Tori over number fields and special values at $s = 1$

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Abstract. We define a Weil-étale complex with compact support for duals (in the sense of the Bloch dualizing cycles complex $\mathbb{Z}^c$) of a large class of $\mathbb{Z}$-constructible sheaves on an integral 1-dimensional proper arithmetic scheme flat over $\mathrm{Spec}(\mathbb{Z})$. This complex can be thought of as computing Weil-étale homology. For those $\mathbb{Z}$-constructible sheaves that are moreover tamely ramified, we define an "additive" complex which we think of as the Lie algebra of the dual of the $\mathbb{Z}$-constructible sheaf. The product of the determinants of the additive and Weil-étale complex is called the fundamental line. We prove a duality theorem which implies that the fundamental line has a natural trivialization, giving a multiplicative Euler characteristic. We attach a natural L-function to the dual of a $\mathbb{Z}$-constructible sheaf; up to a finite number of factors, this L-function is an Artin L-function at $s + 1$. Our main theorem contains a vanishing order formula at $s = 0$ for the L-function and states that, in the tamely ramified case, the special value at $s = 0$ is given up to sign by the Euler characteristic. This generalizes the analytic class number formula for the special value at $s = 1$ of the Dedekind zeta function. In the function field case, this is a theorem of Geisser–Suzuki.

Contents

1. Introduction

1.1. Results

1.1.1. A new L-function. Let $\mathcal{O}$ be an order in a number field K and $X = \mathrm{Spec}(\mathcal{O})$. Denote $\mathbb{Z}_X^c := \mathbb{Z}_X^c(0)$ Bloch's cycle complex (with cohomological indexing), defined on the étale site of X; in particular if X is regular we have $\mathbb{Z}_X^c = \mathbb{G}_m[1]$. A sheaf of abelian groups F on the étale site of X is $\mathbb{Z}$-constructible if on a dense open it is locally constant associated to a finite type abelian group, and moreover the stalks at all geometric points are finite type abelian groups. If F is a $\mathbb{Z}$-constructible sheaf, denote $F^D := R\mathcal{H}om_X(F, \mathbb{Z}_X^c)$. The cohomology of F^D is related to the compactly supported cohomology of F by *Artin–Verdier duality*; thus we think of it as computing homology with coefficients in F. To F^D, we associate an L-function:

Definition. For a complex of étale sheaves M, we put

$$M \,\widehat{\otimes}\, \mathbb{Q}_l := \left(R \lim_n (M \otimes^L \mathbb{Z}/l^n\mathbb{Z}) \right) \otimes \mathbb{Q}$$

computed on the proétale site. For each closed point x of X, let $l = l_x$ be a prime number such that $l \neq \mathrm{char}(\kappa(x))$ and L_x the usual local factor defined using the geometric Frobenius φ: for a finite dimensional $\mathbb{Q}_l$-representation V,

$$L_x(V, s) := \det\left(1 - \varphi N(x)^{-s}|V\right)^{-1}$$

with $N(x) = \mathrm{card}(\kappa(x))$. We define the L-function of F^D as

$$L_X(F^D, s) := \prod_{x \in X_0} L_x\left((i_x^* F^D) \,\widehat{\otimes}\, \mathbb{Q}_{l_x}, s\right).$$

We compute explicitly the local factors and show that they are well defined and that the Euler product converges for $s > 1$. In fact, denote $g : \mathrm{Spec}(K) \to X$ the inclusion of the generic point and $G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$. If V is the rational representation of G_K corresponding to $g^* F \otimes \mathbb{Q}$ and $V^\vee$ is its linear dual, then the L-function of F^D equals up to a finite number of factors the Artin L-function $L_K(V^\vee, s + 1)$. In particular, the L-function extends to a meromorphic function on $\mathbb{C}$. If F is a $\mathbb{Z}$-constructible sheaf on $\mathrm{Spec}(K)$ associated to a finite type integral G_K-representation M, we have moreover $L_X((g_* F)^D, s) = L_K(\mathrm{Hom}_{\mathrm{Ab}}(M, \mathbb{Q}), s + 1)$. We have as a special case $L_X(\mathbb{Z}^D, s) = \zeta_X(s + 1)$ if X is regular. On the other hand, let $i : x \to X$ be the inclusion of a closed point and F is a $\mathbb{Z}$-constructible sheaf on x. In [31] we introduced L-functions for $\mathbb{Z}$-constructible sheaves on X; then

$$L_X\left((i_* F)^D, s\right) = \det\left(I - \varphi N(x)^{-s}|\mathcal{H}om_x(F, \mathbb{Z}) \otimes \mathbb{Q}\right)^{-1}$$

is the L-function of the $\mathbb{Z}$-constructible sheaf $i_* \mathcal{H}om_x(F, \mathbb{Z})$. Finally, if $j : U \hookrightarrow X$ is the inclusion of a dense open subscheme, we have

$$L_X\left((j_! \mathbb{Z})^D, s\right) = \zeta_U(s + 1) \times \prod_{x \in X \setminus U} \frac{\zeta_x(s + 1)}{\zeta_x(s)}.$$

1.1.2. Weil-étale cohomology and special values. We apply the Weil-étale formalism of Flach–B. Morin [15] to give a special value formula at $s = 0$ of this new L-function. The idea of Weil-étale cohomology originates in [27], where Lichtenbaum constructs a Weil-étale topos for varieties over a finite field and links it to the special value at $s = 0$ of zeta functions. Further work was done by Geisser over a finite field [17]. For schemes over $\mathrm{Spec}(\mathbb{Z})$, attempts at the definition of a Weil-étale topos were made by Lichtenbaum [28] and Flach–B. Morin [14] but its cohomology does not behave well in high degree. Another approach was instigated by B. Morin in [33] and refined by Flach–B. Morin in [15]: instead of constructing the Weil-étale topos, one only constructs Weil-étale cohomology complexes in the derived category of abelian groups which fit into a certain distinguished triangle. This distinguished triangle comes heuristically from the pushforward from the Weil-étale topos to the étale topos[1].

Other relevant works on Weil-étale cohomology include Chiu's thesis [10], Beshenov's thesis [2, 3], Tran's article [44] and the author's article [31]. Work related to the study of the Weil-étale cohomology of F^D are Geisser–Suzuki's article [20] and Tran's thesis [43]. As far as the author knows, Tran was the first to observe that for an integral representation M of the Galois group of a number field K, the special value of the Artin L-function of $M \otimes \mathbb{Q}$ at $s = 1$ should be related to Weil-étale cohomology of the dual of the pushforward of M to $\mathrm{Spec}(\mathcal{O}_K)$.

Following the formalism of Flach–B. Morin, we should construct for each F a "multiplicative" complex[2], the Weil-étale complex (with compact support) which we think of as "Weil-étale homology" with coefficients in F, and an "additive" complex[3], an analogue of Milne's correcting factor in special value formulas for zeta functions of varieties over finite fields. The right object to consider is then the fundamental line $\Delta_X(F^D)$, a free abelian group of rank 1 which is defined as the product of the determinants of the additive and multiplicative complexes. In the general situation of an arithmetic scheme, contrary to the case over a finite field, the additive and multiplicative complexes are linked to each other through phenomena happening on complex points and so cannot be studied independently to get a special value formula; instead one of the fundamental insights of [15] is that one has to study them together through the fundamental line[4]. The fundamental line should have a canonical trivialization

$$\Delta_X(F^D) \otimes \mathbb{R} \xrightarrow{\widetilde{=}} \mathbb{R}$$

after base change to $\mathbb{R}$, which enables one to construct a multiplicative Euler characteristic as the covolume of $\Delta_X(F^D)$ inside $\Delta_X(F^D) \otimes \mathbb{R}$, and this Euler characteristic should give the special value up to sign.

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[1]The pushforward had been computed by Geisser [17] in the case over a finite field and by B. Morin in the case of the spectrum of a ring of integers in a number field [32, §8, §9].

[2]Here "multiplicative" suggests that the complex is linked to motivic cohomology: the latter involves the units, the Picard group, etc.

[3]Here "additive" suggests that the complex is linked to coherent phenomena/de Rham cohomology.

[4]This idea has its origin in the formulation of Fontaine–Perrin-Riou of the Bloch–Kato conjecture on special values of L-functions.

1.1.3. The Weil-étale complex. We introduce compactly supported cohomology and Tate compactly supported cohomology complexes[5]

$$R\Gamma_{c,B}(X, F^D), \quad R\widehat{\Gamma}_{c,B}(X, F^D).$$

The two bear to each other the same kind of relationship as ordinary and Tate cohomology: the fiber of the canonical morphism $R\Gamma_{c,B}(X, F^D) \to R\widehat{\Gamma}_{c,B}(X, F^D)$ computes some homology at the archimedean places. Moreover, we prove an Artin–Verdier duality statement for $R\widehat{\Gamma}_{c,B}(X, F^D)$, which suggests that compactly supported cohomology of F^D should be thought of as "étale homology" of F:

Theorem A (Artin–Verdier duality for F^D, see Section 2.2). *There is a natural pairing*

$$R\Gamma(X, F) \otimes^L R\widehat{\Gamma}_{c,B}(X, F^D) \to \mathbb{Q}/\mathbb{Z}[-2].$$

It induces a map

$$R\widehat{\Gamma}_{c,B}(X, F^D) \to R\operatorname{Hom}\big(R\Gamma(X, F), \mathbb{Q}/\mathbb{Z}[-2]\big).$$

If F is $\mathbb{Z}$-constructible, the above map is an isomorphism in degree $\neq -1, 0$, and an isomorphism after profinite completion of the left-hand side in degree $-1, 0$.

The proof proceeds by twisting the usual Artin–Verdier theorem [30, Thm. II.3.1] by a duality at archimedean places. With the Artin–Verdier-like duality theorem in our hands, we can construct a Weil-étale complex using the methodology of [33]: the cohomology groups with compact support of F^D have a finite type part and a torsion of cofinite type part (i.e., the $\mathbb{Q}/\mathbb{Z}$-dual of a finite type abelian group) and Artin–Verdier duality says that the torsion cofinite type part is the $\mathbb{Q}/\mathbb{Z}$-dual of some étale cohomology of F. Taking inspiration from Geisser's and B. Morin's computation of the derived pushforward from the Weil-étale topos to the étale topos [17, 32], there should exist a fundamental distinguished triangle saying that Weil-étale cohomology with compact support of F^D is obtained by replacing the torsion cofinite type part of cohomology with compact support of F^D by a finite type part, using the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. This suggest the existence of a map $\operatorname{Hom}(H^{2-i}(X, F), \mathbb{Q}) \to H^i_{c,B}(X, F^D)$ making the diagram

$$
\begin{array}{ccc}
\operatorname{Hom}\big(H^{2-i}(X, F), \mathbb{Q}\big) & \dashrightarrow & H^i_{c,B}(X, F^D) \\
\downarrow & & \downarrow \\
\operatorname{Hom}\big(H^{2-i}(X, F), \mathbb{Q}/\mathbb{Z}\big) & \xleftarrow{\text{Artin–Verdier}} & \widehat{H}^i_{c,B}(X, F^D)
\end{array}
$$

commute; taking kernels and cokernels will then give something of finite type. We are thus led to consider the existence of a map $R\operatorname{Hom}(R\Gamma(X, F), \mathbb{Q}[-2]) \overset{?}{\to} R\Gamma_{c,B}(X, F^D)$, the

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[5]The subscript B refers to the way we correct the cohomology at infinity, which involves the Tate twist $\mathbb{Z}(1) = 2i\pi\mathbb{Z}$.

cone of which we want to name Weil-étale cohomology with compact support of F^D. We achieve the construction of such a map with good functoriality properties in $D(\mathbb{Z})$ for a large class of $\mathbb{Z}$-constructible sheaves, which we name red sheaves and blue sheaves.

Definition. Let F be a $\mathbb{Z}$-constructible sheaf on X. We say that

- F is red if $H^0_{c,B}(X, F^D)$ is torsion, hence finite.
- F is blue if $H^1(X, F)$ is torsion, hence finite.
- A red-to-blue morphism is a morphism of sheaves $F \to G$ where either F and G are both blue, or are both red, or F is red and G is blue ; a red-to-blue short exact sequence is a short exact sequence with red-to-blue morphisms.

Examples of red sheaves are extensions by zero of locally constant $\mathbb{Z}$-constructible sheaves on a regular open subscheme. Examples of blue sheaves are $\mathbb{Z}$-constructible sheaves supported on a finite closed subscheme. The important point is that there are "enough red and blue sheaves", meaning that any $\mathbb{Z}$-constructible sheaf can be put in a short exact sequence where the first term is red and the last is blue[6].

Theorem B (Existence of the Weil-étale complex, see Section 3). *For every red or blue sheaf F, there exists a Weil-étale complex with compact support $R\Gamma_{W,c}(X, F^D) \in D(\mathbb{Z})$, well defined up to unique isomorphism. It sits in a distinguished triangle*

$$R \operatorname{Hom}\left(R\Gamma(X, F), \mathbb{Q}[-2]\right) \to R\Gamma_{c,B}(X, F^D) \to R\Gamma_{W,c}(X, F^D) \to .$$

It is a perfect complex, functorial in red-to-blue morphisms, and it yields a long exact cohomology sequence for red-to-blue short exact sequences. If $Y = \operatorname{Spec}(\mathcal{O}')$ is the spectrum of an order in a number field with a finite dominant morphism $\pi : Y \to X$, we have a canonical isomorphism

$$R\Gamma_{W,c}\left(X, (\pi_* F)^D\right) \simeq R\Gamma_{W,c}(Y, F^D).$$

The Weil-étale cohomology with compact support of F^D is constructed from the complex $R\Gamma_{c,B}(X, F^D)$, so it can be thought of as some "Weil-étale homology" of F. The idea of such a homology theory is not new: Geisser had defined "arithmetic homology" for curves over finite fields [19], and there is a tentative construction of a Weil-étale complex of F^D over the spectrum of a ring of integers in a totally imaginary number field in Tran's thesis [43]. Finally, let K be a function field associated to a smooth proper curve C over a finite field. Geisser–Suzuki showed in [21] that for a given torus over K, its connected Néron model $\mathcal{T}^\circ$ over C is of the form F^D for F a complex defined in terms of the character group of T, and they linked the special value of the L-function of the torus at $s = 1$ to Weil-étale cohomology of $\mathcal{T}^\circ$; this suggested that our approach might be fruitful (see Section 1.2.2 for a precise comparison).

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[6]Namely, take the short exact localization sequence associated to a regular dense open subscheme on which the sheaf is locally constant.

Having an ∞-categorical construction of the map

$$R\operatorname{Hom}\left(R\Gamma(X, F), \mathbb{Q}[-2]\right) \to R\Gamma_{c,B}(X, F^D)$$

would have been better, but this construction has eluded us. Instead, we use a kind of miraculous vanishing of Ext^1 groups, for red or blue sheaves, which enables to define the map by only specifying the maps it induces in cohomology. This vanishing is not true for arbitrary $\mathbb{Z}$-constructible sheaves, which explains our restriction. It will turn out that to define the Weil-étale Euler characteristic this is not a problem.

1.1.4. The additive complex. We now have the multiplicative part of the fundamental line, so we turn to the additive part, which we dub the tangent space of F^D:

Definition (The tangent space of F^D). Let F be a $\mathbb{Z}$-constructible sheaf on X and denote $g : \operatorname{Spec}(K) \to X$ the inclusion of the generic point. We identify $g^* F$ with a discrete G_K-module M. Denote also $\mathcal{O}_K$ the ring of integers in K.

- We say that F is tamely ramified if for each $x \in X_0$, the wild ramification group at x acts trivially on M. Let K^t be the maximal tamely ramified extension of K, and $G_K^t := \operatorname{Gal}(K^t/K)$; if F is tamely ramified, M carries a natural action of G_K^t.

- Suppose that F is tamely ramified. The tangent space of F^D is the complex

$$\operatorname{Lie}_X(F^D) := R\operatorname{Hom}_{G_K^t}\left(M, \mathcal{O}_{K^t}[1]\right).$$

- Suppose that F is tamely ramified and red or blue. The fundamental line is

$$\Delta_X(F^D) := \det_{\mathbb{Z}} R\Gamma_{W,c}(X, F^D) \otimes \det_{\mathbb{Z}} \operatorname{Lie}_X(F^D)^{-1}.$$

Our definition of the tangent space is made so that in particular, when $X = \operatorname{Spec}(\mathcal{O}_K)$ is regular, the tangent space of $\mathbb{Z}^D = \mathbb{Z}_X^c = \mathbb{G}_m[1]$ is $R\Gamma(X, \mathbb{G}_a[1]) = \mathcal{O}_K[1]$, and the tangent space of the dual of a sheaf supported on a finite closed subscheme is 0. This is as expected from the L-function we introduce: indeed for $\mathbb{Z}^D$ it gives the zeta function at $s + 1$ so its special value at 0 is the special value of the Dedekind zeta function at 1 and should involve the contribution from $\mathcal{O}_K$ (the discriminant); this was shown already (in the Weil-étale formalism) by Flach–B. Morin in the case of $\mathbb{Z}(1) = (\mathbb{Z}^D)[-2]$ for X regular [15]. On the other hand, for the dual of a sheaf supported on a finite closed subscheme we find an L-function as in [31, §6.4], for which there is no "additive" contribution for the special value at 0. The restriction to tamely ramified sheaves is justified by a theorem of Noether [38], which implies that $\mathcal{O}_{K^t}$ is cohomologically trivial. We are still investigating how to remove the tamely unramified hypothesis. The definition was inspired by [21], which uses the Lie algebra of the Néron model of a torus T over K to form the additive part of the fundamental line giving the special value at $s = 1$ of the L-function of T (see Section 1.2.2). Our definition is a generalization of this construction in the tamely ramified case that includes finite groups of multiplicative type, as shows the following:

Proposition (see Corollary 4.7). *Suppose that F is tamely ramified and that g^*F is torsion-free. Denote T the torus over K with character group g^*F and $\mathcal{T}$ its Néron model over* $\mathrm{Spec}(\mathcal{O}_K)$. *Then there is a canonical isomorphism*

$$\mathrm{Lie}_X(F^D) \simeq \mathrm{Lie}(\mathcal{T})[1].$$

1.1.5. Trivialization of the fundamental line. Now that we have our fundamental line, we should seek a canonical trivialization to obtain the Euler characteristic. We propose a contravariant generalization of the complex $R\Gamma_c(X, \mathbb{R}(1))$ of [15][7]:

Definition. The map $\log|-| : \mathbb{C}^\times \to \mathbb{R}$ induces a natural map

$$\mathrm{Log} : R\Gamma(X, F^D)_\mathbb{R} \to R\,\mathrm{Hom}_{G_\mathbb{R}, X(\mathbb{C})} \big(\alpha^* F, \mathbb{R}[1] \big).$$

For F a $\mathbb{Z}$-constructible sheaf, we define the Deligne compactly supported complex with coefficients in F^D by

$$R\Gamma_{c,\mathscr{D}}(X, F_\mathbb{R}^D) := \mathrm{fib} \big(R\Gamma(X, F^D)_\mathbb{R} \xrightarrow{\mathrm{Log}} R\,\mathrm{Hom}_{G_\mathbb{R}, X(\mathbb{C})} \big(\alpha^* F, \mathbb{R}[1] \big) \big).$$

Then, again following [15], we introduce Weil–Arakelov complexes:

Definition. Let F be a $\mathbb{Z}$-constructible sheaf on X. We define Weil–Arakelov complex of F^D as the complex:

$$R\Gamma_{\mathrm{ar},c}(X, F_\mathbb{R}^D) := R\Gamma_{c,\mathscr{D}}(X, F_\mathbb{R}^D)[-1] \oplus R\Gamma_{c,\mathscr{D}}(X, F_\mathbb{R}^D).$$

The determinant of the Weil–Arakelov complex has a canonical trivialization; we will obtain the trivialization of the fundamental line by relating the fundamental line with the Weil–Arakelov complex through a duality theorem and the rational splitting of Weil-étale cohomology:

Theorem C (Duality theorem for $\mathbb{R}$-coefficients, see Section 5). *There is a natural pairing*

$$\big(R\Gamma(X, F) \otimes \mathbb{R} \big) \otimes_\mathbb{R}^L R\Gamma_{c,\mathscr{D}}(X, F_\mathbb{R}^D) \to \mathbb{R}[0].$$

It induces a map

$$R\Gamma_{c,\mathscr{D}}(X, F_\mathbb{R}^D) \to R\,\mathrm{Hom}\big(R\Gamma(X, F), \mathbb{R} \big)$$

which is an isomorphism for $F \in D^+(X_{\mathrm{et}})$. If moreover F is a bounded complex with $\mathbb{Z}$-constructible cohomology groups, both sides are perfect complexes of $\mathbb{R}$-vector spaces.

Proposition (Rational splitting of Weil-étale cohomology, see Proposition 3.12). *Let F be a red or blue sheaf. The defining distinguished triangle of Weil-étale cohomology splits*

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[7]The complex of Flach–B. Morin is the mapping fiber of the Beilinson regulator between motivic cohomology (tensored with $\mathbb{R}$) and (real) Deligne cohomology on the complex points, in weight 1; see [15, §2.1].

rationally to give an isomorphism

$$R\Gamma_{W,c}(X, F^D) \otimes \mathbb{Q} \xrightarrow{\simeq} R\operatorname{Hom}\left(R\Gamma(X, F), \mathbb{Q}[-1]\right) \oplus R\Gamma_{c,B}(X, F^D) \otimes \mathbb{Q}$$

natural in red-to-blue morphisms and red-to-blue short exact sequences, and compatible with finite dominant morphisms between spectra of orders in number fields.

The duality theorem and the rational splitting imply that there is a distinguished triangle

$$R\Gamma_{\mathrm{ar},c}(X, F_{\mathbb{R}}^D) \to R\Gamma_{W,c}(X, F^D) \otimes \mathbb{R} \to \operatorname{Lie}_X(F^D) \otimes \mathbb{R}$$

which gives the natural trivialization

$$
\begin{aligned}
\lambda : \Delta_X(F^D)_{\mathbb{R}} &= \det_{\mathbb{R}}\left(R\Gamma_{W,c}(X, F^D) \otimes \mathbb{R}\right) \otimes \det_{\mathbb{R}}\left(\operatorname{Lie}_X(F^D) \otimes \mathbb{R}\right)^{-1} \\
&\xrightarrow{\simeq} \det_{\mathbb{R}}\left(R\Gamma_{\mathrm{ar},c}(X, F_{\mathbb{R}}^D)\right) \\
&\xrightarrow{\simeq} \mathbb{R}.
\end{aligned}
$$

1.1.6. The Euler characteristic and the special value theorem.

Definition. • Let F be a tamely ramified red or blue sheaf. The Weil-étale Euler characteristic of F^D is the positive real number $\chi_X(F^D)$ such that

$$\lambda\left(\Delta_X(F^D)\right) = \chi_X(F^D)^{-1} \cdot \mathbb{Z} \hookrightarrow \mathbb{R}.$$

• Let F be a tamely ramified $\mathbb{Z}$-constructible sheaf. There exists a short exact sequence $0 \to F' \to F \to F'' \to 0$ with F' red and tamely ramified and F'' blue and tamely ramified; define

$$\chi_X(F^D) = \chi_X\left((F')^D\right)\chi_X\left((F'')^D\right).$$

It does not depend on the chosen sequence.

The constructed Euler characteristic is multiplicative thanks to the functoriality properties of our constructions. We have an explicit computation:

Proposition (see Proposition 7.9). *Let F be a tamely ramified $\mathbb{Z}$-constructible sheaf and suppose that X is regular. We have*

$$\chi_X(F^D) = \frac{(2\pi)^{r_2(F)} 2^{r_1(F)} \left[H^0(X, F)_{\mathrm{tor}}\right]\left[\operatorname{Ext}_X^1(F, \mathbb{G}_m)_{\mathrm{tor}}\right] R(F^D)}{\left[H^1(X, F)_{\mathrm{tor}}\right]\left[\operatorname{Hom}_X(F, \mathbb{G}_m)_{\mathrm{tor}}\right]\left[\operatorname{Ext}_{G_K^t}^1(F_\eta, \mathcal{O}_{K^t})\right][N_2]\operatorname{Disc}(F)}$$

where $r_1(F)$ and $r_2(F)$ are some positive integers, N_2 is a certain finite 2-torsion abelian group, $R(F^D)$ is a regulator-type real number and $\operatorname{Disc}(F)$ is a square-root-of-discriminant-type real number.

Now that we have an Euler characteristic at our disposition, the general method following work of Tran [44] (also used in [20, 31]) is to reduce the special value formula *via* Artin induction to computations in specific cases. We obtain:

Theorem D (Special value formula, see Section 7.2). *Let F be a $\mathbb{Z}$-constructible sheaf. We have the vanishing order formula*

$$\mathrm{ord}_{s=0} L_X(F^D, s) = \sum (-1)^i i \cdot \dim_{\mathbb{R}} H^i_{\mathrm{ar},c}(X, F^D_{\mathbb{R}}).$$

If F is tamely ramified and red or blue, we have the special value formula

$$\lambda^{-1}\big(L_X^*(F^D, 0)^{-1} \cdot \mathbb{Z}\big) = \Delta_X(F^D).$$

In general, if F is a tamely ramified $\mathbb{Z}$-constructible sheaf, we have the special value formula

$$L_X^*(F^D, 0) = \pm \chi_X(F^D).$$

For $F = \mathbb{Z}$ we find the analytic class formula for the Dedekind zeta function, so this can be seen as a wide generalization of the analytic class number formula. In the singular case, it does not give a special value formula at $s = 1$ of the zeta function: indeed if Z is the singular locus and $\pi : Y \to X$ is the normalization,

$$L_X(\mathbb{Z}^D, s) = \zeta_{X \setminus Z}(s + 1) \times \prod_{z \in Z} \prod_{\pi(y)=z} \frac{\zeta_y(s+1)}{\zeta_y(s)}.$$

Using a remark made by Chen in their thesis [9], we observe that the case of a constructible sheaf (which states that $\chi_X(F^D) = 1$) follows formally using [41, Cor. 1] from the case of a sheaf supported on a closed subscheme and the multiplicativity properties of the Euler characteristic (see Theorem 7.12). Applying this remark to the construction in [31], this gives a quick proof of Tate's formula for Euler characteristics in global fields (see [30, Thm. I.5.1, Thm. II.2.13] and [31, Prop. 6.23] for the reduction to Tate's formula).

1.2. Comparison with other works

1.2.1. [43]. Our work represents a significant improvement on Tran's thesis. The construction of Tran, which takes place over $X = \mathrm{Spec}(\mathcal{O}_K)$ in the totally imaginary case, involves a complex D_{F^D} for so-called "strongly $\mathbb{Z}$-constructible sheaves" with a naturally attached real number $\chi_{\mathrm{Tran}}(F^D)$. In the previous chapters of his thesis, Tran had constructed a complex D_F with a naturally attached real number[8] $\chi_{\mathrm{Tran}}(F)$ such that, for M a finite type torsion-free discrete G_K-module corresponding to an étale sheaf Y on $\mathrm{Spec}(K)$ and $g : \mathrm{Spec}(K) \to X$ the canonical morphism, $L_K^*(M, 0) = \pm \chi_{\mathrm{Tran}}(g_* Y)$. Let $d = \mathrm{rank}_{\mathbb{Z}} Y$; Tran computes

$$\chi_{\mathrm{Tran}}\big((g_* Y)^D\big) = \frac{\chi_{\mathrm{Tran}}(F)(2\pi)^{nd/2}}{\sqrt{|\Delta_K|}^d}$$

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[8]We do not say Euler characteristic because the multiplicativity is not proven.

and essentially deduces from the functional equation the formula

$$L_K^*(M, 1) = \pm \frac{\chi_{\mathrm{Tran}}\big((g_*Y)^D\big)}{N_{K/\mathbb{Q}}\big(\mathfrak{f}(M)\big)}$$

with $\mathfrak{f}(M)$ the Artin conductor. The quantity $N_{K/\mathbb{Q}}(\mathfrak{f}(M))$ is shown to be related to two natural integral structures on the Lie algebra $\mathrm{Lie}(D(M))$ of the torus $D(M)$ with character group M.[9]

Tran's thesis only works out the case of a totally imaginary number field, which avoids the handling of factors of 2 linked to real places. Our construction takes places over an order in an arbitrary number field; in particular we have to be careful with the 2-torsion, which is achieved through our compactly supported cohomology, and with singularities, which are handled by using the dualizing complex $\mathbb{Z}^c$ instead of $\mathbb{G}_m[1]$. Though we restrict ourselves to tamely ramified $\mathbb{Z}$-constructible coefficients, we obtain a multiplicative Euler characteristic, given *on the nose* by the fundamental line, which describes the special value at $s = 0$ of the L-function $L_X(F^D, s)$ (which includes as a special case Artin L-functions at $s = 1$ for tamely ramified finite type discrete G_K-modules). We also have to place some restrictions on the $\mathbb{Z}$-constructible sheaves to obtain a construction of Weil-étale complexes, but these restrictions are significantly weaker than the "strongly $\mathbb{Z}$-constructible" condition of Tran and give a well-behaved complex that has good functorial properties.

