Modified energies for the periodic generalized KdV equation and applications

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Abstract. We construct modified energies for the generalized KdV equation. As a consequence, we obtain quasi-invariance of the high-order Gaussian measures, along with L^q regularity on the corresponding Radon–Nikodym density, as well as new bounds on the growth of the Sobolev norms of the solutions.

Dedicated to the memory of Professor Jean Ginibre

1. Introduction

1.1. Integrable KdV models and their conservation laws

The Korteweg–de Vries (KdV) and modified Korteweg–de Vries (mKdV) equations are canonical integrable partial differential equations which read

$$\partial_t u - \partial_x^3 u + \partial_x(u^{p+1}) = 0, \quad p = 1, 2, \tag{1.1}$$

with $p = 1$ for the KdV equation and $p = 2$ for mKdV. The well-known Miura transform may be used to connect solutions of KdV and mKdV. Both equations (1.1) have a Lax pair formulation and, as a consequence, possess infinitely many conservation laws (see e.g. [20, 42, 51] and references therein). One important consequence of these conservation laws is a priori global in time bound for the Sobolev norm H^k of the solutions of (1.1): for every $k \in \mathbb{N}$, there exists a first integral that is written

$$\int (\partial_x^k u)^2 \, dx + \text{lower-order terms},$$

where integration holds on the line $\mathbb{R}$ or on a torus $\mathbb{R}/(2\pi\mathbb{Z})$ depending on which background (1.1) is considered. Following [51], another interesting consequence of the aforementioned conservation laws is the existence of invariant measures along the flow associated with (1.1); such measures are also absolutely continuous with respect to Gaussian measures (see Section 1.4 for their definition).

1.2. Generalized KdV models and their almost conservation laws

The Lax pair formulation is no longer known to hold if the power p is replaced by $p \neq 1, 2$ in (1.1). Therefore a natural question is whether conservation laws for KdV and mKdV may still hold in the case of a general power p. We introduce in this paper a family of energies which are not exact conservation laws, however will be useful to get new results for the long time behavior of the solutions to (1.2), as we shall specify in the sequel.

Consequently, we consider the Cauchy problem, which we call the generalized KdV (gKdV) equation,

$$\begin{cases} \partial_t u - \partial_x^3 u + \partial_x(u^{p+1}) = 0, & p \geq 3, \ p \in 2\mathbb{N}, \\ u(0, x) = \varphi(x) \in H^k(\mathbb{T}), \end{cases} \tag{1.2}$$

where $\mathbb{T}$ denotes the one-dimensional torus $\mathbb{R}/(2\pi\mathbb{Z})$. Even if (1.2) does not have a Lax pair formulation it still has a Hamiltonian formulation, with conserved Hamiltonian $\mathcal{H}(u)$:

$$\mathcal{H}(u) = \frac{1}{2} \int_{\mathbb{T}} (\partial_x u)^2 \, dx + \frac{1}{p+2} \int_{\mathbb{T}} u^{p+2} \, dx.$$

That $\mathcal{H}$ is preserved along the flow of (1.2) and some of its truncations will play a key role in our analysis.

Equation (1.2) is the so-called defocusing model. The focusing model can be obtained by a change of sign in front of the nonlinear term in (1.2) (which also changes the sign in front of the second term defining $\mathcal{H}$). Most of the results that we will obtain below can be extended to either odd p or to the focusing gKdV for even p, provided that a uniform bound on the H^1 norm of the solution is assumed. These extensions do not require any new significant argument and therefore we restrict ourselves to (1.2).

The Cauchy problem (1.1) has a long and interesting history. The first results dealt mainly with (1.2) on the spatial domain $\mathbb{R}$ rather than the torus $\mathbb{T}$. A classical reference is the work by Saut [41]. In the fundamental paper [21], Kato initiated the use of the dispersive properties in the analysis of (1.2). In [22–24] Kenig–Ponce–Vega implemented tools from harmonic analysis allowing a deeper view of the dispersive effect. However, these tools were not efficient in the periodic case. The dispersive effect in the periodic case was understood by Bourgain in [1]. The methods developed by Bourgain had far-reaching consequences in the field of dispersive PDEs. We will benefit from Bourgain's seminal work in this paper too. In the case $p = 4$ and focusing nonlinearity, when (1.2) is posed on the real line, there are remarkable results by Martel–Merle–Raphaël on solutions developing singularities in finite time; see [27–29] and the references therein. It is not clear how to extend the results of [27–29] to the periodic case because on $\mathbb{R}$ the blow-up point may escape to infinity, and therefore its localization seems a considerable challenge. For recent progress in this direction, we refer to [30].

Throughout the paper we shall also need to analyze Fourier truncations of (1.2). For a given even integer $p \geq 3$ and $M \in \mathbb{N}$, we consider the following truncated version of the

gKdV equation (1.2):

$$\begin{cases} \partial_t u - \partial_x^3 u + \pi_M \partial_x ((\pi_M u)^{p+1}) = 0, \\ u(0, x) = \varphi(x), \end{cases} \tag{1.3}$$

where for $M \in \mathbb{N}$ we denote by π_M the Dirichlet projector defined by $\pi_M (\sum_{n \in \mathbb{Z}} c_n e^{inx})$ $= \sum_{|n| \leq M} c_n e^{inx}$. Thanks to the aforementioned references on the Cauchy problem for (1.2), solving (1.3) will not be an issue here because we know from [1, 3, 7, 44] that (1.3) is globally well posed in $H^s(\mathbb{T})$, $s \geq 1$ (and even locally well posed for $s \geq \frac{1}{2}$), where $H^s(\mathbb{T})$ is defined through the Fourier transform as $((1 + |n|^s)c_n)_n \in l^2(\mathbb{Z})$. In fact, for $M < \infty$ an easier argument can be applied by using that for frequencies $\leq M$ equation (1.3) becomes an ODE (with a Lipschitz vector field) for which global well-posedness holds thanks to the L^2 conservation law, while for frequencies $> M$ equation (1.3) becomes a linear equation. However, if one wishes to have $H^s(\mathbb{T})$ bounds uniformly with respect to M, the analysis of [1,3,7,44] cannot be avoided. Let us denote by $\Phi_M(t)$ the flow of (1.3) and $\Phi(t)$ the flow of (1.2). We shall not specify the dependence on p in this notation.

Our aims are twofold: first, we provide new a priori polynomial bounds on the growth of high-order Sobolev norm for solutions to (1.2) (see Theorem 1.1); second, we gain knowledge about the transport of Gaussian measures along the flow associated with (1.2) (see Theorem 1.4).

Let us introduce some notation that will be used in the sequel: for every $s \in \mathbb{R}$ and for every $p \in [1, \infty]$ denote $H^s = H^s(\mathbb{T})$ and $L^p = L^p(\mathbb{T})$; H^{s^-} will denote any Sobolev space $H^{s-\varepsilon}$ with $\varepsilon > 0$ small enough. Similarly, given a real number α, denote by α^+ any real number larger than α. We shall denote $\mathbb{N} = \{0, 1, 2, \ldots\}$, and for every $k \in \mathbb{N} \setminus \{0\}$ we shall use the following specific Sobolev norm: $\|\varphi\|_{H^k}^2 = \|\varphi\|_{L^2}^2 + \|\partial_x^k \varphi\|_{L^2}^2$. We recall that $\Phi(t)$ and $\Phi_M(t)$ denote respectively the nonlinear flows associated with (1.2) and (1.3). We shall also sometimes use the notation $\Phi(t) = \Phi_\infty(t)$. The Dirichlet projector on frequencies smaller than or equal to M is denoted by π_M and we also sometimes use the notation π_∞ to denote the identity operator.

We can now state our main results.

1.3. Growth of Sobolev norms

The first main result of this paper is the following one.

Theorem 1.1. *Let $k > 1$ be an integer. Then for every $\varphi \in H^k$ and for every $\varepsilon > 0$ there exists a constant $C > 0$ such that*

$$\|\Phi(t)\varphi\|_{H^k} \leq \|\varphi\|_{H^k} + C t^{\frac{k-1}{2}+\varepsilon} \quad \forall t > 0. \tag{1.4}$$

Although Theorem 1.1 is stated for $t > 0$, it can be extended to every $t \in \mathbb{R}$ by the reversibility of the flow associated with (1.2). Moreover, the constant C may be chosen

uniformly with respect to bounded sets in H^k, and, even better, in H^1, once ε has been chosen (through the use of Proposition 2.1).

The line of research leading to results such as the one in Theorem 1.1 was initiated in [5]. We improve results obtained in [44] (where the growth had exponent $2k$) and [31] (where the growth was lowered to $k - 1 + \varepsilon$.)

It would be very interesting to construct solutions of the defocusing gKdV such that the H^k norms do not remain bounded in time for $k > 1$. Unfortunately, such results are rare in the context of canonical dispersive models (with the notable exception of [19]). Our approach is restricted so far to one-dimensional models and it would be very interesting to extend it to higher dimensions; however, we believe it does apply to the one-dimensional nonlinear Schrödinger (NLS) equation, modulo some additional difficulties related to complex-valued functions. This will be addressed elsewhere.

The proof of Theorem 1.1 stems from constructing suitable modified energies at the level of H^k norms: while we are unable to obtain invariant energies, we quantify how far they are from being exact conservation laws. The closer we are to conservation laws, the better are polynomial bounds for corresponding nonlinear solutions.

Theorem 1.2. *Let $k > 1$ be an integer. Then for every $T > 0$ there exist functionals*

$$\mathcal{E}_k \colon H^k \to \mathbb{R}, \quad \mathcal{G}_k^T \colon H^k \to \mathbb{R}$$

such that

- *for all $\varphi \in H^k$ and all $T > 0$ we have*

$$\mathcal{E}_k(\Phi(T)\varphi) - \mathcal{E}_k(\varphi) = \mathcal{G}_k^T(\varphi); \tag{1.5}$$

- *the energy $\mathcal{E}_k$ has the structure*

$$\mathcal{E}_k(u) = \|\partial_x^k u\|_{L^2}^2 + \mathcal{R}_k(u), \tag{1.6}$$

with $\mathcal{R}_k(u)$ satisfying for all $u \in H^k$,

$$|\mathcal{R}_k(u)| \leq C + C \|u\|_{H^k}^{\frac{2k-4}{k-1}} \|u\|_{H^1}^{p+\frac{2}{k-1}}; \tag{1.7}$$

- *for every fixed $R > 0$ and $T > 0$, there exists a constant $C > 0$ such that, for all $\varphi \in H^k$ such that $\|\varphi\|_{H^1} < R$, we have*

$$|\mathcal{G}_k^T(\varphi)| \leq C + C \|\varphi\|_{H^k}^{(\frac{2k-4}{k-1})^+}. \tag{1.8}$$

We emphasize that Theorem 1.2 immediately holds, in a stronger form, for $k = 0, 1$, with energies $\mathcal{E}_0(u) = \|u\|_{L^2}^2$, $\mathcal{E}_1(u) = \mathcal{H}(u)$; these energies are indeed conserved along the flow associated with (1.2). However, for $k > 1$, such exact conservation laws are not available and hence the r.h.s. in (1.5) is nonzero.

The proof of Theorem 1.2 relies on a key improvement in some ideas developed in [38] in the context of NLS, together with bounds resulting from dispersive estimates such

as Bourgain's L^6 Strichartz inequality for (linear) KdV. Let us now briefly sketch the main point: adapting the argument in [38] would allow us to get suitable energies $\widetilde{\mathcal{E}}_k$ such that $\widetilde{\mathcal{E}}_k(\Phi(T)\varphi) - \widetilde{\mathcal{E}}_k(\varphi) = \widetilde{\mathcal{G}}_k^T(\varphi)$, where $\widetilde{\mathcal{G}}_k^T$ would satisfy $|\widetilde{\mathcal{G}}_k^T(\varphi)| \leq C + C\|\varphi\|_{H^k}^{\alpha(k)}$, but with $\alpha(k) > (\frac{2k-4}{k-1})^+$. Here, we get a significantly smaller power of the H^k norm of the initial datum in (1.8) for $\mathcal{G}_k^T$. This improvement on $\alpha(k)$, which in turn is crucial in order to obtain the growth of the Sobolev norms in Theorem 1.1, follows from a refinement of the construction of the energies $\mathcal{E}_k$ compared with $\widetilde{\mathcal{E}}_k$. In particular, once we compute the variation of the energies introduced in this paper along solutions, we get that $\mathcal{G}_k^T(\varphi)$ can be expressed as the space-time integral of multilinear expressions of densities in which the worse single terms (namely those that carry the maximal number of derivatives) have at least five factors involving at least one derivative. This key property of distributing derivatives on several factors was out of reach with our previous constructions of modified energies. For details we refer to Section 3. Then the dispersive effect, through the L^6 Strichartz bound, allows us to transform the aforementioned distribution of derivatives in terms of powers of Sobolev norms of the initial datum, as discussed above.

We should point out that, in the context of gKdV, modified energies similar to those we get in Theorem 1.2 have already appeared, at the level of $k = 2$, in [26], where they are used in connection with N-soliton asymptotics.

For details on how Theorem 1.2 implies Theorem 1.1 we refer to [38]; however for the sake of completeness we briefly outline the argument. By local Cauchy theory, for any given $\varphi \in H^k$ with $k > 1$, there exists $T = T(\|\varphi\|_{H^1})$ and a constant $C = C(\|\varphi\|_{H^1})$ such that $\sup_{t \in (0,T)} \|\Phi(t)\varphi\|_{H^k} \leq C\|\varphi\|_{H^k}$. From $\sup_t \|\Phi(t)\varphi\|_{H^1} < \infty$ (recall we are considering the defocusing equation), we may use the estimate above uniformly, selecting as initial condition $\Phi(s)\varphi$ for arbitrary s and hence

$$\sup_{t \in (s,s+T)} \|\Phi(t)\varphi\|_{H^k} \leq C\|\Phi(s)\varphi\|_{H^k} \quad \forall s > 0. \tag{1.9}$$

Combining (1.5) with (1.6) and recalling the conservation of the L^2 norm, we get, for our chosen T,

$$\|\Phi(t+T)\varphi\|_{H^k}^2 - \|\Phi(t)\varphi\|_{H^k}^2 = \mathcal{G}_k^T(\Phi(t)\varphi) - \mathcal{R}_k(\Phi(t+T)\varphi) + \mathcal{R}_k(\Phi(t)\varphi).$$

In turn, (1.8) and (1.7) (recalling that the H^1 norm on the r.h.s. is uniformly bounded) together imply

$$\|\Phi(t+T)\varphi\|_{H^k}^2 - \|\Phi(t)\varphi\|_{H^k}^2 \leq C + C\|\Phi(t+T)\varphi\|_{H^k}^{(\frac{2k-4}{k-1})^+} + C\|\Phi(t)\varphi\|_{H^k}^{(\frac{2k-4}{k-1})^+}$$
$$\leq C + C\|\Phi(t)\varphi\|_{H^k}^{(\frac{2k-4}{k-1})^+}$$

for all $t > 0$, where we used (1.9) in the last step. Choosing $t = nT$ and defining $\alpha_n = \|\Phi_{nT}(\varphi)\|_{H^k}^2$ we get a discrete Gronwall-type inequality

$$\alpha_{n+1} - \alpha_n \leq C + C\alpha_n^{(\frac{k-2}{k-1})^+};$$

this yields $\alpha_n \le Cn^{(k-1)^+}$ which in turn implies (1.4) for $t = nT$. Using (1.9), this extends to all t.

1.4. Quasi-invariant Gaussian measures

Next we focus on the quasi-invariance of certain Gaussian measures under $\Phi(t)$. For $k \ge 1$, we denote by μ_k the measure induced by the map

$$\omega \mapsto \sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{(1 + n^2)^{k/2}} e^{inx},$$

where $g_n = \overline{g_{-n}}$, $g_0 = 0$ and $(g_n)_{n>0}$ is a sequence of independent, identically distributed complex Gaussian random variables. We can see μ_k as a probability measure on $H^{(k-\frac{1}{2})^-}$. We aim to understand how μ_k is transported under the flow of (1.2). This is a delicate task because infinite-dimensional measures become easily mutually singular. As $\mathcal{H}$ is preserved along the flow of (1.2), studying the transport of $\chi_R(\mathcal{H}(u)) \, d\mu_k(u)$ is more natural, where $R > 0$ is a fixed energy level and $\chi_R \colon \mathbb{R} \to \mathbb{R}$ is a continuous function vanishing outside $[-R, R]$. Such cutoffs were first introduced in the field of dispersive PDEs in [25].

For $M < \infty$, verifying that the transport of the measure $\chi_R(\mathcal{H}(\pi_M u)) \, d\mu_k(u)$ by $\Phi_M(T)$ is absolutely continuous with respect to its initial value will be relatively easy, with a Radon–Nikodym derivative given by

$$g_{T,M}(u) = e^{\|\pi_M u\|^2_{H^k} - \|\pi_M(\Phi_M(T)u)\|^2_{H^k}}. \tag{1.10}$$

The main difficulty is passing to the limit $M \to \infty$ in $f_{T,M}$. The issue is that each term in the exponential in (1.10) strongly diverges in the limit $M \to \infty$ and therefore subtle cancellations should be exploited. A suitable adaptation of the modified energies introduced in Theorem 1.2 can be useful to overcome this difficulty; more precisely we shall need the following result.

Theorem 1.3. *Let $k > 1$ be an integer. Then for every $T > 0$, $M \in \mathbb{N} \cup \{\infty\}$ there exist functionals*

$$\overline{\mathcal{E}}_k \colon H^k \to \mathbb{R}, \quad \overline{\mathcal{G}}^T_{k,M} \colon H^{(k-\frac{1}{2})^-} \to \mathbb{R}$$

such that

- *for every $M \in \mathbb{N} \cup \{\infty\}$, $T > 0$ and $\varphi \in H^k$ we have*

$$\overline{\mathcal{E}}_k(\pi_M(\Phi_M(T)\varphi)) - \overline{\mathcal{E}}_k(\pi_M \varphi) = \overline{\mathcal{G}}^T_{k,M}(\pi_M \varphi); \tag{1.11}$$

- *the energy $\overline{\mathcal{E}}_k$ has the structure*

$$\overline{\mathcal{E}}_k(u) = \|\partial^k_x u\|^2_{L^2} + \overline{\mathcal{R}}_k(u),$$

where $\overline{\mathcal{R}}_k \colon H^{(k-\frac{1}{2})^-} \to \mathbb{R}$ satisfies for all $u \in H^{(k-\frac{1}{2})^-}$ the bound

$$|\overline{\mathcal{R}}_k(u)| \le C + C \|u\|^{(\frac{4k-8}{2k-3})^+}_{H^{(k-\frac{1}{2})^-}} \|u\|^{p + \frac{2}{2k-3}}_{H^1}; \tag{1.12}$$

- *for every fixed $R > 0$ and $T > 0$, there exists a constant $C > 0$, uniform w.r.t. to $M \in \mathbb{N} \cup \{\infty\}$, such that for all $\varphi \in H^{(k-\frac{1}{2})^-}$ such that $\|\varphi\|_{H^1} < R$ we have*

$$|\bar{\mathscr{G}}^T_{k,M}(\pi_M \varphi))| \leq C + C \|\varphi\|_{H^{(k-\frac{1}{2})^-}}^{(\frac{4k-8}{2k-3})^+}; \tag{1.13}$$

- *for every $\varphi \in H^{(k-\frac{1}{2})^-}$ and for every $T > 0$ we have the convergence*

$$\bar{\mathscr{G}}^T_{k,M}(\pi_M \varphi) \xrightarrow{M \to \infty} \bar{\mathscr{G}}^T_{k,\infty}(\varphi). \tag{1.14}$$

Let us compare Theorems 1.2 and 1.3: we point out that both energies $\mathscr{E}_k$ and $\bar{\mathscr{E}}_k$ coincide; however, we elected to use different notation, as the corresponding lower-order part of $\mathscr{E}_k$, namely $\mathscr{R}_k$, introduced in Theorem 1.2, is defined on H^k, while $\bar{\mathscr{R}}_k$ introduced in Theorem 1.3 is defined on the larger space $H^{(k-\frac{1}{2})^-}$. It will be clear in the proof that $\mathscr{R}_k(u) = \bar{\mathscr{R}}_k(u)$ for $u \in H^k$ but $\bar{\mathscr{R}}_k$ to be defined on $H^{(k-\frac{1}{2})^-}$ is crucial for our purpose: when dealing with the analysis of transported Gaussian measures μ_k, working at regularity H^k is not sufficient and one needs to go below at lower regularity $H^{(k-\frac{1}{2})^-}$, as $\mu_k(H^{(k-\frac{1}{2})^-}) = 1$ and $\mu_k(H^k) = 0$. The same comment applies to functionals $\mathscr{G}^T_k$ and $\bar{\mathscr{G}}^T_{k,\infty}$ appearing in Theorems 1.2 and 1.3. While they coincide on H^k, we consider the second one as defined on the larger space $H^{(k-\frac{1}{2})^-}$. Estimates (1.12) and (1.13) will be crucial in the sequel and should be compared with (1.7) and (1.8). Notice how the power of the H^1 norm differs in (1.7) and (1.12), while in (1.13) we lose more derivatives but we gain a smaller power compared to (1.8): in the r.h.s. of (1.12) and (1.13) we have the $H^{(k-\frac{1}{2})^-}$ norm of the initial datum to a power that is less than 2, and this is crucial for using some standard Gaussian analysis. This is the key to proving, beyond quasi-invariance, L^q regularity for the density of the transported Gaussian measure (see Theorem 1.4).

Let us also comment on the introduction, in Theorem 1.3, unlike in Theorem 1.2, of a family of functionals $\bar{\mathscr{G}}^T_{k,M}$ for $M \in \mathbb{N}$. In order to study properties of the transported Gaussian measures along the infinite-dimensional flow associated with (1.2), we first need to analyze the variation of the energy $\bar{\mathscr{E}}_k$ along the flow $\Phi_M(t)$ associated with (1.3) (see (1.11)). The functionals $\bar{\mathscr{G}}^T_{k,M}$ turn out to be strongly related to $\bar{\mathscr{G}}^T_{k,\infty}$ (and hence to the functional $\mathscr{G}^T_k$ in Theorem 1.2). Actually they do look alike except for the Dirichlet projector π_M that appears in $\bar{\mathscr{G}}^T_{k,M}$ for $M \in \mathbb{N}$. However, the nice properties enjoyed by projectors π_M allow us to estimate $\bar{\mathscr{G}}^T_{k,M}$ uniformly w.r.t. M.

As Theorem 1.2, Theorem 1.3 relies similarly on a key improvement in [39]. The energies that we introduce in Theorem 1.3 allow us to improve on the power that we get on the r.h.s. in (1.12) and (1.13) when compared to what we would get by simply adapting the construction used in [39] in the NLS context.

We can now give the precise statement of our quasi-invariance result for gKdV.

Theorem 1.4. *Let $k > 1$ be an integer. The Gaussian measure μ_k is quasi-invariant by the flow $\Phi(T)$ for every $T > 0$. Moreover, for every fixed $R > 0$ and $T > 0$ we have, for*

A a Borel subset of $H^{(k-\frac{1}{2})^-}$,

$$\int_{\Phi(T)A} \chi_R(\mathcal{H}(u)) \, d\mu_k(u) = \int_A g_T(u) \chi_R(\mathcal{H}(u)) \, d\mu_k(u),$$

where $g_T(u)\chi_R(\mathcal{H}(u)) \in L^q(\mu_k)$ with $q \in [1, \infty)$ for $k > 2$ and $q = \infty$ for $k = 2$. In addition, we have

$$g_T(u) = \exp\big(-\bar{\mathcal{R}}_k(u) + \bar{\mathcal{R}}_k(\Phi(T)u) - \bar{\mathcal{G}}_{k,\infty}^T(u)\big),$$

where $\bar{\mathcal{R}}_k$ and $\bar{\mathcal{G}}_{k,\infty}^T$, as introduced in Theorem 1.3, are well-defined quantities on the support of μ_k. More precisely, with $g_{T,M}$ defined in (1.10), we have

$$\lim_{M \to \infty} \|g_{T,M}(u)\chi_R(\mathcal{H}(\pi_M u)) - g_T(u)\chi_R(\mathcal{H}(u))\|_{L^q(d\mu_k(u))} = 0.$$

We point out that with such quasi-invariance we also get L^q regularity for the Radon–Nikodym density. Exactly as for Theorem 1.1, the result in Theorem 1.4 can be extended to the case $T < 0$ by using the reversibility of the flow associated with (1.2). However, for simplicity we shall focus on the case $T > 0$.

