t}.$$

For $\frac{t}{2N\sqrt{\gamma}}\in[\frac{1}{M},M]$, writing $\eta_c^0=-\frac{y}{2t}+\gamma\Theta(\frac{y}{2t},\frac{t}{2N\sqrt{\gamma}},\gamma)$ yields, for $\Theta(\frac{y}{2t},\frac{t}{2N\sqrt{\gamma}},\gamma)$,

$$\Theta+\frac{1}{2}\Big(\frac{t}{2N\sqrt{\gamma}}\Big)^2(q\nabla q)\Big(-\frac{y}{2t}+\gamma\Theta\Big)=0, \tag{25}$$

which further reads, with $\Theta=(\Theta^{(1)},\dots,\Theta^{(d-1)})$ and for all $1\leq l\leq d-1$, as

$$\Theta^{(l)}+\Big(\frac{t}{2N\sqrt{\gamma}}\Big)^2\sum_{j,k,p}q_{j,k}q_{p,l}\Big(-\frac{y_j}{2t}+\gamma\Theta^{(j)}\Big)\Big(-\frac{y_k}{2t_k}+\gamma\Theta^{(k)}\Big)\Big(-\frac{y_p}{2t}+\gamma\Theta^{(p)}\Big)=0.$$

As $\gamma \le \varepsilon_0 \ll 1$, this equation has a unique solution, which is a smooth function of $(\frac{y}{2t}, \frac{t}{2N\sqrt{\gamma}}, \gamma)$ and $\Theta^{(l)} = (\frac{t}{2N\sqrt{\gamma}})^2(\sum_{j,k,p} q_{j,k}q_{p,l}(\frac{y_j}{2t})(\frac{y_k}{2t})(\frac{y_p}{2t})) + \gamma\widetilde{\Theta}^{(l)}$, where $\widetilde{\Theta} = (\widetilde{\Theta}^{(1)}, \dots, \widetilde{\Theta}^{(d-1)})$ is a smooth function of $(\frac{y}{2t}, \frac{t}{2N\sqrt{\gamma}}, \gamma)$. For $N = 1$, t may take (very) small values but does not vanish where $\Phi_{N=1,a,\gamma}$ may be stationary and therefore (25) still holds and $\frac{|y|}{2t} \in [\frac{1}{4}, 2]$, hence we obtain Θ in the same way.

We now prove that for all $N \ge 1$, η_c is a function of $s+\sigma$, $(\sigma - s)^2$, $(\sigma - s)\frac{(x-a)}{\gamma}$, $\frac{y}{2t}$ and $\frac{t}{2N\sqrt{\gamma}}$. This will be useful later on, especially in the proof of the upcoming Proposition 6. Inserting (21) in (22) yields

$$\begin{aligned}
&\eta_c + \frac{\gamma}{2}\Big(\frac{t}{2N\sqrt{\gamma}}q^{1/2}(\eta_c) - \frac{\sigma+s}{2N}\Big)^2 \nabla q(\eta_c)\\
&\quad = -\frac{y}{2t} - \frac{\gamma^{3/2}}{2t}\frac{\nabla q(\eta_c)}{2q^{1/2}(\eta_c)}\\
&\qquad \times\Big[\frac{\sigma^3}{3} + \sigma\frac{x}{\gamma} + \frac{s^3}{3} + s\frac{a}{\gamma} - \frac{(s+\sigma)}{3}\Big(\frac{t}{2N\sqrt{\gamma}}q^{1/2}(\eta_c) - \frac{\sigma+s}{2N}\Big)^2\Big]. \qquad (26)
\end{aligned}$$

It follows that η_c is a function of $(s+\sigma)$ and $\frac{\sigma^3}{3} + \sigma\frac{x}{\gamma} + \frac{s^3}{3} + s\frac{a}{\gamma}$ and writing the last term in the form $\frac{(s+\sigma)^3}{3} - 4(s+\sigma)((s+\sigma)^2 - (s-\sigma)^2) + (s+\sigma)\frac{(x+a)}{2\gamma} + (\sigma - s)\frac{(x-a)}{2\gamma}$ allows us to conclude. Now taking the derivative with respect to σ in (26) yields

$$\begin{aligned}
&\partial_\sigma\eta_c\Big(\mathbb{I}_{d-1} + O(\gamma) + O\Big(\frac{\gamma^{\frac{3}{2}}}{t}\Big)\Big)\\
&\quad = \frac{\gamma\nabla q(\eta_c)}{2N} + \frac{\gamma^{\frac{3}{2}}\nabla q(\eta_c)}{4tq^{\frac{1}{2}}(\eta_c)}\Big[\sigma^2 + \frac{x}{\gamma} + \frac{\alpha_c^{\frac{1}{2}}}{3}\Big(\frac{s+\sigma}{N} - \alpha_c^{\frac{1}{2}}\Big)\Big], \qquad (27)
\end{aligned}$$

where the second and third terms in brackets in the first line of (27) are smooth, bounded functions of η_c, $\frac{t}{2N\sqrt{\gamma}}$, $(s+\sigma)$ and $\frac{\sigma^3}{3} + \sigma\frac{x}{\gamma} + \frac{s^3}{3} + s\frac{a}{\gamma}$ with coefficients γ and $\gamma^{3/2}/t$, respectively. First, let $N \ge 2$. Then, using $\frac{t}{2N\sqrt{\gamma}} \in [\frac{1}{M}, M]$, we find $\gamma^{3/2}/t \sim \gamma/N$ and therefore $\partial_\sigma\eta_c = O(\gamma^{3/2}/t)$. In the same way we obtain $\partial_s\eta_c = O(\gamma^{3/2}/t)$. Now let $N = 1$. Then $\gamma^{3/2}/t \gtrsim \gamma$ whenever the phase may be stationary, and therefore we still find $\partial_\sigma\eta_c = O(\gamma^{3/2}/t)$ and $\partial_s\eta_c = O(\gamma^{3/2}/t)$. Therefore, $\Theta_1 := \frac{t}{\gamma^{3/2}}\partial_\sigma\eta_c$ (and $\Theta_2 := \frac{t}{\gamma^{3/2}}\partial_s\eta_c$) is a smooth and uniformly bounded vector-valued function depending on

$$\sigma + s, \quad \sigma^2 + \frac{x}{\gamma}, \quad \frac{\sigma^3}{3} + \sigma\frac{x}{\gamma} + \frac{s^3}{3} + s\frac{a}{\gamma}, \quad \Big(\frac{t}{2N\sqrt{\gamma}}, \frac{y}{2t}, \gamma\Big)$$

(and, respectively, on $\sigma + s, s^2 + \frac{a}{\gamma}, \frac{\sigma^3}{3} + \sigma\frac{x}{\gamma} + \frac{s^3}{3} + s\frac{a}{\gamma}$ and $(\frac{t}{2N\sqrt{\gamma}}, \frac{y}{2t}, \gamma)$). In the following we write $\Theta_j = \Theta_j(\sigma, s, \frac{t}{2N\sqrt{\gamma}}, \frac{x}{\gamma}, \frac{a}{\gamma}, \frac{y}{2t}, \gamma)$ for $j \in \{1, 2\}$. ■

Lemma 6. *For all $N \ge 1$, the critical point α_c is such that*

$$\sqrt{\alpha_c} = \frac{t}{2N\sqrt{\gamma}}q^{1/2}(\eta_c^0) - \frac{\sigma}{2N}(1 - \gamma\mathcal{E}_1) - \frac{s}{2N}(1 - \gamma\mathcal{E}_2), \qquad (28)$$

where $\mathcal{E}_j$ are smooth, uniformly bounded functions:

$$\mathcal{E}_1 := \left\langle \int_0^1 \Theta_1\Big(o\sigma, os, \frac{t}{2N\sqrt{\gamma}}, \frac{x}{\gamma}, \frac{a}{\gamma}, \frac{y}{2t}, \gamma\Big)do, \int_0^1 \frac{\nabla q}{2q^{1/2}}(o\eta_c^0 + (1-o)\eta_c)do \right\rangle, \tag{29}$$

$$\mathcal{E}_2 := \left\langle \int_0^1 \Theta_2\Big(o\sigma, os, \frac{t}{2N\sqrt{\gamma}}, \frac{x}{\gamma}, \frac{a}{\gamma}, \frac{y}{2t}, \gamma\Big)do, \int_0^1 \frac{\nabla q}{2q^{1/2}}(o\eta_c^0 + (1-o)\eta_c)do \right\rangle. \tag{30}$$

Proof. Rewrite (21) as

$$\sqrt{\alpha_c} = \frac{t}{2N\sqrt{\gamma}} q^{1/2}(\eta_c^0) - \frac{(\sigma+s)}{2N} + \frac{t}{2N\sqrt{\gamma}}(q^{1/2}(\eta_c) - q^{1/2}(\eta_c^0)).$$

As we have

$$\eta_c - \eta_c^0 = \frac{\gamma^{3/2}}{t}\left\langle (\sigma, s), \int_0^1 (\Theta_1, \Theta_2)\Big(o\sigma, os, \frac{t}{2N\sqrt{\gamma}}, \frac{x}{\gamma}, \frac{a}{\gamma}, \frac{y}{2t}, \gamma\Big)do \right\rangle$$

and

$$q^{1/2}(\eta_c) - q^{1/2}(\eta_c^0) = (\eta_c - \eta_c^0)\int_0^1 \Big(\frac{\nabla q}{2q^{1/2}}\Big)(o\eta_c^0 + (1-o)\eta_c)do, \tag{31}$$

defining $\mathcal{E}_j$ as in (29) and (30) yields (28). ∎

Corollary 1. *There exist $C \neq 0$ (independent of h, a, γ) and $\tilde{\psi} \in C_0^\infty([\frac{1}{4}, 2])$ with $\tilde{\psi} = 1$ on the support of ψ such that*

$$G_{h,\gamma}(t,x,y) = \frac{C}{h^d}\Big(\frac{h}{t}\Big)^{(d-1)/2} \tilde{\psi}\Big(\frac{|y|}{2t}\Big) \sum_{\frac{t}{\sqrt{\gamma}} \sim N \lesssim \frac{1}{\sqrt{\gamma}}} V_{N,h,\gamma}(t,x,y) + O(h^\infty),$$

$$V_{N,h,\gamma}(t,x,y) = \frac{\gamma^2}{h}\frac{1}{\sqrt{\lambda_\gamma N}} \int e^{\frac{i}{h}\phi_{N,a,\gamma}(\sigma,s,t,x,y)} \varkappa(\sigma,s,t,x,y;h,\gamma,1/N)\, d\sigma\, ds,$$

with phase $\phi_{N,a,\gamma}(\sigma,s,t,x,y) = \Phi_{N,a,\gamma}(\eta_c,\alpha_c,\sigma,s,t,x,y)$ and symbol $\varkappa(\cdot;h,\gamma,1/N)$.

This immediately follows from the stationary phase in α and η, with a leading-order term for $\varkappa$ being $q(\eta_c)\psi(|\eta_c|)\psi_2(\alpha_c)e^{iNB(q^{1/2}(\eta_c)\lambda_\gamma\alpha_c^{3/2})}$. Notice that this main contribution to the symbol $\varkappa(\cdot;h,\gamma,1/N)$ has a harmless dependence on the parameters h, a, γ, $1/N$, as $\varkappa(\cdot;h,\gamma,1/N)$ reads as an asymptotic expansion with small parameters $(\lambda_\gamma N)^{-1} = h/(N\gamma^{3/2})$ in α and (h/t) in η, and all terms in the expansions are smooth functions of α_c, η_c. Using Remark 3, we may introduce cutoffs $\chi(\sigma/(2\sqrt{\alpha_c}))$ and $\chi(s/(2\sqrt{\alpha_c}))$, supported for $|(\sigma,s)| \le 2\sqrt{\alpha_c}$ in $V_{N,h,\gamma}$ without changing its contribution modulo $O(h^\infty)$ (as for $|(s,\sigma)|/\sqrt{\alpha} \ge 3/2$, the phase functions are nonstationary).

We are left with integrals with respect to the variables s, σ to estimate $\|V_{N,h,\gamma}(t,\cdot)\|_{L^\infty}$. We first compute higher-order derivatives of the critical value $\Phi_{N,a,\gamma}(\eta_c,\alpha_c,s,\sigma,t,y,x)$, with

$$\partial_\sigma(\Phi_{N,a,\gamma}(\eta_c,\alpha_c,s,\sigma,\cdot)) = \gamma^{3/2}q^{1/2}(\eta_c)\Big(\sigma^2 + \frac{x}{\gamma} - \alpha_c\Big), \tag{32}$$

$$\partial_s(\Phi_{N,a,\gamma}(\eta_c,\alpha_c,s,\sigma,\cdot)) = \gamma^{3/2}q^{1/2}(\eta_c)\Big(s^2 + \frac{a}{\gamma} - \alpha_c\Big). \tag{33}$$

Higher-order derivatives of $\phi_{N,a,\gamma}(\sigma,s,\cdot) := \Phi_{N,a,\gamma}(\eta_c,\alpha_c,\sigma,s,\cdot)$ involve derivatives of critical points α_c, η_c with respect to σ, s:

$$\partial^2_{\sigma,\sigma}(\Phi_{N,a,\gamma}(\eta_c,\alpha_c,\cdot)) = \partial_\sigma\eta_c \frac{\nabla q(\eta)}{2q(\eta)}\Big|_{\eta=\eta_c}\partial_\sigma\phi_{N,a,\gamma} + \gamma^{3/2}q^{1/2}(\eta_c)(2\sigma - 2\sqrt{\alpha_c}\partial_\sigma\sqrt{\alpha_c}), \tag{34}$$

$$\partial^2_{s,s}(\Phi_{N,a,\gamma}(\eta_c,\alpha_c,\cdot)) = \partial_s\eta_c \frac{\nabla q(\eta)}{2q(\eta)}\Big|_{\eta=\eta_c}\partial_s\phi_{N,a,\gamma} + \gamma^{3/2}q^{1/2}(\eta_c)(2s - 2\sqrt{\alpha_c}\partial_s\sqrt{\alpha_c}), \tag{35}$$

$$\partial^2_{\sigma,s}(\Phi_{N,a,\gamma}(\eta_c,\alpha_c,\cdot)) = \partial_\sigma\eta_c \frac{\nabla q(\eta)}{2q(\eta)}\Big|_{\eta=\eta_c}\partial_s\phi_{N,a,\gamma} - \gamma^{3/2}q^{1/2}(\eta_c)(2\sqrt{\alpha_c}\partial_\sigma\sqrt{\alpha_c}), \tag{36}$$

and therefore, when $\partial_s\phi_{N,a,\gamma} = \partial_\sigma\phi_{N,a,\gamma} = 0$, we have

$$\begin{aligned}
\partial^2_{\sigma,\sigma}\phi_{N,a,\gamma}(\eta_c,\alpha_c,s,\sigma,\cdot)|_{\partial_s\phi_{N,a,\gamma}=\partial_\sigma\phi_{N,a,\gamma}=0} &= 2\gamma^{3/2}q^{1/2}(\eta_c)(\sigma - \sqrt{\alpha_c}\partial_\sigma\sqrt{\alpha_c}),\\
\partial^2_{s,s}\phi_{N,a,\gamma}(\eta_c,\alpha_c,s,\sigma,\cdot)|_{\partial_s\phi_{N,a,\gamma}=\partial_\sigma\phi_{N,a,\gamma}=0} &= 2\gamma^{3/2}q^{1/2}(\eta_c)(s - \sqrt{\alpha_c}\partial_s\sqrt{\alpha_c}),\\
\partial^2_{\sigma,s}\phi_{N,a,\gamma}(\eta_c,\alpha_c,s,\sigma,\cdot)|_{\partial_s\phi_{N,a,\gamma}=\partial_\sigma\phi_{N,a,\gamma}=0} &= -2\gamma^{3/2}q^{1/2}(\eta_c)\sqrt{\alpha_c}\partial_\sigma\sqrt{\alpha_c}.
\end{aligned}$$

Remark 4. At critical points we have $\partial_\sigma\sqrt{\alpha_c} = \partial_s\sqrt{\alpha_c}$: derivatives of α_c depend on η_c which solves (22); from (22), $\partial_\sigma\eta_c$ (and $\partial_s\eta_c$) depends upon $(s+\sigma)$, $\sigma^2 + \frac{x}{\gamma}$ and $\sigma^3/3 + \sigma\frac{x}{\gamma} + s^3/3 + s\frac{a}{\gamma}$ (and upon $(s+\sigma)$, $s^2 + \frac{a}{\gamma}$ and $\sigma^3/3 + \sigma\frac{x}{\gamma} + s^3/3 + s\frac{a}{\gamma}$ respectively); at the critical points σ, s we have $\sigma^2 + \frac{x}{\gamma} = s^2 + \frac{a}{\gamma} = \alpha_c$ and we find $\partial_\sigma\eta_c = \partial_s\eta_c$.

3.1.1. "Tangential" waves $a \in [\frac{1}{8}\gamma, 8\gamma]$. We abuse notation and write $G_{h,a} = G_{h,\gamma\sim a}$, $\lambda = a^{3/2}/h = \lambda_{\gamma\sim a}$ and from Corollary 1, with $\phi_{N,a}(\sigma,s,t,x,y) = \Phi_{N,a,a}(\eta_c,\alpha_c,\sigma,s,t,x,y)$,

$$G_{h,a}(t,x,y) = \frac{C}{h^d}\Big(\frac{h}{t}\Big)^{(d-1)/2}\tilde\psi\Big(\frac{|y|}{2t}\Big)\sum_{\frac{t}{\sqrt{a}}\sim N\lesssim\frac{1}{\sqrt{a}}} V_{N,h,a}(t,x,y) + O(h^\infty), \tag{37}$$

$$V_{N,h,a}(t,x,y) = \frac{a^2}{h}\frac{1}{\sqrt{\lambda N}}\int e^{\frac{i}{h}\phi_{N,a}(\sigma,s,t,x,y)}\varkappa(\sigma,s,t,x,y,h,a,1/N)\,d\sigma\,ds. \tag{38}$$

As in the proof of Lemma 4, only values $N \lesssim \lambda$ are of interest: indeed, as $\gamma \gtrsim (ht)^{1/2}$, we then obtain $N \lesssim t/\sqrt{\gamma} \lesssim \gamma^{3/2}/h = \lambda_\gamma$. It will turn out that one needs to separate the cases $N < \lambda^{1/3}$ and $\lambda^{1/3} \lesssim N$. Fix t and set $T = \frac{t}{\sqrt{a}}$: if $\lambda^{1/3} \lesssim T \sim N$, then $\phi_{N,a}$ behaves like the phase of a product of two Airy functions and can be bounded using mainly their respective asymptotic behaviors. When $N \sim T \lesssim \lambda^{1/3}$, $\phi_{N,a}$ may have degenerate critical points up to order 3. We claim that for any t such that $T := \frac{t}{\sqrt{a}} \ll \lambda^{1/3}$ and for any $N \sim T$ there exists a locus of points

$$\mathcal{Y}_N(T) := \{Y \in \mathbb{R}^{d-1} | K_a(\tfrac{Y}{4N}, \tfrac{T}{4N}) = 1\},$$

where K_a is the smooth function to be defined in (39) such that, for all $y \in \sqrt{a}\mathcal{Y}_N(T)$, we have

$$\|G_{h,a}(t,\cdot)\|_{L^\infty(\Omega)} = |G_{h,a}(t,a,a,y)|\big|_{y\in\sqrt{a}\mathcal{Y}_N(t/\sqrt{a})} \sim \frac{1}{h^d}\Big(\frac{h}{t}\Big)^{(d-1)/2} a^{1/4}\Big(\frac{h}{t}\Big)^{1/4}$$

for all $(ht)^{1/2} \lesssim a \lesssim \varepsilon_0$. Optimality then follows.