1.2.2. [21]. Our work can be seen as adapting and generalizing to the number field case a part of Geisser–Suzuki's work on special values of 1-motives over a function field K, specifically the part about tori. The analogy between our work and theirs is as follows: let K be a global field, let T be a torus over K with character group M, let X be either $\mathrm{Spec}(\mathcal{O}_K)$ or the smooth complete curve with function field K and let $g : \mathrm{Spec}(K) \to X$ be the canonical morphism. In the function field case, Geisser–Suzuki consider the connected Néron model $\mathcal{T}^\circ$ of T over X. They prove that $\mathcal{T}^\circ \simeq R\mathcal{H}om_{X}(\tau^{\leq 1} Rg_*M, \mathbb{G}_m)$ as an étale sheaf. The cup product with a generator $e \in H_W^1(X, \mathbb{Z}) \simeq \mathbb{Z}$ induces a trivialization on $\det_{\mathbb{R}}(R\Gamma_W(X, \mathcal{T}^\circ) \otimes \mathbb{R})$ while the complex $R\Gamma_{\mathrm{Zar}}(X, \mathcal{L}ie(\mathcal{T}^\circ)) \otimes \mathbb{R}$ is trivial; this induces a trivialization

$$\lambda_e : \Big(\big(\det_{\mathbb{Z}} R\Gamma_W(X, \mathcal{T}^\circ)\big)^{-1} \otimes \det_{\mathbb{Z}} R\Gamma_{\mathrm{Zar}}\big(X, \mathcal{L}ie(\mathcal{T}^\circ)\big)\Big) \otimes \mathbb{R} \xrightarrow{\sim} \mathbb{R}.$$

After some reformulation, the special value formula for the L-function of T is[10]

Theorem ([21, Thm. 4.6]).

$$\lambda_e^{-1}\big(L^*(T, 0) \cdot \mathbb{Z}\big) = \big(\det_{\mathbb{Z}} R\Gamma_W(X, \mathcal{T}^\circ)\big)^{-1} \otimes \det_{\mathbb{Z}} R\Gamma_{\mathrm{Zar}}\big(X, \mathcal{L}ie(\mathcal{T}^\circ)\big).$$

---

[9]It seems to us that there is a mistake there, as Tran claims that $\mathrm{Lie}(D(M)) = \mathrm{Hom}_{\mathbb{Z}}(M, K)$ while the correct formula should be $\mathrm{Lie}(D(M)) = \mathrm{Hom}_{G_K}(M, K^{\mathrm{sep}})$. It is not clear to us what impact this potential mistake has on his results.

[10]We use the L-function $L(T, s)$ of the 1-motive of $[0 \to T]$, which is related to the Hasse–Weil L-function $L_{\mathrm{HW}}(T, s)$ by $L(T, s) = L_{\mathrm{HW}}(T, s + 1)$. This is why our formula takes place at $s = 0$ while the formula in [21] is for $s = 1$.

In the number field case, denote $F := \tau^{\leq 1} Rg_* M$ so that

$$F^D = R\mathcal{H}om_X(\tau^{\leq 1} Rg_* M, \mathbb{G}_m)[1] = \mathcal{T}^{\circ}[1].$$

The sheaf $g_* M$ is red and moreover $R^1 g_* M$ is constructible, so the conclusions of both thm. A and thm. B hold without modification for F and we have a Weil-étale complex with compact support $R\Gamma_{W,c}(X, \mathcal{T}^{\circ}) := R\Gamma_{W,c}(X, F^D)[-1]$. We have $L_X(F^D, s) = L(T, s)^{-1}$. Finally when M is tamely ramified, since $R^1 g_* M$ is supported on a finite closed subscheme we have

$$\text{Lie}_X(F^D) = \text{Lie}_X\left((g_* M)^D\right) = \text{Lie}(\mathcal{T})[1] = \text{Lie}(\mathcal{T}^{\circ})[1] = R\Gamma_{\text{Zar}}\left(X, \mathcal{L}ie(\mathcal{T}^{\circ})\right)[1]$$

so our formula is the direct analogue of the previous one:

Theorem.

$$\lambda^{-1}\left(L^*(T,0) \cdot \mathbb{Z}\right) = \left(\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathcal{T}^{\circ})\right)^{-1} \otimes \det_{\mathbb{Z}} R\Gamma_{\text{Zar}}\left(X, \mathcal{L}ie(\mathcal{T}^{\circ})\right).$$

Since we consider all coefficients F^D for F a $\mathbb{Z}$-constructible sheaf, we gain some flexibility which allows us to consider the case where X is singular. The main technical difficulty in our situation seems to be the definition of Weil-étale cohomology, while in the function field case Weil-étale cohomology is a well-established construction by work of Lichtenbaum and Geisser [17, 27].

1.2.3. [23]. Jordan–Poonen proved an analytic class number formula for the zeta function of a 1-dimensional affine reduced arithmetic scheme. Our work is related but different in nature, as the functions we consider are not the same: for $j : U \to X$ an open immersion, we have $L_X((j_! \mathbb{Z})^D, s) = \zeta_U(s + 1)$ only when $U = X$ and X is regular. Let $X = \text{Spec}(\mathcal{O})$ be the spectrum of an order in a number field K, let $j : U \to X$ be an open subscheme and let S_f be the finite places missing in the normalization of U. Denote $A := \mathcal{O}_U(U)$. Their special value formula for U is

$$\zeta_U^*(1) = \frac{2^{r_1}(2\pi)^{r_2} h(A) R(A) \prod_{x \in S_f}\left((1 - N(x)^{-1})/\log N(x)\right)}{\omega(A)\sqrt{|\Delta_A|}}$$

with r_1, r_2 the number of real and complex places of K, $h(A) := [\text{Pic}(U)]$, $R(A)$ is the covolume of $A^\times \subset \mathcal{O}_{K,S}^\times$ under the usual logarithmic embedding, $\omega(A) = [(A^\times)_{\text{tor}}]$ and Δ_A is the discriminant of A, that is $\det(\text{Tr}(e_i e_j))$ for a $\mathbb{Z}$-basis e_i of A.

Meanwhile our formula is computed explicitly as

$$L_X^*\left((j_! \mathbb{Z})^D, 0\right) = \frac{2^{r_1}(2\pi)^{r_2} h_U R_U}{\omega\sqrt{|\Delta_K|}}$$

with $h_U = [\text{CH}_0(U)]$, R_U the regulator from [31] and ω the number of roots of unity in K.

1.3. Notations

1.3.1. Schemes. The étale site of a scheme Y will be denoted Y_{et} and the pro-étale site Y_{proet} [4]. There is a morphism of topoi $\nu : \mathrm{Sh}(Y_{\mathrm{proet}}) \to \mathrm{Sh}(Y_{\mathrm{et}})$ such that ν^* is fully faithful. A sheaf of abelian groups on Y_{et} will be called an étale sheaf for short.

An arithmetic scheme is a scheme separated of finite type over $\mathrm{Spec}(\mathbb{Z})$; an arithmetic curve is a dimension 1 arithmetic scheme. In this paper, we will consider proper integral arithmetic curves that are flat over $\mathrm{Spec}(\mathbb{Z})$. Such a scheme is the spectrum of an order $\mathcal{O}$ in a number field K. For the rest of the paper, we fix $X = \mathrm{Spec}(\mathcal{O})$. The generic point will be denoted $g : \eta = \mathrm{Spec}(K) \to X$. We write $G_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ for the Galois group of K. If v is a finite place (resp. an archimedean place) of K, we will denote K_v the henselian local field (resp. complete local field) at v and $g_v : \eta_v = \mathrm{Spec}(K_v) \to X$ the canonical morphism. If x is a closed point of X, we will denote $i_x : x \to X$ the inclusion (or i when the context is clear), $G_x = \mathrm{Gal}(\kappa(x)^{\mathrm{sep}}/\kappa(x))$ the Galois group of the residue field at x and $N(x) = [\kappa(x)]$ the cardinality of the residue field at x. When the context allows, we will abuse notation and write v for a regular closed point of X corresponding to a finite place v of K. If U is an open subscheme of X, we will denote $j : U \to X$ the corresponding morphism. We denote by X_0 the set of closed points of X.

For F an étale sheaf on X, we denote $F_\eta := g^* F$ and $F_x = i_x^* F$ for x a closed point, and we identify those with a discrete G_K-module resp. discrete G_x-module. We will make no distinction between an étale sheaf on the spectrum of a field and the associated Galois module.

1.3.2. Dualizing complex. Denote $z_0(X, i)$ the free abelian group generated by closed integral subschemes of $X \times \Delta^i$, of relative dimension i, which intersect all faces properly. The dualizing complex $\mathbb{Z}_X^{\mathrm{c}} := \mathbb{Z}_X^{\mathrm{c}}(0)$ is the complex of étale sheaves with $z_0(-, -i)$ in degree i and differentials the sum of face maps [18]. Denote

$$\mathbb{G}_X := \left[g_* \mathbb{G}_m \to \bigoplus_{x \in X_0} i_{x,*} \mathbb{Z} \right].$$

Deninger's dualizing complex [12]. In our case we have $\mathbb{Z}_X^{\mathrm{c}} \simeq \mathbb{G}_X[1]$ [34].

For F an étale sheaf on X, we denote $F^D := R\mathcal{H}om_X(F, \mathbb{Z}_X^{\mathrm{c}})$ the derived internal hom in the derived category of étale shaves on X.

1.3.3. Group cohomology. If G is a finite group, we denote $R\Gamma(G, -)$ the derived functor of G-invariants. Let $P^\bullet$ be the standard complete resolution of $\mathbb{Z}$. We denote furthermore $R\widehat{\Gamma}(G, -) := R\mathrm{Hom}_G(P^\bullet, -)$. This latter functor computes Tate cohomology.

Let G be a profinite group and $H \subset G$ is an open subgroup. If M is a discrete H-module, then the induction of M to G is the G-module $\mathrm{ind}_H^G M := \mathrm{Cont}_H(G, M)$ of H-equivariant continuous maps $G \to M$. Let L/K be a finite extension of fields and F an étale sheaf on $\mathrm{Spec}(L)$ corresponding to a discrete G_L-module M. If π denotes the map $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$ then $\pi_* F$ corresponds to the discrete G_K-module $\mathrm{ind}_{G_L}^{G_K} M$.

If M is an abelian group, we will denote $M^\vee := R\operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ its (derived) linear dual, $M^* := \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ its Pontryagin dual, and $M^\dagger := \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q})$[11]. If V is a vector space over a field E, we will denote $V^\vee$ its linear dual.

1.3.4. Complex points. We endow the complex points $X(\mathbb{C})$ with the analytic topology; in our case the complex points are the embeddings $K \to \mathbb{C}$ and the topology is discrete. We denote $\operatorname{Sh}(G_{\mathbb{R}}, X(\mathbb{C}))$ the topos of $G_{\mathbb{R}}$-equivariant sheaves on $X(\mathbb{C})$. There is a morphism of topoi $\alpha : \operatorname{Sh}(G_{\mathbb{R}}, X(\mathbb{C})) \to \operatorname{Sh}(X_{\text{et}})$. We denote $\mathbb{Z}(1) := 2i\pi\mathbb{Z} \in \operatorname{Sh}(G_{\mathbb{R}}, X(\mathbb{C}))$, and for M a $G_{\mathbb{R}}$-equivariant sheaf of abelian groups on $X(\mathbb{C})$, $M(1) := M \otimes \mathbb{Z}(1)$ with diagonal action and

$$M^\vee := R\mathcal{H}om_{G_{\mathbb{R}}, X(\mathbb{C})}(M, \mathbb{Z})$$
$$M^\vee(1) := R\mathcal{H}om_{G_{\mathbb{R}}, X(\mathbb{C})}(M, \mathbb{Z}(1)).$$

The $G_{\mathbb{R}}$-equivariant cohomology of a complex $C \in D(\operatorname{Sh}(G_{\mathbb{R}}, X(\mathbb{C})))$ is defined as

$$R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), C) := R\Gamma(G_{\mathbb{R}}, R\Gamma(X(\mathbb{C}), C)).$$

We also denote

$$R\widehat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), C) := R\widehat{\Gamma}(G_{\mathbb{R}}, R\Gamma(X(\mathbb{C}), C))$$

the Tate $G_{\mathbb{R}}$-equivariant cohomology. The norm map N induces a fiber sequence

$$\mathbb{Z} \otimes^L_{\mathbb{Z}[G]} R\Gamma(X(\mathbb{C}), C) \xrightarrow{N} R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), C) \to R\widehat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), C).$$

Remark. Since $X(\mathbb{C})$ is discrete, a $G_{\mathbb{R}}$-equivariant sheaf on $X(\mathbb{C})$ is the data of

- For each embedding $\sigma : K \to \mathbb{C}$, an abelian group F_σ
- An isomorphism $F_\sigma \to F_{\bar\sigma}$ for σ complex and an action of $G_{\mathbb{R}}$ on F_σ for σ real.

Choose an embedding σ_v for each archimedean place and put $F_v = F_{\sigma_v}$. We then have

$$R\Gamma(X(\mathbb{C}), F) = \prod_{v \text{ real}} F_v \times \prod_{v \text{ complex}} \operatorname{ind}^{G_{\mathbb{R}}} F_v$$

as a $G_{\mathbb{R}}$-module, and thus

$$R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), F) = \prod_{v \text{ archimedean}} R\Gamma(G_{K_v}, F_v).$$

This breaks down if $X(\mathbb{C})$ is not discrete.

If F is an étale sheaf on X, we will denote F_v its pullback to $\operatorname{Spec}(K_v)$; thus

$$(\alpha^* F)_v = F_v.$$

---

[11]"The dagger kills torsion."

1.3.5. Determinants. Let $A = \mathbb{Z}$ or $\mathbb{R}$. A complex C in the derived category $D(A)$ is perfect if it is bounded with finite type cohomology groups. We will use the determinant construction of Knudsen–Mumford [25], and the subsequent work of Breuning, Burns, and Knudsen, in particular [5,6]. Denote Proj_A the exact category of projective finite type A-modules, $\mathrm{Gr}^b(\mathrm{Mod}_A^{\mathrm{ft}})$ the bounded graded abelian category of finite type A-modules, $D_{\mathrm{perf}}(A)$ the derived category of perfect complexes and $\mathscr{P}_A$ the Picard groupoid of graded A-lines. The usual determinant functor

$$\det_A \in \det(\mathrm{Proj}_A, \mathscr{P}_A), \quad M \mapsto (\Lambda^{\mathrm{rank}_A M} M, \mathrm{rank}_A M)$$

extends to a determinant functor $g_A \in \det(\mathrm{Gr}^b(\mathrm{Modft}_A), \mathscr{P}_A)$. Moreover, the graded cohomology functor $H : D_{\mathrm{perf}}(A) \to \mathrm{Gr}^b(\mathrm{Modft}_A)$ induces a functor

$$H^* : \det\left(\mathrm{Gr}^b(\mathrm{Modft}_A), \mathscr{P}_A\right) \to \det\left(D_{\mathrm{perf}}(A), \mathscr{P}_A\right),$$

and we put $\det_A := H^* g_A$[12]. There is a canonical isomorphism

$$(\det_{\mathbb{Z}} C) \otimes \mathbb{R} \simeq \det_{\mathbb{R}}(C \otimes \mathbb{R}).$$

1.3.6. Derived ∞-categories. We will use the theory of stable ∞-categories, see [29]. If $\mathcal{X}$ is a topos, we will denote $\mathcal{D}(\mathcal{X})$ the derived ∞-category associated to abelian objects in $\mathcal{X}$. It is a stable ∞-category whose homotopy category is the usual derived (1-)category $D(\mathcal{X})$. In particular we will denote $\mathcal{D}(X)$ the derived ∞-category of étale sheaves on X, $\mathcal{D}(\mathbb{Z})$ the derived ∞-category of abelian groups and $\mathcal{D}(G_{\mathbb{R}}, X(\mathbb{C}))$ the derived ∞-category of $G_{\mathbb{R}}$-equivariant sheaves on $X(\mathbb{C})$.

A stable ∞-category has all finite limits and colimits. Moreover, pushouts are pullbacks and reciprocally. The homotopy category of a stable ∞-category has a canonical structure of triangulated category. If $\mathcal{C}$ is a stable ∞-category and $A \in \mathcal{C}$, we denote $A[1]$ the shift of A, which is given by the following pushout:

$$\begin{array}{ccc} A & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & A[1] \end{array}$$

If $f : A \to B$ is a morphism in $\mathcal{C}$, we define $\mathrm{fib}(f)$ and $\mathrm{cofib}(f)$ by the following pullback and pushout diagrams:

$$\begin{array}{ccc} \mathrm{fib}(f) & \longrightarrow & A \\ \downarrow & & \downarrow f \\ 0 & \longrightarrow & B \end{array} \qquad \begin{array}{ccc} A & \overset{f}{\longrightarrow} & B \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{cofib}(f) \end{array}$$

A sequence $A \to B \to C$ is called a fiber sequence if $A = \mathrm{fib}(B \to C)$ (or equivalently $C = \mathrm{cofib}(A \to B)$). A fiber sequence induces a distinguished triangle in the homotopy category.

---

[12]This convention is chosen to simplify proofs later on, but of course all determinant functors extending $\det_A \in \det(\mathrm{Proj}_A)$ are isomorphic.

2. Cohomology with compact support of F^D

2.1. Definitions

The $G_{\mathbb{R}}$-equivariant sheaf $\alpha^* \mathbb{Z}_X^c = \alpha^* \mathbb{G}_X[1]$ is $\overline{\mathbb{Q}}^\times[1]$, with action of $G_{\mathbb{R}}$ via decomposition groups at archimedean places. We have a canonical morphism $\alpha^* \mathbb{Z}_X^c \to \mathbb{C}^\times[1]$ and the short exact sequence $0 \to 2i\pi\mathbb{Z} \to \mathbb{C} \to \mathbb{C}^\times \to 0$ gives a map $\mathbb{C}^\times[1] \to 2i\pi\mathbb{Z}[2]$, whence a composite arrow $\alpha^* \mathbb{Z}_X^c \to \mathbb{Z}(1)[2]$. Let F, G be abelian sheaves on X_{et}. The functor α^* is strict monoidal, hence from the arrow

$$\alpha^* \big(R\mathcal{H}om_X(F, G) \otimes^L F \big) \to \alpha^* G$$

obtained by applying α^* to the natural map, we obtain by adjunction a canonical map

$$\alpha^* R\mathcal{H}om_X(F, G) \to R\mathcal{H}om_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^* F, \alpha^* G).$$

In particular, there is a natural transformation

$$\alpha^*(F^D) \to R\mathcal{H}om_{G_{\mathbb{R}}, X(\mathbb{C})}\big(\alpha^* F, \mathbb{C}^\times[1]\big) \to (\alpha^* F)^\vee(1)[2]$$

hence maps

$$R\Gamma(X, F^D) \to R\Gamma_{G_{\mathbb{R}}}\big(X(\mathbb{C}), (\alpha^* F)^\vee(1)[2]\big) \to R\widehat{\Gamma}_{G_{\mathbb{R}}}\big(X(\mathbb{C}), (\alpha^* F)^\vee(1)[2]\big).$$

Definition 2.1. The corrected cohomology with compact support of F^D is the fiber of the first morphism:

$$R\Gamma_{c,B}(X, F^D) := \mathrm{fib}\,\big(R\Gamma(X, F^D) \to R\Gamma_{G_{\mathbb{R}}}\big(X(\mathbb{C}), (\alpha^* F)^\vee(1)[2]\big)\big).$$

The Tate corrected cohomology with compact support of F^D is the fiber of the composite morphism:

$$R\widehat{\Gamma}_{c,B}(X, F^D) := \mathrm{fib}\,\big(R\Gamma(X, F^D) \to R\widehat{\Gamma}_{G_{\mathbb{R}}}\big(X(\mathbb{C}), (\alpha^* F)^\vee(1)[2]\big)\big).$$

We also recall the definition of Tate cohomology with compact support of F from [31, §2][13] as the fiber:

$$R\widehat{\Gamma}_c(X, F) := \mathrm{fib}\,\big(R\Gamma(X, F) \to R\widehat{\Gamma}_{G_{\mathbb{R}}}\big(X(\mathbb{C}), \alpha^* F\big)\big).$$

Remark. The above is only defined for sheaves of the specific form F^D. It is not a functor on the whole category of sheaves. We think of it as a contravariant functor in F. As we will see later, Tate corrected cohomology with compact support of F^D is in an "Artin–Verdier-like" duality with étale cohomology of F, so we can think of it as a étale homology of F.

Proposition 2.2. *Let F be an étale sheaf on X. The canonical map $\alpha^* \mathbb{Z}_X^c \to \mathbb{Z}(1)[2]$ induces an isomorphism*

$$R\widehat{\Gamma}_c(X, F^D) \xrightarrow{\simeq} R\widehat{\Gamma}_{c,B}(X, F^D).$$

---

[13]The definition is not original but the terminology is, see [30, §II.2, Cohomology with compact support].

Proof. We have to show that

$$R\widehat{\Gamma}\big(G_{\mathbb{R}}, \alpha^*(F^D)\big) \xrightarrow{\simeq} R\widehat{\Gamma}\big(G_{\mathbb{R}}, (\alpha^*F)^\vee(1)[2]\big).$$

It is enough to show it at real places. For v a real place, denote $g_v : \mathrm{Spec}(K_v) \to X$ the canonical map. As g_v is proétale, we have $g_v^*(F^D) = R\mathcal{H}om_{G_{\mathbb{R}}}(g_v^*F, \overline{\mathbb{Q}}^\times[1])$, and we thus want to show

$$R\widehat{\Gamma}\big(G_{\mathbb{R}}, R\mathcal{H}om_{G_{\mathbb{R}}}\big(g_v^*F, \overline{\mathbb{Q}}^\times[1]\big)\big) \xrightarrow{\simeq} R\widehat{\Gamma}\big(G_{\mathbb{R}}, R\mathcal{H}om_{G_{\mathbb{R}}}\big(g_v^*F, 2i\pi\mathbb{Z}[2]\big)\big).$$

All terms now depend only on $M := g_v^*F$ so we can reason by Artin induction on any finite type $G_{\mathbb{R}}$-module M.

We first treat the case $M = \mathbb{Z}$: from the exact sequence $0 \to 2i\pi\mathbb{Z} \to \mathbb{C} \to \mathbb{C}^\times \to 0$ we find an isomorphism

$$R\widehat{\Gamma}\big(G_{\mathbb{R}}, \mathbb{C}^\times[1]\big) \xrightarrow{\simeq} R\widehat{\Gamma}\big(G_{\mathbb{R}}, \mathbb{Z}(1)[2]\big).$$

Moreover, $\overline{\mathbb{Q}}^\times$ and $\mathbb{C}^\times$ have the same Tate cohomology: both are zero in odd degree and in even degree, both equal $\mathbb{Z}/2\mathbb{Z}$ *via* the sign map because a real algebraic number which is a norm is the square of a real algebraic number, hence the norm of an algebraic number.

If $M = \mathrm{ind}_{\{0\}}^{G_{\mathbb{R}}} \mathbb{Z}$ is induced, we have that $R\mathcal{H}om_{G_{\mathbb{R}}}(M, N) \simeq \mathrm{ind}_{\{0\}}^{G_{\mathbb{R}}} N$ is induced for any $G_{\mathbb{R}}$-module N, so it has trivial Tate cohomology.

Finally, if M is finite $R\mathcal{H}om_{G_{\mathbb{R}}}(M, \overline{\mathbb{Q}}^\times)$ and $R\mathcal{H}om_{G_{\mathbb{R}}}(M, \mathbb{C}^\times)$ are canonically isomorphic because $\overline{\mathbb{Q}}^\times$ and $\mathbb{C}^\times$ have the same torsion. Since $\mathbb{C}$ is uniquely divisible and has no torsion, we have moreover $R\mathcal{H}om_{G_{\mathbb{R}}}(M, \mathbb{C}^\times) \xrightarrow{\simeq} R\mathcal{H}om_{G_{\mathbb{R}}}(M, 2i\pi\mathbb{Z}[1])$. We thus obtain the required isomorphism on Tate cohomology.

By Artin induction we obtain the required isomorphism for any finite type $G_{\mathbb{R}}$-module M; by taking filtered colimits, we obtain the statement for any $G_{\mathbb{R}}$-module M since Tate cohomology, being a derived Hom, commutes with derived limits. ∎

2.2. Artin–Verdier duality for F^D

For a complex of abelian groups, we will denote $R\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}^\bullet(-, \mathbb{Q}/\mathbb{Z})$ by $(-)^*$. Artin–Verdier duality gives a map

$$R\Gamma(X, F^D) = R\mathrm{Hom}_X\big(F, \mathbb{G}_X[1]\big) \xrightarrow{AV} R\widehat{\Gamma}_c(X, F)^*[-2]$$

which is "almost" an isomorphism. We want to modify this duality at the complex points to obtain an Artin–Verdier duality relating $R\widehat{\Gamma}_{c,B}(X, F^D)$ and $R\Gamma(X, F)$. We have fiber sequences

$$R\widehat{\Gamma}_{G_{\mathbb{R}}}\big(X(\mathbb{C}), (\alpha^*F)^\vee(1)[1]\big) \to R\widehat{\Gamma}_{c,B}(X, F^D) \to R\Gamma(X, F^D),$$
$$R\widehat{\Gamma}_c(X, F) \to R\Gamma(X, F) \to R\widehat{\Gamma}_{G_{\mathbb{R}}}\big(X(\mathbb{C}), \alpha^*F\big).$$

To obtain an Artin–Verdier duality statement, we will construct in the next three paragraphs pairings

$$R\widehat{\Gamma}_{G_{\mathbb{R}}}\big(X(\mathbb{C}), (\alpha^* F)^{\vee}(1)\big) \otimes^L R\widehat{\Gamma}_{G_{\mathbb{R}}}\big(X(\mathbb{C}), \alpha^* F\big) \to \mathbb{Q}/\mathbb{Z}[-3],$$
$$R\Gamma(X, F) \otimes^L R\widehat{\Gamma}_{c,B}(X, F^D) \to \mathbb{Q}/\mathbb{Z}[-2]$$

such that we have a morphism of fiber sequences given by the adjoint maps:

$$
\begin{array}{ccccc}
R\widehat{\Gamma}_{G_{\mathbb{R}}}\big(X(\mathbb{C}), (\alpha^* F)^{\vee}(1)[1]\big) & \longrightarrow & R\widehat{\Gamma}_{c,B}(X, F^D) & \longrightarrow & R\Gamma(X, F^D) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle AV} \\
R\widehat{\Gamma}_{G_{\mathbb{R}}}\big(X(\mathbb{C}), \alpha^* F\big)^*[-2] & \longrightarrow & R\Gamma(X, F)^*[-2] & \longrightarrow & R\widehat{\Gamma}_c(X, F)^*[-2]
\end{array}
\tag{2.1}
$$

2.2.1. Construction of the pairing for Tate cohomology on $X(\mathbb{C})$.
Let us construct the first pairing: Tate cohomology $R\widehat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), -) = R\widehat{\Gamma}_{G_{\mathbb{R}}}(-) \circ R\Gamma(X(\mathbb{C}), -)$ is lax-monoidal [37, Thm. I.3.1], hence the natural evaluation map $(\alpha^* F)^{\vee}(1)[1] \otimes^L \alpha^* F \to 2i\pi\mathbb{Z}[1]$ gives a map

$$R\widehat{\Gamma}_{G_{\mathbb{R}}}\big(X(\mathbb{C}), (\alpha^* F)^{\vee}(1)\big) \otimes^L R\widehat{\Gamma}_{G_{\mathbb{R}}}\big(X(\mathbb{C}), \alpha^* F\big) \to R\widehat{\Gamma}_{G_{\mathbb{R}}}\big(X(\mathbb{C}), 2i\pi\mathbb{Z}\big).$$

Since $\widehat{H}^i_{G_{\mathbb{R}}}(X(\mathbb{C}), 2i\pi\mathbb{Z}) = 0$ for i even, there is a canonical isomorphism[14]

$$R\widehat{\Gamma}_{G_{\mathbb{R}}}\big(X(\mathbb{C}), 2i\pi\mathbb{Z}\big) \simeq \bigoplus_i \widehat{H}^i_{G_{\mathbb{R}}}\big(X(\mathbb{C}), 2i\pi\mathbb{Z}\big)[-i].$$

