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Locally free twisted sheaves of infinite rank

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Abstract. We study twisted vector bundles of infinite rank on gerbes, giving a new point of view on Grothendieck's famous problem on the equality of the Brauer group and cohomological Brauer group. We show that the relaxed version of the question has an affirmative answer in many, but not all, cases, including for any algebraic space with the resolution property and any algebraic space obtained by pinching two closed subschemes of a projective scheme. We also discuss some possible theories of infinite rank Azumaya algebras, consider a new class of "very positive" infinite rank vector bundles on projective varieties, and show that an infinite rank vector bundle on a curve in a surface can be lifted to the surface away from finitely many points.

1. Introduction

1.1. Background and main results

One of the fundamental invariants of a scheme is its Brauer group. By analogy with the Brauer group of a field, which classifies central simple algebras up to Morita equivalence, the *Brauer group* $\mathrm{Br}(X)$ of a scheme X classifies equivalence classes of *Azumaya algebras*, which are étale twists of matrix algebras $\mathrm{Mat}_{n\times n}(\mathcal{O}_X)$. Just as with fields, any Azumaya algebra $\mathcal{A}$ has an associated étale cohomology class $[\mathcal{A}]$ in the *cohomological Brauer group* $\mathrm{Br}'(X) := \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})_{\mathrm{tors}}$. This defines a functorial embedding

$$\mathrm{Br}(X) \subseteq \mathrm{Br}'(X)$$

called the *Brauer map*.

This paper originated with the problem of determining whether the Brauer map is an isomorphism, which is one of the central problems at the interface of modern algebra and algebraic geometry. Whereas the Brauer map is an isomorphism when X is the spectrum of a field (i.e., every torsion étale cohomology class over a field is represented by some central division algebra), for arbitrary schemes this is not necessarily true. By a theorem of Gabber [6], the Brauer map is known to be an isomorphism when X admits an ample line bundle, but the question of surjectivity of the Brauer map remains open more generally, e.g. for schemes admitting an ample family of line bundles (which include smooth separated schemes over a field).

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Using the language of *twisted sheaves* [25], the $\mathrm{Br} = \mathrm{Br}'$ question may be rephrased as follows. By Giraud's theory of non-abelian cohomology [13], the cohomological Brauer group $\mathrm{Br}'(X)$ classifies $\mathbf{G}_{\mathrm{m}}$-gerbes $\mathcal{X} \to X$ of finite order. For a $\mathbf{G}_{\mathrm{m}}$-gerbe $\mathcal{X}$ over a scheme X, the condition that the class $[\mathcal{X}]$ comes from an Azumaya algebra of rank r^2 is equivalent to saying that $\mathcal{X}$ admits a twisted vector bundle of rank r. The existence of such vector bundles has strong consequences: for example, if there is a twisted vector bundle of rank r on $\mathcal{X}$, then $[\mathcal{X}]$ is r-torsion in $\mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})$.

If $[\mathcal{X}]$ is arbitrary (not necessarily torsion), we can ask whether $\mathcal{X}$ admits twisted locally free $\mathcal{O}_{\mathcal{X}}$-modules of *infinite* rank. Let

$$\mathrm{LPBr}(X) \subseteq \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})$$

be the submonoid consisting of classes $[\mathcal{X}]$ such that $\mathcal{X}$ admits a twisted locally free $\mathcal{O}_{\mathcal{X}}$-module (not necessarily of finite rank); this is roughly equivalent to saying that $[\mathcal{X}]$ comes from an Azumaya algebra of possibly infinite rank. Our main theorem asserts the existence of twisted vector bundles of infinite rank on all $\mathbf{G}_{\mathrm{m}}$-gerbes over a large class of algebraic spaces (in greater generality than the known answers to the $\mathrm{Br} = \mathrm{Br}'$ question).

Theorem 1.1.1. *For an algebraic space X, the inclusion* $\mathrm{LPBr}(X) \subseteq \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})$ *is an equality if at least one of the following hold.*

(i) *X has the resolution property.*

(ii) *X admits an ample family of line bundles.*

(iii) *X is an algebraic surface (i.e., finite type and separated over a field, and* $\dim X = 2$*).*

(iv) *X is a quotient Y/G over some base scheme S, where Y is an S-algebraic space satisfying at least one of the conditions* (i)–(iii)*, and G is an S-group scheme acting freely on Y and such that $G \to S$ is an affine, flat, finitely presented morphism with geometrically irreducible fibers.*

At first glance, it seems vector bundles of infinite rank would be more difficult to understand than those of finite rank. In fact, infinite rank vector bundles have a simple local structure (often by arguments involving the Eilenberg swindle), enough to suggest the following as the key guiding principle of our main results:

Principle 1.1.2. Vector bundles of infinite rank have simpler structure than vector bundles of finite rank.

The prime example of this is Bass's theorem on the triviality of projective modules of countably infinite rank over Noetherian rings.

Theorem 1.1.3 (Bass [2]). *Let A be a ring such that $A/J(A)$ is Noetherian, where $J(A)$ is the Jacobson radical. If M is a projective A-module of countably infinite rank, then M is free.*

As we will see, local uniqueness theorems such as these yield global existence results that are quite strong in comparison to their finite-rank analogues (which are often false).

Remark 1.1.4. One can also read Bass's theorem as telling us that the various naïve notions of "infinite rank vector bundle" – that is, locally free sheaves or simply locally projective sheaves – coincide.

To prove Theorem 1.1.1, we first consider the affine case, for which we prove a twisted version of Bass's theorem. Then we show that in each case X admits a cover $\pi : X' \to X$ where X' is affine and π is a surjective *affine-pure* morphism, which is roughly characterized by the property that the pushforward of a locally free module by such a morphism remains locally free (in case (ii), Jouanolou's trick provides an example of such a map). The cases (i), (ii), (iii) are special cases of (iv), which reduces to the claim that $G \to S$ is affine-pure under the stated assumptions. This is enough to conclude since the $\mathrm{LPBr}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})$ property descends under affine-pure morphisms.

For non-affine schemes, it is not true that all infinite rank vector bundles are trivial (in fact, we give examples of infinite rank vector bundles on $\mathbf{P}^1$ that are not decomposable as direct sums of line bundles). In response to this difficulty, we introduce a class of infinite rank vector bundles which we call *very positive vector bundles*. We show that, for projective schemes over an infinite field, very positive vector bundles exist and are unique up to isomorphism; we view this as another example of Principle 1.1.2. This uniqueness forces them to descend through simple pushouts, allowing us to prove that $\mathrm{LPBr}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})$ for a class of schemes not included in Theorem 1.1.1.

Theorem 1.1.5. *Let X be the colimit of a diagram $Z \rightrightarrows Y$ where Z and Y are smooth and projective over an infinite field and the two arrows are closed immersions with disjoint images. Then* $\mathrm{LPBr}(X) = \mathrm{H}^2_{\acute{e}t}(X, \mathbf{G}_{\mathrm{m}})$.

In our investigations, one essential subtlety lies in what precisely one means by "vector bundle of infinite rank". The bundles we consider here are essentially the discrete parts of the types of infinite vector bundles described by Drinfeld in [7]. A consequence of this restriction is that we lose dualizability; this is why the subset $\mathrm{LPBr}(X)$ of $\mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})$ that we study is only a submonoid. It is an interesting question to consider what we would happen if one allows the compact duals as in Drinfeld's theory, in which case one seems to lose tensor products, so that one would no longer have a submonoid but merely a subset closed under inversion.

The $\mathrm{Br} = \mathrm{Br}'$ and $\mathrm{LPBr}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})$ problems are two instances of a broader desire in algebraic geometry to find appropriate geometric or ring-theoretic objects corresponding to cohomology classes. For $\mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})$, there is an earlier, different approach due to Taylor [32] who defined the *bigger Brauer group* $\widetilde{\mathrm{Br}}(X)$. These are the classes representable by certain quasi-coherent $\mathcal{O}_X$-algebras that are neither unital nor locally free in general. By work of Heinloth–Schröer [21], it is known that

$$\widetilde{\mathrm{Br}}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})$$

for Noetherian algebraic stacks X whose diagonal is quasi-affine. They show that all $\mathbf{G}_{\mathrm{m}}$-gerbes over such X admit a twisted coherent sheaf that locally contains an invertible summand.

1.2. Outline

We provide an outline of the paper. In Section 2 we discuss twisted sheaves and define the submonoid $\mathrm{LPBr}(X)$ over an arbitrary locally ringed site. We define infinite rank Azumaya algebras (Section 2.3) and Brauer–Severi varieties (Section 2.4) and discuss their relationship to twisted vector bundles. After giving some sufficient conditions for a morphism to be affine-pure (Section 3.1), we prove the main theorem in Section 3.2. We modify the example of a scheme for which $\mathrm{Br} \neq \mathrm{Br}'$ (due to Edidin–Hassett–Kresch–Vistoli [8]) to produce a scheme for which $\mathrm{LPBr}(X) = 0$ but $\mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}}) = \mathbf{Z}$. In Section 4, we introduce very positive vector bundles and prove that the proper schemes admitting a very positive vector bundle are exactly the projective ones (Section 4.2) and that very positive vector bundles are unique up to isomorphism (Section 4.3). We also prove that every finite rank vector bundle over a projective scheme admits a finite-length “forward resolution” by the very positive vector bundle (Section 4.4). In Section 5, we prove that infinite invertible matrices lift under any surjective ring map, which implies in particular that all infinite rank vector bundles lift from a curve to an ambient surface away from finitely many points (again supporting Principle 1.1.2). In Section A, we prove infinite rank analogues of some results used in the definition of the Brauer map, including the Skolem–Noether theorem (generalizing a result of Courtemanche–Dugas [5]) and the fact that endomorphism algebras of projective modules are central algebras.

2. Twisted sheaves and Azumaya algebras of possibly infinite rank

2.1. Terminology about modules

In this paper we will use the following terminology regarding modules (which may be different from existing definitions).

Definition 2.1.1. Let $\mathcal{C}$ be a locally ringed site, let $\mathcal{F}$ be an $\mathcal{O}_{\mathcal{C}}$-module.

We say that $\mathcal{F}$ is *locally free* if for every object U of $\mathcal{C}$ there is a covering $\{U_i \to U\}_{i\in I}$ such that for every i the restriction $\mathcal{F}|_{U_i}$ is a free $\mathcal{O}_{U_i}$-module.

We say that $\mathcal{F}$ is *locally projective* if for every object U of $\mathcal{C}$ there is a covering $\{U_i \to U\}_{i\in I}$ such that for every i the restriction $\mathcal{F}|_{U_i}$ is a direct summand of a free $\mathcal{O}_{U_i}$-module.

We say a locally projective $\mathcal{O}_{\mathcal{C}}$-module $\mathcal{F}$ has *positive rank* if for every object U of $\mathcal{C}$ there is a covering $\{U_i \to U\}_{i\in I}$ such that for every $i \in I$ there is a surjection $\mathcal{F}|_{U_i} \to \mathcal{O}_{U_i}$.

We say a locally projective $\mathcal{O}_{\mathcal{C}}$-module $\mathcal{F}$ is *countably generated* for every object U of $\mathcal{C}$ there is a covering $\{U_i \to U\}_{i\in I}$ such that for every $i \in I$ there is a countable set I_i and a surjection $\mathcal{O}_{U_i}^{\oplus I_i} \to \mathcal{F}|_{U_i}$ of $\mathcal{O}_{U_i}$-modules.

Remark 2.1.2. We note that every locally projective $\mathcal{O}_{\mathcal{C}}$-module $\mathcal{F}$ is quasi-coherent. Indeed, after localizing on $\mathcal{C}$ if necessary, we may assume that $\mathcal{O}_{\mathcal{C}}^{\oplus I} \simeq \mathcal{F} \oplus \mathcal{G}$ for some $\mathcal{O}_{\mathcal{C}}$-module $\mathcal{G}$. Then $\mathcal{F}$ is the cokernel of the composition $\mathcal{O}_{\mathcal{C}}^{\oplus I} \to \mathcal{G} \to \mathcal{O}_{\mathcal{C}}^{\oplus I}$.

Remark 2.1.3. For an algebraic space X, we are primarily interested in classes in the étale cohomology group $\mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})$. Thus whenever we discuss modules, cohomology, and gerbes on an algebraic space X, we will assume without further mention that the underlying site $\mathcal{C}$ is the small étale site $X_{\text{ét}}$ of X. (Recall that we have $\mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}}) \simeq \mathrm{H}^2_{\text{fppf}}(X, \mathbf{G}_{\mathrm{m}})$ by [18, Théorème 11.7 (1)].)

For a scheme X and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, the following Theorem 2.1.4 shows that $\mathcal{F}$ is locally projective for the fpqc topology if and only if it is locally projective for the Zariski topology.

Theorem 2.1.4 (Raynaud–Gruson [30], The Stacks project [33, Tag 05A9]). *Let A be a ring, let M be an A-module, let $A \to B$ be a faithfully flat ring map. If $M \otimes_A B$ is a projective B-module, then M is a projective A-module.*

Question 2.1.5 (Infinite-rank Hilbert Theorem 90). Let X be a scheme, let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_X$-module. If there exists an fppf (resp. étale) cover $X' \to X$ such that $\mathcal{E}|_{X'}$ is a free $\mathcal{O}_{X'}$-module, then does there exist an étale (resp. Zariski) cover $X'' \to X$ such that $\mathcal{E}|_{X''}$ is a free $\mathcal{O}_{X''}$-module? In other words, are the maps

$$\mathrm{H}^1_{\mathrm{Zar}}(X, \mathrm{GL}_I) \to \mathrm{H}^1_{\text{ét}}(X, \mathrm{GL}_I) \to \mathrm{H}^1_{\text{fppf}}(X, \mathrm{GL}_I)$$

isomorphisms for any index set I? By Theorem 1.1.3, we know that if X is Noetherian and I is countable, then the answer is “yes”. See also [33, Tag 05VF].

2.2. Twisted sheaves

We briefly recall some background about gerbes and twisted sheaves (see [26, Section 3.1.1] for a reference).

Definition 2.2.1. Let $\mathcal{C}$ be a locally ringed site, let $\alpha \in \mathrm{H}^2(\mathcal{C}, \mathbf{G}_{\mathrm{m}})$ be a class, and let $\mathcal{X} \to \mathcal{C}$ be a $\mathbf{G}_{\mathrm{m}}$-gerbe with $[\mathcal{X}] = \alpha$. An $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$ is said to be 1*-twisted* if the natural inertial action $\mathbf{G}_{\mathrm{m}} \times \mathcal{F} \to \mathcal{F}$ equals the action obtained by restricting the module action $\mathcal{O}_{\mathcal{X}} \times \mathcal{F} \to \mathcal{F}$ [26, Definition 3.1.1.1].

Remark 2.2.2. Let $f : Y \to X$ be a morphism of algebraic spaces and let $\mathcal{X} \to X$ be a $\mathbf{G}_{\mathrm{m}}$-gerbe. If $\mathcal{G}$ is a 1-twisted $\mathcal{O}_{Y \times_X \mathcal{X}}$-module, then the pushforward $f_*\mathcal{G}$ is a 1-twisted $\mathcal{O}_{\mathcal{X}}$-module. If $\mathcal{F}$ is a 1-twisted $\mathcal{O}_{\mathcal{X}}$-module, then the pullback $f^*\mathcal{F}$ is a 1-twisted $\mathcal{O}_{Y \times_X \mathcal{X}}$-module.

Remark 2.2.3. We can equivalently describe twisted sheaves using hypercoverings (this is discussed in [25, Section 2.1.3]). Given a class $\alpha \in \mathrm{H}^2(\mathcal{C}, \mathbf{G}_{\mathrm{m}})$, there exists a hypercover $U_\bullet$ of the final object $U \in \mathcal{C}$ and a class $\alpha' \in \check{\mathrm{H}}^2(U_\bullet, \mathbf{G}_{\mathrm{m}})$ mapping to α under the natural map

$$\check{\mathrm{H}}^2(U_\bullet, \mathbf{G}_\mathrm{m}) \to \mathrm{H}^2(\mathcal{C}, \mathbf{G}_\mathrm{m}).$$

Suppose that $a \in \Gamma(U_2, \mathbf{G}_\mathrm{m})$ is a 2-cocycle representing α' and that $\mathcal{X}$ is a $\mathbf{G}_\mathrm{m}$-gerbe such that $\alpha = [\mathcal{X}]$. Then the condition that $\alpha' \mapsto \alpha$ is equivalent to saying that there exists an object $s \in \mathcal{X}(U_0)$ and an isomorphism $\sigma \in \mathrm{Mor}_{\mathcal{X}(U_1)}(p_1^* s, p_2^* s)$ such that $p_{23}^*\sigma \circ p_{12}^*\sigma = \iota_a \circ p_{13}^*\sigma$ in $\mathrm{Mor}_{\mathcal{X}(U_2)}((p_1 p_{12})^* s, (p_2 p_{13})^* s)$, where ι_a is the image of a under the isomorphism $\Gamma(U_2, \mathbf{G}_\mathrm{m}) \simeq \mathrm{Aut}_{\mathcal{X}(U_2)}((p_2 p_{13})^* s)$.

An *α-twisted sheaf* is a pair $(\mathcal{F}_0, \varphi_1)$ where $\mathcal{F}_0$ is an $\mathcal{O}_{U_0}$-module and $\varphi_1 : p_1^*\mathcal{F}_0 \to p_2^*\mathcal{F}_0$ is an $\mathcal{O}_{U_1}$-module isomorphism satisfying the condition $p_{23}^*\varphi_1 \circ p_{12}^*\varphi_1 = a \cdot p_{13}^*\varphi_1$ on U_2. Given a 1-twisted $\mathcal{O}_\mathcal{X}$-module $\mathcal{F}$, the canonical isomorphism

$$\sigma_\mathcal{F} : p_1^* s^* \mathcal{F} \to p_2^* s^* \mathcal{F}$$

satisfies $p_{23}^*\sigma_\mathcal{F} \circ p_{12}^*\sigma_\mathcal{F} = a \cdot p_{13}^*\sigma_\mathcal{F}$ on U_2, hence $(s^*\mathcal{F}, \sigma_\mathcal{F})$ is an α-twisted sheaf. As verified in [24, Section 2.1.3], the assignment $\mathcal{F} \mapsto (s^*\mathcal{F}, \sigma_\mathcal{F})$ induces an equivalence of categories between the category of 1-twisted quasi-coherent $\mathcal{O}_\mathcal{X}$-modules and the category of quasi-coherent α-twisted sheaves. Furthermore, for each of the properties P in Definition 2.1.1, the $\mathcal{O}_\mathcal{X}$-module $\mathcal{F}$ satisfies P if and only if the $\mathcal{O}_{U_0}$-module $\mathcal{F}_0$ satisfies P.

Definition 2.2.4. For a locally ringed site $\mathcal{C}$, we define $\mathrm{LPBr}(\mathcal{C})$ (resp. $\mathrm{LFBr}(\mathcal{C})$) to be the subset of classes $\alpha \in \mathrm{H}^2(\mathcal{C}, \mathbf{G}_\mathrm{m})$ such that any $\mathbf{G}_\mathrm{m}$-gerbe $\mathcal{X} \to \mathcal{C}$ with $[\mathcal{X}] = \alpha$ admits a countably generated locally projective (resp. locally free) 1-twisted $\mathcal{O}_\mathcal{X}$-module of positive rank.

Lemma 2.2.5. *For a locally ringed site $\mathcal{C}$, the subsets* $\mathrm{LPBr}(\mathcal{C})$ *and* $\mathrm{LFBr}(\mathcal{C})$ *are additive submonoids of* $\mathrm{H}^2(\mathcal{C}, \mathbf{G}_\mathrm{m})$.