When dealing with the wave flow in [15], a parametrix is also obtained as a sum of reflected waves: due to the finite speed of propagation, the main contribution at fixed t is provided by waves located between the $(N-1)$th and $(N+1)$th reflections, where $N = [\frac{t}{\sqrt{a}}]$. For each $N \ll \lambda^{1/3}$, the worst bound occurs at a unique time t_N, at $x = a$ and for a unique y_N. In contrast, for the Schrödinger flow, for all $t/\sqrt{a} \ll \lambda^{1/3}$ and all $N \sim t/\sqrt{a}$, we have $|V_{N,h,a}(t,a,y)|\big|_{y\in\sqrt{a}\mathcal{Y}_N(t/\sqrt{a})} \sim \|G_{h,a}(t,\cdot)\|_{L^\infty}$, where $\mathcal{Y}_N(t/\sqrt{a}) \cap \mathcal{Y}_{N'}(t/\sqrt{a}) = \emptyset$ for $N \neq N'$. In other words, the worst bound is reached for any t but only for a small interval in y.

We denote $\alpha_c^0 = \alpha_c|_{s=\sigma=0}$, with α_c obtained in (28). Recall from Lemma 5 (with γ replaced by a) that $\eta_c^0 = -\frac{y}{2t} + a\Theta(\frac{y}{2t}, \frac{t}{2N\sqrt{a}}, a)$ is a smooth function of $(\frac{y}{2t}, \frac{t}{2N\sqrt{a}}, a)$, hence so is $\sqrt{\alpha_c^0} = \frac{t}{2N\sqrt{a}} q^{1/2}(\eta_c^0)$. Let $T = \frac{t}{\sqrt{a}}$, $Y = \frac{y}{\sqrt{a}}$ and define

$$K_a\Big(\frac{Y}{4N}, \frac{T}{2N}\Big) = \sqrt{\alpha_c^0\Big(\frac{Y}{4N}\frac{2N}{T}, \frac{T}{2N}, a\Big)}.$$

Then K_a is smooth in all variables and

$$K_a\Big(\frac{Y}{4N}, \frac{T}{2N}\Big) = \frac{|Y|}{4N} q^{1/2}\Big(-\frac{Y}{|Y|} + a\frac{T}{2N}\frac{4N}{|Y|}\Theta\Big(\frac{Y}{4N}\frac{2N}{T}, \frac{T}{2N}, a\Big)\Big). \tag{39}$$

Proposition 4. *For $\lambda^{1/3} \lesssim T \sim N$, $\frac{x}{a} \le 1$, we have*

$$|V_{N,h,a}(t,x,y)| \lesssim \frac{h^{1/3}}{(N/\lambda^{1/3})^{1/2} + \lambda^{1/6}\sqrt{4N}|K_a(\frac{Y}{4N}, \frac{T}{2N}) - 1|^{1/2}}.$$

Proposition 5. *For $1 \le N < \lambda^{1/3}$ and $|K_a(\frac{Y}{4N}, \frac{T}{2N}) - 1| \gtrsim 1/N^2$, $\frac{x}{a} \le 1$ we have*

$$|V_{N,h,a}(t,x,y)| \lesssim \frac{h^{1/3}}{(1 + 2N|K_a(\frac{Y}{4N}, \frac{T}{2N}) - 1|^{1/2})}.$$

Proposition 6. *For $1 \le N < \lambda^{1/3}$ and $|K_a(\frac{Y}{4N}, \frac{T}{2N}) - 1| \le \frac{1}{4N^2}$, $\frac{x}{a} \le 1$ we have*

$$|V_{N,h,a}(t,x,y)| \lesssim \frac{h^{1/3}}{(N/\lambda^{1/3})^{1/4} + N^{1/3}|K_a(\frac{Y}{4N}, \frac{T}{2N}) - 1|^{1/6}}. \tag{40}$$

Moreover, at $x = a$ and $K_a(\frac{Y}{4N}, \frac{T}{2N}) = 1$ we have

$$|V_{N,h,a}(t,a,y)| \sim \frac{h^{1/3}}{(N/\lambda^{1/3})^{1/4}}.$$

We postpone the proofs of Propositions 4, 5 and 6 to Section 4 and we complete the proof of Theorem 1 in the case $(ht)^{1/2} \lesssim a \sim \gamma \leq \varepsilon_0 < 1$. Therefore, let $\sqrt{a} \lesssim t \lesssim 1$ be fixed and let $N_t \geq 1$ be the unique positive integer such that $T = \frac{t}{\sqrt{a}} > N_t \geq \frac{t}{\sqrt{a}} - 1 = T - 1$, hence $N_t = [T]$, where $[T]$ denotes the integer part of T. If N_t is bounded then the number of $V_{N,h,a}$ with $N \sim N_t$ in the sum (37) is also bounded and we can easily conclude, adding the (worst) bound from Proposition 6 a finite number of times. Assume $N_t \geq 2$ is large enough. We introduce the following notation: for $k \in \mathbb{Z}$ let $I_{N_t,k} := [4(N_t + k) - 2, 4(N_t + k) + 2)$. As $\alpha_c, \eta_c \in [\frac{1}{2}, \frac{3}{2}]$ and $\sqrt{\alpha_c} = \frac{T}{2N} q^{1/2}(\eta_c) - \frac{(\sigma+s)}{2N}$ with $|(\sigma, s)| \leq 2\sqrt{\alpha_c}$ on the support of χ (see Remark 3), we deduce (using (23)) that, for M defined in (24), we have $2N \in [\frac{T}{M}, MT] \subset [\frac{N_t}{M}, M(N_t + 1)]$. Using (37), we then bound $G_{h,a}(t, \cdot)$ as

$$\|G_{h,a}(t,\cdot)\|_{L^\infty(0\leq x\leq a,y)} \lesssim \frac{1}{h^d}\left(\frac{h}{t}\right)^{(d-1)/2} \sup_{x\leq a,y} \sum_{\frac{N_t}{M}\leq 2N\leq M(N_t+1)} |V_{N,h,a}(t,x,y)|.$$

It will follow from the proof of Proposition 6 that the worst dispersive bounds for $V_{N,h,a}$ occurs at $x = a$ (when $\phi_{N,a}$ may have a critical point of order 3). Therefore, we will seek bounds for $G_{h,a}$ especially at $x = a$.

For a fixed y on the support of $\tilde{\psi}(\frac{|y|}{2t})$, recall $Y = \frac{y}{\sqrt{a}}$, then $\frac{1}{4} \leq \frac{|Y|}{2T} \leq 2$, and therefore $|Y| \in [\frac{T}{2}, 4T] \subset [\frac{N_t}{2}, 4(N_t + 1)]$. Using (39) and the fact that $q^{1/2}$ is homogeneous of order 1, it follows that $K_a(\frac{Y}{4N}, \frac{T}{2N})$ is close to 1 when $q^{1/2}(-Y + 2aT\Theta(\frac{Y}{2T}, \frac{T}{2N}, a))$ is sufficiently close to $4N$. As $2 < N_t \leq T \leq 1/\sqrt{a}$, $|Y|/T \in [\frac{1}{2}, 4]$, Θ is bounded and $0 < a \leq \varepsilon_0$ is small, then, for m_0 and M_0 defined in (20),

$$q^{1/2}\Big(-Y + 2aT\Theta\Big(\frac{Y}{2T}, \frac{T}{2N}, a\Big)\Big) \subset [\tfrac{1}{2}N_t(m_0 - \varepsilon_0), 4(N_t + 1)(M_0 + \varepsilon_0)].$$

Setting $k_1 = -N_t(1 - (m_0 - \varepsilon_0)/8)$, $k_2 = (N_t + 1)(M_0 + \varepsilon_0 - 1) + 1$, we have $N_t + k \sim N_t$ and $[\frac{1}{2}N_t(m_0 - \varepsilon_0), 4(N_t + 1)(M_0 + \varepsilon_0)] \subset \bigcup_{k_1\leq k\leq k_2} I_{N_t,k}$. Let

$$\tilde{I}_{N_t,k} := (4(N_t + k) - 1, 4(N_t + k) + 1) \subset I_{N_t,k}.$$

As $I_{N_t,k}$ are disjoint intervals, write

$$\begin{aligned}
&\sup_{x,y}\Bigg(\sum_{\frac{N_t}{M}\leq 2N\leq M(N_t+1)} |V_{N,h,a}(t,x,y)|\Bigg)\\
&\quad= \sup_{k_1\leq k\leq k_2}\Bigg(\sup_{\substack{q^{1/2}(-Y+2aT\Theta(\frac{Y}{2T},\frac{T}{2N},a))\\ \in I_{N_t,k}}}\Bigg(\sum_{\frac{N_t}{M}\leq 2N\leq M(N_t+1)} |V_{N,h,a}(t,a,y)|\Bigg)\Bigg)\\
&\quad\geq \sup_{k_1\leq k\leq k_2}\Bigg(\sup_{\substack{q^{1/2}(-Y+2aT\Theta(\frac{Y}{2T},\frac{T}{2N},a))\\ \in \tilde{I}_{N_t,k}}}\Bigg(\sum_{\frac{N_t}{M}\leq 2N\leq M(N_t+1)} |V_{N,h,a}(t,a,y)|\Bigg)\Bigg). \qquad (41)
\end{aligned}$$

We will use the equality for our upper bound while the last inequality will be relevant for the optimality, through a corresponding lower bound.

Proposition 7. *There exists $C > 0$ (independent of h, a) such that, if $N_t := [\frac{t}{\sqrt{a}}] \gg \lambda^{1/3}$,*

$$\|G_{h,a}(t,\cdot)\|_{L^\infty(\Omega_d)} \leq \frac{C}{h^d}\Big(\frac{h}{t}\Big)^{(d-1)/2}\Big(\frac{ht}{a}\Big)^{1/2}.$$

Proof. If $\lambda^{1/3} \ll N_t$, then $N_t + k \gg \lambda^{1/3}$ for all $k \in [k_1, k_2]$ and we estimate the L^∞ norms of $G_{h,a}(t,\cdot)$ using the first equality in (41) and Proposition 4: if $k_y \in [k_1, k_2]$ is such that $q^{1/2}(-Y) \in I_{N_t,k_y}$, then

$$4NK_a\Big(\frac{Y}{4N}, \frac{T}{2N}\Big) = q^{1/2}\Big(-Y + 2aT\Theta\Big(\frac{Y}{2T}, \frac{T}{2N}, a\Big)\Big) \in \bigcup_{|k'-k_y|\leq 1} I_{N_t,k'}$$

(using that a is small) and therefore the second line in (41) can be (uniformly) bounded as

$$\begin{aligned}
&\sup_{k_1\leq k\leq k_2}\Bigg(\sup_{\substack{4NK_a(\frac{Y}{4N},\frac{T}{2N})\\ \in I_{N_t,k}}}\Bigg(\sum_{2N\in[\frac{N_t}{M},M(N_t+1)]}|V_{N,h,a}(t,a,y)|\Bigg)\Bigg)\\
&\leq \sup_{|k'-k_y|\leq 1}\Bigg(\sup_{\substack{4NK_a(\frac{Y}{4N},\frac{T}{2N})\\ \in I_{N_t,k'}}}\Bigg(\sum_{2N\in[\frac{N_t}{M},M(N_t+1)]}|V_{N,h,a}(t,a,y)|\Bigg)\Bigg)\\
&\leq \sup_{\substack{4NK_a(\frac{Y}{4N},\frac{T}{2N})\\ \in\bigcup_{|k'-k_y|\leq 1} I_{N_t,k'}}}\Bigg(\sum_{2N\in[\frac{N_t}{M},M(N_t+1)}\frac{h^{1/3}}{(N/\lambda^{1/3})^{1/2} + \lambda^{1/6}|4NK_a(\frac{Y}{4N},\frac{T}{2N}) - 4N|^{1/2}}\Bigg).
\end{aligned}\tag{42}$$

As $4NK_a(\frac{Y}{4N}, \frac{T}{2N}) \in \bigcup_{|k'-k_y|\leq 1} I_{N_t,k'}$, we find, for $N = N_t + k_y + j$ and $|j| \geq 2$, that

$$\Big|4NK_a\Big(\frac{Y}{4N}, \frac{T}{2N}\Big) - 4N\Big| \geq |j| - 1,$$

and therefore the last line in (42) can be bounded by

$$\frac{h^{\frac{1}{3}}}{(N_t + k_y)^{\frac{1}{2}}}\Bigg(3\lambda^{\frac{1}{6}} + \sum_{|N-(N_t+k_y)|=|j|\geq 2}\frac{\lambda^{\frac{1}{6}}}{(1 + j/(N_t + k_y))^{1/2} + \lambda^{\frac{1}{3}}|(|j| - 1)/(N_t + k_y)|^{\frac{1}{2}}}\Bigg).\tag{43}$$

The sums over $N = N_t + k_y \pm (j + 1)$, $j \geq 1$, read

$$\begin{aligned}
&\frac{h^{1/3}(N_t + k_y)^{1/2}}{\lambda^{1/6}(N_t + k_y)}\sum_{\substack{N=N_t+k_y\pm(j+1)\\ j\geq 1}}\frac{1}{(1 \pm (j + 1)/(N_t + k_y))^{1/2}\lambda^{-1/3} + |j/(N_t + k_y)|^{1/2}}\\
&\leq h^{1/3}\frac{(N_t + k_y)^{1/2}}{\lambda^{1/6}}\sum_{\pm}\int_0^{1-\frac{1+N_t/(2M)}{N_t+k_y}}\frac{dx}{\sqrt{x} + \lambda^{-1/3}(1 \pm (N_t + k_y)^{-1} \pm x)^{1/2}},
\end{aligned}$$

where the last integral is taken on $[0, 1 - \frac{1+N_t/(2M)}{N_t+k_y}]$ as $N = N_t + k_y \pm (j+1) \geq \frac{N_y}{2M}$. As $k_y \geq k_1$, we have $N_t + k_y \geq N_t(1 + (m_0 - \varepsilon_0)/8)$ and using (24),

$$\frac{N_t}{2M(N_t + k_y)} \leq \frac{4}{M(m_0 - \varepsilon_0)} \leq \frac{1}{\sqrt{3/2}}.$$

Both integrals (with $\pm$ signs) are bounded by $\frac{1}{2}$, so the contribution coming from the sum over $|N - (N_t + k_y)| \geq 2$ in (43) is $h^{1/3}(N_t + k_y)^{1/2}/\lambda^{1/6}$. As $N_t + k_y \leq (N_t + 1)(M_0 + \varepsilon_0 - 1)$, where M_0 is fixed, depending only on q, and $N_t \in [\frac{t}{\sqrt{a}} - 1, \frac{t}{\sqrt{a}}]$, we obtain

$$\sup_{\substack{4NK_a(\frac{Y}{4N}, \frac{T}{2N}) \\ \in \bigcup_{|k'-k_y|\leq 1} I_{N_t,k'}}} \Big(\sum_{2N \in [\frac{N_t}{M}, M(N_t+1)]} |V_{N,h,a}(t, a, y)| \Big) \leq \sqrt{M_0} h^{1/3} \Big(\frac{t/\sqrt{a}}{\lambda^{1/3}} \Big)^{1/2}$$
$$\lesssim \Big(\frac{ht}{a} \Big)^{1/2},$$

which concludes the proof of Proposition 7. ∎

We need to introduce one more piece of notation. If y is such that

$$q^{1/2}\Big(-Y + 2aT\Theta\Big(\frac{Y}{2T}, \frac{T}{2N}, a\Big)\Big) \in \tilde{I}_{N_t,k} \quad \text{for some } k_1 \leq k \leq k_2,$$

then k is unique and we denote it $k_y^\#$. If $2(N_t + k_y^\#) \in [\frac{N_t}{M}, M(N_t + 1)]$, we have either $\lambda^{1/3} \lesssim N_t + k_y^\#$, or $N_t + k_y^\# < \lambda^{1/3}$.