Theorem 1.4 fits in the line of research aiming to describe macroscopical (statistical dynamics) properties of Hamiltonian PDEs. In particular, it implies a stability property of the corresponding infinite-dimensional Liouville equation (see [45, Corollary 1.3]). The earliest reference we are aware of is [13], followed by [2, 4, 25, 51]. Inspired by the work on invariant measures for the Benjamin–Ono equation [10, 48–50], quasi-invariance of Gaussian measure for several dispersive models was obtained in recent years; see [9, 11, 12, 15, 16, 18, 33–37, 39, 43, 47]. The method to identify the densities in Theorem 1.4 is inspired by recent works [9, 14]. In Theorem 1.4, we provide much more information on the densities when compared to [39], which used modified energies on the nonlinear Schrödinger equation. It should be underlined that a key novelty in the proof of Theorem 1.4 with respect to [14] and [39] is that we crucially use dispersive estimates in the analysis.

We hope that the approach developed here may allow us to identify the Radon–Nikodym derivatives in the quasi-invariance obtained in [39], up to additional difficulties related to complex-valued functions in the context of NLS.

Let us finally comment on the case $k = 1$. In this case the Gibbs measure, which is absolutely continuous with respect to μ_1, is an invariant measure. This is precisely the result obtained in [40] in the case $p = 3$ and [6] in the general case $p \geq 3$ (see also [32] for weak solutions in the case $p > 3$).

1.5. An informal conclusion

The results of this paper and previous works of the second and third authors [10, 48–50] can be summarized as follows. In the case of integrable models, exact conservation laws for all Sobolev regularities imply existence of invariant measures; the modified energies

we construct in the context of nonintegrable models imply the existence of quasi-invariant measures. Concerning the deterministic behavior of the solutions, exact conservation laws imply uniform bounds on Sobolev norms of solutions, while the modified energies we construct imply polynomial bounds on Sobolev norms of solutions.

1.6. Organization of the paper

In Section 2 we get dispersive estimates that play a crucial role in the analysis. For completeness, the proof of the key nonlinear estimate is presented in the appendix. In Section 3 we prove Theorem 1.2 and in Section 4 we prove Theorem 1.3. Section 5 is devoted to the proof of Theorem 1.4.

2. Dispersive estimates

The aim of this section is to collect useful results on the flows associated with (1.2) and (1.3). First, global existence and uniqueness of solutions for the truncated flows follow by a straightforward ODE argument, along with conservation of L^2 mass. From now on we assume without further comment existence and uniqueness of global flows $\Phi_M(t)$ for $M \in \mathbb{N}$. The Cauchy problem associated with (1.2) is much more involved. In particular, we quote [1, 3, 7, 44] whose analyses imply that for every $s \geq 1$ there exists a unique global solution associated with the initial datum $\varphi \in H^s$; moreover, we have continuous dependence on the initial datum. The analysis in [7] allows local Cauchy theory to be treated down to low regularity $H^{\frac{1}{2}}$.

It will later be important to have a series of uniform bounds with respect to M (in particular, suitable L^6 bounds), as well as some delicate convergences in suitable topologies of the finite-dimensional flows to the infinite-dimensional one. To the best of our knowledge, these properties do not follow in a straightforward way from the aforementioned works and their proofs require some further arguments. Indeed, in our analysis we shall borrow many ideas from the references above (in particular [7]), which in conjunction with new ingredients will imply several properties for the flows $\Phi_M(t)$ with $M \in \mathbb{N} \cup \{\infty\}$.

In order to provide a precise statement we first define our resolution space. First recall $X^{s,b}$ spaces, introduced in the fundamental work [1]. For real numbers s, b and a function u on $\mathbb{R} \times \mathbb{T}$, we define the $X^{s,b}$ norm associated with the KdV dispersive relation by

$$\|u\|_{X^{s,b}} = \|\langle n \rangle^s \langle \tau + n^3 \rangle^b \hat{u}(\tau, n)\|_{L^2_{n,\tau}},$$

where $\hat{u}(\tau, n)$, $\tau \in \mathbb{R}$, $n \in \mathbb{Z}$ is the space-time Fourier transform of u. For $T > 0$, we denote by $X^{s,b}_T$ the restriction space of a function on $(-T, T) \times \mathbb{T}$ equipped with the norm

$$\|u\|_{X^{s,b}_T} = \inf_{\substack{\tilde{u} \in X^{s,b} \\ \tilde{u}|_{(-T,T)} = u}} \|\tilde{u}\|_{X^{s,b}}.$$

In our analysis one needs to take $b = 1/2$ and we are led to work in the space Y^s equipped with the norm

$$\|u\|_{Y^s} = \|u\|_{X^{s,\frac{1}{2}}} + \|\langle n \rangle^s \hat{u}(\tau, n)\|_{l^2_n L^1_\tau}.$$

One can introduce as above the restriction spaces Y^s_T. We recall the embedding $Y^s_T \subset \mathcal{C}([0, T]; H^s)$.

Proposition 2.1. *Let $s \geq 1$ and $T > 0$; then*

$$\forall \varphi \in H^s, \quad \exists! \, \Phi(t)\varphi \in Y^s_T, \text{ a solution to } (1.2) \tag{2.1}$$

and we have convergence of the truncated flow, for all compact $\mathcal{K} \subset H^s$,

$$\sup_{\varphi \in \mathcal{K}} \|\pi_M(\Phi_M(t)\varphi) - \Phi(t)\varphi\|_{L^\infty([0,T];H^s)} \xrightarrow{M \to \infty} 0. \tag{2.2}$$

For every $\varepsilon > 0$ and $R > 0$ there exists $C > 0$ independent of $M \in \mathbb{N} \cup \{\infty\}$ such that

$$\forall \varphi \in H^s \text{ s.t. } \|\varphi\|_{H^1} < R, \quad \|\pi_M \Phi_M(t)\varphi\|_{H^s} \leq C \|\varphi\|_{H^s}, \tag{2.3}$$

$$\forall \varphi \in H^{s+\varepsilon} \text{ s.t. } \|\varphi\|_{H^1} < R, \quad \|\pi_M(\Phi_M(t)\varphi)\|_{L^6((0,T);W^{s,6})} \leq C \|\varphi\|_{H^{s+\varepsilon}}, \tag{2.4}$$

$$\forall \varphi \in H^{s+\varepsilon}, \quad \|\pi_M(\Phi_M(t)\varphi) - \Phi(t)\varphi\|_{L^6((0,T);W^{s,6})} \xrightarrow{M \to \infty} 0. \tag{2.5}$$

We remark that (2.2) (with $\mathcal{K} = \{\varphi\}$) and (2.4) imply (2.5): using the interpolation inequality, for $\delta > 0$,

$$\|u\|_{W^{s,6}} \leq C \|u\|_{W^{s+\delta,6}}^{\frac{s}{s+\delta}} \|u\|_{L^6}^{\frac{\delta}{s+\delta}},$$

we get, by time integration, the Hölder inequality in time and the Sobolev embedding $H^s \subset L^6$,

$$\|u\|_{L^6(0,T);W^{s,6})} \leq C \|u\|_{L^6((0,T);W^{s+\delta,6})}^{\frac{s}{s+\delta}} \|u\|_{L^\infty(0,T);H^s)}^{\frac{\delta}{s+\delta}}. \tag{2.6}$$

Next one can choose $u = \pi_M(\Phi_M(t)\varphi) - \Phi(t)\varphi$ and we get (2.5) since the second term on the r.h.s. in (2.6) converges to zero by (2.2) and the first term on the r.h.s. is bounded provided we choose $\delta = \frac{\varepsilon}{2}$ and apply (2.4), replacing s by $s + \frac{\varepsilon}{2}$ (there is room to play with $\varepsilon > 0$).

Hence we are reduced to proving (2.1), (2.2), (2.3) and (2.4). The main idea is to perform a gauge transform on gKdV, work on the gauged equation and then transfer results back to the original flow. To prove (2.2) it will be of importance to have a lemma about the continuity of time-dependent translations for time-dependent functions (see Lemma 2.1 below).

2.1. The gauged gKdV equation

We now present the gauge transform following [7]. Set $u_M(t, x) = \pi_M(\Phi_M(t)\varphi)$ and introduce a change of unknown,

$$v_M(t, x) = u_M\left(t, x + (p + 1)\int_0^t \int_\mathbb{T} u_M^p \, dx \, dt\right),$$

taming the derivative loss in the nonlinearity. By invariance of the Lebesgue norm by translation,

$$u_M(t, x) = v_M\left(t, x - (p + 1)\int_0^t \int_{\mathbb{T}} v_M^p \, dx \, dt\right). \tag{2.7}$$

Therefore, v_M is a solution to

$$\begin{cases} \partial_t v_M - \partial_x^3 v_M + \pi_M \partial_x(v_M^{p+1}) - (p + 1)\left(\int v_M^p \, dx\right)\partial_x v_M = 0, \\ v_M(0, x) = \pi_M \varphi(x). \end{cases}$$

Let Π be the orthogonal projector on the nonzero frequencies defined by $\Pi f(x) = f(x) - \int_{\mathbb{T}} f \, dx$ for $x \in \mathbb{T}$, and observe that $\pi_M v_M = v_M$. Therefore, we can write

$$\pi_M \partial_x(v_M^{p+1}) - (p + 1)\left(\int_{\mathbb{T}} v_M^p \, dx\right)\partial_x v_M = (p + 1)\pi_M(v_M^p \partial_x v_M)$$

$$- (p + 1)\left(\int_{\mathbb{T}} v_M^p \, dx\right)\pi_M \partial_x v_M$$

$$= (p + 1)\pi_M(\partial_x v_M \Pi v_M^p).$$

Next, notice that since $\Pi(\partial_x v_M) = \partial_x v_M$, we can write

$$\int_{\mathbb{T}} \partial_x v_M \Pi v_M^p \, dx = \int_{\mathbb{T}} v_M^p \Pi \partial_x v_M \, dx = \int_{\mathbb{T}} v_M^p \partial_x v_M \, dx = \frac{1}{p + 1}\int_{\mathbb{T}} \partial_x(v_M^{p+1}) \, dx = 0;$$

therefore we have $(p + 1)\pi_M(\partial_x v_M \Pi v_M^p) = (p + 1)\pi_M \Pi(\partial_x v_M \Pi v_M^p)$ and the equation for v_M can be written as

$$\begin{cases} \partial_t v_M - \partial_x^3 v_M + (p + 1)\pi_M \Pi(\partial_x v_M \Pi v_M^p) = 0, \\ v_M(0, x) = \pi_M \varphi(x). \end{cases} \tag{2.8}$$

The projector Π in the nonlinear term of (2.8) is of fundamental importance because it allows cancellation of resonant nonlinear interactions in Bourgain spaces. Next we denote by $\Phi_M^{\mathscr{G}}(t)\varphi$ the flow associated with

$$\begin{cases} \partial_t v - \partial_x^3 v + (p + 1)\pi_M \Pi((\partial_x \pi_M v)\Pi(\pi_M v)^p) = 0, \\ v(0, x) = \varphi(x), \end{cases}$$

and by $\Phi_\infty^{\mathscr{G}}(t)\varphi$ the flow associated with

$$\begin{cases} \partial_t v - \partial_x^3 v + (p + 1)\Pi(\partial_x v \Pi v^p) = 0, \\ v(0, x) = \varphi(x). \end{cases} \tag{2.9}$$

We abuse notation, writing $\Phi_\infty^{\mathscr{G}}(t) = \Phi^{\mathscr{G}}(t)$. Notice that the solution to (2.8) is provided by $\pi_M(\Phi_M^{\mathscr{G}}(t)\varphi)$.

In order to prove Proposition 2.1, we work with the flow $\Phi_M^{\mathscr{G}}(t)$ and then go back to the original flow $\Phi_M(t)$.

Proposition 2.2. *Let $s \geq 1$ and $T > 0$; then*

$$\forall \varphi \in H^s, \quad \exists! \, \Phi^{\mathcal{G}}(t)\varphi \in Y_T^s, \text{ a solution to (2.9).} \tag{2.10}$$

For every $R > 0$, the map from $H^s \cap \{\varphi \in H^s \text{ s.t. } \|\varphi\|_{H^1} < R\}$ to Y_T^s,

$$\varphi \rightarrow \pi_M \Phi_M^{\mathcal{G}}(t)\varphi \text{ is uniformly Lipschitz w.r.t. } M \in \mathbb{N} \cup \{\infty\}, \tag{2.11}$$

and there exists $C > 0$, independent of $M \in \mathbb{N} \cup \{\infty\}$, such that

$$\forall \varphi \in H^s \text{ s.t. } \|\varphi\|_{H^1} < R, \quad \|\pi_M \Phi_M^{\mathcal{G}}(t)\varphi\|_{L^\infty([0,T];H^s)} \leq C \|\varphi\|_{H^s}. \tag{2.12}$$

We also have the following convergence, for all compact $\mathcal{K} \subset H^s$:

$$\sup_{\varphi \in K} \|\pi_M(\Phi_M^{\mathcal{G}}(t)\varphi) - \Phi^{\mathcal{G}}(t)\varphi\|_{L^\infty([0,T];H^s)} \xrightarrow{M \rightarrow \infty} 0. \tag{2.13}$$

Finally, for all $\varepsilon > 0$ there exists $C > 0$, independent of $M \in \mathbb{N} \cup \{\infty\}$, such that

$$\forall \varphi \in H^{s+\varepsilon} \text{ s.t. } \|\varphi\|_{H^1} < R, \quad \|\pi_M(\Phi_M^{\mathcal{G}}(t)\varphi)\|_{L^6((0,T);W^{s,6})} \leq C \|\varphi\|_{H^{s+\varepsilon}}. \tag{2.14}$$

Notice that in the statement of Proposition 2.2 we claim Lipschitz continuity of the flow. This property will be crucial in order to prove (2.13). Once we are done with this proposition, we can go back to the original flow $\Phi_M(t)$ by using the following result concerning a time-dependent version of the continuity of the translation operator.

Lemma 2.1. *Let $s \geq 0$ and $\mathcal{W} \subset \mathcal{C}([0, T]; H^s)$ be compact. Assume that for every $w(t, x) \in \mathcal{W}$ there exists a sequence $w_M(t, x) \in \mathcal{C}([0, T]; H^s)$ and functions $\tau_M^w, \tau^w \in \mathcal{C}([0, T]; \mathbb{R})$ such that*

$$\sup_{w \in \mathcal{K}} \|w_M(t, x) - w(t, x)\|_{L^\infty([0,T];H^s)} \xrightarrow{M \rightarrow \infty} 0 \tag{2.15}$$

and

$$\sup_{w \in \mathcal{K}} \|\tau_M^w(t) - \tau^w(t)\|_{L^\infty([0,T];\mathbb{R})} \xrightarrow{M \rightarrow \infty} 0. \tag{2.16}$$

Then we have

$$\sup_{w \in \mathcal{K}} \|w_M(t, x + \tau_M^w(t)) - w(t, x + \tau^w(t))\|_{L^\infty((0,T);H^s)} \xrightarrow{M \rightarrow \infty} 0. \tag{2.17}$$

Proof of Proposition 2.1. Together, Proposition 2.2 and Lemma 2.1 imply Proposition 2.1. From (2.7),

$$[\pi_M(\Phi_M(t)\varphi)](x) = [\pi_M(\Phi_M^{\mathcal{G}}(t)\varphi)](x + \tau_M^\varphi(t)),$$

where

$$\tau_M^\varphi(t) = -(p + 1) \int_0^t \int_{\mathbb{T}} [\Phi_M^{\mathcal{G}}(t)\varphi]^{p+1} \, dx \, dt, \tag{2.18}$$

hence (2.1) follows by (2.10). As $W^{s,6}$ is translation invariant, (2.4) follows by (2.14), and (2.3) follows by (2.12). Moreover, (2.2) follows by (2.13) in conjunction with Lemma 2.1, where we choose $\mathcal{W} = \{\Phi^{\mathcal{G}}(t)\varphi \text{ s.t. } \varphi \in \mathcal{K}\}$, and if we denote $w = \Phi^{\mathcal{G}}(t)\varphi$ then the corresponding translation parameters $\tau^w_M(t)$ that appear in Lemma 2.1 are given by (2.18). Notice that the compactness of $\{\Phi^{\mathcal{G}}(t)\varphi \mid \varphi \in \mathcal{K}\}$ in $\mathcal{C}([0,T];H^s)$, required in Lemma 2.1, comes as a by-product of continuity of the flow map $\Phi^{\mathcal{G}}(t)$ (see (2.11)) along with the embedding $Y^s_T \subset \mathcal{C}([0,T];H^s)$. Uniform convergence of the translation parameters as $M \to \infty$ follows by (2.13) in conjunction with Sobolev embedding $H^s \subset L^{p+1}$. ∎

Proof of Lemma 2.1. By translation invariance of the H^s norm and by (2.15) we get

$$\sup_{w \in \mathcal{W}} \|w_M(t, x + \tau^w(t)) - w(t, x + \tau^w(t))\|_{L^\infty([0,T];H^s)} \xrightarrow{M \to \infty} 0;$$

hence, it will be enough to prove

$$\sup_{w \in \mathcal{W}} \|w_M(t, x + \tau^w_M(t)) - w_M(t, x + \tau^w(t))\|_{L^\infty([0,T];H^s)} \xrightarrow{M \to \infty} 0 \qquad (2.19)$$

in order for (2.17) to hold. Again by translation invariance of the H^s norm, (2.19) is equivalent to

$$\sup_{w \in \mathcal{W}} \|w_M(t, x + \tau^w_M(t) - \tau^w(t)) - w_M(t, x)\|_{L^\infty([0,T];H^s)} \xrightarrow{M \to \infty} 0.$$

Assume by contradiction there exist $w^k \in \mathcal{W}$ and times $t^k_{M(k)} \in [0,T]$ with $M(k) \xrightarrow{k \to \infty} \infty$ such that

$$\|w^k_M(t^k_{M(k)}, x + \tau^{w^k}_M(t^k_{M(k)}) - \tau^{w_k}(t^k_{M(k)})) - w^k_M(t^k_{M(k)}, x)\|_{H^s} > \varepsilon_0 > 0.$$

By (2.15) we get

$$\|w^k(t^k_{M(k)}, x + \tau^{w^k}_{M(k)}(t^k_{M(k)}) - \tau^{w^k}(t^k_{M(k)})) - w^k(t^k_{M(k)}, x)\|_{H^s} > \frac{\varepsilon_0}{2} > 0.$$

We claim that this cannot be: by compactness of $\mathcal{W}$ we can assume that

$$w^k \xrightarrow{k \to \infty} w^* \in \mathcal{W} \quad \text{in } \mathcal{C}([0,T];H^s),$$

and therefore

$$\|w^*(t^k_{M(k)}, x + \tau^{w^k}_{M(k)}(t^k_{M(k)}) - \tau^{w^k}(t^k_{M(k)})) - w^*(t^k_{M(k)}, x)\|_{H^s} > \frac{\varepsilon_0}{4} > 0.$$

Next, up to a subsequence, $t^k_{M(k)} \xrightarrow{k \to \infty} t^*$ and by continuity of the function w^* w.r.t. time, we get

$$\|w^*(t^*, x + \tau^{w^k}_{M(k)}(t^k_{M(k)}) - \tau^{w^k}(t^k_{M(k)})) - w^*(t^*, x)\|_{H^s} > \frac{\varepsilon_0}{8} > 0. \qquad (2.20)$$

But by (2.16) we have

$$|\tau_{M(k)}^{w^k}(t_{M(k)}^k) - \tau^{w_k}(t_{M(k)}^k)| \xrightarrow{k\to\infty} 0$$

and (2.20) contradicts the boundedness of the translation operator in (time-independent) H^s. ∎

Proposition 2.2 follows from the proposition below, which holds for finite time intervals. The defocusing character of our equation is crucial in order to extend these local-in-time properties to large times.

Proposition 2.3. *Let $s \geq 1$ and $R > 0$; then there exists $T > 0$ such that*

$$\forall \varphi \in H^s \cap \{\varphi \in H^s \text{ s.t. } \|\varphi\|_{H^1} < R\}, \quad \exists! \, \Phi^{\mathscr{G}}(t)\varphi \in Y_T^s, \text{ a solution to (2.9);} \quad (2.21)$$

the solution maps from $H^s \cap \{\varphi \in H^s \text{ s.t. } \|\varphi\|_{H^1} < R\}$ to Y_T^s,

$$\varphi \to \pi_M \Phi_M^{\mathscr{G}}(t)\varphi \text{ are uniformly Lipschitz w.r.t. } M \in \mathbb{N} \cup \{\infty\}. \quad (2.22)$$

Moreover, there exists a constant $C > 0$ independent of $M \in \mathbb{N} \cup \{\infty\}$ such that

$$\forall \varphi \in H^s \text{ s.t. } \|\varphi\|_{H^1} < R, \quad \|\pi_M \Phi_M(t)\varphi\|_{L^\infty([0,T];H^s)} \leq C \|\varphi\|_{H^s}. \quad (2.23)$$

We also have, for all compact $\mathcal{K} \subset H^s$, the convergence

$$\sup_{\varphi \in \mathcal{K}} \|\pi_M(\Phi_M^{\mathscr{G}}(t)\varphi) - \Phi^{\mathscr{G}}(t)\varphi\|_{L^\infty([0,T];H^s)} \xrightarrow{M\to\infty} 0, \quad (2.24)$$

and for every $\varepsilon > 0$ there exists $C > 0$ independent of $M \in \mathbb{N} \cup \{\infty\}$ such that

$$\forall \varphi \in H^{s+\varepsilon} \text{ s.t. } \|\varphi\|_{H^1} < R, \quad \|\pi_M(\Phi_M^{\mathscr{G}}(t)\varphi)\|_{L^6((0,T);W^{s,6})} \leq C \|\varphi\|_{H^{s+\varepsilon}}. \quad (2.25)$$

Proof of Proposition 2.2. We now prove that Proposition 2.3 implies Proposition 2.2. The quantity

$$\frac{1}{2} \int \left(\pi_M(\Phi_M^{\mathscr{G}}(t)\varphi)\right)^2 + \left(\partial_x \pi_M(\Phi_M^{\mathscr{G}}(t)\varphi)\right)^2 + \frac{1}{p+2} \int \left(\pi_M(\Phi_M^{\mathscr{G}}(t)\varphi)\right)^{p+2} \quad (2.26)$$

is conserved by (1.3) for $M \in \mathbb{N} \cup \{\infty\}$. In particular, if we set $M = \infty$ and $R > 0$ we get

$$\sup_{\substack{t \in [0, T_\varphi^{\max}) \\ \varphi \in H^s \text{ s.t. } \|\varphi\|_{H^1} < R}} \|\Phi^{\mathscr{G}}(t)\varphi\|_{H^1} = K(R) < \infty,$$

where $T_\varphi^{\max}$ is the maximal time of existence of a solution in the iteration space Y_T^s. The solution $\Phi^{\mathscr{G}}(t)\varphi$ retains Sobolev regularity H^s and the corresponding H^1 norm stays below $K(R)$ up to $T_\varphi^{\max}$. Hence we have $T_\varphi^{\max} = \infty$ as we can iterate the local existence

result provided by (2.21) infinitely many times, where R is replaced by $K(R)$ and T is the corresponding time of local existence in (2.21). We also have, by (2.26),

$$\sup_{\substack{M \in \mathbb{N} \cup \{\infty\},\ t \in [0,\infty) \\ \varphi \in H^s \text{ s.t. } \|\varphi\|_{H^1} < R}} \|\pi_M \Phi_M^{\mathscr{G}}(t)\varphi\|_{H^1} = \tilde{K}(R) < \infty,$$

and arguing as above one easily checks that (2.11) follows by iterations of (2.22), with $R = \tilde{K}(R)$. By a similar iteration argument, (2.14) follows by (2.25) and (2.12) follows by (2.23). Finally (2.24) and (2.11) imply (2.13) following a general argument from [47]. ∎

Proof of Proposition 2.3. Denote by $S(t)$ the linear group associated with the linear KdV equation, namely $S(t) = e^{t\partial_x^3}$. Then (2.8) is rewritten, in integral form, as

$$v_M(t) = S(t)(\pi_M \varphi) + (p + 1) \int_0^t S(t - \tau)\pi_M \Pi(\partial_x v_M(\tau)\Pi v_M^p(\tau))\, d\tau. \quad (2.27)$$

The analysis of [7, pp. 183–186 and pp. 197–200] may be used to obtain that, for $s \geq 1$,

$$\left\| \int_0^t S(t - \tau)\pi_M \Pi(\partial_x w(\tau)\Pi w^p(\tau))\, d\tau \right\|_{Y_T^s} \leq C T^\kappa \|w\|_{Y_T^1}^p \|w\|_{Y_T^s}, \quad (2.28)$$

where $\kappa > 0$ and $T \in (0, 1)$. We refer to the appendix for the proof of (2.28). Notice that (2.28) is a slightly modified version compared with the one available in the literature: we gain a power of T, which is very important later. By a similar argument one proves a multilinear estimate for $s \geq 1$:

$$\left\| \int_0^t S(t - \tau)\pi_M \Pi(\partial_x w_{p+1}(\tau)\Pi(w_1(\tau) \times \cdots \times w_p(\tau)))\, d\tau \right\|_{Y_T^s}$$
$$\leq C T^\kappa \sum_{i=1}^{p+1} \left(\|w_i\|_{Y_T^s} \prod_{\substack{j=1,\ldots,p+1 \\ j \neq i}} \|w_j\|_{Y_T^1} \right), \quad (2.29)$$

and existence and uniqueness follow by a classical fixed point argument in the space Y_T^s.