Proof. The identity character $\mathbf{G}_\mathrm{m} \to \mathbf{G}_\mathrm{m}$ gives a 1-twisted invertible sheaf on the trivial gerbe $\mathrm{B}\mathbf{G}_\mathrm{m}$, so $0 \in \mathrm{LFBr}(\mathcal{C})$. To prove that $\mathrm{LPBr}(\mathcal{C})$ and $\mathrm{LFBr}(\mathcal{C})$ are closed under addition, we use the language of α-twisted sheaves as in Remark 2.2.3. Let $\alpha_1, \alpha_2 \in \mathrm{H}^2(\mathcal{C}, \mathbf{G}_\mathrm{m})$ be classes and let $U_\bullet$ be a hypercovering of $\mathcal{C}$ such that both α_1, α_2 are representable on $U_\bullet$. If $(\mathcal{F}_{i,0}, \varphi_{i,1})$ is an α_i-twisted sheaf for $i = 1, 2$, then $(\mathcal{F}_{1,0} \otimes_{\mathcal{O}_{U_0}} \mathcal{F}_{2,0}, \varphi_{1,1} \otimes \varphi_{2,1})$ is an $(\alpha_1 + \alpha_2)$-twisted sheaf. If $\alpha_1, \alpha_2 \in \mathrm{LPBr}(\mathcal{C})$ (resp. $\alpha_1, \alpha_2 \in \mathrm{LFBr}(\mathcal{C})$) and the $\mathcal{F}_{1,0}, \mathcal{F}_{2,0}$ are countably generated locally projective (resp. locally free) of positive rank, then $\mathcal{F}_{1,0} \otimes_{\mathcal{O}_{U_0}} \mathcal{F}_{2,0}$ is also countably generated locally projective (resp. locally free) of positive rank. ∎

Remark 2.2.6. Suppose $(\mathcal{F}_0, \varphi_1)$ is a locally projective α-twisted $\mathcal{O}_\mathcal{C}$-module. If $\mathcal{F}_0$ has finite rank, then $(\mathcal{H}om_{\mathcal{O}_\mathcal{C}}(\mathcal{F}_0, \mathcal{O}_\mathcal{C}), \varphi_1^\vee)$ is a locally projective $(-\alpha)$-twisted $\mathcal{O}_\mathcal{C}$-module. However, if $\mathcal{F}_0$ does not have finite rank, then $\mathcal{H}om_{\mathcal{O}_\mathcal{C}}(\mathcal{F}_0, \mathcal{O}_\mathcal{C})$ need not be quasi-coherent (in particular, not locally projective).

Question 2.2.7. Does there exist an algebraic space X such that $\mathrm{LPBr}(X)$ is not a subgroup of $\mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_\mathrm{m})$ (i.e., does not contain additive inverses)?

Question 2.2.8. For which algebraic spaces X is the inclusion $\mathrm{LPBr}(X) \to \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})$ an equality?

Remark 2.2.9. For any algebraic space X, let $\mathrm{Br}(X)$ (resp. $\mathrm{Br}'(X)$) be the Brauer group (resp. cohomological Brauer group) of X. We have inclusions

$$\begin{array}{ccccc} \mathrm{Br}(X) & & \longrightarrow & & \mathrm{Br}'(X) \\ \downarrow & & & & \downarrow \\ \mathrm{LFBr}(X) & \longrightarrow & \mathrm{LPBr}(X) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}}) \end{array}$$

of additive submonoids of $\mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})$. The classes in $\mathrm{Br}(X)$ correspond to $\mathbf{G}_{\mathrm{m}}$-gerbes $\mathcal{X}$ admitting a 1-twisted locally free $\mathcal{O}_{\mathcal{X}}$-module of finite rank. By a theorem of Gabber [6], the inclusion $\mathrm{Br}(X) \subseteq \mathrm{Br}'(X)$ is known to be an equality if X is a quasi-compact scheme admitting an ample line bundle.

Lemma 2.2.10. *For a locally Noetherian algebraic space X, we have*

$$\mathrm{LFBr}(X) = \mathrm{LPBr}(X).$$

Proof. Let $\mathcal{X} \to X$ be a $\mathbf{G}_{\mathrm{m}}$-gerbe admitting a countably generated locally projective 1-twisted $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{E}$ of positive rank. Let X' be a locally Noetherian scheme with an étale surjection $X' \to X$ such that $\mathcal{X}_{X'}$ is trivial. For any section $s : X' \to \mathcal{X}_{X'}$, the restriction $s^*(\mathcal{E}|_{\mathcal{X}_{X'}})$ is a locally projective $\mathcal{O}_{X'}$-module of positive rank. For an affine open subscheme $U = \operatorname{Spec} A$ of X', the A-module $\Gamma(U, \mathcal{E}|_U)$ is projective by Theorem 2.1.4. If $\Gamma(U, \mathcal{E}|_U)$ has finite rank, then $\mathcal{E}|_U$ is Zariski-locally free; otherwise, since A is Noetherian, $\Gamma(U, \mathcal{E}|_U)$ is free by Bass' theorem [2]. ∎

2.3. Azumaya algebras

Definition 2.3.1 (Azumaya algebras). Let $\mathcal{C}$ be a locally ringed site. Given an $\mathcal{O}_{\mathcal{C}}$-algebra $\mathcal{A}$, a *trivialization* of $\mathcal{A}$ is a pair

$$(\mathcal{E}, \varphi)$$

where $\mathcal{E}$ is a locally free $\mathcal{O}_{\mathcal{C}}$-module of positive rank, and $\varphi : \mathcal{A} \to \mathcal{E}nd_{\mathcal{O}_{\mathcal{C}}}(\mathcal{E})$ is an $\mathcal{O}_U$-algebra isomorphism. We say that $\mathcal{A}$ is an *Azumaya $\mathcal{O}_{\mathcal{C}}$-algebra* if for every object $U \in \mathcal{C}$ there exist

(1) a covering $\{U_i \to U\}_{i \in I}$ and

(2) a trivialization $(\mathcal{E}_i, \varphi_i)$ of the restriction $\mathcal{A}|_{U_i}$ for all $i \in I$

such that

(∗) for all $i_1, i_2 \in I$, there exists a covering $\{V_\lambda \to U_{i_1} \times_U U_{i_2}\}_{\lambda \in I_{i_1,i_2}}$ such that $\mathcal{E}_{i_1}|_{V_\lambda}$ and $\mathcal{E}_{i_2}|_{V_\lambda}$ are free and the $\mathcal{O}_{V_\lambda}$-algebra isomorphism

$$\varphi_{i_2}|_{V_\lambda} \circ (\varphi_{i_1}|_{V_\lambda})^{-1} : \mathcal{E}nd_{V_\lambda}(\mathcal{E}_{i_1}|_{V_\lambda}) \to \mathcal{E}nd_{V_\lambda}(\mathcal{E}_{i_2}|_{V_\lambda})$$

is induced by an $\mathcal{O}_{V_\lambda}$-module isomorphism $\mathcal{E}_{i_1}|_{V_\lambda} \to \mathcal{E}_{i_2}|_{V_\lambda}$.

Remark 2.3.2. According to Definition 2.3.1, an Azumaya $\mathcal{O}_X$-algebra $\mathcal{A}$ is not quasi-coherent unless it has finite rank. If the structure sheaf $\mathcal{O}_{\mathcal{C}}$ is locally Noetherian, then condition Definition 2.3.1(i) is automatically satisfied by Theorem A.1.3.

Definition 2.3.3 (The Brauer map). Let $\mathcal{A}$ be an Azumaya $\mathcal{O}_{\mathcal{C}}$-algebra. We define the *gerbe of trivializations* of $\mathcal{A}$ to be the category $\mathcal{G}_{\mathcal{A}}$ whose objects are tuples $(U, \mathcal{E}, \varphi)$ where $U \in \mathcal{C}$ is an object and $(\mathcal{E}, \varphi)$ is a trivialization of $\mathcal{A}|_U$, and a morphism

$$(U_1, \mathcal{E}_1, \varphi_1) \to (U_2, \mathcal{E}_2, \varphi_2)$$

is a pair (f, ρ) where $f : U_1 \to U_2$ is a morphism in $\mathcal{C}$ and $\rho : f^*\mathcal{E}_2 \to \mathcal{E}_1$ is an $\mathcal{O}_{U_1}$-module isomorphism such that the diagram

$$\begin{array}{ccc} \mathcal{A}|_{U_1} & = & f^*\mathcal{A}|_{U_2} \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle f^*\varphi_2} \\ \mathcal{E}nd_{\mathcal{O}_{U_1}}(\mathcal{E}_1) & \xrightarrow[c_\rho]{} & \mathcal{E}nd_{\mathcal{O}_{U_1}}(f^*\mathcal{E}_2) \end{array}$$

commutes, where c_ρ is the conjugation-by-ρ map. Then $\mathcal{G}_{\mathcal{A}}$ is a stack fibered in groupoids over $\mathcal{C}$, and it is a $\mathbf{G}_{\mathrm{m}}$-gerbe by Definition 2.3.1(2), (i) and Lemma A.2.1. The association $\mathcal{A} \mapsto \mathcal{G}_{\mathcal{A}}$ defines an extension of the usual Brauer map.

Proposition 2.3.4. *Let X be an algebraic space, let $\alpha \in \mathrm{H}^2_{\acute{e}t}(X, \mathbf{G}_{\mathrm{m}})$ be a class. The following conditions are equivalent.*

(i) *We have $\alpha = [\mathcal{G}_{\mathcal{A}}]$ for some Azumaya $\mathcal{O}_X$-algebra $\mathcal{A}$.*

(ii) *The class α is in* $\mathrm{LFBr}(X)$.

Proof. Let $\mathcal{G}$ be the $\mathbf{G}_{\mathrm{m}}$-gerbe corresponding to α.

(i)⇒(ii): Let $\mathcal{A}$ be an Azumaya algebra on X and suppose that $\mathcal{G}$ is the gerbe of trivializations of $\mathcal{A}$. The assignment $(U, \mathcal{E}, \varphi) \mapsto (U, \mathcal{E})$ and $(f, \rho) \mapsto (f, \rho)$ defines a 1-twisted locally free $\mathcal{O}_{\mathcal{G}}$-module.

(ii)⇒(i): Suppose there exists a 1-twisted locally free $\mathcal{O}_{\mathcal{G}}$-module $\mathcal{E}$. The endomorphism sheaf $\mathcal{E}nd_{\mathcal{O}_{\mathcal{G}}}(\mathcal{E})$ is a 0-twisted $\mathcal{O}_{\mathcal{G}}$-algebra, hence is isomorphic to the pullback of some $\mathcal{O}_X$-algebra $\mathcal{A}$. There exists an étale cover $X' \to X$ such that $\mathcal{G}' := X' \times_X \mathcal{G}$ admits a 1-twisted line bundle $\mathcal{L}'$. If we denote by $\mathcal{E}'$ the $\mathcal{O}_{X'}$-module whose pullback to $\mathcal{G}'$ is $\mathcal{E}|_{\mathcal{G}'} \otimes_{\mathcal{O}_{\mathcal{G}'}} \mathcal{L}'^{-1}$, then we have an $\mathcal{O}_{X'}$-algebra isomorphism $\varphi : \mathcal{A}|_{X'} \to \mathcal{E}nd_{\mathcal{O}_{X'}}(\mathcal{E}')$. Let $X'' := X' \times_X X'$ and consider the $\mathcal{O}_{X''}$-algebra isomorphism

$$p_2^*\varphi \circ (p_1^*\varphi)^{-1} : \mathcal{E}nd_{\mathcal{O}_{X''}}(p_1^*\mathcal{E}') \to \mathcal{E}nd_{\mathcal{O}_{X''}}(p_2^*\mathcal{E}') \tag{2.3.4.1}$$

where $p_1, p_2 : X'' \to X$ denote the two projections. If we take a cover $\{V_\lambda \to X''\}_{\lambda \in \Lambda}$ which trivializes $p_1^*\mathcal{L}' \otimes_{\mathcal{O}_{\mathcal{G}''}} (p_2^*\mathcal{L}')^\vee$, then the restriction of (2.3.4.1) to V_λ is induced by an $\mathcal{O}_{\mathcal{G} \times_X V_\lambda}$-module isomorphism $(p_1^*\mathcal{L}')|_{\mathcal{G} \times_X V_\lambda} \to (p_2^*\mathcal{L}')|_{\mathcal{G} \times_X V_\lambda}$. Thus $\mathcal{A}$ is an Azumaya algebra. ∎

2.4. Brauer–Severi varieties

Let I be an index set, let $\mathbf{Z}^{\oplus I}$ be a free abelian group with basis indexed by I, and define *projective I-space* to be

$$\mathbf{P}^I_{\operatorname{Spec}\mathbf{Z}} := \operatorname{Proj}_{\operatorname{Spec}\mathbf{Z}} \operatorname{Sym}^\bullet_{\mathbf{Z}} \mathbf{Z}^{\oplus I}.$$

As in Bass' theorem, from which we know that infinite-rank vector bundles on finite-type schemes are simple, finite-rank vector bundles on infinite-dimensional varieties are also simple: by work of Barth–van de Ven [1], Tyurin [34], Sato [31], it is known that finite rank vector bundles on $\mathbf{P}^{\mathbf{N}}_{\mathbf{C}}$ are direct sums of line bundles.

We define a Brauer–Severi variety to be an étale twist of $\mathbf{P}^I$ for some I:

Definition 2.4.1. Let $X \to S$ be a morphism of schemes. We say that X is a *Brauer–Severi scheme* over S if there exists an étale surjection $S' \to S$, an index set I, and an isomorphism $X \times_S S' \simeq \mathbf{P}^I_{S'}$ of S'-schemes.

Definition 2.4.2. Let $\mathrm{GL}_I : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Grp}$ be the sheaf of groups $S \mapsto \operatorname{Aut}_{\mathcal{O}_S\text{-mod}}(\mathcal{O}_S^{\oplus I})$ and let PGL_I be the quotient $\mathrm{GL}_I / \mathbf{G}_{\mathrm{m}}$. A module automorphism of $\mathcal{O}_S^{\oplus I}$ induces a graded algebra automorphism of $\operatorname{Sym}^\bullet_{\mathcal{O}_S} \mathcal{O}_S^{\oplus I}$, hence a scheme automorphism of $\mathbf{P}^I_S$. Thus we obtain a natural homomorphism

$$\mathrm{PGL}_I \to \operatorname{Aut}_{\mathrm{sch}}(\mathbf{P}^I) \tag{2.4.2.1}$$

of sheaves of groups on Sch. (Note that neither GL_I nor PGL_I are representable if I is infinite.)

Lemma 2.4.3. *The map* (2.4.2.1) *is an isomorphism.*

Proof. Let S be a scheme and let $\varphi : \mathbf{P}^I_S \to \mathbf{P}^I_S$ be an S-scheme automorphism. By [17, Proposition 4.2.3], a morphism $T \to \mathbf{P}^I_{\operatorname{Spec}\mathbf{Z}}$ is given by an invertible $\mathcal{O}_T$-module $\mathcal{L}$ and an I-indexed collection of global sections $\sigma_i \in \Gamma(T, \mathcal{L})$ which are globally generating, i.e.,

$$T = \bigcup_{i \in I} T_{\sigma_i}.$$

By Lemma 2.4.4, after taking a Zariski cover of S, we may assume that $S = \operatorname{Spec} A$ is affine and that φ is the morphism corresponding to the line bundle $\mathcal{O}_{\mathbf{P}^I_S}(n)$ and globally generating sections $\{\sigma_i\}_{i\in I}$. Since φ induces an automorphism of $\operatorname{Pic}(\mathbf{P}^I_S)$, we have $n = 1$ (note that $\Gamma(\mathbf{P}^I_S, \mathcal{O}_{\mathbf{P}^I_S}(-1)) = 0$). Let $\{\sigma_i\}_{i\in I}$ and $\{\tau_i\}_{i\in I}$ be the I-indexed collections of sections corresponding to φ and φ^{-1}, respectively. For any I-indexed collection of sections $v_i \in \Gamma(\mathbf{P}^I_S, \mathcal{O}_{\mathbf{P}^I_S}(1))$, let $\left[\{v_i\}_{i\in I}\right]$ denote the $I \times 1$ column vector whose ith entry is v_i. Then there exist matrices $\mathsf{M}, \mathsf{N} \in \operatorname{Mat}_{I\times I}(A)$ such that

$$\left[\{\sigma_i\}_{i\in I}\right] = \mathsf{M}\left[\{x_i\}_{i\in I}\right] \quad \text{and} \quad \left[\{\tau_i\}_{i\in I}\right] = \mathsf{N}\left[\{x_i\}_{i\in I}\right].$$

Since $\mathsf{NM}\left[\{x_i\}_{i\in I}\right]$ corresponds to the identity morphism, by [17, Proposition 4.2.3] there exists a unit $u \in A^\times$ such that $\mathsf{NM} = u \cdot \mathrm{id}_I$, thus $\mathsf{M} \in \mathrm{GL}_I(A)$. ∎

In Lemma 2.4.4, we compute the Picard group of infinite-dimensional projective spaces over an arbitrary scheme (the case $S = \operatorname{Spec} k$ for a field k was proved by Tyurin [34, Proposition 1.1 (1)]).

Lemma 2.4.4. *For any scheme S, the map*

$$\operatorname{Pic}(S) \oplus \Gamma(S, \underline{\mathbf{Z}}) \to \operatorname{Pic}(\mathbf{P}^I_S) \tag{2.4.4.1}$$

sending $(\mathcal{L}, n) \mapsto \mathcal{L} \otimes \mathcal{O}_{\mathbf{P}^I_S}(n)$ is surjective.

Proof. Let $\mathcal{L}$ be a line bundle on $\mathbf{P}^I_S$. Let Λ denote the set of finite subsets of I, so that I is the filtered union $I = \varinjlim_{\lambda \in \Lambda} \lambda$. For each $\lambda \in \Lambda$, we have a projection map $\mathbf{Z}^{\oplus I} \to \mathbf{Z}^{\oplus \lambda}$ which induces a closed immersion $\mathbf{P}^\lambda_S \to \mathbf{P}^I_S$. We know that (2.4.4.1) is an isomorphism when I is finite, thus there exist invertible $\mathcal{O}_S$-modules $\mathcal{M}_\lambda$ and integers n_λ such that

$$\mathcal{L}|_{\mathbf{P}^\lambda_S} \simeq \mathcal{M}_\lambda|_{\mathbf{P}^\lambda_S} \otimes_{\mathcal{O}_{\mathbf{P}^\lambda_S}} \mathcal{O}_{\mathbf{P}^\lambda_S}(n_\lambda)$$

on $\mathbf{P}^\lambda_S$. Since we have linear transition maps $\mathbf{P}^\lambda_S \to \mathbf{P}^{\lambda'}_S$, we have in fact $n_\lambda = n_{\lambda'}$ and $\mathcal{M}_\lambda \simeq \mathcal{M}_{\lambda'}$ for any two $\lambda, \lambda' \in \Lambda$. Thus, after taking a Zariski cover of S and replacing $\mathcal{L}$ by $\mathcal{L} \otimes_{\mathcal{O}_{\mathbf{P}^I_S}} (\mathcal{M}^{-1}|_{\mathbf{P}^I_S} \otimes_{\mathcal{O}_{\mathbf{P}^I_S}} \mathcal{O}_{\mathbf{P}^I_S}(-n))$ for this common invertible $\mathcal{O}_S$-module $\mathcal{M}$ and integer n, we may assume that in fact $\mathcal{L}|_{\mathbf{P}^\lambda_S}$ is trivial for all λ, and that $S = \operatorname{Spec} A$ is affine. Let $U_i := \mathrm{D}_+(x_i)$ denote the distinguished affine open subscheme of $\mathbf{P}^I_S$ associated to $i \in I$ and set $U_\lambda := \bigcap_{i \in \lambda} U_i$ for all λ. By Lemma 2.4.5, the restriction $\mathcal{L}|_{U_i}$ is trivial for all $i \in I$. Then $\mathcal{L}$ is described by a collection of units $u_{i_1,i_2} \in \Gamma(U_{\{i_1,i_2\}}, \mathbf{G}_{\mathrm{m}})$ for $i_1, i_2 \in I$, corresponding to the transition map between trivializations of $\mathcal{L}$ on $U_{\{i_1,i_2\}}$. Fix two distinct indices $i_1^\circ, i_2^\circ \in I$ and let λ be a finite subset of I containing the indices of all variables x_i that appear in $u_{i_1^\circ,i_2^\circ}$. Then $\mathcal{L}|_{U_{\{i_1^\circ,i_2^\circ\}}}$ is isomorphic to the pullback of a line bundle on $\mathbf{P}^\lambda_S$ via the projection map $\mathbf{P}^I_S \setminus \mathrm{V}_+(\{x_i\}_{i \in I \setminus \lambda}) \to \mathbf{P}^\lambda_S$. Since $\mathcal{L}|_{\mathbf{P}^\lambda_S}$ is trivial by the above, the restriction $\mathcal{L}|_{U_{\{i_1^\circ,i_2^\circ\}}}$ is trivial. Hence, we may assume that $u_{i_1^\circ,i_2^\circ} = 1$. For all $i' \in I \setminus \{i_1^\circ, i_2^\circ\}$, the sequence

$$0 \to \Gamma(U_{i'}, \mathcal{O}_{\mathbf{P}^I_S}) \to \Gamma(U_{\{i',i_1^\circ\}}, \mathcal{O}_{\mathbf{P}^I_S}) \oplus \Gamma(U_{\{i',i_2^\circ\}}, \mathcal{O}_{\mathbf{P}^I_S}) \to \Gamma(U_{\{i',i_1^\circ,i_2^\circ\}}, \mathcal{O}_{\mathbf{P}^I_S})$$

is exact, so we may assume that $u_{i',i_1^\circ} = u_{i',i_2^\circ} = 1$ for all $i' \in I$, and (by the same argument) that $u_{i,i'} = 1$ for all $i, i' \in I$. ∎

Lemma 2.4.5. *Let A be a filtered colimit of subrings $A = \varinjlim_{\lambda \in \Lambda} A_\lambda$ such that, for all λ, the inclusion $A_\lambda \to A$ admits a (ring-theoretic) retraction $A \to A_\lambda$. Suppose M is an invertible A-module such that $M \otimes_A A_\lambda$ is a trivial A_λ-module for all λ. Then M is trivial.*

Proof. Since M is a finitely presented A-module, there exists some $\lambda \in \Lambda$ and a finitely presented A_λ-module M_λ such that $M_\lambda \otimes_{A_\lambda} A \simeq M$. Tensoring by $- \otimes_A A_\lambda$ then implies that M_λ is trivial, hence M is trivial. ∎

Using Lemma 2.4.3, we may extend the dictionary between Azumaya algebras and twisted vector bundles (2.3.4) to include Brauer–Severi varieties.