Remark 5. When $N_t + k_y^\# < \lambda^{1/3}$, Proposition 6 may apply only for $N = N_t + k_y^\#$, as for $k_y^\# \neq k \in [k_1, k_2]$ and $n = N_t + k$ we must have

$$\begin{aligned}
&\Big|q^{1/2}\Big(-Y + 2aT\Theta\Big(\frac{Y}{2T}, \frac{T}{2N}, a\Big)\Big) - 4n\Big| \\
&\quad \geq 4|n - (N_t + k_y^\#)| \\
&\qquad - \Big|q^{1/2}\Big(-Y + 2aT\Theta\Big(\frac{Y}{2T}, \frac{T}{2N}, a\Big)\Big) - 4(N_t + k_y^\#)\Big| \\
&\quad \geq 3 \gg \frac{1}{n}.
\end{aligned}$$

Proposition 8. *There exists $C > 0$ (independent of h, a) such that, if $N_t := [\frac{t}{\sqrt{a}}] \ll \lambda^{1/3}$,*

$$\|G_{h,a}(t, \cdot)\|_{L^\infty(\Omega_d)} \sim \frac{C}{h^d}\Big(\frac{h}{t}\Big)^{(d-1)/2}\Big(\frac{ha}{t}\Big)^{1/4}. \tag{44}$$

Proof. If y is such that $q^{1/2}(-Y) \in I_{N_t,k_y}$ for $k_y \in [k_1, k_2]$, then, using $a \leq \varepsilon_0$,

$$\begin{aligned}
&\Big|q^{1/2}\Big(-Y + 2aT\Theta\Big(\frac{Y}{2T}, \frac{T}{2N}, a\Big)\Big) - 4n\Big| \\
&\quad \geq 4|n - (N_t + k_y)| \\
&\qquad - \Big|q^{1/2}\Big(-Y + 2aT\Theta\Big(\frac{Y}{2T}, \frac{T}{2N}, a\Big)\Big) - 4(N_t + k_y)\Big|
\end{aligned}$$

for all $n \neq N_t + k_y$; the second term on the right-hand side is smaller than 2, while the first one is at least 4; therefore the assumption of Proposition 6 cannot hold for $n \neq N_t + k_y$. For all such n we then use Proposition 5 to obtain

$$\begin{aligned}
&\sup_{q^{1/2}(-Y)\in I_{N_t,k_y}} \Bigg(\sum_{\substack{2n\in[\frac{N_t}{M},M(N_t+1)]\\ n\neq N_t+k_y}} |V_{n,h,a}(t,a,y)| \Bigg)\\
&\lesssim h^{1/3} \sum_{\substack{2n\in[\frac{N_t}{M},M(N_t+1)]\\ n\neq N_t+k_y}} \frac{1}{1+|n(q^{1/2}(-Y+2aT\Theta(\frac{Y}{2T},\frac{T}{2n},a))-4n)|^{1/2}}\\
&\lesssim h^{1/3} \sum_{\substack{n=N_t+k_y+j\\ 1\leq|j|\lesssim N_t}} \frac{1}{1+(N_t+k_y+j)^{1/2}|j|^{1/2}}\\
&\leq h^{1/3} \sum_{\pm} \int_0^{1-\frac{1+N_t/(2M)}{N_t+k_y}} \frac{dx}{x^{1/2}(1\pm x)^{1/2}+(N_t+k_y)^{-1}},
\end{aligned} \tag{45}$$

where the last two integrals are uniform bounds for the sum over $N < N_t + k_y$ and $N > N_t + k_y$, respectively; when $N > N_t + k_y$, the integral over $[0, 1]$ is bounded by a uniform constant; when $N < N_t + k_y$, write $x = \sin^2\theta$, $\theta \in [0, \frac{\pi}{2})$, therefore $1 - x = \cos^2\theta$, $dx = 2\sin\theta\cos\theta$: the corresponding integral is also bounded by at most π.

We are left with $N = N_t + k_y$. If $q^{1/2}(-Y + 2aT\Theta(\frac{Y}{2T}, \frac{T}{2N}, a)) \notin \tilde{I}_{N_t,k_y}$, then we use Proposition 5 again. If, on the contrary, $q^{1/2}(-Y + 2aT\Theta(\frac{Y}{2T}, \frac{T}{2N}, a)) \in \tilde{I}_{N_t,k_y}$, then $k_y^{\#} = k_y \in [k_1, k_2]$ and we may apply Proposition 6 with $N = N_t + k_y^{\#}$ provided that

$$\left| q^{1/2}\Big(-Y + 2aT\Theta\Big(\frac{Y}{2T}, \frac{T}{2N}, a\Big)\Big) - 4N \right| \lesssim \frac{1}{N}.$$

We then have

$$\begin{aligned}
\sup_{\substack{q^{1/2}(-Y)\\ \in I_{N_t,k_y}}} |V_{N_t+k_y,h,a}(t,a,y)| &\lesssim \frac{h^{\frac{1}{3}}}{(N/\lambda^{\frac{1}{3}})^{\frac{1}{4}}}\\
&\quad + \frac{h^{\frac{1}{3}}}{(1+|N(q^{\frac{1}{2}}(2aT\Theta(\frac{Y}{2T},\frac{T}{2N},a)-Y)-4N)|^{\frac{1}{2}})}\\
&\lesssim \Big(\frac{ha}{t}\Big)^{1/4} + h^{1/3}.
\end{aligned}$$

As for $N_t \sim \frac{t}{\sqrt{a}} \ll \frac{\sqrt{a}}{h^{1/3}} = \lambda^{1/3}$ we have $h^{1/3} \ll (\frac{ha}{t})^{1/4}$, it follows that at fixed t, the supremum of the sum over $V_{N,h,a}(t,x,y)$ is reached for y such that $q^{1/2}(-Y + 2aT\Theta(\frac{Y}{2T}, \frac{T}{2N}, a)) = 4N$ with $N = N_t + k_y^{\#}$ and at $x = a$. As the contribution from (45) in the sum over $n \neq N_t + k_y$ is $\sim h^{1/3}$, we obtain an upper bound for $G_{h,a}(t,\cdot)$. The last line of (41) and $h^{1/3} \ll (\frac{ha}{t})^{1/4}$ provide a similar lower bound for $G_{h,a}$ and therefore (44) holds true, concluding the proof of Proposition 8. ∎

Proposition 9. *There exists $C > 0$ (independent of h, a) such that, if $N_t := [\frac{t}{\sqrt{a}}] \sim \lambda^{1/3}$,*

$$\|G_{h,a}(t,\cdot)\|_{L^\infty(\Omega_d)} \le \frac{C}{h^d}\Big(\frac{h}{t}\Big)^{(d-1)/2}\Big(\Big(\frac{ha}{t}\Big)^{1/4} + \Big(\frac{ht}{a}\Big)^{1/2} + h^{1/3}\Big), \tag{46}$$

which yields $\|G_{h,a}(t,\cdot)\|_{L^\infty(\Omega_d)} \lesssim \frac{1}{h^d}(\frac{h}{t})^{(d-1)/2}h^{1/3}$.

Proof. As $N_t \sim \lambda^{1/3}$ and $k \sim N_t$, we separately consider y such that $N_t + k_y < \lambda^{1/3}$ and y such that $N_t + k_y \ge \lambda^{1/3}$; we then proceed as in the previous cases using Propositions 4, 5 and 6. For such N_t, $(\frac{ha}{t})^{1/4} \sim h^{1/3} \sim (\frac{th}{a})^{1/2}$, and the uniform bound $h^{1/3}$ for the three terms summed on the right-hand side of (46) follows. ■

Gathering Propositions 7, 8 and 9, we obtain, for our tangential contribution, the upper bound from Theorem 1, using that we are in the restricted range $(ht)^{1/2} \lesssim a$.

3.1.2. Transverse waves. Let $\gamma > 8a$ and recall $\lambda_\gamma := \frac{\gamma^{3/2}}{h}$.

Proposition 10. *Let $t > h$ and $\varepsilon_0 > \gamma > 8a$. Then*

$$\|G_{h,\gamma}(t,\cdot)\|_{L^\infty(x\le a,y)} \lesssim \begin{cases} \frac{1}{h^d}\Big(\frac{h}{t}\Big)^{\frac{d-1}{2}}\Big(\frac{th}{\gamma}\Big)^{1/2} & \text{if } \frac{t}{\sqrt{\gamma}} \gtrsim \lambda_\gamma^{1/3}, \\ \frac{1}{h^d}\Big(\frac{h}{t}\Big)^{\frac{d-1}{2}} h^{1/3} & \text{if } \frac{a}{\gamma} \lesssim \frac{t}{\sqrt{\gamma}} \lesssim \lambda_\gamma^{1/3}, \\ \frac{1}{h^d}\Big(\frac{h}{t}\Big)^{\frac{d}{2}} & \text{if } h < t \text{ and } \frac{t}{\sqrt{\gamma}} \le \frac{1}{3\sqrt{3/2}M_0}\frac{a}{\gamma}. \end{cases} \tag{47}$$

Moreover, for $h < t < a$ we have $\|G_h^{\varepsilon_0}(t,\cdot)\|_{L^\infty(x\le a,y)} \lesssim \frac{1}{h^d}(\frac{h}{t})^{d/2}$, while for $a \lesssim t \le T_0$,

$$\sum_{\gamma\in\Gamma_3(a)} \|G_{h,\gamma}(t,\cdot)\|_{L^\infty(x\le a,y)} \lesssim \begin{cases} \frac{1}{h^d}\Big(\frac{h}{t}\Big)^{\frac{d-1}{2}} h^{1/3}\log_2\Big(\frac{\varepsilon_0}{a}\Big) & \text{if } a \lesssim t \le \frac{a}{h^{1/3}}\ (< \frac{\gamma}{h^{1/3}}), \\ \frac{1}{h^d}\Big(\frac{h}{t}\Big)^{\frac{d-1}{2}}\Big[\Big(\frac{ht}{a}\Big)^{\frac12} + h^{1/3}\log_2\Big(\frac{\varepsilon_0}{a}\Big)\Big] & \text{if } t \ge \frac{a}{h^{1/3}}. \end{cases} \tag{48}$$

Proof. According to Proposition 3, if $\frac{t}{\sqrt{\gamma}} \le \frac{1}{3\sqrt{3/2}M_0}\frac{a}{\gamma}$ then $V_{N,h,\gamma}(t,\cdot) = O(h^\infty)$ for all $a \le \gamma \le \varepsilon_0$ and all $N \ge 1$, hence $G_{h,\gamma}(t,\cdot) = V_{0,h,\gamma}(t,\cdot)$. The last line in (47) follows using the proof of Proposition 2 applied to $V_{0,h,\gamma}(t,\cdot)$. If $h < t \lesssim a$, then $\frac{t}{\sqrt{\gamma}} \ll \frac{a}{\gamma}$ for all $a \le \gamma \le \varepsilon_0$, so $G_h^{\varepsilon_0}(t,\cdot) = \sum_\gamma G_{h,\gamma}(t,\cdot) = \sum_\gamma V_{0,h,\gamma}(t,\cdot)$ and we use Proposition 2.

Let $\frac{t}{\sqrt{\gamma}} \gtrsim \frac{a}{\gamma}$. Let $T = \frac{t}{\sqrt{\gamma}}$, $Y = \frac{y}{\sqrt{\gamma}}$ and let K_γ be given by (39) (with a replaced by γ). Let $V_{N,h,\gamma}$ be as in Corollary 1. Then $G_{h,\gamma}(t,x,y) = \sum_{N\sim\frac{t}{\sqrt{\gamma}}} V_{N,h,\gamma}(t,x,y)$. For $x \le a$, $8a < \gamma$ and $1 \le N \sim T$, we have

$$|V_{N,h,\gamma}(t,x,y)| \lesssim \frac{\gamma^2}{h} \times \frac{1}{\sqrt{N\lambda_\gamma}} \times \frac{1}{\lambda_\gamma}. \tag{49}$$

Indeed, as long as $x \le a$, we easily see that, for each N, the phase function of $V_{N,h,\gamma}$ has nondegenerate critical points with respect to both σ, s and the estimates (49) follow. Summing over $N \gtrsim \lambda_\gamma^{1/3}$ as in the proof of Proposition 7 yields the first line of (47). Summing over $N \lesssim \lambda_\gamma^{1/3}$ as in the proof of Proposition 8 yields the second line of (47).

Let $a \lesssim t \lesssim a/h^{1/3}$. Then $t \le \gamma/h^{1/3}$ for all $8a \le \gamma \le \varepsilon_0$. Summing for $\gamma_j = 2^j a$ yields the first line in (48), as $j \le \log_2(\frac{\varepsilon_0}{a})$. Now let $a/h^{1/3} \lesssim t \le T_0$. Then for $a \le \gamma \lesssim th^{1/3}$, $|G_{h,\gamma}(t,\cdot)|$ is bounded by the term in the first line of (47), while for $th^{1/3} \le \gamma \le \varepsilon_0$, $|G_{h,\gamma}(t,\cdot)|$ is bounded by the term in the second line of (47). The sum for $\gamma_j = 2^j a$ over $0 \le j \le \log_2(\frac{\max(\varepsilon_0, th^{1/3})}{a})$ yields the first contribution in the second line of (48) and the sum over $\frac{\max(\varepsilon_0, th^{1/3})}{a} < j \le \log_2(\frac{\varepsilon_0}{a})$ yields the second one. ∎

We then obtain the upper bound in Theorem 1 from Propositions 9 and 10, using again that we are in the regime $(ht)^{\frac{1}{2}} \lesssim a$.

3.1.3. Optimality for $\sqrt{a} \le t \ll \frac{a}{h^{1/3}}$ ($\le \frac{\gamma}{h\hat{}1/3}$). The equivalence bound in Theorem 1 follows easily from the next lemma, considering the reductions we performed earlier.

Lemma 7. *For $\sqrt{a} \le t \ll \frac{a}{h^{1/3}}$ ($\le \frac{\gamma}{h^{1/3}}$) we have*

$$\|G_h^{\varepsilon_0}(t,\cdot)\|_{L^\infty(\Omega_d)} \sim \frac{1}{h^d}\left(\frac{h}{t}\right)^{(d-1)/2}\left(\frac{ah}{t}\right)^{1/4}.$$

Proof. Write, for $1 \le \frac{t}{\sqrt{a}} \ll \lambda^{1/3} = \frac{\sqrt{a}}{h^{1/3}}$,

$$\|G_h^{\varepsilon_0}(t,\cdot)\|_{L^\infty(\Omega_d)} \ge \|G_{h,a}(t,\cdot)\|_{L^\infty(\Omega_d)} - \sum_{\gamma\in\Gamma_0(a)} \|G_{h,\gamma_j}(t,\cdot)\|_{L^\infty(\Omega_d)}.$$

From (44) we have $\|G_{h,a}(t,\cdot)\|_{L^\infty(\Omega_d)} \sim \frac{1}{h^d}(\frac{h}{t})^{(d-1)/2}(\frac{ah}{t})^{1/4}$ and from the first line of (48) we have

$$\sum_{\gamma\in\Gamma_0(a)} \|G_{h,\gamma_j}(t,\cdot)\|_{L^\infty(\Omega_d)} \le \frac{1}{h^d}\left(\frac{h}{t}\right)^{(d-1)/2} h^{1/3}\log_2\left(\frac{\varepsilon_0}{a}\right).$$

We now remark that $(\frac{ah}{t})^{1/4} \gg h^{1/3}\log_2(\frac{\varepsilon_0}{a})$ for all t such that $1 \le \frac{t}{\sqrt{a}} \le \lambda^{1/3-\epsilon} = \frac{\sqrt{a}}{h^{1/3}}\lambda^{-\epsilon}$, $\epsilon > 0$: as in the regime we consider here we have $a \ge h^{2/3-\epsilon}$, then $\lambda = \frac{a^{3/2}}{h} \ge h^{-3\epsilon/2}$, hence $\lambda^{-\epsilon} \le h^{3\epsilon^2/2}$ and we obtain $t \le \frac{a}{h^{1/3}}h^{3\epsilon^2/2}$, which further yields

$$\left(\frac{ah}{t}\right)^{1/4} \ge h^{1/3-3\epsilon^2/8} \gg h^{1/3}\log_2\left(\frac{1}{h}\right) \gtrsim h^{1/3}\log_2\left(\frac{\varepsilon_0}{a}\right)$$

(again using $a \ge h^{2/3-\epsilon}$). This concludes our proof. ∎

3.2. Case $a \lesssim \max(h^{2/3-\epsilon}, (ht)^{1/2})$ for (small) $\epsilon > 0$

3.2.1. The sum over $8\max(h^{2/3-\epsilon}, (ht)^{1/2}) \le \gamma \le \varepsilon_0$. This part is easy to deal with, as it is transverse and we can apply the estimates obtained in the previous section (with

a replaced by $(ht)^{1/2}$). Indeed, we have $8a \le \gamma$ and as in this regime we can use the parametrix (16), we obtain

$$\Bigg\| \sum_{\substack{8a\lesssim 8\max(h^{2/3-\epsilon},(ht)^{1/2})\\ \le\gamma\le\varepsilon_0}} G_{h,\gamma}(t,\cdot)\Bigg\|_{L^\infty(\Omega_d)} \lesssim \frac{1}{h^d}\Big(\frac{h}{t}\Big)^{(d-1)/2} \frac{(ht)^{1/2}}{\max(h^{2/3-\epsilon},(ht)^{1/2})^{1/2}}. \quad (50)$$

When $t \ge h^{1/3-2\epsilon}$ then $\max(h^{2/3-\epsilon},(ht)^{1/2}) = (ht)^{1/2}$ and the last factor in (50) equals $(ht)^{1/4}$. When $t \le h^{1/3-2\epsilon}$ the last factor in (50) is bounded by $(ht)^{1/2}/h^{(2/3-\epsilon)/2} \le h^{1/3-\epsilon/2}$.

3.2.2. The sum over $a^\natural \lesssim \gamma \lesssim \max(h^{2/3-\epsilon},(ht)^{1/2})$. This part will be dealt with entirely by the spectral sum, using formula (11), and the next lemma.

Lemma 8 (See [13]). *There exists C_0 such that for $L \ge 1$ the following holds true:*

$$\begin{aligned} &\sup_{b\in\mathbb{R}}\Big(\sum_{1\le k\le L}\omega_k^{-1/2}\,\mathrm{Ai}^2(b-\omega_k)\Big) \le C_0 L^{1/3},\\ &\sup_{b\in\mathbb{R}_+}\Big(\sum_{1\le k\le L}\omega_k^{-1/2}\,\mathrm{Ai}'^2(b-\omega_k)\Big) \le C_0 L. \end{aligned} \quad (51)$$

Write, for $\gamma_{\max} := \sup(h^{2/3-\epsilon},(ht)^{1/2})$, $\gamma_{\min} := \sup(a,h^{2/3})$,

$$\begin{aligned} \sum_{\gamma_{\min}\le\gamma\le\gamma_{\max}} & G_{h,\gamma}(t,x,a,y)\\ = \sum_{\substack{k\sim\lambda_\gamma\\ \gamma_{\min}\le\gamma\le\gamma_{\max}}} & \frac{h^{1/3}}{h^d}\int e^{\frac{i}{h}\langle y,\eta\rangle}\psi(|\eta|)e^{\frac{i}{h}t(|\eta|^2+\omega_k h^{2/3}q^{2/3}(\eta))}\frac{q^{1/3}(\eta)}{L'(\omega_k)}\\ & \times\psi_2(h^{2/3}\omega_k/(q^{1/3}(\eta)\gamma))\,\mathrm{Ai}(xq^{1/3}(\eta)/h^{2/3}-\omega_k)\\ & \times\mathrm{Ai}(aq^{1/3}(\eta)/h^{2/3}-\omega_k)\,d\eta + O(h^\infty), \end{aligned} \quad (52)$$

where we used that ψ_2 and ψ are supported on $[\frac12,\frac32]$ to deduce $k \sim \omega_k^{3/2} \sim \lambda_\gamma q^{1/2}(\eta) \sim \lambda_\gamma$ on the support of $\psi_2(h^{2/3}\omega_k/(q^{1/3}(\eta)\gamma))\psi(|\eta|)$; the term $O(h^\infty)$ comes from the (finite) sum over $1 \le k \ll \lambda_\gamma$ and $\lambda_\gamma \ll k \lesssim 1/h$. Notice that if $t \le h^{1/3-2\epsilon}$ then $(ht)^{1/2} \le h^{2/3-\epsilon}$, which yields $\gamma_{\max} = h^{2/3-\epsilon}$, hence for such t we have to consider only values $a \le h^{2/3-\epsilon}$. For $t \le h^{1/3-2\epsilon}$ and $\gamma \le \gamma_{\max} = h^{2/3-\epsilon}$, $\lambda_\gamma \lesssim h^{-3/2\epsilon}$ for small $\epsilon > 0$ and we cannot perform stationary phase arguments with the parameter λ_γ; formula (16) becomes useless and we have to resort to (11). We consider separately the situations $t \ge h^{1/3-2\epsilon}$ and $t \le h^{1/3-2\epsilon}$, although the arguments in the corresponding proofs are of the same nature and rely on (11).