Let r_1 be the number of real places of the function field of K. There is an identification $\widehat{H}^3(G_{\mathbb{R}}, X(\mathbb{C}), 2i\pi\mathbb{Z}) = (\mathbb{R}^{\times}/\mathbb{R}^{\times}_{>0})^{r_1}$ comes from the natural map $\mathbb{C}^{\times} \to 2i\pi\mathbb{Z}[1]$. We compose the previous pairing with a projection and a sum map to get the pairing

$$R\widehat{\Gamma}_{G_{\mathbb{R}}}\big(X(\mathbb{C}), (\alpha^* F)^{\vee}(1)\big) \otimes^L R\widehat{\Gamma}_{G_{\mathbb{R}}}\big(X(\mathbb{C}), \alpha^* F\big) \to R\widehat{\Gamma}_{G_{\mathbb{R}}}\big(X(\mathbb{C}), 2i\pi\mathbb{Z}\big)$$

$$\xrightarrow{H^3} \widehat{H}^3_{G_{\mathbb{R}}}\big(X(\mathbb{C}), 2i\pi\mathbb{Z}\big)[-3]$$

$$= (\mathbb{R}^{\times}/\mathbb{R}^{\times}_{>0})^{r_1}[-3]$$

$$\xrightarrow{\Sigma} \mathbb{Q}/\mathbb{Z}[-3]. \tag{2.2}$$

2.2.2. Construction of the pairing for $R\widehat{\Gamma}_{c,B}(X, F^D)$.
In the following diagrams, we will denote $R\Gamma$, $R\Gamma_{G_{\mathbb{R}}}$, $R\widehat{\Gamma}_{G_{\mathbb{R}}}$, $R\widehat{\Gamma}_c$, and $R\widehat{\Gamma}_{c,B}$ for $R\Gamma(X, -)$, $R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), -)$, $R\widehat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), -)$, $R\widehat{\Gamma}_c(X, -)$, and $R\widehat{\Gamma}_{c,B}(X, -)$, respectively, and we will also write $\otimes$ for the derived tensor product. The natural lax monoidality pairings of $R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), -)$ and $R\widehat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), -)$ are compatible with each other [37, Thm. I.3.1] and also with the natural pairing for $R\Gamma(X, -)$ via the map $R\Gamma(X, -) \to R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), \alpha^*(-))$,[15] so the

---

[14]Indeed, there is always such an isomorphism (non-canonical) in the homotopy category $D(\mathbb{Z})$, and there is a canonical one here because there are no Ext^1 between two consecutive cohomology groups.

following diagram commutes:

$$
\begin{array}{ccc}
R\Gamma(F) \otimes R\Gamma(F^D) & \longrightarrow & R\Gamma(\mathbb{Z}_X^c) \\
\downarrow & & \downarrow \\
R\Gamma(F) \otimes R\Gamma_{G_{\mathbb{R}}}(F_{\mathbb{C}}^\vee(1)[2]) \to R\Gamma_{G_{\mathbb{R}}}(\alpha^* F) \otimes R\Gamma_{G_{\mathbb{R}}}(F_{\mathbb{C}}^\vee(1)[2]) \to R\Gamma_{G_{\mathbb{R}}}(2i\pi\mathbb{Z}[2]) \\
\downarrow & \downarrow & \downarrow \\
R\Gamma(F) \otimes R\widehat{\Gamma}_{G_{\mathbb{R}}}(F_{\mathbb{C}}^\vee(1)[2]) \to R\widehat{\Gamma}_{G_{\mathbb{R}}}(\alpha^* F) \otimes R\widehat{\Gamma}_{G_{\mathbb{R}}}(F_{\mathbb{C}}^\vee(1)[2]) \to R\widehat{\Gamma}_{G_{\mathbb{R}}}(2i\pi\mathbb{Z}[2])
\end{array}
$$

We thus obtain the left dotted map in the following commutative diagram where the top row is a fiber sequence:

$$
\begin{array}{ccc}
R\Gamma(X,F) \otimes^L R\widehat{\Gamma}_{c,B}(X,F^D) \to R\Gamma(X,F) \otimes^L R\Gamma(X,F^D) \to R\Gamma(X,F) \otimes^L R\widehat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), F_{\mathbb{C}}^\vee(1)[2]) \\
\downarrow \quad\quad\quad\quad\quad\quad \downarrow \quad\quad\quad\quad\quad\quad \downarrow \\
R\widehat{\Gamma}_{c,B}(X,\mathbb{Z}^D) \longrightarrow R\Gamma(X,\mathbb{Z}_X^c) \longrightarrow R\widehat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), 2i\pi\mathbb{Z}[2])
\end{array}
$$

We have

$$
\tau^{\geq 2} R\widehat{\Gamma}_{c,B}(X,\mathbb{Z}^D) = \tau^{\geq 2} R\widehat{\Gamma}_c(X,\mathbb{Z}_X^c) = \mathbb{Q}/\mathbb{Z}[-2]
$$

by Proposition 2.2 and [30, Prop. II.2.6, Lem. II.6.1], hence we get a pairing

$$
R\Gamma(X,F) \otimes^L R\widehat{\Gamma}_{c,B}(X,F^D) \to R\widehat{\Gamma}_{c,B}(X,\mathbb{Z}^D) \xrightarrow{\tau^{\geq 2}} \mathbb{Q}/\mathbb{Z}[-2]. \qquad (2.3)
$$

2.2.3. The morphism of fiber sequences. The following cube is commutative by compatibility of the involved pairings:

$$
\begin{array}{c}
R\Gamma(F) \otimes R\Gamma(F^D) \xrightarrow{\hspace{4cm}} R\Gamma(F) \otimes R\widehat{\Gamma}_{G_{\mathbb{R}}}((\alpha^* F)^\vee(1)[2]) \\
\searrow \quad\quad\quad\quad\quad\quad\quad\quad \searrow \\
R\Gamma(\mathbb{Z}_X^c) \xrightarrow{\hspace{4cm}} R\widehat{\Gamma}_{G_{\mathbb{R}}}(2i\pi\mathbb{Z}[2]) \\
\downarrow \quad\quad\quad\quad\quad \downarrow \quad\quad\quad\quad\quad \downarrow \quad\quad\quad\quad\quad \downarrow \\
R\widehat{\Gamma}_{G_{\mathbb{R}}}(\alpha^* F) \otimes R\Gamma(F^D) \xrightarrow{\hspace{2cm}} R\widehat{\Gamma}_{G_{\mathbb{R}}}(\alpha^* F) \otimes R\widehat{\Gamma}_{G_{\mathbb{R}}}((\alpha^* F)^\vee(1)[2]) \\
\searrow \quad\quad\quad\quad\quad\quad\quad\quad \searrow \\
R\widehat{\Gamma}_{G_{\mathbb{R}}}(\alpha^* \mathbb{Z}_X^c) \xrightarrow{\hspace{4cm}} R\widehat{\Gamma}_{G_{\mathbb{R}}}(2i\pi\mathbb{Z}[2])
\end{array}
$$

---

[15]Since the lax monoidal structure for $R\Gamma(X,-)$ and $R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}),-)$ come from the strict monoidal structure on their left adjoint, this is a formal consequence of the commutative triangle of right adjoints with strict monoidal left adjoints

$$
\begin{array}{ccc}
\mathcal{D}(G_{\mathbb{R}}, X(\mathbb{C})) & \xrightarrow{\alpha_*} & \mathcal{D}(X_{\mathrm{et}}) \\
\quad {}_{R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}),-)} \searrow & & \swarrow {}_{R\Gamma(X,-)} \\
& \mathcal{D}(\mathbb{Z}) &
\end{array}
$$

hence by adding the fibers of horizontal and vertical maps we deduce a morphism of 3×3 diagrams depicted in Diagram 1. Doing so, we recover in the middle layer the pairing (2.3).

Let A, A', B, B', C, C' be objects of $D(\mathrm{Ab})$, with maps $A \to A'$, $B' \to B$, $C \to C'$. Given two pairings $A \otimes^L B \to C$ and $A' \otimes^L B' \to C'$, the commutativity of the two following induced diagrams is equivalent:

$$
\begin{array}{ccc}
A & \longrightarrow & A' \\
\downarrow & & \downarrow \\
R\mathcal{H}om(B,C) & \longrightarrow & R\mathcal{H}om(B',C')
\end{array}
\qquad
\begin{array}{ccccc}
A \otimes^L B' & \longrightarrow & A \otimes^L B & \longrightarrow & C \\
& \searrow & & & \downarrow \\
& & A' \otimes^L B' & \longrightarrow & C'
\end{array}
$$

We now prove that the natural diagram (2.1) commutes. By the above, this reformulates to the commutativity of a subdiagram of (1), together with the following commutative diagram:

$$
R\widehat{\Gamma}\big(2i\pi\mathbb{Z}[1]\big) = R\widehat{\Gamma}(\mathbb{C}^\times) \longrightarrow R\widehat{\Gamma}_{c,B}(\mathbb{Z}^D) \xleftarrow{\;\simeq\;} R\widehat{\Gamma}_c(\mathbb{Z}_X^c)
$$

with maps $\Sigma \circ H^2$ and $\tau^{\geq 2}$ and $\tau^{\geq 2}$ into $\mathbb{Q}/\mathbb{Z}[-2]$.

which follows from the proof of [30, Prop. II.2.6].

Theorem 2.3 (Artin–Verdier duality for F^D). *The pairing*

$$
R\Gamma(X,F) \otimes^L R\widehat{\Gamma}_{c,B}(X,F^D) \to \mathbb{Q}/\mathbb{Z}[-2]
$$

induces a map

$$
R\widehat{\Gamma}_{c,B}(X,F^D) \to R\Gamma(X,F)^*[-2]
$$

which is an isomorphism in degree $\neq -1, 0$, and an isomorphism after profinite completion of the left-hand side in degree -1, 0. In particular if F is constructible then the map is an isomorphism.

Remark. The statement also holds more generally for a bounded complex $F \in \mathcal{D}^b(X)$ such that $H^0(F)$ is $\mathbb{Z}$-constructible and $H^i(F)$ is constructible for $i \neq 0$ (by filtering with the truncations). If we reformulate the theorem as the map $R\widehat{\Gamma}_{c,B}(X,F^D) \otimes \widehat{\mathbb{Z}} \to R\Gamma(X,F)^*[-2]$ being an isomorphism, this generalizes to bounded complexes with $\mathbb{Z}$-constructible cohomology groups.

Proof. By the diagram (2.1) and Artin–Verdier duality for singular schemes [30, Thm. II.6.2], it suffices to show that the pairing

$$
R\widehat{\Gamma}_{G_\mathbb{R}}\big(X(\mathbb{C}), (\alpha^* F)^\vee(1)[1]\big) \otimes^L R\widehat{\Gamma}_{G_\mathbb{R}}\big(X(\mathbb{C}), \alpha^* F\big) \to \mathbb{Q}/\mathbb{Z}[-2]
$$

is a perfect pairing between complexes of abelian groups with finite cohomology groups. To prove that the pairing is perfect, it suffices to do it at every real place, that is for the Galois cohomology of $G_\mathbb{R}$; thus it suffices to prove the next proposition. ∎
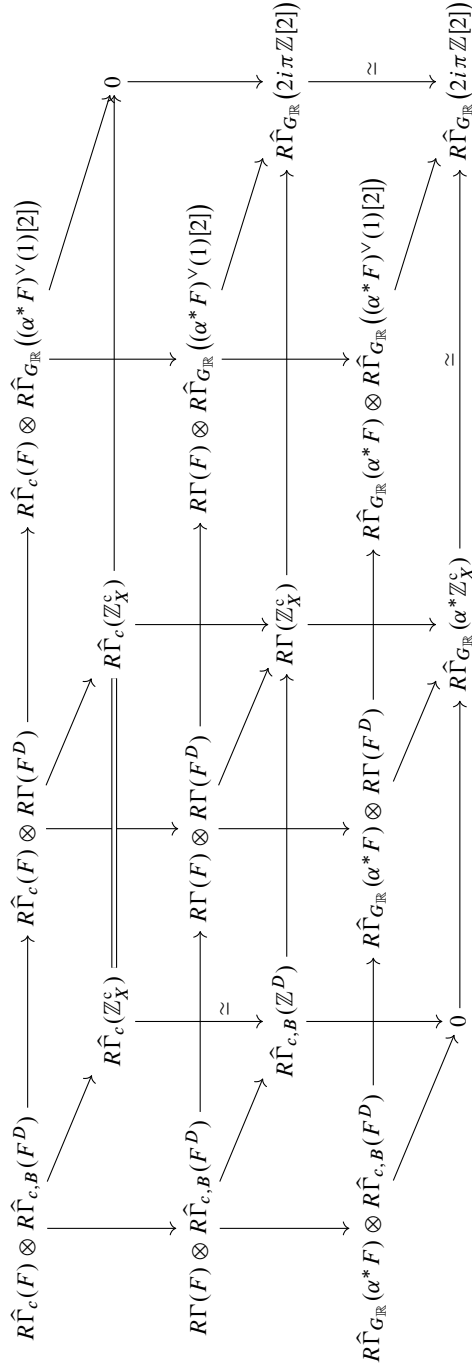
$$
\begin{array}{ccccccc}
R\widehat{\Gamma}_c(F) \otimes R\widehat{\Gamma}_{c,B}(F^D) & \longrightarrow & R\widehat{\Gamma}_c(F) & \longrightarrow & R\widehat{\Gamma}_c(F) \otimes R\widehat{\Gamma}_{G_\mathbb{R}}((\alpha^*F)^\vee(1)[2]) & \longrightarrow & 0 \\[4pt]
\nearrow R\widehat{\Gamma}_c(\mathbb{Z}_X^c) & = & R\widehat{\Gamma}_c(\mathbb{Z}_X^c) & & & & \\[4pt]
R\Gamma(F) \otimes R\widehat{\Gamma}_{c,B}(F^D) & \longrightarrow & R\Gamma(F) \otimes R\Gamma(F^D) & \longrightarrow & R\Gamma(F) \otimes R\widehat{\Gamma}_{G_\mathbb{R}}((\alpha^*F)^\vee(1)[2]) & \longrightarrow & R\widehat{\Gamma}_{G_\mathbb{R}}(2i\pi\mathbb{Z}[2]) \\[4pt]
\nearrow R\widehat{\Gamma}_{c,B}(\mathbb{Z}^D) & \overset{\simeq}{\longrightarrow} & R\Gamma(\mathbb{Z}_X^c) & & & & \downarrow \simeq \\[4pt]
R\widehat{\Gamma}_{G_\mathbb{R}}(\alpha^*F) \otimes R\widehat{\Gamma}_{c,B}(F^D) & \longrightarrow & R\widehat{\Gamma}_{G_\mathbb{R}}(\alpha^*F) \otimes R\Gamma(F^D) & \longrightarrow & R\widehat{\Gamma}_{G_\mathbb{R}}(\alpha^*F) \otimes R\widehat{\Gamma}_{G_\mathbb{R}}((\alpha^*F)^\vee(1)[2]) & \longrightarrow & R\widehat{\Gamma}_{G_\mathbb{R}}(2i\pi\mathbb{Z}[2]) \\[4pt]
\nearrow 0 & \longrightarrow & R\widehat{\Gamma}_{G_\mathbb{R}}(\alpha^*\mathbb{Z}_X^c) & & & &
\end{array}
$$

Diagram 1.

Proposition 2.4. *The pairing*

$$R\widehat{\Gamma}(G_{\mathbb{R}}, M^{\vee}(1)) \otimes^L R\widehat{\Gamma}(G_{\mathbb{R}}, M) \to \mathbb{Q}/\mathbb{Z}[-3]$$

is perfect for M any discrete $G_{\mathbb{R}}$-module of finite type.

Proof. We will reduce to Theorem A.1 for the finite group $G_{\mathbb{R}} = \mathbb{Z}/2\mathbb{Z}$. Denote ε the augmentation $\mathbb{Z}[G_{\mathbb{R}}] \to \mathbb{Z}$ and $I_{G_{\mathbb{R}}} = (s - 1)\mathbb{Z}$ the augmentation ideal. We have

$$H^1(G_{\mathbb{R}}, 2i\pi\mathbb{Z}) = \operatorname{Ext}^1_{G_{\mathbb{R}}}(\mathbb{Z}, 2i\pi\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z},$$

and the non-zero class t corresponds to a morphism $\mathbb{Z} \to 2i\pi\mathbb{Z}[1]$ in the derived category coming from the equivalence class of the non-split exact sequence

$$0 \to 2i\pi\mathbb{Z} = I_{G_{\mathbb{R}}} \to \mathbb{Z}[G_{\mathbb{R}}] \xrightarrow{\varepsilon} \mathbb{Z} \to 0.$$

Since $I_{G_{\mathbb{R}}} = 2i\pi\mathbb{Z}$ as a $G_{\mathbb{R}}$-module, we obtain an isomorphism

$$R\widehat{\Gamma}(G_{\mathbb{R}}, M) \xrightarrow[(\mathrm{id}_M \otimes t)_*]{\simeq} R\widehat{\Gamma}(G_{\mathbb{R}}, M(1))[1].$$

Note that $M^{\vee}(1) = (M^{\vee})(1)$. From the above isomorphism we deduce an isomorphism of pairings

$$
\begin{array}{ccccccc}
R\widehat{\Gamma}(G_{\mathbb{R}}, M^{\vee}) \otimes^L R\widehat{\Gamma}(G_{\mathbb{R}}, M) & \longrightarrow & R\widehat{\Gamma}(G_{\mathbb{R}}, \mathbb{Z}) & \longrightarrow & \widehat{H}^2(G_{\mathbb{R}}, \mathbb{Z})[-2] & \longrightarrow & \mathbb{Q}/\mathbb{Z}[-2] \\
\downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{\simeq} & & \| \\
R\widehat{\Gamma}(G_{\mathbb{R}}, M^{\vee}(1))[1] \otimes^L R\widehat{\Gamma}(G_{\mathbb{R}}, M) & \to & R\widehat{\Gamma}(G_{\mathbb{R}}, 2i\pi\mathbb{Z})[1] & \to & \widehat{H}^3(G_{\mathbb{R}}, 2i\pi\mathbb{Z})[-2] & \to & \mathbb{Q}/\mathbb{Z}[-2]
\end{array}
$$

which shows that it suffices to check that the natural pairing

$$R\widehat{\Gamma}(G_{\mathbb{R}}, M^{\vee}) \otimes^L R\widehat{\Gamma}(G_{\mathbb{R}}, M) \to R\widehat{\Gamma}(G_{\mathbb{R}}, \mathbb{Z}) \to \widehat{H}^2(G_{\mathbb{R}}, \mathbb{Z})[-2] \to \mathbb{Q}/\mathbb{Z}[-2]$$

is perfect. In a similar way, the non-zero class $u \in H^2(G_{\mathbb{R}}, \mathbb{Z})$ corresponds to a map $\mathbb{Z} \to \mathbb{Z}[2]$ whose fiber comes from induced modules[16], and we also get an isomorphism of pairings

$$
\begin{array}{ccccccc}
R\widehat{\Gamma}(G_{\mathbb{R}}, M^{\vee}) \otimes^L R\widehat{\Gamma}(G_{\mathbb{R}}, M) & \longrightarrow & R\widehat{\Gamma}(G_{\mathbb{R}}, \mathbb{Z}) & \longrightarrow & \widehat{H}^0(G_{\mathbb{R}}, \mathbb{Z})[0] & \longrightarrow & \mathbb{Q}/\mathbb{Z}[0] \\
\downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{\simeq} & & \| \\
R\widehat{\Gamma}(G_{\mathbb{R}}, M^{\vee}) \otimes^L R\widehat{\Gamma}(G_{\mathbb{R}}, M)[2] & \to & R\widehat{\Gamma}(G_{\mathbb{R}}, \mathbb{Z})[2] & \to & \widehat{H}^2(G_{\mathbb{R}}, \mathbb{Z})[0] & \to & \mathbb{Q}/\mathbb{Z}[0]
\end{array}
$$

so equivalently it suffices to check that the natural pairing

$$R\widehat{\Gamma}(G_{\mathbb{R}}, M^{\vee}) \otimes^L R\widehat{\Gamma}(G_{\mathbb{R}}, M) \to R\widehat{\Gamma}(G_{\mathbb{R}}, \mathbb{Z}) \to \widehat{H}^0(G_{\mathbb{R}}, \mathbb{Z})[0] \to \mathbb{Q}/\mathbb{Z}[0]$$

is perfect; this is Theorem A.1. ∎

---

[16]This is just a reformulation of the 2-periodicity of the Tate cohomology of cyclic groups.

We finish with a study of the behaviour of our corrected compactly supported cohomology with respect to finite dominant morphisms:

Proposition 2.5. *Let $Y = \operatorname{Spec}(\mathcal{O}')$ be the spectrum of an order in a number field with a finite dominant morphism $\pi : Y \to X$ and let F be a $\mathbb{Z}$-constructible sheaf on Y. We have canonical isomorphisms*

$$R\Gamma_{c,B}\big(X, (\pi_* F)^D\big) \xrightarrow{\simeq} R\Gamma_{c,B}(Y, F^D),$$
$$R\widehat{\Gamma}_{c,B}\big(X, (\pi_* F)^D\big) \xrightarrow{\simeq} R\widehat{\Gamma}_{c,B}(Y, F^D)$$

compatible with the maps $R\Gamma_{c,B}(X,-) \to R\widehat{\Gamma}_{c,B}(X,-)$ resp. $R\Gamma_{c,B}(Y,-) \to R\widehat{\Gamma}_{c,B}(Y,-)$.

Proof. We prove it for $R\Gamma_{c,B}(X, (-)^D)$. The functor π_* is exact and we have

$$R\pi^! \mathbb{Z}_X^c = R\pi^! \mathbb{G}_X[1] = \mathbb{G}_Y[1] = \mathbb{Z}_Y^c$$

by the finite base change theorem, hence $(\pi_* F)^D = \pi_*(F^D)$. Denote by

$$\pi' : \operatorname{Sh}\big(G_{\mathbb{R}}, Y(\mathbb{C})\big) \to \operatorname{Sh}\big(G_{\mathbb{R}}, X(\mathbb{C})\big)$$

the morphism of topoi induced by π, and α' the morphism $\operatorname{Sh}(G_{\mathbb{R}}, Y(\mathbb{C})) \to \operatorname{Sh}(Y_{\mathrm{et}})$. Consider the commutative square

$$
\begin{array}{ccc}
\operatorname{Sh}\big(G_{\mathbb{R}}, Y(\mathbb{C})\big) & \xrightarrow{\ \alpha'\ } & \operatorname{Sh}(Y_{\mathrm{et}}) \\
\downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\
\operatorname{Sh}\big(G_{\mathbb{R}}, X(\mathbb{C})\big) & \xrightarrow{\ \alpha\ } & \operatorname{Sh}(X_{\mathrm{et}})
\end{array}
$$

It induces a canonical map $\alpha^* \pi_* F \to \pi'_* \alpha'^* F$. We claim that it is an isomorphism; indeed it suffices to check it on points of $X(\mathbb{C})$, and then it follows from the computation of the stalks of a finite morphism in the étale case and in the topological case[17].

Since $X(\mathbb{C})$ and $Y(\mathbb{C})$ are finite discrete, we have $\pi'^! = \pi'^*$, so $R\pi'^! = \pi'^*$, $R\pi'^* 2i\pi\mathbb{Z} = 2i\pi\mathbb{Z}$, and $R\pi'^* \mathbb{C}^\times = \mathbb{C}^\times$. The counit $\pi'_* \pi'^* \to \mathrm{id}$ is given on stalks by the sum map. Denote π_x the base change of π to a point $x \in X$ and $g : \eta \to X$, $g' : \eta' \to Y$, $i_x : x \to X$, $i_y : y \to Y$ the inclusion of the generic points, resp. of closed points, of X and Y. The counit $\pi_* \mathbb{Z}_Y^c \to \mathbb{Z}_X^c$ is given by the morphism of complexes (with left term in degree -1)

$$
\begin{array}{ccc}
\pi_* g'_* \mathbb{G}_m = g_* \pi_{\eta,*} \mathbb{G}_m & \xrightarrow{\ \sum \operatorname{ord}_y\ } & \bigoplus_{y \in Y_0} \pi_* i_{y,*}\mathbb{Z} = \bigoplus_{x \in X_0} i_{x,*}(\pi_{x,*}\mathbb{Z}) \\
\downarrow & & \downarrow \\
g_* \mathbb{G}_m & \xrightarrow{\ \sum \operatorname{ord}_x\ } & \bigoplus_{x \in X_0} i_{x,*}\mathbb{Z}
\end{array}
$$

---

[17]In the topological case, by finite morphism we mean a universally closed separated continuous map with finite discrete fibers; we then use [42, Tag 09V4].

where the left arrow is obtained by applying g_* to the counit of the adjunction $\pi_{\eta,*} \dashv \pi_\eta^*$, which is simply the sum map. Thus we have an identification $\alpha^*(\pi_* \mathbb{Z}_Y^c \to \mathbb{Z}_X^c) = \pi_*' \overline{\mathbb{Q}}^\times[1] \to \overline{\mathbb{Q}}^\times[1]$[18]. Combining this with the equality $\alpha^* \pi_* F = \pi_*' \alpha'^* F$ and the identification $R\pi'^!(\overline{\mathbb{Q}}^\times \to 2i\pi\mathbb{Z}[2]) = \overline{\mathbb{Q}}^\times \to 2i\pi\mathbb{Z}[1]$, we obtain a commutative diagram[19]:

$$
\begin{array}{ccccc}
R\operatorname{Hom}_Y(F, \mathbb{Z}_Y^c) & \xrightarrow{\alpha'^*} & R\operatorname{Hom}_{G_\mathbb{R}, Y(\mathbb{C})}\left(\alpha^* F, \overline{\mathbb{Q}}^\times[1]\right) & \longrightarrow & R\operatorname{Hom}_{G_\mathbb{R}, Y(\mathbb{C})}\left(\alpha^* F, 2i\pi\mathbb{Z}[2]\right) \\
\simeq \downarrow & & \downarrow \simeq & & \downarrow \simeq \\
R\operatorname{Hom}_X(\pi_* F, \mathbb{Z}_X^c) & \xrightarrow[\alpha^*]{} & R\operatorname{Hom}_{G_\mathbb{R}, X(\mathbb{C})}\left(\pi_*' \alpha^* F, \overline{\mathbb{Q}}^\times[1]\right) & \longrightarrow & R\operatorname{Hom}_{G_\mathbb{R}, X(\mathbb{C})}\left(\pi_*' \alpha^* F, 2i\pi\mathbb{Z}[2]\right)
\end{array}
$$

hence we obtain in the following diagram of fiber sequences the induced arrow, which is an isomorphism:

$$
\begin{array}{ccccc}
R\Gamma_{c,B}\left(Y, (-)^D\right) & \longrightarrow & R\Gamma\left(Y, (-)^D\right) & \longrightarrow & R\Gamma_{G_\mathbb{R}}\left(Y(\mathbb{C}), (-)_\mathbb{C}^\vee(1)[2]\right) \\
\simeq \downarrow & & \downarrow \simeq & & \downarrow \simeq \\
R\Gamma_{c,B}\left(X, (\pi_*-)^D\right) & \longrightarrow & R\Gamma\left(X, (\pi_*-)^D\right) & \longrightarrow & R\Gamma_{G_\mathbb{R}}\left(X(\mathbb{C}), (\pi_*-)_\mathbb{C}^\vee(1)[2]\right)
\end{array}
\qquad \blacksquare
$$

Proposition 2.6. *In the same setting, there is a commutative diagram of pairings:*

$$
\begin{array}{ccc}
R\Gamma(Y, F) \otimes^L R\widehat{\Gamma}_{c,B}(Y, F^D) & \longrightarrow & R\widehat{\Gamma}_{c,B}(Y, \mathbb{Z}^D) \\
\simeq \downarrow & & \downarrow \\
R\Gamma(X, \pi_* F) \otimes^L R\widehat{\Gamma}_{c,B}\left(X, (\pi_* F)^D\right) & \longrightarrow & R\widehat{\Gamma}_{c,B}(X, \mathbb{Z}^D)
\end{array}
$$

where the arrow $R\widehat{\Gamma}_{c,B}(Y, \mathbb{Z}^D) \to R\widehat{\Gamma}_{c,B}(X, \mathbb{Z}^D)$ is induced from the map

$$
R\Gamma(Y, F^D) \to R\Gamma(X, F^D)
$$

(coming from the counit $\varepsilon : \pi_* R\pi^! \mathbb{Z}_X^c = \pi_* \mathbb{Z}^D \to \mathbb{Z}^D$) and from the map

$$
R\Gamma_{G_\mathbb{R}}\left(Y(\mathbb{C}), 2i\pi\mathbb{Z}[2]\right) \to R\Gamma_{G_\mathbb{R}}\left(X(\mathbb{C}), 2i\pi\mathbb{Z}[2]\right).
$$

Proof. By compatibility of the pairings for $R\Gamma_{G_\mathbb{R}}(X(\mathbb{C}), -)$ and $R\widehat{\Gamma}_{G_\mathbb{R}}(X(\mathbb{C}), -)$, we can reduce to checking the statement with $R\Gamma_{c,B}$ instead of $R\widehat{\Gamma}_{c,B}$. This will follow formally (by taking the fiber) if we prove that the cube of Diagram 2 is commutative.