Applying (2.28) with $s = 1$, $w = v_N$ and recalling (2.27), we obtain $\|v_M\|_{Y_T^1} \leq C\|\varphi\|_{H^1}$ provided T is small enough, depending only on a bound for φ in H^1. Applying (2.28) once again, we get

$$\|v_M\|_{Y_T^s} \leq C\|\varphi\|_{H^s} + C T^\kappa (C\|\varphi\|_{H^1})^p \|v_M\|_{Y_T^s},$$

which implies

$$\|v_M\|_{Y_T^s} \leq C\|\varphi\|_{H^s}$$

by possibly taking T smaller but still depending only on an H^1 bound for φ. By the embedding $Y_T^s \subset L^\infty([0, T]; H^s)$, (2.23) follows and we also get

$$\|v_M\|_{X_T^{s,\frac{1}{2}}} \leq C\|\varphi\|_{H^s}. \quad (2.30)$$

Now we invoke the Strichartz estimate of [1, (8.37)]:

$$\|S(t)g\|_{L^6((0,T);L^6)} \le C \|g\|_{H^\sigma}, \quad \sigma > 0,$$

which together with the transfer principle from [17, Lemma 2.3] yields

$$\|w\|_{L^6((0,T);L^6)} \le C \|w\|_{X_T^{\sigma,b}}, \quad b > \frac{1}{2}. \tag{2.31}$$

Next let $w \in X_T^{\frac{1}{3},\frac{1}{3}}$; then we may assume without loss of generality that w is a global space-time function such that $\|w\|_{X^{\frac{1}{3},\frac{1}{3}}} \le 2\|w\|_{X_T^{\frac{1}{3},\frac{1}{3}}}$. By the Sobolev embedding $H^{\frac{1}{3}} \subset L^6$ and $S(t)$ being an isometry on H^s,

$$\|w\|_{L^6(\mathbb{R};L^6(\mathbb{T}))} \le C \|S(-t)w(t,.)\|_{L^6(\mathbb{R};H^{\frac{1}{3}}(\mathbb{T}))} \le C \|\langle D\rangle_x^{\frac{1}{3}}(S(-t)w(t,.))\|_{L^6(\mathbb{R};L^2(\mathbb{T}))},$$

and by the Minkowski inequality and Sobolev embedding (which we now exploit w.r.t. the time variable),

$$\cdots \le C \|\langle D\rangle_x^{\frac{1}{3}}(S(-t)w(t,.))\|_{L^2(\mathbb{T};L^6(\mathbb{R}))}$$
$$\le C \|\langle D\rangle_x^{\frac{1}{3}} S(-t)w(t,.)\|_{L^2(\mathbb{T};H^{\frac{1}{3}}(\mathbb{R}))}$$
$$= C \|\langle D\rangle_t^{\frac{1}{3}} \langle D\rangle_x^{\frac{1}{3}} (S(-t)w(t,.))\|_{L^2(\mathbb{R}\times\mathbb{T})}$$
$$= C \|w\|_{X^{\frac{1}{3},\frac{1}{3}}} \le 2C \|w\|_{X_T^{\frac{1}{3},\frac{1}{3}}},$$

so that $\|w\|_{L^6((0,T);L^6)} \le C \|w\|_{X_T^{\frac{1}{3},\frac{1}{3}}}$. Interpolation with (2.31) yields

$$\forall \varepsilon > 0, \quad \|w\|_{L^6((0,T);L^6)} \le C \|w\|_{X_T^{\varepsilon,\frac{1}{2}}}.$$

By choosing $w = v_M$ and recalling (2.30) where we replace s by $s + \varepsilon$,

$$\|v_M\|_{L^6((0,T);W^{s,6})} \le C \|v_M\|_{X_T^{s+\varepsilon,\frac{1}{2}}} \le C \|\varphi\|_{H^{s+\varepsilon}} \quad \forall \varepsilon > 0,$$

and we get (2.25). The proof of (2.22) follows by (2.29) by considering the difference of two solutions.

Finally,

$$\pi_M \Pi(\partial_x v_M \Pi v_M^p) - \Pi(\partial_x v \Pi v^p) = \pi_M \Pi(\partial_x v_M \Pi(v_M^p - v^p)) + (\partial_x v_M - \partial_x v)\Pi v^p$$
$$- (1 - \pi_M)\Pi(\partial_x v \Pi v^p),$$

where v_M, v are solutions to (2.8) and (2.9). Therefore, using (2.29), where we choose p factors w_i equal to either v_M, v and one factor equal to $v - v_M$, writing the fixed point equation solved by $v - v_M$, and recalling (2.22), we get (see e.g. [47, Proposition 2.7] for details), with $\mathcal{K}$ being compact in H^s,

$$\sup_{\varphi \in \mathcal{K}} \|\pi_M \Phi_M^{\mathscr{G}}(t)\varphi - \Phi^{\mathscr{G}}(t)\varphi\|_{Y_T^s} \xrightarrow{M\to\infty} 0.$$

Therefore, we get (2.24) by using the continuous embedding $Y_T^s \subset L^\infty([0,T];H^s)$. ∎

3. The energy $\mathcal{E}_k(u)$ and proof of Theorem 1.2

Recall $p \in 2\mathbb{N}$ is the integer involved in the nonlinearity in (1.2) and the following notation will be used without any further comment. For any time-independent function $u(x)$ we write $\int u = \int_{\mathbb{T}} u(x)\, dx$ and, similarly, for any time-dependent function $w(t,x)$ and $T > 0$ we write $\int_0^T \int w = \int_0^T \int_{\mathbb{T}} w(t,x)\, dx\, dt$.

We now introduce suitable densities that will be needed to construct the modified energy $\mathcal{E}_k(u)$. We start with densities that represent the "worse contributions" that will appear when we compute the time derivative of $\mathcal{E}_k(u)$ along solutions to (1.2). The specific structure of $\mathcal{E}_k(u)$ will allow us to erase the aforementioned "worse" contributions due to algebraic cancellations.

Definition 3.1. For all $(j_0, j_1, j_2) \in \mathbb{N}^3$, let $\mathcal{I}_{j_0,j_1,j_2}(u) = u^{p-2}(\partial_x^{j_0} u)^2 \partial_x^{j_1} u \partial_x^{j_2} u$. For every integer $k > 1$ we define sets of densities

$$\Upsilon_k = \left\{ \mathcal{I}_{j_0,j_1,j_2}(u) \text{ s.t. } (j_0, j_1, j_2) \in \mathcal{A}^k \right\}, \quad \Xi_k = \left\{ \mathcal{I}_{j_0,j_1,j_2}(u) \text{ s.t. } (j_0, j_1, j_2) \in \mathcal{B}^k \right\},$$

where sets of indices $\mathcal{A}^k$, $\mathcal{B}^k$ are defined as

$$\mathcal{A}^k = \{(j_0, j_1, j_2) \in \mathbb{N}^3 \text{ s.t. } j_2 \le j_1 \le j_0, 2j_0 + j_1 + j_2 = 2k - 2\},$$
$$\mathcal{B}^k = \{(j_0, j_1, j_2) \in \mathbb{N}^3 \text{ s.t. } j_2 \le j_1 \le j_0, 2j_0 + j_1 + j_2 = 2k + 1\}.$$

We next introduce "good densities", namely densities that appear once we compute the time derivative of $\mathcal{E}_k(u)$ along solutions to (1.2), but are harmless since they can be handled thanks to the dispersive effect of (1.2).

Definition 3.2. Let $\mathcal{J}_{i_0,\dots,i_m}(u) = \prod_{l=0}^m \partial_x^{i_l} u$. For every $k > 1$ we define

$$\Theta_k = \left\{ \mathcal{J}_{j_0,\dots,j_{2p+1}}(u) \text{ s.t. } (j_0, \dots, j_{2p+1}) \in \mathcal{C}^k \right\},$$
$$\Omega_k = \left\{ \mathcal{J}_{j_0,\dots,j_{p+1}}(u) \text{ s.t. } (j_0, \dots, j_{p+1}) \in \mathcal{D}^k \right\},$$

with

$$\mathcal{C}^k = \Big\{ (j_0, \dots, j_{2p+1}) \in \mathbb{N}^{2p+2} \text{ s.t. } j_{2p+1} \le \cdots \le j_0, \sum_{l=0}^{2p+1} j_l \le 2k - 1,$$
$$j_2 \ge 1, \ j_0 \le k - 1 \Big\},$$
$$\mathcal{D}^k = \Big\{ (j_0, \dots, j_{p+1}) \in \mathbb{N}^{p+2} \text{ s.t. } j_{p+1} \le \cdots \le j_0, \sum_{l=0}^{p+1} j_l = 2k + 1, j_4 \ge 1 \Big\}.$$

Remark 3.1. Roughly speaking, densities in Θ_k are homogeneous of order $2p + 2$, the sum of the derivatives on each factor is $2k - 1$ and derivatives are distributed on at least three factors. Densities in Ω_k are homogeneous of order $p + 2$, the sum of the derivatives is $2k + 1$ and derivatives are shared on at least five factors.

Definition 3.3. For any $(j_0, j_1, j_2) \in \mathcal{A}^k$ we define

$$\mathcal{I}^*_{j_0,j_1,j_2}(u) = (p-2)u^{p-3}\partial_x^3 u (\partial_x^{j_0} u)^2 \partial_x^{j_1} u \partial_x^{j_2} u + 2u^{p-2}\partial_x^{j_0} u \partial_x^{j_0+3} u \partial_x^{j_1} u \partial_x^{j_2} u$$
$$+ u^{p-2}(\partial_x^{j_0} u)^2 \partial_x^{j_1+3} u \partial_x^{j_2} u + u^{p-2}(\partial_x^{j_0} u)^2 \partial_x^{j_1} u \partial_x^{j_2+3} u$$

and

$$
\begin{aligned}
\mathcal{I}^{**}_{j_0,j_1,j_2}(u) = {} & (p-2)u^{p-3}\partial_x(u^{p+1})(\partial_x^{j_0}u)^2\partial_x^{j_1}u\partial_x^{j_2}u \\
& + 2u^{p-2}\partial_x^{j_0}u\partial_x^{j_0+1}(u^{p+1})\partial_x^{j_1}u\partial_x^{j_2}u + u^{p-2}(\partial_x^{j_0}u)^2\partial_x^{j_1+1}(u^{p+1})u\partial_x^{j_2}u \\
& + u^{p-2}(\partial_x^{j_0}u)^2\partial_x^{j_1}u\partial_x^{j_2+1}(u^{p+1}).
\end{aligned}
$$

Remark 3.2. The expression $\mathcal{I}^*_{j_0,j_1,j_2}(u)$ is obtained by considering the time derivative of the density $\mathcal{I}_{j_0,j_1,j_2}(u)$ and by replacing $\partial_t u$ with $\partial_x^3 u$. Similarly, $\mathcal{I}^{**}_{j_0,j_1,j_2}(u)$ is constructed by considering the time derivative of the density $\mathcal{I}_{j_0,j_1,j_2}(u)$ and by replacing $\partial_t u$ with $\partial_x(u^{p+1})$. Then, if $v(t,x)$ is a solution to (1.2),

$$
\frac{d}{dt}\int \mathcal{I}_{j_0,j_1,j_2}(v) = \int \mathcal{I}^*_{j_0,j_1,j_2}(v) - \int \mathcal{I}^{**}_{j_0,j_1,j_2}(v). \tag{3.1}
$$

The next proposition, on the structure of $\mathcal{E}_k(u)$, will be key in obtaining Theorem 1.2. We shall assume in the next proposition that $v(t,x)$ is smooth in order to justify all the necessary computations. Then this smoothness assumption may be removed once we integrate (3.3) in time and use a density argument.

Proposition 3.1. *Let $k > 1$. Then for all $(j_0, j_1, j_2) \in \mathcal{A}^k$ there exists $\lambda_{j_0,j_1,j_2} \in \mathbb{R}$, for all $(j_0, \dots, j_{2p+1}) \in \mathcal{C}^k$ there exists $\mu_{j_0,\dots,j_{2p+1}} \in \mathbb{R}$ and for all $(j_0, \dots, j_{p+1}) \in \mathcal{D}^k$ there exists $\nu_{j_0,\dots,j_{p+1}} \in \mathbb{R}$ such that, with*

$$
\mathcal{E}_k(u) = \|\partial_x^k u\|_{L^2}^2 - \sum_{(j_0,j_1,j_2)\in\mathcal{A}^k} \lambda_{j_0,j_1,j_2} \int \mathcal{I}_{j_0,j_1,j_2}(u), \tag{3.2}
$$

the following identity holds for any smooth solution $v(t,x)$ to (1.2):

$$
\begin{aligned}
\frac{d}{dt}\mathcal{E}_k(v) = {} & \sum_{(j_0,\dots,j_{2p+1})\in\mathcal{C}^k} \mu_{j_0,\dots,j_{2p+1}} \int \mathcal{J}_{j_0,\dots,j_{2p+1}}(v) \\
& + \sum_{(j_0,\dots,j_{p+1})\in\mathcal{D}^k} \nu_{j_0,\dots,j_{p+1}} \int \mathcal{J}_{j_0,\dots,j_{p+1}}(v) \\
& - \sum_{(j_0,j_1,j_2)\in\mathcal{A}^k} \lambda_{j_0,j_1,j_2} \int \mathcal{I}^{**}_{j_0,j_1,j_2}(v).
\end{aligned} \tag{3.3}
$$

Remark 3.3. The energy $\mathcal{E}_k(u)$ has leading-order term $\|\partial_x^k u\|_{L^2}^2$, plus a lower-order remainder that is a linear combination of densities belonging to Υ_k, whose coefficients λ_{j_0,j_1,j_2} are well prepared in such a way that on the r.h.s. in (3.3) we get a linear combination of densities belonging to Θ_k and Ω_k. In particular, densities $\mathcal{I}^{**}_{j_0,j_1,j_2}(v)$ can be expressed as linear combinations of terms belonging to Θ_k computed along $v(t,x)$.

Proof of Theorem 1.2. We prove how Proposition 3.1 implies Theorem 1.2. We first define the quantity

$$
\mathcal{R}_k(u) = \sum_{(j_0,j_1,j_2)\in\mathcal{A}^k} \lambda_{j_0,j_1,j_2} \int \mathcal{I}_{j_0,j_1,j_2}(u) \tag{3.4}
$$

and we show (1.7). We have by the Hölder and Sobolev embedding,

$$
\left| \int (\partial_x^{j_0} u)^2 \partial_x^{j_1} u \partial_x^{j_2} u u^{p-2} \right| \le \|\partial_x^{j_0} u\|_{L^2}^2 \|\partial_x^{j_1} u\|_{L^\infty} \|\partial_x^{j_2} u\|_{L^\infty} \|u\|_{L^\infty}^{p-2}
$$

$$
\le C \|u\|_{H^{j_0}}^2 \|u\|_{H^{j_1+1}} \|u\|_{H^{j_2+1}} \|u\|_{H^1}^{p-2}
$$

$$
\le C \|u\|_{H^k}^{2\theta_0 + \theta_1 + \theta_2} \|u\|_{H^1}^{2-2\theta_0 - \theta_1 - \theta_2 + p}
$$

$$
= C \|u\|_{H^k}^{\frac{2k-4}{k-1}} \|u\|_{H^1}^{p + \frac{2}{k-1}}, \tag{3.5}
$$

where we performed interpolation with $\theta_0 k + (1 - \theta_0) = j_0$, $\theta_l k + (1 - \theta_l) = j_l + 1$ for $l = 1, 2$ and hence $2\theta_0 + \theta_1 + \theta_2 = (2k - 4)/(k - 1)$. Next we define the functional

$$
\mathcal{G}_k^T(\varphi) = \sum_{(j_0,\ldots,j_{2p+1}) \in \mathcal{C}^k} \mu_{j_0,\ldots,j_{2p+1}} \int_0^T \int \mathcal{J}_{j_0,\ldots,j_{2p+1}}(\Phi(\tau)\varphi) \, d\tau
$$

$$
+ \sum_{(j_0,\ldots,j_{p+1}) \in \mathcal{D}^k} \nu_{j_0,\ldots,j_{p+1}} \int_0^T \int \mathcal{J}_{j_0,\ldots,j_{p+1}}(\Phi(\tau)\varphi) \, d\tau
$$

$$
- \sum_{(j_0,j_1,j_2) \in \mathcal{A}^k} \lambda_{j_0,j_1,j_2} \int_0^T \int \mathcal{I}_{j_0,j_1,j_2}^{**}(\Phi(\tau)\varphi) \, d\tau, \tag{3.6}
$$

and we shall prove (1.8). Notice that the identity (1.5) follows after integration in time of the identity (3.3) provided that φ is regular enough that all computations may be justified. Then by a straightforward density argument we get identity (1.5) for $\varphi \in H^k$ by adapting the proof of (1.8) that we give below. By looking at the structure of the densities involved in the definition of (3.6) we deduce (1.8) provided that for any $R > 0$ and $T > 0$ there exists $C > 0$ such that for all $\varphi \in H^k$ with $\|\varphi\|_{H^1} < R$, we have the following bound for $v(t, x) = \Phi(t)\varphi$:

$$
\left| \int_0^T \int \prod_{l=0}^{p+1} \partial_x^{j_l} v \right| \le C + C \|\varphi\|_{H^k}^{(\frac{2k-4}{k-1})^+}, \quad j_{p+1} \le \cdots \le j_0, \sum_{l=0}^{p+1} j_l = 2k+1,
$$
$$
j_4 \ge 1, \quad (3.7)
$$

$$
\left| \int_0^T \int \prod_{l=0}^{2p+1} \partial_x^{j_l} v \right| \le C + C \|\varphi\|_{H^k}^{(\frac{2k-4}{k-1})^+}, \quad j_{2p+1} \le \cdots \le j_0, \sum_{l=0}^{2p+1} j_l \le 2k-1,
$$
$$
j_2 \ge 1. \ (3.8)
$$

One easily checks that, by integration by parts, for every $(j_0, j_1, j_2) \in \mathcal{A}_k$, $\int \mathcal{I}_{j_0,j_1,j_2}^{**}(u)$ can be expressed as the integral of a linear combination of densities belonging to Θ_k: if $(j_0, j_1, j_2) \ne (k - 1, 0, 0)$ then $\mathcal{I}_{j_0,j_1,j_2}^{**}(u)$ is a linear combination of densities with homogeneity $2p + 2$ and the top-order derivative is at most $k - 1$; for $(j_0, j_1, j_2) = (k - 1, 0, 0)$ we have

$$
\mathcal{I}_{k-1,0,0}^{**}(u) = 2\partial_x^k(u^{p+1})\partial_x^{k-1} u u^p + p(\partial_x^{k-1} u)^2 \partial_x(u^{p+1}) u^{p-1}.
$$

The second term on the r.h.s. belongs to Θ_k, and the first term on the r.h.s. can be written, by using the Leibniz rule to develop $\partial_x^k(u^{p+1})$, as a linear combination of terms belonging to Θ_k plus a multiple of $\partial_x^k u \partial_x^{k-1} u u^{2p}$. We conclude using $\int \partial_x^k u \partial_x^{k-1} u u^{2p} = \frac{1}{2} \int \partial_x (\partial_x^{k-1} u)^2 u^{2p} = -p \int (\partial_x^{k-1} u)^2 \partial_x u u^{2p-1}$, and the last one is the integral of a density in Θ_k. Hence, all terms on the r.h.s. of (3.3) can be expressed as linear combinations of terms in Θ_k and Ω_k and therefore (3.7) and (3.8) are enough to conclude.

Next we prove (3.7) and (3.8) by splitting the proof into four subcases.

Proof of (3.7), $j_5 = 0$. By the Hölder inequality we get

$$\left| \int_0^T \int \prod_{l=0}^{p+1} \partial_x^{j_l} v \right| \leq \prod_{l=0}^4 \|\partial_x^{j_l} v\|_{L^6((0,T);L^6)} \|v\|_{L^6((0,T);L^6)} \|v\|_{L^\infty((0,T);L^\infty)}^{p-4} \tag{3.9}$$

and by Sobolev embedding, conservation of the Hamiltonian and (2.4) we proceed as follows:

$$(\cdots) \leq C \left(\prod_{l=0}^4 \|\varphi\|_{H^{j_l}^+} \right) \|\varphi\|_{H^1}^{p-3} \leq C \prod_{l=0}^4 (\|\varphi\|_{H^k}^{\theta_l} \|\varphi\|_{H^1}^{1-\theta_l}) \|\varphi\|_{H^1}^{p-3}$$

$$= C \|\varphi\|_{H^k}^{\sum_{l=0}^4 \theta_l} \|\varphi\|_{H^1}^{p+2-\sum_{l=0}^4 \theta_l},$$

where $\theta_l k + (1-\theta_l) = j_l^+$ for $j = 0, \ldots, 4$, namely $\theta_l = \frac{j_l^+ - 1}{k-1}$, and

$$\theta_0 + \theta_1 + \theta_2 + \theta_3 + \theta_4 = \left(\frac{2k-4}{k-1} \right)^+.$$

Summarizing, we get (3.7) by using the uniform a priori bound on $\|\varphi\|_{H^1}$.

Proof of (3.7), $j_5 \neq 0$. We follow the previous argument closely. By the Hölder inequality, Sobolev embedding, (2.4) and interpolation, we get

$$\left| \int_0^T \int \prod_{l=0}^{p+1} \partial_x^{j_l} v \right| \leq \prod_{l=0}^5 \|\partial_x^{j_l} v\|_{L^6((0,T);L^6)} \prod_{l=6}^{p+1} \|\partial_x^{j_l} v\|_{L^\infty((0,T);L^\infty)}$$

$$\leq C \prod_{l=0}^5 \|\varphi\|_{H^{j_l}^+} \prod_{l=6}^{p+1} \|v\|_{L^\infty((0,T);H^{j_l^+ + 1})}$$

$$\leq C \prod_{l=0}^5 \|\varphi\|_{H^{j_l}^+} \prod_{l=6}^{p+1} \|\varphi\|_{H^{j_l^+ + 1}}$$

$$\leq C \prod_{l=0}^5 (\|\varphi\|_{H^1}^{1-\theta_l} \|\varphi\|_{H^k}^{\theta_l}) \prod_{l=6}^{p+1} (\|\varphi\|_{H^1}^{1-\eta_l} \|\varphi\|_{H^k}^{\eta_l})$$

$$= C \|\varphi\|_{H^k}^{\sum_{l=0}^5 \theta_l + \sum_{l=6}^{p+1} \eta_l} \|\varphi\|_{H^1}^{p+2-\sum_{l=0}^5 \theta_l - \sum_{l=6}^{p+1} \eta_l}, \tag{3.10}$$

where $\theta_l k + (1 - \theta_l) = j_l^+$ for $l = 0, \ldots, 5$ and $\eta_l k + (1 - \eta_l) = j_l^+ + 1$ for $l = 6, \ldots, p+1$, hence

$$\sum_{l=0}^{5} \theta_l + \sum_{l=6}^{p+1} \eta_l = \left(\frac{2k-5}{k-1}\right)^+.$$

The estimates above, along with the uniform bound assumed on $\|\varphi\|_{H^1}$, imply

$$\left| \int_0^T \int \prod_{l=0}^{p+1} \partial_x^{j_l} v \right| \le C \, \|\varphi\|_{H^k}^{\left(\frac{2k-5}{k-1}\right)^+},$$

hence we get (3.7) (indeed, in this subcase we get even better bounds as the corresponding powers are smaller than the ones that appear in (3.7)).