3.2.3. Let $t \ge h^{1/3-2\epsilon}$, in which case $(ht)^{1/2} \ge h^{2/3-\epsilon}$. We will bring the Airy functions into the symbol and apply the stationary phase in $\eta \in \mathbb{R}^{d-1}$. The sum over k is taken over $1 \le k \lesssim (ht)^{3/4}/h$ and on the support of ψ_2 we have $k^{2/3} \sim \omega_k \sim \lambda_\gamma^{2/3}$ with $\gamma \le \gamma_{\max} := (ht)^{1/2}$.

Proposition 11. *For $t \geq h^{1/3-2\epsilon}$, the following dispersive estimate holds:*

$$\Big\| \sum_{a^\natural \leq \gamma \leq (ht)^{1/2}} G_{h,\gamma}(t,\cdot) \Big\|_{L^\infty(\Omega_d)} \lesssim \frac{1}{h^d} \Big(\frac{h}{t}\Big)^{(d-1)/2} (ht)^{1/4}.$$

Proof. Let $z = y/t$ and let $\frac{t}{h}$ be the large parameter in the integrals in the fourth line of (52) whose phase function is, for each $\omega_k \sim \lambda_\gamma^{2/3}$, of the form $\langle z, \eta \rangle + |\eta|^2 + \omega_k h^{2/3} q^{2/3}(\eta)$. For each $\omega_k \lesssim \gamma_{\max}/h^{2/3} = (ht)^{1/2}/h^{2/3}$, the corresponding critical point η_c satisfies $z + 2\eta_c + O(\omega_k h^{2/3}) = 0$ and using $\omega_k h^{2/3} \leq \varepsilon_0$, we obtain that the Hessian behaves like $2\mathbb{I}_{d-1} + O(\varepsilon_0)$. In order to apply the stationary phase with symbol

$$q^{1/3}(\eta)\psi(|\eta|)\psi_2\Big(\frac{\omega_k}{q^{1/3}(\eta)\lambda_\gamma^{2/3}}\Big) \operatorname{Ai}\Big(q^{1/3}(\eta)\lambda_\gamma^{2/3}\frac{x}{\gamma} - \omega_k\Big) \operatorname{Ai}\Big(q^{1/3}(\eta)\lambda_\gamma^{2/3}\frac{a}{\gamma} - \omega_k\Big)$$

we check that there exists some $\nu > 0$ such that for all $j \geq 1$ and for all α with $|\alpha| = j$,

$$\Big|\partial_\eta^\alpha \Big(\operatorname{Ai}\Big(q^{1/3}(\eta)\lambda_\gamma^{2/3}\frac{x}{\gamma} - \omega_k\Big)\Big)\Big| \leq C_j \Big(\frac{t}{h}\Big)^{j(1-2\nu)/2}.$$

In particular, this allows us to deduce that, for η on the support of ψ we have

$$\partial^2_{ij}\Big(q^{\frac13}(\eta)\psi_2\Big(\frac{\omega_k}{q^{\frac13}(\eta)\lambda_\gamma^{\frac23}}\Big) \operatorname{Ai}\Big(q^{\frac13}(\eta)\lambda_\gamma^{\frac23}\frac{x}{\gamma} - \omega_k\Big) \operatorname{Ai}\Big(q^{\frac13}(\eta)\lambda_\gamma^{\frac23}\frac{a}{\gamma} - \omega_k\Big)\Big) \lesssim \Big(\frac{t}{h}\Big)^{1-2\nu}$$

and ensures that the stationary phase can be applied with the Airy factors as part of the symbol. As one has, for all $l \geq 0$, $\sup_{b \geq 0} |b^l \operatorname{Ai}^{(l)}(b - \omega_k)| \leq C_l \omega_k^{3l/2}$, it is sufficient to check that for $t \geq h^{1/3-2\epsilon}$ and $k \leq (ht)^{3/4}/h$, we have

$$\omega_k^{3/2} \lesssim \Big(\frac{t}{h}\Big)^{(1-2\nu)/2}. \tag{53}$$

As $\omega_k \sim k^{2/3} \lesssim \lambda_{\gamma_{\max}}^{2/3} \sim ((ht)^{3/4}/h)^{2/3}$ for $k \leq (ht)^{3/4}/h$, (53) holds if we prove $t^{1/2}(t/h)^{1/4} = (ht)^{3/4}/h \lesssim (t/h)^{(1-2\nu)/2}$, which is obviously true as it reduces to $t \lesssim (t/h)^{1/2-2\nu}$ for some $\nu > 0$ (recall that we consider here only values $t \lesssim 1$). The sum of the main contributions of the symbols obtained after applying the stationary phase in η equals

$$\begin{aligned}
&\Big| \sum_{k \lesssim (ht)^{3/4}/h} \omega_k^{-1/2} \operatorname{Ai}\Big(x \frac{q^{1/3}(\eta_c(z, \omega_k h^{2/3}))}{h^{2/3}} - \omega_k\Big) \operatorname{Ai}\Big(a \frac{q^{1/3}(\eta_c(z, \omega_k h^{2/3}))}{h^{2/3}} - \omega_k\Big)\Big| \\
&\quad \leq \Big| \sum_{k \lesssim (ht)^{3/4}/h} \omega_k^{-1/2} \operatorname{Ai}^2\Big(x \frac{q^{1/3}(\eta_c(z, \omega_k h^{2/3}))}{h^{2/3}} - \omega_k\Big)\Big|^{1/2} \\
&\qquad \times \Big| \sum_{k \lesssim (ht)^{3/4}/h} \omega_k^{-1/2} \operatorname{Ai}^2\Big(a \frac{q^{1/3}(\eta_c(z, \omega_k h^{2/3}))}{h^{2/3}} - \omega_k\Big)\Big|^{1/2} \lesssim \lambda_{\gamma_{\max}}^{1/3},
\end{aligned}$$

where we applied Cauchy–Schwarz followed by (51) from Lemma 8 with $L \sim \lambda_{\gamma_{\max}} = (ht)^{3/4}/h$. However, this is not enough to conclude: we also need to prove that lower-order terms in the symbol obtained after the stationary phase do sum and provide smaller contributions. This can be done using the second inequality in (51), as well as the equation satisfied by the Airy function. ■

3.2.4. Let $t \le h^{1/3-2\epsilon}$, with (small) $\epsilon > 0$. Then $\max(h^{2/3-\epsilon}, (ht)^{1/2}) = h^{2/3-\epsilon}$ and we consider only γ such that $a^\natural \lesssim \gamma \lesssim h^{2/3-\epsilon}$, as the sum over $\gamma > h^{2/3-\epsilon} > (ht)^{1/2}$ can be handled as in (50). Then $\lambda_{\gamma_{\max}} = (h^{2/3-\epsilon})^{3/2}/h = h^{-3\epsilon/2}$.

Proposition 12. *Let $0 < \epsilon < \frac{1}{6}$ and $d \ge 1$. For $h^{1/3+\epsilon} \lesssim t \le h^{1/3-2\epsilon}$ we have*

$$\Big\| \sum_{a^\natural \lesssim \gamma \lesssim h^{2/3-\epsilon}} G_{h,\gamma}(t,\cdot) \Big\|_{L^\infty(\Omega_d)} \lesssim \frac{1}{h^d}\Big(\frac{h}{t}\Big)^{\frac{d-1}{2}} h^{\frac{1}{3}-\frac{\epsilon}{2}}.$$

For $0 < t \lesssim h^{1/3+\epsilon}$ we have $\| \sum_{a^\natural \lesssim \gamma \lesssim h^{2/3-\epsilon}} G_{h,\gamma}(t,\cdot)\|_{L^\infty(\Omega_d)} \lesssim \frac{1}{h^d}(\frac{h}{t})^{d/2}$.

Proof. Let $0 < \epsilon < \frac{1}{6}$ and $\nu = \nu(\epsilon, d) > 0$ such that $\epsilon(\frac{1}{d} + \frac{3}{1-2\nu}) = \frac{2}{3d}$ with $d \ge 1$. Moreover, we set $t(h,\epsilon) := h^{1-\frac{3\epsilon}{1-2\nu}}$. This implies $t(h,\epsilon) = h^{1+\frac{\epsilon}{d}-\frac{2}{3d}}$, $t(h,\epsilon) \ll h^{1/3+\epsilon}$ (as $\epsilon < 2/3$) and also $h^{-\frac{3\epsilon}{2}} \lesssim (\frac{t}{h})^{\frac{1}{2}-\nu}$ for all $t \ge t(h,\epsilon)$. We apply the stationary phase with the Airy factors in the symbol as the condition (53) is satisfied for all $k \lesssim \lambda_{\gamma_{\max}}$ and we obtain the first bound for $t \ge t(h,\epsilon)$. As $\epsilon < 1/6$, we obtain $h^{1/3-\epsilon/2} \ll h^{1/4} \le (h/t)^{1/4}$.

Now let $t \lesssim t(h,\epsilon)$ and $L := 8h^{-3\epsilon/2}$. Then the sum over k in (52) is limited to $k \le L$. Using (51) yields

$$\Big| \sum_{a^\natural \lesssim \gamma \lesssim h^{2/3-\epsilon}} G_{h,\gamma}(t,\cdot) \Big| \lesssim \frac{h^{\frac{1}{3}}}{h^d} L^{\frac{1}{3}}.$$

As $t \lesssim t(h,\epsilon)$, then $\frac{1}{t(h,\epsilon)} \lesssim \frac{1}{t}$ and as $(\frac{h}{t(h,\epsilon)})^{d/2} = (h^{-\epsilon/d+2/(3d)})^{d/2} = h^{1/3-\epsilon/2}$ we find

$$h^{1/3} L^{1/3} = 2h^{1/3-\epsilon/2} = 2\Big(\frac{h}{t(h,\epsilon)}\Big)^{d/2} \lesssim 2\Big(\frac{h}{t}\Big)^{d/2},$$

which concludes our proof. ■

Gathering the bounds from Propositions 11 and 12, we therefore complete the proof of the upper bound of Theorem 1, now using that we are in the remaining regime $a \lesssim \max(h^{2/3-\epsilon}, (ht)^{1/2})$.

4. Refined estimates for degenerate oscillatory integrals

In this section we prove Propositions 4, 5 and 6. Here we need to analyze in detail the structure of higher-order derivatives of the phase functions $\phi_{N,a}$. The proof of Proposition 4 closely follows that of [15, Proposition 7] (in the case $x \le a$); the proofs of

Propositions 5 and 6 become much more delicate in the case of Schrödinger flow, due to the presence of the critical point η_c which is a function depending on s, σ. As these propositions are crucial in the proof of Theorem 1, we provide a detailed proof.

Let $V_{N,h,a}$ be defined in (38) and let $N < \lambda^{1/3}$. Using Remark 3, we assume (without changing the contribution of $V_{N,h,a}$ modulo $O(h^\infty)$) that its symbol $\varkappa$ is supported on $|(\sigma, s)| \le 2\sqrt{\alpha_c}$. Fix T, $N \in [\frac{T}{M}, MT]$ with $M > 8$ large enough and let $X = \frac{x}{a} \le 1$, $Y = \frac{y}{\sqrt{a}}$ with $\frac{Y}{2T} \in [\frac{1}{4}, 2]$.

Proof of Proposition 4

We start with the case where $\lambda^{1/3} \lesssim N$ and we closely follow the proof of [15, Proposition 7]. We will prove the following:

$$\left|\int_{\mathbb{R}^2} e^{\frac{i}{h}\phi_{N,a}} \varkappa(\sigma, s, t, x, y, h, a, 1/N)\, ds\, d\sigma\right| \lesssim \frac{\lambda^{-2/3}}{1 + \lambda^{1/3}|K_a^2(\frac{Y}{4N}, \frac{T}{2N}) - 1|^{1/2}}. \tag{54}$$

We rescale variables with $\sigma = \lambda^{-1/3} p$ and $s = \lambda^{-1/3} q$ and define

$$A = \lambda^{2/3}\Big(K_a^2\Big(\frac{Y}{4N}, \frac{T}{2N}\Big) - X\Big) \quad \text{and} \quad B = \lambda^{2/3}\Big(K_a^2\Big(\frac{Y}{4N}, \frac{T}{2N}\Big) - 1\Big), \tag{55}$$

and we are reduced to proving that the following holds uniformly in (A, B):

$$\left|\int_{\mathbb{R}^2} e^{iG_{N,a,\lambda}(p,q,t,x,y)} \varkappa(\lambda^{-1/3} p, \lambda^{-1/3} q, t, x, y, h, a, 1/N)\, dp\, dq\right| \lesssim \frac{1}{1 + |B|^{1/2}}, \tag{56}$$

where the rescaled phase is

$$G_{N,a,\lambda}(p, q, t, x, y) := \frac{1}{h}\big(\phi_{N,a}(\lambda^{-1/3} p, \lambda^{-1/3} q, t, x, y) - \phi_{N,a}(0, 0, t, x, y)\big).$$

Replacing γ by a in first-order derivatives of $\phi_{N,a,\gamma}$ ((32) and (33)) yields

$$\begin{aligned}
\partial_p G_{N,a,\lambda} &= \frac{1}{h}\frac{\partial\sigma}{\partial p}\partial_\sigma(\phi_{N,a})|_{(\sigma,s)=(\lambda^{-1/3}p,\lambda^{-1/3}q)} = q^{1/2}(\eta_c)(p^2 - \lambda^{2/3}(\alpha_c - X)),\\
\partial_q G_{N,a,\lambda} &= \frac{1}{h}\frac{\partial s}{\partial q}\partial_s(\phi_{N,a})|_{(\sigma,s)=(\lambda^{-1/3}p,\lambda^{-1/3}q)} = q^{1/2}(\eta_c)(q^2 - \lambda^{2/3}(\alpha_c - 1)).
\end{aligned}$$

From (28), in our new variables, α_c has the expansion

$$\alpha_c|_{(\lambda^{-1/3}p,\lambda^{-1/3}q)} = \Big(K_a\Big(\frac{Y}{4N}, \frac{T}{2N}\Big) - \lambda^{-1/3}\frac{p}{2N}(1 - a\mathcal{E}_1) - \lambda^{-1/3}\frac{q}{2N}(1 - a\mathcal{E}_2)\Big)^2,$$

where f_j are smooth functions of $(\sigma, s) = \lambda^{-1/3}(p, q)$ and of $\frac{T}{2N}$, X, $\frac{Y}{4N}$. With this notation and with $K_a = K_a(\frac{Y}{4N}, \frac{T}{2N})$, we rewrite the first-order derivatives of $G_{N,a,\lambda}$ as

$$\begin{aligned}
\partial_p G_{N,a,\lambda} = q^{1/2}(\eta_c)\Big(p^2 - A &+ \frac{\lambda^{1/3}}{N} K_a(p(1 - a\mathcal{E}_1) + q(1 - a\mathcal{E}_2))\\
&- \frac{1}{4N^2}(p(1 - a\mathcal{E}_1) + q(1 - a\mathcal{E}_2))^2\Big),
\end{aligned}$$

$$\partial_q G_{N,a,\lambda} = q^{1/2}(\eta_c)\Big(q^2 - B + \frac{\lambda^{1/3}}{N} K_a(p(1-a\mathcal{E}_1) + q(1-a\mathcal{E}_2)) \\ - \frac{1}{4N^2}(p(1-a\mathcal{E}_1) + q(1-a\mathcal{E}_2))^2\Big).$$

As $\lambda^{1/3} \leq N$, if A, B are bounded, then (56) obviously holds for $|(p,q)|$ bounded and by integration by parts if $|(p,q)|$ is large. So we can assume that $|(A,B)| \geq r_0$ with $r_0 \gg 1$. Set $(A,B) = r(\cos(\theta), \sin(\theta))$ and again rescale $(p,q) = r^{1/2}(\tilde{p},\tilde{q})$: we aim for

$$\Big|\int_{\mathbb{R}^2} e^{ir^{3/2}\widetilde{G}_{N,a,\gamma}} \varkappa(\lambda^{-1/3} r^{1/2}\tilde{p}, \lambda^{-1/3} r^{1/2}\tilde{q}, t, x, y, h, a, 1/N)\, d\tilde{p}\, d\tilde{q}\Big| \lesssim \frac{1}{r^{5/4}}, \quad (57)$$

where r is our large parameter, and $\widetilde{G}_{N,a,\lambda}(\tilde{p},\tilde{q},t,x,y) = r^{-3/2} G_{N,a,\lambda}(r^{1/2}p, r^{1/2}q, t, x, y)$. Let us compute, using the formulas for the first-order derivatives of $G_{N,a,\lambda}$,

$$\frac{\partial_{\tilde{p}}\widetilde{G}_{N,a,\lambda}}{q^{1/2}(\eta_c)} = \tilde{p}^2 - \cos\theta + \frac{\lambda^{\frac{1}{3}} K_a}{N r^{\frac{1}{2}}}(\tilde{p}(1-a\mathcal{E}_1) + \tilde{q}(1-a\mathcal{E}_2)) \\ - \frac{(\tilde{p}(1-a\mathcal{E}_1) + \tilde{q}(1-a\mathcal{E}_2))^2}{4N^2},$$

$$\frac{\partial_{\tilde{q}}\widetilde{G}_{N,a,\lambda}}{q^{1/2}(\eta_c)} = \tilde{q}^2 - \sin\theta + \frac{\lambda^{\frac{1}{3}} K_a}{N r^{\frac{1}{2}}}(\tilde{p}(1-a\mathcal{E}_1) + \tilde{q}(1-a\mathcal{E}_2)) \\ - \frac{(\tilde{p}(1-a\mathcal{E}_1) + \tilde{q}(1-a\mathcal{E}_2))^2}{4N^2},$$

where, abusing notation, $\mathcal{E}_j$ is now $\mathcal{E}_j(r^{1/2}\lambda^{-1/3}(\tilde{q},\tilde{p}), \frac{T}{2N}, \frac{Y}{4N})$. On the support of $\varkappa(\cdots)$ we have $|(\tilde{p},\tilde{q})| \lesssim \lambda^{1/3} r^{-1/2} \lesssim \lambda^{1/3} r_0^{-1/2}$: for $\lambda^{1/3} \lesssim N$, the last term in both derivatives is $O(r_0^{-1})$, while the next to last term is $r_0^{-1/2} O(\tilde{p},\tilde{q})$; indeed, using boundedness of $\mathcal{E}_{1,2}$ and K_a, we obtain

$$\Big|\frac{\lambda^{1/3}}{N} K_a \frac{(\tilde{p}(1-a\mathcal{E}_1) + \tilde{q}(1-a\mathcal{E}_2))}{r^{1/2}}\Big| \lesssim r_0^{-1/2}|\tilde{p} + \tilde{q}|.$$

Hence, when $|(\tilde{p},\tilde{q})| > \widetilde{C}$ with $\widetilde{C}$ sufficiently large, the corresponding part of the integral is $O(r^{-\infty})$ by integration by parts. So we are left with restricting our integral to a compact region in $(\tilde{p},\tilde{q})$.