The lax monoidality of $\pi_*, \pi_*', \alpha^*, \alpha'^*$ induce maps

$$
\begin{aligned}
c_\alpha &: \alpha^* R\mathcal{H}om_X(-,-) \to R\mathcal{H}om_{G_\mathbb{R}, X(\mathbb{C})}(\alpha^*-, \alpha^*-) \\
c_{\alpha'} &: \alpha'^* R\mathcal{H}om_Y(-,-) \to R\mathcal{H}om_{G_\mathbb{R}, Y(\mathbb{C})}(\alpha'^*-, \alpha'^*-) \\
c_\pi &: \pi_* R\mathcal{H}om_Y(-,-) \to R\mathcal{H}om_X(\pi_*-, \pi_*-) \\
c_{\pi'} &: \pi_*' R\mathcal{H}om_{G_\mathbb{R}, Y(\mathbb{C})}(-,-) \to R\mathcal{H}om_{G_\mathbb{R}, X(\mathbb{C})}(\pi_*'-, \pi_*'-).
\end{aligned}
$$

---

[18]That is, the pullback of the counit is the counit between the pullbacks.

[19]The commutation can be seen from the observations made by writing the adjunction arrows as the composition of applying the adjoint functor and postcomposing with the counit.
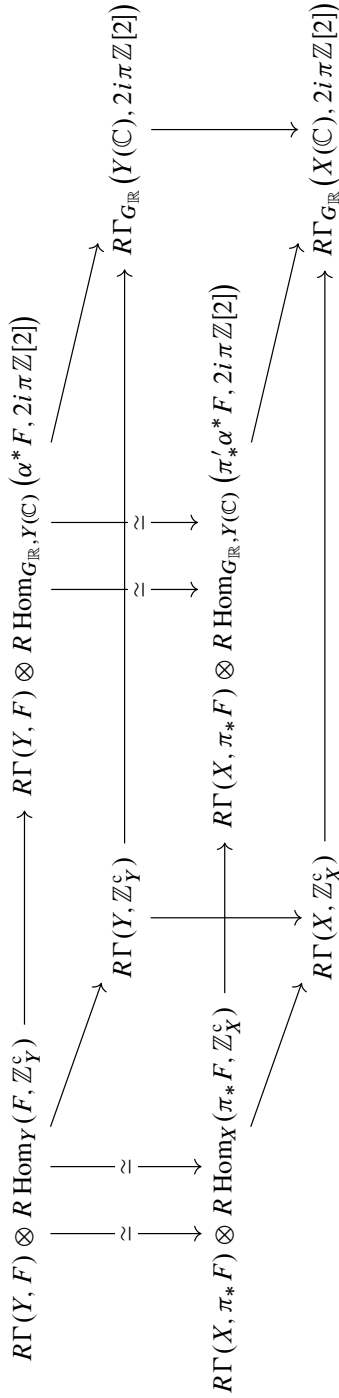
$$
\begin{array}{ccc}
R\Gamma(Y,F) \otimes R\mathrm{Hom}_Y(F,\mathbb{Z}_Y^c) & \xrightarrow{\simeq} & R\Gamma(Y,F) \otimes R\mathrm{Hom}_{G_{\mathbb{R}},Y(\mathbb{C})}(\alpha^*F, 2i\pi\mathbb{Z}[2]) \\
\downarrow & & \downarrow \\
R\Gamma(Y,\mathbb{Z}_Y^c) & \longrightarrow & R\Gamma_{G_{\mathbb{R}}}(Y(\mathbb{C}), 2i\pi\mathbb{Z}[2]) \\
\\
R\Gamma(X,\pi_*F) \otimes R\mathrm{Hom}_X(\pi_*F,\mathbb{Z}_X^c) & \xrightarrow{\simeq} & R\Gamma(X,\pi_*F) \otimes R\mathrm{Hom}_{G_{\mathbb{R}},Y(\mathbb{C})}(\pi_*'\alpha^*F, 2i\pi\mathbb{Z}[2]) \\
\downarrow & & \downarrow \\
R\Gamma(X,\mathbb{Z}_X^c) & \longrightarrow & R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), 2i\pi\mathbb{Z}[2])
\end{array}
$$

Diagram 2.

The following diagram commutes by naturality; the composite rectangle is given by the adjunction isomorphism $\pi_* F^D \simeq (\pi_* F)^D$:

$$
\begin{array}{ccc}
\pi_* F^D & \xrightarrow{\ \eta_\alpha\ } & R\alpha_* \alpha^* \pi_* F^D = R\alpha_* \pi'_* \alpha'^* F^D \\
{\scriptstyle c_\pi}\downarrow & & \downarrow{\scriptstyle R\alpha_* \alpha^*(c_\pi)} \\
R\mathcal{H}om_X(\pi_* F, \pi_* \mathbb{Z}_Y^c) & \xrightarrow{\ \eta_\alpha\ } & R\alpha_* \alpha^* R\mathcal{H}om_X(\pi_* F, \pi_* \mathbb{Z}_Y^c) \\
{\scriptstyle \varepsilon_*}\downarrow & & \downarrow{\scriptstyle R\alpha_* \alpha^*(\varepsilon_*)} \\
(\pi_* F)^D & \xrightarrow{\ \eta_\alpha\ } & R\alpha_* \alpha^* (\pi_* F)^D
\end{array}
\tag{2.4}
$$

We have $\pi_* \alpha^* = \alpha'^* \pi'_*$ so the following diagram commutes formally for any sheaf F and G:

$$
\begin{array}{c}
\pi'_* \alpha'^* R\mathcal{H}om_Y(F, G) \\
\end{array}
$$

$$
\begin{array}{ccc}
\alpha^* \pi_* R\mathcal{H}om_Y(F, G) & & \pi'_* R\mathcal{H}om_{G_\mathbb{R}, Y(\mathbb{C})}(\alpha'^* F, \alpha'^* G) \\
{\scriptstyle \alpha_*(c_\pi)}\downarrow & & \downarrow{\scriptstyle c_{\pi'}} \\
\alpha^* R\mathcal{H}om_X(\pi_* F, \pi_* G) & & R\mathcal{H}om_X(\pi'_* \alpha'^* F, \pi'_* \alpha'^* G) \\
\end{array}
$$

$$
\begin{array}{c}
R\mathcal{H}om_X(\alpha^* \pi_* F, \alpha^* \pi_* G)
\end{array}
$$

with arrows labelled $\pi'_*(c_{\alpha'})$ on upper right, c_α on lower left and lower right.

Thus the top square in the following diagram commutes:

$$
\begin{array}{ccc}
\alpha^* \pi_* F^D = R\alpha_* \pi'_* \alpha'^* F^D & \xrightarrow{\pi'_*(c_{\alpha'})} & \pi'_* R\mathcal{H}om_{G_\mathbb{R}, Y(\mathbb{C})}\big(\alpha^* F, \bar{\mathbb{Q}}^\times[1]\big) \\
{\scriptstyle \alpha^*(c_\pi)}\downarrow & & \downarrow{\scriptstyle c_{\pi'}} \\
\alpha^* R\mathcal{H}om_X(\pi_* F, \pi_* \mathbb{Z}_Y^c) & \xrightarrow{\ c_\alpha\ } & R\mathcal{H}om_{G_\mathbb{R}, X(\mathbb{C})}\big(\pi'_* \alpha^* F, \pi'_* \bar{\mathbb{Q}}^\times[1]\big) \\
{\scriptstyle \alpha^*(\varepsilon_*)}\downarrow & & \downarrow{\scriptstyle \varepsilon_*} \\
\alpha^* R\mathcal{H}om_X(\pi_* F, \mathbb{Z}_X^c) & \xrightarrow{\ c_\alpha\ } & R\mathcal{H}om_{G_\mathbb{R}, X(\mathbb{C})}\big(\pi'_* \alpha^* F, \bar{\mathbb{Q}}^\times[1]\big)
\end{array}
\tag{2.5}
$$

The bottom square also commutes, because

$$
\alpha^*(\pi_* \mathbb{Z}_Y^c \to \mathbb{Z}_X^c) = \pi'_* \bar{\mathbb{Q}}^\times[1] \to \bar{\mathbb{Q}}^\times[1].
$$

The commutative square

$$
\begin{array}{ccc}
\pi'_* \bar{\mathbb{Q}}^\times[1] & \longrightarrow & \pi'_* 2i\pi\mathbb{Z}[2] \\
\downarrow & & \downarrow \\
\bar{\mathbb{Q}}^\times[1] & \longrightarrow & 2i\pi\mathbb{Z}[2]
\end{array}
$$

implies that the following diagram is commutative:

$$
\begin{array}{ccc}
\pi'_* R\mathcal{H}om_{G_\mathbb{R}, Y(\mathbb{C})}(\alpha^* F, \overline{\mathbb{Q}}^\times[1]) & \longrightarrow & \pi'_* R\mathcal{H}om_{G_\mathbb{R}, Y(\mathbb{C})}(\alpha^* F, 2i\pi\mathbb{Z}[2]) \\
\downarrow{\scriptstyle c_{\pi'}} & & \downarrow{\scriptstyle c_{\pi'}} \\
R\mathcal{H}om_{G_\mathbb{R}, X(\mathbb{C})}(\pi'_* \alpha^* F, \pi'_* \overline{\mathbb{Q}}^\times[1]) & \longrightarrow & R\mathcal{H}om_{G_\mathbb{R}, X(\mathbb{C})}(\pi'_* \alpha^* F, \pi'_* 2i\pi\mathbb{Z}[2]) \\
\downarrow{\scriptstyle \varepsilon_*} & & \downarrow{\scriptstyle \varepsilon_*} \\
R\mathcal{H}om_{G_\mathbb{R}, X(\mathbb{C})}(\pi'_* \alpha^* F, \overline{\mathbb{Q}}^\times[1]) & \longrightarrow & R\mathcal{H}om_{G_\mathbb{R}, X(\mathbb{C})}(\pi'_* \alpha^* F, 2i\pi\mathbb{Z}[2])
\end{array}
\quad (2.6)
$$

Apply $R\alpha_*$ to diagrams (2.5) and (2.6) and paste them next to diagram (2.4) to obtain the following diagram:

where we have rewritten the terms on the right using

$$
R\alpha_* R\mathcal{H}om_{G_\mathbb{R}, X(\mathbb{C})}(\alpha^* -, -) = R\mathcal{H}om(-, R\alpha_* -).
$$

The back and side faces are obtained by composing the top and front faces; by properties of adjunctions, the dotted maps are the adjunction isomorphisms for $\pi_* \dashv R\pi^!$ resp. $\pi'_* \dashv R\pi'^!$. By the $\otimes \dashv$ Hom-adjunction, the above diagram is equivalent to the following cube:

Finally, applying the lax monoidal functor $R\Gamma(X, -)$ recovers the sought-after cube. ∎

2.3. Computations

Proposition 2.7. *Let F be a $\mathbb{Z}$-constructible sheaf. We have that*

$$
\begin{aligned}
H^i_{c,B}(X, F^D) &\simeq \hat{H}^i_{c,B}(X, F^D) \quad \text{for } i \geq 1; \\
H^0_{c,B}(X, F^D) &\to \hat{H}^0_{c,B}(X, F^D) \quad \text{is surjective.}
\end{aligned}
$$

If $\alpha^* F$ is torsion-free, we have that

$$H^i_{c,B}(X, F^D) \simeq \hat{H}^i_{c,B}(X, F^D) \quad \text{for } i \geq 0;$$
$$H^{-1}_{c,B}(X, F^D) \to \hat{H}^{-1}_{c,B}(X, F^D) \quad \text{is surjective.}$$

Proof. Let $T(F^D)$ be the cofiber of $R\Gamma_{c,B}(X, F^D) \to R\hat{\Gamma}_{c,B}(X, F^D)$. Let s be the generator of $G_\mathbb{R}$ and denote $N = 1 + s$ the norm map. By the stable 3×3 lemma [31, Lem. 2.2], we find that

$$
\begin{aligned}
T(F^D) &= \mathrm{fib}\left(R\Gamma_{G_\mathbb{R}}\left(X(\mathbb{C}), (\alpha^* F)^\vee(1)[2]\right) \xrightarrow{N} R\hat{\Gamma}_{G_\mathbb{R}}\left(X(\mathbb{C}), (\alpha^* F)^\vee(1)[2]\right)\right) \\
&= \mathbb{Z} \otimes^L_{\mathbb{Z}[G_\mathbb{R}]} R\Gamma\left(X(\mathbb{C}), (\alpha^* F)^\vee(1)[2]\right) \\
&= \prod_{v \text{ archimedean}} \mathbb{Z} \otimes^L_{\mathbb{Z}[G_{K_v}]} F_v^\vee(1)[2]
\end{aligned}
$$

computes homology at the archimedean places. Let M be a $G_\mathbb{R}$-module of finite type. Then $M^\vee(1)[2] = R\,\mathrm{Hom}_\mathbb{Z}(M, 2i\pi\mathbb{Z})[2]$ is concentrated in degree -2 if M is torsion-free and in degree $[-2, -1]$ in general. It follows that $\mathbb{Z} \otimes^L_{\mathbb{Z}[G_\mathbb{R}]} M^\vee(1)[2]$ is concentrated in degree ≤ -2 if M is torsion-free and in degree ≤ -1 in general. ∎

Proposition 2.8. *Let F be a $\mathbb{Z}$-constructible sheaf. Then $R\hat{\Gamma}_{c,B}(X, F^D)$ is concentrated in degree ≤ 2. If F is constructible, the complex has finite cohomology groups. In general, we have*

$$
\hat{H}^i_{c,B}(X, F^D) = \begin{cases} \text{finite} & i \leq -2 \\ \text{finite type} & i = -1, 0 \\ \text{torsion of cofinite type} & i = 1, 2. \end{cases}
$$

Proof. The vanishing in degree > 2 comes from Theorem 2.3. The remaining claims follow from the defining long exact cohomology sequence and [30, Lem. II.3.6], because the complex $R\hat{\Gamma}_{G_\mathbb{R}}(X(\mathbb{C}), (\alpha^* F)^\vee(1)[2])$ has finite 2-torsion cohomology groups. ∎

Proposition 2.9. *Let F be a $\mathbb{Z}$-constructible sheaf. Then $R\Gamma_{c,B}(X, F^D)$ is concentrated in degree $[-1, 2]$. If F is constructible, the complex is perfect. In general, we have*

$$
H^i_{c,B}(X, F^D) = \begin{cases} \text{finite type} & i = -1, 0 \\ \text{torsion of cofinite type} & i = 1, 2. \end{cases}
$$

Proof. By Proposition 2.7 and the previous proposition, we have

$$H^i_{c,B}(X, F^D) = \hat{H}^i_{c,B}(X, F^D) = 0$$

for $i > 2$ and $H^i_{c,B}(X, F^D) = \hat{H}^i_{c,B}(X, F^D)$ is torsion of cofinite type for $i = 1, 2$; finally for $i = -1, 0$ the difference between $H^i_{c,B}(X, F^D)$ and $\hat{H}^i_{c,B}(X, F^D)$ is given by a group of finite type so $H^i_{c,B}(X, F^D)$ is finite type by the previous proposition. The vanishing in degree < -1 is clear. ∎

We now compute some special cases.

Proposition 2.10. *Suppose that $X = \mathrm{Spec}(\mathcal{O}_K)$ is regular. We have*

$$H^i(X, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ \textit{torsion} & i > 1. \end{cases}$$

Let $j : U \to X$ be an open subscheme of X with $U \neq X$. We have

$$H^i(X, j_!\mathbb{Z}) = \begin{cases} 0 & i = 0 \\ \left(\prod_{v \in X \setminus U} \mathbb{Z}\right)/\mathbb{Z} & i = 1 \\ \textit{torsion} & i > 1. \end{cases}$$

Proof. The result for $i > 1$ follows from [30, Lem. II.2.10]. Since X is normal, $\pi_1^{\mathrm{proet}}(X) = \pi_1^{\mathrm{et}}(X)$ is profinite so $H^1(X, \mathbb{Z}) = \mathrm{Hom}_{cont}(\pi_1^{\mathrm{et}}(X), \mathbb{Z}) = 0$. ∎

Proposition 2.11. *Suppose that $X = \mathrm{Spec}(\mathcal{O}_K)$ is regular. We have*

$$H_{c,B}^i(X, \mathbb{Z}^D) = \begin{cases} \textit{finite type of rank } [K : \mathbb{Q}] - 1 & i = -1 \\ \textit{finite} & i = 0 \\ 0 & i = 1 \\ \mathbb{Q}/\mathbb{Z} & i = 2. \end{cases}$$

Moreover we have an exact sequence

$$0 \to \mathbb{Z}^{r_2} \to H_{c,B}^{-1}(X, \mathbb{Z}^D) \to \mathcal{O}_K^\times \to (\mathbb{Z}/2\mathbb{Z})^{r_1} \to H_{c,B}^0(X, \mathbb{Z}^D) \to \mathrm{Pic}(X) \to 0$$

and $H_{c,B}^0(X, \mathbb{Z}^D)$ is the narrow ideal class group $\mathrm{Pic}^+(X)$.

Proof. The exact sequence comes from the long exact cohomology sequence of the defining fiber sequence. By Proposition 2.7, we have

$$\tau^{\geq 0} R\Gamma_{c,B}(X, \mathbb{Z}^D) \xrightarrow{\simeq} \tau^{\geq 0} R\widehat{\Gamma}_{c,B}(X, \mathbb{Z}^D) \simeq \tau^{\geq 0} R\widehat{\Gamma}_c(X, \mathbb{G}_m[1])$$

hence the result for $i \geq 0$ follows from [30, Prop. II.2.6, Rem. II.2.8 (a)]. ∎

Remark. • We have $H_{G_\mathbb{R}}^1(X(\mathbb{C}), 2i\pi\mathbb{Z}) \simeq \widehat{H}_{G_\mathbb{R}}^0(X(\mathbb{C}), \mathbb{C}^\times) = (\mathbb{R}^\times/\mathbb{R}_+^\times)^{r_1}$. Since the canonical map $H^0(X, \mathbb{G}_m) \to H_{G_\mathbb{R}}^1(X(\mathbb{C}), 2i\pi\mathbb{Z})$ factors through $H_{G_\mathbb{R}}^0(X(\mathbb{C}), \mathbb{C}^\times)$ by definition, it is given by

$$\mathcal{O}_K^\times \xrightarrow{(\mathrm{sign}_v)_v} \bigoplus_{v \text{ real}} \mathbb{Z}/2\mathbb{Z}.$$

Its kernel is $\mathcal{O}_{K,+}^\times$, the group of totally positive units; it has same rank as $\mathcal{O}_K^\times$. Thus we have a short exact sequence

$$0 \to \mathbb{Z}^{r_2} \to H_{c,B}^{-1}(X, \mathbb{Z}^D) \to \mathcal{O}_{K,+}^\times \to 0.$$

• We can also recover the result for $i \geq 1$ by Artin–Verdier duality (Theorem 2.3) and Proposition 2.7 since $H^i(X, \mathbb{Z}) = 0, \mathbb{Z}, 0$ for $i < 0, i = 0, i = 1$. On the other hand, from the result for $i = 0$ we recover $H^2(X, \mathbb{Z}) \simeq \mathrm{Pic}^+(X)^*$.

• If X is singular, the exact sequence becomes

$$0 \to \mathbb{Z}^{r_2} \to H^{-1}_{c,B}(X, \mathbb{Z}^D) \to \mathrm{CH}_0(X, 1) \to (\mathbb{Z}/2\mathbb{Z})^{r_1} \to H^0_{c,B}(X, \mathbb{Z}^D) \to \mathrm{CH}_0(X) \to 0$$

and $H^1_{c,B}(X, \mathbb{Z}^D) = H^1(X, \mathbb{Z})^*$ can be non-zero and non-finite, depending on the singularities of X. Moreover, $H^0_{c,B}(X, \mathbb{Z}^D)$ can be seen as a "narrow Chow group".

Proposition 2.12. *Suppose that $X = \mathrm{Spec}(\mathcal{O}_K)$ is regular and let $j : U \to X$ be an open subscheme with $U \neq X$, say $U = \mathrm{Spec}(\mathcal{O}_{K,S})$ with S a set of places containing the archimedean places and at least one finite place. Denote S_f the set of finite places in S and s_f its cardinality. We have*

$$H^i_{c,B}\left(X, (j_!\mathbb{Z})^D\right) = \begin{cases} \text{finite type of rank } [K : \mathbb{Q}] - 1 & i = -1 \\ \text{finite} & i = 0 \\ \mathrm{Ker}\left(\bigoplus_{v \in S_f} \mathrm{Br}(K_v) \xrightarrow{\Sigma} \mathbb{Q}/\mathbb{Z}\right) \simeq (\mathbb{Q}/\mathbb{Z})^{s_f - 1} & i = 1 \\ 0 & i = 2. \end{cases}$$

Moreover, we have an exact sequence

$$0 \to \mathbb{Z}^{r_2} \to H^{-1}_{c,B}\left(X, (j_!\mathbb{Z})^D\right) \to \mathcal{O}^\times_{K,S} \to (\mathbb{Z}/2\mathbb{Z})^{r_1} \to H^0_{c,B}\left(X, (j_!\mathbb{Z})^D\right) \to \mathrm{Pic}(U) \to 0$$

and $H^0_{c,B}(X, (j_!\mathbb{Z})^D)$ is the narrow S-class group $\mathrm{Pic}^+(U)$.

Proof. By Proposition 2.7 and Artin–Verdier duality we obtain the result for $i = 1, 2$ from the computation of $H^i(X, j_!\mathbb{Z})$. We have $(j_!\mathbb{Z})^D = Rj_*\mathbb{Z}^c_U = j_*\mathbb{G}_m[1]$ [30, Lem. II.1.4] and $(j_!\mathbb{Z})^\vee_{\mathbb{C}}(1) = 2i\pi\mathbb{Z}$ so the exact sequence comes from the long exact cohomology sequence associated to the defining fiber sequence.

We have $R\widehat{\Gamma}_{c,B}(X, (j_!\mathbb{Z})^D) \xleftarrow{\sim} R\widehat{\Gamma}_c(X, j_*\mathbb{G}_m[1])$ by Proposition 2.2. The divisor short exact sequence

$$0 \to j_*\mathbb{G}_m \to g_*\mathbb{G}_m \to \bigoplus_{v \in U_0} i_{v,*}\mathbb{Z} \to 0$$

and Proposition 2.7 gives, as in [30, Rem. II.2.8 (a)], the identification of

$$H^0_{c,B}\left(X, (j_!\mathbb{Z})^D\right) \simeq \widehat{H}^0_{c,B}\left(X, (j_!\mathbb{Z})^D\right) \simeq \widehat{H}^0_c\left(X, j_*\mathbb{G}_m[1]\right)$$

with the narrow S-class group. ∎

Remark. • The map $H^0(U, \mathbb{G}_m) \to H^1_{G_\mathbb{R}}(X(\mathbb{C}), 2i\pi\mathbb{Z})$ is given by

$$\mathcal{O}^\times_{K,S} \xrightarrow{(\mathrm{sign}_v)_{v \text{ real}}} \bigoplus_{v \text{ real}} \mathbb{Z}/2\mathbb{Z}.$$

Its kernel is $\mathcal{O}_{K,S,+}^{\times}$, the group of totally positive S-units; it has same rank as $\mathcal{O}_{K,S}^{\times}$. Thus there is a short exact sequence

$$0 \to \mathbb{Z}^{r_2} \to H_c^{-1}\big(X, (j_!\mathbb{Z})^D\big) \to \mathcal{O}_{K,S}^{\times,+} \to 0.$$

• For $i = 1, 2$, we can also prove the result directly by considering the following snake diagram:

$$
\begin{array}{ccccccc}
& & 0 & \longrightarrow & H_{c,B}^1\big(X, (j_!\mathbb{Z})\big)^D & & \\
& & & & \downarrow & & \\
0 \longrightarrow & 0 & \longrightarrow & H^2(U, \mathbb{G}_m) & = & H^2(U, \mathbb{G}_m) & \longrightarrow 0 \\
& & & & & & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to \prod_{v \in S_f} \mathrm{Br}(K_v) & \to & \prod_{v \in S} \mathrm{Br}(K_v) & \to & \prod_{v \text{ archimedean}} \mathrm{Br}(K_v) & \to 0 \\
\| & & \downarrow \Sigma & & \downarrow & \\
\prod_{v \in S_f} \mathrm{Br}(K_v) & \xrightarrow{\Sigma} & \mathbb{Q}/\mathbb{Z} & \longrightarrow & H_{c,B}^2\big(X, (j_!\mathbb{Z})^D\big) & \longrightarrow 0
\end{array}
$$

• We recover from Artin–Vertier duality an identification $H^2(X, j_!\mathbb{Z}) \simeq \mathrm{Pic}^+(U)^*$.

Let $i : x \to X$ be a closed point of X and M a discrete G_x-module of finite type. Since $Ri^!\mathbb{Z}_X^c = \mathbb{Z}[0]$, we have $(i_*M)^D = i_*M^\vee$, while $(i_*M)_{\mathbb{C}} = 0$. Thus

$$R\Gamma_{c,B}\big(X, (i_*M)^D\big) = R\Gamma(G_x, M^\vee)$$

and we obtain the following:

Proposition 2.13. *We have*

$$H_{c,B}^i\big(X, (i_*M)^D\big) = \begin{cases} 0 & i = -1 \\ \textit{finite type} & i = 0 \\ \textit{finite} & i = 1 \\ \textit{torsion of cofinite type} & i = 2. \end{cases}$$

If M is finite, $R\Gamma_{c,B}(X,(i_*M)^D)$ has finite cohomology groups and $H_{c,B}^0(X,(i_*M)^D)=0$.

Proof. If M is finite, we have $M^\vee = M^*[-1]$. On the other hand, if M is torsion-free we have $M^\vee = \mathrm{Hom}_{\mathrm{Ab}}(M, \mathbb{Z})$. Thus in both cases we reduce to cohomology of discrete $G_x \simeq \hat{\mathbb{Z}}$-modules of finite type, for which the result is known. For arbitrary M, we conclude with the short exact sequence $0 \to M_{\mathrm{tor}} \to M \to M/\mathrm{tor} \to 0$. ∎

Let $D_F = R\,\mathrm{Hom}(R\Gamma(X, F), \mathbb{Q}[-2])$, and denote $(-)^\dagger = \mathrm{Hom}(-, \mathbb{Q})$.

Proposition 2.14. *Let F be a $\mathbb{Z}$-constructible sheaf. Then D_F is concentrated in degree* [1, 2].

Proof. We have $H^i(D_F) = H^{2-i}(X, F)^\dagger$. We thus have to show that $H^i(X, F)$ is torsion for $i \neq 0, 1$. But $H^i(X, F)$ differs from $\hat{H}_c^i(X, F)$ by a finite group since X is proper, and the latter is torsion for $i \neq 0, 1$ [30, Thm. II.6.2]. ∎

3. Weil-étale cohomology with compact support of F^D

3.1. Construction of the Weil-étale complex

Following [15], the Weil-étale complex with compact support should be the cone of a map α_F making the following diagram commute

$$
\begin{array}{ccc}
R\operatorname{Hom}\big(R\Gamma(X, F), \mathbb{Q}[-2]\big) & \dashrightarrow \alpha_F \dashrightarrow & R\Gamma_{c,B}(X, F^D) \\
& & \downarrow \\
& & R\hat{\Gamma}_{c,B}(X, F^D) \\
& & \downarrow \\
& & R\operatorname{Hom}\big(R\Gamma(X, F), \mathbb{Q}/\mathbb{Z}[-2]\big)
\end{array}
$$

Let F, G be two $\mathbb{Z}$-constructible sheaves on X and let us compute the morphism group $\operatorname{Hom}_{D(\mathbb{Z})}(D_G, R\Gamma_{c,B}(X, F^D))$. Recall the Verdier spectral sequence [45, Chap. III, §4.6.10]:

$$
E_2^{p,q} = \prod_{i \in \mathbb{Z}} \operatorname{Ext}_{\mathbb{Z}}^p\big(H^i(K), H^{i+q}(L)\big) \Rightarrow \operatorname{Ext}_{D(\mathbb{Z})}^n(K, L). \tag{3.1}
$$

Using the vanishing results of the previous section, the above degenerates to a short exact sequence

$$
\begin{array}{l}
0 \longrightarrow \prod_{i=1,2} \operatorname{Ext}^1\big(H^{2-i}(X, G)^\dagger, H_{c,B}^{i-1}(X, F^D)\big) \longrightarrow \operatorname{Hom}_{D(\mathbb{Z})}\big(D_G, R\Gamma_{c,B}(X, F^D)\big) \\
\\
\qquad \prod_{i=1,2} \operatorname{Hom}\big(H^{2-i}(X, G)^\dagger, H_{c,B}^i(X, F^D)\big) \longrightarrow 0
\end{array} \tag{3.2}
$$

Since $H_{c,B}^1(X, F^D)$ is torsion and $H^0(X, G)^\dagger$ is a $\mathbb{Q}$-vector space, we obtain

$$
\operatorname{Ext}^1\big(H^0(X, G)^\dagger, H_{c,B}^1(X, F^D)\big) = 0.
$$

Therefore the left term is $\operatorname{Ext}^1(H^1(X, G)^\dagger, H_{c,B}^0(X, F^D))$. Similarly to the approach of [31], this motivates the following definition:

Definition 3.1. Let F be a $\mathbb{Z}$-constructible sheaf on X. We say that

- F is red if $H_{c,B}^0(X, F^D)$ is torsion, hence finite.

