Publ. RIMS, Kyoto Univ.
12 (1976), 49–53

Some Results on Formal Power Series and Differentiable Functions

By

Masahiro SHIOTA

§ 1. Introduction

In [2] we see that any formal power series in two variables with coefficients in **R** or **C** (in this paper only the real case will be considered,) can be transformed to a polynomial by some automorphism change of the variables. In [3] Whitney shows an example which is a convergent series in three variables but which cannot be transformed to a polynomial. In this paper we give a formal power series example in three variables that is never transformed to be convergent (§ 2).

A formal power series is the Taylor expansion of some C^∞ function at the origin by *E*. Borel theorem. The followings refine it.

Theorem 1. *Let f be a formal power series in the variables $x=(x_1,\cdots,x_n)$. Let K be a positive real. There exists a C^∞ function g defined on $|x|<K$ with the Taylor expansion at 0 $Tg=f$ and which is analytic except when $x=0$.*

Theorem 2. *There exists a homomorphism S from the* **R***-algebra $\mathcal{F}$ of formal power series in one variable x to the* **R***-algebra $\mathcal{E}$ of germs of C^∞ function in one variable x at 0 such that the composition $T\circ S$ is the identity homomorphism of $\mathcal{F}$.*

There is a question in Malgrange [1] whether any homomorphism between the $\boldsymbol{R}$-algebras of C^∞ function germs is a morphism (see § 4). Theorem 2 gives a counter-example to it (Corollary).

Recieved March 20, 1975.

§ 2. An Example

The example of Whitney is an analytic function f in three variables of the form $xy(y-x)(y-(3+z)x)(y-\nu(z)x)$ where ν is a transcendental function with $\nu(0)=4$. If we replace the transcendental function above by a non-convergent formal power series, then f cannot be transformed to a convergent one by any automorphism of the algebra of formal power series.

Proof. Suppose it is not so, then there exist formal power series in (X, Y, Z)-variables $x(X, Y, Z)$, $y(X, Y, Z)$, $z(X, Y, Z)$ such that $f(x(X, Y, Z), y(X, Y, Z), z(X, Y, Z))$ is analytic and that the determinant of Jacobian $D(x, y, z)/D(X, Y, Z)$ does not vanish at 0. Moreover we can assume $\frac{D(x,y,z)}{D(X,Y,Z)}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ at $(X, Y, Z)=(0, 0, 0)$. We know Zariski-Nagata Theorem and the fact that the formal power series ring and the convergent power series ring are unique factorization rings. Therefore there exist formal power series $g_1, \cdots, g_5$ in (X, Y, Z)-variables such that $g_i(0)=1$ for each i, $g_1\cdots g_5=1$, and g_1x, $g_2y,\cdots,g_5(y-\nu(z)x)$ are convergent. Let G_i $i=1,\cdots,5$ be C^∞ functions in (X, Y, Z)-variables such that $TG_i=g_i$ and $G_1\cdots G_5=1$. We assume $g_1x,\cdots$ converge in a neighbourhood U of $(X, Y, Z)=(0, 0, 0)$. Let $\phi_1=g_1x/G_1$, $\cdots$, $\phi_5=g_5(y-\nu(z)x)/G_5$ in U. Clearly $T\phi_1=x,\cdots, T\phi_5=y-\nu(z)x$, and the Taylor expansions of $\phi_3-(\phi_2-\phi_1)$, $\phi_4-(\phi_2-(3+S)\phi_1)$ and $\phi_5-(\phi_2-R(S)\phi_1)$ are zeros at 0. Here S, R are C^∞ functions in (X, Y, Z)-, z-variables respectively such that $TS=z$, $TR=\nu$. From the assumption, (g_1x, g_2y, Z) is an analytic local coordinates system around 0. Hence $\phi_1^{-1}(0)\cap\phi_2^{-1}(0)$ is an analytic curve. These imply that the functions $\phi_3-(\phi_2-\phi_1),\cdots$ are zero identically on the curve $\phi_1^{-1}(0)\cap\phi_2^{-1}(0)$. On the other hand (ϕ_1, ϕ_2, Z) also is a local coordinates system around 0. Thus, we find C^∞ functions ψ_{ij} $i=3, 4, 5$ $j=1, 2$ flat at 0 such that

$$\phi_3-(\phi_2-\phi_1)=\psi_{31}\phi_1+\psi_{32}\phi_2\ ;$$

$$\phi_4-(\phi_2-(3+S)\phi_1)=\psi_{41}\phi_1+\psi_{42}\phi_2\ ;$$

$$\phi_5-(\phi_2-R(S)\phi_1)=\psi_{51}\phi_1+\psi_{52}\phi_2\ .$$

Hence the intersection of any two $\phi_i^{-1}(0)$ are the same one $\phi_1^{-1}(0)$