Proof of (3.8), $j_3 = 0$. Again, by the Hölder inequality we get

$$\left| \int_0^T \int \prod_{l=0}^{2p+1} \partial_x^{j_l} v \right| \le \prod_{l=0}^{2} \|\partial_x^{j_l} v\|_{L^4((0,T);L^4)} \|v\|_{L^4((0,T);L^4)} \|v\|_{L^\infty((0,T);L^\infty)}^{2p-2}$$

$$\le C \prod_{l=0}^{2} \|\partial_x^{j_l} v\|_{L^6((0,T);L^6)} \|v\|_{L^6((0,T);L^6)} \|v\|_{L^\infty((0,T);L^\infty)}^{2p-2} \quad (3.11)$$

and by Sobolev embedding, conservation of the Hamiltonian and (2.4),

$$(\cdots) \le C \left(\prod_{l=0}^{2} \|\varphi\|_{H^{j_l^+}} \right) \|\varphi\|_{H^1}^{2p-4} \le C \prod_{l=0}^{2} (\|\varphi\|_{H^k}^{\theta_l} \|\varphi\|_{H^1}^{1-\theta_l}) \|\varphi\|_{H^1}^{2p-1}$$

$$= C \|\varphi\|_{H^k}^{\sum_{l=0}^{2} \theta_l} \|\varphi\|_{H^1}^{2p+2-\sum_{l=0}^{2} \theta_l},$$

where $\theta_l k + (1 - \theta_l) = j_l^+$ for $l = 0, 1, 2$ and hence $\theta_0 + \theta_1 + \theta_2 = (\frac{2k-4}{k-1})^+$. Combining the above estimates with the uniform bound on $\|\varphi\|_{H^1}$ we get

$$\left| \int_0^T \int \prod_{l=0}^{2p+1} \partial_x^{j_l} v \right| \le C \, \|\varphi\|_{H^k}^{\left(\frac{2k-4}{k-1}\right)^+}.$$

Proof of (3.8), $j_3 \ne 0$. By the Hölder inequality, Sobolev embedding, (2.4) and interpolation,

$$\left| \int_0^T \int \prod_{l=0}^{2p+1} \partial_x^{j_l} v \right| \le \prod_{l=0}^{3} \|\partial_x^{j_l} v\|_{L^4((0,T);L^4)} \prod_{l=4}^{2p+1} \|\partial_x^{j_l} v\|_{L^\infty((0,T);L^\infty)}$$

$$\le C \prod_{l=0}^{3} \|\partial_x^{j_l} v\|_{L^6((0,T);L^6)} \prod_{l=4}^{2p+1} \|\partial_x^{j_l} v\|_{L^\infty((0,T);L^\infty)}$$

$$\le C \prod_{l=0}^{3} \|\varphi\|_{H^{j_l^+}} \prod_{l=4}^{2p+1} \|v\|_{L^\infty((0,T);H^{j_l^+ + 1})}$$

$$\leq C \prod_{l=0}^{3} \|\varphi\|_{H^{j_l^+}} \prod_{l=4}^{2p+1} \|\varphi\|_{H^{j_l^+ +1}}$$

$$\leq C \prod_{l=0}^{3} (\|\varphi\|_{H^1}^{1-\theta_l} \|\varphi\|_{H^k}^{\theta_l}) \prod_{l=4}^{2p+1} (\|\varphi\|_{H^1}^{1-\eta_l} \|\varphi\|_{H^k}^{\eta_l})$$

$$\leq C \|\varphi\|_{H^k}^{\sum_{l=0}^{3} \theta_l + \sum_{l=4}^{2p+1} \eta_l} \|\varphi\|_{H^1}^{2p+2-\sum_{l=0}^{5} \theta_l - \sum_{l=6}^{p+1} \eta_l}, \qquad (3.12)$$

where $\theta_l k + (1 - \theta_l) = j_l^+$ for $l = 0, 1, 2, 3$ and $\eta_l k + (1 - \eta_l) = j_l^+ + 1$ for $l = 4, \ldots, 2p + 1$, then

$$\sum_{l=0}^{3} \theta_l + \sum_{l=4}^{2p+1} \eta_l = \left(\frac{2k-5}{k-1}\right)^+.$$

Combining the above estimates with the uniform bound on $\|\varphi\|_{H^1}$,

$$\left| \int_0^T \int \prod_{l=0}^{2p+1} \partial_x^{j_l} v \right| \leq C \|\varphi\|_{H^k}^{(\frac{2k-5}{k-1})^+}. \qquad \blacksquare$$

3.1. Proof of Proposition 3.1

We first define an ordering on the sets $\mathcal{A}^k$ and $\mathcal{B}^k$.

Definition 3.4. Given $(i_0, i_1, i_2), (l_0, l_1, l_2) \in \mathcal{A}^k$ or $\mathcal{B}^k$, we define

$$(i_0, i_1, i_2) \prec (l_0, l_1, l_2) \Leftrightarrow \begin{cases} \text{either } i_0 < l_0 \\ \text{or } i_0 = l_0, i_1 < l_1, \end{cases}$$

$$(i_0, i_1, i_2) \preceq (l_0, l_1, l_2) \Leftrightarrow \begin{cases} \text{either } (i_0, i_1, i_2) \prec (l_0, l_1, l_2) \\ \text{or } (i_0, i_1, i_2) = (l_0, l_1, l_2). \end{cases}$$

For any given $(\bar{j}_0, \bar{j}_1, \bar{j}_2) \in \mathcal{A}_k$ we define

$$\mathcal{A}_k^{\prec(\bar{j}_0, \bar{j}_1, \bar{j}_2)} = \{(j_0, j_1, j_2) \in \mathcal{A}_k \mid (j_0, j_1, j_2) \prec (\bar{j}_0, \bar{j}_1, \bar{j}_2)\},$$

as well as $\mathcal{A}_k^{\preceq(\bar{j}_0, \bar{j}_1, \bar{j}_2)}$, $\mathcal{A}_k^{\succ(\bar{j}_0, \bar{j}_1, \bar{j}_2)}$, $\mathcal{A}_k^{\succeq(\bar{j}_0, \bar{j}_1, \bar{j}_2)}$ by modifying the order accordingly.

The next propositions will be of importance:

Proposition 3.2. *For every $(j_0, j_1, j_2) \in \mathcal{A}^k$ there exists $\delta_{j_0, j_1, j_2} \in \mathbb{R}$ and for every $(j_0, \ldots, j_{p+1}) \in \mathcal{D}^k$ there exists $\gamma_{j_0, \ldots, j_{p+1}} \in \mathbb{R}$ such that*

$$\int \partial_x^k u \partial_x^{k+1} (u^{p+1}) = \sum_{(j_0, j_1, j_2) \in \mathcal{A}^k} \delta_{j_0, j_1, j_2} \int \mathcal{I}_{j_0+1, j_1+1, j_2}(u)$$

$$+ \sum_{(j_0, \ldots, j_{p+1}) \in \mathcal{D}^k} \gamma_{j_0, \ldots, j_{p+1}} \int \mathcal{J}_{j_0, \ldots, j_{p+1}}(u).$$

Proof. The map $(j_0, j_1, j_2) \to (j_0 + 1, j_1 + 1, j_2)$ is one to one from $\mathcal{A}_k$ to $\mathcal{B}_k$, hence the statement is equivalent to

$$\int \partial_x^k u \partial_x^{k+1}(u^{p+1}) = \sum_{(j_0, j_1, j_2) \in \mathcal{B}^k} \delta_{j_0, j_1, j_2} \int \mathcal{I}_{j_0, j_1, j_2}(u)$$

$$+ \sum_{(j_0, \ldots, j_{p+1}) \in \mathcal{D}^k} \gamma_{j_0, \ldots, j_{p+1}} \int \mathcal{J}_{j_0, \ldots, j_{p+1}}(u). \qquad (3.13)$$

Expanding the $(k + 1)$st derivative on the l.h.s. by the chain rule, we get integrals of linear combinations of the densities

$$\prod_{\substack{j_0 \geq j_1 \geq \cdots \geq j_{p+1} \geq 0 \\ j_0 + \cdots + j_{p+1} = 2k+1}} \partial_x^{j_l} u.$$

We shall denote by l.o.t. all the densities belonging to Ω_k, namely densities that can be absorbed in the second term on the r.h.s. in (3.13). Hence we are reduced to considering only terms like

$$\int \partial_x^{j_0} u \partial_x^{j_1} u \partial_x^{j_2} u \partial_x^{j_3} u u^{p-2},$$

$$j_0 + j_1 + j_2 + j_3 = 2k + 1, \quad j_0 \geq j_1 \geq j_2 \geq j_3, \qquad (3.14)$$

and we will prove that, up to l.o.t., they can be expressed as a linear combination of terms

$$\int \mathcal{I}_{i_0, i_1, i_2}(u), \quad (i_0, i_1, i_2) \in \mathcal{B}^k.$$

For every density defined in (3.14) we also define $\Delta(j_0, j_1, j_2, j_3) = j_0 - j_1$, so that by integration by parts we are done if $\Delta(j_0, j_1, j_2, j_3) \in \{0, 1\}$. We claim that if $\Delta(j_0, j_1, j_2, j_3) > 1$ then, up to l.o.t., we can express the terms in (3.14) as a linear combination of densities such that $\Delta(j_0, j_1, j_2, j_3) \in \{0, 1\}$. In order to do so, assume that we have a density like (3.14) such that $\Delta(j_0, j_1, j_2, j_3) > 1$. Then by integration by parts, more precisely by moving one derivative from the higher-order derivative $\partial_x^{j_0} u$ on the other factors we get, up to l.o.t., a linear combination of terms

$$\int \partial_x^{\tilde{j}_0} u \partial_x^{\tilde{j}_1} u \partial_x^{\tilde{j}_2} u \partial_x^{\tilde{j}_3} u u^{p-2},$$

$$\tilde{j}_0 + \tilde{j}_1 + \tilde{j}_2 + \tilde{j}_3 = 2k + 1, \quad \tilde{j}_0 \geq \tilde{j}_1 \geq \tilde{j}_2 \geq \tilde{j}_3,$$

$$\Delta(\tilde{j}_0, \tilde{j}_1, \tilde{j}_2, \tilde{j}_3) \in \{\Delta(j_0, j_1, j_2, j_3) - 1, \Delta(j_0, j_1, j_2, j_3) - 2\}.$$

We can further iterate this argument, and conclude after a finite number of steps. ∎

Our next proposition will be used in the proof of Proposition 3.1 and we postpone its proof for now.

Proposition 3.3. *Let $(j_0, j_1, j_2) \in \mathcal{A}^k$. There exists $\alpha_{j_0, j_1, j_2} \in \mathbb{R} \setminus \{0\}$ such that*

$$\int \mathcal{I}^*_{j_0, j_1, j_2}(u) = \alpha_{j_0, j_1, j_2} \int \mathcal{I}_{j_0+1, j_1+1, j_2}(u)$$

$$+ \sum_{\substack{(i_0, i_1, i_2) \in \mathcal{B}^k \\ (i_0, i_1, i_2) \prec (j_0+1, j_1+1, j_2)}} \eta_{i_0, i_1, i_2} \int \mathcal{I}_{i_0, i_1, i_2}(u)$$

$$+ \sum_{(j_0, \ldots, j_{p+1}) \in \mathcal{D}^k} \varepsilon_{j_0, \ldots, j_{p+1}} \int \mathcal{J}_{j_0, \ldots, j_{p+1}}(u), \tag{3.15}$$

where $\varepsilon_{j_0, \ldots, j_{p+1}}, \eta_{i_0, i_1, i_2} \in \mathbb{R}$.

Proof of Proposition 3.1. Propositions 3.2 and 3.3 will imply Proposition 3.1. If $v(t, x)$ is a solution to (1.2) then

$$\frac{d}{dt} \|\partial_x^k v\|_{L^2}^2 = 2 \int \partial_t \partial_x^k v \partial_x^k v = 2 \int \partial_x^{k+3} v \partial_x^k v - 2 \int \partial_x^{k+1}(v^{p+1}) \partial_x^k v$$

$$= -2 \int \partial_x^{k+1}(v^{p+1}) \partial_x^k v.$$

Hence, by Proposition 3.2,

$$\frac{d}{dt} \|\partial_x^k v\|_{L^2}^2 = \sum_{(j_0, j_1, j_2) \in \mathcal{A}^k} \beta_{j_0, j_1, j_2} \int \mathcal{I}_{j_0+1, j_1+1, j_2}(v) + \text{l.o.t.}, \tag{3.16}$$

where $\beta_{j_0, j_1, j_2} = -2\delta_{j_0, j_1, j_2}$ and l.o.t. denotes the integral of a linear combination of densities belonging to Θ_k and Ω_k. Next we prove that we can select the coefficients λ_{j_0, j_1, j_2} in such a way that

$$\sum_{(j_0, j_1, j_2) \in \mathcal{A}_k} \beta_{j_0, j_1, j_2} \int \mathcal{I}_{j_0+1, j_1+1, j_2}(v)$$

$$- \sum_{(j_0, j_1, j_2) \in \mathcal{A}_k} \lambda_{j_0, j_1, j_2} \int \mathcal{I}^*_{j_0, j_1, j_2}(v) = \text{l.o.t}, \tag{3.17}$$

where again l.o.t. denotes the integral of a linear combination of densities belonging to Θ_k and Ω_k. Of course, (3.17) allows us to conclude the proof thanks to (3.1) and by recalling that terms $\int \mathcal{I}^{**}_{j_0, j_1, j_2}(v) \, dx$ can be absorbed in l.o.t. (more precisely those densities are a linear combination of terms belonging to Θ_k).

We proceed with (3.17): we claim that for every $(j_0, j_1, j_2) \in \mathcal{A}_k$ we can select a real number λ_{j_0, j_1, j_2} such that the following holds: for any given $(\bar{j}_0, \bar{j}_1, \bar{j}_2) \in \mathcal{A}_k$ there exists a function

$$\rho^{(\bar{j}_0, \bar{j}_1, \bar{j}_2)} \colon \mathcal{A}_k^{\prec(\bar{j}_0, \bar{j}_1, \bar{j}_2)} \ni (j_0, j_1, j_2) \to \rho^{(\bar{j}_0, \bar{j}_1, \bar{j}_2)}_{j_0, j_1, j_2} \in \mathbb{R}$$

such that

$$\sum_{(j_0,j_1,j_2)\in\mathcal{A}_k^{\succeq(\bar{j}_0,\bar{j}_1,\bar{j}_2)}} \int \left(\beta_{j_0,j_1,j_2}\mathcal{I}_{j_0+1,j_1+1,j_2}(u) - \lambda_{j_0,j_1,j_2}\mathcal{I}^*_{j_0,j_1,j_2}(u)\right)$$

$$= \sum_{(j_0,j_1,j_2)\in\mathcal{A}_k^{\prec(\bar{j}_0,\bar{j}_1,\bar{j}_2)}} \rho^{(\bar{j}_0,\bar{j}_1,\bar{j}_2)}_{j_0,j_1,j_2} \int \mathcal{I}_{j_0+1,j_1+1,j_2}(u) + \text{l.o.t.} \tag{3.18}$$

If we prove the claim then (3.17) holds by choosing $(\bar{j}_0, \bar{j}_1, \bar{j}_2) \in \mathcal{A}_k$ such that, for all $(j_0, j_1, j_2) \in \mathcal{A}_k$, $(\bar{j}_0, \bar{j}_1, \bar{j}_2) \preceq (j_0, j_1, j_2)$.

We first select $\lambda_{(k-1,0,0)}$ where $(k-1,0,0) = \max\{(j_0, j_1, j_2) \in \mathcal{A}_k\}$; in a second step we define $\lambda_{(j_0', j_1', j_2')}$, where $(j_0', j_1', j_2') = \max\{(j_0, j_1, j_2) \in \mathcal{A}_k^{\prec(k-1,0,0)}\}$. Then we proceed in a similar way until we get to defining $\lambda_{(j_0'', j_1'', j_2'')}$, where $(j_0'', j_1'', j_2'') = \min\{(j_0, j_1, j_2) \in \mathcal{A}_k\}$. We start by choosing $\lambda_{k-1,0,0}$. Notice that by (3.15), if we impose the condition $\lambda_{k-1,0,0}\alpha_{k-1,0,0} = \beta_{k-1,0,0}$, then we get

$$\beta_{k-1,0,0}\int \mathcal{I}_{k,1,0}(u) - \lambda_{k-1,0,0}\int \mathcal{I}^*_{k-1,0,0}(u)$$

$$= \sum_{(j_0,j_1,j_2)\in\mathcal{A}_k^{\prec(k-1,0,0)}} \rho^{(k-1,0,0)}_{j_0,j_1,j_2} \int \mathcal{I}_{j_0+1,j_1+1,j_2}(u) + \text{l.o.t.}$$

Assume that we have selected λ_{j_0,j_1,j_2} for every $(j_0, j_1, j_2) \succeq (\bar{j}_0, \bar{j}_1, \bar{j}_2)$ and assume that (3.18) holds. In the next step we select $\lambda_{\tilde{j}_0,\tilde{j}_1,\tilde{j}_2}$ where $(\tilde{j}_0, \tilde{j}_1, \tilde{j}_2) = \max\{(j_0, j_1, j_2) \in \mathcal{A}_k^{\prec(\bar{j}_0,\bar{j}_1,\bar{j}_2)}\}$. Then, by using the same construction as above, we can select $\lambda_{\tilde{j}_0,\tilde{j}_1,\tilde{j}_2}$, with the condition $\lambda_{\tilde{j}_0,\tilde{j}_1,\tilde{j}_2}\alpha_{\tilde{j}_0,\tilde{j}_1,\tilde{j}_2} = \rho^{(\bar{j}_0,\bar{j}_1,\bar{j}_2)}_{\tilde{j}_0,\tilde{j}_1,\tilde{j}_2}$. This concludes the proof. ∎

Proof of Proposition 3.3. To streamline our proof, for a given density $\mathcal{D}(u)$, we say that $\int \mathcal{D}(u) \equiv 0$ if and only if $\mathcal{D}(u)$ is a linear combination of densities in Ω_k and densities $\mathcal{I}_{i_0,i_1,i_2}(u) \in \Xi_k$ such that $(i_0, i_1, i_2) \prec (j_0 + 1, j_1 + 1, j_2)$. If, moreover, $\mathcal{D}_1(u)$ and $\mathcal{D}_2(u)$ are two densities then we say $\int \mathcal{D}_1(u) \equiv \int \mathcal{D}_2(u)$ if and only if $\int(\mathcal{D}_1(u) - \mathcal{D}_2(u)) \equiv 0$. Hence we aim to prove $\int \mathcal{I}^*_{j_0,j_1,j_2}(u) \equiv \alpha_{j_0,j_1,j_2}\int \mathcal{I}_{j_0+1,j_1+1,j_2}(u)$, where $\alpha_{j_0,j_1,j_2} \neq 0$.

We may write

$$\mathcal{I}^*_{j_0,j_1,j_2}(u) = 2A + B + C + (p-2)D \tag{3.19}$$

with

$$A = \int u^{p-2}\partial_x^{j_0+3}u\partial_x^{j_0}u\partial_x^{j_1}u\partial_x^{j_2}u, \qquad B = \int u^{p-2}(\partial_x^{j_0}u)^2\partial_x^{j_1+3}u\partial_x^{j_2}u,$$

$$C = \int u^{p-2}(\partial_x^{j_0}u)^2\partial_x^{j_1}u\partial_x^{j_2+3}u, \qquad D = \int u^{p-3}\partial_x^3 u(\partial_x^{j_0}u)^2\partial_x^{j_1}u\partial_x^{j_2}u. \tag{3.20}$$

We split the proof into three cases, each with subcases. Define $\Delta(j_0, j_1, j_2) = j_0 - j_1$.

First case: $\Delta(j_0, j_1, j_2) \geq 2$. First we prove that

$$B \equiv C \equiv D \equiv 0. \tag{3.21}$$

The condition $\Delta(j_0, j_1, j_2) \geq 2$ is equivalent to $j_1 + 3 \leq j_0 + 1$, hence we get $B \equiv 0$ as at most one derivative involved in the density of B is of order $j_0 + 1$ and all the remaining factors have lower-order derivatives. By a similar argument we get $C \equiv 0$. Regarding D, $\Delta(j_0, j_1, j_2) \geq 2$ implies $j_0 \geq 2$ and hence $D \equiv 0$ as again at most one derivative involved in the density of D can be of order $j_0 + 1$ and all the other factors involve lower-order derivatives.

In order to deal with A notice that by integration by parts we can move one derivative from $\partial_x^{j_0+3} u$ to the other factors in the density of A:

$$
\begin{aligned}
A = -\int &\left(u^{p-2} \partial_x^{j_0+2} u \, \partial_x^{j_0+1} u \, \partial_x^{j_1} u \, \partial_x^{j_2} u + u^{p-2} \partial_x^{j_0+2} u \, \partial_x^{j_0} u \, \partial_x^{j_1+1} u \, \partial_x^{j_2} u \right. \\
&\left. + u^{p-2} \partial_x^{j_0+2} u \, \partial_x^{j_0} u \, \partial_x^{j_1} u \, \partial_x^{j_2+1} u \right) \\
&- (p-2) \int u^{p-3} \partial_x u \, \partial_x^{j_0+2} u \, \partial_x^{j_0} u \, \partial_x^{j_1} u \, \partial_x^{j_2} u \\
&= (\mathrm{I}_A + \mathrm{II}_A + \mathrm{III}_A) + \mathrm{IV}_A. \tag{3.22}
\end{aligned}
$$

As $\mathrm{I}_A = -\frac{1}{2} \int u^{p-2} \partial_x((\partial_x^{j_0+1} u)^2) \partial_x^{j_1} u \, \partial_x^{j_2} u$, by integration by parts,

$$
\begin{aligned}
\mathrm{I}_A = &\frac{1}{2} \int u^{p-2} (\partial_x^{j_0+1} u)^2 \partial_x^{j_1+1} u \, \partial_x^{j_2} u + \frac{1}{2} \int u^{p-2} (\partial_x^{j_0+1} u)^2 \partial_x^{j_1} u \, \partial_x^{j_2+1} u \\
&+ \frac{p-2}{2} \int u^{p-3} \partial_x u (\partial_x^{j_0+1} u)^2 \partial_x^{j_1} u \, \partial_x^{j_2} u.
\end{aligned}
$$

For II_A, again by integration by parts we move one derivative from factor $\partial_x^{j_0+2} u$ onto the others:

$$
\begin{aligned}
\mathrm{II}_A = &\int u^{p-2} (\partial_x^{j_0+1} u)^2 \partial_x^{j_1+1} u \, \partial_x^{j_2} u + \underbrace{\int u^{p-2} \partial_x^{j_0+1} u \, \partial_x^{j_0} u \, \partial_x^{j_1+2} u \, \partial_x^{j_2} u}_{\equiv 0} \\
&+ \underbrace{\int u^{p-2} \partial_x^{j_0+1} u \, \partial_x^{j_0} u \, \partial_x^{j_1+1} u \, \partial_x^{j_2+1} u}_{\equiv 0} \\
&+ (p-2) \underbrace{\int u^{p-3} \partial_x u \, \partial_x^{j_0+1} u \, \partial_x^{j_0} u \, \partial_x^{j_1+1} u \, \partial_x^{j_2} u}_{\equiv 0} .
\end{aligned}
$$

Indeed, the second, third and fourth terms on the r.h.s. are equivalent to zero as, with $\Delta(j_0, j_1, j_2) \geq 2$, in the corresponding densities we have at most one derivative of order $j_0 + 1$ and all the others are lower order.

Similarly, for the third term III_A we get by integration by parts,

$$\mathrm{III}_A = \int u^{p-2}(\partial_x^{j_0+1}u)^2\partial_x^{j_1}u\partial_x^{j_2+1}u + \underbrace{\int u^{p-2}\partial_x^{j_0+1}u\partial_x^{j_0}u\partial_x^{j_1+1}u\partial_x^{j_2+1}u}_{\equiv 0}$$

$$+ \underbrace{\int u^{p-2}\partial_x^{j_0+1}u\partial_x^{j_0}u\partial_x^{j_1}u\partial_x^{j_2+2}u}_{\equiv 0} + (p-2)\underbrace{\int u^{p-3}\partial_x u\partial_x^{j_0+1}u\partial_x^{j_0}u\partial_x^{j_1}u\partial_x^{j_2+1}u}_{\equiv 0},$$

where similarly to II_A we get that the second, third and fourth terms on the r.h.s. are equivalent to zero from $\Delta(j_0, j_1, j_2) \geq 2$.

Summarizing, we get from (3.22) the following equivalence:

$$A \equiv \frac{3}{2}\int u^{p-2}(\partial_x^{j_0+1}u)^2\partial_x^{j_1+1}u\partial_x^{j_2}u + \frac{3}{2}\int u^{p-2}(\partial_x^{j_0+1}u)^2\partial_x^{j_1}u\partial_x^{j_2+1}u$$

$$+ \frac{p-2}{2}\int u^{p-3}\partial_x u(\partial_x^{j_0+1}u)^2\partial_x^{j_1}u\partial_x^{j_2}u$$

$$- (p-2)\int u^{p-3}\partial_x u\partial_x^{j_0+2}u\partial_x^{j_0}u\partial_x^{j_1}u\partial_x^{j_2}u. \tag{3.23}$$

Notice that no integration by parts was performed on IV_A as its contribution will depend on the value of j_2. Next we consider two subcases.

First subcase: $\Delta(j_0, j_1, j_2) \geq 2$, $j_2 > 0$. We have $\int u^{p-3}\partial_x u\partial_x^{j_0+2}u\partial_x^{j_0}u\partial_x^{j_1}u\partial_x^{j_2}u \equiv 0$ as the density under the integral has five factors with a nontrivial derivative. Similarly we have $\int u^{p-3}\partial_x u(\partial_x^{j_0+1}u)^2\partial_x^{j_1}u\partial_x^{j_2}u \equiv 0$, hence by (3.23) we get

$$A \equiv \frac{3}{2}\int u^{p-2}(\partial_x^{j_0+1}u)^2\partial_x^{j_1+1}u\partial_x^{j_2}u + \frac{3}{2}\int u^{p-2}(\partial_x^{j_0+1}u)^2\partial_x^{j_1}u\partial_x^{j_2+1}u.$$

Summarizing, we obtain that, for $j_1 = j_2$,

$$A \equiv 3\int \mathcal{I}_{j_0+1,j_1+1,j_2}(u), \tag{3.24}$$

while for $j_1 > j_2$,

$$A \equiv \frac{3}{2}\int \mathcal{I}_{j_0+1,j_1+1,j_2}(u). \tag{3.25}$$

In any case by (3.24), (3.25), (3.21) and (3.19) we get the desired conclusion in this subcase.