- F is blue if $H^1(X, F)$ is torsion, hence finite; this happens if and only if

$$\hat{H}^1_{c,B}(X, F^D) = H^1_{c,B}(X, F^D)$$

  is finite (Theorem 2.3).

- A red-to-blue morphism is a morphism of sheaves $F \to G$ where either F and G are both blue, or are both red, or F is red and G is blue ; a red-to-blue short exact sequence is a short exact sequence with red-to-blue morphisms.

Remark. • If F is constructible, F is blue and red.

• The constant sheaf $\mathbb{Z}$ is red; this follows from the finiteness of $\mathrm{CH}_0(X)$[20]. If X is regular or more generally unibranch (so that each point has a singleton preimage in the normalization of X), the sheaf $\mathbb{Z}$ is also blue.

• If $j : U \to X$ is an open inclusion with $U \neq X$, the sheaf $j_!\mathbb{Z}$ is red, and it is blue if X is unibranch and $|X \setminus U| = 1$.

• If F is $\mathbb{Z}$-constructible and supported on a closed subscheme, then F is blue. This reduces to the case of a single discrete finite type $\hat{\mathbb{Z}}$-module, where it follows from $H^1(\hat{\mathbb{Z}}, \mathbb{Z}) = 0$ and the Hochschild–Serre spectral sequence for a normal open subgroup acting trivially.

Proposition 3.2. *Let $j : U \to X$ be an open immersion such that U is regular and G a locally constant $\mathbb{Z}$-constructible sheaf on U. Then $j_!G$ is red.*

Proof. Indeed, the result is true if G is torsion so we can suppose that G is torsion-free. Let $g : \eta = \mathrm{Spec}(K) \to U$ be the generic point. Since G is locally constant, we have $G = g_*g^*G$.[21] By Artin induction for the G_K-module g^*G, we find normal open subgroups H_k, H_l of G_K, an integer $n \in \mathbb{N}^\times$ and a finite G_K-module N such that we have a short exact sequence

$$0 \to (g^*G)^n \oplus \bigoplus \mathrm{ind}_{G_K}^{H_k} \mathbb{Z} \to \bigoplus \mathrm{ind}_{G_K}^{H_l} \mathbb{Z} \to N \to 0.$$

Denote $\pi'_k : V_k \to U, \pi'_l : V_l \to U$ the normalizations of U in $(K^{\mathrm{sep}})^{H_k}, (K^{\mathrm{sep}})^{H_l}$. By applying g_*, we find a short exact sequence

$$0 \to G^n \oplus \bigoplus \pi'_{k,*}\mathbb{Z} \to \bigoplus \pi'_{l,*}\mathbb{Z} \to Q \to 0$$

where Q is a subsheaf of the constructible sheaf g_*N and hence is constructible. Denote by $\pi_k : Y_k \to X$ (resp. $\pi_l : Y_l \to X$) the normalization of X in $(K^{\mathrm{sep}})^{H_k}$ (resp. $(K^{\mathrm{sep}})^{H_l}$) and $j_k : V_k \to Y_k$ the open inclusion (resp. $j_l : V_l \to Y_l$). The points in Y_k (resp. Y_l) above U are exactly the points of V_k (resp. V_l) so

$$j_!\pi'_{k,*}\mathbb{Z} = \pi_{k,*}j_{k,!}\mathbb{Z}, \quad j_!\pi'_{l,*}\mathbb{Z} = \pi_{l,*}j_{l,!}\mathbb{Z}.$$

By Proposition 2.5 we conclude with the preceding remark. ∎

---

[20]See the remark after Proposition 2.11 and the remark after [31, Cor. 7.5].
[21]The regularity hypothesis appears here: in general $g_*\mathbb{Z} \neq \mathbb{Z}$.

Remark. It follows that there are "enough red-and-blues"; by this, we mean that any $\mathbb{Z}$-constructible sheaf F sits in a short exact sequence

$$0 \to R \to F \to B \to 0$$

with R red and B blue. Indeed, it suffices to take $R = j_! F_{|U}$ and $B = i_* i^* F$ for $j : U \to X$ a regular open subscheme such that $F_{|U}$ is locally constant and $i : Z \to X$ its closed complement. As in [31], this remark will allow us to bootstrap the construction of the Weil-étale Euler characteristic from red or blue sheaves to arbitrary $\mathbb{Z}$-constructible sheaves by enforcing multiplicativity.

If F is red or G is blue, the left term of the short exact sequence (3.2) vanishes and we obtain:

Proposition 3.3. *Suppose that F is red or that G is blue. We have*

$$\operatorname{Hom}\left(D_G, R\Gamma_{c,B}(X, F^D)\right) = \prod_{i=1,2} \operatorname{Hom}\left(H^{2-i}(X, G)^\dagger, H^i_{c,B}(X, F^D)\right).$$

In particular, suppose that F is red or blue. For $i = 1, 2$, we have by Theorem 2.3 isomorphisms

$$H^i_{c,B}(X, F^D) \xrightarrow{\simeq} \hat{H}^i_{c,B}(X, F^D) \xrightarrow{\simeq} H^{2-i}(X, F)^*.$$

Hence we obtain a canonical element $\alpha_F \in \operatorname{Hom}(D_F, R\Gamma_{c,B}(X, F^D))$, given in cohomology in degree $i = 1, 2$ by

$$H^i(\alpha_F) : H^{2-i}(X, F)^\dagger \to H^{2-i}(X, F)^* \xleftarrow{\simeq} H^i_{c,B}(X, F^D).$$

Definition 3.4. Let F be red or blue. We define

$$R\Gamma_{W,c}(X, F^D) := \operatorname{Cone}(\alpha_F)$$

the Weil-étale complex with compact support with coefficients in F^D.

Remark. We want to emphasize that our construction is only done in the homotopy category, as a mapping cone. Nevertheless for red or blue sheaves we will obtain sufficient functoriality properties; this is unusual.

3.2. Computation of Weil-étale cohomology

Proposition 3.5. *Let F be a red or blue sheaf. Then $R\Gamma_{W,c}(X, F^D)$ is a perfect complex of abelian groups concentrated in degree $[-1, 2]$. Moreover, we have*

$$H^i_{W,c}(X, F^D) = \begin{cases} H^{-1}_c(X, F^D) & i = -1 \\ \left(H^0(X, F)_{\mathrm{tor}}\right)^* & i = 2 \end{cases}$$

and short exact sequences

$$0 \to H^0_{c,B}(X, F^D) \to H^0_{W,c}(X, F^D) \to \operatorname{Hom}\left(H^1(X, F), \mathbb{Z}\right) \to 0,$$

$$0 \to \left(H^1(X, F)_{\mathrm{tor}}\right)^* \to H^1_{W,c}(X, F^D) \to \operatorname{Hom}\left(H^0(X, F), \mathbb{Z}\right) \to 0.$$

Proof. The groups $H^i(X, F)$ are of finite type for $i = 0, 1$: they differ from $\hat{H}^i_c(X, F)$ by a finite group since X is proper, and the latter are of finite type for $i = 0, 1$ [30, Thm. II.3.1, Thm. II.6.2]. The claim then follows from the distinguished triangle

$$R\operatorname{Hom}\left(R\Gamma(X, F), \mathbb{Q}[-2]\right) \to R\Gamma_{c,B}(X, F^D) \to R\Gamma_{W,c}(X, F^D) \to$$

and Propositions 2.9 and 2.14. ∎

We compute some special cases, using Propositions 2.11 and 2.12.

Proposition 3.6. *Suppose that $X = \operatorname{Spec}(\mathcal{O}_K)$ is regular. We have*

$$H^i_{W,c}(X, \mathbb{Z}^D) = \begin{cases} \mathbb{Z} & i = 1 \\ 0 & i = 2 \end{cases}$$

and an exact sequence

$$0 \to \mathbb{Z}^{r_2} \to H^{-1}_{W,c}(X, \mathbb{Z}^D) \to \mathcal{O}_K^\times \to (\mathbb{Z}/2\mathbb{Z})^{r_1} \to H^0_{W,c}(X, \mathbb{Z}^D) \to \operatorname{Pic}(X) \to 0.$$

Moreover $H^0_{W,c}(X, \mathbb{Z}^D) = \operatorname{Pic}^+(X)$.

Remark. • If X is singular, we still have $H^2_{W,c}(X, \mathbb{Z}^D) = 0$; the abelian group $H^1_{W,c}(X, \mathbb{Z}^D)$ is of rank 1 but it can have torsion coming from $H^1(X, \mathbb{Z})$ if X is not unibranch.

• The complexes $R\Gamma_{W,c}(X, \mathbb{Z}^D)$ is equal to the complex $R\Gamma_{W,c}(X, \mathbb{Z}(1))[2]$ of [15] and the above computation reproduces the one in [15].

Proposition 3.7. *Suppose that $X = \operatorname{Spec}(\mathcal{O}_K)$ is regular and let $j : U \to X$ be an open immersion with $U \neq X$; we keep the notations from Proposition 2.12. Then*

$$H^1_{W,c}\left(X, (j_!\mathbb{Z})^D\right) = H^2_{W,c}\left(X, (j_!\mathbb{Z})^D\right) = 0$$

and we have short exact sequences

$$0 \to \mathbb{Z}^{r_2} \to H^{-1}_{W,c}\left(X, (j_!\mathbb{Z})^D\right) \to \mathcal{O}_{K,S,+}^\times \to 0$$

and

$$0 \to \operatorname{Pic}^+(U) \to H^0_{W,c}\left(X, (j_!\mathbb{Z})^D\right) \to \ker\left(\mathbb{Z}^{s_f} \xrightarrow{\Sigma} \mathbb{Z}\right) \to 0.$$

Proposition 3.8. *Let $i : x \to X$ be a closed point and M a finite type discrete G_x-module. Then*

$$H^i_{W,c}\left(X, (i_*M)^D\right) = \begin{cases} 0 & i = -1 \\ H^0(G_x, M^\vee) = \operatorname{Hom}_{G_x}(M, \mathbb{Z}) & i = 0 \\ \left(H^0(G_x, M)_{\mathrm{tor}}\right)^* & i = 2 \end{cases}$$

and we have a short exact sequence

$$0 \to H^1(G_x, M)^* \to H^1_{W,c}(X, (i_* M)^D) \to \mathrm{Hom}(H^0(G_x, M), \mathbb{Z}) \to 0.$$

Proposition 3.9. *If F is constructible then*

$$R\Gamma_{c,B}(X, F^D) \xrightarrow{\sim} R\Gamma_{W,c}(X, F^D).$$

3.3. Functoriality properties

Theorem 3.10. *Let F be red or blue. The complex $R\Gamma_{W,c}(X, F^D)$ is well defined up to unique isomorphism, is functorial in red-to-blue morphisms, and gives long exact cohomology sequences for red-to-blue short exact sequences.*

Proof. For each red or blue sheaf F we fix a choice $R\Gamma_{W,c}(X, F^D)$ of cone of α_F. Let $f : F \to G$ be a red-to-blue morphism. Let us consider the following diagram with distinguished rows:

$$
\begin{array}{ccccc}
D_F & \xrightarrow{\alpha_F} & R\Gamma_{c,B}(X, F^D) & \longrightarrow & R\Gamma_{W,c}(X, F^D) \longrightarrow \\
{\scriptstyle f^*} \uparrow & & {\scriptstyle f^*} \uparrow & & \\
D_G & \xrightarrow{\alpha_G} & R\Gamma_{c,B}(X, G^D) & \longrightarrow & R\Gamma_{W,c}(X, G^D) \longrightarrow
\end{array}
$$

We claim that the left square commutes. Indeed, by Proposition 3.3 it suffices to check it in cohomology in degree 1 and 2, in which case it follows from the functoriality of the maps

$$D_{(-)} \to R\mathrm{Hom}_{\mathbb{Z}}(R\Gamma(X, -), \mathbb{Q}/\mathbb{Z}[-2])$$
$$R\widehat{\Gamma}_{c,B}(X, (-)^D) \to R\mathrm{Hom}_{\mathbb{Z}}(R\Gamma(X, -), \mathbb{Q}/\mathbb{Z}[-2])$$
$$R\Gamma_{c,B}(X, (-)^D) \to R\widehat{\Gamma}_{c,B}(X, (-)^D).$$

We obtain an induced morphism $f^*: R\Gamma_{W,c}(X, G^D) \to R\Gamma_{W,c}(X, F^D)$ in $D(\mathbb{Z})$ completing the diagram to a morphism of distinguished triangles. Let us show that this induced morphism is uniquely determined by the left square. It suffices to show that the natural map

$$\mathrm{Hom}(R\Gamma_{W,c}(X, G^D), R\Gamma_{W,c}(X, F^D)) \to \mathrm{Hom}(R\Gamma_{c,B}(X, G^D), R\Gamma_{W,c}(X, F^D))$$

is injective. We have an exact sequence

$$\mathrm{Hom}(D_G[1], R\Gamma_{W,c}(X, F^D)) \to \mathrm{Hom}(R\Gamma_{W,c}(X, G^D), R\Gamma_{W,c}(X, F^D))$$
$$\to \mathrm{Hom}(R\Gamma_{c,B}(X, G^D), R\Gamma_{W,c}(X, F^D)).$$

Since D_G is a complex of $\mathbb{Q}$-vector spaces, the multiplication-by-n map on D_G is a quasi-isomorphism for every integer n. The same must hold for $\mathrm{Hom}(D_G[1], R\Gamma_{W,c}(X, F^D))$

by functoriality, so the latter is a $\mathbb{Q}$-vector space. Moreover, by Proposition 3.5 and (3.1), the abelian group $\mathrm{Hom}(R\Gamma_{W,c}(X, G^D), R\Gamma_{W,c}(X, F^D))$ is of finite type so the image of

$$\mathrm{Hom}\left(D_G[1], R\Gamma_{W,c}(X, F^D)\right) \quad \text{inside} \quad \mathrm{Hom}\left(R\Gamma_{W,c}(X, G^D), R\Gamma_{W,c}(X, F^D)\right)$$

must be 0 and the claim follows. This proves the functoriality in red-to-blue morphisms; by running the same argument with the morphism $\mathrm{id} : F \to F$ for a choice of two different cones of α_F, we obtain the uniqueness up to unique isomorphism.

Let $0 \to F \to G \to H \to 0$ be a red-to-blue short exact sequence. Denote $u : H \to F[1]$ the induced map. By the previous argument, we obtain a diagram

$$
\begin{array}{ccccc}
D_F & \xrightarrow{\alpha_F} & R\Gamma_{c,B}(X, F^D) & \longrightarrow & R\Gamma_{W,c}(X, F^D) \longrightarrow \\
{\scriptstyle f^*}\big\uparrow & & {\scriptstyle f^*}\big\uparrow & & {\scriptstyle f^*}\big\uparrow \\
D_G & \xrightarrow{\alpha_G} & R\Gamma_{c,B}(X, G^D) & \longrightarrow & R\Gamma_{W,c}(X, G^D) \longrightarrow \\
{\scriptstyle g^*}\big\uparrow & & {\scriptstyle g^*}\big\uparrow & & {\scriptstyle g^*}\big\uparrow \\
D_H & \xrightarrow{\alpha_H} & R\Gamma_{c,B}(X, H^D) & \longrightarrow & R\Gamma_{W,c}(X, H^D) \longrightarrow \\
{\scriptstyle u^*}\big\uparrow & & {\scriptstyle u^*}\big\uparrow & & {\scriptstyle u^*}\big\vdots \\
D_F[-1] & \xrightarrow{\alpha_F[-1]} & R\Gamma_{c,B}(X, F^D)[-1] & \longrightarrow & R\Gamma_{W,c}(X, F^D)[-1] \longrightarrow
\end{array}
$$

To construct the unique dotted arrow u^* making the diagram commute, repeat the previous argument using Proposition 2.14 and (3.1).

The right column provides a triangle that is not necessarily distinguished. We will show that it induces a long exact cohomology sequence. Fix a prime p and an integer $n \in \mathbb{N}^\times$. Observe that D_F is a complex of $\mathbb{Q}$-vector spaces, so that $D_F \otimes^L \mathbb{Z}/p^n\mathbb{Z} = 0$. It follows that

$$R\Gamma_{c,B}(X, F^D) \otimes^L \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{\simeq} R\Gamma_{W,c}(X, F^D) \otimes^L \mathbb{Z}/p^n\mathbb{Z}.$$

Denote by $(-)_p^\wedge = R\lim(- \otimes^L \mathbb{Z}/p^n\mathbb{Z})$ the derived p-completion. By passing to the derived limit, we find

$$R\Gamma_{c,B}(X, F^D)_p^\wedge \xrightarrow{\simeq} R\Gamma_{W,c}(X, F^D)_p^\wedge = R\Gamma_{W,c}(X, F^D) \otimes^L \mathbb{Z}_p$$

naturally in F where the right equality holds because $R\Gamma_{W,c}(X, F^D)$ is a perfect complex[22]. Since derived p-completion is an exact functor, it follows that

$$R\Gamma_{W,c}\left(X, (-)^D\right) \otimes^L \mathbb{Z}_p$$

is an exact functor too. Now $\mathbb{Z}_p$ is flat and the family $(\mathbb{Z}_p)_{p \text{ prime}}$ is faithfully flat so it suffices to show that the long cohomology sequence is exact after tensoring with $\mathbb{Z}_p$, for each prime p; this follows from the exactness of $R\Gamma_{W,c}(X, (-)^D) \otimes^L \mathbb{Z}_p$. ∎

---

[22]This follows from [42, Tag 0EEU] and [42, Tag 00MA] by filtering with the truncations.

Proposition 3.11. *Let* $Y = \mathrm{Spec}(\mathcal{O}')$ *be the spectrum of an order in a number field with a finite dominant morphism* $\pi : Y \to X$ *and let* F *be a red or blue* $\mathbb{Z}$*-constructible sheaf on* Y*. Then* $\pi_* F$ *is red or blue and we have a unique isomorphism*

$$R\Gamma_{W,c}(Y, F^D) \xrightarrow{\simeq} R\Gamma_{W,c}\big(X, (\pi_* F)^D\big).$$

Proof. We can run an argument similar to the previous proof: it suffices to check that the square

$$
\begin{array}{ccc}
R\Gamma(Y, F)^\dagger[-2] & \xrightarrow{\alpha_F} & R\Gamma_{c,B}(Y, F^D) \\
\uparrow{\simeq} & & \uparrow{\simeq} \\
R\Gamma(X, \pi_* F)^\dagger[-2] & \xrightarrow[\alpha_{\pi_* F}]{} & R\Gamma_{c,B}\big(X, (\pi_* F)^D\big)
\end{array}
$$

is commutative. This can be checked in cohomology in degree 2 and 3, whence from the definition of α_F we reduce to checking that

$$
\begin{array}{ccc}
\widehat{H}^i_{c,B}(Y, F^D) & \longrightarrow & H^{2-i}(Y, F)^* \\
\uparrow{\simeq} & & \uparrow{\simeq} \\
\widehat{H}^i_{c,B}\big(X, (\pi_* F)^D\big) & \longrightarrow & H^{2-i}(X, \pi_* F)^*
\end{array}
$$

is commutative for $i = 2, 3$. This would be implied by the commutativity of the following diagram:

$$
\begin{array}{ccccccc}
R\Gamma(Y, F) \otimes^L R\widehat{\Gamma}_{c,B}(Y, F^D) & \longrightarrow & R\widehat{\Gamma}_{c,B}(Y, \mathbb{Z}^D) & \xrightarrow{\tau^{\geq 2}} & \mathbb{Q}/\mathbb{Z}[-2] \\
\downarrow{\simeq} & & \downarrow & & \| \\
R\Gamma(X, \pi_* F) \otimes^L R\widehat{\Gamma}_{c,B}\big(X, (\pi_* F)^D\big) & \longrightarrow & R\widehat{\Gamma}_{c,B}(X, \mathbb{Z}^D) & \xrightarrow{\tau^{\geq 2}} & \mathbb{Q}/\mathbb{Z}[-2]
\end{array}
$$

By Proposition 2.6, the left square is commutative. Moreover, we can identify the canonical map $R\widehat{\Gamma}_{c,B}(Y, \mathbb{Z}^D) \to R\widehat{\Gamma}_{c,B}(X, \mathbb{Z}^D)$ as the composition

$$R\widehat{\Gamma}_{c,B}(Y, \mathbb{Z}^D) \simeq R\widehat{\Gamma}_c(Y, \mathbb{Z}^c_Y) \simeq R\widehat{\Gamma}_c(X, \pi_* R\pi^! \mathbb{Z}^c_X) \to R\widehat{\Gamma}_c(X, \mathbb{Z}^c_X) \simeq R\widehat{\Gamma}_{c,B}(X, \mathbb{Z}^D)$$

coming from the counit $\varepsilon : \pi_* R\pi^! \to \mathrm{id}$. It induces in cohomology in degree 2 the identity $\mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ [30, Lem. II.3.10], [22, Thm. 8.9], so the right square is commutative. ∎

The Weil-étale complex splits rationally:

Proposition 3.12. *Let* F *be a red or blue sheaf. There is an isomorphism*

$$R\Gamma_{W,c}(X, F^D)_{\mathbb{Q}} \xrightarrow{\simeq} R\,\mathrm{Hom}\big(R\Gamma(X, F), \mathbb{Q}[-1]\big) \oplus R\Gamma_{c,B}(X, F^D)_{\mathbb{Q}}$$

natural in red-to-blue morphisms and red-to-blue short exact sequences.

Proof. Consider the distinguished triangle

$$R\Gamma_{c,B}(X, (-)^D)_{\mathbb{Q}} \to R\Gamma_{W,c}(X, (-)^D)_{\mathbb{Q}} \xrightarrow{p} D_{(-)}[1].$$

It suffices to show that p has a section; but if F is a red sheaf or G is a blue sheaf, considerations on the Verdier spectral sequence (3.1) show that

$$\mathrm{Hom}_{D(\mathbb{Z})}\left(D_F[1], R\Gamma_{c,B}(X, G^D)_{\mathbb{Q}}\right) = 0,$$

$$\mathrm{Hom}_{D(\mathbb{Z})}\left(D_F[1], R\Gamma_{c,B}(X, G^D)_{\mathbb{Q}}[1]\right) = 0$$

so that composition with p_G induces an isomorphism

$$\mathrm{Hom}_{D(\mathbb{Z})}\left(D_F[1], R\Gamma_{W,c}(X, G^D)_{\mathbb{Q}}\right) \xrightarrow[p_{G,*}]{\simeq} \mathrm{Hom}_{D(\mathbb{Z})}\left(D_F[1], D_G[1]\right).$$

We then put $s_F = p_{F,*}^{-1}(\mathrm{id})$; the functoriality is easily checked from the above isomorphism. ∎

Proposition 3.13. *Let* $Y = \mathrm{Spec}(\mathcal{O}')$ *be the spectrum of an order in a number field with a finite dominant morphism* $\pi : Y \to X$ *and let* F *be a red or blue sheaf on* Y. *Then the rational splitting of Weil-étale cohomology is compatible with* π_*, *i.e., the following square commutes*

$$
\begin{array}{ccc}
R\Gamma_{W,c}(Y, F^D)_{\mathbb{Q}} & \xrightarrow{\ \simeq\ } & D_F[1] \oplus R\Gamma_c(Y, F^D)_{\mathbb{Q}} \\
\Big\downarrow{\simeq} & & \Big\downarrow{\simeq} \\
R\Gamma_{W,c}(X, (\pi_* F)^D)_{\mathbb{Q}} & \xrightarrow{\ \simeq\ } & D_{\pi_* F}[1] \oplus R\Gamma_c(X, (\pi_* F)^D)_{\mathbb{Q}}
\end{array}
$$

Proof. It suffices to prove that in the following square, the square with the sections commutes:

$$
\begin{array}{ccc}
R\Gamma_{W,c}(Y, F^D)_{\mathbb{Q}} & \overset{s_F}{\underset{p_F}{\rightleftarrows}} & D_F[1] \\
\Big\downarrow{\gamma} & & \Big\downarrow{\delta} \\
R\Gamma_{W,c}(X, (\pi_* F)^D)_{\mathbb{Q}} & \overset{p_{\pi_* F}}{\underset{s_{\pi_* F}}{\rightleftarrows}} & D_{\pi_* F}[1]
\end{array}
$$

As in the previous proof, there is an isomorphism

$$\mathrm{Hom}_{D(\mathbb{Z})}\left(D_F[1], R\Gamma_{W,c}(X, (\pi_* F)^D)_{\mathbb{Q}}\right) \xrightarrow[p_{\pi_* F,*}]{\simeq} \mathrm{Hom}_{D(\mathbb{Z})}\left(D_F[1], D_{\pi_* F}[1]\right).$$

We compute

$$p_{\pi_* F} \gamma s_F = \delta p_F s_F = \delta = p_{\pi_* F} s_{\pi_* F} \delta.$$ ∎

4. The tangent space and the fundamental line

Denote $g : \eta = \mathrm{Spec}(K) \to X$ the inclusion of the generic point, G_K the absolute Galois group of K, K^t the maximal tamely ramified extension of K, $G_K^t = \mathrm{Gal}(K^t/K)$ and $\mathcal{O}_{K^t}$ the ring of integers of K^t. For each finite place v of K, denote $P_v \subset I_v \subset D_v \subset G_K$ the wild ramification, inertia, decomposition subgroups of G_K at v[23]. Denote N the smallest closed normal subgroup of G_K containing P_v for each finite place v. We have $G_K^t = G_K/N$ by Proposition B.4. For a sheaf F on X, denote $F_\eta := g^*F$, seen as a discrete G_K-module.

Definition 4.1. Let F be a $\mathbb{Z}$-constructible sheaf on X.

*   We say that F is tamely ramified if for each finite place v of K, P_v acts trivially on F_η; then F_η carries a natural action of G_K^t.

*   Suppose that F is tamely ramified. The tangent space of F^D is the complex

$$\mathrm{Lie}_X(F^D) := R\,\mathrm{Hom}_{G_K^t}\big(F_\eta, \mathcal{O}_{K^t}[1]\big).$$

Remark. • Let us motivate the above definition. Following the work of Flach–B. Morin [15], the complex $\mathrm{Lie}_X(F^D)$ should be the "additive complex" giving the additive part of the fundamental line. It should behave like Milne's correcting factor in the special value formula for $L_X(F^D, s)$ at $s = 0$. Since $L_X(\mathbb{Z}^D, s) = \zeta_X(s + 1)$ when $X = \mathrm{Spec}(\mathcal{O}_K)$ is regular, we must have $\mathrm{Lie}_X(\mathbb{G}_m) = \mathcal{O}_K$: this is the additive complex for $\mathbb{Z}(1)[1]$ on X. We want Lie_X to be an exact functor[24]. Recall the L-function of a $\mathbb{Z}$-constructible sheaf $L_X(-, s)$ introduced in [31, §6.4]. If $i : x \to X$ is the inclusion of a closed point and M is a finite type G_x-module, we have

$$L_X\big((i_*M)^D, s\big) = L_X(i_*M^\vee, s).$$

Since the special value at $s = 0$ of $L_X(i_*M^\vee, s)$ does not involve an additive complex, it is expected that $\mathrm{Lie}_X((i_*M)^D) = 0$. Therefore, $\mathrm{Lie}_X(F^D)$ should only depend on F_η. In [21], Geisser–Suzuki consider a torus T over a function field K associated to a proper smooth curve over a finite field C. They write down a special value formula for the L-function of T (defined in terms of the rational l-adic Tate module) at $s = 1$ in terms of Weil-étale cohomology and of the Lie algebra of the connected Néron model $\mathcal{T}^0$ on C of T. Moreover, they show that on the étale site, $\mathcal{T}^0 = R\mathcal{H}om(\widetilde{\mathcal{Y}}, \mathbb{G}_m)$ where $\widetilde{\mathcal{Y}}$ is a bounded complex with $\mathbb{Z}$-constructible cohomology sheaves related to the character group of T. This inspired our construction.