$\cap\phi_2^{-1}(0)$, and the intersection is described as $(X, Y, Z)=(X(Z), Y(Z), Z)$ where $X(Z)$ and $Y(Z)$ are analytic in Z-variable. We see easily that the Taylor expansions at 0 of the cross ratios of $(\phi_1^{-1}(0), \phi_2^{-1}(0), \phi_3^{-1}(0), \phi_4^{-1}(0))$ and $(\phi_1^{-1}(0), \phi_2^{-1}(0), \phi_3^{-1}(0), \phi_5^{-1}(0))$ are $1/(3+z(X(Z), Y(Z), Z))$ and $1/\nu(z(X(Z), Y(Z), Z))$, respectively.

In the same way we find analytic functions χ_{ij} $i=3, 4, 5$ $j=1, 2$ such that $\chi_{ij}(0)\neq 0$ and

$$G_i\phi_i=\chi_{i1}G_1\phi_1-\chi_{i2}G_2\phi_2 \quad i=3,\ 4,\ 5,$$

and we see that the cross ratios above are $\chi_{31}\chi_{42}/\chi_{41}\chi_{32}$ and $\chi_{31}\chi_{52}/\chi_{51}\chi_{32}$, respectively. Hence they are analytic, but both $z(X(Z), Y(Z), Z)$ and $\nu(z(X(Z), Y(Z), Z))$ are not convergent by the assumption. That is a contradiction.

§ 3. Proof of Theorem 1

We prove only the case $n=K=1$. In the general case there is nothing to prove moreover.

Let f be a formal power series $\sum a_n x^n$ where a_n are reals. It is enough to find sufficiently large reals m_n such that $\sum a_n(1-\exp(-1/m_n x^2))x^n$ converges on

(1) the real interval $[-1, 1]$ with its each derivatives; and

(2) any compact subset of the complex domain $0<|x|<1$.

Proof of (1). For $n\geqq 2$ and $k\leqq n/3$, we have

$$\begin{aligned}&\text{the } k\text{-th derivative of } (1-\exp(-1/mx^2))x^n\\ &=n\cdots(n-k+1)(1-\exp(-1/mx^2))x^{n-k}+P(x, m)\exp(-1/mx^2).\end{aligned}$$

Here $P(x, m)$ is a polynomial in x and uniformly converges to 0 when $m\to+\infty$. We can see that $(1-\exp(-1/mx^2))x^{n-k}$ and $\exp(-1/mx^2)$ are monotonous in the intervals $[-1, 0]$ and $[0, 1]$ Hence these functions take the maximal values at $x=-1$ or 1. Now, it follows that the k-th derivative of $(1-\exp(-1/mx^2))x^n$ uniformly converges to 0 when $m\to+\infty$ for $n\geqq 2$ and $k\leqq n/3$. This proves (1).

Proof of (2) is also easy.

§ 4. Homomorphism

Proof of Theorem 2. Let X be the ordered set consisting of the pairs (A, ϕ). Here A is a subring of $\mathcal{F}$ containing $\mathbf{R}$ and ϕ is a homomorphism from A to $\mathcal{E}$ such that the composition $T\circ\phi$ is the identity of A. Order two elements (A, ϕ), (B, ψ) of X as follows

$$(A, \phi)\leqq(B, \psi) \text{ if } A\subset B \text{ and } \psi|_A=\phi\,.$$

Apply Zorn's lemma, and X has a maximal element (A, ϕ).

Now, we prove that A of the maximal is itself $\mathcal{F}$. Assume that A is a proper subset of $\mathcal{F}$, and that ζ is an element in $\mathcal{F}$ but not in A. There are two cases,

(1). ζ is algebraic over A;

(2). ζ is not so.