Proposition 2.4.6. *For any scheme S and index set I, there is a bijective correspondence between isomorphism classes of Azumaya $\mathcal{O}_S$-algebras of rank I and Brauer–Severi schemes of relative dimension I.*

Proof. Both are étale PGL_I-torsors on S, by Definition 2.3.1 and Lemma 2.4.3 respectively. ■

3. Pushing forward twisted sheaves via affine-pure morphisms

3.1. Affine-pure morphisms

Definition 3.1.1. A morphism $f : X \to Y$ of algebraic spaces is said to be *affine-pure* if f is affine and $f_*\mathcal{O}_X$ is a locally projective $\mathcal{O}_Y$-module.

Lemma 3.1.2. [3, Lemma 3.5 (γ)] *Let $B \to A$ be a ring map such that A is a projective B-module, and let M be a projective A-module. Then M is projective as a B-module.*

Proof. There exist a split B-linear surjection $B^{\oplus I} \to A$ and a split A-linear surjection $A^{\oplus J} \to M$, hence a split B-linear surjection $(B^{\oplus I})^{\oplus J} \to M$. ■

Lemma 3.1.3. *Let A be a ring, let M be a projective A-module, set $S := \operatorname{Spec} A$ and let $\mathcal{E}$ be the quasi-coherent $\mathcal{O}_S$-module corresponding to M. Then $\mathcal{E}$ has positive rank if and only if $M \otimes_A \kappa(\mathfrak{p}) \neq 0$ for all primes $\mathfrak{p}$ of A.*

Proof. The "only if" implication is clear. For the "if" implication, choose an index set I and a A-linear surjection $\pi : A^{\oplus I} \to M$. Then π admits an A-linear section $\xi : M \to A^{\oplus I}$. Denoting by $\{\mathrm{e}_i\}_{i\in I}$ the basis of $A^{\oplus I}$, set $\alpha_i := \xi(\pi(\mathrm{e}_i)) = (\alpha_{i,j})_{i\in I}$, so that $\{\alpha_i\}_{i\in I}$ constitutes a set of generators for M viewed as a A-submodule of $A^{\oplus I}$. Let $\mathfrak{p}$ be a prime ideal of A. Since $M \otimes_A \kappa(\mathfrak{p})$ is the image of $\xi\pi \otimes \mathrm{id}_{\kappa(\mathfrak{p})}$, by hypothesis there exists some $(i, j) \in I^2$ such that $\alpha_{i,j} \notin \mathfrak{p}$. After inverting $\alpha_{i,j}$, we may assume that $\alpha_{i,j}$ is a unit. Then the composition $p_i\xi\pi : A^{\oplus I} \to A$ is surjective, where p_i is the projection onto the ith coordinate of $A^{\oplus I}$. Hence $p_i\xi : M \to A$ is surjective. ■

Lemma 3.1.4. *Let $f : X \to Y$ be an affine-pure morphism of algebraic spaces, let $\mathcal{E}$ be a locally projective $\mathcal{O}_X$-module. Then $f_*\mathcal{E}$ is a locally projective $\mathcal{O}_Y$-module. If in addition f is surjective and $\mathcal{E}$ has positive rank, then $f_*\mathcal{E}$ has positive rank.*

Proof. We may work étale-locally on Y so that we may assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine. Then the first claim follows from Theorem 2.1.4 and Lemma 3.1.2. For the second claim, we use Lemma 3.1.3. Let $\mathfrak{p}$ be a prime of A lying over the prime $\mathfrak{q}$ of B. The surjection $A \otimes_B \kappa(\mathfrak{q}) \to \kappa(\mathfrak{p})$ induces a surjection $M \otimes_B \kappa(\mathfrak{q}) \to M \otimes_A \kappa(\mathfrak{p})$ of A-modules, so if $M \otimes_A \kappa(\mathfrak{p}) \neq 0$ then $M \otimes_B \kappa(\mathfrak{q}) \neq 0$ as well. ■

Lemma 3.1.5. *Let $f : X \to Y$ be a morphism of algebraic spaces.*

(i) *For any morphism $Y' \to Y$, let $f' : X' \to Y'$ be the base change of f. If f is affine-pure, then f' is affine-pure.*

(ii) *Suppose there is an fpqc cover $\{Y_i \to Y\}_{i \in I}$ such that the base change $f_i : X_i \to Y_i$ is affine-pure for each $i \in I$. Then f is affine-pure.*

(iii) *Let $g : Y \to Z$ be a morphism of algebraic spaces. If f and g are affine-pure, then $g \circ f$ is affine-pure.*

Proof. (i) The property "affine" is stable under arbitrary base change. Since f is affine, the map $(f_* \mathcal{O}_X)|_{Y'} \to (f')_* \mathcal{O}_{Y'}$ is an isomorphism.

(ii) The property "affine" is fpqc local on the base. For each i, the pushforward $(f_i)_* \mathcal{O}_{X_i}$ is a locally projective $\mathcal{O}_{Y_i}$-module. As above, the map $(f_* \mathcal{O}_X)|_{Y_i} \to (f_i)_* \mathcal{O}_{X_i}$ is an isomorphism, thus $f_* \mathcal{O}_X$ is a locally projective $\mathcal{O}_Y$-module by Theorem 2.1.4.

(iii) The property "affine" is stable under composition. The pushforward $f_* \mathcal{O}_X$ is a locally projective $\mathcal{O}_Y$-module by assumption, so $g_* f_* \mathcal{O}_X$ is a locally projective $\mathcal{O}_Z$-module by Lemma 3.1.4. ∎

We will use the following lemma, which is an analogue of [11, Chapter II, Lemma 4] in the infinite rank case:

Lemma 3.1.6. *Let $f : X \to Y$ be a surjective, affine-pure, finitely presented morphism. If $\alpha \in \mathrm{H}^2_{ét}(Y, \mathbf{G}_{\mathrm{m}})$ is an element such that $f^* \alpha$ is in* $\mathrm{LPBr}(X)$*, then α is in* $\mathrm{LPBr}(Y)$.

Proof. Let $\mathcal{Y} \to Y$ be a $\mathbf{G}_{\mathrm{m}}$-gerbe corresponding to α, set $\mathcal{X} := \mathcal{Y} \times_Y X$ with projection map $\pi : \mathcal{X} \to \mathcal{Y}$. The morphism π is surjective, affine-pure (by Lemma 3.1.5 (i)), and finitely presented. Let $\mathcal{E}$ be a countably generated locally projective 1-twisted $\mathcal{O}_{\mathcal{X}}$-module of positive rank. Then $\pi_* \mathcal{E}$ is a locally projective 1-twisted $\mathcal{O}_{\mathcal{Y}}$-module which is of positive rank by Lemma 3.1.4 and is countably generated since f is finitely presented. ∎

Lemma 3.1.7. *Let S be an algebraic space and let $f : X \to Y$ be a morphism of algebraic spaces over S. If f satisfies at least one of the following conditions, then f is affine-pure.*

(a) *f is affine, flat, of finite presentation and all geometric fibers are integral,*

(b) *f is affine, flat, of finite presentation and all geometric fibers are Cohen–Macaulay and irreducible,*

(c) *f is an fppf G-torsor for an S-group scheme G such that $G \to S$ is an affine-pure morphism,*

(d) *f is an fppf G-torsor for an S-group scheme G such that $G \to S$ is an affine, flat, finitely presented morphism, all of whose geometric fibers are irreducible.*

Proof. (a) This is a restatement of [33, Tag 05FT] which generalizes [30, I, Proposition 3.3.1] (the case when f is smooth).

(b) By [30, I, Théorème 3.3.5], it is enough to prove that $\mathcal{O}_X$ is "pure over Y" as defined in [30, I, Définition 3.3.3]. For this, we choose a point $y \in Y$ and replace Y by the Henselization Spec $\mathcal{O}^{\mathrm{h}}_{Y,y}$ which reduces the task to showing that, for any point $x' \in X$

with image $y' := f(x)$ such that x' is an associated point of its fiber $X_{y'}$, the closure $\overline{\{x'\}}$ of x' in X intersects the special fiber X_y. Let x be the generic point of X_y. By going-down for flatness, there exists some $x'' \in X_{y'}$ such that $x \in \overline{\{x''\}}$. Since $X_{y'}$ is Cohen–Macaulay, we deduce by [33, Tag 031Q] that x' is the generic point of $X_{y'}$. Thus we have $x'' \in \overline{\{x'\}}$, which implies $x \in \overline{\{x'\}}$.

(c) By definition, there exists an fppf cover $Y' \to Y$ and an isomorphism of Y'-schemes $X|_{Y'} \simeq G_{Y'}$. Thus f is affine-pure by Lemma 3.1.5 (i) and (ii).

(d) By (c), it suffices to show that $G \to S$ is affine-pure. By [19, Exp. VIA, Proposition 1.1.1], the geometric fibers of $G \to S$ are Cohen–Macaulay, so this follows from (b). ∎

Lemma 3.1.8. *Let X be an algebraic space having at least one of the following properties:*

(a) *X has an ample family of invertible modules* [4, II, Définition 2.2.4], [33, Tag 0FXR],

(b) *X has an ample invertible module,*

(c) *X is quasi-projective over an affine scheme,*

(d) *X is quasi-affine,*

(e) *X is quasi-compact and quasi-separated and has the resolution property* [35], *namely every quasi-coherent $\mathcal{O}_X$-module of finite type $\mathcal{F}$ admits a surjective $\mathcal{O}_X$-linear map from a finite locally free $\mathcal{O}_X$-module,*

(f) *X is an algebraic surface (i.e., finite type and separated over a field, and* $\dim X = 2$*),*

(g) *there exists a base scheme S, an S-algebraic space Y satisfying at least one of the conditions* (a)–(f)*, an S-group scheme G acting freely on Y and such that $G \to S$ is affine-pure and finitely presented, and X is the quotient Y/G.*

Then there exists an affine scheme T and a surjective, affine-pure, finitely presented morphism $T \to X$.

Proof. (a) This follows from Thomason's extension of the Jouanolou trick [36, Proposition 4.4] which implies the existence of an affine scheme T and a smooth surjection $T \to X$ such that there exists a Zariski cover $U \to X$ and an isomorphism $T \times_X U \simeq \mathbf{A}^n_U$ of U-schemes. The map $\mathbf{A}^n_{\mathbf{Z}} \to \operatorname{Spec} \mathbf{Z}$ is affine-pure so $T \to X$ is affine-pure by Lemma 3.1.5 (i), (ii).

(b) This is a special case of (a).

(c) This is a special case of (b).

(d) This is a special case of (c).

(e) By [15, Theorem 1.1] we can write $X = X'/\operatorname{GL}_N$ for a quasi-affine scheme X'. Then $X' \to X$ is surjective and affine-pure by Lemma 3.1.7 (d). By case (d), we can find an affine scheme T with a surjective, affine-pure morphism $T \to X'$. Then Lemma 3.1.5 (iii) implies that $T \to X' \to X$ is affine-pure.

(f) This follows from (e) by [27, Theorem 41] (see [14, Corollary 5.3] for the scheme case).

(g) This follows from Lemmas 3.1.7 (d) and 3.1.5 (iii). ∎

Lemma 3.1.9. *For any algebraic space X as in Lemma 3.1.8 and α in $\mathrm{H}^2_{ét}(X, \mathbf{G}_{\mathrm{m}})$, there exists an affine scheme T and a surjective, affine-pure, finitely presented morphism $T \to X$ such that α is Zariski-locally trivial on T.*

Proof. By Lemma 3.1.8, there exists an affine scheme T' and a surjective affine-pure morphism $T' \to X$. By [33, Tag 01FW] there is an étale surjection $T'' \to T'$ such that $\alpha|_{T''}$ is trivial. We refine the cover $T'' \to T'$ so that T'' is affine. Then [33, Tag 02LH] implies that there exists a surjective, finite locally free $T \to T'$ that factors through $T'' \to T'$ Zariski-locally on T', i.e., $\alpha|_T$ is Zariski-locally trivial. The composite $T \to T' \to X$ is surjective affine-pure by Lemma 3.1.5 (iii). ∎

Question 3.1.10. Given an affine scheme X and a class $\alpha \in \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})$, does there exist a surjective, affine-pure morphism $T \to X$ such that $\alpha|_T$ is trivial?

3.2. Proof of the main theorem

In this section we prove Theorem 3.2.5 and give some applications and questions.

Proposition 3.2.1. *Let A be a Noetherian ring, set $X := \operatorname{Spec} A$, and let $\mathcal{X} \to X$ be a $\mathbf{G}_{\mathrm{m}}$-gerbe. Let $\mathcal{F}$ be a countably generated locally projective 1-twisted $\mathcal{O}_{\mathcal{X}}$-module of positive rank. Then $\mathcal{F}$ is a projective object of the category of quasi-coherent 1-twisted $\mathcal{O}_{\mathcal{X}}$-modules. Moreover, every quasi-coherent 1-twisted $\mathcal{O}_{\mathcal{X}}$-module is a quotient of a direct sum of copies of $\mathcal{F}$.*

Proof. For the first claim, it is enough to show that

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{H}) = 0 \tag{3.2.1.1}$$

for any quasi-coherent 1-twisted $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{H}$. We will show

$$\mathcal{E}xt^1_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{H}) = 0 \tag{3.2.1.2}$$

and

$$\mathrm{H}^1\big(X, \mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{H})\big) = 0 \tag{3.2.1.3}$$

which implies (3.2.1.1) by the local-to-global spectral sequence for Ext.

For (3.2.1.2), let X' be a finitely presented affine X-scheme with an X-morphism $X' \to \mathcal{X}$. If $\mathcal{F}$ has finite rank, then we may replace X' by a Zariski cover so that $\mathcal{F}|_{X'}$ is a free $\mathcal{O}_{X'}$-module of finite rank. Then we have $\mathrm{Ext}^1_{\mathcal{O}_{X'}}(\mathcal{F}|_{X'}, \mathcal{H}|_{X'}) = 0$. If $\mathcal{F}$ has countably infinite rank, then $\mathcal{F}|_{X'}$ is in fact a free $\mathcal{O}_{X'}$-module by Bass' theorem, so

$$\mathrm{Ext}^1_{\mathcal{O}_{X'}}(\mathcal{F}|_{X'}, \mathcal{H}|_{X'}) \simeq \prod_{\mathbf{N}} \mathrm{H}^1(X', \mathcal{H}|_{X'}) = 0.$$

For (3.2.1.3), write

$$\mathcal{F} = \mathrm{colim}_{i \in I} \mathcal{F}_i$$

as a filtered colimit of coherent 1-twisted $\mathcal{O}_{\mathcal{X}}$-modules [25, Proposition 2.2.1.5]. Since $\mathcal{F}$ is countably generated, we can assume the index set I is countable. After further refinement, we may assume $I = \mathbf{N}$ with the usual ordering, i.e., we have

$$\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to \cdots \to \mathcal{F}$$

with colimit $\mathcal{F}$. Set $\mathcal{K}_i := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_i, \mathcal{H})$ and $\mathcal{K} := \lim_{i \in \mathbf{N}} \mathcal{K}_i \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H})$.

Take an étale surjection $W \to X$ where $W = \mathrm{Spec}\, B$ is affine and such that $\alpha|_W$ is trivial. Let L be a 1-twisted invertible $\mathcal{O}_{\mathcal{X}_W}$-module. Then $\mathcal{F}' := \mathcal{F}|_{\mathcal{X}_W} \otimes_{\mathcal{O}_{\mathcal{X}_W}} L^{-1}$ is a 0-twisted locally projective $\mathcal{O}_{\mathcal{X}_W}$-module of countably infinite rank, hence $\mathcal{F}'$ is free by Bass' theorem. By [33, Tag 059Z], $\mathcal{F}'$ is a Mittag-Leffler module, hence

$$\{\mathcal{H}om_{\mathcal{O}_W}(\mathcal{F}_i|_W, \mathcal{H}|_W)\}_{i \in \mathbf{N}}$$

is a Mittag-Leffler system by [33, Tag 059E] (1)⇒(4). Since $W \to X$ is faithfully flat and $\mathcal{H}om_{\mathcal{O}_W}(\mathcal{F}_i|_W, \mathcal{H}|_W) \simeq \mathcal{K}_i|_W$, the system $\{\mathcal{K}_i\}_{i \in \mathbf{N}}$ is Mittag-Leffler. Thus, we may construct another inverse system $\{\mathcal{K}'_i\}_{i \in \mathbf{N}}$ such that $\mathcal{K} \simeq \lim_{i \in \mathbf{N}} \mathcal{K}'_i$ and the transition maps $\mathcal{K}'_{i+1} \to \mathcal{K}'_i$ are surjective. Then we have $\mathrm{H}^1(X, \mathcal{K}) = \mathrm{H}^1(X, \lim \mathcal{K}'_i) = 0$ by [33, Tag 0A0J (3)].

For the second claim, since every quasi-coherent 1-twisted $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{G}$ is the filtered colimit of coherent 1-twisted $\mathcal{O}_{\mathcal{X}}$-submodules, it suffices to show that every coherent 1-twisted $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{G}$ admits a surjection from a direct sum of copies of $\mathcal{F}$. We show that, for a closed point $i : u \to X$, there exists an index set I and a morphism $\mathcal{F}^{\oplus I} \to \mathcal{G}$ which is surjective in a neighborhood of u. By Nakayama's lemma, this is equivalent to requiring that the fiber $\mathcal{F}|_{\mathcal{X}_u}^{\oplus I} \to \mathcal{G}|_{\mathcal{X}_u}$ is surjective. Let $\mathcal{I}_u \subset \mathcal{O}_X$ be the ideal sheaf of i and consider the exact sequence

$$0 \to \mathcal{I}_u\mathcal{G} \to \mathcal{G} \to \mathcal{G}/\mathcal{I}_u\mathcal{G} \to 0$$

of $\mathcal{O}_{\mathcal{X}}$-modules. By the first claim, we have $\mathrm{Ext}^1(\mathcal{F}, \mathcal{I}_u\mathcal{G}) = 0$, hence

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G}/\mathcal{I}_u\mathcal{G})$$

is surjective. Hence it suffices to find an $\mathcal{O}_{\mathcal{X}}$-linear surjection $\mathcal{F}^{\oplus I} \to \mathcal{G}/\mathcal{I}_u\mathcal{G}$ for some index set I, i.e., an $\mathcal{O}_{\mathcal{X}_u}$-linear surjection $\mathcal{F}^{\oplus I}/\mathcal{I}_u\mathcal{F}^{\oplus I} \to \mathcal{G}/\mathcal{I}_u\mathcal{G}$. Since $\mathcal{X}_u$ is a $\mathbf{G}_{\mathrm{m}}$-gerbe over the spectrum of a field, by Wedderburn's theorem [12, Theorem 2.1.3] there is a 1-twisted vector bundle $\mathcal{E}$ on $\mathcal{X}_u$ of minimal positive rank (equal to the index of the Brauer class $[\mathcal{X}_u] \in \mathrm{Br}(\kappa(u))$) and every 1-twisted vector bundle on $\mathcal{X}_u$ is isomorphic to a direct sum of this $\mathcal{E}$. This gives the desired result. ∎

Lemma 3.2.2 (Eilenberg swindle). *Let X be a Noetherian affine scheme, let $\mathcal{X} \to X$ be a $\mathbf{G}_{\mathrm{m}}$-gerbe, let $\mathcal{F}, \mathcal{G}$ be two countably generated locally projective 1-twisted $\mathcal{O}_{\mathcal{X}}$-modules of positive rank. Then $\mathcal{F}^{\oplus \mathbf{N}} \simeq \mathcal{G}^{\oplus \mathbf{N}}$.*

Proof. Set $\mathcal{F}' := \mathcal{F}^{\oplus \mathbf{N}}$ and $\mathcal{G}' := \mathcal{G}^{\oplus \mathbf{N}}$. By Proposition 3.2.1, there exists an index set I and a surjective $\mathcal{O}_{\mathcal{X}}$-linear map $(\mathcal{F}')^{\oplus I} \to \mathcal{G}$. Since $\mathcal{G}$ is countably generated, we may assume $I = \mathbf{N}$. Thus we obtain a surjection $\mathcal{F}' \simeq ((\mathcal{F}')^{\oplus \mathbf{N}})^{\oplus \mathbf{N}} \to \mathcal{G}^{\oplus \mathbf{N}} = \mathcal{G}'$. Since $\mathcal{G}'$ is a projective object in the category of quasi-coherent 1-twisted $\mathcal{O}_{\mathcal{X}}$-modules by Proposition 3.2.1, we obtain a decomposition

$$\mathcal{G}' \oplus \mathcal{Q} \simeq \mathcal{F}'$$

which gives an isomorphism

$$\mathcal{G}' \oplus \mathcal{F}' \simeq \mathcal{G}' \oplus (\mathcal{G}' \oplus \mathcal{Q}) = (\mathcal{G}' \oplus \mathcal{G}') \oplus \mathcal{Q} \simeq \mathcal{G}' \oplus \mathcal{Q} \simeq \mathcal{F}'$$

where we use that $\mathcal{G}' \oplus \mathcal{G}' \simeq \mathcal{G}'$. By symmetry, we have

$$\mathcal{G}' \simeq \mathcal{G}' \oplus \mathcal{F}'$$

which implies $\mathcal{F}' \simeq \mathcal{G}'$. ∎

Question 3.2.3. In Lemma 3.2.2 (which is a twisted analogue of Bass' theorem), can we do without the infinite direct sum, i.e., is it necessarily true that in fact $\mathcal{F} \simeq \mathcal{G}$?