We remark that, from $X \leq 1$, we have $A \geq B$ (and $A = B$ if and only if $X = 1$), e.g. $\cos\theta \geq \sin\theta$ and therefore $\theta \in (-\frac{3\pi}{4}, \frac{\pi}{4})$. We proceed differently upon the size of $B = r\sin\theta$. If $\sin\theta < -C/r^{1/2}$ for some $C > 0$ sufficiently large then $\partial_{\tilde{q}}\widetilde{G}_{N,a,\lambda} > c/(2r^{1/2})$ for some $C > c > 0$ and the phase is nonstationary. Indeed, in this case

$$\frac{\partial_{\tilde{q}}\widetilde{G}_{N,a,\lambda}}{q^{1/2}(\eta_c)} \geq \tilde{q}^2 + \frac{C}{2r^{1/2}} + \frac{\lambda^{1/3} K_a}{N r^{1/2}}(\tilde{p}(1-a\mathcal{E}_1) + \tilde{q}(1-a\mathcal{E}_2)) \\ - \frac{(\tilde{p}(1-a\mathcal{E}_1) + \tilde{q}(1-a\mathcal{E}_2))^2}{4N^2}$$

and using that $\tilde{p}, \tilde{q}$ are bounded, that on the support of $\varkappa$ we have $|r^{1/2}(\tilde{p}, \tilde{q})| \lesssim \lambda^{1/3}$ and that $\frac{1}{N} \lesssim \frac{1}{\lambda^{1/3}} \ll 1$, we then have, for some C large enough,

$$\frac{\lambda^{1/3}}{N}(\tilde{p}+\tilde{q})\Big(\frac{K_a}{r^{1/2}} - \frac{(\tilde{p}(1-a\mathcal{E}_1)+\tilde{q}(1-a\mathcal{E}_2))}{4N\lambda^{1/3}}\Big) \lesssim \frac{C}{4r^{1/2}}.$$

We recall that on the support of $\psi_2(\alpha)$ we had $\alpha \in [\frac{1}{2}, \frac{3}{2}]$ and the critical point α_c is such that (21) holds (with γ replaced by a in this case), hence $K_a = K_a(\frac{Y}{4N}, \frac{T}{2N})$ introduced in (39) stays close to 1 as the main contribution to α_c. It follows that $\partial_{\tilde{q}} \widetilde{G}_{N,a,\lambda} > C/(2r^{1/2})$ and integrations by parts yield a bound $O(r^{-n})$ for all $n \geq 1$.

Next, let $\sin\theta > -C/r^{1/2}$ and assume $A > 0$ (since otherwise the nonstationary phase applies), which in turn implies $A > r_0/2$. Indeed, $\cos\theta \geq \sin\theta > -C/r^{1/2}$ implies $\theta \in (-\frac{C}{\sqrt{r_0}}, \frac{\pi}{4})$ and therefore in this regime $\cos\theta \geq \frac{\sqrt{2}}{2}$. Consider first the case $|\sin\theta| < C/r^{1/2}$. The nondegenerate stationary phase always applies in $\tilde{p}$, at two (almost) opposite values of $\tilde{p}$, such that $|\tilde{p}_\pm| \sim |\pm\sqrt{\cos\theta}| \geq 1/4$, and the integral in (57) is rewritten

$$\begin{aligned} r\int_{\mathbb{R}^2} & e^{ir^{3/2}\widetilde{G}_{N,a,\lambda}} \varkappa(\lambda^{-1/3}r^{1/2}\tilde{p}, \lambda^{-1/3}r^{1/2}\tilde{q}, t, x, y, h, a, 1/N)\, d\tilde{p}\, d\tilde{q} \\ &= \frac{r}{r^{3/4}}\Big(\int_{\mathbb{R}} e^{ir^{3/2}\widetilde{G}^+_{N,a,\lambda}} \varkappa^+(\tilde{q}, t, x, y, h, a, 1/N)\, d\tilde{q} \\ &\qquad + \int_{\mathbb{R}} e^{ir^{3/2}\widetilde{G}^-_{N,a,\lambda}} \varkappa^-(\tilde{q}, h, a, 1/N)\, d\tilde{q}\Big). \end{aligned} \tag{58}$$

Indeed, the phase is stationary in $\tilde{p}$ when

$$\tilde{p}^2 = \cos\theta - \frac{\lambda^{1/3}K_a}{Nr^{1/2}}(\tilde{p}(1-a\mathcal{E}_1)+\tilde{q}(1-a\mathcal{E}_2)) + \frac{(\tilde{p}(1-a\mathcal{E}_1)+\tilde{q}(1-a\mathcal{E}_2))^2}{4N^2},$$

and from $\cos\theta \geq \frac{\sqrt{2}}{2}$ and $\frac{1}{r} \leq \frac{1}{r_0} \ll 1$, there are exactly two disjoint solutions to $\partial_{\tilde{p}} \widetilde{G}_{N,a,\lambda} = 0$, which we denote $\tilde{p}_\pm = \pm\sqrt{\cos\theta} + O(r^{-1/2})$. We compute, at critical points,

$$\partial^2_{\tilde{p},\tilde{p}} \widetilde{G}_{N,a,\lambda}|_{p_\pm} = q^{1/2}(\eta_c)\Big(2\tilde{p} + \frac{\lambda^{1/3}K_a}{Nr^{1/2}}(1+O(a))\Big) + O(N^{-2})|_{\tilde{p}_\pm},$$

where we used $\tilde{p}, \tilde{q}$ bounded and $\partial_{\tilde{p}}\mathcal{E}_j = O(\frac{r^{1/2}\lambda^{-1/3}}{N})$ to deduce that all the terms except the first one are small. As $\lambda^{1/3} \lesssim N$, $r^{-1/2} \ll 1$, K_a bounded, close to 1, for $\tilde{p} \in \{\tilde{p}_\pm\}$ we get $\partial^2_{\tilde{p},\tilde{p}} \widetilde{G}_{N,a,\lambda}|_{\tilde{p}_\pm} \sim 2\tilde{p}_\pm + O(r^{-1/2})$, and as $|\tilde{p}_\pm| \geq \frac{1}{4} - O(r^{-1/2})$, the stationary phase applies. The critical values of the phase at $\tilde{p}_\pm$, denoted $\widetilde{G}^\pm_{N,a,\lambda}$, are such that

$$\begin{aligned} \partial_{\tilde{q}} \widetilde{G}^\pm_{N,a,\lambda}(\tilde{q}, \cdot) &:= \partial_{\tilde{q}} \widetilde{G}_{N,a,\lambda}(\tilde{q}, \tilde{p}_\pm, \cdot) \\ &= q^{1/2}(\eta_c)\Big(\tilde{q}^2 - \sin\theta + \frac{\lambda^{1/3}K_a(\tilde{p}(1-a\mathcal{E}_1)+\tilde{q}(1-a\mathcal{E}_2))}{Nr^{1/2}} \\ &\qquad - \frac{(\tilde{p}(1-a\mathcal{E}_1)+\tilde{q}(1-a\mathcal{E}_2))^2}{4N^2}|_{\tilde{p}=\tilde{p}_\pm}\Big). \end{aligned} \tag{59}$$

As $|\sin\theta| < C/r^{1/2}$, the phases $\widetilde{G}^{\pm}_{N,a,\lambda}$ may be stationary but degenerate; taking two derivatives in (59), one easily checks that $|\partial^3_{\tilde{q}}\widetilde{G}^{\pm}_{N,a,\lambda}| \geq q^{1/2}(\eta_c)(2 - O(r_0^{-1/2}))$. Hence we get, by the Van der Corput lemma,

$$\left|\int_{\mathbb{R}} e^{ir^{3/2}\widetilde{G}^{\pm}_{N,a,\lambda}} \varkappa^{\pm}(\tilde{q}, t, x, y, h, a, 1/N)\, d\tilde{q}\right| \lesssim (r^{3/2})^{-1/3}. \tag{60}$$

Using (58) and (60) eventually yields

$$\left|r\int_{\mathbb{R}^2} e^{ir^{3/2}\widetilde{G}_{N,a,\lambda}} \varkappa(\lambda^{-1/3}r^{1/2}\tilde{p}, \lambda^{-1/3}r^{1/2}\tilde{q}, t, x, y, h, a, 1/N)\, d\tilde{p}\, d\tilde{q}\right| \lesssim r^{-1/4}.$$

Notice moreover that $|B| = |r\sin\theta| \leq Cr^{1/2}$; hence from $r^2 = A^2 + B^2$, we have $A \sim r$ (large) and $r^{-1/4} \lesssim 1/(1 + |B|^{1/2})$: (56) holds true and, replacing B by $\lambda^{2/3}(K_a^2 - 1)$, it yields (54). Substitution with (55) and using $a^2 = (h\lambda)^{4/3}$, we obtain from (54),

$$\begin{aligned}
|V_{N,h,a}(t, x, y)| &\leq \frac{a^2}{h}\frac{1}{\sqrt{\lambda N}}\frac{\lambda^{-\frac{2}{3}}}{(1 + \lambda^{\frac{1}{3}}|K_a^2 - 1|^{\frac{1}{2}})} \\
&= \frac{2h^{\frac{1}{3}}}{2\sqrt{N/\lambda^{\frac{1}{3}}} + \lambda^{\frac{1}{6}}\sqrt{K_a + 1}|4NK_a - 4N|^{\frac{1}{2}}}.
\end{aligned}$$

In the last case $\sin\theta > C/r^{1/2}$ ($A \geq B \geq Cr^{1/2}$), the stationary phase holds in $(\tilde{p}, \tilde{q})$: the determinant of the Hessian is at least $C\sqrt{\cos\theta}\sqrt{\sin\theta}$ and we get

$$|(\text{LHS}) \text{ of } (57)| \lesssim \frac{1}{(\sqrt{\cos\theta}\sqrt{\sin\theta})^{1/2} r^{3/2}} \lesssim \frac{1}{r}\frac{1}{(r\sqrt{\cos\theta}\sqrt{\sin\theta})^{1/2}} \lesssim \frac{1}{r}\frac{1}{|AB|^{1/4}},$$

so in this case our estimate is slightly better than (54), as we have

$$\left|\int_{\mathbb{R}^2} e^{\frac{i}{h}\phi_{N,a}} \varkappa(s, \sigma, t, x, y, h, a, 1/N)\, ds\, d\sigma\right| \lesssim \frac{1}{\lambda^{2/3}|AB|^{1/4}} \leq \frac{1}{\lambda^{2/3}|B|^{1/2}}.$$

This completes the proof of Proposition 4 as it eventually yields

$$|V_{N,h,a}(t, x, y)| \lesssim \frac{(h\lambda)^{4/3}}{h}\frac{\lambda^{-1/2}}{N^{1/2}}\frac{1}{\lambda^{2/3}|B|^{1/2}} \sim h^{1/3}\frac{\lambda^{1/6}}{N^{1/2}}\frac{1}{\lambda^{1/3}|K_a^2 - 1|^{1/2}}.$$

Proof of Propositions 5 and 6

The main differences between the proof of Proposition 5 and that of [15, Proposition 5] occur from the additional critical point η_c, which is not considered in the case of the wave equation. Similarly, the proof of Proposition 6 follows the same path as [15, Proposition 6], but one has to deal carefully with contributions coming from the higher-order derivatives of η_c. Let $1 \leq N < \lambda^{1/3}$: we aim to prove

$$\left|\int_{\mathbb{R}^2} e^{\frac{i}{h}\phi_{N,a}} \varkappa(\sigma, s, t, x, y, h, a, 1/N)\, ds\, d\sigma\right| \lesssim N^{1/4}\lambda^{-3/4}.$$

As N is bounded by $\lambda^{1/3}$, ignoring the last two terms in the first-order derivatives of $\phi_{N,a}$, as we did in the previous case, is no longer possible. Set $\Lambda = \lambda/N^3$ to be the new large parameter. Rescale variables $\sigma = p'/N$ and $s = q'/N$ again and now set

$$\Lambda G_{N,a}(p', q', t, x, y) = \frac{1}{h}\big(\phi_{N,a}(\sigma, s, t, x, y) - \phi_{N,a}(0, 0, t, x, y)\big).$$

We are reduced to proving $|\int_{\mathbb{R}^2} e^{i\Lambda G_{N,a}} \varkappa(p'/N, q'/N, \dots)\, dp'\, dq'| \lesssim \Lambda^{-3/4}$. Compute

$$\begin{aligned}\nabla_{(p',q')} G_{N,a} &= \frac{N^3}{h}\Big(\frac{\partial\sigma}{\partial p'}\partial_\sigma \phi_{N,a}, \frac{\partial s}{\partial q'}\partial_s \phi_{N,a}\Big)\Big|_{(p'/N, q'/N)} \\ &= q^{1/2}(\eta_c)\big(p'^2 + N^2(X - \alpha_c), q'^2 + N^2(1-\alpha_c)\big),\end{aligned} \tag{61}$$

where, using (28),

$$\alpha_c(\sigma, s, \cdot)|_{(\sigma = p'/N, s = q'/N)} = \Big(K_a - \frac{p'}{2N^2}(1 - af_1) - \frac{q'}{2N^2}(1 - af_2)\Big)^2.$$

Recall that $K_a = \sqrt{\alpha_c^0}$ and stays close to 1 on the support of the symbol. We define $A' = (K_a^2 - X)N^2$ and $B' = (K_a^2 - 1)N^2$. First-order derivatives of $G_{N,a,\lambda}$ read

$$\begin{aligned}\partial_{p'} G_{N,a} &= q^{1/2}(\eta_c)\Big(p'^2 - A' + K_a(p'(1 - a\mathcal{E}_1) + q'(1 - a\mathcal{E}_2)) \\ &\qquad - \frac{1}{4N^2}(p'(1 - a\mathcal{E}_1) + q'(1 - a\mathcal{E}_2))^2\Big), \\ \partial_{q'} G_{N,a} &= q^{1/2}(\eta_c)\Big(q'^2 - B' + K_a(p'(1 - a\mathcal{E}_1) + q'(1 - a\mathcal{E}_2)) \\ &\qquad - \frac{1}{4N^2}(p'(1 - a\mathcal{E}_1) + q'(1 - a\mathcal{E}_2))^2\Big).\end{aligned}$$

Unlike the previous case, the two last terms are no longer disposable. We start with $|(A', B')| \geq r_0$ for some large, fixed r_0, in which case we can follow the same approach as in the previous case. Again, set $A' = r\cos\theta$ and $B' = r\sin\theta$. If $|(p', q')| < r_0/2$, then the corresponding integral is nonstationary and we get decay by integration by parts. We change variables $(p', q') = r^{1/2}(\tilde{p}', \tilde{q}')$ with $r_0 \leq r \lesssim N^2$ and aim to prove

$$\begin{aligned}&\Big| r \int_{\mathbb{R}^2} e^{i r^{3/2}\Lambda \widetilde{G}_{N,a}} \varkappa(r^{1/2}\tilde{p}'/N, r^{1/2}\tilde{q}'/N, t, x, y, h, a, 1/N)\, d\tilde{p}'\, d\tilde{q}' \Big| \\ &\qquad \lesssim r^{-1/4}\Lambda^{-5/6},\end{aligned} \tag{62}$$

The new phase is $\widetilde{G}_{N,a}(\tilde{p}', \tilde{q}', t, x, y) = r^{-3/2} G_{N,a}(r^{1/2}\tilde{p}', r^{1/2}\tilde{q}', t, x, y)$. We compute

$$\begin{aligned}\frac{\partial_{\tilde{p}'}\widetilde{G}_{N,a}}{q^{1/2}(\eta_c)} &= \tilde{p}'^2 - \cos\theta + \frac{K_a}{r^{1/2}}(\tilde{p}'(1 - a\mathcal{E}_1) + \tilde{q}'(1 - a\mathcal{E}_2)) \\ &\quad - \frac{(\tilde{p}'(1 - a\mathcal{E}_1) + \tilde{q}'(1 - a\mathcal{E}_2))^2}{4N^2},\end{aligned}$$

$$\frac{\partial_{\tilde q'}\widetilde{G}_{N,a}}{q^{1/2}(\eta_c)}=\tilde q'^2-\sin\theta+\frac{K_a}{r^{1/2}}(\tilde q'(1-a\mathcal{E}_1)+\tilde q'(1-a\mathcal{E}_2))$$
$$-\frac{(\tilde p'(1-a\mathcal{E}_1)+\tilde q'(1-a\mathcal{E}_2))^2}{4N^2}.$$