The case (1). Let $A[\zeta]$ and $A[t]$ be the ring generated by ζ over A and the polynomial ring in t-variable with coefficients in A respectively, and let θ be the homomorphism from $A[t]$ to $A[\zeta]$ naturally defined by $\theta(t)=\zeta$. Let $P(t)$ be an element of ker θ whose degree as a t-polynomial takes the minimal in ker θ. For any element Q of ker θ, dividing Q by P we have $QQ'=PP'+R$ with $Q'\in A$, $P', R\in A[t]$. Since $R\in\ker\theta$ and degree $R\lneqq$degree P, we see $R=0$. Hence we have the equality (a) $QQ'=PP'$ for some $Q'\in A-\{0\}$ and $P'\in A[t]$. Let $P(t)=a_1t^n+\cdots+a_{n+1}$. We may assume $\zeta=x^s$ for an integer s through some change of the variable x. Let $\phi_*P(t)$ denote $\phi(a_1)t^n+\cdots+\phi(a_{n+1})$. We shall define an extension homomorphism Φ of ϕ from $A[t]$ to $\mathcal{E}$ such that $T\circ\Phi(t)=x^s$ and $\Phi(\ker\theta)=0$. This follows from (a) if we choose a germ $g(x)$ flat at 0 such that $\phi_*P(x^s+g)=0$. Let y be a variable. Then $\phi_*P(x^s+y)$ is a polynomial $b_1y^n+\cdots+b_{n+1}$ in y with coefficients in $\mathcal{E}$. We see that b_{n+1} is flat at 0 and that $b_n=(\partial\phi_*P/\partial t)(x^s)$ is not flat. Because we have degree $(\partial p/\partial t)(t)=$degree $P(t)-1$ and therefore $\partial p/\partial t\notin\ker\,\theta$. Put $y=x^Nz$ for a sufficiently large N and a new variable z. We can divide $b_1y^n+\cdots+b_{n+1}$ by x^Nb_n. The quotient is $c_1z^n+\cdots+z+c_{n+1}$, here c_i are in $\mathcal{E}$ and c_{n+1} is flat at 0. Applying the implicit function theorem, we give a germ $z(x)$ flat at 0 such that $c_1z^N(x)+\cdots+z(x)+c_{n+1}\equiv0$. The germ $g(x)=x^Nz(x)$ is what we want. Now we have defined a homomorphism Φ. It is clear that Φ induces a homomorphism ρ from $A[\zeta]$ to $\mathcal{E}$ such that the composition $T\circ\rho$ is the identity of $A[\zeta]$. This contradicts the maximality of (A, ϕ). Hence A is $\mathcal{F}$.

The proof of the case (2) is trivial from the proof of (1). Theorem 2 follows.

Remark. Even if we treat only the homomorphisms where the image of a convergent power series is naturally defined, there are infinitely many homomorphisms. We can prove this from the fact that any non-convergent formal power series is algebraically independent over the convergent series ring.

Definition [1]. An endomorphism u of $\mathcal{E}$ is called a *morphism* if there exists a germ ϕ with $\phi(0)=0$ such that for any $f\in\mathcal{E}$, we have $u(f)=f\circ\phi$.

The following answers the question in [1].

Corollary. *The composed homomorphism $S\circ T$ is not a morphism. Here S is defined in Theorem 2.*

Proof. Suppose it is a morphism induced by some ϕ. The first derivative of ϕ takes a non-zero value at 0. Hence $S\circ T$ is an automorphism, on the other hand we have $S\circ T(f)=0$ for f flat at 0.

Remark. The general preparation theorem in [1] does not hold in the homomorphism case. That is, this $S\circ T$ is quasifinite but not finite.

References

[1] Malgrange, B., *Ideals of differentiable functions*, Oxford University Press, 1966.
[2] Shiota, M., On germs of differentiable functions in two variables, ***Publ. RIMS, Kyoto Univ.***, **9** (1974), 319–324.
[3] Whitney, H., Local properties of analytic varieties, ***Differential and Combinatorial Topology***, Princeton Univ. Press, 1965.

Added in proof: The author was informed that Theorem 2 and its corollary were also obtained independently by K. Reichard (*Manuscripta Math.* **15** (1975), 243-250) and M. van der Put (to appear in *Compositio Math.*).