Remark 3.2.4. In Lemma 3.2.2, if one of $\mathcal{F}$ or $\mathcal{G}$ is assumed to be of finite rank, then we may argue as follows, without the need for Proposition 3.2.1. If $\mathcal{G}$ has finite rank, then we have isomorphisms

$$\mathcal{F}^{\oplus \mathbf{N}} \simeq \mathcal{F} \otimes \mathcal{O}_X^{\oplus \mathbf{N}} \overset{*}{\simeq} \mathcal{F} \otimes (\mathcal{G}^\vee \otimes \mathcal{G})^{\oplus \mathbf{N}} \simeq (\mathcal{F} \otimes \mathcal{G}^\vee)^{\oplus \mathbf{N}} \otimes \mathcal{G} \overset{*}{\simeq} \mathcal{O}_X^{\oplus \mathbf{N}} \otimes \mathcal{G} \simeq \mathcal{G}^{\oplus \mathbf{N}}$$

of $\mathcal{O}_{\mathcal{X}}$-modules where those labeled $*$ follow from Bass [2].

Theorem 3.2.5. *For X as in Lemma 3.1.8, we have* $\mathrm{LPBr}(X) = \mathrm{H}^2_{\acute{e}t}(X, \mathbf{G}_{\mathrm{m}})$.

Proof. Let $\alpha \in \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})$ and let $\mathcal{X} \to X$ be a $\mathbf{G}_{\mathrm{m}}$-gerbe corresponding to α. By Lemma 3.1.9 there exists an affine scheme T and a surjective affine-pure morphism $T \to X$ such that $\alpha|_T$ is Zariski-locally trivial. Moreover, by Lemma 3.1.6, if $\alpha|_T \in \mathrm{LPBr}(T)$ then $\alpha \in \mathrm{LPBr}(X)$. Thus we may reduce to the case when X is affine and α is Zariski-locally trivial. By absolute Noetherian approximation [33, Tags 01ZA, 09YQ] we may also assume that X is of finite type over $\mathbf{Z}$.

Suppose $X = \operatorname{Spec} A$ and $f_1, \ldots, f_n \in A$ are elements which generate the unit ideal of A and such that every $\mathcal{X}_{f_i}$ has a countably generated 1-twisted locally projective $\mathcal{O}_{\mathcal{X}_{f_i}}$-module $\mathcal{E}_i$ of positive rank. We follow an idea of Gabber to produce a countably generated 1-twisted locally projective $\mathcal{O}_{\mathcal{X}}$-module of positive rank. Let $a_1, \ldots, a_n \in A$ be elements such that $a_1 f_1 + \cdots + a_n f_n = 1$. After replacing f_i by $a_i f_i$, we may assume that $f_1 + \cdots + f_n = 1$. By Lemma 3.2.2 we have $\mathcal{E}_1^{\oplus \mathbf{N}}|_{\mathcal{X}_{f_1} \cap \mathcal{X}_{f_2}} \simeq \mathcal{E}_2^{\oplus \mathbf{N}}|_{\mathcal{X}_{f_1} \cap \mathcal{X}_{f_2}}$ so we may glue to obtain a countably generated 1-twisted locally projective $\mathcal{O}_{\mathcal{X}_{f_1} \cup \mathcal{X}_{f_2}}$-module. Since $X_{f_1 + f_2} \subseteq X_{f_1} \cup X_{f_2}$, restriction gives a 1-twisted $\mathcal{O}_{\mathcal{X}_{f_1+f_2}}$-module. We conclude by induction on n. ∎

Remark 3.2.6. In Theorem 3.2.5, we may also argue using Remark 3.2.4, since we may assume that the $\mathcal{E}_i$ are in fact invertible $\mathcal{O}_{\mathcal{X}_{f_i}}$-modules: Set $f_i' := f_1 + \cdots + f_i$, and suppose $\mathcal{E}_i'$ is a countably generated 1-twisted locally projective $\mathcal{O}_{\mathcal{X}_{f_i'}}$-module. By Remark 3.2.4, the restrictions of $\mathcal{E}_i'^{\oplus \mathbf{N}}$ and $\mathcal{E}_{i+1}^{\oplus \mathbf{N}}$ to $\mathcal{X}_{f_i'} \cap \mathcal{X}_{f_{i+1}}$ are isomorphic.

Corollary 3.2.7. *Let X be a Noetherian scheme admitting a cover $X = U_1 \cup U_2$ such that both U_1, U_2 are as in Lemma 3.1.8 and $U_1 \cap U_2$ is affine. Then*

$$\mathrm{LPBr}(X) = \mathrm{H}^2_{ét}(X, \mathbf{G}_{\mathrm{m}}).$$

Proof. Let $\mathcal{X} \to X$ be a $\mathbf{G}_{\mathrm{m}}$-gerbe. By Theorem 3.2.5, for $i = 1, 2$ there exists a countably generated 1-twisted locally projective $\mathcal{O}_{\mathcal{X}_{U_i}}$-module $\mathcal{E}_i$. By Lemma 3.2.2, there is an isomorphism $\mathcal{E}_1^{\oplus \mathbf{N}}|_{U_1 \cap U_2} \simeq \mathcal{E}_2^{\oplus \mathbf{N}}|_{U_1 \cap U_2}$ on $U_1 \cap U_2$, so we may glue $\mathcal{E}_1^{\oplus \mathbf{N}}$ and $\mathcal{E}_2^{\oplus \mathbf{N}}$. ∎

Proposition 3.2.8. *Let k be a field, let X_0 be a separated finite type k-scheme with the resolution property, let X be a finite-order thickening of X_0, i.e., there exists a closed immersion $X_0 \to X$ whose ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ is nilpotent. Then* $\mathrm{LPBr}(X) = \mathrm{H}^2_{ét}(X, \mathbf{G}_{\mathrm{m}})$.

Proof. By Lemma 3.1.6, it suffices to show that there exists an affine scheme U and an affine-pure morphism $U \to X$. As in Lemma 3.1.8(e), we may write $X_0 = W_0 / \mathrm{GL}_n$ for some free action of GL_n on a quasi-affine scheme W_0. The Jouanolou trick gives an affine scheme U_0 and a smooth surjective morphism $U_0 \to W_0$ whose fibers are affine spaces. The composition $U_0 \to W_0 \to X_0$ is smooth, surjective, affine-pure, and has geometrically irreducible fibers. Since U_0 is affine and $U_0 \to X_0$ is smooth, by [22, Chapitre III, Théorème 2.1.7] there exists a scheme U admitting a flat morphism $U \to X$ such that $U \times_X X_0 \simeq U_0$. Since U is a nilpotent thickening of an affine scheme, it itself is an affine scheme [16, Proposition 5.1.9]. The map $U \to X$ is surjective, smooth (by [33, Tag 06AG (17)]), and has geometrically irreducible fibers. Thus $U \to X$ is affine-pure by Lemma 3.1.7(a). ∎

Example 3.2.9. Let k be a separably closed field, set $X_0 := \mathbf{P}^2_k$, let $\mathcal{I}$ be a coherent $\mathcal{O}_{X_0}$-module and let X be a finite-order thickening of X_0 by $\mathcal{I}$. Mathur [28, Corollary 4] has shown that every $\mathbf{G}_{\mathrm{m}}$-gerbe over X corresponding to a torsion class in $\mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})_{\text{tors}}$ admits a 1-twisted finite locally free module. Then Proposition 3.2.8 (or Lemmas 3.1.8(f) and 3.1.6) shows that every $\mathbf{G}_{\mathrm{m}}$-gerbe over X admits a 1-twisted locally projective module. In case k has characteristic 0, we may choose e.g. $\mathcal{I} = \mathcal{O}_{X_0}(-3)$ as in [20, Exercise III.5.9] so that $\mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})$ does contain non-torsion elements: the exact sequence

$$1 \to 1 + \mathcal{I} \to \mathbf{G}_{m,X} \to \mathbf{G}_{m,X_0} \to 1$$

gives the description $\mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}}) = \mathrm{H}^2(X_0, \mathcal{I})/\mathbf{Z} = k/\mathbf{Z}$.

Remark 3.2.10. Let X be a separated Noetherian scheme. Given any $\alpha \in \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})$, we can always find an open subset $U \subset X$ such that $X \setminus U$ has codimension at least 2 in X and such that $\alpha|_U$ is contained in $\mathrm{LPBr}(U)$. Indeed, by [33, Tag 09NN], we may find two affine open subschemes U_1, U_2 of X whose union contains all the codimension 1 points of X; then we apply Corollary 3.2.7.

Example 3.2.11. The scheme X in [8, Corollary 3.11] also satisfies

$$\mathrm{LPBr}(X) \neq \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}}).$$

We recall the construction. Let k be an algebraically closed field, set

$$R := k[[x, y, z]]/(xy - z^2),$$

let $S_1 = S_2 = \operatorname{Spec} R$, let $U := \operatorname{Spec} R \setminus \{(x, y, z)R\}$ be the punctured spectrum of $\operatorname{Spec} R$, and let X be the gluing of S_1 and S_2 along the identity morphism on U. The Mayer–Vietoris exact sequence gives a coboundary map $\partial : \mathrm{H}^1_{\text{ét}}(U, \mathbf{G}_{\mathrm{m}}) \to \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})$ which is an isomorphism since $\mathrm{H}^i_{\text{ét}}(\operatorname{Spec} R, \mathbf{G}_{\mathrm{m}}) = 0$ for $i > 0$. Here $\mathrm{H}^1_{\text{ét}}(U, \mathbf{G}_{\mathrm{m}}) = \operatorname{Pic}(U) = \mathbf{Z}/(2)$, and under the isomorphism ∂ the unique nonzero class $\alpha \in \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})$ corresponds to the invertible $\mathcal{O}_U$-module $\mathcal{L}$ which (uniquely) extends to the coherent $\mathcal{O}_X$-module corresponding to the R-module $M = \langle x, z\rangle R$. Let $\mathcal{X}$ be the $\mathbf{G}_{\mathrm{m}}$-gerbe corresponding to α. Since R is strictly Henselian, the restriction of $\mathcal{X}$ to S_i is trivial. Let $\mathcal{F}_i$ be a 1-twisted line bundle on $\mathcal{X}_{S_i}$. Thus if α were contained in $\mathrm{LPBr}(X)$, this would imply there exist 1-twisted locally free $\mathcal{O}_{\mathcal{X}|_{S_i}}$-modules $\mathcal{E}_i$ and an isomorphism

$$\mathcal{E}_1|_U \otimes_{\mathcal{O}_{\mathcal{X}_U}} \mathcal{L}|_{\mathcal{X}_U} \simeq \mathcal{E}_2|_U$$

on U. Tensoring by $(\mathcal{F}_i)^{-1}$ for either $i = 1, 2$ gives an $\mathcal{O}_U$-module isomorphism

$$\mathcal{L}^{\oplus I} \simeq \mathcal{O}_U^{\oplus J}$$

where I, J are index sets. By Hartog's theorem, this would give an R-module isomorphism

$$M^{\oplus I} \simeq R^{\oplus J}$$

but no nonempty direct sum of copies of M is a free R-module (since this would imply that M is projective, being a direct summand of a free module). Thus $\mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}}) = \mathrm{Br}'(X) = \mathbf{Z}/(2)$ and $\mathrm{LPBr}(X) = 0$.

Example 3.2.12. In Example 3.2.11, we may instead take $R := k[[x, y, z, w]]/(xy - zw)$, in which case $\operatorname{Pic}(U) = \mathbf{Z}$ and so we obtain an example with

$$\mathrm{Br}(X) = \mathrm{Br}'(X) = \mathrm{LPBr}(X) = 0$$

while $\mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}}) = \mathbf{Z}$.

Remark 3.2.13. A long-standing open question of Totaro [35] is to determine whether every quasi-compact separated algebraic space X has the resolution property. To find a counterexample, it would be enough (by Theorem 3.2.5 and Lemma 3.1.8(e)) to find a separated Noetherian scheme X which satisfies $\mathrm{LPBr}(X) \neq \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})$.

4. Very positive vector bundles and pushouts

In this section we introduce *very positive vector bundles* (Definition 4.3.1), which are vector bundles of infinite rank that are "infinitely ample" (to be made precise in Definition 4.2.1). These bundles have strong uniqueness properties, and we use this to study

Question 2.2.8 for certain non-projective varieties that arise as pushouts of projective varieties.

4.1. Pinching and LPBr

In this section we study $\mathrm{LPBr}(X)$ for a class of examples of a proper, non-projective k-scheme X. We begin with the following example, which is later generalized in Example 4.3.10.

Example 4.1.1. Let k be an algebraically closed field. A standard way to make a nonprojective proper variety over k is to choose

(1) a smooth projective variety Y,

(2) an integer $n > 0$,

(3) automorphisms $g_i : Y \to Y$ for $i = 1, \dots, n$, and

(4) pairwise distinct points $s_1, \dots, s_n, t_1, \dots, t_n$ in $\mathbf{P}^1(k)$

and let X be the "pinching" of $\mathbf{P}^1 \times Y$ along the two closed immersions

$$\coprod_{i=1,\dots,n} Y \rightrightarrows \mathbf{P}^1 \times Y$$

sending $y \mapsto (s_i, y), (t_i, g_i(y))$ on the ith component Y. More precisely, we define X to be the pushout of the diagram

$$\begin{array}{ccc} \coprod_{i=1,\dots,n} Y \sqcup \coprod_{i=1,\dots,n} Y & \longrightarrow & \coprod_{i=1,\dots,n} Y \\ \downarrow & & \\ \mathbf{P}^1 \times Y & & \end{array}$$

which always exists in the category of ringed spaces and is a scheme in our situation by result of Ferrand [10, Théorème 7.1]. If there does not exist an ample line bundle $\mathcal{L}$ on Y such that $g_i^*\mathcal{L} \simeq \mathcal{L}$ for all i, then X is not projective (for example when Y is an abelian variety and g_1 is translation by a nontorsion point).

Proposition 4.1.2. *For X as in Example 4.1.1, we have* $\mathrm{LPBr}(X) = \mathrm{H}^2_{\acute{e}t}(X, \mathbf{G}_\mathrm{m})$.

Proof. Let $\nu : \mathbf{P}^1 \times Y \to X$ be the projection, which we may identify with the normalization morphism of X. Then by [10, Scolie 4.3 (a)(iii)] we have an exact sequence

$$1 \to \mathbf{G}_{m,X} \to \nu_*(\mathbf{G}_{m,\mathbf{P}^1 \times Y}) \to \prod_{i=1,\dots,n} \mathbf{G}_{m,Y} \to 1$$

in the étale topology on X. The long exact sequence in cohomology gives a map

$$\prod_{i=1,\dots,n} \mathrm{Pic}(Y) \to \mathrm{H}^2_{\acute{e}t}(X, \mathbf{G}_\mathrm{m}) \tag{4.1.2.1}$$

which is surjective since the next terms are $\mathrm{H}^2_{\acute{e}t}(\mathbf{P}^1 \times Y, \mathbf{G}_\mathrm{m}) \to \prod_{i=1,\dots,n} \mathrm{H}^2_{\acute{e}t}(Y, \mathbf{G}_\mathrm{m})$ which is the diagonal embedding, in particular injective.

Given line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_n$ on Y, the image of $(\mathcal{L}_1, \ldots, \mathcal{L}_n)$ under (4.1.2.1) is in $\mathrm{LPBr}(X)$ if and only if there exists a locally projective $\mathcal{O}_{\mathbf{P}^1 \times Y}$-module $\mathcal{E}$ such that

$$\mathcal{E}_{s_i} \simeq g_i^*(\mathcal{E}_{t_i}) \otimes_{\mathcal{O}_Y} \mathcal{L}_i \tag{4.1.2.2}$$

for all i.

Let $G \subset \mathrm{Aut}_k(Y)$ be the subgroup generated by $g_1, \ldots, g_n$ and define the quasi-coherent $\mathcal{O}_Y$-module

$$\mathcal{V} := \bigoplus_{r \geq 0} \bigoplus_{h_1,\ldots,h_r \in G} \bigoplus_{1 \leq i_1,\ldots,i_r \leq n} \bigoplus_{e_1,\ldots,e_r \in \mathbf{Z}} (h_1^* \mathcal{L}_{i_1}^{e_1} \otimes_{\mathcal{O}_Y} \cdots \otimes_{\mathcal{O}_Y} h_r^* \mathcal{L}_{i_r}^{e_r})$$

and

$$\mathcal{E} := \mathcal{V}|_{\mathbf{P}^1 \times Y}$$

which satisfies (4.1.2.2) since $g_i^* \mathcal{V} \simeq \mathcal{V}$ and $\mathcal{V} \otimes_{\mathcal{O}_Y} \mathcal{L}_i \simeq \mathcal{V}$ for all i. ∎

4.2. Conditions on global generation and vanishing higher cohomology

We generalize the above example by defining certain infinite rank vector bundles on projective k-schemes.

Definition 4.2.1. Let X be a scheme. Let

$$\mathcal{E}_\bullet = \{\mathcal{E}_0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \cdots\} \tag{4.2.1.1}$$

be a sequence of locally split injections of finite locally free $\mathcal{O}_X$-modules of positive rank. We define the following conditions.

- **(G)** For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ of finite type, there exists some $n' \in \mathbf{N}$ such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}_n$ is globally generated for all $n \geq n'$.
- **($\mathbf{V}_\ell$)** For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ of finite type, there exists some $n' \in \mathbf{N}$ such that $\mathrm{H}^q(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}_n) = 0$ for all $q \geq \ell + 1$ and all $n \geq n'$.
- **($\mathbf{V}'_\ell$)** For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ of finite type, we have

$$\varinjlim_{n \in \mathbf{N}} \mathrm{H}^q(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}_n) = 0 \quad \text{for all } q \geq \ell + 1.$$

Remark 4.2.2. The primary way of getting a system $\mathcal{E}_\bullet$ satisfying **(G)** and **($\mathbf{V}'_0$)** is as follows. Let X be a scheme admitting an ample line bundle $\mathcal{L}$. After replacing $\mathcal{L}$ by a tensor power, we may assume there exist sections $s_1, \ldots, s_m \in \Gamma(X, \mathcal{L})$ such that each X_{s_i} is affine and $X = \bigcup_{i=1,\ldots,m} X_{s_i}$. Let $\varphi : \mathcal{O}_X \to \mathcal{L}^{\oplus m}$ be the map $1 \mapsto (s_1, \ldots, s_m)$. Set

$$\mathcal{E}_n := (\mathcal{L}^{\oplus m})^{\otimes n} \simeq (\mathcal{L}^{\otimes n})^{\oplus m^n}$$

and let the transition map $\mathcal{E}_n \to \mathcal{E}_{n+1}$ be $\varphi \otimes \mathrm{id}_{\mathcal{E}_n}$, which is split when restricted to any X_{s_i}. We thus obtain a system

$$\mathcal{E}_\bullet = \{\mathcal{O}_X \to \mathcal{L}^{\oplus m} \to (\mathcal{L}^{\otimes 2})^{\oplus m^2} \to (\mathcal{L}^{\otimes 3})^{\oplus m^3} \to \cdots\}$$

of locally split injections of finite locally free $\mathcal{O}_X$-modules. The system $\mathcal{E}_\bullet$ satisfies (**G**) by [33, Tag 01Q3] and ($\mathbf{V}_0'$) by [33, Tag 01XR].

Lemma 4.2.3. *Let X be a quasi-compact semi-separated scheme. The following are equivalent.*

(1) *There exists an ample line bundle $\mathcal{L}$ on X.*

(2′) *There exists a system $\mathcal{E}_\bullet$ as in Definition 4.2.1 satisfying* (**G**) *and* ($\mathbf{V}_0'$).