To the extent it is possible to do so, we follow the previous case $\lambda^{1/3}\lesssim N$. From $X\leq 1$, $A'\geq B'$ implying $\cos\theta\geq\sin\theta$. If $|(\tilde p',\tilde q')|\geq\widetilde{C}$ for some large $\widetilde{C}\geq 1$, then $(\tilde p'_c,\tilde q'_c)$ are such that $\tilde p'^2_c\geq\tilde q'^2_c$ and if $\widetilde{C}$ is sufficiently large the nonstationary phase applies (pick any $\widetilde{C}>4$). Therefore, we are reduced to bounding $|(\tilde p',\tilde q')|$. We sort out cases, depending upon $B'=r\sin\theta$: if $\sin\theta<-\frac{C}{\sqrt{r}}$ for some sufficiently large constant $C>0$, then

$$\frac{\partial_{\tilde q'}\widetilde{G}_{N,a}}{q^{1/2}(\eta_c)}\geq\tilde q'^2+\frac{C}{r^{1/2}}+\frac{K_a}{r^{1/2}}(\tilde p'(1-a\mathcal{E}_1)+\tilde q'(1-a\mathcal{E}_2))$$
$$-\frac{(\tilde p'(1-a\mathcal{E}_1)+\tilde q'(1-a\mathcal{E}_2))^2}{4N^2},$$

and $\mathcal{E}_{1,2}$ are bounded, N is sufficiently large in this case (indeed, recall that $r_0\leq r\lesssim N^2$ so that $\frac{1}{\sqrt{r}}\geq\frac{1}{N}$); then the nonstationary phase applies as the sum of the last three terms in the previous inequality is greater than $C/(2r^{1/2})$ if C is large enough. If $|\sin\theta|\leq\frac{C}{\sqrt{r}}$ then, again, $\theta\in(-\frac{C}{\sqrt{r_0}},\frac{\pi}{4})$ and $\cos\theta\geq\frac{\sqrt{2}}{2}$. We have $|B'|=|r\sin\theta|\leq C\sqrt{r}$; if $|B'|<C$, then $1+|B'|\lesssim r^{1/2}$, while $|A'|\sim r$. The stationary phase applies in $\tilde p'$ with nondegenerate critical points $\tilde p'_\pm$ and yields a factor $(r^{3/2}\Lambda)^{-1/2}$; the critical value of the phase function at these critical points, which we denote $\widetilde{G}^\pm_{N,a}$, is always such that $|\partial^3_{\tilde q'}\widetilde{G}^\pm_{N,a}|\geq q^{1/2}(\eta_c)(2-O(\frac{1}{r_0^{1/2}}))$ and the integral in $\tilde q'$ is bounded by $(r^{3/2}\Lambda)^{-1/3}$ by Van der Corput. Therefore, we obtain (62) which yields, using that $|B'|=|N^2(K_a^2-1)|\leq r^{1/2}$,

$$\begin{aligned}&|V_{N,a,h}(t,x,y)|\\&\quad=\frac{h^{1/3}\lambda^{4/3}}{\sqrt{\lambda N}N^2}\left|r\int_{\mathbb{R}^2}e^{ir^{3/2}\Lambda\widetilde{G}_{N,a}}\varkappa(r^{1/2}\tilde p'/N,r^{1/2}\tilde q'/N,t,x,y,h,a,1/N)\,d\tilde p'\,d\tilde q'\right|\\&\quad\lesssim\frac{h^{1/3}\lambda^{5/6}}{N^{5/2}}r^{-1/4}\Big(\frac{\lambda}{N^3}\Big)^{-5/6}\\&\quad\lesssim\frac{h^{1/3}}{(1+|B'|^{1/2})}\sim\frac{h^{1/3}}{(1+N|K_a(\frac{Y}{4N},\frac{T}{2N})-1|^{1/2})}.\end{aligned}$$

If $\sin\theta>\frac{C}{\sqrt{r}}$, then $B'=r\sin\theta>C\sqrt{r}$ and therefore $N^2|K_a^2-1|>Cr^{1/2}$. We do the stationary phase in both variables with large parameter $r^{3/2}\Lambda$ as the determinant of the Hessian at critical points is at least $C\sqrt{\cos\theta\sin\theta}$, and obtain, for the left-hand-side term in (62), a bound

$$\frac{cr}{(\sqrt{\sin\theta}\sqrt{\cos\theta})^{1/2}r^{3/2}\Lambda}=\frac{1}{\Lambda}\frac{1}{|A'B'|^{1/4}}\leq\frac{1}{\Lambda}\frac{1}{|B'|^{1/2}}.$$

We just proved that for $N < \lambda^{1/3}$ and not too small $N^2|K_a(\frac{Y}{4N}, \frac{T}{2N}) - 1|$,

$$|V_{N,h,a}(t,x,y)| \lesssim \frac{h^{1/3}}{\lambda^{1/6}\sqrt{N}|K_a(\frac{Y}{4N}, \frac{T}{2N}) - 1|^{1/2}}.$$

We now move to the most delicate case $|(A', B')| \leq r_0$. For $|(p', q')|$ large, the phase is nonstationary and integrations by parts provide $O(\Lambda^{-\infty})$ decay. So we may replace $\varkappa$ by a cutoff, which we still call $\varkappa$, compactly supported in $|(p', q')| < R$. We proceed by identifying one variable where the usual stationary phase applies and then evaluating the remaining one-dimensional oscillatory integral using Van der Corput (with different decay rates depending on the lower bounds on derivatives, of order at most 4). Using (61), we compute derivatives of $G_{N,a}$,

$$\partial_{p'} G_{N,a} = q^{1/2}(\eta_c)(p'^2 + N^2(X - \alpha_c)), \quad \partial_{q'} G_{N,a} = q^{1/2}(\eta_c)(q'^2 + N^2(1 - \alpha_c)).$$

The second-order derivatives of $G_{N,a}$ follow from (34), (35) and (36):

$$\begin{aligned}
\partial^2_{p'p'} G_{N,a} &= q^{1/2}(\eta_c)(2p' - N^2\partial_{p'}\alpha_c) + \frac{\partial_{p'}\eta_c \nabla q(\eta_c)}{2q^{1/2}(\eta_c)}(p'^2 + N^2(X - \alpha_c)),\\
\partial^2_{q'q'} G_{N,a} &= q^{1/2}(\eta_c)(2q' - N^2\partial_{q'}\alpha_c) + \frac{\partial_{q'}\eta_c \nabla q(\eta_c)}{2q^{1/2}(\eta_c)}(q'^2 + N^2(1 - \alpha_c)),\\
\partial^2_{q'p'} G_{N,a} &= q^{1/2}(\eta_c)(-N^2\partial_{q'}\alpha_c) + \frac{\partial_{q'}\eta_c \nabla q(\eta_c)}{2q^{1/2}(\eta_c)}(p'^2 + N^2(X - \alpha_c))\\
= \partial^2_{p'q'} G_{N,a} &= q^{1/2}(\eta_c)(-N^2\partial_{p'}\alpha_c) + \frac{\partial_{p'}\eta_c \nabla q(\eta_c)}{2q^{1/2}(\eta_c)}(q'^2 + N^2(1 - \alpha_c)).
\end{aligned}$$

At critical points, where $\partial_{p'} G_{N,a} = \partial_{q'} G_{N,a} = 0$, the determinant of the Hessian reads

$$\det \mathrm{Hess}_{(p',q')} G_{N,a}|_{\nabla_{(p',q')} G_{N,a} = 0} = q(\eta_c)\big(4p'q' - N^2(p' + q')\partial_{p'}\alpha_c\big).$$

If $|\det \mathrm{Hess}_{(p',q')} G_{N,a}| > c > 0$ for some small $c > 0$ we can apply the usual stationary phase in both variables p', q'. We expect the worst contributions to occur in a neighborhood of the critical points where $|\det \mathrm{Hess}_{(p',q')} G_{N,a}| \leq c$ for some c sufficiently small. We turn variables with $\xi_1 = (p' + q')/2$ and $\xi_2 = (p' - q')/2$. Then $p' = \xi_1 + \xi_2$ and $q' = \xi_1 - \xi_2$, and we also let $\mu := A' + B' = N^2(2K_a^2 - 1 - X)$, $\nu := A' - B' = N^2(1 - X)$. The most degenerate situation will turn out to be $\nu = \mu = 0$ and $\xi_1 = 0$, $\xi_2 = 0$. Let $g_{N,a}(\xi_1, \xi_2) = G_{N,a}(\xi_1 + \xi_2, \xi_1 - \xi_2)$.

Case $c \lesssim |\xi_1|$. For ξ_1 outside a small neighborhood of 0, the nondegenerate stationary phase applies in ξ_2 and the critical value $g_{N,a}(\xi_1, \xi_{2,c})$ may have degenerate critical points of order at most 2. The phase $g_{N,a}$ is stationary in ξ_2 whenever $\partial_{p'} G_{N,a} = \partial_{q'} G_{N,a}$ and from Remark 4, we then have $\partial_{p'}\eta_c = \partial_{q'}\eta_c$ and $\partial_{p'}\alpha_c = \partial_{q'}\alpha_c$. We have

$$\partial^2_{\xi_2,\xi_2} g_{N,a}(\xi_1, \xi_2) = (\partial^2_{p'p'} G_{N,a} - 2\partial^2_{p'q'} G_{N,a} + \partial^2_{q'q'} G_{N,a})(p', q')|_{\xi_1,\xi_2}.$$

Using the explicit form of the second-order derivatives of $G_{N,a}$ given above, at $p' = \xi_1 + \xi_2, q' = \xi_1 - \xi_2$ such that $p'^2 + N^2(X - \alpha_c) = q'^2 + N^2(1 - \alpha_c)$ and with $\partial_{p'}\eta_c = \partial_{q'}\eta_c$, we obtain

$$\partial^2_{\xi_2,\xi_2} g_{N,a}(\xi_1, \xi_2)|_{\partial_{\xi_2} g_{N,a}=0} = 2q^{1/2}(\eta_c)(p' + q') = 4q^{1/2}(\eta_c)\xi_1.$$

As $q(\eta_c) = |\eta_c| q(\eta_c/|\eta_c|) \in [\frac{1}{2}m_0^2, \frac{3}{2}M_0^2]$ with m_0, M_0 defined in (20), the stationary phase applies in ξ_2. We denote by $\xi_{2,c}$ the critical point, such that

$$\partial_{\xi_2} g_{N,a}(\xi_1, \xi_2) = (\partial_{p'} G_{N,a} - \partial_{q'} G_{N,a})(p', q')|_{p'=\xi_1+\xi_2, q'=\xi_1-\xi_2} = 0,$$

which can be rewritten as $(\xi_1 + \xi_{2,c})^2 + N^2(X - \alpha_c) = (\xi_1 - \xi_{2,c})^2 + N^2(1 - \alpha_c)$, which, in turn, yields $4\xi_1\xi_{2,c} = N^2(1 - X) = \nu$ and therefore $\xi_{2,c} = \frac{\nu}{4\xi_1}$. We will now compute higher-order derivatives of the critical value of $g_{N,a}(\xi_1, \xi_{2,c})$ with respect to ξ_1.

Lemma 9. *For $|N| \geq 1$, the phase $g_{N,a}(\xi_1, \xi_{2,c})$ may have degenerate critical points of order at most 2.*

Proof. Again, at $\xi_{2,c}$, Remark 4 implies $\partial_{p'}\eta_c = \partial_{q'}\eta_c$ and $\partial_{p'}\alpha_c = \partial_{q'}\alpha_c$. In turn, the functions $\Theta_{1,2}$ in Lemma 5 coincide as well, hence the functions $\mathcal{E}_{1,2}$ defined in (29),(30) coincide also at $\xi_{2,c}$. We abuse notation with $\mathcal{E}_{1,2}$ as functions of $(p'/N, q'/N) = (\xi_1 + \xi_2)/N, (\xi_1 - \xi_2)/N$. Set $\mathcal{E} := \mathcal{E}_1|_{p'^2+N^2X=q'^2+N^2} = \mathcal{E}_2|_{p'^2+N^2X=q'^2+N^2}$ in (28). Then $\sqrt{\alpha_c}|_{\partial_{\xi_2} g_{N,a}=0} = K_a - \frac{\xi_1}{N^2}(1 - a\mathcal{E})$ and therefore

$$\begin{aligned}
&\partial_{\xi_1}(g_{N,a}(\xi_1, \xi_{2,c})) \\
&\quad = \partial_{\xi_1} g_{N,a}(\xi_1, \xi_{2,c}) + \frac{\partial\xi_{2,c}}{\partial\xi_1}\partial_{\xi_2} g_{N,a}(\xi_1, \xi_2)|_{\xi_2=\xi_{2,c}} \\
&\quad = (\partial_{p'} G_{N,a} + \partial_{q'} G_{N,a})(p', q')|_{\xi_1,\xi_{2,c}} \\
&\quad = q^{1/2}(\eta_c)\Big(2\xi_1^2\Big(1 - \frac{1}{N^2}(1 - a\mathcal{E})\Big) + 2\frac{\nu^2}{16\xi_1^2} - \mu + 4K_a\xi_1(1 - a\mathcal{E})\Big). \qquad (63)
\end{aligned}$$

Taking a derivative of (63) with respect to ξ_1 yields

$$\begin{aligned}
&\partial^2_{\xi_1,\xi_1}(g_{N,a}(\xi_1, \xi_{2,c})) \\
&\quad = q^{1/2}(\eta_c)\Big[4\xi_1\Big(1 - \frac{1}{N^2}\Big(1 - a\Big(\mathcal{E} + \frac{1}{2}\xi_1\partial_{\xi_1}\mathcal{E}\Big)\Big)\Big) \\
&\qquad\qquad - \frac{\nu^2}{8\xi_1^3} + 4K_a(1 - a(\mathcal{E} + \xi_1\partial_{\xi_1}\mathcal{E}))\Big] \\
&\qquad + \Big(\partial_{p'}(q^{1/2}(\eta_c)) + \partial_{q'}(q^{1/2}(\eta_c)) \\
&\qquad\quad + \frac{\partial\xi_{2,c}}{\partial\xi_1}(\partial_{p'}(q^{1/2}(\eta_c)) - \partial_{q'}(q^{1/2}(\eta_c)))\Big)\frac{\partial_{\xi_1} g_{N,a}(\xi_1, \xi_{2,c})}{q^{1/2}(\eta_c)},
\end{aligned}$$

where the last line vanishes when $\partial_{\xi_1} g_{N,a}(\xi_1,\xi_{2,c}) = 0$. In the same way we compute

$$\partial^3_{\xi_1,\xi_1,\xi_1}(g_{N,a}(\xi_1,\xi_{2,c}))|_{\partial_{\xi_1}(g_{N,a}(\xi_1,\xi_{2,c}))=\partial^2_{\xi_1,\xi_1}(g_{N,a}(\xi_1,\xi_{2,c}))=0}$$
$$= q^{1/2}(\eta_c)\Big(4\Big(1-\frac{1}{N^2}\Big)+\frac{3\nu^2}{8\xi_1^4}+O(a)\Big).$$

First, let $|N| \geq 2$. Then we immediately see that the third-order derivative takes positive values and stays bounded from below by a fixed constant, $\partial^3_{\xi_1,\xi_1,\xi_1}(g_{N,a}(\xi_1,\xi_{2,c})) \geq 2$, and therefore the critical points may be degenerate (when $\partial^2_{\xi_1,\xi_1}(g_{N,a}(\xi_1,\xi_{2,c})) = 0$) of order at most 2. Now let $|N| = 1$ when the coefficient of $2\xi_1^2$ in (63) is $O(a)$. Assume that for $c \lesssim |\xi_1|$, the first two derivatives vanish. Then $\frac{\nu^2}{8\xi_1^3} = 4K_a + O(a)$ and therefore the third derivative cannot vanish since its main contribution is $\frac{3\nu^2}{8\xi_1^4}$. ∎

Case $|\xi_1| \lesssim c$, for small $0 < c < 1/2$. First, the (usual) stationary phase applies in ξ_1:

$$\partial_{\xi_1} g_{N,a}(\xi_1,\xi_2) = q^{1/2}(\eta_c)\big((\xi_1+\xi_2)^2 + N^2(X-\alpha_c) + (\xi_1-\xi_2)^2 + N^2(1-\alpha_c)\big),$$

and using (28), we write again, with $K_a = K_a(\frac{Y}{4N}, \frac{T}{2N}) = \frac{T}{2N} q^{1/2}(\eta_c^0)$,

$$\sqrt{\alpha_c} = K_a - \frac{(\sigma+s)}{2N} + \frac{T}{2N}(q^{1/2}(\eta_c) - q^{1/2}(\eta_c^0)),$$

where in the new variables $\sigma + s = 2\xi_1/N$. Using (31), we have $(q^{1/2}(\eta_c) - q^{1/2}(\eta_c^0)) = \frac{a}{NT} O(\xi_1,\xi_2)$ and with $|\xi_1| \leq c < \frac{1}{2}$ small, $a \leq \varepsilon_0$ and $\alpha_c \in [\frac{1}{2}, \frac{3}{2}]$, from $K_a = \sqrt{\alpha_c} + O(c/N^2)$ we have $K_a \in [\frac{1}{4}, 2]$ for all $N \geq 1$. The derivative of $g_{N,a}(\xi_1,\xi_2)$ becomes

$$\begin{aligned}
&\partial_{\xi_1} g_{N,a}(\xi_1,\xi_2)\\
&= q^{1/2}(\eta_c)\Big\{2\xi_1^2 + 2\xi_2^2 - \mu - 2N^2\Big[\Big(K_a - \frac{\xi_1}{N^2} + \frac{a}{N^2}O(\xi_1,\xi_2)\Big)^2 - K_a^2\Big]\Big\}\\
&= q^{1/2}(\eta_c)\Big(2\xi_1^2\Big(1-\frac{1}{N^2}\Big) + 2\xi_2^2 - \mu + 4K_a\xi_1 + aO(\xi_1,\xi_2)\Big).
\end{aligned}$$

At the critical point, the second derivative with respect to ξ_1 is

$$\partial^2_{\xi_1,\xi_1} g_{N,a}(\xi_1,\xi_2)|_{\partial_{\xi_1} g_{N,a}(\xi_1,\xi_2)=0} = q^{1/2}(\eta_c)\Big(4\xi_1\Big(1-\frac{1}{N^2}\Big) + 4K_a + O(a)\Big),$$

and as $K_a \in [\frac{1}{4}, 2]$, the leading-order term is $4q^{1/2}(\eta_c)K_a$. The stationary phase applies for any $|N| \geq 1$ and provides a factor $\Lambda^{-1/2}$. We are left with the integral with respect to ξ_2. We first compute the critical point $\xi_{1,c}$, a solution to $\partial_{\xi_1} g_{N,a}(\xi_1,\xi_2) = 0$, as a function of ξ_2:

$$2\xi_{1,c}^2 + 2\xi_2^2 = \mu + 2N^2\Big[K_a^2 - \Big(K_a^2 - \frac{\xi_1}{N^2} + \frac{T}{2N}(q^{1/2}(\eta_c) - q^{1/2}(\eta_c^0))\Big)^2|_{\xi_1,\xi_2}\Big], \tag{64}$$

where, using (31), $\frac{T}{2N}(q^{1/2}(\eta_c) - q^{1/2}(\eta_c^0)) = O(\frac{a}{N^2})$. From $|\xi_{1,c}| \leq c$, $|\mu/2 - \xi_2^2| \lesssim c$, as, if $|\mu/2 - \xi_2^2| > 4c$, equation (64) has no real solution $\xi_{1,c}$ such that $|\xi_{1,c}| \leq c$.