Second subcase: $\Delta(j_0, j_1, j_2) \geq 2$, $j_2 = 0$. By (3.23), we get

$$A \equiv \frac{3}{2}\int u^{p-1}(\partial_x^{j_0+1}u)^2\partial_x^{j_1+1}u + \frac{3}{2}\int u^{p-2}(\partial_x^{j_0+1}u)^2\partial_x^{j_1}u\partial_x u$$

$$+ \frac{p-2}{2}\int u^{p-2}\partial_x u(\partial_x^{j_0+1}u)^2\partial_x^{j_1}u - (p-2)\int u^{p-2}\partial_x u\partial_x^{j_0+2}u\partial_x^{j_0}u\partial_x^{j_1}u$$

$$= \mathrm{I}_A + \mathrm{II}_A + \mathrm{III}_A + \mathrm{IV}_A. \tag{3.26}$$

For the term IV_A, we move one derivative from factor $\partial_x^{j_0+2}u$ onto the other factors,

$$IV_A = (p-2)\int u^{p-2}\partial_x u(\partial_x^{j_0+1}u)^2\partial_x^{j_1}u + (p-2)\underbrace{\int u^{p-2}\partial_x u\partial_x^{j_0+1}u\partial_x^{j_0}u\partial_x^{j_1+1}u}_{\equiv 0}$$

$$+ (p-2)\underbrace{\int u^{p-2}\partial_x^2 u\partial_x^{j_0+1}u\partial_x^{j_0}u\partial_x^{j_1}u}_{\equiv 0}$$

$$+ (p-2)^2\underbrace{\int u^{p-3}(\partial_x u)^2\partial_x^{j_0+1}u\partial_x^{j_0}u\partial_x^{j_1}u}_{\equiv 0}\,.$$

Again, the second, third and fourth terms on the r.h.s. are equivalent to zero. Indeed, at most one derivative involved in the densities is of order j_0+1 and all the others are lower order: from $\Delta(j_0,j_1,j_2)\geq 2$ we get $j_0+1\geq 3$ and that settles the third term on the r.h.s.; for the fourth term the argument is even easier and for the second term, $\Delta(j_0,j_1,j_2)\geq 2$ implies $j_1+1 < j_0+1$. Hence we get from (3.26),

$$A \equiv \frac{3}{2}\int u^{p-1}(\partial_x^{j_0+1}u)^2\partial_x^{j_1+1}u + \frac{3}{2}\int u^{p-2}(\partial_x^{j_0+1}u)^2\partial_x^{j_1}u\partial_x u$$

$$+ \frac{3}{2}(p-2)\int u^{p-2}\partial_x u(\partial_x^{j_0+1}u)^2\partial_x^{j_1}u.$$

Notice that if $j_1 > 0$ then the second and third terms on the r.h.s. in (3.26) are equivalent to zero: the corresponding densities involve two derivatives of order j_0+1 but the remaining derivatives necessarily have order below j_1+1, and we conclude that

$$A \equiv \frac{3}{2}\int \mathcal{I}_{j_0+1,j_1+1,j_2}(u). \tag{3.27}$$

If $j_1 = 0$, densities are multiples of $\int u^{p-1}(\partial_x^{j_0+1}u)^2\partial_x u$. By computing the sum of the coefficients we get

$$A \equiv \frac{3}{2}p\int \mathcal{I}_{j_0+1,j_1+1,j_2}(u). \tag{3.28}$$

Combining (3.27), (3.28), (3.21) and (3.19) we get the desired conclusion in this subcase.

Second case: $\Delta(j_0,j_1,j_2) = 1$. As $j_0 = j_1+1$ we have by integration by parts (see (3.20) for the definition of B),

$$B = \int u^{p-2}(\partial_x^{j_0}u)^2\partial_x^{j_0+2}u\partial_x^{j_2}u$$

$$= -\underbrace{\int u^{p-2}(\partial_x^{j_0}u)^2\partial_x^{j_0+1}u\partial_x^{j_2+1}u}_{\equiv 0} - 2\int u^{p-2}\partial_x^{j_0}u(\partial_x^{j_0+1}u)^2\partial_x^{j_2}u$$

$$- (p-2)\underbrace{\int u^{p-3}\partial_x u(\partial_x^{j_0}u)^2\partial_x^{j_0+1}u\partial_x^{j_2}u}_{\equiv 0}\,,$$

where the first and third terms on the r.h.s. are equivalent to zero, as one can easily check that only one derivative involved in the corresponding densities has order $j_0 + 1$ and all the others are lower order. Then

$$B \equiv -2 \int \mathcal{I}_{j_0+1, j_1+1, j_2}(u). \tag{3.29}$$

Moreover, we have (see (3.20)) $A = \int u^{p-2} \partial_x^{j_0+3} u \partial_x^{j_0} u \partial_x^{j_0-1} u \partial_x^{j_2} u$, and if we move one derivative by integration by parts, from $\partial_x^{j_0+3} u$ onto the other factors we get

$$A = - \int u^{p-2} \left(\partial_x^{j_0+2} u \partial_x^{j_0+1} u \partial_x^{j_0-1} u \partial_x^{j_2} u + u^{p-2} \partial_x^{j_0+2} u (\partial_x^{j_0} u)^2 \partial_x^{j_2} u \right.$$
$$\left. + u^{p-2} \partial_x^{j_0+2} u \partial_x^{j_0} u \partial_x^{j_0-1} u \partial_x^{j_2+1} u \right)$$
$$- (p-2) \int u^{p-3} \partial_x u \partial_x^{j_0+2} u \partial_x^{j_0} u \partial_x^{j_0-1} u \partial_x^{j_2} u$$
$$= (\mathrm{I}_A + \mathrm{II}_A + \mathrm{III}_A) + \mathrm{IV}_A. \tag{3.30}$$

We have $\mathrm{I}_A = -\frac{1}{2} \int u^{p-2} \partial_x((\partial_x^{j_0+1} u)^2) \partial_x^{j_0-1} u \partial_x^{j_2} u$ and by integration by parts,

$$\mathrm{I}_A = \frac{1}{2} \int u^{p-2} (\partial_x^{j_0+1} u)^2 \partial_x^{j_0} u \partial_x^{j_2} u + \frac{1}{2} \int u^{p-2} (\partial_x^{j_0+1} u)^2 \partial_x^{j_0-1} u \partial_x^{j_2+1} u$$
$$+ \frac{p-2}{2} \int u^{p-3} \partial_x u (\partial_x^{j_0+1} u)^2 \partial_x^{j_0-1} u \partial_x^{j_2} u.$$

Regarding II_A, integration by parts moves one derivative from $\partial_x^{j_0+2} u$ onto the other factors and we get

$$\mathrm{II}_A = 2 \int u^{p-2} (\partial_x^{j_0+1} u)^2 \partial_x^{j_0} u \partial_x^{j_2} u + \underbrace{\int u^{p-2} \partial_x^{j_0+1} u (\partial_x^{j_0} u)^2 \partial_x^{j_2+1} u}_{\equiv 0}$$
$$+ (p-2) \underbrace{\int u^{p-3} \partial_x u \partial_x^{j_0+1} u (\partial_x^{j_0} u)^2 \partial_x^{j_2} u}_{\equiv 0}.$$

Again, the second and third terms on the r.h.s. are equivalent to zero, as $\Delta(j_0, j_1, j_2) = 1$ implies at most one derivative in the corresponding densities has order $j_0 + 1$ and all the others are lower order.

Similarly, by integration by parts, we rewrite III_A as

$$\mathrm{III}_A = \int u^{p-2} (\partial_x^{j_0+1} u)^2 \partial_x^{j_0-1} u \partial_x^{j_2+1} u + \underbrace{\int u^{p-2} \partial_x^{j_0+1} u (\partial_x^{j_0} u)^2 \partial_x^{j_2+1} u}_{\equiv 0}$$
$$+ \int u^{p-2} \partial_x^{j_0+1} u \partial_x^{j_0} u \partial_x^{j_0-1} u \partial_x^{j_2+2} u$$
$$+ (p-2) \underbrace{\int u^{p-3} \partial_x u \partial_x^{j_0+1} u \partial_x^{j_0} u \partial_x^{j_0-1} u \partial_x^{j_2+1} u}_{\equiv 0}.$$

Again, on the r.h.s. we have terms equivalent to zero as, exactly as above, in the corresponding densities only one derivative has order $j_0 + 1$ and all the others are lower order.

By integration by parts, moving one derivative from $\partial_x^{j_0+2} u$ onto other factors, we get

$$
\mathrm{IV}_A = (p-2) \int u^{p-3} \partial_x u (\partial_x^{j_0+1} u)^2 \partial_x^{j_0-1} u \partial_x^{j_2} u
$$

$$
+ (p-2) \underbrace{\int u^{p-3} \partial_x u \partial_x^{j_0+1} u (\partial_x^{j_0} u)^2 \partial_x^{j_2} u}_{\equiv 0}
$$

$$
+ (p-2) \underbrace{\int u^{p-3} \partial_x u \partial_x^{j_0+1} u \partial_x^{j_0} u \partial_x^{j_0-1} u \partial_x^{j_2+1} u}_{\equiv 0}
$$

$$
+ (p-2) \int u^{p-3} \partial_x^2 u \partial_x^{j_0+1} u \partial_x^{j_0} u \partial_x^{j_0-1} u \partial_x^{j_2} u
$$

$$
+ (p-2)(p-3) \underbrace{\int u^{p-4} (\partial_x u)^2 \partial_x^{j_0+1} u \partial_x^{j_0} u \partial_x^{j_0-1} u \partial_x^{j_2} u}_{\equiv 0} .
$$

On the r.h.s. we get several terms equivalent to zero as, once again, the corresponding densities have one derivative of order $j_0 + 1$ and all the others are lower order. Summarizing, we get from (3.30) the equivalence

$$
A \equiv \frac{1}{2} \int u^{p-2} (\partial_x^{j_0+1} u)^2 \partial_x^{j_0} u \partial_x^{j_2} u + \frac{1}{2} \int u^{p-2} (\partial_x^{j_0+1} u)^2 \partial_x^{j_0-1} u \partial_x^{j_2+1} u
$$

$$
+ \frac{p-2}{2} \int u^{p-3} \partial_x u (\partial_x^{j_0+1} u)^2 \partial_x^{j_0-1} u \partial_x^{j_2} u + 2 \int u^{p-2} (\partial_x^{j_0+1} u)^2 \partial_x^{j_0} u \partial_x^{j_2} u
$$

$$
+ \int u^{p-2} (\partial_x^{j_0+1} u)^2 \partial_x^{j_0-1} u \partial_x^{j_2+1} u + \int u^{p-2} \partial_x^{j_0+1} u \partial_x^{j_0} u \partial_x^{j_0-1} u \partial_x^{j_2+2} u
$$

$$
+ (p-2) \int u^{p-3} \partial_x u (\partial_x^{j_0+1} u)^2 \partial_x^{j_0-1} u \partial_x^{j_2} u
$$

$$
+ (p-2) \int u^{p-3} \partial_x^2 u \partial_x^{j_0+1} u \partial_x^{j_0} u \partial_x^{j_0-1} u \partial_x^{j_2} u. \tag{3.31}
$$

Moreover, by (3.20) and integration by parts,

$$
D = \int u^{p-3} \partial_x^3 u (\partial_x^{j_0} u)^2 \partial_x^{j_0-1} u \partial_x^{j_2} u
$$

$$
= -2 \int u^{p-3} \partial_x^2 u \partial_x^{j_0+1} u \partial_x^{j_0} u \partial_x^{j_0-1} u \partial_x^{j_2} u - \underbrace{\int u^{p-3} \partial_x^2 u (\partial_x^{j_0} u)^3 \partial_x^{j_2} u}_{\equiv 0}
$$

$$
- \underbrace{\int u^{p-3} \partial_x^2 u (\partial_x^{j_0} u)^2 \partial_x^{j_0-1} u \partial_x^{j_2+1} u}_{\equiv 0} - (p-3) \underbrace{\int u^{p-4} \partial_x u \partial_x^2 u (\partial_x^{j_0} u)^2 \partial_x^{j_0-1} u \partial_x^{j_2} u}_{\equiv 0} ,
$$

where three terms on the r.h.s. are equivalent to zero as the corresponding densities have at most one derivative of order $j_0 + 1$ and all the others are lower order. Hence we get

$$D \equiv -2 \int u^{p-3} \partial_x^2 u \partial_x^{j_0+1} u \partial_x^{j_0} u \partial_x^{j_0-1} u \partial_x^{j_2} u. \tag{3.32}$$

Next we split into three subcases.

First subcase: $\Delta(j_0, j_1, j_2) = 1$, $j_1 > j_2$. Here all terms except the first and the fourth on the r.h.s. in (3.31) are equivalent to zero and

$$A \equiv \frac{5}{2} \int \mathcal{I}_{j_0+1, j_1+1, j_2}(u). \tag{3.33}$$

In fact, densities involved at the second, third, fifth and seventh terms on the r.h.s. in (3.31) involve two derivatives of order $j_0 + 1$ but all the remaining factors involve derivatives of order less than j_0. Notice also that sixth and eighth terms on the r.h.s. in (3.31) are equivalent to zero as, in the corresponding densities, only one derivative has order $j_0 + 1$ and all the remaining ones are lower order. Moreover, $j_2 + 3 \le j_0 + 1$ and (3.20) yield

$$C \equiv 0 \tag{3.34}$$

as at most one derivative in the corresponding density is of order $j_0 + 1$ and all the others have lower order. Notice also that

$$D \equiv 0 \tag{3.35}$$

from (3.32), as necessarily $j_0 > 1$, hence only one derivative in the density is of order $j_0 + 1$ and all the others are lower order. We conclude by combining (3.33), (3.34), (3.35), (3.29) and (3.19).

Second subcase: $\Delta(j_0, j_1, j_2) = 1$, $j_1 = j_2 > 0$. All terms on the r.h.s. in (3.31) are equivalent to zero, except the first, second, fourth, fifth, sixth, and hence

$$A \equiv 5 \int \mathcal{I}_{j_0+1, j_1+1, j_2}(u). \tag{3.36}$$

In fact, one can check that the remaining terms on the r.h.s. in (3.31) are equivalent to zero as either the corresponding densities involve at most one derivative of order $j_0 + 1$ and the others are lower order, or two derivatives are of order $j_0 + 1$ but the other derivatives are of order less than j_0. Moreover, we have from (3.20),

$$C = \int u^{p-2} (\partial_x^{j_0} u)^2 \partial_x^{j_0-1} u \partial_x^{j_0+2} u$$

$$= -\underbrace{\int u^{p-2} (\partial_x^{j_0} u)^2 \partial_x^{j_0} u \partial_x^{j_0+1} u}_{\equiv 0} - 2 \int u^{p-2} \partial_x^{j_0} u \partial_x^{j_0-1} u (\partial_x^{j_0+1} u)^2$$

$$- (p-2) \underbrace{\int u^{p-3} \partial_x u (\partial_x^{j_0} u)^2 \partial_x^{j_0-1} u \partial_x^{j_0+1} u}_{\equiv 0},$$

where we used integration by parts and noticed that the first and third terms on the r.h.s. are equivalent to zero as the corresponding densities have only one derivative of order $j_0 + 1$ and the others are lower order. Hence,

$$C \equiv -2 \int \mathcal{I}_{j_0+1, j_1+1, j_2}(u). \tag{3.37}$$

Notice also that by (3.32) and $j_0 > 1$, we get

$$D \equiv 0 \tag{3.38}$$

as in the corresponding density at most one derivative is of order $j_0 + 1$. We conclude this subcase by combining (3.36), (3.37), (3.38), (3.29) and (3.19).

Third subcase: $\Delta(j_0, j_1, j_2) = 1$, $j_1 = j_2 = 0$. On the r.h.s. in (3.31) no term is equivalent to zero; indeed, all of them are equivalent to $\int \mathcal{I}_{j_0+1, j_1+1, j_2}(u) \, dx$, hence

$$A \equiv \frac{5}{2} p \int \mathcal{I}_{j_0+1, j_1+1, j_2}(u). \tag{3.39}$$

Notice also that in this subcase, by (3.20) we get by integration by parts,

$$\begin{aligned}
C &= \int u^{p-1} (\partial_x u)^2 \partial_x^3 u \\
&= -2 \int u^{p-1} \partial_x u (\partial_x^2 u)^2 - (p-1) \underbrace{\int u^{p-2} (\partial_x u)^3 \partial_x^2 u}_{\equiv 0},
\end{aligned}$$

where the second term on the r.h.s. is equivalent to zero as only one derivative has order $j_0 + 1$ and all the other factors involve derivatives of order less than $j_0 + 1$. Hence we get

$$C \equiv -2 \int \mathcal{I}_{j_0+1, j_1+1, j_2}(u). \tag{3.40}$$

Moreover, by (3.32) we get

$$D \equiv -2 \int u^{p-1} (\partial_x^2 u)^2 \partial_x u \, dx = -2 \int \mathcal{I}_{j_0+1, j_1+1, j_2}(u). \tag{3.41}$$

We conclude this subcase by combining (3.39), (3.40), (3.41), (3.29) and (3.19).

Third case: $\Delta(j_0, j_1, j_2) = 0$. First of all, we have by (3.20),

$$D = \int u^{p-3} \partial_x^3 u (\partial_x^{j_0} u)^2 \partial_x^{j_0} u \partial_x^{j_2} u.$$

For $j_2 > 0$ we get $D \equiv 0$, as the density on the r.h.s. involves five factors with nontrivial derivatives. For $j_2 = 0$ we have, using $2j_0 + j_1 + j_2 = 2k - 2$, that j_1 is an even number and as $j_1 = j_0$ we necessarily have $j_0 \geq 2$, and hence

$$D = \int u^{p-2} \partial_x^3 u (\partial_x^{j_0} u)^2 \partial_x^{j_0} u \equiv 0,$$

where in the density above at most one derivative has order $j_0 + 1$ due to the fact that $j_0 \geq 2$ and all the other factors have lower-order derivatives. Summarizing, we have

$$D \equiv 0. \tag{3.42}$$

Notice also that in this case, by (3.20) we have

$$A \equiv B, \tag{3.43}$$

hence we focus on A. By definition of A (see (3.20)) and integration by parts, we get

$$
\begin{aligned}
A &= \int u^{p-2} \partial_x^{j_0+3} u (\partial_x^{j_0} u)^2 \partial_x^{j_2} u \\
&= -2 \int u^{p-2} \partial_x^{j_0+2} u \partial_x^{j_0+1} u \partial_x^{j_0} u \partial_x^{j_2} u - \int u^{p-2} \partial_x^{j_0+2} u (\partial_x^{j_0} u)^2 \partial_x^{j_2+1} u \\
&\quad - (p-2) \int u^{p-3} \partial_x u \partial_x^{j_0+2} u (\partial_x^{j_0} u)^2 \partial_x^{j_2} u \\
&= \mathrm{I}_A + \mathrm{II}_A + \mathrm{III}_A.
\end{aligned}
\tag{3.44}
$$

We have $\mathrm{I}_A = -\int u^{p-2} \partial_x ((\partial_x^{j_0+1} u)^2) \partial_x^{j_0} u \partial_x^{j_2} u$ and, by integration by parts,

$$
\begin{aligned}
\mathrm{I}_A &= \int u^{p-2} (\partial_x^{j_0+1} u)^2 \partial_x^{j_0+1} u \partial_x^{j_2} u \\
&\quad + \int u^{p-2} (\partial_x^{j_0+1} u)^2 \partial_x^{j_0} u \partial_x^{j_2+1} u \\
&\quad + (p-2) \underbrace{\int u^{p-3} \partial_x u (\partial_x^{j_0+1} u)^2 \partial_x^{j_0} u \partial_x^{j_2} u}_{\equiv 0}.
\end{aligned}
$$

The third term on the r.h.s. is equivalent to zero as the density involves two factors with derivatives of order $j_0 + 1$ and the other factors involve derivatives of order less than $j_1 + 1 = j_0 + 1$. By integration by parts, we move one derivative from $\partial_x^{j_0+2} u$ on the other factors and write the second term on the r.h.s. in (3.44) as

$$
\begin{aligned}
\mathrm{II}_A &= 2 \int u^{p-2} (\partial_x^{j_0+1} u)^2 \partial_x^{j_0} u \partial_x^{j_2+1} u \\
&\quad + \int u^{p-2} \partial_x^{j_0+1} u (\partial_x^{j_0} u)^2 \partial_x^{j_2+2} u \\
&\quad + (p-2) \underbrace{\int u^{p-3} \partial_x u \partial_x^{j_0+1} u (\partial_x^{j_0} u)^2 \partial_x^{j_2+1} u}_{\equiv 0},
\end{aligned}
$$

where the third term on the r.h.s. is equivalent to zero as the corresponding density involves five factors with nontrivial derivatives. Similarly, by integration by parts we write the third

term on the r.h.s. in (3.44) as

$$\mathrm{III}_A = 2(p-2) \underbrace{\int u^{p-3} \partial_x u (\partial_x^{j_0+1} u)^2 \partial_x^{j_0} u \partial_x^{j_2} u}_{\equiv 0}$$

$$+ (p-2) \underbrace{\int u^{p-3} \partial_x u \partial_x^{j_0+1} u (\partial_x^{j_0} u)^2 \partial_x^{j_2+1} u}_{\equiv 0}$$

$$+ (p-2) \underbrace{\int u^{p-3} \partial_x^2 u \partial_x^{j_0+1} u (\partial_x^{j_0} u)^2 \partial_x^{j_2} u}_{\equiv 0}$$

$$+ (p-2)(p-3) \underbrace{\int u^{p-4} (\partial_x u)^2 \partial_x^{j_0+1} u (\partial_x^{j_0} u)^2 \partial_x^{j_2} u}_{\equiv 0}.$$

The first term on the r.h.s. is equivalent to zero as it is equal to the third term on the r.h.s. of I_A above. Concerning the second and fourth terms on the r.h.s., they are equivalent to zero as in the corresponding densities there are at least five factors involving nontrivial derivatives. Finally, the third term on the r.h.s. is zero as at most two factors may have $j_0 + 1$ derivatives and all the other terms have derivatives of order less than $j_0 + 1 = j_1 + 1$. Summarizing, we get from (3.44),

$$A \equiv \int \mathcal{I}_{j_0+1,j_1+1,j_2}(u) + 3 \int u^{p-2} (\partial_x^{j_0+1} u)^2 \partial_x^{j_0} u \partial_x^{j_2+1} u$$

$$+ \int u^{p-2} \partial_x^{j_0+1} u (\partial_x^{j_0} u)^2 \partial_x^{j_2+2} u. \tag{3.45}$$

Next we split into two subcases.

First subcase: $\Delta(j_0, j_1, j_2) = 0$, $j_0 = j_1 = j_2$. We have from (3.20),

$$A = B = C = \int u^{p-2} \partial_x^{j_0+3} u \partial_x^{j_0} u \partial_x^{j_0} u \partial_x^{j_0} u \tag{3.46}$$

and, moreover, by (3.45) we get

$$A \equiv \int \mathcal{I}_{j_0+1,j_1+1,j_2}(u) + 3 \int u^{p-2} (\partial_x^{j_0+1} u)^3 \partial_x^{j_0} u + \int u^{p-2} \partial_x^{j_0+1} u (\partial_x^{j_0} u)^2 \partial_x^{j_0+2} u$$

$$= 4 \int \mathcal{I}_{j_0+1,j_1+1,j_2}(u) + \frac{1}{2} \int u^{p-2} (\partial_x^{j_0} u)^2 \partial_x ((\partial_x^{j_0+1} u)^2)$$

$$= 4 \int \mathcal{I}_{j_0+1,j_1+1,j_2}(u) - \int u^{p-2} \partial_x^{j_0} u (\partial_x^{j_0+1} u)^3$$

$$- \frac{p-2}{2} \underbrace{\int u^{p-3} \partial_x u (\partial_x^{j_0} u)^2 (\partial_x^{j_0+1} u)^2}_{\equiv 0},$$

where again we used integration by parts, and the third term on the r.h.s. is equivalent to zero as the corresponding density involves five factors with nontrivial derivatives. Summarizing, we get

$$A \equiv 3 \int \mathcal{I}_{j_0+1, j_1+1, j_2}(u). \tag{3.47}$$

We conclude by (3.47), (3.42), (3.46) and (3.19).