If X is proper over a Noetherian affine scheme, then (1) *and* (2′) *are equivalent to*

(2) *There exists a system $\mathcal{E}_\bullet$ as in Definition 4.2.1 satisfying* (**G**) *and* ($\mathbf{V}_0$).

Proof. (1)⇒(2′): See Remark 4.2.2. If in addition X is proper over a Noetherian affine scheme, then we have ($\mathbf{V}_0$) by [20, Chapter III, Theorem 5.2].

(2′)⇒(1): We show that for any $x \in X$ there exists a globally generated line bundle $\mathcal{L}$ and a section $s \in \Gamma(X, \mathcal{L})$ such that X_s is an affine open neighborhood of x; then the result follows from [33, Tag 09NC]. Since X is quasi-compact, we may assume that x is a closed point. Let $i : \operatorname{Spec} \kappa(x) \to X$ be the closed immersion. Let $U \subset X$ be an affine open neighborhood of x. Set $Z := X \setminus U$ and $Z' := Z \cup \{x\}$, and let $\mathcal{I}, \mathcal{I}'$ be the ideal sheaves corresponding to the reduced closed subscheme structures on Z, Z' respectively. By [33, Tag 01PG], we may write

$$\mathcal{I}' := \varinjlim_{\lambda \in \Lambda} \mathcal{I}'_\lambda$$

where each $\mathcal{I}'_\lambda$ is a quasi-coherent ideal sheaf of finite type. We have an exact sequence

$$0 \to \mathcal{I}' \to \mathcal{I} \to \mathcal{I}/\mathcal{I}' \to 0$$

where $\mathcal{I}/\mathcal{I}'$ is isomorphic to $i_*\mathcal{O}_{\operatorname{Spec} \kappa(x)}$. For any n and $\lambda \in \Lambda$, we have an exact sequence

$$0 \to \mathcal{I}'_\lambda \otimes_{\mathcal{O}_X} \mathcal{E}_n \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{E}_n \to \mathcal{I}/\mathcal{I}'_\lambda \otimes_{\mathcal{O}_X} \mathcal{E}_n \to 0$$

on X.

Choose some n and choose a nonzero element $s_\circ \in \Gamma(X, \mathcal{I}/\mathcal{I}' \otimes_{\mathcal{O}_X} \mathcal{E}_n)$. Find some $\lambda \in \Lambda$ such that $s_\circ$ lifts to some $s_1 \in \Gamma(X, \mathcal{I}/\mathcal{I}'_\lambda \otimes_{\mathcal{O}_X} \mathcal{E}_n)$. By ($\mathbf{V}_0'$), there exists some $n' \geq n$ so that the image of s_1 in $\mathrm{H}^1(X, \mathcal{I}'_\lambda \otimes_{\mathcal{O}_X} \mathcal{E}_{n'})$ is 0. Lift to an element $s_1' \in \Gamma(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{E}_{n'})$. Find $n'' \geq n'$ such that $\mathcal{E}_{n''}$ is globally generated, set $r := \operatorname{rank} \mathcal{E}_{n''}$, let $s_1'' \in \Gamma(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{E}_{n''})$ be the image of s_1' and choose sections $s_2'', \dots, s_r'' \in \Gamma(X, \mathcal{E}_{n''})$ such that the images of $s_1'', s_2'', \dots, s_r''$ in $i^*\mathcal{E}_{n''}$ constitute a basis for $i^*\mathcal{E}_{n''}$ as a $\kappa(x)$-vector space, then consider $s_1'' \otimes \cdots \otimes s_r'' \in \Gamma(X, T^r\mathcal{E}_{n''})$ and its image

$$s := s_1'' \wedge \cdots \wedge s_r'' \in \Gamma(X, \det \mathcal{E}_{n''})$$

under the quotient $T^r\mathcal{E}_{n''} \to \det \mathcal{E}_{n''}$. Then X_s is the same as the nonvanishing locus of $s|_U \in \Gamma(U, \det \mathcal{E}_{n''}|_U)$ in U, hence it is affine. By construction, s maps to a generator of $i^*(\det \mathcal{E}_{n''})$ as a $\kappa(x)$-vector space, hence x is contained in X_s. The determinant $\det \mathcal{E}_{n''}$ is globally generated since it is a quotient of the tensor power $T^r\mathcal{E}_{n''}$. ∎

Remark 4.2.4. Assume the notation of Definition 4.2.1.

(1) Condition (**G**) implies that the colimit $\varinjlim_{n\in\mathbf{N}}(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{E}_n)$ is globally generated for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ of finite type, but the converse is not true. Set $X=\mathbf{P}^1$ and let $\mathcal{G}_\bullet$ be the system

$$\mathcal{O}_X \to \mathcal{O}_X(1)^{\oplus 2} \to \mathcal{O}_X(2)^{\oplus 4} \to \mathcal{O}_X(3)^{\oplus 8} \to \cdots$$

as in Remark 4.2.2, and consider $\mathcal{E}_\bullet := \bigoplus_{n\in\mathbf{N}}\mathcal{G}_\bullet(-1)[-n]$ where "$[n]$" denotes shift. Then $\varinjlim_{n\in\mathbf{N}}\mathcal{E}_n$ is globally generated but $\mathcal{E}_n=\bigoplus_{0\le i\le n}\mathcal{O}(i-1)^{\oplus 2^i}$ is not globally generated for any n.

(2) Condition ($\mathbf{V}_\ell$) implies ($\mathbf{V}'_\ell$), but the converse is not true. For this, set $X=\mathbf{A}^2\setminus\{(0,0)\}$ and apply the construction of Remark 4.2.2.

Lemma 4.2.5. *Let $f: Y\to X$ be a closed immersion of Noetherian schemes and let $\mathcal{E}_\bullet$ be a system of vector bundles as in Definition 4.2.1. If $\mathcal{E}_\bullet$ satisfies any of* (**G**), ($\mathbf{V}_\ell$), ($\mathbf{V}'_\ell$), *then $f^*\mathcal{E}_\bullet$ satisfies the same property.*

Proof. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module of finite type. We have isomorphisms

$$\mathcal{G}\otimes_{\mathcal{O}_Y} f^*\mathcal{E}_n \simeq f^*(f_*\mathcal{G}\otimes_{\mathcal{O}_X}\mathcal{E}_n), \tag{4.2.5.1}$$

$$\varinjlim_{n\in\mathbf{N}}(\mathcal{G}\otimes_{\mathcal{O}_Y} f^*\mathcal{E}_n) \simeq \varinjlim_{n\in\mathbf{N}} f^*(f_*\mathcal{G}\otimes_{\mathcal{O}_X}\mathcal{E}_n) \simeq f^*(\varinjlim_{n\in\mathbf{N}}(f_*\mathcal{G}\otimes_{\mathcal{O}_X}\mathcal{E}_n)), \tag{4.2.5.2}$$

$$\mathrm{H}^p(Y,\mathcal{G}\otimes_{\mathcal{O}_Y} f^*\mathcal{E}_n) \simeq \mathrm{H}^p(X, f_*(\mathcal{G}\otimes_{\mathcal{O}_Y} f^*\mathcal{E}_n)) \overset{\dagger}{\simeq} \mathrm{H}^p(X, f_*\mathcal{G}\otimes_{\mathcal{O}_X}\mathcal{E}_n), \tag{4.2.5.3}$$

where † follows from the projection formula. The claim about (**G**) follows from (4.2.5.1) and (4.2.5.2) because the pullback of a globally generated sheaf is again globally generated, and the claims about ($\mathbf{V}_\ell$) and ($\mathbf{V}'_\ell$) follow from (4.2.5.3). ∎

4.3. Very positive vector bundles and uniqueness

Definition 4.3.1. Let k be a field, let X be a projective scheme over k. Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_X$-module. We say $\mathcal{E}$ is a *very positive vector bundle* if there exists a sequence

$$\mathcal{E}_\bullet = \{\mathcal{E}_0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \cdots\}$$

of locally split injections of finite locally free $\mathcal{O}_X$-modules of positive rank such that

(1) $\mathcal{E}=\varinjlim_{n\in\mathbf{N}}\mathcal{E}_n$,

(2) $\operatorname{rank}\mathcal{E}_n < \operatorname{rank}\mathcal{E}_{n+1}$ for all $n\in\mathbf{N}$, and

(3) the system $\mathcal{E}_\bullet$ satisfies (**G**) and ($\mathbf{V}_0$) as in Definition 4.2.1.

Lemma 4.3.2. *Every projective k-scheme X has a very positive vector bundle.*

Proof. This is a consequence of Lemma 4.2.3. ∎

Remark 4.3.3. In Definition 4.3.1, for every point $x \in X$, the stalk $\mathcal{E}_x$ is a free $\mathcal{O}_{X,x}$-module by Kaplansky's theorem [2, Corollary 3.3]. However, it is not clear whether $\mathcal{E}$ is even fppf-locally free. By construction, the very positive vector bundles in Lemma 4.2.3 are Zariski-locally free (of countably infinite rank).

Remark 4.3.4. In general, the pullback of a very positive vector bundle need not be a very positive vector bundle. Indeed, set $X := \mathbf{P}^1$ and $Y := \mathbf{P}^1 \times \mathbf{P}^1$, let $f : Y \to X$ be the second projection and let $\mathcal{E}$ be a very positive vector bundle on X. Then

$$\mathrm{H}^1\big(Y, \mathcal{O}(-2,0) \otimes_{\mathcal{O}_Y} f^*\mathcal{E}_n\big) \neq 0 \quad \text{for } n \gg 0$$

because it contains $\mathrm{H}^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-2)) \otimes_k \mathrm{H}^0(X, \mathcal{E}_n)$.

Remark 4.3.5. The quotient of a very positive vector bundle need not be a very positive vector bundle. Set $X = \mathbf{P}^2$. Let

$$\mathcal{O}_X \to \mathcal{O}_X(1)^{\oplus 3} \to \cdots \to \mathcal{O}_X(n)^{\oplus 3^n} \to \cdots \tag{4.3.5.1}$$

be the system obtained by taking $\mathcal{L} = \mathcal{O}_X(1)$ in Remark 4.2.2, let $\{\mathcal{E}_n\}_{n\in\mathbf{N}}$ be the system obtained by twisting (4.3.5.1) by $\mathcal{O}_X(-3)$, let $\{\mathcal{F}_n\}_{n\in\mathbf{N}}$ be the constant system defined by $\mathcal{F}_n := \mathcal{O}_X(-3)$ for all n, and let $\mathcal{Q}_n := \mathcal{E}_n/\mathcal{F}_n$ be the quotient. Then we have exact sequences

$$\cdots \to \mathrm{H}^1(X, \mathcal{E}_n) \to \mathrm{H}^1(X, \mathcal{Q}_n) \to \mathrm{H}^2(X, \mathcal{F}_n) \to \mathrm{H}^2(X, \mathcal{E}_n) \to \cdots$$

for all n. Here $\mathrm{H}^1(X, \mathcal{E}_n) = \mathrm{H}^2(X, \mathcal{E}_n) = 0$ for $n \gg 0$ but $\mathrm{H}^2(X, \mathcal{F}_n) \neq 0$ for all n, so $\mathrm{H}^1(X, \mathcal{Q}_n) \neq 0$ for all $n \gg 0$; hence $\mathcal{Q}$ does not satisfy condition (3) in Definition 4.3.1.

Example 4.3.6. As is well known, there are no nontrivial finite rank vector bundles on the punctured spectrum of a regular local ring of dimension 2. This fact may be used to produce 1-twisted finite rank vector bundles on $\mathbf{G}_{\mathrm{m}}$-gerbes over smooth surfaces. Here we explain an example which illustrates a subtlety in the analogous approach for infinite rank twisted sheaves.

On $\mathbf{P}^1$, we have the very positive vector bundle

$$\mathcal{V} := \mathrm{colim}(\mathcal{O}_{\mathbf{P}^1} \to \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus 2} \to \mathcal{O}_{\mathbf{P}^1}(2)^{\oplus 4} \to \mathcal{O}_{\mathbf{P}^1}(3)^{\oplus 8} \to \cdots)$$

as in Remark 4.2.2. Set $U := \mathbf{A}^2 \setminus \{(0,0)\}$ and let $\pi : U \to \mathbf{P}^1$ be the projection map. Then

$$\mathcal{W} := \pi^*\mathcal{V} \simeq \mathrm{colim}(\mathcal{O}_U \to \mathcal{O}_U^{\oplus 2} \to \mathcal{O}_U^{\oplus 4} \to \mathcal{O}_U^{\oplus 8} \to \cdots)$$

is not a free $\mathcal{O}_U$-module. Indeed, we have

$$\langle x, y\rangle \cdot \Gamma(U, \mathcal{W}) = \Gamma(U, \mathcal{W})$$

since any section in the image of $\Gamma(U, \mathcal{O}_U^{\oplus 2^i}) \to \Gamma(U, \mathcal{W})$ can be written as a linear combination $xs + yt$ for sections s, t in the image of $\Gamma(U, \mathcal{O}_U^{\oplus 2^{i+1}}) \to \Gamma(U, \mathcal{W})$. Here we use that taking global sections commutes with filtered colimits since U is quasi-compact [33, Tag 0738].

The above also shows that $\mathcal{V}$ is a locally free $\mathcal{O}_{\mathbf{P}^1}$-module which is not a direct sum of line bundles (otherwise $\mathcal{W}$ would be as well, but line bundles on U are trivial).

Lemma 4.3.7. *Let k be an infinite field, let X be a proper k-scheme, let $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{F}$ be finite locally free $\mathcal{O}_X$-modules of ranks r_1, r_2, r_3, s, respectively. Suppose given an exact sequence*

$$0 \to \mathcal{E}_1 \xrightarrow{f_1} \mathcal{E}_2 \xrightarrow{f_2} \mathcal{E}_3 \to 0 \tag{4.3.7.1}$$

of $\mathcal{O}_X$-modules and let

$$a_1 : \mathcal{E}_1 \to \mathcal{F}$$

be a locally split morphism of $\mathcal{O}_X$-modules. If $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{F})$ is globally generated and $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{F}) = 0$ and $s > \max\{r_2, \dim X + r_3 - 1\}$, there exists a locally split $\mathcal{O}_X$-linear map $a_2 : \mathcal{E}_2 \to \mathcal{F}$ such that $a_2 f_1 = a_1$.

Proof. Applying $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{F})$ to (4.3.7.1), the obstruction to the existence of an a_2' satisfying $a_2' f_1 = a_1$ is an element of $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{F})$, which is 0 by assumption; however, such an a_2' may not be locally split. Any other a_2'' satisfying $a_2'' f_1 = a_1$ is of the form $a_2'' = a_2' + a_3 f_2$ for a unique $a_3 : \mathcal{E}_3 \to \mathcal{F}$. If a_3 is locally split, then a_2'' is also locally split since the composition $a_2'' f_1$ is locally split (we may locally choose a section of f_2). Hence we are reduced to the task of producing a morphism $a_3 : \mathcal{E}_3 \to \mathcal{F}$ which is locally split.

Set $\mathcal{H}_3 := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{F})$ and $H_3 := \mathrm{Spec}_X \mathrm{Sym}^\bullet_{\mathcal{O}_X} \mathcal{H}_3^\vee$. Then H_3 represents the functor $(\mathrm{Sch}/X)^{\mathrm{op}} \to \mathrm{Set}$ sending $T \mapsto \mathrm{Hom}_{\mathcal{O}_T}(\mathcal{E}_3|_T, \mathcal{F}|_T) = \Gamma(T, \mathcal{H}_3|_T)$. Let $\xi_{\mathrm{univ}} : \mathcal{E}_3|_{H_3} \to \mathcal{F}|_{H_3}$ be the universal morphism and let $K \subset H_3$ be the closed subscheme defined by the condition that $\xi_{\mathrm{univ}}|_T : \mathcal{E}_3|_T \to \mathcal{F}|_T$ is not locally split. The sequence $K \to H_3 \to X$ is locally on X isomorphic to the pullback of $V_{r_3} \to \mathbf{A}^{r_3 s}_{\mathbf{Z}} \to \mathrm{Spec}\,\mathbf{Z}$, where V_{r_3} is the determinantal variety defined by the maximal minors of a $r_3 \times s$ matrix with indeterminate coefficients. By a theorem of Eagon–Northcott [9, Exercise 10.10], the codimension of V_{r_3} in $\mathbf{A}^{r_3 s}_{\mathbf{Z}}$ (and of K in H_3) is $s - r_3 + 1$.

Set $N := \dim_k \Gamma(X, \mathcal{H}_3)$ and let $\mathcal{O}_X^{\oplus N} \to \mathcal{H}_3$ be a surjection. We have a morphism $\mathbf{A}^N_k \times_k X \to H_3$ which is locally on H_3 isomorphic to $\mathbf{A}^{N - r_3 s}_{\mathbf{Z}} \to \mathrm{Spec}\,\mathbf{Z}$. Set $I := (\mathbf{A}^N_k \times_k X) \times_{H_3} K$. Then the codimension of I in $\mathbf{A}^N_k \times_k X$ is also $s - r_3 + 1$, hence $\dim I = N + \dim X - (s - r_3 + 1)$. Since $s > \dim X + r_3 - 1$, the projection $I \to \mathbf{A}^N_k$ is not surjective. Since k is infinite, there exists a rational point $p \in \mathbf{A}^N_k(k)$ which is not in the image of $I \to \mathbf{A}^N_k$, for which the corresponding fiber $\xi_{\mathrm{univ}}|_p : \mathcal{E}_2 \to \mathcal{F}$ is locally split. ∎

Theorem 4.3.8. *Let k be an infinite field, let X be a projective k-scheme. Any two very positive vector bundles on X are isomorphic.*

Proof. Suppose

$$\mathcal{E} = \mathrm{colim}(\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to \cdots), \quad \mathcal{F} = \mathrm{colim}(\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to \cdots) \tag{4.3.8.1}$$

are very positive vector bundles on X. We inductively find a sequence of integers $n_1 < n_2 < n_3 < \cdots$ and maps

$$\begin{aligned} a_{2i-1} &: \mathcal{E}_{n_{2i-1}} \to \mathcal{F}_{n_{2i}}, \\ a_{2i} &: \mathcal{F}_{n_{2i}} \to \mathcal{E}_{n_{2i+1}} \end{aligned}$$

for $i \geq 1$ such that

(i) each a_ℓ is injective and $\operatorname{coker}(a_\ell)$ is locally free of positive rank, and

(ii) the compositions

$$\begin{aligned} a_{2i} \circ a_{2i-1} &: \mathcal{E}_{n_{2i-1}} \to \mathcal{E}_{n_{2i+1}}, \\ a_{2i+1} \circ a_{2i} &: \mathcal{F}_{n_{2i}} \to \mathcal{F}_{n_{2i+2}} \end{aligned}$$

are equal to the transition maps in (4.3.8.1).

Given a collection of such morphisms $\{a_\ell\}_{\ell \in \mathbf{N}}$, condition (ii) implies the existence of morphisms $f : \mathcal{E} \to \mathcal{F}$ and $g : \mathcal{F} \to \mathcal{E}$ such that $f \circ g = \mathrm{id}_{\mathcal{F}}$ and $g \circ f = \mathrm{id}_{\mathcal{E}}$.

For the induction hypothesis, suppose given integers $n_{\ell-1} < n_\ell$ and a morphism

$$a_{\ell-1} : \mathcal{E}_{n_{\ell-1}} \to \mathcal{F}_{n_\ell}$$

satisfying (i). (Here, by symmetry we have assumed ℓ is even, and we view the base case as a map $a_0 : 0 \to \mathcal{E}_{n_1}$.) We find an integer $n_{\ell+1}$ and a morphism

$$a_\ell : \mathcal{F}_{n_\ell} \to \mathcal{E}_{n_{\ell+1}}$$

satisfying (i) and making the diagram

$$\begin{array}{ccc} \mathcal{E}_{n_{\ell-1}} & \xrightarrow{a_{\ell-1}} & \mathcal{F}_{n_\ell} \\ \downarrow & & \downarrow a_\ell \\ \mathcal{E}_{n_{\ell+1}} & = & \mathcal{E}_{n_{\ell+1}} \end{array} \qquad (4.3.8.2)$$

commute.