Lemma 10. *For all* $|N| \geq 1$ *and* $|\mu/2 - \xi_2^2| \leq 4c$, (64) *has one real-valued solution,*

$$\xi_{1,c} = (\mu/2 - \xi_2^2)\Xi_0 + a\big((\mu/2 - \xi_2^2)\Xi_1 + \xi_2^2\Xi_2 + \xi_2\frac{\nu}{N^2}\Xi_3\big), \tag{65}$$

where $K_{a=0} = \frac{|Y|}{4N}q^{1/2}(-Y/|Y|)$ *and* $\Xi_0 = \Xi_0(\mu/2 - \xi_2^2, K_{a=0}, 1/N^2)$ *is defined as*

$$\Xi_0(\mu/2 - \xi_2^2, K_{a=0}, 1/N^2) = \Big(K_{a=0} + \sqrt{K_{a=0}^2 + (\mu/2 - \xi_2^2)(1 - 1/N^2)}\Big)^{-1} \tag{66}$$

and where $\Xi_{1,2,3}$ *are smooth functions of* $(\xi_2, \mu/2 - \xi_2^2, \nu/N^2, K_a, 1/N, a)$ *such that* $|\partial_{\xi_2}^k \Xi_j| \leq C_k$, *for all* $k \geq 0$, *where* C_k *are positive constants.*

Proof. For $a = 0$, (64) has a unique, explicit solution $\xi_{1,c}|_{a=0}$,

$$\xi_{1,c}|_{a=0} = (\mu/2 - \xi_2^2)\big(K_{a=0} + \sqrt{K_{a=0}^2 + (\mu/2 - \xi_2^2)(1 - 1/N^2)}\big)^{-1},$$

that we rename $(\mu/2 - \xi_2^2)\Xi_0$ with Ξ_0 defined in (66). Now let $a \neq 0$. Using Lemma 5 with $s + \sigma = (p' + q')/N = 2\xi_1/N$, $\sigma - s = (p' - q')/N = 2\xi_2/N$, $(a - x)/a = \nu/N^2$, the critical point η_c is a function of ξ_1/N, ξ_2^2/N^2 and $\xi_2\nu/N^3$. Write $\xi_{1,c}$ as $\xi_{1,c} = (\mu/2 - \xi_2^2)\Xi_0 + a\Xi$ for some unknown function Ξ; introducing this in (64) allows us to obtain Ξ as a sum of smooth functions with factors $\mu/2 - \xi_2^2$, ξ_2^2 and $\xi_2\nu/N^2$ as follows: $\Xi = (\mu/2 - \xi_2^2)\Xi_1 + \xi_2^2\Xi_2 + \xi_2\frac{\nu}{N^2}\Xi_3$, where Ξ_j are smooth functions of $\mu/2 - \xi_2^2$, ξ_2^2/N^2 and $\xi_2\nu/N^3$. ∎

Let $\tilde{g}_{N,a}(\xi_2) := g_{N,a}(\xi_{1,c}, \xi_2)$: the first derivative of $\tilde{g}_{N,a}$ with respect to ξ_2 vanishes when $(\partial_{p'}G_{N,a} - \partial_{q'}G_{N,a})(p', q')|_{(\xi_{1,c},\xi_2)} = 0$, which is equivalent to $4\xi_{1,c}\xi_2 = \nu$. We compute, using $\partial_{\xi_2}\tilde{g}_{N,a} = \nu - 4\xi_{1,c}\xi_2$ and $\xi_{1,c}$ given in (65), $\partial^2_{\xi_2\xi_2}\tilde{g}_{N,a} = -4(\xi_2\partial_{\xi_2}\xi_{1,c} + \xi_{1,c})$. Then the critical points ξ_2 are degenerate if

$$\begin{aligned}(\mu/2 - \xi_2^2)\Xi_0 &+ a\Big((\mu/2 - \xi_2^2)\Xi_1 + \xi_2^2\Xi_2 + \xi_2\frac{\nu}{N^2}\Xi_3\Big)\\ &= 2\xi_2^2\Xi_0(1 - (\mu/2 - \xi_2^2)\tilde{\Xi}_0) + a\Big(2\xi_2^2\Big(\Xi_1 - \Xi_2 - \frac{1}{2}\xi_2\partial_{\xi_2}\Xi_2 - \frac{\nu}{N^2}\partial_{\xi_2}\Xi_3\Big)\\ &\qquad - \xi_2(\mu/2 - \xi_2^2)\partial_{\xi_2}\Xi_1 - \xi_2\frac{\nu}{N^2}\Xi_3\Big),\end{aligned} \tag{67}$$

where the sum of the terms in the second and third lines of (67) equals $\xi_2\partial_{\xi_2}\Xi_0$. We have thus set

$$\tilde{\Xi}_0(\mu/2 - \xi_2^2, K_a, 1/N^2) := \frac{(1 - 1/N^2)\Xi_0(\mu/2 - \xi_2^2, K_a, 1/N^2)}{2\sqrt{K_a^2 + (\mu/2 - \xi_2^2)(1 - 1/N^2)}}.$$

Consider $a = 0$ in (67) for a moment. Then critical points are degenerate if

$$\mu/2 - \xi_2^2 = 2\xi_2^2\big(1 - (\mu/2 - \xi_2^2)\tilde{\Xi}_0(\mu/2 - \xi_2^2, K_0, 1/N^2)\big). \tag{68}$$

Recall that $K_a \in [\frac{1}{4}, 2]$ and that $|\mu/2 - \xi_2^2| \le 4c$ with c small enough. Rewrite (68) as

$$(\mu/2-\xi_2^2)\Big(2+\frac{1}{1-(\mu/2-\xi_2^2)\widetilde{\Xi}_0}\Big)=\mu,$$

which may have solutions only if μ is also small enough, $|\mu| \le 10c$. Let $z = \mu/2 - \xi_2^2$; for $|z| \le 4c$ and $|\mu| \le 10c$ with c small enough, we may now seek the solution to (68) as $z = \mu Z_0(\mu, K_0, 1/N^2)$ and obtain $Z_0(\mu, K_0, 1/N^2)$ explicitly, with $Z_0(0, K_0, 1/N^2) = \frac{1}{3}$. Solutions to (67) for $a = 0$ are therefore functions of $\sqrt{\mu}$ which both vanish at $\mu = 0$. They are written $\xi_{2,\pm}|_{a=0} = \pm\frac{\sqrt{\mu}}{\sqrt{6}}(1 + \mu\zeta(\mu, K_0, 1/N^2))$ for some smooth function ζ.

Now let $a \ne 0$: solutions ξ_2 to (67) are functions of $\sqrt{\mu}$, ν/N^2, a that coincide at $\mu = \nu = 0$ (they both vanish). Actually, as Ξ_1 is a function of $\mu/2 - \xi_2^2, \xi_2^2, \xi_2\nu/N^2$, $\xi_2\partial_{\xi_2}\Xi_1$ is also a function of $\mu/2 - \xi_2^2, \xi_2^2, \xi_2\nu/N^2$ and we write

$$\begin{aligned}\mu/2-\xi_2^2 &= 2\xi_2^2(1-(\mu/2-\xi_2^2)\widetilde{\Xi}_0(\mu/2-\xi_2^2,K_a,1/N^2))\\ &\quad + a\Big(\xi_2^2F_1(\xi_2^2,\xi_2\nu/N^2,\mu)+\xi_2\frac{\nu}{N^2}F_2(\xi_2^2,\xi_2\nu/N^2,\mu)\Big)\end{aligned}\tag{69}$$

for some smooth functions $F_{1,2}$. Notice that, as $|\mu/2 - \xi_2^2| \le 4c$ and a is small, (69) may have real solutions ξ_2 only for $|\xi_2^2| \le 4c$. For such small ξ_2, equation (69) has at most two distinct solutions (that coincide at $\mu = \nu = 0$), which read

$$\begin{aligned}&\xi_{2,\pm}=\pm\frac{\sqrt{\mu}}{\sqrt{6}}\Big(1+\mu\zeta\Big(\mu,K_a,\frac{1}{N^2}\Big)\Big)+a<\Big(\sqrt{\mu},\frac{\nu}{N^2}\Big),\\ &(\zeta_{1,\pm},\zeta_{2,\pm})>\Big(\sqrt{\mu},\frac{\nu}{N^2},K_a,a\Big),\end{aligned}\tag{70}$$

for some smooth functions $\zeta, \zeta_{j,\pm}$. We compute the third derivative of $\tilde{g}_{N,a}$ at $\xi_{2,\pm}$ defined in (70) whenever the second derivative vanishes. Using (67) yields

$$\begin{aligned}&\partial^3_{\xi_2,\xi_2,\xi_2}\tilde{g}_{N,a}(\xi_{1,c},\xi_2)|_{\xi_2=\xi_{2,\pm}}\\ &\quad= -4(2\partial_{\xi_2}\xi_{1,c}+\xi_2\partial^2_{\xi_2,\xi_2}\xi_{1,c})|_{\xi_{2,\pm}}\\ &\quad= 16\xi_2\Xi_0\big(1-(\mu/2-\xi_2^2)\widetilde{\Xi}_0(\mu/2-\xi_2^2,K_a,1/N^2)\big)\\ &\qquad + 8a\Big(2\xi_2\Big(\Xi_1-\Xi_2-\frac{1}{2}\xi_2\partial_{\xi_2}\Xi_2-\frac{\nu}{N^2}\partial_{\xi_2}\Xi_3\Big)\\ &\qquad\qquad -(\mu/2-\xi_2^2)\partial_{\xi_2}\Xi_1-\frac{\nu}{N^2}\Xi_3\Big)\Big|_{\xi_{2,\pm}}\\ &\qquad + 8\xi_2\Xi_0(1+O(\mu/2-\xi_2^2)+O(a))|_{\xi_{2,\pm}},\end{aligned}\tag{71}$$

where the last line in (71) is $-4\xi_{2,\pm}\partial^2_{\xi_2,\xi_2}\xi_{1,c}$: we do not expand this formula as $\xi_{2,\pm}$ is sufficiently small for what we need. The second and third lines of (71) come from the formula for $-8\partial_{\xi_2}\xi_{1,c}$, already obtained in (67) (where $\partial_{\xi_2}\xi_{1,c}$ comes with a factor ξ_2). As the third derivative of $\tilde{g}_{N,a}$ is evaluated at $\xi_{2,\pm}$ we can replace (69) in (71) and obtain

$$\partial^3_{\xi_2,\xi_2,\xi_2}\tilde{g}_{N,a}(\xi_{1,c},\xi_2)|_{\xi_2=\xi_{2,\pm}}=\frac{12\xi_{2,\pm}}{K_a}(1+O(\xi_{2,\pm}^2)+O(a))+O(a\nu/N^2).$$

It follows that at $\mu=\nu=0$ the order of degeneracy is higher as $\xi_{2,\pm}|_{\mu=\nu=0}=0$ and $\partial^3_{\xi_2,\xi_2,\xi_3}\tilde{g}_{N,a}|_{\xi_{2,\pm},\mu=\nu=0}=0$. We now write

$$\tilde{g}_{N,a}(\xi_2)=\tilde{g}_{N,a}(\xi_{2,\pm})+(\xi_2-\xi_{2,\pm})\partial_{\xi_2}\tilde{g}_{N,a}(\xi_{2,\pm})+\frac{(\xi_2-\xi_{2,\pm})^3}{6}\partial^3_{\xi_2,\xi_2,\xi_2}\tilde{g}_{N,a}(\xi_{2,\pm})$$
$$+O((\xi_2-\xi_{2,\pm})^4),$$

where $\partial^4_{\xi_2^4}\tilde{g}_{N,a}$ does not cancel at $\xi_{2,\pm}$ as it stays close to $12/K_a\in[6,48]$. We are to have $\partial_{\xi_2}\tilde{g}_{N,a}(\xi_{2,\pm})=0$, from which $\nu=4\xi_{1,c}|_{\xi_{2,\pm}}\xi_{2,\pm}$, which is written

$$\nu=4\Big(\pm\frac{\sqrt{\mu}}{\sqrt{6}}(1+\mu\zeta(\mu))+a\Big(\sqrt{\mu}\zeta_{1,\pm}+\frac{\nu}{N^2}\zeta_{2,\pm}\Big)\Big)$$
$$\times\Big((\mu/2-\xi^2_{2,\pm})\Xi_0+a\Big((\mu/2-\xi^2_{2,\pm})\Xi_1+\xi^2_{2,\pm}\Xi_2+\xi_{2,\pm}\frac{\nu}{N^2}\Xi_3\Big)\Big)$$

and replacing (70) in (65) yields

$$\nu=\pm\frac{\sqrt{2}\mu^{3/2}}{3\sqrt{3}K_a}(1+O(a)),$$

which is at leading order the equation of a cusp. At degenerate critical points $\xi_{2,\pm}$ where

$$\nu=\pm\frac{\sqrt{2}\mu^{3/2}}{3\sqrt{3}K_a}(1+O(a)),$$

the phase integral behaves like

$$I=\int_{\xi_2}\rho(\xi_2)e^{\mp i\Lambda\frac{\sqrt{2}\sqrt{\mu}}{K_a\sqrt{3}}(\xi_2-\xi_{2,\pm})^3}\,d\xi_2,$$

and we may conclude in a small neighborhood of the set $\{\xi_2^2+|\mu|+|\nu|^{2/3}\lesssim c\}$ (as outside this set, the nonstationary phase applies) by using the Van der Corput lemma on the remaining oscillatory integral in ξ_2 with phase $\tilde{g}_{N,a}(\xi_2)$. In fact, on this set, $\partial^4_{\xi_2}\tilde{g}_{N,a}$ is bounded from below, which yields an upper bound $\Lambda^{-1/4}$, uniformly in all parameters. When $\mu\neq 0$, the third-order derivative of the phase is bounded from below by $\frac{|\xi_2|}{K_a}$: either $|\mu/6-\xi_2^2|\leq|\mu|/12$ and then $|\partial^3_{\xi_2}\tilde{g}_{N,a}|$ is bounded from below by $|\mu|^{1/2}/(12K_a)$, or $|\mu/6-\xi_2^2|\geq|\mu|/12$ in which case $|\partial^2_{\xi_2}\tilde{g}_{N,a}|$ is bounded from below by $|\mu|/(12K_a)$. Hence, using that $K_a\in[\frac{1}{4},2]$, we find

$$|\partial^3_{\xi_2}\tilde{g}_{N,a}|+|\partial^3_{\xi_2}\tilde{g}_{N,a}|\gtrsim\sqrt{|\mu|}$$

(recall that here μ is small so $\sqrt{|\mu|}\geq|\mu|$) which yields an upper bound $(\sqrt{|\mu|}\Lambda)^{-1/3}$. Eventually we obtain

$$|I|\lesssim\inf\Big\{\frac{1}{\Lambda^{1/4}},\frac{1}{|\mu|^{1/6}\Lambda^{1/3}}\Big\}.$$

From $\mu=A'+B'$ and $\nu=A'-B'\sim\pm|\mu|^{3/2}$ and $|\mu|^{3/2}\ll|\mu|$ for $\mu<1$, we deduce that $A'\sim B'$ and therefore $|\mu|\sim 2|B'|$, which is our desired bound (40) after unraveling all notation, as the nondegenerate stationary phase in ξ_1 had already provided a factor $\Lambda^{-1/2}$.

5. The defocusing cubic nonlinear Schrödinger equation

We now turn to nonlinear applications and consider (1) with $\kappa = 1$. One could state local-in-time results for the focusing case $k = -1$ and extend them to global in time provided we require a standard smallness condition on the mass of the data (identifying the threshold is, however, a delicate issue). Our underlying manifold Ω is compact and such that, in local charts intersecting its boundary, the Laplace–Beltrami operator is, to first order, our model operator. For such a suitable manifold Ω, with $d \geq 3$, we may gather available results and state homogeneous Strichartz estimates. In the next theorem, Sobolev spaces should be understood as defined through the spectral resolution of the Dirichlet Laplacian and they are known to match the ones defined by classical interpolation.

Theorem 5. *Let $d \geq 2$, (q,r) be such that $\frac{1}{q} \leq (\frac{d}{2} - \frac{1}{4})(\frac{1}{2} - \frac{1}{r})$, $s = \frac{d}{2} - \frac{2}{q} - \frac{d}{r}$; there exist $C(d) > 0$, $T > 0$ such that, for v a solution to (1) with data $v_0 \in H^{s+1/q}(\Omega)$,*

$$\|v\|_{L^q([0,T],L^r(\Omega))} \leq C(d) T^{1/q} \|v_0\|_{H^{s+1/q}(\Omega)}. \tag{72}$$

Both Lebesgue and Sobolev spaces may be localized in coordinate charts. As such, the proof of Theorem 5 follows by standard arguments, localizing the linear solution to each patch, and using either [7] (on patches that do not intersect the boundary), Theorem 2 (on patches that do intersect the convex boundary) or [10] (if one considers a concave boundary on one end). Because one works on semiclassical times, source terms that are produced by cutoffs are dealt with through energy estimates, a well-established procedure that goes back to [27]. We refer to [5] or [16] for detailed implementations of the above strategy. Notice that (72) subsumes the underlying semiclassical estimate: assume v_0 is spectrally localized at frequency h^{-1}, and $T \lesssim h$, then $\|v_0\|_{H^{s+1/q}(\Omega)} \sim h^{-1/q} \|v_0\|_{H^s(\Omega)}$. We also need a suitable semiclassical version of the double endpoint inhomogeneous Strichartz estimate which follows from similar arguments: let

$$\begin{aligned} -i\partial_t u + \Delta_g u &= f, \\ u|_{t=0} &= 0, \\ u|_{\mathbb{R}\times\partial\Omega} &= 0, \end{aligned} \tag{73}$$

with f supported in a time interval I such that $|I| \lesssim h$, $I \subset \mathbb{R}_+$. Then we have the following proposition:

Proposition 13. *Let $d \geq 3$, r be such that $\frac{1}{2} = (\frac{d}{2} - \frac{1}{4})(\frac{1}{2} - \frac{1}{r})$, $\sigma = -d/r + d/(2d/(d-2))$, $r' = r/(r-1)$; there exists $C(d) > 0$ such that, for u a solution to (73),*

$$\|\psi(h^2\Delta_g)u\|_{L^2(I,L^r(\Omega))} \leq C(d) h^{-2\sigma} \|\psi(h^2\Delta_g) f\|_{L^2(I,L^{r'}(\Omega))}.$$

This should be understood as a weaker version of [7, Lemma 3.4]: the loss of regularity σ corresponds to a Sobolev embedding from $L^{2d/(d-2)}$ (the endpoint Strichartz exponent on a boundaryless manifold) to L^r for a spectrally localized function. The value of r will be irrelevant in the forthcoming argument: any $r < +\infty$ would do as long as σ is chosen accordingly (preserving scale invariance).