Second subcase: $j_0 = j_1 > j_2$. From (3.45) we get

$$A \equiv \int \mathcal{I}_{j_0+1, j_1+1, j_2}(u) + 3 \underbrace{\int u^{p-2}(\partial_x^{j_0+1} u)^2 \partial_x^{j_0} u \, \partial_x^{j_2+1} u}_{\equiv 0}$$

$$+ \underbrace{\int u^{p-2} \partial_x^{j_0+1} u (\partial_x^{j_0} u)^2 \partial_x^{j_2+2} u}_{\equiv 0},$$

where the second term on the r.h.s. is equivalent to zero as the density involves two derivatives of order $j_0 + 1$ and all the other derivatives have order less than $j_0 + 1$, while the density of the third term on the r.h.s. involves at most two derivatives of order $j_0 + 1$ and all the other derivatives have order at most j_0. Summarizing, we get

$$A \equiv \int \mathcal{I}_{j_0+1, j_1+1, j_2}(u). \tag{3.48}$$

Next, $2j_0 + j_1 + j_2 = 2k - 2$ implies that $j_1 + j_2$ is even and as $j_1 > j_2$ we get

$$j_2 + 2 \le j_1 = j_0. \tag{3.49}$$

On the other hand, by (3.20) we have

$$C = \int u^{p-2}(\partial_x^{j_0} u)^2 \partial_x^{j_0} u \, \partial_x^{j_2+3} u \equiv 0, \tag{3.50}$$

where on the r.h.s. the density involves at most one derivative of order $j_0 + 1$ (see (3.49)) and all the other factors have derivatives of order less than j_0. We conclude by (3.42), (3.43), (3.48), (3.50) and (3.19). ∎

4. Proof of Theorem 1.3

The proof relies heavily on Proposition 3.1. We set $\bar{\mathcal{R}}_k(u)$ to be the density from the r.h.s. of (3.4) and $\bar{\mathcal{G}}_{k,\infty}^T(\varphi)$ to be the r.h.s. in (3.6). First, we will prove (1.12). For $M = \infty$, identity (1.11) follows from integrating the identity from Proposition 3.1 and a density argument like the one we used to prove (1.5). Second, we will prove (1.13) in the case $M = \infty$. In a further step, we will define, for $M \in \mathbb{N}$, the functionals $\bar{\mathcal{G}}_{k,M}^T(\varphi)$ in such a way that (1.11) occurs and we will prove the bound (1.13) for $M \in \mathbb{N}$. Constructing $\bar{\mathcal{G}}_{k,M}^T(\varphi)$ will require a slight modification of $\bar{\mathcal{G}}_{k,\infty}^T(\varphi)$. Finally, we will prove (1.13) for $M \in \mathbb{N}$ to conclude with the proof of (1.14).

4.1. Proof of (1.12)

The proof of (1.12) follows the same ideas used in the proof of (1.7). Recall that the structure of $\bar{\mathcal{R}}_k(u)$ is provided in (3.4) and hence by repeating the same computations as in (3.5), except for a modification of the interpolation estimates at the last step, we get

$$\left| \int (\partial_x^{j_0} u)^2 \partial_x^{j_1} u \partial_x^{j_2} u u^{p-2} \right| \leq C \|u\|_{H^{j_0}}^2 \|u\|_{H^{j_1+1}} \|u\|_{H^{j_2+1}} \|u\|_{H^1}^{p-2}$$

$$\leq C \|u\|_{H^{(k-\frac{1}{2})^-}}^{2\theta_0+\theta_1+\theta_2} \|u\|_{H^1}^{2-2\theta_0-\theta_1-\theta_2+p},$$

where $\theta_0(k-\frac{1}{2})^- + (1-\theta_0) = j_0$, $\theta_l(k-\frac{1}{2})^- + (1-\theta_l) = j_l + 1$ for $l = 1, 2$. We conclude (1.12) since we have

$$2\theta_0 + \theta_1 + \theta_2 = \left(\frac{4k-8}{2k-3} \right)^+.$$

4.2. Proof of (1.13) in the case $M = \infty$

Since the expression $\bar{\mathcal{G}}_{k,\infty}^T(\varphi)$ is defined by the r.h.s. in (3.6) we can adapt the proof of (1.8) in order to get (1.13) in the case $M = \infty$. In fact it is sufficient to prove the following version of (3.7) and (3.8) for any $R, T > 0$ and for all $\varphi \in H^k$ such that $\|\varphi\|_{H^1} < R$:

$$\left| \int_0^T \int \prod_{l=0}^{p+1} \partial_x^{j_l} v \right| \leq C + C \|\varphi\|_{H^{(k-\frac{1}{2})^-}}^{\left(\frac{4k-8}{2k-3} \right)^+}, \quad j_{p+1} \leq \cdots \leq j_0, \sum_{l=0}^{p+1} j_l = 2k+1,$$
$$j_4 \geq 1, \quad (4.1)$$

$$\left| \int_0^T \int \prod_{l=0}^{2p+1} \partial_x^{j_l} v \right| \leq C + C \|\varphi\|_{H^{(k-\frac{1}{2})^-}}^{\left(\frac{4k-8}{2k-3} \right)^+}, \quad j_{2p+1} \leq \cdots \leq j_0, \sum_{l=0}^{2p+1} j_l \leq 2k-1,$$
$$j_2 \geq 1. \quad (4.2)$$

The proof of (4.1) follows, exactly as for the proof of (3.7), by considering two subcases: $j_5 = 0$ and $j_5 \neq 0$. In the first case we start from the estimate (3.9) and the subsequent computation, which imply

$$\left| \int_0^T \int \prod_{l=0}^{p+1} \partial_x^{j_l} v \right| \leq C \left(\prod_{l=0}^4 \|\varphi\|_{H^{j_l^+}} \right) \|\varphi\|_{H^1}^{p-3},$$

and hence by interpolation we can continue the estimate as follows:

$$(\cdots) \leq C \left(\prod_{l=0}^4 \|\varphi\|_{H^{(k-\frac{1}{2})^-}}^{\sum_{l=0}^4 \theta_l} \right) \|\varphi\|_{H^1}^{p+2-\sum_{l=0}^4 \theta_l},$$

where $\theta_l(k-\frac{1}{2})^- + (1-\theta_l) = j_l^+$ for $l = 0, \ldots, 4$, namely $\theta_l = \frac{j_l^+ - 1}{k - \frac{3}{2}}$ and

$$\theta_0 + \theta_1 + \theta_2 + \theta_3 + \theta_4 = \left(\frac{4k-8}{2k-3} \right)^+.$$

Hence we get (4.1) in the case $j_5 = 0$. In order to establish (4.1) in the case $j_5 \neq 0$ we argue as in the proof of (3.7) for $j_5 \neq 0$ and hence following (3.10) we get

$$\left| \int_0^T \int \prod_{l=0}^{p+1} \partial_x^{j_l} v \right| \leq C \prod_{l=0}^{5} \|\varphi\|_{H^{j_l^+}} \prod_{l=6}^{p+1} \|v\|_{L^\infty((0,T);H^{j_l^++1})}$$

$$\leq C \prod_{l=0}^{5} \|\varphi\|_{H^{j_l^+}} \prod_{l=6}^{p+1} \|\varphi\|_{H^{j_l^++1}},$$

and we can continue the estimate as follows by interpolation:

$$(\cdots) \leq C \prod_{l=0}^{5} (\|\varphi\|_{H^1}^{1-\theta_l} \|\varphi\|_{H^{(k-\frac{1}{2})^-}}^{\theta_l}) \prod_{l=6}^{p+1} (\|\varphi\|_{H^1}^{1-\eta_l} \|\varphi\|_{H^{(k-\frac{1}{2})^-}}^{\eta_l})$$

$$= C \|\varphi\|_{H^{(k-\frac{1}{2})^-}}^{\sum_{l=0}^{5} \theta_l + \sum_{l=6}^{p+1} \eta_l} \|\varphi\|_{H^1}^{p+2-\sum_{l=0}^{5} \theta_l - \sum_{l=6}^{p+1} \eta_l},$$

where $\theta_l(k - \frac{1}{2})^- + (1 - \theta_l) = j_l^+$ for $l = 0, \ldots, 5$, $\eta_l(k - \frac{1}{2})^- + (1 - \eta_l) = j_l^+ + 1$ for $l = 6, \ldots, p + 1$. We conclude (4.1) for $j_5 \neq 0$ since

$$\sum_{l=0}^{5} \theta_l + \sum_{l=6}^{p+1} \eta_l = \left(\frac{4k - 10}{2k - 3} \right)^+$$

(notice that in fact we get in this case a stronger version of (4.1) since we get on the r.h.s. an exponent even smaller than $(\frac{4k-8}{2k-3})$).

In order to establish (4.2) we argue as in the proof of (3.8) and hence we split into the following subcases: $j_3 = 0$ and $j_3 \neq 0$. In the case $j_3 = 0$ we can argue as in (3.11) and the subsequent computations in order to get

$$\left| \int_0^T \int \prod_{l=0}^{2p+1} \partial_x^{j_l} v \right| \leq C \left(\prod_{l=0}^{2} \|\varphi\|_{H^{j_l^+}} \right) \|\varphi\|_{H^1}^{2p-4},$$

and hence by interpolation we can continue the estimate as follows:

$$(\cdots) \leq C \prod_{l=0}^{2} (\|\varphi\|_{H^{(k-\frac{1}{2})^-}}^{\theta_l} \|\varphi\|_{H^1}^{1-\theta_l}) \|\varphi\|_{H^1}^{2p-1} = C \|\varphi\|_{H^{(k-\frac{1}{2})^-}}^{\sum_{l=0}^{2} \theta_l} \|\varphi\|_{H^1}^{2p+2-\sum_{l=0}^{2} \theta_l},$$

where $\theta_l(k - \frac{1}{2})^- + (1 - \theta_l) = j_l^+$ for $l = 0, 1, 2$. We get (4.2) in the case $j_3 = 0$ since $\theta_0 + \theta_1 + \theta_2 = (\frac{4k-8}{2k-3})^+$. In order to prove (4.2) in the case $j_3 \neq 0$, we can argue as in (3.12) and we get

$$\left| \int_0^T \int \prod_{l=0}^{2p+1} \partial_x^{j_l} v \right| \leq C \prod_{l=0}^{3} \|\varphi\|_{H^{j_l^+}} \prod_{l=4}^{2p+1} \|\varphi\|_{H^{j_l^++1}},$$

and we can continue by interpolation as follows:

$$(\cdots) \leq C \prod_{l=0}^{3} (\|\varphi\|_{H^1}^{1-\theta_l} \|\varphi\|_{H^{(k-\frac{1}{2})-}}^{\theta_l}) \prod_{l=4}^{2p+1} (\|\varphi\|_{H^1}^{1-\eta_l} \|\varphi\|_{H^{(k-\frac{1}{2})-}}^{\eta_l})$$

$$\leq C \|\varphi\|_{H^{(k-\frac{1}{2})-}}^{\sum_{l=0}^{3} \theta_l + \sum_{l=4}^{2p+1} \eta_l} \|\varphi\|_{H^1}^{2p+2 - \sum_{l=0}^{5} \theta_l - \sum_{l=6}^{p+1} \eta_l},$$

where $\theta_l (k - \frac{1}{2})^- + (1 - \theta_l) = j_l^+$ for $l = 0, 1, 2, 3$ and $\eta_l (k - \frac{1}{2})^- + (1 - \eta_l) = j_l^+ + 1$ for $l = 4, \ldots, 2p + 1$, then $\sum_{l=0}^{3} \theta_l + \sum_{l=4}^{2p+1} \eta_l = (\frac{4k-10}{2k-3})^+$. Then we conclude (4.2) in the case $j_3 \neq 0$ (in fact we get an even stronger version since we have $(\frac{4k-10}{2k-3})^+ < (\frac{4k-8}{2k-3})$).

4.3. Definition of $\bar{\mathcal{G}}_{k,M}^T$ for $M \in \mathbb{N}$ and proof of (1.13) with constant uniform w.r.t. M

We first introduce the energies $\bar{\mathcal{G}}_{k,M}^T$ in such a way that (1.11) occurs, and we shall also establish (1.13) for $M \in \mathbb{N}$. We set $v_M(t, x) = \pi_M(v(t, x))$, where $v(t, x)$ is the unique solution to (1.3) for a given $M \in \mathbb{N}$. We retain the notation from Section 3, except that we need to alter $\mathcal{I}_{j_0, j_1, j_2}^{**}(u)$ as follows:

Definition 4.1. For any $(j_0, j_1, j_2) \in \mathcal{A}^k$ and for every $M \in \mathbb{N}$ we define

$$\mathcal{I}_{j_0, j_1, j_2, M}^{**}(u) = (p-2) u^{p-3} \pi_M (\partial_x (u^{p+1})) (\partial_x^{j_0} u)^2 \partial_x^{j_1} u \partial_x^{j_2} u$$
$$+ 2 u^{p-2} \partial_x^{j_0} u \pi_M (\partial_x^{j_0+1} (u^{p+1})) \partial_x^{j_1} u \partial_x^{j_2} u$$
$$+ u^{p-2} (\partial_x^{j_0} u)^2 \pi_M (\partial_x^{j_1+1} (u^{p+1})) u \partial_x^{j_2} u$$
$$+ u^{p-2} (\partial_x^{j_0} u)^2 \partial_x^{j_1} u \pi_M (\partial_x^{j_2+1} (u^{p+1})).$$

The main difference between $\mathcal{I}_{j_0, j_1, j_2}^{**}(u)$ and $\mathcal{I}_{j_0, j_1, j_2, M}^{**}(u)$ is the projector π_M. The reason for this modification is that we have the following identity (compare with (3.1)):

$$\frac{d}{dt} \int \mathcal{I}_{j_0, j_1, j_2}(v_M) = \int \mathcal{I}_{j_0, j_1, j_2}^*(v_M) - \int \mathcal{I}_{j_0, j_1, j_2, M}^{**}(v_M), \qquad (4.3)$$

where $v_M(t, x) = \pi_M(v(t, x))$ and $v(t, x)$ is a solution to (1.3). Next we notice that we have the following version of (3.16), where the solution $v(t, x)$ to (1.2) is replaced by $v_M(t, x)$:

$$\frac{d}{dt} \|\partial_x^k v_M\|_{L^2}^2 = \sum_{(j_0, j_1, j_2) \in \mathcal{A}^k} \beta_{j_0, j_1, j_2} \int \mathcal{I}_{j_0+1, j_1+1, j_2}(v_M) + \text{l.o.t.}, \qquad (4.4)$$

where l.o.t. denotes the integral of a linear combination of densities belonging to Θ_k and Ω_k computed along the solution $v_M(t, x)$. In fact we have

$$\frac{d}{dt} \|\partial_x^k v_M\|_{L^2}^2 = 2 \int \partial_t \partial_x^k v_M \partial_x^k v_M = 2 \int \partial_x^{k+3} v_M \partial_x^k v_M - 2 \int \partial_x^{k+1} \pi_M (v_M^{p+1}) \partial_x^k v_M$$

$$= -2 \int \partial_x^{k+1} (v_M^{p+1}) \partial_x^k v_M,$$

where we used that π_M is symmetric and $\pi_M(v_M) = v_M$. Hence, the proof of (4.4) follows that of (3.16). Next recall that by (3.17), which is available for any generic function $u(x)$, we get

$$\sum_{(j_0,j_1,j_2)\in\mathcal{A}_k} \beta_{j_0,j_1,j_2} \int \mathcal{I}_{j_0+1,j_1+1,j_2}(v_M) - \sum_{(j_0,j_1,j_2)\in\mathcal{A}_k} \lambda_{j_0,j_1,j_2} \int \mathcal{I}^*_{j_0,j_1,j_2}(v_M) = \text{l.o.t.},$$

where l.o.t. denotes the integral of a linear combination of densities belonging to Θ_k and Ω_k. By combining this identity with (4.4) and (4.3) we get (by defining the energy $\overline{\mathcal{E}}_k$ as in (3.2))

$$\frac{d}{dt}\overline{\mathcal{E}}_k(v_M) = \sum_{(j_0,\dots,j_{2p+1})\in\mathcal{C}^k} \mu_{j_0,\dots,j_{2p+1}} \int \mathcal{J}_{j_0,\dots,j_{2p+1}}(v_M)$$

$$+ \sum_{(j_0,\dots,j_{p+1})\in\mathcal{D}^k} \nu_{j_0,\dots,j_{p+1}} \int \mathcal{J}_{j_0,\dots,j_{p+1}}(v_M)$$

$$- \sum_{(j_0,j_1,j_2)\in\mathcal{A}^k} \lambda_{j_0,j_1,j_2} \int \mathcal{I}^{**}_{j_0,j_1,j_2,M}(v_M), \tag{4.5}$$

where on the r.h.s. we have the integral of a linear combination of densities belonging to Θ_k and Ω_k, computed along the solution $v_M(t,x)$, plus the integral of a linear combination of the expressions $\mathcal{I}^{**}_{j_0,j_1,j_2,M}(v_M(t,x))$ where $(j_0, j_1, j_2) \in \mathcal{A}_k$. Notice that we cannot expand $\int \mathcal{I}^{**}_{j_0,j_1,j_2,M}(v_M(t,x))\,dx$ as the integral of a linear combination of terms belonging to Θ_k computed along $v_M(t,x)$ because of π_M appearing in $\mathcal{I}^{**}_{j_0,j_1,j_2,M}(u)$.

Based on the r.h.s. in (4.5) we define

$$\overline{\mathcal{G}}^T_{k,M}(\varphi) = \sum_{(j_0,\dots,j_{2p+1})\in\mathcal{C}^k} \mu_{j_0,\dots,j_{2p+1}} \int_0^T \int \mathcal{J}_{j_0,\dots,j_{2p+1}}(\pi_M(\Phi_M(t)\varphi)$$

$$+ \sum_{(j_0,\dots,j_{p+1})\in\mathcal{D}^k} \nu_{j_0,\dots,j_{p+1}} \int_0^T \int \mathcal{J}_{j_0,\dots,j_{p+1}}(\pi_M(\Phi_M(t)\varphi)$$

$$- \sum_{(j_0,j_1,j_2)\in\mathcal{A}^k} \lambda_{j_0,j_1,j_2} \int_0^T \int \mathcal{I}^{**}_{j_0,j_1,j_2,M}(\pi_M(\Phi_M(t)\varphi), \tag{4.6}$$

and we prove (1.13) for $M \in \mathbb{N}$ with constant uniform w.r.t. M. On the r.h.s. of (4.6) we have three types of densities. The first two groups of densities on the r.h.s. computed along $v_M(t,x)$ can be estimated by using (4.1) and (4.2), where $v(t,x)$ is replaced by $v_M(t,x)$. Indeed, the proof can be done *mutatis mutandis* by replacing $v(t,x)$ with $v_M(t,x)$.

Hence we only need to estimate the third group of densities on the r.h.s. in (4.6) as follows:

$$\left|\int_0^T \int \mathcal{I}^{**}_{j_0,j_1,j_2,M}(v_M(t,x))\right| \leq C + C\|\varphi\|_{H^{(k-\frac{1}{2})-}}^{(\frac{4k-8}{2k-3})^+}, \quad (j_0, j_1, j_2) \in \mathcal{A}^k, \tag{4.7}$$

with a constant uniform w.r.t. M, in order to conclude (1.13). Notice that $\mathcal{I}^{**}_{j_0,j_1,j_2,M}(v_M)$ with $(j_0, j_1, j_2) \in \mathcal{A}^k$ can be expressed as a linear combination of densities belonging to Θ_k, except that the projector π_M can appear in front of a group of factors. The proof of this fact is obvious if $(j_0, j_1, j_2) \neq (k-1, 0, 0)$. Assuming $(j_0, j_1, j_2) = (k-1, 0, 0)$, we have

$$\int \mathcal{I}^{**}_{k-1,0,0,M}(v_M) = 2 \int \partial_x^k \pi_M(v_M^{p+1}) \partial_x^{k-1} v_M v_M^p$$
$$+ p \int (\partial_x^{k-1} v_M)^2 v_M^{p-1} \pi_M \partial_x(v_M^{p+1}). \tag{4.8}$$

The density of the second integral on the r.h.s. belongs to Θ_k, with factors v_M up to the projector π_M. For the first term on the r.h.s., recall that by Leibniz rule we have

$$(p+1)\partial_x^{k-1} u u^p = \partial_x^{k-1}(u^{p+1}) + \sum_{\substack{(\alpha_1,\ldots,\alpha_{p+1}) \\ \alpha_1+\cdots+\alpha_{p+1}=k-1 \\ \alpha_j < k-1}} c_{\alpha_1,\ldots,\alpha_{p+1}} \prod_{j=1}^{p+1} \partial_x^{\alpha_j} u,$$

hence the first term on the r.h.s. in (4.8) can be expressed as a multiple of

$$\int \partial_x^k \pi_M(v_M^{p+1}) \partial_x^{k-1}(v_M^{p+1}) + \sum_{\substack{(\alpha_1,\ldots,\alpha_{p+1}) \\ \alpha_1+\cdots+\alpha_{p+1}=k-1 \\ \alpha_j < k-1}} c_{\alpha_1,\ldots,\alpha_{p+1}} \int \partial_x^k \pi_M(v_M^{p+1}) \prod_{j=1}^{p+1} \partial_x^{\alpha_j} v_M$$

$$= \int \partial_x^k \pi_M(v_M^{p+1}) \partial_x^{k-1} \pi_M(v_M^{p+1})$$

$$- \sum_{\substack{(\alpha_1,\ldots,\alpha_{p+1}) \\ \alpha_1+\cdots+\alpha_{p+1}=k-1 \\ \alpha_j < k-1}} c_{\alpha_1,\ldots,\alpha_{p+1}} \int \partial_x^{k-1} \pi_M(v_M^{p+1}) \partial_x\left(\prod_{j=1}^{p+1} \partial_x^{\alpha_j} v_M\right), \tag{4.9}$$

where we used that π_M is a projector. Now, the first term on the r.h.s. in (4.9) is zero, as an integral of an exact derivative, and the other terms, after expanding the derivative of a product, can be expressed as the integral of densities belonging to Θ_k up to the projector π_M. Summarizing, in order to get (4.7) it is sufficient to estimate

$$\left| \int_0^T \int \left(\prod_{j=0}^m \partial_x^{\gamma_j} v_M\right) \pi_M\left(\prod_{i=m+1}^{2p+1} \partial_x^{\beta_i} v_M\right)\right| \leq C + C\|\varphi\|_{H^{(k-\frac{1}{2})-}}^{(\frac{4k-8}{2k-3})^+}, \tag{4.10}$$

where $\sum_{j=0}^m \gamma_j + \sum_{i=m+1}^{2p+1} \beta_i = 2k-1$ and at least three numbers in $\{\gamma_j, \beta_i\}$ are nonzero. We can assume the following alternative: the four factors with higher derivatives belong to the first product outside the projector, the four factors with higher derivatives belong to the second product on which we have the action of the projector or the four factors

with higher derivatives are distributed between the two groups. Hence, depending on the structure of the density appearing on the l.h.s. in (4.10) and by assuming that the numbers γ_j are decreasing as well as the numbers β_i, we can associate with the density on the l.h.s. in (4.10) a number $J \in \{0, 1, 2, 3, 4\}$ defined as

$$J = 4 \text{ in the case } \{\gamma_k, k = 0, 1, 2, 3\}$$
$$\geq \max\{\gamma_j, \beta_i \mid j = 4, \ldots, m, \ i = m+1, \ldots, 2p+1\},$$

$$J = 0 \text{ in the case } \min\{\beta_k, k = m+1, m+2, m+3, m+4\}$$
$$\geq \max\{\gamma_j, \beta_i \mid j = 0, \ldots, m, \ i = m+5, \ldots, 2p+1\},$$

$$0 < J < 4 \text{ in the case } \min\{\gamma_h, \beta_k \mid h = 0, \ldots, J-1, \ k = m+1, \ldots, m+4-J\}$$
$$\geq \max\{\gamma_j, \beta_i \mid j = J, \ldots, m, \ i = m+4-J, \ldots, 2p+1\}.$$

In the case $J = 4$ we estimate

$$\left| \int_0^T \int \left(\prod_{j=0}^m \partial_x^{\gamma_j} v_M \right) \pi_M \left(\prod_{i=m+1}^{2p+1} \partial_x^{\beta_i} v_M \right) \right|$$
$$\leq \left\| \prod_{j=0}^m \partial_x^{\gamma_j} v_M \right\|_{L^1((0,T);L^1)} \left\| \pi_M \left(\prod_{i=m+1}^{2p+1} \partial_x^{\beta_i} v_M \right) \right\|_{L^\infty((0,T);L^\infty)},$$

and by Sobolev embedding and the Hölder inequality, we proceed with

$$(\cdots) \leq C \prod_{j=0}^3 \|\partial_x^{\gamma_j} v_M\|_{L^4((0,T);L^4)} \prod_{j=4}^m \|\partial_x^{\gamma_j} v_M\|_{L^\infty((0,T);L^\infty)}$$
$$\times \left\| \pi_M \prod_{i=m+1}^{2p+1} \partial_x^{\beta_i} v_M \right\|_{L^\infty((0,T);H^1)}$$
$$\leq C \prod_{j=0}^3 \|\partial_x^{\gamma_j} v_M\|_{L^4((0,T);L^4)} \prod_{j=4}^m \|\partial_x^{\gamma_j} v_M\|_{L^\infty((0,T);H^1))}$$
$$\times \prod_{i=m+1}^{2p+1} \|\partial_x^{\beta_i} v_M\|_{L^\infty((0,T);H^1)},$$

where we used that H^1 is an algebra and continuity of π_M on H^1. The proof of (4.10) can now be completed exactly following the proof of (4.2). A similar argument works for $J = 0$. In the case $0 < J < 4$, we alter the argument above as follows

$$\left| \int_0^T \int \left(\prod_{j=0}^m \partial_x^{\gamma_j} v_M \right) \pi_M \left(\prod_{i=m+1}^{2p+1} \partial_x^{\beta_i} v_M \right) \right|$$
$$\leq \left\| \prod_{j=0}^m \partial_x^{\gamma_j} v_M \right\|_{L^{\frac{4}{J}}((0,T);L^{\frac{4}{J}})} \left\| \pi_M \left(\prod_{i=m+1}^{2p+1} \partial_x^{\beta_i} v_M \right) \right\|_{L^{\frac{4}{4-J}}((0,T);L^{\frac{4}{4-J}})}.$$

By continuity of π_M on L^p with $p \in (1, \infty)$ we can remove the projector, and then we can apply the Hölder inequality twice in order to place the four higher derivatives in L^4 and the other ones in L^∞. Then we proceed exactly following the proof of (4.2).