Set $\mathcal{Q} := \operatorname{coker}(a_{\ell-1})$. By Definition 4.3.1(3) for $\mathcal{E}$, we may choose $n_{\ell+1} \gg n_\ell$ so that $\operatorname{rank} \mathcal{E}_{n_{\ell+1}} > \max\{\operatorname{rank} \mathcal{F}_{n_\ell}, \dim X + \operatorname{rank} \mathcal{Q} - 1\}$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{E}_{n_{\ell+1}}) \simeq \mathcal{Q}^\vee \otimes_{\mathcal{O}_X} \mathcal{E}_{n_{\ell+1}}$ is globally generated and $\operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{E}_{n_{\ell+1}}) = \mathrm{H}^1(\mathcal{O}_X, \mathcal{Q}^\vee \otimes \mathcal{E}_{n_{\ell+1}}) = 0$. Then we conclude using Lemma 4.3.7. ■

Corollary 4.3.9. *Let k be an infinite field, let $\mathcal{E}$ be a very positive vector bundle on a projective k-scheme X.*

(a) *For every finite locally free $\mathcal{O}_X$-module $\mathcal{V}$ of positive rank, we have $\mathcal{E} \otimes \mathcal{V} \simeq \mathcal{E}$.*

(b) *For every invertible $\mathcal{O}_X$-module $\mathcal{L}$, we have $\mathcal{E} \otimes \mathcal{L} \simeq \mathcal{E}$.*

(c) *For every automorphism g of X, we have $g^*\mathcal{E} \simeq \mathcal{E}$.*

Proof. For (a), it suffices by Theorem 4.3.8 to check that $\mathcal{E} \otimes \mathcal{V}$ is a very positive vector bundle. We have

$$\mathcal{E} \otimes \mathcal{V} = \operatorname{colim}(\mathcal{E}_1 \otimes \mathcal{V} \to \mathcal{E}_2 \otimes \mathcal{V} \to \mathcal{E}_3 \otimes \mathcal{V} \to \cdots)$$

since tensor products commute with colimits. To show that $\{\mathcal{E}_n \otimes \mathcal{V}\}_{n\in\mathbf{N}}$ satisfies condition (3) of Definition 4.3.1 for a given coherent $\mathcal{O}_X$-module $\mathcal{F}$, we use that the system $\{\mathcal{E}_n\}_{n\in\mathbf{N}}$ satisfies condition (3) for the coherent $\mathcal{O}_X$-module $\mathcal{F} \otimes \mathcal{V}$.

Claim (b) is a special case of (a), and (c) is clear. ∎

Example 4.3.10. Let k be an infinite field, let Z, Y be smooth projective k-schemes, let $a_1, a_2 : Z \to Y$ be two closed immersions such that $a_1(Z) \cap a_2(Z) = \emptyset$. The coequalizer of a_1, a_2 exists as a scheme X, and the canonical map $\nu : Y \to X$ may be identified with the normalization morphism.

Proposition 4.3.11. *For X as in Example 4.3.10, we have* $\operatorname{LPBr}(X) = \mathrm{H}^2_{\acute{e}t}(X, \mathbf{G}_{\mathrm{m}})$.

Proof. Let $\mathcal{X} \to X$ be a $\mathbf{G}_{\mathrm{m}}$-gerbe with corresponding class $\alpha \in \mathrm{H}^2_{\acute{e}t}(X, \mathbf{G}_{\mathrm{m}})$, and let $\beta := \nu^*\alpha$ and $\gamma := a_1^*\beta = a_2^*\beta$ be the pullbacks to Y and Z. Since Y is projective and smooth, we have $\operatorname{Br}(Y) = \operatorname{Br}'(Y) = \mathrm{H}^2_{\acute{e}t}(Y, \mathbf{G}_{\mathrm{m}})$ so there exists a 1-twisted finite locally free $\mathcal{O}_{\mathcal{X}_Y}$-module $\mathcal{E}$. We have similarly $\operatorname{Br}(Z) = \operatorname{Br}'(Z) = \mathrm{H}^2_{\acute{e}t}(Z, \mathbf{G}_{\mathrm{m}})$. Let $f : P \to Z$ be a Brauer–Severi scheme corresponding to γ. We note that $\mathcal{X}_P \to P$ is the trivial $\mathbf{G}_{\mathrm{m}}$-gerbe. Consider the following diagram:

$$\begin{array}{ccccc} P & & & & \\ \downarrow{\scriptstyle f} & & & & \\ Z & \overset{a_1}{\underset{a_2}{\rightrightarrows}} & Y & \xrightarrow{\nu} & X \end{array}$$

Let $\mathcal{V}$ be a very positive vector bundle on P. Then by Corollary 4.3.9 (a) we have an isomorphism

$$f^*a_1^*\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{X}_P}} \mathcal{V}|_{\mathcal{X}_P} \simeq f^*a_2^*\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{X}_P}} \mathcal{V}|_{\mathcal{X}_P}$$

of 1-twisted $\mathcal{O}_{\mathcal{X}_P}$-modules. Pushing forward gives an isomorphism

$$a_1^*\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{X}_Z}} f_*\mathcal{V}|_{\mathcal{X}_Z} \simeq a_2^*\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{X}_Z}} f_*\mathcal{V}|_{\mathcal{X}_Z} \tag{4.3.11.1}$$

of 1-twisted $\mathcal{O}_{\mathcal{X}_Z}$-modules. The pushforward $f_*\mathcal{V}$ is very positive by Lemma 4.3.12. Let $\mathcal{W}$ be a very positive vector bundle on Y. Then $a_1^*\mathcal{W}, a_2^*\mathcal{W}$ are very positive by Lemma 4.2.5, hence we have

$$f_*\mathcal{V} \simeq a_1^*\mathcal{W} \simeq a_2^*\mathcal{W}$$

by Theorem 4.3.8. Substituting this into (4.3.11.1) gives an isomorphism

$$a_1^*(\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{X}_Y}} \mathcal{W}|_{\mathcal{X}_Y}) \simeq a_2^*(\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{X}_Y}} \mathcal{W}|_{\mathcal{X}_Y})$$

which means $\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{X}_Y}} \mathcal{W}|_{\mathcal{X}_Y}$ descends to $\mathcal{X}$. ∎

Lemma 4.3.12. *Let k be an infinite field, let $f : X \to Y$ be a flat proper morphism between projective k-schemes. If $\mathcal{E}$ is a very positive vector bundle on X, then $f_*\mathcal{E}$ is a very positive vector bundle on Y.*

Proof. We separate the proof into several steps.

Part 1 (writing $f_*\mathcal{E}$ as a colimit). Let $\mathcal{O}_X(1)$ and $\mathcal{O}_Y(1)$ be very ample line bundles on X and Y, respectively. Since $\mathcal{O}_X(1)$ is f-relatively ample (by [17, Proposition 4.6.13 (v)]), we may replace $\mathcal{O}_X(1)$ by $\mathcal{O}_X(n)$ for $n \gg 0$ to assume that the map

$$f_*(\mathcal{O}_X(1)) \otimes_{\mathcal{O}_Y} f_*(\mathcal{O}_X(m)) \to f_*(\mathcal{O}_X(m+1)) \tag{4.3.12.1}$$

is surjective for all $m \geq 1$. We have

$$f_*(\mathcal{O}_X(m_1) \otimes_{\mathcal{O}_X} f^*(\mathcal{O}_Y(m_2))) \simeq f_*(\mathcal{O}_X(m_1)) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(m_2)$$

by the projection formula, so after replacing $\mathcal{O}_X(1)$ by $\mathcal{O}_X(1) \otimes_{\mathcal{O}_X} f^*(\mathcal{O}_Y(m_2))$ for some $m_2 \gg 0$ we may assume that $f_*(\mathcal{O}_X(1))$ is globally generated (this does not change that (4.3.12.1) is surjective since we tensor both sides by $\mathcal{O}_Y(m_2)$, and the new line bundle is still ample by [17, Proposition 4.6.13 (ii)]). By surjectivity of (4.3.12.1), the pushforward $f_*(\mathcal{O}_X(n))$ is globally generated for all n. After replacing $\mathcal{O}_X(1)$ by $\mathcal{O}_X(1) \otimes_{\mathcal{O}_X} f^*(\mathcal{O}_Y(1))$ once more, we may assume that $f_*(\mathcal{O}_X(n)) \simeq \mathcal{V}_n \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(n)$ for some globally generated coherent $\mathcal{O}_Y$-module $\mathcal{V}_n$.

By Remark 4.2.2 and Theorem 4.3.8, we may assume that $\mathcal{E} = \varinjlim_{n \in \mathbf{N}} \mathcal{E}_n$ with $\mathcal{E}_n = \mathcal{O}_X(n)^{\oplus r_n}$ for some r_n. Then $f_*\mathcal{E} \simeq \varinjlim_{n \in \mathbf{N}} f_*\mathcal{E}_n$ and it remains to show that the system $\{f_*\mathcal{E}_n\}_{n \in \mathbf{N}}$ satisfies the conditions of Definition 4.3.1.

Part 2 ($f_*\mathcal{E}_n$ are locally free). Since $\mathcal{O}_X(1)$ is f-relatively ample, there exists some N_1 such that $\mathbf{R}^i f_*\mathcal{E}_n = 0$ for $n \geq N_1$ and $i \geq 1$. By cohomology and base change, the pushforward $f_*\mathcal{E}_n$ is finite locally free for $n \geq N_1$.

Part 3 ($f_*\mathcal{E}_n \to f_*\mathcal{E}_{n+1}$ are locally split). Let $\mathcal{Q}_n$ be the cokernel of $\mathcal{E}_n \to \mathcal{E}_{n+1}$ so that we have an exact sequence

$$0 \to \mathcal{E}_n \to \mathcal{E}_{n+1} \to \mathcal{Q}_n \to 0$$

of $\mathcal{O}_X$-modules. The pushforward

$$0 \to f_*\mathcal{E}_n \to f_*\mathcal{E}_{n+1} \to f_*\mathcal{Q}_n \to 0$$

is exact since $\mathbf{R}^1 f_*\mathcal{E}_n = 0$. Since $\mathcal{E}_n \to \mathcal{E}_{n+1}$ is locally split, the $\mathcal{O}_X$-module $\mathcal{Q}_n$ is finite locally free, hence flat over S. Moreover $\mathbf{R}^i f_*\mathcal{Q}_n = 0$ for $n \geq N_1$ and $i \geq 1$ as well, hence $f_*\mathcal{Q}_n$ is finite locally free, so $f_*\mathcal{E}_n \to f_*\mathcal{E}_{n+1}$ is locally split for $n \geq N_1$.

Part 4 ($f_*\mathcal{E}_\bullet$ satisfies $(\mathbf{V}_0)$*).* Let $\mathcal{G}$ be a coherent $\mathcal{O}_Y$-module. We show that

$$\mathrm{H}^i(Y, f_*\mathcal{E}_n \otimes_{\mathcal{O}_X} \mathcal{G}) = 0 \quad \text{for } i \geq 1 \text{ and } n \gg 0.$$

We have the Leray spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(Y, \mathbf{R}^q f_*(\mathcal{E}_n \otimes_{\mathcal{O}_X} f^*\mathcal{G})) \Rightarrow \mathrm{H}^{p+q}(X, \mathcal{E}_n \otimes_{\mathcal{O}_X} f^*\mathcal{G})$$

with differentials $\mathrm{E}_2^{p,q} \to \mathrm{E}_2^{p+2,q-1}$. By Serre vanishing [20, Chapter III, Theorem 5.2] there exists some $N_2 \geq N_1$ such that $\mathbf{R}^q f_*(\mathcal{E}_n \otimes_{\mathcal{O}_X} f^*\mathcal{G}) = 0$ for all $q \geq 1$ and $n \geq N_2$. Thus

$$\mathrm{H}^p\big(Y, f_*(\mathcal{E}_n \otimes_{\mathcal{O}_X} f^*\mathcal{G})\big) \to \mathrm{H}^p(X, \mathcal{E}_n \otimes_{\mathcal{O}_X} f^*\mathcal{G}) \tag{4.3.12.2}$$

is an isomorphism for all $p \geq 0$ if $n \geq N_2$. Since $\mathcal{E}_\bullet$ satisfies $(\mathbf{V}_0)$, there exists some $N_3 \geq N_2$ such that $\mathrm{H}^p(X, \mathcal{E}_n \otimes_{\mathcal{O}_X} f^*\mathcal{G}) = 0$ for all $p \geq 1$ and $n \geq N_3$.

For $n \geq N_2$, we have $\mathbf{R}^q f_*\mathcal{E}_n = 0$ and $\mathbf{R}^q f_*(\mathcal{E}_n \otimes_{\mathcal{O}_X} f^*\mathcal{G}) = 0$ for $q \geq 1$, so the projection formula [33, Tag 08EU] simplifies to an isomorphism

$$f_*\mathcal{E}_n \otimes_{\mathcal{O}_Y} \mathcal{G} \to f_*(\mathcal{E}_n \otimes_{\mathcal{O}_X} f^*\mathcal{G})$$

since $f_*\mathcal{E}_n$, $\mathcal{E}_n$, and f are flat. Combining this with (4.3.12.2), we have $\mathrm{H}^p(Y, f_*\mathcal{E}_n \otimes_{\mathcal{O}_X} \mathcal{G}) = 0$ for $p \geq 1$ and $n \geq N_3$.

Part 5 ($f_*\mathcal{E}_\bullet$ satisfies $(\mathbf{G})$*).* We prove that $f_*\mathcal{E}_n \otimes_{\mathcal{O}_X} \mathcal{G}$ is globally generated for $n \gg 0$. Choose a surjection $\mathcal{O}_Y(-N_4)^{\oplus r} \to \mathcal{G}$ for some N_4. Tensoring with $f_*\mathcal{E}_n$ gives a surjection $f_*\mathcal{E}_n \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-N_4)^{\oplus r} \to f_*\mathcal{E}_n \otimes_{\mathcal{O}_Y} \mathcal{G}$. The first term is isomorphic to $(\mathcal{V}_n \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(n - N_4))^{\oplus r}$ where $\mathcal{V}_n$ is as in Part 1; the latter is globally generated for $n > N_4$.

By the above, the system $\{f_*\mathcal{E}_n\}$ satisfies the conditions of Definition 4.3.1, hence $f_*\mathcal{E}$ is a very positive vector bundle. ■

4.4. Derived category

In this section, we prove Proposition 4.4.6 which says that a very positive vector bundle on a projective variety X is a generator for the derived category of X. For this, we prove Proposition 4.4.3, which says that every finite rank vector bundle admits a "forward resolution" by copies of the very positive vector bundle.

Lemma 4.4.1. *Let k be an infinite field, let X be a proper k-scheme, let*

$$\mathcal{F} = \operatorname{colim}(\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to \cdots)$$

be a very positive vector bundle on X. Suppose given finite rank vector bundles $\mathcal{E}_1, \mathcal{E}_2$ and locally split injections $\mathcal{E}_1 \to \mathcal{E}_2$ and $\mathcal{E}_1 \to \mathcal{F}_{n_1}$ for some n_1. Then there exists $n_2 \gg n_1$ and a locally split injection $\mathcal{E}_2 \to \mathcal{F}_{n_2}$ making the diagram

$$\begin{array}{ccc} \mathcal{E}_1 & \longrightarrow & \mathcal{E}_2 \\ \downarrow & & \vdots \\ \mathcal{F}_{n_1} & \longrightarrow & \mathcal{F}_{n_2} \end{array}$$

commute.

Proof. Set $\mathcal{E}_3 := \operatorname{coker}(\mathcal{E}_1 \to \mathcal{E}_2)$. By condition (4) for $\mathcal{E}$, we may choose $n_2 \gg n_1$ so that $\operatorname{rank} \mathcal{F}_{n_2} > \max\{\operatorname{rank} \mathcal{E}_2, \dim S + \operatorname{rank} \mathcal{E}_3 - 1\}$ and $\mathcal{H}om_{\mathcal{O}_S}(\mathcal{E}_3, \mathcal{F}_{n_2}) \simeq \mathcal{E}_3^\vee \otimes_{\mathcal{O}_S} \mathcal{F}_{n_2}$ is globally generated and $\operatorname{Ext}^1_{\mathcal{O}_S}(\mathcal{E}_3, \mathcal{F}_{n_2}) = \mathrm{H}^1(\mathcal{O}_S, \mathcal{E}_3^\vee \otimes \mathcal{F}_{n_2}) = 0$. Then we conclude using Lemma 4.3.7. ∎

Lemma 4.4.2 ("dimension shifting"). *Let k be an infinite field, let X be a proper k-scheme, let $\ell \geq 1$ and let $\mathcal{F}_1^{[\ell]} \to \mathcal{F}_2^{[\ell]} \to \cdots$ be a sequence of locally split injections of finite rank vector bundles on X satisfying $(\mathbf{V}_\ell)$. Let $\mathcal{E}$ be a very positive vector bundle on X. By Lemma 4.4.1, there exists an increasing sequence $m_1 < m_2 < \cdots$ and locally split injections $\mathcal{F}_i^{[\ell]} \to \mathcal{E}_{m_i}$ which commute with the transition maps in $\{\mathcal{F}_i^{[\ell]}\}_{i\in\mathbf{N}}$ and $\{\mathcal{E}_i\}_{i\in\mathbf{N}}$. Set $\mathcal{F}_i^{[\ell-1]} := \mathcal{E}_{m_i}/\mathcal{F}_i^{[\ell]}$; then $\{\mathcal{F}_i^{[\ell-1]}\}_{i\in\mathbf{N}}$ satisfies $(\mathbf{V}_{\ell-1})$.*

Proof. Let $\mathcal{H}$ be a coherent $\mathcal{O}_X$-module. Let n' be such that $i \geq n'$ implies

$$\mathrm{H}^q(X, \mathcal{F}_i^{[\ell]} \otimes_{\mathcal{O}_X} \mathcal{H}) = 0 \quad \text{for } q \geq \ell + 1,$$

and let n'' be such that $i \geq n''$ implies $\mathrm{H}^q(X, \mathcal{E}_{m_i} \otimes_{\mathcal{O}_X} \mathcal{H}) = 0$ for all $q \geq 1$. We have a (locally split) exact sequence

$$0 \to \mathcal{F}_i^{[\ell]} \otimes_{\mathcal{O}_X} \mathcal{H} \to \mathcal{E}_{m_i} \otimes_{\mathcal{O}_X} \mathcal{H} \to \mathcal{F}_i^{[\ell-1]} \otimes_{\mathcal{O}_X} \mathcal{H} \to 0 \tag{4.4.2.1}$$

for all i, hence an exact sequence

$$\mathrm{H}^{q-1}(X, \mathcal{E}_{m_i} \otimes_{\mathcal{O}_X} \mathcal{H}) \to \mathrm{H}^{q-1}(X, \mathcal{F}_i^{[\ell-1]} \otimes_{\mathcal{O}_X} \mathcal{H}) \to \mathrm{H}^q(X, \mathcal{F}_i^{[\ell]} \otimes_{\mathcal{O}_X} \mathcal{H})$$

for all i. Since $\mathrm{H}^q(X, \mathcal{F}_i^{[\ell]} \otimes_{\mathcal{O}_X} \mathcal{H}) = 0$ for all $i \geq n'$ and $q \geq \ell + 1$, if $i \geq \max\{n', n''\}$ then $\mathrm{H}^{q-1}(X, \mathcal{F}_i^{[\ell-1]} \otimes_{\mathcal{O}_X} \mathcal{H}) = 0$ for $q \geq \ell$. This implies $\mathcal{F}_i^{[\ell-1]}$ satisfies $(\mathbf{V}_{\ell-1})$. ∎

Proposition 4.4.3. *In the setup of Lemma 4.4.2, assume that $d := \dim X \geq 1$. For any finite rank vector bundle $\mathcal{F}$ on X, there exists an exact sequence*

$$0 \to \mathcal{F} \to \mathcal{E}^{[d]} \to \cdots \to \mathcal{E}^{[0]} \to 0$$

where each $\mathcal{E}^{[i]}$ is a very positive vector bundle.

Proof. We view $\mathcal{F}$ as a constant system $\{\mathcal{F}_i^{[d]}\}_{i\in\mathbf{N}}$, which satisfies $(\mathbf{V}_d)$ by [20, Chapter III, Theorem 2.7]. By Lemma 4.4.2, we may obtain a sequence of exact sequences

$$0 \to \mathcal{F}^{[\ell]} \to \mathcal{E} \to \mathcal{F}^{[\ell-1]} \to 0$$

where $\mathcal{E}$ is a very positive vector bundle and each $\{\mathcal{F}_i^{[\ell]}\}_{i\in\mathbf{N}}$ satisfies $(\mathbf{V}_\ell)$. Let $\mathcal{H}$ be a coherent $\mathcal{O}_X$-module. In view of (4.4.2.1), whenever $\mathcal{E}_{m_i} \otimes_{\mathcal{O}_X} \mathcal{H}$ is globally generated, so is $\mathcal{F}_i^{[\ell-1]} \otimes_{\mathcal{O}_X} \mathcal{H}$. In particular the system $\{\mathcal{F}_i^{[0]}\}_{i\in\mathbf{N}}$ is a very positive vector bundle. ∎

Definition 4.4.4. Let $\mathcal{D}$ be a triangulated category with arbitrary direct sums. Let E be an object of $\mathcal{D}$.

(1) Let $\langle E\rangle$ be the strictly full triangulated subcategory of $\mathcal{D}$ containing E and closed under taking direct summands.