Remark 6. With $g = dx^2 + (1+x)^{-1}dy^2$, the Laplace–Beltrami operator is written $\triangle_g = (1+x)^{1/2}\partial_x(1+x)^{-1/2}\partial_x + (1+x)\Delta_y$. In our model, we use instead $\Delta_F = \partial_x^2 + (1+x)\Delta_y^2$, as Δ_F allows for explicit computations. The difference $\Delta_g - \Delta_F = -(2(1+x))^{-1}\partial_x$ is a first-order differential operator: as such, on a semiclassical timescale in a neighborhood of the boundary, it may be treated as a lower-order perturbative term; proving semiclassical Strichartz estimates for Δ_F implies the same set of estimates for Δ_g.

We are interested in $d = 3$: for $q = 2$ we have $r = 10$, $s + 1/q = 7/10 < 1$. The crucial point is that $r < +\infty$: in [5], for $q = 2$, $r = +\infty$, $s + 1/q = 1$ and one ends up with two successive logarithmic losses that force us to consider only lower-order nonlinearities. In [20], Strichartz estimates are bypassed and replaced by bilinear estimates: logarithmic losses turn out to be more manageable in that setting and one can implement a suitable version of the Brezis–Gallouët argument, obtaining global existence of solutions to the cubic nonlinear Schrödinger equation but only for $H^s(\Omega)$ data, with $1 < s \leq 3$.

5.1. Global well-posedness in the energy space. Proof of Theorem 3

Recall that we are interested in $-i\partial_t v + \Delta_g v = |v|^2 v$ on Ω, with $v_0 \in H_0^1(\Omega)$. Standard compactness methods provide a weak solution $v \in L_t^\infty(H_0^1(\Omega))$, and, with $E(v) = \|\nabla v\|_2^2/2 + \|v\|_4^4/4$, we have $E(v) \leq E(v_0)$. Moreover, using Duhamel's formula, one may prove that $v \in C_t(L^2(\Omega))$. Thus we aim to prove that v is unique and that time continuity holds in $H_0^1(\Omega)$. To this end, [7] implement an argument of Yudovitch in a clever way, using the inhomogeneous endpoint Strichartz estimate at the semiclassical level. We follow them closely, having only to check that the specific value $r = 6$ that they start with is irrelevant to the crux of the matter. Assume we are on $(0, T)$, with $T < 1$ and Δ_j is the usual spectral localization operator associated to a Littlewood–Paley decomposition: $h = 2^{-j}$, $\Delta_j = \psi(h^2\Delta_g)$, $\sum_{j\geq -1}\Delta_j = \mathrm{Id}$, where $\Delta_{-1} = \phi(\Delta_g)$ with ϕ supported in $B(0, 2)$. After time localization on intervals of size h, applying their inhomogeneous endpoint Strichartz estimate and summing over $O(h^{-1})$ time intervals lead [7] (on a boundaryless manifold) to

$$
\begin{aligned}
\|\Delta_j v\|_{L_T^2 L^6} \lesssim{}& \|\Delta_j v(0)\|_{L^2} + \|\Delta_j v(T)\|_{L^2} + 2^{-j/2}\|\Delta_j v\|_{L_T^2 H^1} \\
&+ \|\Delta_j(|v|^2 v)\|_{L_T^2 L^{6/5}}.
\end{aligned}
$$

Reproducing their argument but using Proposition 13 with the $(2, 10)$ Strichartz pair leads to

$$
\begin{aligned}
2^{-j/5}\|\Delta_j v\|_{L_T^2 L^{10}} \lesssim{}& \|\Delta_j v(0)\|_{L^2} + \|\Delta_j v(T)\|_{L^2} + 2^{-j/2}\|\Delta_j v\|_{L_T^2 H^1} \\
&+ 2^{j/5}\|\Delta_j(|v|^2 v)\|_{L_T^2 L^{10/9}}
\end{aligned}
\tag{74}
$$

(observe that, as noted earlier, this is essentially the same estimate in terms of scaling). We now use Sobolev embedding, $H^1 \hookrightarrow L^5$ and product laws to bound the last term,

$$2^{j/5}\|\Delta_j(|v|^2 v)\|_{L^2_T L^{10/9}} \lesssim \sqrt{T} 2^{-4j/5}(\sup_t \|v\|_{H^1})^3.$$

We use Sobolev but on the left-hand side, with a large p (recall $T \le 1$),

$$2^{-3j(1/10-1/p)-j/5}\|\Delta_j v\|_{L^2_T L^p} \lesssim 2^{-j}\sup_t \|v\|_{H^1} + 2^{-j/2}\|\Delta_j v\|_{L^2_T H^1} + 2^{-4j/5}(\sup_t \|v\|_{H^1})^3,$$

to get

$$\|\Delta_j v\|_{L^2_T L^p} \lesssim 2^{-3j/p}\|\Delta_j v\|_{L^2_T H^1} + 2^{-3j/p-2j/5}\sup_t(\|v\|_{H^1} + \|v\|^3_{H^1}). \tag{75}$$

Summing over j, applying Cauchy–Schwarz in j, we finally get the same bound as in [7]:

$$\|v\|_{L^2_T L^p} \lesssim \sqrt{pT} + 1.$$

From there we may proceed similarly and conclude the uniqueness of weak solutions by estimating the difference between two solutions in L^2, the above estimate and an elementary differential inequality.

Once we have uniqueness, the inequality for $E(v)$ is immediately upgraded to conservation, and continuity in H^1 follows as the potential part is itself continuous by interpolation between L^2 and L^6.

This achieves the proof of Theorem 3 in the defocusing case. The focusing case may be handled in a similar way, up to a smallness condition if one wants a global result to preserve coercivity of $E(v)$ and we therefore skip it.

For preservation of H^2 regularity, one may proceed exactly as in [7], using the Brezis–Gallouët argument. One should notice, however, that the resulting bound on the Sobolev norm is a double exponential (to be compared to the triple exponential from [20]).

5.2. Exponential growth for higher Sobolev norms H^m, $m > 1$. Proof of Theorem 4

The double exponential growth for higher Sobolev norms was reduced to a single exponential, for solutions on a generic compact, boundaryless manifold, in [21], using modified energy methods. We now check that the elegant treatment of modified energies in [21] is not spoiled if we replace the endpoint Strichartz estimate from [7] by our endpoint estimate.

In [21], bounds on the H^{2k} norms follow from computing the time derivative of

$$\mathcal{E}_{2k}(v) = \|\partial_t^k v\|^2_{L^2(M^d)} - \frac{1}{2}\int_{M^d} |\partial_t^{k-1}\nabla_g(|v|^2)|_g^2 - \int_{M^d} |\partial_t^{k-1}(|v|^2 v)|^2, \tag{76}$$

where all norms over the boundaryless manifold M^d are understood with respect to the metric volume form. One gets

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_{2k}(v) = 2\int_{M^d} \partial_t^k(|v|^2)\partial_t^{k-1}(|\nabla_g v|_g^2) + \Re\sum_{j=0}^{k-2} c_j \int_{M^d} \partial_t^k(|v|^2)\partial_t^j(\Delta_g v)\partial_t^{k-1-j}\bar{v} \\ + \Re\sum_{j=0}^{k-1} c_j \int_{M^d} \partial_t^j(|v|^2)\partial_t^{k-j} v\partial_t^{k-1}(|v|^2\bar{v}) \\ + \Im\sum_{j=1}^{k-1} c_j \int_{M^d} \partial_t^j(|v|^2)\partial_t^{k-j} v\partial_t^k\bar{v}, \qquad (77)\end{aligned}$$

where c_j are harmless numerical constants. We may perform the exact same computation on our manifold Ω: there are no boundary terms due to the Dirichlet boundary condition, provided that all the time derivatives that are involved do satisfy such a condition. For the linear equation, that would follow from the data being in the domain of the suitable power of the Dirichlet Laplacian. In the nonlinear setting, one has to require suitable compatibility conditions as soon as the regularity is bigger than $s = 4$. Hence we restrict ourselves to $k = 1, 2$ in what follows, although the computation can be carried out for any k provided that the successive time derivatives have zero trace on the boundary. For the linear equation, H^2 regularity may be preserved if we work with data in the domain of the Dirichlet Laplacian, $H^2(\Omega) \cap H_0^1(\Omega)$; similarly, H^4 regularity requires $v_0 \in H^4(\Omega) \cap H_0^1(\Omega)$ and $\Delta_g v_0 \in H_0^1(\Omega)$, where the latter condition is a natural compatibly condition to match traces at $t = 0$.

In the nonlinear setting, we may use the equation to check that having the data in the natural domain of the operator (for H^2 and H^4 regularity) allows for compatibility at $t = 0$: define $i\partial_t v(t = 0) = \Delta_g v_0 - |v_0|^2 v_0 \in L^2$ if $u \in H^2$. Similarly, $-\partial_t^2 v(t = 0) = \Delta^2 v_0 - 2(\Delta_g v - |v_0|^2 v_0)|v_0|^2 + (\Delta_g \bar{v}_0 - |v_0|^2\bar{v}_0)v_0^2 \in L^2$ if $v_0 \in H^4 \cap H_0^1$ and $\Delta v_0 \in H_0^1$. The next iteration would yield a term $\Delta_g^2(|v_0|^2 v_0)$, which has no particular reason to have zero trace on the boundary unless we ask for it.

Remark 7. If we insist that the operator near the boundary should be Δ_F and not the corresponding Laplace–Beltrami operator (from Remark 6), then one may (formally) check that $\partial_x^2 v|_{\{x=0\}} = 0$, and proceed with further regularity, iterating like the usual Laplacian in the half-space where normal derivatives of even order vanish. However, such a model is somewhat artificial and the Laplace–Beltrami version is a better match if one is trying to be closer to the boundary of a ball.

We proceed with the growth analysis: first, we remark that one may replace the Sobolev norms by the L^2 norm of time derivatives:

$$\begin{aligned}\|\partial_t v\|_{L^2(\Omega)}^2 = \int_\Omega i\partial_t v(-i\partial_t\bar{v}) = \int_\Omega (\Delta_g v - |v|^2 v)((\Delta_g\bar{v} - |v|^2\bar{v}) \\ = \|\Delta_g u\|_{L^2(\Omega)}^2 + \text{l.o.t.}\end{aligned}$$

where the lower-order terms are controlled using at most one factor $u \in H^2$. Similarly, one may write

$$\|\partial_t^2 v\|_{L^2(\Omega)}^2 = \int_\Omega i\partial_t\partial_t v(-i\partial_t\partial_t \bar{v}) = \int_\Omega (\partial_t \Delta_g v - \partial_t(|v|^2 v))((\partial_t \Delta_g \bar{v} - \partial_t(|v|^2\bar{v}))$$
$$= \|\Delta_g^2 u\|_{L^2(\Omega)}^2 + \text{l.o.t.}$$

after further substitutions through the equation.

We will therefore address the L^2 norm of $\partial_t u$, replacing it with $\mathcal{E}_2(v)$, which we define with (76), replacing M^d by Ω: $\partial_t^2 u$ (through $\mathcal{E}_4(v)$) can be dealt with similarly (for the top-order term, the remaining terms being lower order). The modified energy bound in [21] proceeds with

$$\frac{d}{dt}\mathcal{E}_2(v) \lesssim \int_{M^d} |\nabla^2 v|\,|\nabla v|^2|v| \lesssim \|v\|_{H^2}\|\nabla v\|_{L^6}^2\|v\|_{L^6},$$

using the equation to eliminate all time derivatives and ignoring lower-order terms; one then uses a suitable version of Proposition 13, involving the endpoint Strichartz estimate, for the gradient term, which is therefore bounded by (the square of) an $H^{3/2}$ with an (important) additional power of time; by interpolation one recovers another H^2 norm and this leads to exponential growth for the H^2 norm by standard arguments.

From the computation of the time derivative (77) (with M^d replaced by Ω), we similarly get

$$\frac{d}{dt}\mathcal{E}_2(v) \lesssim \int_\Omega |\nabla^2 v|\,|\nabla v|^2|v|, \tag{78}$$

ignoring, once again, lower-order terms (where spatial derivatives are distributed over at least four factors v rather than just three of them). We now modify how we distribute norms on the integral over Ω in (78): one should heuristically think that we aim to place each ∇v in $L^4_{t,x}$ (hence controlled by $H^{3/2}$) and $v \in L^\infty_{t,x}$. We need to slightly perturb this choice, however, in order to avoid a log loss: with a large p, we have, by our Strichartz estimate (75) (at the scaling level of H_0^1 data) and Sobolev embedding (starting from L^6_x),

$$\Delta_j v \in 2^{3j/p} L_t^2 L^p \quad \text{and} \quad \Delta_j v \in 2^{-3(1/6-1/p)j} L_t^\infty L^p.$$

Hence we get, balancing regularities with $(1-e)\frac{3}{p} = e(\frac{1}{2}-\frac{3}{p})$ $(e = 6/p)$, $v \in L_t^{\frac{2}{1-6/p}} L^p$ in terms of the energy $E(v)$.

Then we will get $|\nabla v|^2|v| \in L_t^1 L^2$, with $v \in L_t^{\frac{2}{1-6/p}} L^p$, provided one may estimate both factors $\nabla v \in L_t^r L^q$, where (r,q) is close to $(4,4)$ and such that

$$1 = \frac{1}{2}\Big(1 - \frac{6}{p}\Big) + \frac{2}{r} \quad \text{and} \quad \frac{1}{2} = \frac{1}{p} + \frac{2}{q} \quad \Big(\Rightarrow \frac{1}{r} + \frac{3}{q} = 1\Big).$$

We now prove such an estimate on ∇v: start over with (74) but shift regularity on v from H^1 to $H^{3/2}$:

$$2^{-j/5+j/2}\|\Delta_j v\|_{L_T^2 L^{10}} \lesssim 2^{j/2}(\|\Delta_j v(0)\|_{L^2} + \|\Delta_j v(T)\|_{L^2}) + 2^{-j/2}\|\nabla \Delta_j v\|_{L_T^2 H^{1/2}}$$
$$+ 2^{j/5+j/2}\|\Delta_j(|v|^2 v)\|_{L_T^2 L^{10/9}},$$

multiply by $2^{j/2}$ and estimate the nonlinear term placing one factor in $H^{3/2}$ and others in $H^1 \hookrightarrow L^5$,

$$2^{-j/5}\|\nabla\Delta_j v\|_{L^2_T L^{10}} \lesssim \|\nabla\Delta_j v(0)\|_{L^2} + \|\nabla\Delta_j v(T)\|_{L^2} + \|\nabla\Delta_j v\|_{L^2_T H^{1/2}} \\ + 2^{-3j/10}\sqrt{T}\sup_t(\|v\|_{H^{3/2}}\|v\|^2_{H^1}),$$

from which we easily obtain

$$\sup_j(2^{-j/5}\|\nabla\Delta_j v\|_{L^2_T L^{10}}) \lesssim E(v)^{1/2} + \sqrt{T}(1+E(v))\sup_t\|v\|_{H^{3/2}}.$$

Then pick Q $(> q)$ such that $1/q = (2/r)(1/Q) + (1-2/r)(1/2)$ and use Sobolev embedding to go from L^{10} to L^Q:

$$\sup_j(2^{-j/5-3j(1/10-1/Q)}\|\nabla\Delta_j v\|_{L^2_T L^Q}) \lesssim E(v)^{1/2} + \sqrt{T}(1+E(v))\sup_t\|v\|_{H^{3/2}}.$$

On the other hand, from $\nabla v \in L^\infty_T H^{1/2}$ we have

$$\sup_j(2^{j/2}\|\nabla\Delta_j v\|_{L^\infty_T L^2}) \lesssim \sup_{0\le t\le T}\|v\|_{H^{3/2}}.$$

Choosing $\theta = 2/r > 1/2$, the regularities cancel as scaling dictates: from $3/q + 1/r = 1$ we check that indeed

$$\Big(\frac{1}{2} - \frac{3}{Q}\Big)\frac{2}{r} = \frac{1}{2}\Big(1 - \frac{2}{r}\Big).$$

We therefore get the desired $L^r L^q$ estimate for ∇v:

$$\|\nabla v\|_{L^r_T L^q} \lesssim \big(E(v)^{1/2} + \sqrt{T}(1+E(v))\sup_{0\le t\le T}\|v\|_{H^{3/2}}\big)^\theta\big(\sup_{0\le t\le T}\|v\|^{1-\theta}_{H^{3/2}}\big).$$

Then there exists $C(E)$ (which may change from line to line) such that

$$\|\nabla v\|_{L^r_T L^q} \lesssim C(E(v))\big(\sup_{0\le t\le T}\|v\|^{1-\theta}_{H^{3/2}} + T^{1/4}\sup_{0\le t\le T}\|v\|_{H^{3/2}}\big).$$

Gathering all our estimates, we get that, for $0 < T < 1$,

$$\begin{aligned}\|v(\cdot,T)\|^2_{H^2} - \|v(\cdot,0)\|^2_{H^2} &\le \int_0^T\int_M |\nabla^2 v|\,|\partial v|^2|v| \\ &\lesssim \|v\|_{L^\infty_T H^2}\|v|\nabla v|^2\|_{L^1_T L^2} \\ &\lesssim \|v\|_{L^\infty_T H^2}\|\nabla v\|^2_{L^r_T L^q}\|v\|_{L_T^{2/(1-6/p)}L^p} \\ &\lesssim C(E)\big(\sup_{0\le t\le T}\|v\|^{2-\theta}_{H^2} + T^{1/2}\sup_{0\le t\le T}\|v\|^2_{H^2}\big),\end{aligned}$$

from which exponential growth follows as in [21]. This concludes the proof of Theorem 4.

Acknowledgments. The author would like to thank Daniel Tataru for helpful and stimulating discussions on this problem, Nikolay Tzvetkov for suggesting that the result of [21] may be relevant for our nonlinear application and Fabrice Planchon for constructive discussions concerning the applications. The author thanks all referees for their careful reading and constructive remarks and suggestions.

Funding. The author was supported by ERC grant ANADEL 757 996.

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Received 16 August 2021; revised 6 August 2022; accepted 10 August 2022.

Oana Ivanovici
Sorbonne Université, CNRS, Laboratoire Jacques-Louis Lions, LJLL, 75005 Paris, France;
oana.ivanovici@math.cnrs.fr