4.4. Proof of (1.14)

Due to the structure of the functionals $\overline{\mathcal{G}}_{k,M}^T$ we first have to prove that, if we consider a given density belonging to Ω_k, then the corresponding space-time integrals computed along $v_M(t, x)$ converge to the same integral computed along $v(t, x)$. This follows from the expression under consideration being multilinear. Indeed, by following the same computations that we did to get (4.1), we estimate the difference of the two expressions we are interested in, by the product of several factors, where each factor involves norms of $v_M(t, x)$, $v(t, x)$ in either H^σ, $\sigma < k - \frac{1}{2}$ or $L^6((0, T); W^{s,6})$, $s < k - \frac{1}{2}$ and one factor involves the norm of $v - v_M$ in either H^σ, $\sigma < k - \frac{1}{2}$ or $L^6((0, T); W^{s,6})$, $s < k - \frac{1}{2}$. Then we conclude that the difference converges to zero by the result in Section 2. The proof of (1.14) will follow provided that we prove the same convergence property as above if we now have a density belonging to Θ_k with the extra property that in the density, when computed along $v_M(t, x)$, the projector π_M appears in front of the product of a group of derivatives of $v_M(t, x)$. By using the identity $\mathrm{Id} = \pi_M + \pi_{>M}$, we remove the projector π_M at the expense of a remainder. Without the remainder, convergence for the corresponding terms follows the same lines as above in the case of a density in Ω_k, as in the proof of (4.2). We now deal with a remainder given by the density belonging to Θ_k computed along $v_M(t, x)$ with the projector $\pi_{>M}$ in front of a product of derivatives of $v_M(t, x)$. In order to show that this remainder goes to zero as $M \to \infty$, notice that every factor involved in the product on which the operator $\pi_{>M}$ acts, can itself be decomposed by using the identity $\mathrm{Id} = \pi_{cM} + \pi_{>cM}$, where c is a suitable small constant. Once the decomposition of each factor is done following this last identity, then we develop the product and notice that we get nontrivial contributions after the application of the projector $\pi_{>M}$ only for the terms where at least one factor involves $\pi_{>cM} v_M(t, x)$. Then we conclude following the same chain of inequalities needed in order to get (4.2). All factors that we get will be bounded according to estimates in Section 2, except one that appears with a projector $\pi_{>cM}$ and needs to be computed in one of the norms H^σ, $\sigma < k - \frac{1}{2}$ or $L^6((0, T); W^{s,6})$, $s < k - \frac{1}{2}$. As we are allowed to lose ε derivatives in our estimates, we deduce by a straightforward argument that we are converging to zero as $M \to \infty$.

5. Proof of Theorem 1.4

We denote by $d\mu_k$ (resp. $d\mu_{k,M}^\perp$) the Gaussian measure induced by the random series

$$\omega \mapsto \sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{(1 + n^2)^{k/2}} e^{inx} \quad \left(\text{resp.} \sum_{n \in \mathbb{Z}, |n| > M} \frac{g_n(\omega)}{(1 + n^2)^{k/2}} e^{inx} \right),$$

and define the following two measures:

$$d\rho_k = \chi_R(\mathcal{H}(u))\,d\mu_k, \quad d\rho_{k,M} = \chi_R(\mathcal{H}(\pi_M u))\,d\mu_k.$$

Next we shall also use the representation

$$d\mu_k = \gamma_M \exp\left(-\|\pi_M u\|_{H^k}^2\right) du_1 \ldots du_M \times d\mu_{k,M}^\perp, \tag{5.1}$$

where γ_M is a suitable renormalization constant and $du_1 \ldots du_M$ is the Lebesgue measure on $\mathbb{C}^M$. Next, changing variable,

$$\rho_{k,M}(\Phi_M(T)A)$$
$$= \gamma_M \int_{\Phi_M(t)A} \chi_R(\mathcal{H}(\pi_M u)) e^{-\|\pi_M u\|_{H^k}^2} du_1 \ldots du_M \times d\mu_{k,M}^\perp$$
$$= \gamma_M \int_A \chi_R(\mathcal{H}(\pi_M(\Phi_M(T)u))) e^{-(\|\pi_M(\Phi_M(t)(u))\|_{H^k}^2} du_1 \ldots du_M \times d\mu_{k,M}^\perp$$
$$= \int_A \chi_R(\mathcal{H}(\pi_M(\Phi_M(T)u))) e^{\|\pi_M u\|_{H^k}^2 - \|\pi_M(\Phi_M(T)u)\|_{H^k}^2} d\mu_k, \tag{5.2}$$

where A is a Borel subset of $H^{(k-\frac{1}{2})^-}$ and we used (5.1) in the last identity; hence

$$\rho_{k,M}(\Phi_M(T)A)$$
$$= \int_A \chi_R(\mathcal{H}(\pi_M(\Phi_M(T)u))) e^{-\bar{\mathcal{R}}_k(\pi_M u) + \bar{\mathcal{R}}_k(\pi_M(\Phi_M(T)u))}$$
$$\quad \times e^{\bar{\mathcal{E}}_k(\pi_M u) - \bar{\mathcal{E}}_k(\pi_M(\Phi_M(T)u))} d\mu_k$$
$$= \int_A \chi_R(\mathcal{H}(\pi_M(\Phi_M(T)u))) e^{-\bar{\mathcal{R}}_k(\pi_M u) + \bar{\mathcal{R}}_k(\pi_M(\Phi_M(T)u))} e^{-\bar{\mathcal{G}}_{k,M}^T(\pi_M u)} d\mu_k, \tag{5.3}$$

where $\bar{\mathcal{E}}_k$, $\bar{\mathcal{G}}_{k,M}^T$ and $\bar{\mathcal{R}}_k$ are the functionals in Theorem 1.3. Set

$$f_{T,M}(u) = \chi_R\big(\mathcal{H}(\pi_M(\Phi_M(T)u))\big) e^{-\bar{\mathcal{R}}_k(\pi_M u) + \bar{\mathcal{R}}_k(\pi_M(\Phi_M(T)u))} e^{-\bar{\mathcal{G}}_{k,M}^T(\pi_M u)},$$
$$f_T(u) = \chi_R\big(\mathcal{H}(\Phi(T)u)\big) e^{-\bar{\mathcal{R}}_k(u) + \bar{\mathcal{R}}_k((\Phi(T)u)} e^{-\bar{\mathcal{G}}_{k,\infty}^T(u)};$$

then by (1.14) and continuity of $\bar{\mathcal{R}}_k$ on $H^{(k-\frac{1}{2})^-}$ (which follows by minor modifications of the proof of (1.12)), we get $f_{T,M}(u) \to f_T(u)$ almost surely w.r.t μ_k, when $M \to \infty$. Moreover, we claim that, for all $q \in [1, \infty)$,

$$\sup_M \|f_{T,M}(u)\|_{L^q(\mu_k)} < \infty, \tag{5.4}$$

and therefore, as μ_k is a finite measure and by using the Egoroff theorem, we can upgrade to convergence in L^q, namely

$$f_{T,M}(u) \xrightarrow{L^q(\mu_k)} f_T(u) \in L^q(\mu_k) \quad \text{as } M \to \infty, \tag{5.5}$$

and we conclude, by the conservation of the Hamiltonian $\mathcal{H}$, $\chi_R(\mathcal{H}(u))g_T(u) = f_T(u) \in L^q(\mu_k)$. Next we prove (5.4). From conservation of the Hamiltonian, $\mathcal{H}(\pi_M(\Phi_M(T)u)) = \mathcal{H}(\pi_M u)$ and due to χ_R there exists $C > 0$, uniform w.r.t. M, such that the support of $f_{T,M}$ is contained in $\{u \in H^{(k-\frac{1}{2})^-} \mid \|\pi_M u\|_{H^1} < C\}$. By combining this fact with (2.3) (where we choose $s = (k - \frac{1}{2})^-$ and $\varphi = \pi_M u$) we get that, with another constant $C > 0$ uniform w.r.t. M, for every u in the support of $f_{T,M}$,

$$\sup_{\tau \in [0,T]} \|\pi_M \Phi_M(\tau)u\|_{H^{(k-\frac{1}{2})^-}} \leq C \|\pi_M u\|_{H^{(k-\frac{1}{2})^-}}.$$

Gathering all together and recalling (1.12) and (1.13) (where we choose $\varphi = \pi_M u$) we get, with $C > 0$ depending on R, $T > 0$ and uniform w.r.t. M,

$$f_{T,M}(u) \leq \exp\left(C + C \|u\|_{H^{(k-\frac{1}{2})^-}}^{(\frac{4k-8}{2k-3})^+}\right).$$

As $(\frac{4k-8}{2k-3}) < 2$, we can apply classical Gaussian bounds in order to get

$$\exp\left(C \|u\|_{H^{(k-\frac{1}{2})^-}}^{(\frac{4k-8}{2k-3})^+}\right) \in L^q(\mu_k)$$

and we conclude (5.4). Going back to (5.2) and (5.3) we get

$$\rho_{k,M}(\Phi_M(T)A) = \int_A f_{T,M}(u)\, d\mu_k, \tag{5.6}$$

and hence by passing to the limit as $M \to \infty$ (at least formally) we get

$$\rho_k(\Phi(T)A) = \int_A f_T(u)\, d\mu_k, \tag{5.7}$$

where $f_T(u) \in L^q(\mu_k)$ and we could conclude the proof. Proving (5.6) $\Rightarrow$ (5.7) requires some work: we will prove the inequality

$$\rho_k(\Phi(T)A) \leq \int_A f_T(u)\, d\mu_k \tag{5.8}$$

and the reversed one

$$\rho_k(\Phi(T)A) \geq \int_A f_T(u)\, d\mu_k, \tag{5.9}$$

which in turn imply (5.7). Assume in the sequel that A is a compact set in $H^{(k-\frac{1}{2})^-}$; by classical approximation arguments this will be sufficient to deal with any generic measurable set. In order to prove (5.6) $\Rightarrow$ (5.8) with A compact, we use the following property of the flows $\Phi_M(t)$, whose proof follows by combining Proposition 2.1 and the arguments in [47]:

$$\forall A \subset H^{(k-\frac{1}{2})^-} \text{ compact}, \forall \varepsilon > 0,$$

$$\exists M_0 \in \mathbb{N} \text{ s.t. } \Phi(T)A \subset \Phi_M(T)(A + B_\varepsilon^{(k-\frac{1}{2})^-}) \quad \forall M > M_0,$$

where B_ε^s denotes the ball of radius $\varepsilon > 0$ in H^s. Then if A compact and $\varepsilon > 0$ are given, we get

$$\rho_{k,M}(\Phi(T)A) \le \rho_{k,M}\left(\Phi_M(T)(A + B_\varepsilon^{(k-\frac{1}{2})^-})\right)$$
$$= \int_{A+B_\varepsilon^{(k-\frac{1}{2})^-}} f_{T,M}(u)\,d\mu_k \quad \forall M > M_0,$$

where we used (5.6). By using (5.5) we can pass to the limit as $M \to \infty$ and we get

$$\limsup_{M\to\infty} \rho_{k,M}(\Phi(T)A) \le \int_{A+B_\varepsilon^{(k-\frac{1}{2})^-}} f_T(u)\,d\mu_k \quad \forall \varepsilon > 0.$$

On the other hand, one can easily check that $\limsup_{M\to\infty} \rho_{k,M}(\Phi(T)A) = \rho_k(\Phi(T)A)$ and also, with A compact, one can prove that

$$\lim_{\varepsilon\to 0} \int_{A+B_\varepsilon^{(k-\frac{1}{2})^-}} f_T(u)\,d\mu_k = \int_A f_T(u)\,d\mu_k$$

and (5.8) follows. In order to prove (5.9) we take advantage of a property of the flows $\Phi_M(t)$ that follows from Proposition 2.1:

$$\forall A \subset H^{(k-\frac{1}{2})^-} \text{ compact}, \forall \varepsilon > 0,$$
$$\exists M_0 \in \mathbb{N} \text{ s.t. } \forall M > M_0, \Phi_M(T)A \subset \Phi(T)A + B_\varepsilon^{(k-\frac{1}{2})^-}.$$

By combining this fact with (5.6) we get for any compact set A,

$$\rho_{k,M}(\Phi(T)A + B_\varepsilon^{(k-\frac{1}{2})^-}) \ge \rho_{k,M}(\Phi_M(T)A) = \int_A f_{T,M}(u)\,d\mu_k \quad \forall M > M_0,$$

and by passing to the limit as $M \to \infty$ on the l.h.s. and r.h.s. we get

$$\rho_k(\Phi(T)A + B_\varepsilon^{(k-\frac{1}{2})^-}) \ge \int_A f_T(u)\,d\mu_k.$$

From compactness of A, we easily get $\lim_{\varepsilon\to 0} \rho_k(\Phi(T)A + B_\varepsilon^{(k-\frac{1}{2})^-}) = \rho_k(\Phi(T)A)$ and (5.9) follows.

A. A nonlinear local-in-time estimate

The aim of this appendix is to provide a proof of (2.28).

A.1. A first reduction

Let $s \ge 1$ and $p \ge 3$ be integers. For the sake of completeness, we prove

$$\left\| \int_0^t S(t-\tau)\pi_M \Pi\left(\Pi((v(\tau))^p)\partial_x v(\tau)\right) d\tau \right\|_{Y_T^s} \le C T^\kappa \|v\|_{Y_T^1}^p \|v\|_{Y_T^s}, \quad \kappa > 0,$$

for $T \leq 1$ and $C > 0$ independent of M and κ. The arguments that we will perform below are standard. Our only goal is to provide a complete argument for a reader unfamiliar with the $X^{s,b}$ machinery, as well as proving how to gain the positive power of T at the r.h.s. (which was of importance for our analysis in Section 2). To the best of our knowledge, the estimate above written in this form is not readily available in the literature, even if we closely follow [7]. As π_M is bounded on Y^s, it suffices to prove

$$\left\| \int_0^t S(t-\tau) \Pi(\partial_x v(\tau) \Pi v^p(\tau)) \, d\tau \right\|_{Y_T^s} \leq C T^\kappa \|v\|_{Y_T^1}^p \|v\|_{Y_T^s}, \quad \kappa > 0.$$

Recalling the definition of restriction spaces, proving the global-in-time estimate will be sufficient:

$$\left\| \psi(t) \int_0^t S(t-\tau) \Pi(\partial_x v(\tau) \Pi v^p(\tau)) \, d\tau \right\|_{Y^s} \leq C T^\kappa \|v\|_{Y_T^1}^p \|v\|_{Y_T^s}, \quad \kappa > 0,$$

where $\psi \in C_0^\infty(\mathbb{R})$ is such that $\psi \equiv 1$ on $[-1, 1]$. Using [7, Lemma 3.1], we obtain

$$\left\| \psi(t) \int_0^t S(t-\tau) \Pi(\partial_x v(\tau) \Pi v^p(\tau)) \, d\tau \right\|_{Y^s} \leq C \|\Pi(\partial_x v \Pi v^p)\|_{Z^s},$$

where by definition $\|u\|_{Z^s} = \|u\|_{X^{s,-\frac{1}{2}}} + \|\langle \tau + n^3 \rangle^{-1} \langle n \rangle^s \hat{u}(\tau, n)\|_{L_n^2 L_t^1}$. Therefore, proving

$$\|\Pi(\partial_x v \Pi v^p)\|_{Z^s} \leq C T^\kappa \|v\|_{Y_T^1}^p \|v\|_{Y_T^s}, \quad \kappa > 0 \tag{A.1}$$

will be enough. Its proof follows by combining the following propositions. The next statement is a slightly modified version of [7, Theorem 3].

Proposition A.1. *For $s \geq 1$ there exists a constant $C > 0$ such that*

$$\|u^p\|_{X^{s-1,\frac{1}{2}}} \leq C \|u\|_{Y^s} \|u\|_{Y^1}^{p-1}. \tag{A.2}$$

We shall also need the following bilinear estimate.

Proposition A.2. *For $s \geq 1$ there exists $C > 0$ such that, for $T \in (0, 1)$,*

$$\|\Pi(\Pi u_1 \Pi u_2)\|_{Z^s} \leq C T^\kappa \left(\|u_1\|_{X_T^{s-1,\frac{1}{2}}} \|u_2\|_{X_T^{0,\frac{1}{2}}} + \|u_1\|_{X_T^{0,\frac{1}{2}}} \|u_2\|_{X_T^{s-1,\frac{1}{2}}} \right), \quad \kappa > 0.$$

Let us see how (A.1) is a consequence of both propositions: from Proposition A.2 we get

$$\|\Pi(\partial_x v \Pi v^p)\|_{Z^s} = \|\Pi(\Pi \partial_x v \Pi v^p)\|_{Z^s}$$
$$\leq C T^\kappa \left(\|v^p\|_{X^{s-1,\frac{1}{2}}} \|\partial_x v\|_{X^{0,\frac{1}{2}}} + \|v^p\|_{X^{0,\frac{1}{2}}} \|\partial_x v\|_{X^{s-1,\frac{1}{2}}} \right).$$

Next, using Proposition A.1 we get

$$\|v^p\|_{X^{s-1,\frac{1}{2}}} \leq C \|v\|_{Y^s} \|v\|_{Y^1}^{p-1}, \quad \|v^p\|_{X^{0,\frac{1}{2}}} \leq C \|v\|_{Y^1}^p,$$

and we get (A.1).

Proof of Proposition A.1. Write

$$\mathcal{F}(u^p)(\tau, n) = \int_{\tau = \tau_1 + \cdots + \tau_p} \sum_{n = n_1 + \cdots + n_p} \prod_{k=1}^{p} \hat{u}(\tau_k, n_k),$$

where $\mathcal{F}$ and $\hat{u}$ denote the space-time Fourier transform (continuous in time and discrete in space),

$$\|u^p\|_{X^{s-1, \frac{1}{2}}}^2 = \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} \langle n \rangle^{2(s-1)} \langle \tau + n^3 \rangle |\mathcal{F}(u^p)(\tau, n)|^2 \, d\tau. \tag{A.3}$$

Notice that the r.h.s. in (A.3) may be bounded with

$$\int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} \langle n \rangle^{2(s-1)} \langle \tau + n^3 \rangle \left(\int_{\tau = \tau_1 + \cdots + \tau_p} \sum_{n = n_1 + \cdots + n_p} \prod_{k=1}^{p} |\hat{u}(\tau_k, n_k)| \right)^2 d\tau.$$

Hence, if we define $w(t, x)$ by $\hat{w}(\tau, n) = |\hat{u}(\tau, n)|$, we get $\|u\|_{X^{s,b}} = \|w\|_{X^{s,b}}$, $\|u\|_{Y^s} = \|w\|_{Y^s}$, and we are reduced to estimating

$$\int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} \langle n \rangle^{2(s-1)} \langle \tau + n^3 \rangle \left(\int_{\tau = \tau_1 + \cdots + \tau_p} \sum_{n = n_1 + \cdots + n_p} \prod_{k=1}^{p} \hat{w}(\tau_k, n_k) \right)^2 d\tau. \tag{A.4}$$

Next we split the domain of integration and we consider first the contribution to (A.4) in the region

$$|\tau + n^3| \leq 10p|\tau_1 + n_1^3|. \tag{A.5}$$

If we define w_1 by $\hat{w}_1(\tau, n) = \langle \tau + n^3 \rangle^{\frac{1}{2}} \hat{w}(\tau, n)$, then the contribution to (A.4) in the region (A.5) can be controlled in the physical space variables as

$$C\|w_1 w^{p-1}\|_{L^2(\mathbb{R}; H^{s-1})}^2$$

$$\leq C(\|w_1\|_{L^2(\mathbb{R}; H^{s-1})}^2 \|w^{p-1}\|_{L^\infty(\mathbb{R}; L^\infty)}^2 + \|w_1\|_{L^2(\mathbb{R}; L^\infty)}^2 \|w^{p-1}\|_{L^\infty(\mathbb{R}; H^{s-1})}^2)$$

$$\leq C(\|w\|_{X^{s-1, \frac{1}{2}}}^2 \|w\|_{L^\infty(\mathbb{R}; H^1)}^{2(p-1)} + \|w_1\|_{L^2(\mathbb{R}; H^1)}^2 \|w\|_{L^\infty(\mathbb{R}; H^{s-1})}^2 \|w\|_{L^\infty(\mathbb{R}; H^1)}^{2(p-2)}),$$

where we have used standard product rules and Sobolev embedding $H^1 \subset L^\infty$. We proceed with

$$(\cdots) \leq C(\|w\|_{X^{s-1, \frac{1}{2}}}^2 \|w\|_{Y^1}^{2(p-1)} + \|w_1\|_{X^{1, \frac{1}{2}}}^2 \|w\|_{Y^{s-1}}^2 \|w\|_{Y^1}^{2(p-2)}),$$

where we used $Y^1 \subset L^\infty(\mathbb{R}; H^1)$, $Y^{s-1} \subset L^\infty(\mathbb{R}; H^{s-1})$. Notice that we have a better estimate, when compared with (A.2), in the region (A.5). Similarly, we can evaluate the contributions to (A.4) of the regions

$$|\tau + n^3| \leq 10p|\tau_k + n_k^3|, \quad 2 \leq k \leq p.$$

Therefore, we may assume that the summation and the integration in (A.4) are performed in the region

$$\max_{1 \le k \le p} |\tau_k + n_k^3| \le \frac{1}{10p} |\tau + n^3|. \tag{A.6}$$

Write

$$(\tau + n^3) - \sum_{k=1}^{p} (\tau_k + n_k^3) = \left(\sum_{k=1}^{p} n_k \right)^3 - \sum_{k=1}^{p} n_k^3;$$

therefore, in the region (A.6) we have

$$\left| \left(\sum_{k=1}^{p} n_k \right)^3 - \sum_{k=1}^{p} n_k^3 \right| \ge |\tau + n^3| - \sum_{k=1}^{p} |\tau_k + n_k^3| \ge \frac{9}{10} |\tau + n^3|,$$

hence

$$\langle \tau + n^3 \rangle \le C \left| \left(\sum_{k=1}^{p} n_k \right)^3 - \sum_{k=1}^{p} n_k^3 \right|.$$

By symmetry we can assume $|n_1| \ge |n_2| \ge \cdots \ge |n_k|$ and by using [7, Lemma 4.1], we obtain

$$\left| \left(\sum_{k=1}^{p} n_k \right)^3 - \sum_{k=1}^{p} n_k^3 \right| \le C |n_1|^2 |n_2|.$$

Consequently, in the region (A.6) we get $\langle \tau + n^3 \rangle \le C \langle n_1 \rangle^2 \langle n_2 \rangle$, and the corresponding contribution to (A.4) can be estimated as

$$C \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} \left(\int_{\tau = \tau_1 + \cdots + \tau_p} \sum_{n = n_1 + \cdots + n_p} \langle n_1 \rangle^s \langle n_2 \rangle^{\frac{1}{2}} \prod_{k=1}^{p} \widehat{w}(\tau_k, n_k) \right)^2 d\tau. \tag{A.7}$$

If we define w_1, w_2 by $\widehat{w}_1(\tau, n) = \langle n \rangle^s \widehat{w}(\tau, n)$, $\widehat{w}_2(\tau, n) = \langle n \rangle^{\frac{1}{2}} \widehat{w}(\tau, n)$, going back to physical space variables, we estimate (A.7) as

$$C \| w_1 w_2 w^{p-2} \|_{L^2(\mathbb{R}; L^2)}^2 \le C \| w_1 \|_{L^\infty(\mathbb{R}; L^2)}^2 \| w_2 \|_{L^4(\mathbb{R}; L^\infty)}^2 \| w \|_{L^4(\mathbb{R}; L^\infty)}^2 \| w \|_{L^\infty(\mathbb{R}; L^\infty)}^{2(p-3)}$$

$$\le C \| w \|_{L^\infty(\mathbb{R}; H^s)}^2 \| w_2 \|_{L^4(\mathbb{R}; W^{\frac{1}{2}, 4})}^2 \| w \|_{L^4(\mathbb{R}; W^{1,4})}^2 \| w \|_{L^\infty(\mathbb{R}; H^1)}^{2(p-3)}.$$

Hence, by using $Y^1 \subset L^\infty(\mathbb{R}; H^1)$ and $Y^s \subset L^\infty(\mathbb{R}; H^s)$, along with the estimate

$$\| u \|_{L^4(\mathbb{R}; L^4)} \le C \| u \|_{X^{0, \frac{1}{3}}} \tag{A.8}$$

established in the fundamental work [1], we proceed with

$$(\cdots) \le C \| w \|_{Y^s}^2 \| w \|_{X^{1, \frac{1}{3}}}^2 \| w \|_{X^{1, \frac{1}{3}}}^2 \| w \|_{Y^1}^{2(p-3)},$$

and this concludes the proof. ∎

Proof of Proposition A.2. We start with proving

$$\|\Pi(\Pi u_1 \Pi u_2)\|_{X^{s,-\frac{1}{2}}} \leq C T^{\kappa}(\|u_1\|_{X_T^{s-1,\frac{1}{2}}} \|u_2\|_{X_T^{0,\frac{1}{2}}} + \|u_1\|_{X_T^{0,\frac{1}{2}}} \|u_2\|_{X_T^{s-1,\frac{1}{2}}}). \quad \text{(A.9)}$$

As a first step we prove an estimate for global-in-time functions:

$$\|\Pi(\Pi u_1 \Pi u_2)\|_{X^{s,-\frac{1}{2}}} \leq C(\|u_1\|_{X^{s-1,\frac{1}{2}}} \|u_2\|_{X^{0,\frac{1}{3}}} + \|u_1\|_{X^{s-1,\frac{1}{3}}} \|u_2\|_{X^{0,\frac{1}{2}}}$$
$$+ \|u_1\|_{X^{0,\frac{1}{3}}} \|u_2\|_{X^{s-1,\frac{1}{2}}} + \|u_1\|_{X^{0,\frac{1}{2}}} \|u_2\|_{X^{s-1,\frac{1}{3}}}). \quad \text{(A.10)}$$

Notice that by comparing (A.9) and (A.10) in the second estimate we gain derivatives but we lose a positive power of the time T. We will prove toward the end how to go from (A.10) to (A.9).