(2) Let $\langle E\rangle^{\text{big}}$ be the strictly full triangulated subcategory of $\mathcal{D}$ containing E and closed under taking arbitrary direct sums and under taking direct summands.

Lemma 4.4.5. *Let X be a smooth projective variety over an infinite field k, and let $\mathcal{E}$ be a very positive vector bundle on X. Then the essential image of $\mathcal{D}^b_{Coh}(X) \to \mathcal{D}^b_{QCoh}(X)$ is contained in $\langle\mathcal{E}\rangle$.*

Proof. Since X is smooth, we can represent any object of $\mathcal{D}^b_{Coh}(X)$ by a bounded complex of finite locally free modules. Hence it suffices to show that any finite locally free $\mathcal{O}_X$-module F is in $\langle\mathcal{E}\rangle$. By Proposition 4.4.3 and Theorem 4.3.8 we may find an exact sequence

$$0 \to F \to \mathcal{E} \to \cdots \to \mathcal{E} \to 0$$

of length $\dim X$. ■

Proposition 4.4.6. *Let X be a smooth projective variety over an infinite field k, and let $\mathcal{E}$ be a very positive vector bundle on X. Then $\mathcal{D}_{QCoh}(X) = \langle\mathcal{E}\rangle^{\text{big}}$. In particular, $\mathcal{E}$ is a generator of $\mathcal{D}_{QCoh}(X)$.*

Proof. It is enough to show that a perfect generator G is contained in $\langle\mathcal{E}\rangle^{\text{big}}$ because we know that $\langle G\rangle^{\text{big}} = \mathcal{D}_{QCoh}(X)$. This follows from the proof of Lemma 4.4.5 and [33, Tag 0BQT]. ■

5. Surjective ring map induces surjection on GL_∞

In this section we prove that infinite invertible matrices lift under any surjective ring map. We discovered this fact while thinking about how to lift twisted vector bundles from a curve to an ambient surface (see Remark 5.2.4).

5.1. Definition and theorem statement

Definition 5.1.1. Let I be an index set. For any ring A, we denote $\mathrm{GL}_I(A)$ the group of invertible elements of $\mathrm{Hom}_A(A^{\oplus I}, A^{\oplus I})$. We may identify elements of $\mathrm{Hom}_A(A^{\oplus I}, A^{\oplus I})$ with $\mathrm{Mat}^{\mathrm{cf}}_{I\times I}(A)$, matrices whose rows and columns are indexed by I and such that every column has only finitely many nonzero entries. Then elements of $\mathrm{GL}_I(A)$ correspond to matrices which admit a two-sided inverse. Given a ring homomorphism $\varphi : A \to B$, we obtain a group homomorphism $\mathrm{GL}_I(A) \to \mathrm{GL}_I(B)$ by applying φ to each element in the matrix, and this gives a functor Ring $\to$ Grp.

Theorem 5.1.2. *Let $A \to B$ be a surjective ring map. The group homomorphism*

$$\mathrm{GL}_{\mathbf{N}}(A) \to \mathrm{GL}_{\mathbf{N}}(B)$$

is surjective, i.e., any automorphism of the free B-module $B^{\oplus\mathbf{N}}$ lifts to an automorphism of the free A-module $A^{\oplus\mathbf{N}}$.

5.2. Applications

Before the proof, we discuss some applications.

Remark 5.2.1. Recall that Theorem 5.1.2 is false when $\mathbf{N}$ is replaced by a finite index set I. For example, if $|I| = 1$, the induced map $\mathbf{Z}^\times \to \mathbf{F}_p^\times$ is not surjective for any prime $p \geq 5$.

Example 5.2.2. Let $A := \mathbf{Z}[x, y]$ and $B := \mathbf{Z}[u^\pm]$ and let $A \to B$ be the ring map sending $(x, y) \mapsto (u, u^{-1})$. Then Theorem 5.1.2 implies there exists an invertible matrix $\mathtt{M} \in \mathrm{GL}_\mathbf{N}(A)$ whose image in $\mathrm{GL}_\mathbf{N}(B)$ is the diagonal matrix $u\mathsf{Id}_\mathbf{N}$.

Remark 5.2.3. One might ask whether there exists a reasonable notion of "determinant" for infinite invertible matrices. By Theorem 5.1.2, we know at least that there does not exist a natural transformation $\mathrm{GL}_\mathbf{N}(-) \to (-)^\times$ between functors Ring $\to$ Grp such that, for all $n \in \mathbf{N}$, the composition $\mathrm{GL}_n(-) \to \mathrm{GL}_\mathbf{N}(-) \to (-)^\times$ is the usual determinant for invertible $n \times n$ matrices. Indeed, every ring A admits a surjection from a polynomial ring $\mathbf{Z}[\{x_i\}_{i \in I}]$, but the only units of the latter are ± 1.

Remark 5.2.4. Let X be a separated Noetherian scheme, let $Y \to X$ be a closed subscheme which admits a covering $Y \subset U_1 \cup U_2$ by two affine open subsets of X. Then any infinite rank vector bundle on Y extends to $U_1 \cup U_2$. In particular, suppose X is a surface (any quasi-projective k-scheme of dimension 2) and Y is a curve in X, and let $\mathcal{E}$ be a countably generated vector bundle on Y. Then Bass' theorem implies that $\mathcal{E}|_{Y \times_X U_i}$ is trivial for all i. On $Y \times_X (U_1 \cap U_2)$, the two trivializations differ by a transition map $\varphi \in \mathrm{GL}_\mathbf{N}(\Gamma(Y \times_X (U_1 \cap U_2), \mathcal{O}_Y))$ which lifts to some $\varphi' \in \mathrm{GL}_\mathbf{N}(\Gamma(U_1 \cap U_2, \mathcal{O}_X))$ by Theorem 5.1.2. This invertible matrix φ' defines a countably generated vector bundle $\mathcal{E}'$ on $U_1 \cup U_2$ whose restriction to Y is $\mathcal{E}$. Here we may choose U_1, U_2 suitably so that the complement $X \setminus (U_1 \cup U_2)$ consists of a finite collection of closed points. Furthermore the vector bundle $\mathcal{E}'$ may not extend to the entire surface X (see Example 4.3.6).

Question 5.2.5. Is the twisted analogue of Theorem 5.1.2 true? Namely, let $Y \to X$ be a closed immersion of affine schemes, let $\mathcal{G} \to X$ be a $\mathbf{G}_\mathrm{m}$-gerbe, and let $\mathcal{E}$ be an infinite rank 1-twisted vector bundle on $\mathcal{G}$. Is the map

$$\mathrm{Aut}_{\mathcal{O}_\mathcal{G}}(\mathcal{E}) \to \mathrm{Aut}_{\mathcal{O}_{\mathcal{G}_Y}}(\mathcal{E}|_{\mathcal{G}_Y})$$

surjective?

5.3. Proof

We begin the proof of Theorem 5.1.2.

Definition 5.3.1. Let I be an index set. An *elementary matrix* indexed by I over a ring A is a matrix in $\mathrm{Mat}_{I \times I}(A)$ of the form $\mathrm{id}_I + \mathtt{M}$ where $\mathtt{M}$ is a matrix in $\mathrm{Mat}_{(I \setminus J) \times J}(A) \simeq \mathrm{Hom}_A(A^{\oplus J}, A^{\oplus I \setminus J})$ for some subset $J \subseteq I$.

Remark 5.3.2. Any elementary matrix $\mathrm{id}_I + \mathsf{M}$ is invertible, namely its inverse is $\mathrm{id}_I - \mathsf{M}$ which is itself an elementary matrix.

Remark 5.3.3. Let I be an index set. For any $i \in \mathbf{N}$, let $\mathsf{e}_i \in \mathbf{Z}^{\oplus \mathbf{N}}$ denote the ith basis vector. Let $\sigma : I \to I$ be a bijection and let $\varphi_\sigma \in \mathrm{GL}_I(\mathbf{Z})$ be the *permutation matrix* corresponding to σ, which sends $\mathsf{e}_i \mapsto \mathsf{e}_{\sigma(i)}$. Since permutation matrices are defined over $\mathbf{Z}$, they lift via any ring map.

Lemma 5.3.4. *Let A be a ring, let $\begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \in A^{\oplus n}$ be a unimodular vector. There exist elementary matrices $\mathsf{A}_1, \ldots, \mathsf{A}_m \in \mathrm{GL}_{n+1}(A)$ such that*

$$\mathsf{A}_1 \cdots \mathsf{A}_m \begin{bmatrix} a_1 & \cdots & a_n & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T$$

in $\mathrm{GL}_{n+1}(A)$.

Proof. If $c_1 a_1 + \cdots + c_n a_n = 1$, perform the column operations C_{n+1} += $c_i \mathrm{C}_i$ for $i = 1, \ldots, n$ (after which $\mathrm{C}_{n+1} = 1$) and C_i -= $a_i \mathrm{C}_{n+1}$ for $i = 1, \ldots, n$. ∎

5.4. Proof of Theorem 5.1.2

We say that a matrix $\mathsf{M} \in \mathrm{GL}_I(B)$ is *liftable* if it is in the image of $\mathrm{GL}_I(A) \to \mathrm{GL}_I(B)$. Any elementary matrix $\mathrm{id}_I + \mathsf{M} \in \mathrm{GL}_I(B)$ lifts to an elementary matrix in $\mathrm{GL}_I(A)$ (here we must be careful to lift any 0s as 0, to ensure that every column contains only finitely many nonzero entries). We note that if $\mathsf{M}' \in \mathrm{GL}_I(A)$ is a lift of $\mathsf{M} \in \mathrm{GL}_I(B)$, then the inverse M'^{-1} is a lift of M^{-1}. Furthermore, if $\mathsf{M}_1, \mathsf{M}_2 \in \mathrm{GL}_I(B)$ are invertible matrices such that M_1 is liftable, then M_2 is liftable if and only if their product $\mathsf{M}_1\mathsf{M}_2$ is liftable.

Part 1. Let $\mathsf{P} \in \mathrm{GL}_{\mathbf{N}}(B)$ be an automorphism of $B^{\oplus \mathbf{N}}$. Write

$$\mathsf{P}(\mathsf{e}_1) = \sum_{i=1,\ldots,n} a_i \mathsf{e}_i \text{ for } a_i \in B.$$

Then we see that $(a_1, \ldots, a_n)$ is a unimodular vector. By Lemma 5.3.4 we can find a liftable invertible matrix $\mathsf{T} \in \mathrm{GL}_{n+1}(B)$ such that the operator $\mathrm{diag}(\mathsf{T}, 1, 1, \ldots) \in \mathrm{GL}_{\mathbf{N}}(B)$ sends e_1 to the same vector as P.

Part 2. By the argument of Part 1, there exist positive integers $r_1, r_2, r_3, \ldots$ and liftable invertible $r_i \times r_i$ matrices $\mathsf{T}_i \in \mathrm{GL}_{r_i}(B)$ such that $\mathrm{diag}(\mathsf{T}_1, \mathsf{T}_2, \mathsf{T}_3, \ldots) \in \mathrm{GL}_{\mathbf{N}}(B)$ sends $\mathsf{e}_1, \mathsf{e}_{r_1+1}, \mathsf{e}_{r_1+r_2+1}, \ldots$ to the same vectors as P does. Moreover $\mathrm{diag}(\mathsf{T}_1, \mathsf{T}_2, \mathsf{T}_3, \ldots)$ is liftable: if $\mathsf{T}'_i \in \mathrm{GL}_{r_i}(A)$ is a lift of T_i, then $\mathrm{diag}(\mathsf{T}'_1, \mathsf{T}'_2, \mathsf{T}'_3, \ldots) \in \mathrm{GL}_{\mathbf{N}}(A)$ is a lift of $\mathrm{diag}(\mathsf{T}_1, \mathsf{T}_2, \mathsf{T}_3, \ldots)$.

Part 3. By Part 2, after replacing P by $\mathrm{diag}(\mathsf{T}_1, \mathsf{T}_2, \mathsf{T}_3, \ldots)^{-1} \cdot \mathsf{P}$, we may assume $\mathsf{P}(\mathsf{e}_i) = \mathsf{e}_i$ for infinitely many $i \in \mathbf{N}$. After possibly rearranging the e_i, we may assume there exists some countable set I such that P may be written in block matrix form as

$$\mathsf{P} = \begin{bmatrix} \mathsf{Id}_{\mathbf{N}} & \mathsf{T} \\ 0 & \mathsf{U} \end{bmatrix}$$

for some $\mathsf{T} \in \mathrm{Mat}_{\mathbf{N}\times I}(B)$ and $\mathsf{U} \in \mathrm{GL}_I(B)$. If we can show that

$$\mathrm{diag}(\mathsf{Id}_{\mathbf{N}}, \mathsf{U}) = \begin{bmatrix} \mathsf{Id}_{\mathbf{N}} & 0 \\ 0 & \mathsf{U} \end{bmatrix}$$

is liftable, then we are done since

$$\begin{bmatrix} \mathsf{Id}_{\mathbf{N}} & \mathsf{T} \\ 0 & \mathsf{U} \end{bmatrix} \begin{bmatrix} \mathsf{Id}_{\mathbf{N}} & 0 \\ 0 & \mathsf{U}^{-1} \end{bmatrix} = \begin{bmatrix} \mathsf{Id}_{\mathbf{N}} & \mathsf{TU}^{-1} \\ 0 & \mathsf{Id}_I \end{bmatrix}$$

is liftable (because it is an elementary matrix).

Part 4 (Whitehead's lemma). For any $\mathsf{A}, \mathsf{B} \in \mathrm{GL}_I(A)$, the matrix $\left[\begin{smallmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{B} \end{smallmatrix}\right]$ is liftable if and only if $\left[\begin{smallmatrix} \mathsf{AB} & 0 \\ 0 & \mathsf{Id}_I \end{smallmatrix}\right]$ is liftable. Namely, perform the following (block) row/column operations by multiplying by appropriate elementary matrices and permutation matrices on the left/right, respectively [23, Chapter I, Lemma 5.1]:

$$\begin{bmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{B} \end{bmatrix} \xrightarrow{\mathsf{C}_1 \mathrel{+}= \mathsf{C}_2\mathsf{B}^{-1}} \begin{bmatrix} \mathsf{A} & 0 \\ \mathsf{Id}_I & \mathsf{B} \end{bmatrix} \xrightarrow{\mathsf{C}_2 \mathrel{-}= \mathsf{C}_1\mathsf{B}} \begin{bmatrix} \mathsf{A} & -\mathsf{AB} \\ \mathsf{Id}_I & 0 \end{bmatrix} \xrightarrow[\mathsf{C}_1 \mathrel{*}= -1]{\mathsf{C}_1 \Leftrightarrow \mathsf{C}_2} \begin{bmatrix} \mathsf{AB} & \mathsf{A} \\ 0 & \mathsf{Id}_I \end{bmatrix} \xrightarrow{\mathsf{R}_1 \mathrel{-}= \mathsf{AR}_2} \begin{bmatrix} \mathsf{AB} & 0 \\ 0 & \mathsf{Id}_I \end{bmatrix}$$

Part 5 (Eilenberg swindle). Let I and U be as in Part 3. If I is empty, there is nothing to show. If I is nonempty, by Part 4 we see that

$$\mathsf{D} := \mathrm{diag}(\ldots, \mathsf{U}, \mathsf{U}^{-1}, \mathsf{U}, \mathsf{U}^{-1})$$

is liftable. Thus we see that

$$\mathrm{diag}(\mathsf{Id}_{\mathbf{N}}, \mathsf{U})$$

is liftable if and only if

$$\mathrm{diag}(\mathsf{D}, \mathsf{U}) = \mathrm{diag}(\ldots, \mathsf{U}, \mathsf{U}^{-1}, \mathsf{U}, \mathsf{U}^{-1}, \mathsf{U})$$

is liftable, which it is by the same argument with U replaced by U^{-1}.

A. Infinite matrix rings

A.1. Skolem–Noether

In this section, we prove Theorem A.1.3, which is closely related to a theorem of Courtemanche–Dugas [5, Lemma 2.2] on the group of automorphisms of endomorphism algebras of projective modules. We remove their hypotheses on idempotents and indecomposable projective modules, but require that the ring is Noetherian and that the automorphism is of the entire sheaf of algebras (as opposed to just the global sections) on the Zariski site. For the finite rank case, see for example [29, Chapter IV, Proposition 1.4].

Lemma A.1.1. *Let R be a Noetherian ring, let I be an index set, let $V := \bigoplus_{i\in I} R\mathsf{e}_i$ be a free R-module with basis $\{\mathsf{e}_i\}_{i\in I}$, set $\mathcal{A} := \mathrm{End}_R(V)$ and let $\varphi : \mathcal{A} \to \mathcal{A}$ be an R-algebra* *automorphism. Let* $\mathsf{E}_i \in \mathcal{A}$ *be the projection onto the* i*th summand, set* $\mathsf{N}_i := \varphi(\mathsf{E}_i) \in \mathcal{A}$ *and let* $W_i := \operatorname{im} \mathsf{N}_i \subseteq V$ *be the image of* N_i*. Then*

(i) *the* W_i *are pairwise isomorphic,*

(ii) *each* W_i *is an invertible* R*-module, and*

(iii) *the natural map*

$$\bigoplus_{i \in I} W_i \to V \tag{A.1.1.1}$$

is an isomorphism.

Proof. We separate the proof into several steps.

Part 1 (proof of (i)*).* For $i_1, i_2 \in I$, let E_{i_1,i_2} be the (i_1, i_2)th matrix unit (the only nonzero entry is a 1 in the (i_1, i_2)th entry) so that $\mathsf{E}_i = \mathsf{E}_{i,i}$. For any unit $\mathsf{U} \in \mathcal{A}^\times \simeq \mathrm{GL}_I(R)$ we have

$$\varphi(\mathsf{U}^{-1} \cdot \mathsf{E}_i \cdot \mathsf{U}) = (\varphi(\mathsf{U}))^{-1} \cdot \mathsf{N}_i \cdot \varphi(\mathsf{U})$$

in $\mathcal{A}$. Since the E_i are pairwise conjugate (if $\mathsf{U}_{i_1,i_2} \in \mathcal{A}^\times$ is the R-automorphism switching e_{i_1} and e_{i_2}, then $\mathsf{E}_{i_2} = \mathsf{U}_{i_1,i_2}^{-1} \cdot \mathsf{E}_{i_1} \cdot \mathsf{U}_{i_1,i_2}$), the N_i are also pairwise conjugate, namely $(\varphi(\mathsf{U}_{i_1,i_2}))^{-1}$ defines an R-module isomorphism $W_{i_1} \simeq W_{i_2}$.

Part 2 (proof of (ii) *when R is a field).* Suppose that R is a field. Since φ is injective, we have $\mathsf{N}_i \neq 0$ and thus $W_i \neq 0$ also. Suppose for the sake of contradiction that $\dim_R(W_i) \geq 2$. Let $W_i = W_i' \oplus W_i''$ be a decomposition of W_i into a direct sum of nonzero subspaces $W_i', W_i'' \subset W_i$. This allows us to write N_i as a sum $\mathsf{N}_i = \mathsf{N}_i' + \mathsf{N}_i''$ of nonzero orthogonal idempotents $\mathsf{N}_i', \mathsf{N}_i'' \in \mathcal{A}$, corresponding to projections onto W_i', W_i'' respectively. Applying φ^{-1} gives a decomposition $\mathsf{E}_i = \varphi^{-1}(\mathsf{N}_i') + \varphi^{-1}(\mathsf{N}_i'')$ where $\varphi^{-1}(\mathsf{N}_i'), \varphi^{-1}(\mathsf{N}_i'')$ are nonzero orthogonal idempotents. Multiplying by $\varphi^{-1}(\mathsf{N}_i')$ gives

$$\mathsf{E}_i \varphi^{-1}(\mathsf{N}_i') = \varphi^{-1}(\mathsf{N}_i')\mathsf{E}_i = \varphi^{-1}(\mathsf{N}_i'),$$

so there exists some $u' \in R$ such that $\varphi^{-1}(\mathsf{N}_i') = u'\mathsf{E}_i$. Similarly there exists some $u'' \in R$ such that $\varphi^{-1}(\mathsf{N}_i'') = u''\mathsf{E}_i$. Substituting gives $1 = u' + u''$ where $u', u'' \in R$ are nonzero orthogonal idempotents, which gives a contradiction since R is a field.

Part 3 (proof of (ii)*, general case).* Since W_i is the image of an idempotent endomorphism of the free R-module V, we deduce that W_i is projective as an R-module and that the formation of W_i commutes with arbitrary base change, i.e., the map

$$\operatorname{im}(\mathsf{N}_i) \otimes_R S \to \operatorname{im}(\mathsf{N}_i \otimes_R S) \tag{A.1.1.2}$$

is an isomorphism for any ring map $R \to S$.