• If F is $\mathbb{Z}$-constructible and $X = \mathrm{Spec}(\mathcal{O}_K)$ is regular, F is tamely ramified if and only if for each $U \subset X$ such that $F_{|U}$ is locally constant, there exist a finite extension L/K such that the normalization $\pi : Y \to X$ of X in L is étale over U with $(\pi^*F)_{|\pi^{-1}(U)}$ constant, and π is tamely ramified above points $x \in X \backslash U$.

---

[23]coming from a choice of embedding $K_v \hookrightarrow K^{\mathrm{sep}}$.

[24]A way to avoid the tame ramification hypothesis may be to relax this condition and only ask that $\det_{\mathbb{Z}}(\mathrm{Lie}_X((-)^D))$ is multiplicative with respect to fiber sequences.

• If X is not regular we nonetheless have $\mathrm{Lie}_X(\mathbb{Z}^D) = \mathcal{O}_K[1]$. This seems strange to us, but as we explained $\mathrm{Lie}_X(F^D)$ depends only on F_η which does not see singular points so it is also more or less expected.

• We have $\mathrm{Lie}_X(F^D)_{\mathbb{Q}} = \mathrm{Hom}_{G_K}(F_\eta, K^{\mathrm{sep}})[1]$.

The restriction to tamely ramified sheaves is justified by the following theorem, essentially due to Noether [38][25]:

Theorem 4.2 ([36, Thm. 6.1.10]). *Let K be a number field or a p-adic field. The discrete G_K^t-module $\mathcal{O}_{K^t}$ is cohomologically trivial. In particular*

$$R\Gamma(G_K^t, \mathcal{O}_{K^t}) = \mathcal{O}_K[0].$$

Corollary 4.3. *Let L/K be a finite tamely ramified Galois extension of number fields or p-adic fields with Galois group G. For $i \geq 1$ denote $\mathcal{O}_L[\![t]\!]_i := 1 + t^i \mathcal{O}_L[\![t]\!]$. Then $\mathcal{O}_L[\![t]\!]_i$ is a cohomologically trivial G-module.*

Proof. We have an isomorphism $\mathcal{O}_L[\![t]\!]_i / \mathcal{O}_L[\![t]\!]_{i+1} \simeq \mathcal{O}_L$ and

$$\mathcal{O}_L[\![t]\!]_i = \lim_{k>i} \mathcal{O}_L[\![t]\!]_i / \mathcal{O}_L[\![t]\!]_k.$$

The transition maps are surjective, and the terms are cohomologically trivial by induction using Theorem 4.2. Moreover, the kernel of $\mathcal{O}_L[\![t]\!]_i / \mathcal{O}_L[\![t]\!]_{k+1} \to \mathcal{O}_L[\![t]\!]_i / \mathcal{O}_L[\![t]\!]_k$ is $\mathcal{O}_L[\![t]\!]_k // \mathcal{O}_L[\![t]\!]_{k+1} \simeq \mathcal{O}_L$ which is cohomologically trivial, so for every subgroup H of G the maps $(\mathcal{O}_L[\![t]\!]_i / \mathcal{O}_L[\![t]\!]_{k+1})^H \to (\mathcal{O}_L[\![t]\!]_i / \mathcal{O}_L[\![t]\!]_k)^H$ are surjective.

Put $A_k = \mathcal{O}_L[\![t]\!]_i / \mathcal{O}_L[\![t]\!]_k$ and let H be a subgroup of G. We compute:

$$\begin{aligned}
R\Gamma\big(H, \mathcal{O}_L[\![t]\!]_i\big) &= R\Gamma(H, \lim A_k) = R\Gamma(H, R\lim A_k) \\
&= R\lim R\Gamma(H, A_k) = R\lim \big(A_k^H[0]\big) = (\lim A_k^H)[0]
\end{aligned}$$

where each limit is a derived limit because the transition maps are surjective. ∎

The following lemma together with the above theorem will enable us to compute the tangent space of F^D more explicitly:

Lemma 4.4. *Let G be a profinite group and let $\mathcal{O}$ be a torsion-free cohomologically trivial discrete G-module. Let M be a finite type discrete G-module. The complex $R\mathrm{Hom}_G(M, \mathcal{O})$ is cohomologically concentrated in degree $[0, 1]$. Moreover, if M is finite it is concentrated in degree 1 and if M is torsion-free it is concentrated in degree 0.*

Proof. To show the vanishing result, we can use the short exact sequence

$$0 \to M_{\mathrm{tor}} \to M \to M/\mathrm{tor} \to 0$$

to reduce to the finite and torsion-free cases; thus it suffices to study those cases:

---

[25]See also [26, Thm. 1.1].

- Suppose first that M is finite. We then have $\mathrm{Hom}_G(M, \mathcal{O}) = 0$ because $\mathcal{O}$ is torsion-free. As $\mathrm{Ext}_{\mathbb{Z}}^p(M, \mathcal{O}) = 0$ for $p \neq 1$, we obtain for $i \geq 1$:

$$\mathrm{Ext}_G^i(M, \mathcal{O}) = H^{i-1}\big(G, \mathrm{Ext}_{\mathbb{Z}}^1(M, \mathcal{O})\big).$$

Let us now show the vanishing for $i > 1$. We can suppose that M is p-primary, and even p-torsion. Let H be any p-Sylow of G. As an H-module, M has a composition series with quotients isomorphic to $\mathbb{Z}/p\mathbb{Z}$ with its trivial action [40, Chap. IX, §1, Thm. 2]. We have

$$H^i\big(H, \mathrm{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/p\mathbb{Z}, \mathcal{O})\big) = H^i(H, \mathcal{O}/p\mathcal{O}) = 0$$

for $i \geq 1$ because $\mathcal{O}$ is cohomologically trivial. It follows that for any $i \geq 1$ we also have $H^i(H, \mathrm{Ext}_{\mathbb{Z}}^1(M, \mathcal{O})) = 0$. Finally, the following triangle commutes:

$$H^i\big(G, \mathrm{Ext}_{\mathbb{Z}}^1(M, \mathcal{O})\big) \xrightarrow[\simeq]{[G:H]} H^i\big(G, \mathrm{Ext}_{\mathbb{Z}}^1(M, \mathcal{O})\big)$$

with Res and Cores arrows to $H^i\big(H, \mathrm{Ext}_{\mathbb{Z}}^1(M, \mathcal{O})\big) = 0$

whence it follows that $H^i(G, \mathrm{Ext}_{\mathbb{Z}}^1(M, \mathcal{O})) = 0$ for $i \geq 1$.

- Suppose now that M is torsion-free, and let us show that $\mathrm{Ext}_G^i(M, \mathcal{O}) = 0$ for $i > 0$. By Artin induction and the previous case it suffices to consider the case of $M = \mathrm{ind}_H^G \mathbb{Z}$ for H an open subgroup. Since $\mathcal{O}$ is cohomologically trivial we have

$$R\,\mathrm{Hom}_G(\mathrm{ind}_H^G \mathbb{Z}, \mathcal{O}) = R\Gamma(H, \mathcal{O}) = \mathcal{O}^H[0]$$

which is concentrated in degree 0. ∎

Corollary 4.5. *Let M be a finite type discrete G_K^t-module. Then $R\,\mathrm{Hom}_{G_K^t}(M, \mathcal{O}_{K^t}[1])$ is a perfect complex cohomologically concentrated in degree $[-1, 0]$. More precisely, $\mathrm{Ext}_{G_K^t}^{-1}(M, \mathcal{O}_{K^t}[1])$ is free of finite rank and $\mathrm{Ext}_{G_K^t}^0(M, \mathcal{O}_{K^t}[1])$ is finite.*

Proof. We apply first Lemma 4.4 to obtain the vanishing outside $[-1, 0]$. Now let H be an open normal subgroup of G_K^t acting trivially on M and corresponding to a finite tamely ramified Galois extension L/K, and put $G := G_K^t/H = \mathrm{Gal}(L/K)$. There is a natural G-action on M. The functor $R\Gamma(H, -)$ is right adjoint to the forgetful functor $G\text{-Mod} \to G_K^t\text{-Mod}$, so we obtain

$$R\,\mathrm{Hom}_G\big(M, \mathcal{O}_{K^t}[1]\big) = R\,\mathrm{Hom}_G\big(M, R\Gamma\big(H, \mathcal{O}_{K^t}[1]\big)\big) = R\,\mathrm{Hom}_G\big(M, \mathcal{O}_L[1]\big)$$

by Theorem 4.2.

Since $\mathcal{O}_L$ is torsion-free, $\mathrm{Hom}_G(M, \mathcal{O}_L)$ is free of finite rank. On the other hand, we can use the short-exact sequence

$$0 \to M_{\mathrm{tor}} \to M \to M/\mathrm{tor} \to 0,$$

the vanishing $\operatorname{Ext}_{\mathbb{Z}}^p(M_{\mathrm{tor}}, \mathcal{O}_L) = 0$ for $p \neq 1$ and Lemma 4.4 (applied to G and $\mathcal{O}_L$) to compute

$$\operatorname{Ext}_G^1(M, \mathcal{O}_L) \xrightarrow{\simeq} \operatorname{Ext}_G^1(M_{\mathrm{tor}}, \mathcal{O}_L) = H^0\big(G, \operatorname{Ext}_{\mathbb{Z}}^1(M_{\mathrm{tor}}, \mathcal{O}_L)\big)$$

which is finite. ∎

Corollary 4.6. *Let F be a tamely ramified $\mathbb{Z}$-constructible sheaf. Then $\operatorname{Lie}_X(F^D)$ is a perfect complex concentrated in degree $[-1, 0]$. Moreover $H^{-1}\operatorname{Lie}_X(F^D)$ is free of finite rank and $H^0\operatorname{Lie}_X(F^D)$ is finite.*

Corollary 4.7. *Suppose that M is a torsion-free discrete G_K^t-module and let T be the torus over K with character group M. Denote $\mathcal{T}$ the lft (locally of finite type) Néron model of T over $\operatorname{Spec}(\mathcal{O}_K)$. We have a canonical, functorial isomorphism*

$$\operatorname{Lie}(\mathcal{T}) \xrightarrow{\simeq} \operatorname{Hom}_{G_K^t}(M, \mathcal{O}_{K^t})$$

where the left term is the Lie algebra of $\mathcal{T}$.

Proof. Both terms are lattices inside $\operatorname{Lie}(T) = \operatorname{Hom}_{G_K}(M, K^{\mathrm{sep}})$. It suffices to show that they are canonically isomorphic after tensoring with $\mathcal{O}_{K_v}$ for all non-archimedean places v. On the one hand we have

$$\operatorname{Lie}(\mathcal{T}) \otimes_{\mathcal{O}_K} \mathcal{O}_{K_v} = \operatorname{Lie}(\mathcal{T}_{\mathcal{O}_{K_v}}) = \operatorname{Lie}\big(\mathcal{N}(T_{K_v})\big)$$

where $\mathcal{N}$ denotes the lft Néron model. On the other hand let $G = \operatorname{Gal}(L/K)$ be a quotient of G_K^t through which the action of G_K^t on M factors. We have

$$\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K_v} \simeq \prod_{w|v} \mathcal{O}_{L_w}$$

and thus

$$\operatorname{Hom}_{G_K^t}(M, \mathcal{O}_{K^t}) \otimes_{\mathcal{O}_K} \mathcal{O}_{K_v} = \operatorname{Hom}_G(M, \mathcal{O}_L) \otimes_{\mathcal{O}_K} \mathcal{O}_{K_v} = \operatorname{Hom}_G(M, \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K_v})$$
$$= \operatorname{Hom}_G\Big(M, \prod_{w|v} \mathcal{O}_{L_w}\Big) = \operatorname{Hom}_{D_{w_0/v}}(M, \mathcal{O}_{L_{w_0}})$$

where w_0 is a fixed place of L above v, so we can reduce to the case where K is a p-adic field. By [8, §A1, Prop. A1.7] we have a canonical functorial isomorphism

$$\operatorname{Lie}(\mathcal{T}) \xrightarrow{\simeq} \big\{v \in \operatorname{Hom}_G(M, \mathcal{O}_L) \mid v \text{ lifts to } \operatorname{Hom}_G\big(M, \mathcal{O}_L[\![t]\!]_1\big)\big\}.$$

It thus suffices to show that $\operatorname{Hom}_G(M, \mathcal{O}_L[\![t]\!]_1) \to \operatorname{Hom}_G(M, \mathcal{O}_L)$ is surjective. We have an exact sequence

$$\operatorname{Hom}_G\big(M, \mathcal{O}_L[\![t]\!]_1\big) \to \operatorname{Hom}_G(M, \mathcal{O}_L) \to \operatorname{Ext}_G^1\big(M, \mathcal{O}_L[\![t]\!]_2\big).$$

Now $\mathcal{O}_L[\![t]\!]_2$ is torsion-free and cohomologically trivial by Corollary 4.3 and M is a torsion-free finite type G-module, so we can apply Lemma 4.4 to obtain

$$\operatorname{Ext}^i_G\left(M, \mathcal{O}_L[\![t]\!]_2\right) = 0 \quad \text{for } i \geq 1.\qquad\blacksquare$$

Remark. As a corollary, a short exact sequence of tori that are split by a tamely ramified extension gives rise to a short exact sequence of the Lie algebras of the respective lft Néron models.

In the following, denote $(-)_{\mathbb{R}} := - \otimes^L \mathbb{R}$.

Proposition 4.8. *Let F be a tamely ramified $\mathbb{Z}$-constructible sheaf. We have*

$$\operatorname{Lie}_X(F^D)_{\mathbb{R}} = R\operatorname{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}\left(\alpha^* F, \mathbb{C}[1]\right).$$

Proof. Let L/K be a finite tamely ramified Galois extension such that G_L acts trivially on F_η, and denote $G := G^t_K/G^t_L = \operatorname{Gal}(L/K)$. We have:

$$\operatorname{Lie}_X(F^D)_{\mathbb{R}} = R\operatorname{Hom}_G\left(F_\eta, \mathcal{O}_L[1]\right)_{\mathbb{R}} \underset{(*)}{=} R\operatorname{Hom}_G\left(F_\eta, \mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{R}[1]\right)$$

$$= R\operatorname{Hom}_G\left(F_\eta, \prod_{v \text{ archimedean}} \prod_{w|v} L_w[1]\right)$$

$$= \prod_{v \text{ archimedean}} R\operatorname{Hom}_G\left(F_\eta, \prod_{w|v} L_w[1]\right).$$

where $(*)$ holds because F_η is a finite type abelian group, hence a finite presentation $\mathbb{Z}[G]$-module. For each archimedean place v, choose a place w_0 in L above v. The group G acts transitively on the places above v, the fields L_w are pairwise isomorphic, and the stabilizer D_{w_0} of w_0 identifies with $\operatorname{Gal}(L_{w_0}/K_v)$. Thus $\prod_{w|v} L_w = \operatorname{ind}^G_{D_{w_0}} L_{w_0}$ and we find

$$\operatorname{Lie}_X(F^D)_{\mathbb{R}} = \prod_{v \text{ archimedean}} R\operatorname{Hom}_{D_{w_0}}\left(F_\eta, L_{w_0}[1]\right).$$

On the other hand, for v an archimedean place we have $F_v = F_\eta$ with the restricted action to $G_{K_v} := \operatorname{Gal}(\mathbb{C}/K_v)$. Therefore the action of G_{K_v} factors through $\operatorname{Gal}(L_{w_0}/K_v)$; as the functor $R\Gamma(\operatorname{Gal}(\mathbb{C}/L_{w_0}), -)$ is right adjoint to the forgetful functor $D_{w_0}\text{-Mod} \to G_{K_v}\text{-Mod}$, we find

$$R\operatorname{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}\left(\alpha^* F, \mathbb{C}[1]\right) = R\Gamma_{G_{\mathbb{R}}}\left(X(\mathbb{C}), R\mathcal{H}om_{X(\mathbb{C})}(\alpha^* F, \mathbb{C}[1])\right)$$

$$= \prod_{v \text{ archimedean}} R\Gamma\left(G_{K_v}, R\operatorname{Hom}_{\mathbb{Z}}\left(F_v, \mathbb{C}[1]\right)\right)$$

$$= \prod_{v \text{ archimedean}} R\operatorname{Hom}_{G_{K_v}}\left(F_v, \mathbb{C}[1]\right)$$

$$= \prod_{v \text{ archimedean}} R\operatorname{Hom}_{D_{w_0}}\left(F_v, L_{w_0}[1]\right)$$

$$= \operatorname{Lie}_X(F^D)_{\mathbb{R}}.\qquad\blacksquare$$

Proposition 4.9. *Let* $Y = \mathrm{Spec}(\mathcal{O}')$ *be the spectrum of an order in a number field with a finite dominant morphism* $\pi : Y \to X$. *If* $K(Y)/K$ *is tamely ramified and* F *is a tamely ramified* $\mathbb{Z}$-*constructible sheaf on* Y, *we have*

$$\mathrm{Lie}_X\big((\pi_* F)^D\big) = \mathrm{Lie}_Y(F^D).$$

Proof. Denote $g' : \mathrm{Spec}(L) \to Y$ the inclusion of the generic point and $\pi' : L \to K$ the map induced by base change of π by $g : \mathrm{Spec}(K) \to X$. By finite base change, we have $g^* \pi_* F = \pi'_* g'^* F$. If $K \subset H \subset G$ are normal subgroups of a profinite group G and M is a discrete H/K-module, we have

$$\mathrm{ind}_{H/K}^{G/K} M = \mathrm{Cont}_{H/K}(G/K, M) \simeq \mathrm{Cont}(G/H, M) \simeq \mathrm{Cont}_H(G, M) = \mathrm{ind}_H^G M$$

where the left term is seen as a G-module, and on the right M has the natural H-module structure. Denote $U : G_K^t\text{-Mod} \to G_K\text{-Mod}$ resp. $U' : G_L^t\text{-Mod} \to G_L\text{-Mod}$ the forgetful functors. The functor π'_* identifies with induction $\mathrm{ind}_{G_L}^{G_K}$. We have $K^t = L^t$ because L/K is tamely ramified so we can apply the above to $\mathrm{Gal}\, K^{\mathrm{sep}}/K^t \subset G_L \subset G_K$. We obtain the formula $U \circ \mathrm{ind}_{G_L^t}^{G_K^t} = \mathrm{ind}_{G_L}^{G_K} \circ U'$, so that

$$\mathrm{Lie}_Y\big((\pi_* F)^D\big) := R\,\mathrm{Hom}_{G_K^t}\big(\mathrm{ind}_{G_L^t}^{G_K^t} g'^* F, \mathcal{O}_{K^t}[1]\big)$$
$$= R\,\mathrm{Hom}_{G_L^t}\big(g'^* F, \mathcal{O}_{K^t}[1]\big) = \mathrm{Lie}_X(F^D)$$

since induction is right *and* left adjoint to the forgetful functor by the finiteness of L/K. ∎

Definition 4.10. Let F be a tamely ramified red or blue sheaf. The fundamental line is the determinant

$$\Delta_X(F^D) = \det_{\mathbb{Z}}\big(R\Gamma_{W,c}(X, F^D)\big) \otimes \det_{\mathbb{Z}}\big(\mathrm{Lie}_X(F^D)\big)^{-1}.$$

5. Deligne compactly supported cohomology and the duality theorem

There is a natural $G_{\mathbb{R}}$-equivariant map of sheaves on $X(\mathbb{C})$ given by the logarithm of the absolute value $\log|\cdot| : \mathbb{C}^\times \to \mathbb{R}$. Let F be a sheaf on X. We denote by Log the natural map

$$\mathrm{Log} : R\,\mathrm{Hom}(F, \mathbb{Z}_X^c) \xrightarrow{\alpha^*} R\,\mathrm{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}\big(\alpha^* F, \overline{\mathbb{Q}}^\times[1]\big)$$
$$\to R\,\mathrm{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}\big(\alpha^* F, \mathbb{C}^\times[1]\big)$$
$$\xrightarrow{(\log|\cdot|)_*[1]} R\,\mathrm{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}\big(\alpha^* F, \mathbb{R}[1]\big).$$

Since the target is a sheaf of $\mathbb{R}$-vector spaces, the map factors as $\mathrm{Log} : R\Gamma(X, F^D)_{\mathbb{R}} = R\,\mathrm{Hom}(F, \mathbb{Z}_X^c)_{\mathbb{R}} \to R\,\mathrm{Hom}_{G_R R, X(\mathbb{C})}(\alpha^* F, \mathbb{R}[1])$.

Definition 5.1. Let F be a $\mathbb{Z}$-constructible sheaf. We define the Deligne compactly supported cohomology with coefficients in F^D by

$$R\Gamma_{c,\mathcal{D}}(X, F_{\mathbb{R}}^D) := \mathrm{fib}\left(R\Gamma(X, F^D)_{\mathbb{R}} \xrightarrow{\mathrm{Log}} R\operatorname{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}\left(\alpha^* F, \mathbb{R}[1]\right)\right).$$

Remark. We have chosen to name the different cohomologies with compact support by what happens at the archimedean places; here the cohomology at the archimedean places is replaced by a contravariant version of Deligne cohomology with real coefficients

$$\mathbb{R}(1)_{\mathcal{D}} = [2i\pi\mathbb{R} \to \mathbb{C}] \xrightarrow{\sim} \mathbb{R}[-1].$$

The commutative diagram with exact rows of $G_{\mathbb{R}}$-equivariant sheaves

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 2i\pi\mathbb{Z} & \longrightarrow & \mathbb{C} & \longrightarrow & \mathbb{C}^\times & \longrightarrow & 0 \\
& & \downarrow & & \| & & \downarrow{\scriptstyle \log|\cdot|} & & \\
0 & \longrightarrow & 2i\pi\mathbb{R} & \longrightarrow & \mathbb{C} & \longrightarrow & \mathbb{R} & \longrightarrow & 0
\end{array}
$$

gives a commutative square

$$
\begin{array}{ccc}
\mathbb{C}^\times & \longrightarrow & 2i\pi\mathbb{Z}[1] \\
\downarrow{\scriptstyle \log|\cdot|} & & \downarrow \\
\mathbb{R} & \longrightarrow & 2i\pi\mathbb{R}[1]
\end{array}
\tag{5.1}
$$

Consider the following diagram

$$
\begin{array}{ccccc}
R\Gamma_{c,B}(X, F^D)_{\mathbb{R}} & \longrightarrow & R\Gamma(X, F^D)_{\mathbb{R}} & \longrightarrow & R\operatorname{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^* F, 2i\pi\mathbb{R})[2] \\
\downarrow{\scriptstyle \mathrm{Log}} & & \downarrow{\scriptstyle \mathrm{Log}} & & \| \\
R\operatorname{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}\left(\alpha^* F, \mathbb{C}[1]\right) \to R\operatorname{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}\left(\alpha^* F, \mathbb{R}[1]\right) \to R\operatorname{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^* F, 2i\pi\mathbb{R})[2]
\end{array}
$$

where the top fiber sequence is the defining fiber sequence of $R\Gamma_{c,B}(X, F^D)$ tensored with $\mathbb{R}$ and the bottom fiber sequence is induced from the short exact sequence $0 \to 2i\pi\mathbb{R} \to \mathbb{C} \to \mathbb{R} \to 0$. We claim that the right square is commutative, hence induces the left map making the whole diagram commute. This follows from the following commutative diagram, induced by the commutative square (5.1), by using the universal property of base change to $\mathbb{R}$-coefficients:

$$
\begin{array}{ccc}
R\Gamma(X, F^D) \xrightarrow{\alpha^*} R\operatorname{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}\left(\alpha^* F, \overline{\mathbb{Q}}^\times[1]\right) & & \\
\searrow & \downarrow & \\
R\operatorname{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}\left(\alpha^* F, \mathbb{C}^\times[1]\right) \longrightarrow R\operatorname{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}\left(\alpha^* F, 2i\pi\mathbb{Z}[2]\right) \\
\downarrow{\scriptstyle (\log|\cdot|)_*[1]} & \| \\
R\operatorname{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}\left(\alpha^* F, \mathbb{R}[1]\right) \longrightarrow R\operatorname{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}\left(\alpha^* F, 2i\pi\mathbb{R}[2]\right)
\end{array}
$$

with Log labeling the diagonal arrow.

By the ∞-categorical nine lemma[26] we obtain

$$R\Gamma_{c,\mathcal{D}}(X, F_\mathbb{R}^D) = \mathrm{fib}\left(R\Gamma_{c,B}(X, F^D)_\mathbb{R} \xrightarrow{\mathrm{Log}} R\,\mathrm{Hom}_{G_\mathbb{R},X(\mathbb{C})}\left(\alpha^* F, \mathbb{C}[1]\right)\right).$$

We now want to prove a duality theorem relating $R\Gamma(X, F)_\mathbb{R}$ and $R\Gamma_{c,\mathcal{D}}(X, F_\mathbb{R}^D)$. Let us construct a natural pairing

$$R\Gamma(X, F)_\mathbb{R} \otimes^L R\Gamma_{c,\mathcal{D}}(X, F_\mathbb{R}^D) \to \mathbb{R}[0].$$

Consider Diagram 3. Its right rectangle commutes. The top and bottom rows are fiber sequences, so we obtain an induced dotted map making the diagram commute. To obtain the desired pairing, it remains to compute $R\Gamma_{c,\mathcal{D}}(X, \mathbb{Z}_\mathbb{R}^D)$. The complex $R\Gamma(X, \mathbb{Z}_X^c) = R\Gamma(X, \mathbb{G}_X)$ is torsion in degree $i \geq 1$ [30, Thm. II.6.2], and we have $R\Gamma_{G_\mathbb{R}}(X(\mathbb{C}), \mathbb{R}[1]) = (\prod_{\sigma:K\to\mathbb{C}} \mathbb{R})^{G_\mathbb{R}}[1]$. The cohomology in degree -1 and 0 is given by the exact sequence

$$0 \to H_{c,\mathcal{D}}^{-1}(X, \mathbb{Z}_\mathbb{R}^D) \to \mathrm{CH}_0(X,1)_\mathbb{R} \xrightarrow{\mathrm{Log}} \left(\prod_{\sigma:K\to\mathbb{C}} \mathbb{R}\right)^{G_\mathbb{R}} \to H_{c,\mathcal{D}}^0(X, \mathbb{Z}_\mathbb{R}^D) \to \mathrm{CH}_0(X)_\mathbb{R} = 0.$$

Denote $f : X \to \mathrm{Spec}(\mathbb{Z})$ the structure map of X, Z its singular locus and U its regular locus. By [31, Thm. 4.4][27], we have[28]

$$\mathrm{CH}_0(X)_\mathbb{R} = \mathrm{Hom}_X(\mathbb{Z}, \mathbb{Z}_X^c)_\mathbb{R} \xrightarrow{\simeq} \mathrm{Hom}\left(H_c^0(X, \mathbb{Z})_\mathbb{R}, \mathbb{R}\right) = 0.$$

Moreover, [31, Thm. 4.4] and the computation of $H^1(X, \mathbb{Z})$ gives a short exact sequence

$$0 \to \mathrm{CH}_0(X,1)_\mathbb{R} \xrightarrow{\mathrm{Log}} \left(\prod_{\sigma:K\to\mathbb{C}} \mathbb{R}\right)^{G_\mathbb{R}} \times \prod_{v\in Z}\prod_{\pi(w)=v} \mathbb{R} \xrightarrow{\Sigma\times\prod_{v\in Z}\Sigma} \mathbb{R}\times\prod_{v\in Z}\mathbb{R} \to 0$$

where Σ denote sum maps. Consider the following snake diagram:

---

[26]See [31, Lem. 2.2].

[27]Which also holds for singular schemes; this can be seen either by applying it on $\mathrm{Spec}(\mathbb{Z})$ to the direct image with compact support along the structural morphism, or by modifying slightly the proof to reduce to the regular case by removing the singular points.