The square of the l.h.s. of (A.10) may be written as

$$\int_{\mathbb{R}} \sum_{n \neq 0} \langle n \rangle^{2s} \langle \tau + n^3 \rangle^{-1} |\mathcal{F}(\Pi u_1 \Pi u_2)(\tau, n)|^2 \, d\tau,$$

and moreover we easily have

$$|\mathcal{F}(\Pi u_1 \Pi u_2)(\tau, n)| \leq \sum_{n_1 \neq 0, n} \int_{\mathbb{R}} |\hat{u}_1(\tau_1, n_1)| \, |\hat{u}_2(\tau - \tau_1, n - n_1)| \, d\tau_1.$$

For $j = 1, 2$, define $w_j(t, x)$ with $\hat{w}_j(\tau, n) = |\hat{u}_j(\tau, n)|$. Then we estimate the l.h.s. of (A.10) as

$$\int_{\mathbb{R}} \sum_{n \neq 0} \frac{\langle n \rangle^{2s}}{\langle \tau + n^3 \rangle} \left(\sum_{n_1 \neq 0, n} \int_{\mathbb{R}} \hat{w}_1(\tau_1, n_1) \hat{w}_2(\tau - \tau_1, n - n_1) \, d\tau_1 \right)^2 d\tau,$$

which in turn by a duality argument is bounded with

$$\sup_{\|v\|_{L^2_{t,x}} \leq 1} \int_{\mathbb{R}} \sum_{n \neq 0} \frac{\langle n \rangle^{s}}{\langle \tau + n^3 \rangle^{\frac{1}{2}}} \sum_{n_1 \neq 0, n} \int_{\mathbb{R}} \hat{w}_1(\tau_1, n_1) \hat{w}_2(\tau - \tau_1, n - n_1) |\hat{v}(\tau, n)| \, d\tau_1 \, d\tau.$$

For $n \neq 0$ and $n_1 \neq 0, n$, we have $\langle n \rangle^s \leq C |n_1|^{\frac{1}{2}} |n - n_1|^{\frac{1}{2}} |n|^{\frac{1}{2}} (\langle n_1 \rangle^{s-1} + \langle n - n_1 \rangle^{s-1})$. Therefore, by using a symmetry argument, it suffices to evaluate

$$\sup_{\|u\|_{L^2_{t,x}} \leq 1} \int_{\mathbb{R}^2} \sum_{n \neq 0, n_1 \neq 0, n} \frac{|n_1|^{\frac{1}{2}} |n - n_1|^{\frac{1}{2}} |n|^{\frac{1}{2}}}{\langle \tau + n^3 \rangle^{\frac{1}{2}}} (\langle n_1 \rangle^{s-1} \hat{w}_1(\tau_1, n_1))$$
$$\times \hat{w}_2(\tau - \tau_1, n - n_1) |\hat{v}(\tau, n)| \, d\tau_1 \, d\tau. \quad \text{(A.11)}$$

The key property for smoothing is the elementary bound

$$\max(\langle \tau + n^3 \rangle^{\frac{1}{2}}, \langle \tau_1 + n_1^3 \rangle^{\frac{1}{2}}, \langle \tau - \tau_1 + (n - n_1)^3 \rangle^{\frac{1}{2}}) \geq C |n_1|^{\frac{1}{2}} |n - n_1|^{\frac{1}{2}} |n|^{\frac{1}{2}}.$$

We will consider a splitting of the expression in (A.11) into three contributions, taking into account which term is the maximum in the above elementary bound. Notice that the contribution of (A.11) in the region

$$\langle \tau + n^3 \rangle^{\frac{1}{2}} \geq C |n_1|^{\frac{1}{2}} |n - n_1|^{\frac{1}{2}} |n|^{\frac{1}{2}} \tag{A.12}$$

may be estimated as

$$C \|w_2 \times v_1 \times \langle D_x \rangle^{s-1} w_1\|_{L^1(\mathbb{R};L^1)} \leq C \|v\|_{L^2_{t,x}} \|\langle D_x \rangle^{s-1} w_1\|_{L^4(\mathbb{R};L^4)} \|w_2\|_{L^4(\mathbb{R};L^4)},$$

where $v_1(t, x)$ is defined with $\hat{v}_1(\tau, n) = |\hat{v}(\tau, n)|$. Now, using (A.8), we write

$$\|\langle D_x \rangle^{s-1} w_1\|_{L^4(\mathbb{R};L^4)} \leq C \|w_1\|_{X^{s-1,\frac{1}{3}}} = C \|u_1\|_{X^{s-1,\frac{1}{3}}},$$

$$\|w_1\|_{L^4(\mathbb{R};L^4)} \leq C \|w_1\|_{X^{0,\frac{1}{3}}} = C \|u_1\|_{X^{0,\frac{1}{3}}}.$$

Hence we can estimate the contribution to (A.11) in the region (A.12) by

$$\|u_1\|_{X^{s-1,\frac{1}{3}}} \|u_2\|_{X^{0,\frac{1}{3}}},$$

up to a multiplicative constant. Next we consider the contribution to (A.11) in the region

$$\langle \tau_1 + n_1^3 \rangle^{\frac{1}{2}} \geq C |n_1|^{\frac{1}{2}} |n - n_1|^{\frac{1}{2}} |n|^{\frac{1}{2}}. \tag{A.13}$$

Let $v_1(t, x)$ be defined by

$$\hat{v}_1(\tau, n) = \langle \tau + n^3 \rangle^{-\frac{1}{2}} |\hat{v}(\tau, n)|$$

and let $w_{11}(t, x)$ be defined as

$$\widehat{w}_{11}(\tau, n) = \langle n \rangle^{s-1} \langle \tau + n^3 \rangle^{\frac{1}{2}} \widehat{w}_1(\tau, n);$$

then we can estimate the contribution of (A.11) in the region (A.13) by

$$C \|w_{11} \times w_2 \times v_1\|_{L^1(\mathbb{R};L^1)} \leq C \|w_{11}\|_{L^2(\mathbb{R};L^2)} \|w_2\|_{L^4(\mathbb{R};L^4)} \|v_1\|_{L^4(\mathbb{R};L^4)}.$$

Using (A.8) again we obtain

$$\|v_1\|_{L^4(\mathbb{R};L^4)} \leq C \|v\|_{X^{0,-\frac{1}{6}}} \leq C \|v\|_{L^2(\mathbb{R};L^2)},$$

$$\|w_2\|_{L^4(\mathbb{R};L^4)} \leq C \|w_2\|_{X^{0,\frac{1}{3}}} \leq C \|u_2\|_{X^{0,\frac{1}{3}}}.$$

Moreover, we have

$$\|w_{11}\|_{L^2(\mathbb{R};L^2)} = \|w_1\|_{X^{s-1,\frac{1}{2}}} = \|u_1\|_{X^{s-1,\frac{1}{2}}},$$

and, summarizing, we control the contribution to (A.11) in the region (A.13) by $\|u_1\|_{X^{s-1,\frac{1}{2}}} \|u_2\|_{X^{0,\frac{1}{3}}}$ up to a multiplicative factor. Finally, consider the contribution to (A.11) in the third region

$$\langle \tau - \tau_1 + (n - n_1)^3 \rangle^{\frac{1}{2}} \geq C |n_1|^{\frac{1}{2}} |n - n_1|^{\frac{1}{2}} |n|^{\frac{1}{2}}. \tag{A.14}$$

Retain $v_1(t, x)$ with $\hat{v}_1(\tau, n) = \langle \tau + n^3 \rangle^{-\frac{1}{2}} |\hat{v}(\tau, n)|$ and let $w_{21}(t, x)$ be defined with $\hat{w}_{21}(\tau, n) = \langle \tau + n^3 \rangle^{\frac{1}{2}} \hat{w}_2(\tau, n)$. Then we can control the contribution to (A.11) in the region (A.14) by

$$
\begin{aligned}
C \| w_{21} &\times v_1 \times \langle D_x \rangle^{s-1} w_1 \|_{L^1(\mathbb{R}; L^1)} \\
&\leq C \| \langle D_x \rangle^{s-1} w_1 \|_{L^4(\mathbb{R}; L^4)} \| w_{21} \|_{L^2(\mathbb{R}; L^2)} \| v_1 \|_{L^4(\mathbb{R}; L^4)} \\
&\leq C \| u_1 \|_{X^{s-1, \frac{1}{3}}} \| u_2 \|_{X^{0, \frac{1}{2}}} \| v \|_{X^{0, -\frac{1}{6}}},
\end{aligned}
$$

where we have used (A.8) again. Hence, the contribution of (A.11) in the region (A.14) can be estimated, up to a multiplicative constant, by $\| u_1 \|_{X^{s-1, \frac{1}{3}}} \| u_2 \|_{X^{0, \frac{1}{2}}}$. Summarizing, we estimate (A.11) by

$$
\| u_1 \|_{X^{s-1, \frac{1}{2}}} \| u_2 \|_{X^{0, \frac{1}{3}}} + \| u_1 \|_{X^{s-1, \frac{1}{3}}} \| u_2 \|_{X^{0, \frac{1}{2}}}
$$

for functions u_1 and u_2 which are not localized in time. Recall that by symmetry, in order to estimate the l.h.s. in (A.10), we need to add further terms where the roles of u_1 and u_2 have been exchanged. Hence we have established (A.10) which, as already said, in some sense is stronger than (A.9), since lower-order conormal derivatives of u_1 and u_2 are involved, but it is weaker than (A.9) since no gain of positive power of T has been obtained in (A.10). We finally deal with this issue: as a consequence of [46, Lemma 2.11] (see also [8, Lemma 3.2]) there exists $\kappa > 0$ such that

$$
\| w \|_{X_T^{s-1, \frac{1}{3}}} \leq C T^\kappa \| w \|_{X_T^{s-1, \frac{1}{2}}}, \quad \| w \|_{X_T^{0, \frac{1}{3}}} \leq C T^\kappa \| w \|_{X_T^{0, \frac{1}{2}}},
$$

and we complete the proof of (A.9) by combining the estimates above with (A.10), along with a suitable choice of extensions for u_1 and u_2, which a priori are defined on the strip of time $(-T, T)$, on the whole space-time with a global norm of the extension which is at most twice the corresponding localized norm. ∎

References

[1] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I; II. *Geom. Funct. Anal.* **3** (1993), no. 2, 107–156; no. 3, 209–262 Zbl 0787.35097 Zbl 0787.35098 MR 1209299 MR 1215780

[2] J. Bourgain, Periodic nonlinear Schrödinger equation and invariant measures. *Comm. Math. Phys.* **166** (1994), no. 1, 1–26   Zbl 0822.35126   MR 1309539

[3] J. Bourgain, On the Cauchy problem for periodic KdV-type equations. In *Proceedings of the Conference in Honor of Jean-Pierre Kahane (Orsay, 1993)*, *J. Fourier Anal. Appl. Spec. Iss.* (1995) 17–86   Zbl 0891.35137   MR 1364878

[4] J. Bourgain, Invariant measures for the 2D-defocusing nonlinear Schrödinger equation. *Comm. Math. Phys.* **176** (1996), no. 2, 421–445   Zbl 0852.35131   MR 1374420

[5] J. Bourgain, On the growth in time of higher Sobolev norms of smooth solutions of Hamiltonian PDE. *Internat. Math. Res. Notices* (1996), no. 6, 277–304   Zbl 0934.35166   MR 1386079

[6] A. Chapouto and N. Kishimoto, Invariance of the Gibbs measures for periodic generalized Korteweg-de Vries equations. 2022, arXiv:2104.07382

[7] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, Multilinear estimates for periodic KdV equations, and applications. *J. Funct. Anal.* **211** (2004), no. 1, 173–218   Zbl 1062.35109   MR 2054622

[8] J. Colliander and T. Oh, Almost sure well-posedness of the cubic nonlinear Schrödinger equation below $L^2(\mathbb{T})$. *Duke Math. J.* **161** (2012), no. 3, 367–414   Zbl 1260.35199   MR 2881226

[9] A. Debussche and Y. Tsutsumi, Quasi-invariance of Gaussian measures transported by the cubic NLS with third-order dispersion on **T**. *J. Funct. Anal.* **281** (2021), no. 3, Paper No. 109032   MR 4247030

[10] Y. Deng, N. Tzvetkov, and N. Visciglia, Invariant measures and long time behaviour for the Benjamin-Ono equation III. *Comm. Math. Phys.* **339** (2015), no. 3, 815–857   Zbl 1379.37136   MR 3385985

[11] J. Forlano and K. Seong, Transport of Gaussian measures under the flow of one-dimensional fractional nonlinear Schrödinger equations. *Comm. Partial Differential Equations* **47** (2022), no. 6, 1296–1337   Zbl 1490.35414   MR 4432957

[12] J. Forlano and W. J. Trenberth, On the transport of Gaussian measures under the one-dimensional fractional nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **36** (2019), no. 7, 1987–2025   Zbl 1427.35253   MR 4020531

[13] L. Friedlander, An invariant measure for the equation $u_{tt} - u_{xx} + u^3 = 0$. *Comm. Math. Phys.* **98** (1985), no. 1, 1–16   Zbl 0576.35082   MR 785258

[14] G. Genovese, R. Lucà, and N. Tzvetkov, Transport of Gaussian measures with exponential cut-off for Hamiltonian PDEs. 2021, arXiv:2103.04408

[15] G. Genovese, R. Lucà, and N. Tzvetkov, Quasi-invariance of low regularity Gaussian measures under the gauge map of the periodic derivative NLS. *J. Funct. Anal.* **282** (2022), no. 1, Paper No. 109263   MR 4323509

[16] G. Genovese, R. Lucà, and D. Valeri, Invariant measures for the periodic derivative nonlinear Schrödinger equation. *Math. Ann.* **374** (2019), no. 3-4, 1075–1138   Zbl 1420.35354   MR 3985108

[17] J. Ginibre, Y. Tsutsumi, and G. Velo, On the Cauchy problem for the Zakharov system. *J. Funct. Anal.* **151** (1997), no. 2, 384–436   Zbl 0894.35108   MR 1491547

[18] T. Gunaratnam, T. Oh, N. Tzvetkov, and H. Weber, Quasi-invariant Gaussian measures for the nonlinear wave equation in three dimensions. *Probab. Math. Phys.* **3** (2022), no. 2, 343–379   Zbl 07560160   MR 4452511

[19] Z. Hani, B. Pausader, N. Tzvetkov, and N. Visciglia, Modified scattering for the cubic Schrödinger equation on product spaces and applications. *Forum Math. Pi* **3** (2015), Paper No. e4   Zbl 1326.35348   MR 3406826

[20] T. Kappeler and J. Pöschel, *KdV & KAM*. Ergeb. Math. Grenzgeb., 3. Folge 45, Springer, Berlin, 2003   MR 1997070

[21] T. Kato, On the Cauchy problem for the (generalized) Korteweg-de Vries equation. In *Studies in applied mathematics*, pp. 93–128, Adv. Math. Suppl. Stud. 8, Academic Press, New York, 1983   MR 759907

[22] C. E. Kenig, G. Ponce, and L. Vega, On the (generalized) Korteweg-de Vries equation. *Duke Math. J.* **59** (1989), no. 3, 585–610   Zbl 0795.35105   MR 1046740

[23] C. E. Kenig, G. Ponce, and L. Vega, Oscillatory integrals and regularity of dispersive equations. *Indiana Univ. Math. J.* **40** (1991), no. 1, 33–69   Zbl 0738.35022   MR 1101221

[24] C. E. Kenig, G. Ponce, and L. Vega, Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle. *Comm. Pure Appl. Math.* **46** (1993), no. 4, 527–620   Zbl 0808.35128   MR 1211741

[25] J. L. Lebowitz, H. A. Rose, and E. R. Speer, Statistical mechanics of the nonlinear Schrödinger equation. *J. Statist. Phys.* **50** (1988), no. 3-4, 657–687   Zbl 0925.35142   MR 939505

[26] Y. Martel, Asymptotic N-soliton-like solutions of the subcritical and critical generalized Korteweg-de Vries equations. *Amer. J. Math.* **127** (2005), no. 5, 1103–1140   Zbl 1090.35158   MR 2170139

[27] Y. Martel, F. Merle, and P. Raphaël, Blow up for the critical generalized Korteweg–de Vries equation. I: Dynamics near the soliton. *Acta Math.* **212** (2014), no. 1, 59–140   Zbl 1301.35137   MR 3179608

[28] Y. Martel, F. Merle, and P. Raphaël, Blow up for the critical gKdV equation. II: Minimal mass dynamics. *J. Eur. Math. Soc. (JEMS)* **17** (2015), no. 8, 1855–1925   Zbl 1326.35320   MR 3372073

[29] Y. Martel, F. Merle, and P. Raphaël, Blow up for the critical gKdV equation III: exotic regimes. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **14** (2015), no. 2, 575–631   Zbl 1331.35307   MR 3410473

[30] Y. Martel and D. Pilod, Finite point blowup for the critical generalized Korteweg-de Vries equation. 2021, arXiv:2107.00268

[31] S. Oh and A. G. Stefanov, Smoothing and growth bound of periodic generalized Korteweg–De Vries equation. *J. Hyperbolic Differ. Equ.* **18** (2021), no. 4, 899–930   Zbl 1492.35279   MR 4378712

[32] T. Oh, G. Richards, and L. Thomann, On invariant Gibbs measures for the generalized KdV equations. *Dyn. Partial Differ. Equ.* **13** (2016), no. 2, 133–153   Zbl 1352.35151   MR 3520810

[33] T. Oh and K. Seong, Quasi-invariant Gaussian measures for the cubic fourth order nonlinear Schrödinger equation in negative Sobolev spaces. *J. Funct. Anal.* **281** (2021), no. 9, Paper No. 109150   MR 4285012

[34] T. Oh, P. Sosoe, and N. Tzvetkov, An optimal regularity result on the quasi-invariant Gaussian measures for the cubic fourth order nonlinear Schrödinger equation. *J. Éc. polytech. Math.* **5** (2018), 793–841   Zbl 1405.35198   MR 3877167

[35] T. Oh, Y. Tsutsumi, and N. Tzvetkov, Quasi-invariant Gaussian measures for the cubic nonlinear Schrödinger equation with third-order dispersion. *C. R. Math. Acad. Sci. Paris* **357** (2019), no. 4, 366–381   Zbl 1421.35341   MR 3955208

[36] T. Oh and N. Tzvetkov, Quasi-invariant Gaussian measures for the cubic fourth order nonlinear Schrödinger equation. *Probab. Theory Related Fields* **169** (2017), no. 3-4, 1121–1168 Zbl 1382.35274 MR 3719064

[37] T. Oh and N. Tzvetkov, Quasi-invariant Gaussian measures for the two-dimensional defocusing cubic nonlinear wave equation. *J. Eur. Math. Soc. (JEMS)* **22** (2020), no. 6, 1785–1826 Zbl 1441.35175 MR 4092899

[38] F. Planchon, N. Tzvetkov, and N. Visciglia, On the growth of Sobolev norms for NLS on 2- and 3-dimensional manifolds. *Anal. PDE* **10** (2017), no. 5, 1123–1147 Zbl 1371.35276 MR 3668586

[39] F. Planchon, N. Tzvetkov, and N. Visciglia, Transport of Gaussian measures by the flow of the nonlinear Schrödinger equation. *Math. Ann.* **378** (2020), no. 1-2, 389–423 Zbl 1448.35475 MR 4150922

[40] G. Richards, Invariance of the Gibbs measure for the periodic quartic gKdV. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **33** (2016), no. 3, 699–766 Zbl 1342.35310 MR 3489633

[41] J.-C. Saut, Sur quelques généralisations de l'équation de Korteweg-de Vries. *J. Math. Pures Appl. (9)* **58** (1979), no. 1, 21–61 Zbl 0449.35083 MR 533234

[42] P. C. Schuur, *Asymptotic analysis of soliton problems*. Lecture Notes in Math. 1232, Springer, Berlin, 1986 Zbl 0643.35003 MR 874343

[43] P. Sosoe, W. J. Trenberth, and T. Xian, Quasi-invariance of fractional Gaussian fields by the nonlinear wave equation with polynomial nonlinearity. *Differential Integral Equations* **33** (2020), no. 7-8, 393–430 Zbl 1463.35360 MR 4122511

[44] G. Staffilani, On solutions for periodic generalized KdV equations. *Internat. Math. Res. Notices* (1997), no. 18, 899–917 Zbl 0932.35177 MR 1481610

[45] C. Sun and N. Tzvetkov, Gibbs measure dynamics for the fractional NLS. *SIAM J. Math. Anal.* **52** (2020), no. 5, 4638–4704 Zbl 1448.35481 MR 4154315

[46] T. Tao, *Nonlinear dispersive equations*. CBMS Reg. Conf. Ser. Math. 106, American Mathematical Society, Providence, RI, 2006 Zbl 1106.35001 MR 2233925

[47] N. Tzvetkov, Quasiinvariant Gaussian measures for one-dimensional Hamiltonian partial differential equations. *Forum Math. Sigma* **3** (2015), Paper No. e28 Zbl 1333.35243 MR 3482274

[48] N. Tzvetkov and N. Visciglia, Gaussian measures associated to the higher order conservation laws of the Benjamin-Ono equation. *Ann. Sci. Éc. Norm. Supér. (4)* **46** (2013), no. 2, 249–299 Zbl 1317.35208 MR 3112200

[49] N. Tzvetkov and N. Visciglia, Invariant measures and long-time behavior for the Benjamin-Ono equation. *Int. Math. Res. Not. IMRN* (2014), no. 17, 4679–4714 Zbl 1301.35141 MR 3257548

[50] N. Tzvetkov and N. Visciglia, Invariant measures and long time behaviour for the Benjamin-Ono equation II. *J. Math. Pures Appl. (9)* **103** (2015), no. 1, 102–141 Zbl 1315.37051 MR 3281949

[51] P. E. Zhidkov, *Korteweg-de Vries and nonlinear Schrödinger equations: Qualitative theory*. Lecture Notes in Math. 1756, Springer, Berlin, 2001 MR 1831831

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