Let $\mathfrak{m}$ be a maximal ideal of R. Since $\mathfrak{m}$ is finitely generated, the natural map

$$\mathcal{A} \otimes_R R/\mathfrak{m} = \operatorname{End}_R(V) \otimes_R R/\mathfrak{m} \to \operatorname{End}_{R/\mathfrak{m}}(V \otimes_R R/\mathfrak{m})$$

is an isomorphism by [33, Tag 059K], hence φ induces an $R/\mathfrak{m}$-algebra automorphism

$$\overline{\varphi} : \operatorname{End}_{R/\mathfrak{m}}(V \otimes_R R/\mathfrak{m}) \to \operatorname{End}_{R/\mathfrak{m}}(V \otimes_R R/\mathfrak{m})$$

which is compatible with φ. Since $W_i \otimes_R R_\mathfrak{m}$ is a projective $R_\mathfrak{m}$-module, it is free by Kaplansky's theorem [2, Corollary 3.3] and its rank is

$$\operatorname{rank}_{R_\mathfrak{m}}(W_i \otimes_R R_\mathfrak{m}) = \dim_{R/\mathfrak{m}}(W_i \otimes_R R/\mathfrak{m}) \overset{1}{=} \dim_{R/\mathfrak{m}}(\operatorname{im}(\mathsf{N}_i \otimes_R R/\mathfrak{m})) \overset{2}{=} 1$$

where equality 1 is by taking $S := R/\mathfrak{m}$ in (A.1.1.2) and equality 2 is by applying Part 2 to $\bar{\varphi}$.

By the above, we conclude that for all maximal ideals $\mathfrak{m}$ of R, the localization $W_i \otimes_R R_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module of rank 1, in particular $W_i \otimes_R R_\mathfrak{m}$ is a finitely generated $R_\mathfrak{m}$-module. Since R is Noetherian, [2, Proposition 4.2 (2)] implies that W_i itself is finitely generated (and projective). This proves that W_i is an invertible R-module.

Part 4 (proof of (iii)*).* We recall the argument of [5, Lemma 2.2]. We first show that, for a fixed $i_0 \in I$, we have $\mathsf{N}_j(\mathsf{e}_{i_0}) = 0$ for all but finitely many $j \in I$. If $\mathsf{N}_j(\mathsf{e}_{i_0}) \neq 0$, then $\mathsf{N}_j \cdot \mathsf{E}_{i_0} \neq 0$ and thus $\mathsf{E}_j \cdot \varphi^{-1}(\mathsf{E}_{i_0}) \neq 0$. We deduce that the only possibilities are for j in the support of $\varphi^{-1}(\mathsf{E}_{i_0})$. By (ii) applied to φ^{-1}, the image $\operatorname{im}(\varphi^{-1}(\mathsf{E}_{i_0}))$ is finitely generated, hence has finite support.

The injectivity of (A.1.1.1) follows from the fact that the N_i are idempotent and $\mathsf{N}_{i_1} \cdot \mathsf{N}_{i_2} = 0$ if $i_1 \neq i_2$. For surjectivity, it suffices to show that for every $i_0 \in I$ the basis element e_{i_0} is in the image of (A.1.1.1). By the above, the set $S_{i_0} := \{i \in I \ : \ \mathsf{N}_i(\mathsf{e}_{i_0}) \neq 0\}$ is finite. Set $v_{i_0} := \mathsf{e}_{i_0} - \sum_{i \in S_{i_0}} \mathsf{N}_i(\mathsf{e}_{i_0})$. If $i \in I \setminus S_{i_0}$, then $\mathsf{N}_i(v_{i_0}) = 0$. If $i \in S_{i_0}$, then $\mathsf{N}_i(v_{i_0}) = \mathsf{N}_i(\mathsf{e}_{i_0}) - \mathsf{N}_i(\mathsf{e}_{i_0}) = 0$ since $\mathsf{N}_{i_1} \cdot \mathsf{N}_{i_2} = 0$ if $i_1 \neq i_2$. Thus $v_{i_0} \in \bigcap_{i \in I}(\ker \mathsf{N}_i)$.

It remains to show that $\bigcap_{i \in I}(\ker \mathsf{N}_i) = 0$. Suppose $v \in V$ is an element such that $\mathsf{N}_i(v) = 0$ for all $i \in I$ and define $\pi : V \to V$ sending $\mathsf{e}_i \mapsto v$ for all $i \in I$. Then $\mathsf{N}_i \cdot \pi = 0$ for all $i \in I$, and $\mathsf{E}_i \cdot \varphi^{-1}(\pi) = 0$ for all $i \in I$, i.e., $\varphi^{-1}(\pi) = 0$, i.e., $\pi = 0$.

This concludes the proof of Lemma A.1.1. ■

Lemma A.1.2. *In the notation of Lemma* A.1.1*, if each W_i is a free R-module, then there exists some $\mathsf{U} \in \mathcal{A}^\times$ such that φ is equal to the conjugation-by-U map.*

Proof (from [5]*).* Choose generators $\mathsf{w}_i \in W_i$ and define $\mathsf{U}' : V \to V$ sending $\mathsf{e}_i \mapsto \mathsf{w}_i$ for all $i \in I$. Since (A.1.1.1) is an isomorphism, we see that U' is an isomorphism. Given $u = \sum_{i \in I} u_i \mathsf{e}_i$, we have $(q_{i_0}\mathsf{U}')(u) = q_{i_0}(\sum_{i \in I} u_i \mathsf{w}_i) = u_{i_0}\mathsf{w}_{i_0}$ and $(\mathsf{U}'\mathsf{E}_{i_0})(u) = \mathsf{U}'(u_{i_0}\mathsf{e}_{i_0}) = u_{i_0}\mathsf{w}_{i_0}$. Thus $q_i = \mathsf{U}'\mathsf{E}_i\mathsf{U}'^{-1}$ for all $i \in I$. After replacing φ by $\varphi \cdot (\mathsf{U}'^{-1}(-)\mathsf{U}')$, we may assume that

$$\varphi(\mathsf{E}_i) = \mathsf{E}_i \tag{A.1.2.1}$$

for all $i \in I$.

Fix i_1, i_2 and set $\varphi(\mathsf{E}_{i_1,i_2})(\mathsf{e}_{j_1}) = \sum_{j_2 \in I} s_{j_1,j_2}^{i_1,i_2} \mathsf{e}_{j_2}$ for some $s_{j_1,j_2}^{i_1,i_2} \in R$. Given $u = \sum_{i \in I} u_i \mathsf{e}_i$, we have

$$\big(\mathsf{E}_{i_1} \cdot \varphi(\mathsf{E}_{i_1,i_2}) \cdot \mathsf{E}_{i_2}\big)(u) = \big(\mathsf{E}_{i_1} \cdot \varphi(\mathsf{E}_{i_1,i_2})\big)(u_{i_2}\mathsf{e}_{i_2}) = \mathsf{E}_{i_1}\Big(u_{i_2} \sum_{i \in I} s_{i_2,i}^{i_1,i_2} \mathsf{e}_i\Big) = u_{i_2} s_{i_2,i_1}^{i_1,i_2} \mathsf{e}_{i_1}$$

and

$$\mathsf{E}_{i_1,i_2}(u) = u_{i_2}\mathsf{e}_{i_1}$$

hence

$$\mathsf{E}_{i_1} \cdot \varphi(\mathsf{E}_{i_1,i_2}) \cdot \mathsf{E}_{i_2} = s_{i_2,i_1}^{i_1,i_2} \mathsf{E}_{i_1,i_2}$$

for all i_1, i_2. Since $\mathsf{E}_{i_1} \cdot \mathsf{E}_{i_1,i_2} \cdot \mathsf{E}_{i_2} = \mathsf{E}_{i_1,i_2}$, applying φ and (A.1.2.1) gives $\varphi(\mathsf{E}_{i_1,i_2}) = s_{i_2,i_1}^{i_1,i_2} \mathsf{E}_{i_1,i_2}$ for all i_1, i_2 (and $s_{j_1,j_2}^{i_1,i_2} = 0$ if $(j_1, j_2) \neq (i_2, i_1)$). We set $s_{i_1,i_2} := s_{i_2,i_1}^{i_1,i_2}$. Applying the above argument to φ^{-1} implies that every s_{i_1,i_2} is a unit. Applying φ to $\mathsf{E}_{i_1,i_2} \cdot \mathsf{E}_{i_2,i_3} = \mathsf{E}_{i_1,i_3}$ gives $s_{i_1,i_2} s_{i_2,i_3} = s_{i_1,i_3}$ (and in particular $s_{i,i} = 1$). We choose an arbitrary $t \in I$, define $\mathsf{U}'' : V \to V$ by $\mathsf{U}'' = \sum_i s_{t,i} \mathsf{E}_i$. After replacing φ by $\varphi \cdot (\mathsf{U}''^{-1}(-)\mathsf{U}'')$, we may assume

$$\varphi(\mathsf{E}_{i_1,i_2}) = \mathsf{E}_{i_1,i_2} \tag{A.1.2.2}$$

for all $i_1, i_2 \in I$.

Let $\mathsf{N} \in \mathcal{A}$ be a matrix and let a_{i_1,i_2} be the (i_1, i_2)th entry of N. Then

$$\mathsf{E}_{i_1} \cdot \mathsf{N} \cdot \mathsf{E}_{i_2} = a_{i_1,i_2} \mathsf{E}_{i_1,i_2}$$

and applying φ gives

$$\mathsf{E}_{i_1} \cdot \varphi(\mathsf{N}) \cdot \mathsf{E}_{i_2} = a_{i_1,i_2} \mathsf{E}_{i_1,i_2}$$

which implies that the (i_1, i_2)th entries of $\varphi(\mathsf{N})$ and N are equal for all $i_1, i_2 \in I$, i.e., $\varphi(\mathsf{N}) = \mathsf{N}$. Since N was arbitrary, we have $\varphi = \mathrm{id}_{\mathcal{A}}$ as desired. ∎

Theorem A.1.3. *Let $\mathcal{C}$ be a locally ringed site with a final object S, let I be an index set, let $V := \bigoplus_{i \in I} \mathcal{O}_{\mathcal{C}} \mathsf{e}_i$ be a free $\mathcal{O}_{\mathcal{C}}$-module with basis $\{\mathsf{e}_i\}_{i \in I}$, set $\mathcal{A} := \mathcal{E}nd_{\mathcal{O}_{\mathcal{C}}}(V)$ and let $\varphi : \mathcal{A} \to \mathcal{A}$ be an $\mathcal{O}_{\mathcal{C}}$-algebra automorphism. Assume that S has a covering $\{U_\lambda \to S\}_{\lambda \in \Lambda}$ such that $\Gamma(U_\lambda, \mathcal{O}_{U_\lambda})$ is a Noetherian ring for all $\lambda \in \Lambda$. Then there exists a covering $\{S_\xi \to S\}_{\xi \in \Xi}$ and units $\mathsf{U}_m \in \Gamma(S_\xi, \mathcal{A}^\times)$ such that $\varphi|_{S_\xi} : \mathcal{A}|_{S_\xi} \to \mathcal{A}|_{S_\xi}$ is the conjugation-by-U_m morphism.*

Proof. Let $\mathsf{E}_i \in \Gamma(S, \mathcal{A})$ be the projection onto the ith summand, and let $W_i \subset V$ be the image of E_i. The formation of W_i is compatible with localization since the projections E_i are. By Lemma A.1.1, the W_i are pairwise-isomorphic invertible $\mathcal{O}_{\mathcal{C}}$-modules. Since $\mathcal{C}$ is locally ringed, there exists a covering $\{S_\xi \to S\}_{\xi \in \Xi}$ such that each $W_i|_{S_\xi}$ is a trivial $\mathcal{O}_{S_\xi}$-module. We conclude by Lemma A.1.2. ∎

A.2. The center of endomorphism rings of projective modules

Given a ring A, it is well known that the center of the matrix ring $\mathrm{Mat}_{n \times n}(A)$ consists of matrices of the form $f\, \mathrm{id}_n$ for some $f \in A$. In this section we prove an extension of this fact to endomorphism rings of projective modules of possibly infinite rank.

Lemma A.2.1. *Let $\mathcal{C}$ be a locally ringed site, let $\mathcal{E}$ be a locally projective $\mathcal{O}_{\mathcal{C}}$-module of positive rank, let $\varphi : \mathcal{E} \to \mathcal{E}$ be an $\mathcal{O}_{\mathcal{C}}$-linear endomorphism. Suppose that, for all $U \in \mathcal{C}$, the restriction $\varphi|_U$ is contained in the center of $\mathrm{End}_{\mathcal{O}_U}(\mathcal{E}|_U)$. There exists a unique $f \in \Gamma(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ such that $\varphi = f\, \mathrm{id}_{\mathcal{E}}$.*

Proof. For any $\mathsf{v} \in \Gamma(\mathcal{C}, \mathcal{E})$, let $s_\mathsf{v} : \mathcal{O}_{\mathcal{C}} \to \mathcal{E}$ denote the $\mathcal{O}_{\mathcal{C}}$-linear map sending $1 \mapsto \mathsf{v}$. We say that v is a *unimodular* element of $\mathcal{E}$ if s_v admits an $\mathcal{O}_{\mathcal{C}}$-linear retraction $\pi_\mathsf{v} : \mathcal{E} \to \mathcal{O}_{\mathcal{C}}$.

In this case, evaluating $\varphi \circ (s_{\mathsf{v}} \circ \pi_{\mathsf{v}}) = (s_{\mathsf{v}} \circ \pi_{\mathsf{v}}) \circ \varphi$ at v implies $\varphi(\mathsf{v}) = \pi_{\mathsf{v}}(\varphi(\mathsf{v})) \cdot \mathsf{v}$, namely $\varphi(\mathsf{v})$ is a scalar multiple of v.

Given two unimodular elements $\mathsf{v}_1, \mathsf{v}_2$ of $\mathcal{E}$ and retractions $\pi_{\mathsf{v}_1}, \pi_{\mathsf{v}_2}$ of $s_{\mathsf{v}_1}, s_{\mathsf{v}_2}$ respectively, evaluating $\varphi \circ (s_{\mathsf{v}_i} \circ \pi_{\mathsf{v}_2}) = (s_{\mathsf{v}_1} \circ \pi_{\mathsf{v}_2}) \circ \varphi$ at v_2 implies $\varphi(\mathsf{v}_1) = \pi_{\mathsf{v}_2}(\varphi(\mathsf{v}_2)) \cdot \mathsf{v}_1$. Applying π_{v_1} gives $\pi_{\mathsf{v}_1}(\varphi(\mathsf{v}_1)) = \pi_{\mathsf{v}_2}(\varphi(\mathsf{v}_2))$. Hence this constant $\pi_{\mathsf{v}}(\varphi(\mathsf{v}))$ is independent of choice of v and retraction π_{v}.

After a localization of $\mathcal{C}$, we may assume that $\mathcal{E}$ is a globally a direct summand of a free module, so that we have $\mathcal{O}_{\mathcal{C}}$-linear maps $\iota : \mathcal{E} \to \mathcal{O}_{\mathcal{C}}^{\oplus I}$ and $\pi : \mathcal{O}_{\mathcal{C}}^{\oplus I} \to \mathcal{E}$ such that $\pi \circ \iota = \mathrm{id}_{\mathcal{E}}$. Let $\mathsf{M} \in \mathrm{Mat}^{\mathrm{cf}}_{I \times I}(\Gamma(\mathcal{C}, \mathcal{O}_{\mathcal{C}}))$ be the matrix corresponding to the $\mathcal{O}_{\mathcal{C}}$-linear map $\iota \circ \pi : \mathcal{O}_{\mathcal{C}}^{\oplus I} \to \mathcal{E} \to \mathcal{O}_{\mathcal{C}}^{\oplus I}$, and let v_i denote the ith column of M. If v_i contains an entry which is a unit, then v_i is unimodular. The assumption that $\mathcal{E}$ has positive rank means that the entries of M generate the unit ideal of $\Gamma(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$: given an object $U \in \mathcal{C}$ for which there exists a surjection $p : \mathcal{E}|_U \to \mathcal{O}_U$, the composition $p \circ \pi$ is a surjection which remains surjective after precomposition by $\iota \circ \pi$. Let $U \in \mathcal{C}$ be an object; by [33, Tag 04ES], there exists a covering $\{U_\lambda \to U\}_{\lambda \in \Lambda}$ such that for each $\lambda \in \Lambda$, there exists an entry of M which is a unit of $\Gamma(U_\lambda, \mathcal{O}_{\mathcal{C}})$; say that v_{i_λ} contains such an entry. By the above, there exists a constant $f_\lambda \in \Gamma(U_\lambda, \mathcal{O}_{\mathcal{C}})$ such that $\varphi(\mathsf{v}_{i_\lambda}) = f_\lambda \mathsf{v}_{i_\lambda}$ in $\Gamma(U_\lambda, \mathcal{E})$. Moreover, we have $f_{\lambda_1}|_{U_{\lambda_1} \times_U U_{\lambda_2}} = f_{\lambda_2}|_{U_{\lambda_1} \times_U U_{\lambda_2}}$ for all $\lambda_1, \lambda_2 \in \Lambda$, so there exists a unique $f \in \Gamma(U, \mathcal{O}_{\mathcal{C}})$ such that $f|_{U_\lambda} = f_\lambda$ for all $\lambda \in \Lambda$.

Now it remains to show that, for any object $U \in \mathcal{C}$ and any element $\mathsf{w} \in \Gamma(U, \mathcal{E})$, there exists a covering $\{U_\lambda \to U\}_{\lambda \in \Lambda}$ such that for each $\lambda \in \Lambda$, the restriction $\mathsf{w}|_{U_\lambda} \in \Gamma(U_\lambda, \mathcal{E})$ is an $\Gamma(U_\lambda, \mathcal{O}_{\mathcal{C}})$-linear combination of unimodular elements of $\mathcal{E}|_{U_\lambda}$. Set $A := \Gamma(U, \mathcal{O}_{\mathcal{C}})$ and let P be the image of the A-linear map $\mathsf{M}|_U : A^{\oplus I} \to A^{\oplus I}$. Since M is idempotent, we deduce that P is a projective A-module. Let $\mathfrak{p}$ be a prime of A. By Kaplansky [2, Corollary 3.3], the localization $P_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module. We may choose finitely many elements $\mathsf{u}'_1, \dots, \mathsf{u}'_n \in P_{\mathfrak{p}}$ such that $\{\mathsf{u}'_1, \dots, \mathsf{u}'_n\}$ is part of an $A_{\mathfrak{p}}$-basis for $P_{\mathfrak{p}}$ and such that w is an $A_{\mathfrak{p}}$-linear combination of the u'_ℓ, say $\mathsf{w} = c'_1 \mathsf{u}'_1 + \cdots + c'_n \mathsf{u}'_n$ with $c'_\ell \in A_{\mathfrak{p}}$. We may find some $a \in A \setminus \mathfrak{p}$ such that $c'_1, \dots, c'_n$ lift to $c''_1, \dots, c''_n \in A_a$ and $\mathsf{u}'_1, \dots, \mathsf{u}'_n$ lift to $\mathsf{u}''_1, \dots, \mathsf{u}''_n \in P_a$. We may multiply a by an element of $A \setminus \mathfrak{p}$ if necessary so that $\mathsf{w} = c''_1 \mathsf{u}''_1 + \cdots + c''_n \mathsf{u}''_n$ in P_a. Since each u'_ℓ is unimodular in $P_{\mathfrak{p}}$, each u''_ℓ contains an entry which is not contained in $\mathfrak{p} A_a$. Thus, we may again multiply a by an element of $A \setminus \mathfrak{p}$ so that these entries become units of A_a. Then each u''_ℓ is unimodular in $\mathcal{E}|_{U'}$ for any morphism $U' \to U$ such that $a|_{U'}$ is invertible in $\Gamma(U', \mathcal{O}_{\mathcal{C}})$.

Since $\mathcal{C}$ is locally ringed, there exists a covering $\{U_{\lambda_{\mathfrak{p}}} \to U\}_{\lambda_{\mathfrak{p}} \in \Lambda_{\mathfrak{p}}}$ such that for each $\lambda_{\mathfrak{p}} \in \Lambda_{\mathfrak{p}}$ either $a|_{U_{\lambda_{\mathfrak{p}}}}$ or $(1-a)|_{U_{\lambda_{\mathfrak{p}}}}$ is invertible in $\Gamma(U_{\lambda_{\mathfrak{p}}}, \mathcal{O}_{\mathcal{C}})$. In this way, since $\operatorname{Spec} A$ is quasi-compact, we obtain elements $a_1, \dots, a_n \in A$ which generate the unit ideal of A and coverings $\{U_{\lambda_\ell} \to U\}_{\lambda_\ell \in \Lambda_\ell}$ such that either $a_\ell|_{U_{\lambda_\ell}}$ or $(1 - a_\ell)|_{U_{\lambda_\ell}}$ is invertible in $\Gamma(U_{\lambda_\ell}, \mathcal{O}_{\mathcal{C}})$. Taking the fiber product of these coverings gives the result. ■

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