[28]This is essentially the finiteness of the class number of a number field.

$$R\Gamma(X,F)_{\mathbb{R}} \otimes^L R\Gamma_{c,\mathcal{D}}(X,F_{\mathbb{R}}^D) \longrightarrow R\Gamma(X,F)_{\mathbb{R}} \otimes^L R\Gamma(X,F^D)_{\mathbb{R}} \longrightarrow R\Gamma(X,F)_{\mathbb{R}} \otimes^L R\operatorname{Hom}_{G_{\mathbb{R}},X(\mathbb{C})}\left(\alpha^*F,\mathbb{R}[1]\right)$$

$$R\Gamma(X,F)_{\mathbb{R}} \otimes^L R\operatorname{Hom}(F,\mathbb{Z}_X^c)_{\mathbb{R}} \qquad R\Gamma_{G_{\mathbb{R}}}\left(X(\mathbb{C}),\alpha^*F\right)_{\mathbb{R}} \otimes^L R\operatorname{Hom}_{G_{\mathbb{R}},X(\mathbb{C})}\left(\alpha^*F,\mathbb{R}[1]\right)$$

$$R\Gamma_{c,\mathcal{D}}(X,\mathbb{Z}_{\mathbb{R}}^D) \longrightarrow R\Gamma(X,\mathbb{Z}_X^c)_{\mathbb{R}} \longrightarrow R\Gamma_{G_{\mathbb{R}}}\left(X(\mathbb{C}),\mathbb{R}[1]\right)$$

Diagram 3.

If we denote by $(\prod_I \mathbb{R})^\Sigma$ the kernel of the sum map $\prod_I \mathbb{R} \to \mathbb{R}$, we obtain identifications

$$H_{c,\mathcal{D}}^{-1}(X, \mathbb{Z}_{\mathbb{R}}^D) = \prod_{v \in Z} \Big(\prod_{\pi(w)=v} \mathbb{R} \Big)^\Sigma, \quad H_{c,\mathcal{D}}^0(X, \mathbb{Z}_{\mathbb{R}}^D) = \mathbb{R}$$

and under the latter identification the map

$$\Big(\prod_{\sigma:K\to\mathbb{C}} \mathbb{R} \Big)^{G_{\mathbb{R}}} \to H_{c,\mathcal{D}}^0(X, \mathbb{Z}_{\mathbb{R}}^D)$$

is simply the restriction of Σ. If X is unibranch, and in particular if it is regular, we have $H_{c,\mathcal{D}}^{-1}(X, \mathbb{Z}_{\mathbb{R}}^D) = 0$.

The desired pairing is now

$$R\Gamma(X, F)_{\mathbb{R}} \otimes^L R\Gamma_{c,\mathcal{D}}(X, F_{\mathbb{R}}^D) \to R\Gamma_{c,\mathcal{D}}(X, \mathbb{Z}_{\mathbb{R}}^D) \xrightarrow{\tau^{\geq 0}} \mathbb{R}[0].$$

Theorem 5.2. *Let $F \in D^b(X_{\text{et}})$ be a bounded complex with $\mathbb{Z}$-constructible cohomology groups. The pairing*

$$R\Gamma(X, F)_{\mathbb{R}} \otimes^L R\Gamma_{c,\mathcal{D}}(X, F_{\mathbb{R}}^D) \to \mathbb{R}[0]$$

is a perfect pairing of perfect complexes of $\mathbb{R}$-vector spaces. If $F \in D^+(X_{\text{et}})$, the map

$$R\Gamma_{c,\mathcal{D}}(X, F_{\mathbb{R}}^D) \to R\operatorname{Hom}\big(R\Gamma(X, F)_{\mathbb{R}}, \mathbb{R}\big)$$

is an isomorphism.

Proof. The map $R\Gamma_{c,\mathcal{D}}(X, (-)_{\mathbb{R}}^D) \to R\operatorname{Hom}(R\Gamma(X, -)_{\mathbb{R}}, \mathbb{R})$ is a natural transformation between exact functors which commute with filtered colimits. Any complex $F \in D^+(X_{\text{et}})$ is a filtered colimit of bounded complexes; a bounded complex has a filtration by truncations with graded pieces shifts of sheaves; and any sheaf is a filtered colimit of $\mathbb{Z}$-constructible sheaves. We thus reduce to the case of a single $\mathbb{Z}$-constructible sheaf. The groups $H^i(X, F)$ differ from $\hat{H}_c^i(X, F)$ by a finite group since X is proper, hence are finite type for $i = 0, 1$ and torsion otherwise, so $R\Gamma(X, F)_{\mathbb{R}}$ is a perfect complex of $\mathbb{R}$-vector spaces.

We now prove the perfectness of the complex $R\Gamma_{c,\mathcal{D}}(X, F_{\mathbb{R}}^D)$ and of the pairing by Artin induction.

- This is trivial if F is constructible.
- If X is regular and $F = \mathbb{Z}$, we have $R\Gamma(X, \mathbb{Z})_{\mathbb{R}} = \mathbb{R}[0]$ and $R\Gamma_{c,\mathcal{D}}(X, \mathbb{Z}_{\mathbb{R}}^D) = \mathbb{R}[0]$ so the pairing is of the form $\mathbb{R} \otimes \mathbb{R} \to \mathbb{R}$, concentrated in degree 0. By construction, we have a commutative square

$$
\begin{array}{ccc}
R\Gamma(X, \mathbb{Z})_{\mathbb{R}} \otimes^L R\Gamma_{G_{\mathbb{R}}}\big(X(\mathbb{C}), \mathbb{R}\big) & \longrightarrow & R\Gamma(X, \mathbb{Z})_{\mathbb{R}} \otimes^L R\Gamma_{c,\mathcal{D}}(X, \mathbb{Z}_{\mathbb{R}}^D) \\
\downarrow & & \downarrow \\
R\Gamma_{G_{\mathbb{R}}}\big(X(\mathbb{C}), \mathbb{R}\big) & \longrightarrow & R\Gamma_{c,\mathcal{D}}(X, \mathbb{Z}_{\mathbb{R}}^D)
\end{array}
$$

so we get a commutative diagram

$$\begin{array}{ccc}
\mathbb{R} \otimes (\prod_{\sigma:K\to\mathbb{C}} \mathbb{R})^{G_\mathbb{R}} & \xrightarrow{\text{id}\otimes\Sigma} & \mathbb{R} \otimes \mathbb{R} \\
\downarrow & & \downarrow \\
(\prod_{\sigma:K\to\mathbb{C}} \mathbb{R})^{G_\mathbb{R}} & \xrightarrow{\Sigma} & \mathbb{R}
\end{array}$$

Since the pairing on the left is multiplication, the pairing on the right is also multiplication, so it is perfect.

- If F is supported on a closed subscheme, without loss of generality we can suppose that there is a closed point $i: x \to X$ and a finite type G_x-module M such that $F = i_* M$. We have

$$R\Gamma(X, i_*M) \simeq R\Gamma(G_x, M),$$

$$R\Gamma_{c,\mathscr{D}}\big(X, (i_*M)_\mathbb{R}^D\big) \xrightarrow{\simeq} R\Gamma\big(X, (i_*M)^D\big)_\mathbb{R} = R\Gamma(X, i_*M^\vee)_\mathbb{R} \simeq R\Gamma(G_x, M^\vee)_\mathbb{R}.$$

The counit $i_* R i^! \mathbb{Z}_X^c = i_* \mathbb{Z} \to \mathbb{Z}_X^c$ induces a map

$$R\Gamma(G_x, \mathbb{Z}) = R\Gamma\big(X, (i_*\mathbb{Z})^D\big) \to R\Gamma(X, \mathbb{Z}_X^c).$$

Since $\alpha^* i_* = 0$, this induces a map

$$R\Gamma(G_x, \mathbb{Z})_\mathbb{R} \to R\Gamma_{c,\mathscr{D}}(X, \mathbb{Z}_\mathbb{R}^D).$$

Upon identifying $\mathbb{Z}$ with $\mathbb{Z}^\vee$, this map is the map

$$R\Gamma_{c,\mathscr{D}}\big(X, (i_*\mathbb{Z})_\mathbb{R}^D\big) \to R\Gamma_{c,\mathscr{D}}(X, \mathbb{Z}_\mathbb{R}^D)$$

induced by $\mathbb{Z} \to i_*\mathbb{Z}$. Denote $j : U := X\backslash x \to X$ the open immersion; the short exact sequence

$$0 \to j_! \mathbb{Z} \to \mathbb{Z} \to i_* \mathbb{Z} \to 0$$

gives a 3×3 diagram:

$$\begin{array}{ccccc}
R\Gamma_{c,\mathscr{D}}\big(X, (j_!\mathbb{Z})_\mathbb{R}^D[-1]\big) & \longrightarrow & R\Gamma_{c,\mathscr{D}}\big(X, (i_*\mathbb{Z})_\mathbb{R}^D\big) & \longrightarrow & R\Gamma_{c,\mathscr{D}}(X, \mathbb{Z}_\mathbb{R}^D) \\
\downarrow & & \| & & \downarrow \\
R\Gamma\big(U, \mathbb{Z}_U^c[-1]\big)_\mathbb{R} & \longrightarrow & R\Gamma(X, i_*\mathbb{Z})_\mathbb{R} & \longrightarrow & R\Gamma(X, \mathbb{Z}_X^c)_\mathbb{R} \\
\downarrow & & \downarrow & & \downarrow \\
R\Gamma_{G_\mathbb{R}}\big(X(\mathbb{C}), \mathbb{R}\big) & \longrightarrow & 0 & \longrightarrow & R\Gamma_{G_\mathbb{R}}\big(X(\mathbb{C}), \mathbb{R}[1]\big)
\end{array}$$

The map

$$R\Gamma_{c,\mathscr{D}}\big(X, (i_*\mathbb{Z})_\mathbb{R}^D\big) \to R\Gamma_{c,\mathscr{D}}(X, \mathbb{Z}_\mathbb{R}^D)$$

is given in degree 0 by a map $\mathbb{R} \to \mathbb{R}$ which is computed as the boundary map δ in the following snake diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H_{c,\mathcal{D}}^{-1}(X, \mathbb{Z}_{\mathbb{R}}^D) & \longrightarrow & H_{c,\mathcal{D}}^{-1}\big(X, (j_!\mathbb{Z})_{\mathbb{R}}^D\big) & \longrightarrow & \mathbb{R} & & \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{CH}_0(X,1)_{\mathbb{R}} & \longrightarrow & \mathrm{CH}_0(U,1)_{\mathbb{R}} & \xrightarrow{\ \mathrm{ord}_v\ } & \mathbb{R} & \longrightarrow & 0 \\
\end{array}
$$

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \big(\prod_{\sigma:K\to\mathbb{C}}\mathbb{R}\big)^{G_{\mathbb{R}}} & = & \big(\prod_{\sigma:K\to\mathbb{C}}\mathbb{R}\big)^{G_{\mathbb{R}}} & \longrightarrow & 0 \longrightarrow 0 \\
& & \downarrow \Sigma & & \downarrow & & \downarrow \\
& & H_{c,\mathcal{D}}^0(X,\mathbb{Z}_{\mathbb{R}}^D) = \mathbb{R} & \longrightarrow & H_{c,\mathcal{D}}^0\big(X,(j_!\mathbb{Z})_{\mathbb{R}}^D\big) & \longrightarrow & 0 \longrightarrow 0
\end{array}
$$

Let $f \in \mathrm{CH}_0(U,1)_{\mathbb{R}}$ such that $\mathrm{ord}_v(f) = 1$; we also have $\mathrm{ord}_w(f) = 0$ for $w \in U$ by definition. Then

$$\delta(1) = \Sigma \circ \mathrm{Log}(f) = -\log N(v)$$

by the product formula. From Diagram 3, we obtain the commutativity of the upper left rectangle in the following commutative diagram.

$$
\begin{array}{ccccc}
R\Gamma(X, i_*M)_{\mathbb{R}} \otimes^L R\Gamma_{c,\mathcal{D}}\big(X, (i_*M)_{\mathbb{R}}^D\big) & \longrightarrow & R\Gamma_{c,\mathcal{D}}(X, \mathbb{Z}_{\mathbb{R}}^D) & \xrightarrow{\ \tau^{\geq 0}\ } & \mathbb{R}[0] \\
\simeq\downarrow & & \downarrow & & \uparrow \\
R\Gamma(X, i_*M)_{\mathbb{R}} \otimes^L R\Gamma\big(X, (i_*M)^D\big)_{\mathbb{R}} & \longrightarrow & R\Gamma(X, \mathbb{Z}_X^{\mathrm{c}})_{\mathbb{R}} & & -\log N(v) \\
\simeq\uparrow & & \uparrow & & \\
R\Gamma(G_x, M)_{\mathbb{R}} \otimes^L R\Gamma(G_x, M^\vee)_{\mathbb{R}} & \longrightarrow & R\Gamma(G_x, \mathbb{Z})_{\mathbb{R}} & \xrightarrow{\ \tau^{\geq 0}\ } & \mathbb{R}[0]
\end{array}
$$

The perfectness of the pairing reduces to that of the natural pairing

$$R\Gamma(G_x, M)_{\mathbb{R}} \otimes^L R\Gamma(G_x, M^\vee)_{\mathbb{R}} \to \mathbb{R}$$

coming from $M \otimes^L M^\vee \to \mathbb{Z}$. It was shown in [31, Paragraph 4.2.4] that this latter pairing is perfect[29].

- Let $Y = \mathrm{Spec}(\mathcal{O}')$ be the spectrum of an order in a number field with a finite dominant morphism $\pi : Y \to X$, and suppose $F = \pi_*G$ for a $\mathbb{Z}$-constructible sheaf G on Y. Denote $\pi' : Y(\mathbb{C}) \to X(\mathbb{C})$ the induced morphism. The counit $\pi_*\mathbb{Z}_Y^{\mathrm{c}} \to \mathbb{Z}_X^{\mathrm{c}}$ is sent

---

[29]This also reduces by Artin induction, after showing compatibility with induction, to the perfectness for $M = \mathbb{Z}$.

under α^* to the counit $\pi'_* \bar{\mathbb{Q}}^\times[1] \to \bar{\mathbb{Q}}^\times[1]$, and we have the canonical maps $\bar{\mathbb{Q}}^\times[1] \to \mathbb{C}^\times[1] \xrightarrow{\log} \mathbb{R}[1]$. We obtain formally a morphism $R\Gamma_{c,\mathfrak{D}}(Y, \mathbb{Z}_\mathbb{R}^D) \to R\Gamma_{c,\mathfrak{D}}(X, \mathbb{Z}_\mathbb{R}^D)$ and isomorphisms $R\Gamma_{c,\mathfrak{D}}(Y, G_\mathbb{R}^D) \xrightarrow{\simeq} R\Gamma_{c,\mathfrak{D}}(X, F_\mathbb{R}^D)$ making the following diagram of pairing commute

$$
\begin{array}{ccccc}
R\Gamma(Y, G)_\mathbb{R} \otimes^L R\Gamma_{c,\mathfrak{D}}(Y, G_\mathbb{R}^D) & \longrightarrow & R\Gamma_{c,\mathfrak{D}}(Y, \mathbb{Z}_\mathbb{R}^D) & \xrightarrow{\tau \geq 0} & \mathbb{R}[0] \\
\downarrow{\simeq} & & \downarrow & & \downarrow \\
R\Gamma(X, F)_\mathbb{R} \otimes^L R\Gamma_{c,\mathfrak{D}}(X, F_\mathbb{R}^D) & \longrightarrow & R\Gamma_{c,\mathfrak{D}}(X, \mathbb{Z}_\mathbb{R}^D) & \xrightarrow{\tau \geq 0} & \mathbb{R}[0]
\end{array}
$$

The rightmost map is determined by the following commutative diagram (coming from the defining long exact cohomology sequences)

$$
\begin{array}{ccc}
(\prod_{\tau:K(Y) \to \mathbb{C}} \mathbb{R})^{G_\mathbb{R}} & \xrightarrow{\Sigma} & H^0_{c,\mathfrak{D}}(Y, \mathbb{Z}_\mathbb{R}^D) = \mathbb{R} \\
\downarrow & & \downarrow \\
(\prod_{\sigma:K \to \mathbb{C}} \mathbb{R})^{G_\mathbb{R}} & \xrightarrow{\Sigma} & H^0_{c,\mathfrak{D}}(X, \mathbb{Z}_\mathbb{R}^D) = \mathbb{R}
\end{array}
$$

where the left map sums components corresponding to embeddings of $K(Y)$ restricting to the same embedding of K. Thus the rightmost map is the identity and the pairing for F is perfect if and only if the pairing for G is perfect. ∎

Remark. Using duality for Deligne cohomology, it should be possible to reduce the theorem to the duality theorem [31, Thm. 4.4] similarly to how we reduced Artin–Verdier duality for F^D to Artin–Verdier duality for F.

6. The L-function of F^D

For any scheme S, we will denote $\nu : \mathrm{Sh}(S_\mathrm{proet}) \to \mathrm{Sh}(S_\mathrm{et})$ the natural morphism of topoi; the left adjoint ν^* is fully faithful [4, Lem. 5.1.2].

Definition 6.1. For each closed point x of X, let l_x be a prime number coprime to the residual characteristic at x and φ be the *geometric* Frobenius in G_x. Denote $-\hat{\otimes}\mathbb{Q}_l = (R \lim(- \otimes^L \mathbb{Z}/l^n\mathbb{Z})) \otimes \mathbb{Q}$ the *completed* tensor product with $\mathbb{Q}_l$, with the derived limit computed on the proétale site.

Let F be a $\mathbb{Z}$-constructible sheaf on X. We define the L-function of F^D by the Euler product

$$
L_X(F^D, s) = \prod_{x \in X_0} \det\left(I - N(x)^{-s}\varphi(\nu^*(i_x^* F^D))\hat{\otimes}\mathbb{Q}_{l_x}\right)^{-1}.
$$

We will show that the L-function does not depend on the choice of the prime numbers $(l_x)_{x \in X_0}$.

6.1. Explicit computation

In this section, we compute more explicitely the L-function of F^D. Denote $\pi : Y \to X$ the normalization of X and let $i : x \to X$ be a closed point. Denote $\bar{x} = \text{Spec}(\kappa(x)^{\text{sep}})$, $G_x = \text{Gal}(\kappa(x)^{\text{sep}}/\kappa(x))$, $\mathcal{O}_x^h$ (resp. $\mathcal{O}_x^{\text{sh}}$) the henselian (resp. strict henselian) local ring at x. For each closed point $y \in Y$ above x, we will consider similarly $G_y = \text{Gal}(\kappa(y)^{\text{sep}}/\kappa(y))$, $\mathcal{O}_y^h$, $\mathcal{O}_y^{\text{sh}}$, and moreover $K_y^h = \text{Frac}(\mathcal{O}_y^h)$ (resp. K_y^{sh}) the henselian local field at y (resp. its maximal unramified extension). Fix an embedding $K_y^h \hookrightarrow K^{\text{sep}}$. This determines an inertia group I_y inside G_K which is the absolute Galois group of K_y^{sh}. If G is a topological group we will denote $\underline{G}$ the associated condensed group.

Let us fix some conventions. If $L(M, s)$ is an L-function attached to some object M[30], defined by an Euler product over closed points of X, we will denote $L_x(M, s)$ for the local factor at a closed point $x \in X$. If N is a discrete G_x-module with free of finite type underlying abelian group, or a rational or l-adic G_x-representation of finite dimension, denote

$$Q_x(N, s) := \det(I - N(x)^{-s}\varphi | N)^{-1}$$

with φ the geometric Frobenius at x.

Theorem 6.2. *Let F be a $\mathbb{Z}$-constructible sheaf on X. The local factor at x of the L-function of F^D is*

$$L_x(F^D, s) = \frac{\prod_{\pi(y)=x} Q_y((F_\eta^\vee)^{I_y}, s+1) Q_x(F_x^\vee, s)}{\prod_{\pi(y)=x} Q_y((F_\eta^\vee)^{I_y}, s)}.$$

The proof is divided into the following several lemmas below. We mention first some important consequences:

Corollary 6.3. • *If F is locally constant around the regular closed point v, then I_v acts trivially on F_η, $F_\eta = F_v$ and we find that the local factor at v of F^D equals the local factor at v of the Artin L-function of $F_\eta \otimes \mathbb{Q}$ at $s + 1$:[31]*

$$L_v(F^D, s) = L_v(F_\eta \otimes \mathbb{Q}, s+1).$$

• *Each local factor is well defined, independently of the choice of a prime number l coprime to the residual characteristic.*

• *The L-function of F^D differs from the Artin L-function $L_K(F_\eta \otimes \mathbb{Q}, s+1)$ by a finite number of factors and is thus well defined.*

• *$L_X(F^D, s)$ is meromorphic.*

Denote T the torus over K with character group $Y := F_\eta/\text{tor}$. For any prime l, define the rational l-adic Tate module of T as

$$V_l(T) := \left(\lim T[l^n] \right) \otimes \mathbb{Q}.$$

---

[30]Which kind of L-function we will try to make clear each time from the context.

[31]Notice that a rational representation of a finite group is self-dual because its character takes real values hence is its own conjugate.

It is a finite dimensional l-adic representation of G_K. We know that

$$V_l(T) \simeq Y^\vee \otimes \mathbb{Q}_l(1) = F_\eta^\vee \otimes \mathbb{Q}_l(1)$$

as l-adic representations.

Definition 6.4 ([16, §8]). The L-function $L_K(T, s)$ of T is the L-function of the 1-motive $[0 \to T]$ over K, defined by the Euler product:

$$L_K(T, s) = \prod_{x \in X_0} \det(I - N(x)^{-s} \varphi | V_{l_x}(T)^{I_x})^{-1}$$

where l_x is a prime number coprime to the residual characteristic at x.

By the above, the L-function of T is also the Artin L-function at $s + 1$ of $Y \otimes \mathbb{Q} = F_\eta \otimes \mathbb{Q}$, and it does not depend on the choice of the family (l_x).

Remark. Our definition differs slightly from the definition in [21]: we use the L-function of the 1-motive $[0 \to T]$, involving $V_l(T)$, while Geisser–Suzuki use the Hasse–Weil L-function of T involving $V_l(T)(-1)$.

Proposition 6.5. *Suppose that v is a regular closed point. Let M be a discrete D_v-module of finite type. There is a canonical G_v-equivariant isomorphism*

$$M^{I_v} \otimes \mathbb{Q} \xrightarrow{\simeq} M_{I_v} \otimes \mathbb{Q}.$$

Proof. The action of D_v on M factors through a finite quotient G; denote H the image of I_v in G. Then the canonical composite map

$$f : M^H \to M \to M_H$$

is G-equivariant hence also G/H-equivariant, and if N denotes the norm $\sum_{h \in H} h$ we have $fN = [H]\mathrm{Id}$ and $Nf = [H]\mathrm{Id}$. The result follows. ∎

Corollary 6.6. • *Suppose that the canonical map $F_v \to F_\eta^{I_v}$ is an isomorphism for a regular closed point v. Then the local factor at v of F^D is $L_v(F_\eta \otimes \mathbb{Q}, s + 1) = L_v(T, s)$.*

• *Suppose that X is regular. We have $L_X((g_* Y)^D, s) = L_K(T, s) = L_K(Y \otimes \mathbb{Q}, s + 1)$.*

• *Suppose that X is regular and denote $\mathcal{T}^0$ the connected (lft) Néron model of T on X, seen as an étale sheaf on X. We can define an L-function $L_X(\mathcal{T}^0, s)$ for $\mathcal{T}^0$ with local factor at $i : x \to X$ given by $Q_x((v^* i^* \mathcal{T}^0) \hat\otimes \mathbb{Q}_l, s)$. Then*

$$L_X(\mathcal{T}^0, s) = L_K(T, s)^{-1}.$$

Proof. We have a canonical G_v-equivariant isomorphism:

$$(F_\eta^\vee)^{I_v} \otimes \mathbb{Q} = \mathrm{Hom}_{I_v}(F_\eta \otimes \mathbb{Q}, \mathbb{Q}) = \mathrm{Hom}_{\mathrm{Ab}}\left((F_\eta)_{I_v} \otimes \mathbb{Q}, \mathbb{Q}\right)$$
$$\simeq \mathrm{Hom}_{\mathrm{Ab}}(F_\eta^{I_v} \otimes \mathbb{Q}, \mathbb{Q}) \simeq F_v^\vee \otimes \mathbb{Q}$$

hence the local factor at v of the L-function of F^D simplifies to $L_v(F_\eta \otimes \mathbb{Q}, s + 1)$.

The second point is an immediate consequence of the first. Let us prove the last point. We have [21, Prop. 2.2]

$$\mathcal{T}^0 = R\mathcal{H}om_X(\tau^{\leq 1} Rg_*Y, \mathbb{G}_m)$$

as étale sheaves. We claim that R^1g_*Y is skyscraper constructible: indeed $(R^1g_*Y)_{\bar{\eta}}$ vanishes, while $(R^1g_*Y)_{\bar{v}} = H^1(K_v^{\text{sh}}, Y)$ which is finite, and zero if the inertia subgroup I_v acts trivially on Y since Y is free as an abelian group[32]. As there are only a finite number of points where the inertia acts non-trivially, we obtain the claim. Therefore R^1g_*Y is killed by an integer N and thus

$$(v^*i^*\mathcal{T}^0)\widehat{\otimes}\mathbb{Q}_l = (v^*i^*(g_*Y)^D)\widehat{\otimes}\mathbb{Q}_l[-1].$$

The result follows. ∎

We now prove Theorem 6.2.

Lemma 6.7. *Let F be an étale sheaf on X, and let $i : x \to X$ be a closed point. There is a fiber sequence*

$$R\mathcal{H}om_x(F_x, \mathbb{Z}) \to i^*F^D \to \prod_{\pi(y)=x} \text{ind}_{G_y}^{G_x} R\text{Hom}_{\text{Spec}(K_y^{\text{sh}})}(F_\eta, \mathbb{G}_m[1]).$$

Remark. We abused notation by identifying what should be the right term with its underlying complex of abelian groups.

Proof. Denote $X(x) := \text{Spec}(\mathcal{O}_x^h)$, $\eta(x) := \text{Spec}(\mathcal{O}_x^h) \times_X \text{Spec}(K)$ and $f : X(x) \to X$, $g : \eta \to X$ the canonical morphisms. We consider the cartesian diagram

$$
\begin{array}{ccc}
\eta(x) & \xrightarrow{g'} & X(x) \\
f' \downarrow & & \downarrow f \\
\eta & \xrightarrow{g} & X
\end{array}
$$

Denote again $i : x \to X(x)$ the closed immersion. Since f is (ind-)étale, we have

$$i^*F^D = i^*f^*R\mathcal{H}om_X(F, \mathbb{G}_X[1]) = i^*R\mathcal{H}om_{X(x)}(F, \mathbb{G}_{X(x)}[1]).$$

By Proposition D.5, we have

$$\eta(x) = \coprod_{\pi(y)=x} \coprod_{\text{Gal}(\kappa(y)/\kappa(x))} \text{Spec}(K_y^{\text{sh}})$$

and thus

$$(Rg'_*\mathbb{G}_m)_{\bar{x}} = R\Gamma(\eta(x), \mathbb{G}_m) = \prod_{\pi(y)=x} \text{ind}_{G_y}^{G_x} R\Gamma(K_y^{\text{sh}}, \mathbb{G}_m) = \prod_{\pi(y)=x} \text{ind}_{G_y}^{G_x}(K_y^{\text{sh}})^\times$$

---

[32]This is the crucial point: if Y is finite then R^1g_*Y is skyscraper with finite stalks but has non-zero stalk almost everywhere; for instance by Hensel's lemma the stalk of $R^1g_*\mathbb{Z}/n\mathbb{Z}$ at a point with characteristic coprime to n is $\mathbb{Z}/n\mathbb{Z}$.

hence $g'_* \mathbb{G}_m = Rg'_* \mathbb{G}_m$. Moreover there is by definition a fiber sequence

$$\mathbb{G}_{X(x)} \to g'_* \mathbb{G}_m \to i_* \mathbb{Z}$$

so we obtain a fiber sequence

$$R\mathcal{H}om_x(F_x, \mathbb{Z}) \to i^* F^D \to i^* Rg'_* R\mathcal{H}om_{\eta(x)}(g'^* F, \mathbb{G}_m[1]).$$

Using again Proposition D.5, we see that the right term is the following complex of abelian groups with its natural G_x-module structure:

$$\left(Rg'_* R\mathcal{H}om_{\eta(x)}(g'^* F, \mathbb{G}_m[1])\right)_{\bar{x}} = \prod_{\pi(y)=x} \text{ind}_{G_y}^{G_x} R\,\text{Hom}_{\text{Spec}(K_y^{\text{sh}})}(F_\eta, \mathbb{G}_m[1]). \quad \blacksquare$$

From now on, we suppose that F is a $\mathbb{Z}$-constructible sheaf on X.

Lemma 6.8. *We have*

$$\left(v^* R\mathcal{H}om_x(F_x, \mathbb{Z})\right) \hat{\otimes} \mathbb{Q}_l = \left(v^* \mathcal{H}om_x(F_x, \mathbb{Z})\right) \otimes \mathbb{Q}_l.$$

Proof. We have

$$v^* R\mathcal{H}om_x(F_x, \mathbb{Z}) = R\mathcal{H}om_{x_{\text{proet}}}(v^* F_x, \mathbb{Z}).$$

The functor $- \otimes^L \mathbb{Z}/l^n\mathbb{Z}$ is the cofiber of the map $(-) \xrightarrow{l^n} (-)$ so it commutes with exact functors, and we find

$$R\mathcal{H}om_{x_{\text{proet}}}(v^* F_x, \mathbb{Z}) \hat{\otimes} \mathbb{Z}_l = R\mathcal{H}om_{x_{\text{proet}}}(v^* F_x, R\lim \mathbb{Z}/l^n\mathbb{Z}).$$

The constant sheaf functor is exact so the transition maps $\mathbb{Z}/l^{n+1}\mathbb{Z} \to \mathbb{Z}/l^n\mathbb{Z}$ are surjective, hence we have $R\lim \mathbb{Z}/l^n\mathbb{Z} = \lim \mathbb{Z}/l^n\mathbb{Z} =: \mathbb{Z}_l$ [4, Prop. 3.1.10].

Under the identification $x_{\text{proet}} \simeq \underline{G_x}\text{-Cond}(Set)$, the sheaf $\mathcal{E}xt^i_{x_{\text{proet}}}(v^* F_x, \mathbb{Z}_l)$ is the condensed abelian group $\mathcal{E}xt^i_{\text{Cond(Ab)}}(v^* F_x, \mathbb{Z}_l)$ with its natural $\underline{G_x}$-action; since both $v^* F_x$ and $\mathbb{Z}_l$ are locally compact abelian groups[33], we find $\mathcal{E}xt^i_{x_{\text{proet}}}(v^* F_x, \mathbb{Z}_l) = 0$ for $i \geq 2$ [11, Remark after 4.9]. Let us show that $\mathcal{E}xt^1_{x_{\text{proet}}}(v^* F_x, \mathbb{Z}_l)$ is killed by some integer N. We have

$$R\mathcal{H}om_{x_{\text{proet}}}(v^* F_x, \mathbb{Z}_l) = R\lim R\mathcal{H}om_{x_{\text{proet}}}(v^* F_x, v^* \mathbb{Z}/l^n\mathbb{Z})$$
$$= R\lim v^* R\mathcal{H}om_x(F_x, \mathbb{Z}/l^n\mathbb{Z})$$

so there is a short exact sequence

$$0 \to R^1 \lim v^* \mathcal{H}om_x(F_x, \mathbb{Z}/l^n\mathbb{Z}) \to \mathcal{E}xt^1_{x_{\text{proet}}}(v^* F_x, \mathbb{Z}_l)$$
$$\to \lim v^* \mathcal{E}xt^1_x(F_x, \mathbb{Z}/l^n\mathbb{Z}) \to 0.$$

---

[33]The former is even discrete.

The underlying abelian group of the discrete G_x-module $\mathcal{H}om_x(F_x, \mathbb{Z}/l^n\mathbb{Z})$ is given by $\mathrm{Hom}_{\mathrm{Ab}}(F_x, \mathbb{Z}/l^n\mathbb{Z})$, so the transition maps $\mathcal{H}om_x(F_x, \mathbb{Z}/l^{n+1}\mathbb{Z}) \to \mathcal{H}om_x(F_x, \mathbb{Z}/l^n\mathbb{Z})$ are eventually surjective: if F_x is torsion-free this is clear, and if F_x is torsion then the groups are constant for n greater than the l-adic valuation of the order of F_x. Thus the left term is zero by Lemma C.2. Let k be the l-adic valuation of $(F_x)_{\mathrm{tor}}$; then

$$\mathcal{E}xt^1_x(F_x, \mathbb{Z}/l^n\mathbb{Z}) = \mathcal{E}xt^1_x\big((F_x)_{\mathrm{tor}}, \mathbb{Z}/l^n\mathbb{Z}\big)$$

is killed by $N = l^k$, and therefore so is $\mathcal{E}xt^1_{x_{\mathrm{proet}}}(\nu^* F_x, \mathbb{Z}_l)$.

From the previous point, it follows by tensoring with $\mathbb{Q}$ that

$$R\mathcal{H}om_{x_{\mathrm{proet}}}(\nu^* F_x, \mathbb{Z})\widehat{\otimes}\mathbb{Q}_l = \mathcal{H}om_{x_{\mathrm{proet}}}(\nu^* F_x, \mathbb{Z}_l) \otimes \mathbb{Q}.$$

There are canonical maps

$$\mathcal{H}om_{x_{\mathrm{proet}}}(\nu^* F_x, \mathbb{Z}_l) \otimes \mathbb{Q} \to \mathcal{H}om_{x_{\mathrm{proet}}}(\nu^* F_x, \mathbb{Q}_l) \leftarrow \mathcal{H}om_{x_{\mathrm{proet}}}(\nu^* F_x, \mathbb{Z}) \otimes \mathbb{Q}_l.$$

We have $\mathrm{Ab}(\mathrm{Sh}(x_{\mathrm{proet}})) = \mathrm{Ab}(\mathrm{Sh}((BG_x)_{\mathrm{proet}})) = \underline{G_x} - \mathrm{Cond}(\mathrm{Ab}) = \mathbb{Z}[\underline{G_x}] - \mathrm{Mod}$, the category of modules under the condensed ring $\mathbb{Z}[\underline{G_x}]$ (Proposition C.7). To conclude it suffices to show that both maps are isomorphisms. Let G be a finite quotient of G_x. We check it first for $F_x = \mathbb{Z}[G]$. This is easily done using next lemma.

By a standard argument it thus suffices to show that $\nu^* F_x$ is (globally) of finite presentation as a $\mathbb{Z}[\underline{G}]$-module for some finite quotient G of G_x; this is clear as F_x is discrete and of finite type as an abelian group. ∎

Lemma 6.9. *Let G be a finite quotient of G_x. Then $\mathbb{Z}[G]$ is a left $\underline{G_x}$-module such that*

$$\mathcal{H}om_{x_{\mathrm{proet}}}\big(\mathbb{Z}[\underline{G}], -\big) = \mathbb{Z}[\underline{G}] \otimes -.$$

Proof. We have $\mathbb{Z}[\underline{G}] = \underline{\mathbb{Z}[G]} = \oplus_G \underline{\mathbb{Z}}$, thus

$$\mathcal{H}om_{x_{\mathrm{proet}}}\big(\mathbb{Z}[\underline{G}], -\big) = \oplus_G \mathcal{H}om_{x_{\mathrm{proet}}}(\underline{\mathbb{Z}}, -) = \oplus_G \mathrm{Id} = \mathbb{Z}[\underline{G}] \otimes -. \qquad ∎$$

Denote $R\mathrm{Hom}_{Y_{\mathrm{proet}}}$ the enriched $R\,\mathrm{Hom}$ on the proétale site of a scheme, a complex of condensed abelian groups with underlying complex of abelian groups $R\,\mathrm{Hom}_Y$.

Lemma 6.10. *We have*

$$\big(\nu^* R\,\mathrm{Hom}_{\mathrm{Spec}(K_y^{\mathrm{sh}})}\big(F_\eta, \mathbb{G}_m[1]\big)\big)\widehat{\otimes}\mathbb{Z}_l = R\underline{\mathrm{Hom}}_{\mathrm{Spec}(K_y^{\mathrm{sh}})_{\mathrm{proet}}}\big(\nu^* F_\eta, \mathbb{Z}_l(1)[2]\big).$$

Proof. We have

$$\nu^* R\,\mathrm{Hom}_{\mathrm{Spec}(K_y^{\mathrm{sh}})}\big(F_\eta, \mathbb{G}_m[1]\big) = R\underline{\mathrm{Hom}}_{\mathrm{Spec}(K_y^{\mathrm{sh}})_{\mathrm{proet}}}\big(\nu^* F_\eta, \nu^* \mathbb{G}_m[1]\big).$$

Thus we find

$$\nu^* R\,\mathrm{Hom}_{\mathrm{Spec}(K_y^{\mathrm{sh}})}\big(F_\eta, \mathbb{G}_m[1]\big)\widehat{\otimes}\mathbb{Z}_l = R\,\underline{\mathrm{Hom}}_{\mathrm{Spec}(K_y^{\mathrm{sh}})_{\mathrm{proet}}}\big(\nu^* F_\eta, \big(\nu^* \mathbb{G}_m[1]\big)\widehat{\otimes}\mathbb{Z}_l\big).$$

Since l is invertible on $\mathrm{Spec}(K_y^{\mathrm{sh}})$, we have $\mathbb{G}_m \otimes^L \mathbb{Z}/l^n\mathbb{Z} = \mu_{l^n}[1]$ and thus

$$\big(\nu^* \mathbb{G}_m[1]\big)\widehat{\otimes}\mathbb{Z}_l = R\lim \nu^* \mu_{l^n}[2] = \mathbb{Z}_l(1)[2]. \qquad ∎$$

Lemma 6.11. *Let H be an open normal subgroup of I_y acting trivially on F_η, and let $G := I_y/H$. Denote $R\underline{\mathrm{Hom}}_G$ the enriched R Hom between condensed $\underline{G}$-modules. There is a fiber sequence*

$$R\underline{\mathrm{Hom}}_G\big(\nu^* F_\eta, \mathbb{Z}_l(1)\big) \to R\underline{\mathrm{Hom}}_{\mathrm{Spec}(K_y^{\mathrm{sh}})_{\mathrm{proet}}}\big(\nu^* F_\eta, \mathbb{Z}_l(1)\big) \to R\underline{\mathrm{Hom}}_G(\nu^* F_\eta, \mathbb{Z}_l)[-1].$$

Proof. We have $\mathrm{Spec}(K_y^{\mathrm{sh}})_{\mathrm{proet}} = (BI_y)_{\mathrm{proet}} = I_y\text{-Cond}(Set)$ by Proposition C.7. The functor $R\underline{\Gamma}(H, -) := R\underline{\mathrm{Hom}}_H(\mathbb{Z}, -)$ is right adjoint to the forgetful functor

$$D\big(I_y\text{-Cond}(Ab)\big) \to D\big(\underline{G}\text{-Cond}(Ab)\big),$$

so we have

$$R\underline{\mathrm{Hom}}_{\mathrm{Spec}(K_y^{\mathrm{sh}})_{\mathrm{proet}}}\big(\nu^* F_\eta, \mathbb{Z}_l(1)\big) = R\underline{\mathrm{Hom}}_G\big(\nu^* F_\eta, R\underline{\Gamma}(H, \mathbb{Z}_l(1))\big).$$

We compute

$$R\underline{\Gamma}\big(H, \mathbb{Z}_l(1)\big) = R \lim R\underline{\Gamma}(H, \nu^* \mu_{l^n}) = R \lim \nu^* R\Gamma(H, \mu_{l^n}).$$

Denote F the finite extension of K_y^{sh} corresponding to H. Since l is prime to the residual characteristic of K_y^{sh}, all l^n-th roots of unity are contained in K_y^{sh} and thus also in H. Hensel's lemma together with the short exact sequence $0 \to \mathcal{O}_F^\times \to F^\times \to \mathbb{Z} \to 0$ then gives $H^i(H, \mu_{l^n}) = \mu_{l^n}, \mathbb{Z}/l^n\mathbb{Z}$ for $i = 0, 1$, while $H^i(H, \mu_{l^n}) = 0$ for $i \geq 2$ because I_y is of cohomological dimension 1 and therefore also H [22, Thm. 8.11 (b), Prop. 5.10]. Thus we have a fiber sequence of discrete G-modules

$$\mu_{l^n} \to R\Gamma(H, \mu_{l^n}) \to \mathbb{Z}/l^n\mathbb{Z}[-1].$$

The transition maps on both sides are surjective. Applying the fully faithful exact functor ν^*, we can compute the $R \lim$ using that the transition maps are again surjective and we obtain a fiber sequence in $D(\underline{G}\text{-Cond}(Ab))$:

$$\mathbb{Z}_l(1) \to R\underline{\Gamma}\big(H, \mathbb{Z}_l(1)\big) \to \mathbb{Z}_l[-1]. \qquad\blacksquare$$

Since K_y^{sh} has all l^n-th roots of unity, as $\underline{G}$-modules we have $\mathbb{Z}_l \simeq \mathbb{Z}_l(1)$.

Lemma 6.12. *We have*

$$R\underline{\mathrm{Hom}}_G(\nu^* F_\eta, \mathbb{Z}_l) \otimes \mathbb{Q} = \nu^* \mathrm{Hom}_G(F_\eta, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_l = \nu^*(F_\eta^\vee)^{I_y} \otimes \mathbb{Q}_l,$$
$$R\underline{\mathrm{Hom}}_G\big(\nu^* F_\eta, \mathbb{Z}_l(1)\big) \otimes \mathbb{Q} = \nu^*(F_\eta^\vee)^{I_y} \otimes \mathbb{Q}_l(1).$$

Proof. The proof of both statements is similar, so we treat only the first one.

We first show that $\underline{\mathrm{Ext}}_G^i(\nu^* F_\eta, \mathbb{Z}_l)$ is killed by some integer N for all $i \geq 1$. There is a spectral sequence giving short exact sequences

$$0 \to R^1 \lim \nu^* \mathrm{Ext}_G^{i-1}(F_\eta, \mathbb{Z}/l^n\mathbb{Z}) \to \underline{\mathrm{Ext}}_G^i(\nu^* F_\eta, \mathbb{Z}_l) \to \lim \nu^* \mathrm{Ext}_G^i(F_\eta, \mathbb{Z}/l^n\mathbb{Z}) \to 0.$$

Denote $\mathcal{H}om_G$ is the internal Hom for discrete G-modules. There is moreover a spectral sequence giving a long exact sequence

$$\cdots \to H^i(G, \mathcal{H}om_G(F_\eta, \mathbb{Z}/l^n\mathbb{Z})) \to \mathrm{Ext}^i_G(F_\eta, \mathbb{Z}/l^n\mathbb{Z}) \to H^{i-1}(G, \mathcal{E}xt^1_G(F_\eta, \mathbb{Z}/l^n\mathbb{Z}))$$

$$\to H^{i+1}(G, \mathcal{H}om_G(F_\eta, \mathbb{Z}/l^n\mathbb{Z})) \longrightarrow \cdots$$

For $i \geq 2$, since

$$\mathcal{H}om_G(F_\eta, \mathbb{Z}/l^n\mathbb{Z}) = \mathrm{Hom}_{\mathrm{Ab}}(F_\eta, \mathbb{Z}/l^n\mathbb{Z})$$

and

$$\mathcal{E}xt^1_G(F_\eta, \mathbb{Z}/l^n\mathbb{Z}) = \mathrm{Ext}^1_{\mathrm{Ab}}(F_\eta, \mathbb{Z}/l^n\mathbb{Z})$$

are of finite type, both $H^i(G, \mathcal{H}om_G(F_\eta, \mathbb{Z}/l^n\mathbb{Z}))$ and $H^{i-1}(G, \mathcal{E}xt^1_G(F_\eta, \mathbb{Z}/l^n\mathbb{Z}))$ are finite and killed by $[G]$, thus $\mathrm{Ext}^i_G(F_\eta, \mathbb{Z}/l^n\mathbb{Z})$ is killed by $[G]^2$. For $i = 1$, the group $H^1(G, \mathcal{H}om_G(F_\eta, \mathbb{Z}/l^n\mathbb{Z}))$ is finite killed by $[G]$ and $H^0(G, \mathcal{E}xt^1_G(F_\eta, \mathbb{Z}/l^n\mathbb{Z}))$ is finite killed by $[(F_\eta)_{\mathrm{tor}}]$, so $\mathrm{Ext}^1_G(F_\eta, \mathbb{Z}/l^n\mathbb{Z})$ is finite killed by $[G][(F_\eta)_{\mathrm{tor}}]$. Finally, for $i = 0$ we have that $\mathrm{Hom}_G(F_\eta, \mathbb{Z}/l^n\mathbb{Z})$ is a finite group.

Suppose first that $i \geq 2$. Then $\mathrm{Ext}^{i-1}_G(F_\eta, \mathbb{Z}/l^n\mathbb{Z})$ is a system of finite groups so it satisfies the Mittag-Leffler condition, thus $R^1 \lim \nu^* \mathrm{Ext}^{i-1}_G(F_\eta, \mathbb{Z}/l^n\mathbb{Z}) = 0$ by Lemmas C.2 and C.3. On the other hand, $\mathrm{Ext}^i_G(F_\eta, \mathbb{Z}/l^n\mathbb{Z})$ is a system of finite group killed by $[G]^2$, thus $\underline{\mathrm{Ext}}^i_G(\nu^*F_\eta, \mathbb{Z}_l) = \lim \nu^* \mathrm{Ext}^i_G(F_\eta, \mathbb{Z}/l^n\mathbb{Z})$ is killed by $[G]^2$. We now treat the case $i = 1$. Let us show that $R^1 \lim \nu^* \mathrm{Hom}_G(F_\eta, \mathbb{Z}/l^n\mathbb{Z}) = 0$. Again this is a system of finite groups, which thus satisfies the Mittag-Leffler condition, and we can apply Lemmas C.2 and C.3. It follows that $\underline{\mathrm{Ext}}^1_G(\nu^*F_\eta, \mathbb{Z}_l) = \lim \nu^* \mathrm{Ext}^1_G(F_\eta, \mathbb{Z}/l^n\mathbb{Z})$ is killed by $[G][(F_\eta)_{\mathrm{tor}}]$.

We deduce that

$$R\underline{\mathrm{Hom}}_G(\nu^*F_\eta, \mathbb{Z}_l) \otimes \mathbb{Q} = \underline{\mathrm{Hom}}_G(\nu^*F_\eta, \mathbb{Z}_l) \otimes \mathbb{Q}.$$

There are canonical maps

$$\underline{\mathrm{Hom}}_G(\nu^*F_\eta, \mathbb{Z}_l) \otimes \mathbb{Q} \to \underline{\mathrm{Hom}}_G(\nu^*F_\eta, \mathbb{Q}_l) \leftarrow \underline{\mathrm{Hom}}_G(\nu^*F_\eta, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_l$$

$$= \nu^* \mathrm{Hom}_G(\nu^*F_\eta, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_l.$$

Since F_η is a finite type, hence finite presentation abelian group and G is finite, F_η is a finite presentation G-module so ν^*F_η is (globally) of finite presentation as a $\underline{G}$-module[34]. By a standard argument, to show that the canonical maps are isomorphisms we reduce to the case of $\mathbb{Z}[G]$. But $\underline{\mathrm{Hom}}_G(\mathbb{Z}[G], -)$ is the forgetful functor $U : \underline{G} - \mathrm{Cond}(\mathrm{Ab}) \to \mathrm{Cond}(\mathrm{Ab})$, thus everything follows from the identifications

$$(U\mathbb{Z}_l) \otimes \mathbb{Q} = \mathbb{Z}_l \otimes \mathbb{Q} = \mathbb{Q}_l = U\mathbb{Q}_l = \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}_l = U\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}_l.$$

The second equality in each case follows from the identity

$$\mathrm{Hom}_G(F_\eta, \mathbb{Q}) = (F_\eta^\vee)^G \otimes \mathbb{Q} = (F_\eta^\vee)^{I_y} \otimes \mathbb{Q}. \qquad \blacksquare$$

---

[34]Note that as G is finite, $\mathbb{Z}[G] = \underline{\mathbb{Z}}[G]$.

6.2. Functoriality

Proposition 6.13. *Let* $0 \to F \to G \to H \to 0$ *be a short exact sequence of* $\mathbb{Z}$-*constructible sheaves on* X. *Then*

$$L_X(G^D, s) = L_X(F^D, s) L_X(H^D, s).$$

Proof. If $i : v \to X$ is a closed point, the short exact sequence gives a fiber triangle

$$(v^* i^* H^D) \widehat{\otimes} \mathbb{Q}_l \to (v^* i^* G^D) \widehat{\otimes} \mathbb{Q}_l \to (v^* i^* F^D) \widehat{\otimes} \mathbb{Q}_l.$$

Local factors are multiplicative with respect to short exact sequences, hence

$$L_v(G^D, s) = L_v(F^D, s) L_v(H^D, s). \qquad \blacksquare$$

Proposition 6.14. *Let* $\pi : Y \to X$ *be a finite morphism between spectra of orders in number fields and let* F *be a* $\mathbb{Z}$-*constructible sheaf on* Y. *We have*

$$L_X\big((\pi_* F)^D, s\big) = L_Y(F^D, s).$$

Proof. This follows readily from Theorem 6.2 using the compatibility of local factors with induction [35, §VII.10, Prop. 10.4 (iv) and its proof]. $\qquad \blacksquare$

We mention the following consequence of [31]:

Proposition 6.15. *Let* $X = \mathrm{Spec}(\mathcal{O})$ *be the spectrum of an order in a number field* K, *with open subscheme* $j : U \to X$. *Let* ω *be the number of roots of units in* K, Δ_K *its discriminant,* r_1 *and* r_2 *respectively the number of real and complex places of* K, *and* R_U *the regulator introduced in* [31] *for irreducible* 1-*dimensional arithmetic schemes. We have*

$$L_X^*\big((j_! \mathbb{Z})^D, 0\big) = \frac{2^{r_1} (2\pi)^{r_2} [\mathrm{CH}_0(U)] R_U}{\omega \sqrt{|\Delta_K|}}.$$

Proof. Denote $f : V \to X$ the regular locus of U seen as an open subscheme of X and Z its closed complement in U. Denote also $\pi : Y \to X$ the normalization of X, $f' : V \to Y$ the inclusion and K the function field of X and Y. We have $f = \pi f'$ and π induces an isomorphism between $\pi^{-1}(V)$ and V. Finally, denote T the closed complement of V in Y. We have for the L-functions of $\mathbb{Z}$-constructible sheaves:

$$L_X(j_! \mathbb{Z}) = \zeta_U = \zeta_Y \frac{\prod_{v \in Z} \zeta_v}{\prod_{w \in T} \zeta_w}.$$

Hence by [31, Thm. C], we get the relation

$$\frac{[\mathrm{CH}_0(U)] R_U}{\omega} = \frac{h_K R_K \prod_{w \in T} \log N(w)}{\omega \prod_{v \in Z} \log N(v)}$$

which simplifies to

$$[\mathrm{CH}_0(U)] R_U = \frac{h_K R_K \prod_{w \in T} \log N(w)}{\prod_{v \in Z} \log N(v)}.$$

On the other hand, we have similarly

$$L_X\big((j_!\mathbb{Z})^D\big) = L_Y(\mathbb{Z}^D)\frac{\prod_{v\in Z}\zeta_v}{\prod_{w\in T}\zeta_w}$$

hence

$$L_X^*\big((j_!\mathbb{Z})^D,0\big) = \zeta_Y^*(1)\frac{\prod_{v\in Z}\zeta_v^*(0)}{\prod_{w\in T}\zeta_w^*(0)}$$

$$= \frac{2^{r_1}(2\pi)^{r_2}h_K R_K \prod_{w\in T}\log N(w)}{\omega\sqrt{|\Delta_K|}\prod_{v\in Z}\log N(v)}$$

$$= \frac{2^{r_1}(2\pi)^{r_2}[\mathrm{CH}_0(U)]R_U}{\omega\sqrt{|\Delta_K|}}. \qquad\blacksquare$$

Remark. When X is singular we do not necessarily have $L_X(\mathbb{G}_X[1],s) = \zeta_X(s+1)$, as can be seen from Theorem 6.2. This formula is to be compared with the formula for $\zeta_U^*(1)$:

$$\zeta_U^*(1) = \frac{2^{r_1}(2\pi)^{r_2}h_K R_K}{\omega\sqrt{|\Delta_K|}}\frac{\prod_{v\in Z}1/(1-N(v)^{-1})}{\prod_{w\in T}1/(1-N(w)^{-1})}$$

$$= \frac{2^{r_1}(2\pi)^{r_2}[\mathrm{CH}_0(U)]R_U}{\omega\sqrt{|\Delta_K|}}\frac{\prod_{v\in Z}\log N(v)/(1-N(v)^{-1})}{\prod_{w\in T}\log N(w)/(1-N(w)^{-1})}$$

and also with the formula from [23].

Remark. We could have also obtained this using Theorem 7.13. Suppose that U is a proper subscheme of X. Denote S_f the finite places in the normalization $\pi : Y \to X$ that are not above U, and Z the closed complement of U. We can compute explicitly $R((j_!\mathbb{Z})^D)$, by considering the following snake diagram:
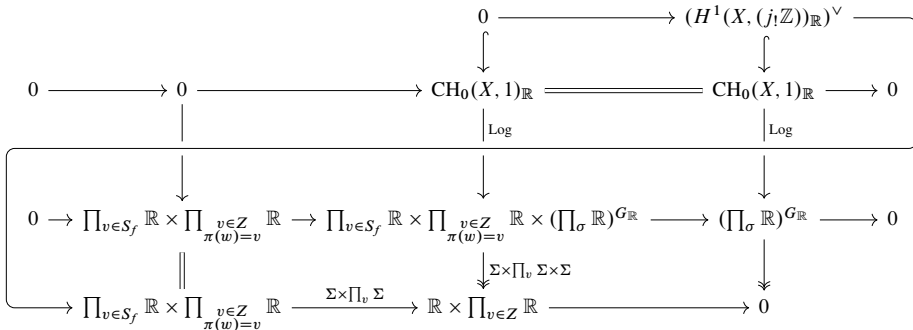
If $U = X$, we get a similar diagram with $H^0(X,\mathbb{Z})_{\mathbb{R}}^{\vee}$ instead of 0 at the bottom, enabling us to compute $R(\mathbb{Z}^D)$. We then obtain the formula using Proposition 7.9.

7. The Weil-étale Euler characteristic and the special value theorem

7.1. Definitions

Definition 7.1. Let F be a $\mathbb{Z}$-constructible sheaf on X. We define Weil–Arakelov cohomology with coefficients in F^D as the complex:

$$R\Gamma_{\mathrm{ar},c}(X, F_{\mathbb{R}}^D) := R\Gamma_{c,\mathcal{D}}(X, F_{\mathbb{R}}^D)[-1] \oplus R\Gamma_{c,\mathcal{D}}(X, F_{\mathbb{R}}^D).$$

The determinant of Weil–Arakelov cohomology has a canonical trivialization

$$\det_{\mathbb{R}} R\Gamma_{\mathrm{ar},c}(X, F_{\mathbb{R}}^D) \xrightarrow{\simeq} \det_{\mathbb{R}} \left(R\Gamma_{c,\mathcal{D}}(X, F_{\mathbb{R}}^D)\right)^{-1} \otimes \det_{\mathbb{R}} \left(R\Gamma_{c,\mathcal{D}}(X, F_{\mathbb{R}}^D)\right) \xrightarrow{\simeq} \mathbb{R}.$$

We now put an integral structure on the determinant of Weil–Arakelov cohomology. Let F be a tamely ramified red or blue sheaf. Recall that

$$\mathrm{Lie}_X(F^D)_{\mathbb{R}} = R\mathrm{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}\left(\alpha^* F, \mathbb{C}[1]\right).$$

Consider the map

$$R\Gamma_{W,c}(X, F^D)_{\mathbb{R}} = R\mathrm{Hom}\left(R\Gamma(X, F), \mathbb{R}[-1]\right) \oplus R\Gamma_{c,B}(X, F^D)_{\mathbb{R}}$$
$$\xrightarrow{p} R\Gamma_{c,B}(X, F^D)_{\mathbb{R}}$$
$$\xrightarrow{\mathrm{Log}} R\mathrm{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}\left(\alpha^* F, \mathbb{C}[1]\right).$$

By the duality Theorem 5.2 and the remark after Definition 5.1, its mapping fiber is

$$R\mathrm{Hom}\left(R\Gamma(X, F), \mathbb{R}[-1]\right) \oplus R\Gamma_{c,\mathcal{D}}(X, F_{\mathbb{R}}^D) \simeq R\Gamma_{\mathrm{ar},c}(X, F_{\mathbb{R}}^D).$$

We thus have a distinguished triangle

$$R\Gamma_{\mathrm{ar},c}(X, F_{\mathbb{R}}^D) \to R\Gamma_{W,c}(X, F^D)_{\mathbb{R}} \to \mathrm{Lie}_X(F^D)_{\mathbb{R}}$$

and we find by taking determinants a natural trivialization

$$\lambda : \Delta_X(F^D)_{\mathbb{R}}$$
$$= \det_{\mathbb{R}} \left(R\Gamma_{W,c}(X, F^D)_{\mathbb{R}}\right) \otimes \det_{\mathbb{R}} \left(\mathrm{Lie}_X(F^D)_{\mathbb{R}}\right)^{-1} \xrightarrow{\simeq} \det_{\mathbb{R}} \left(R\Gamma_{\mathrm{ar},c}(X, F_{\mathbb{R}}^D)\right) \xrightarrow{\simeq} \mathbb{R}.$$

Definition 7.2. Let F be a tamely ramified red or blue sheaf. The Weil-étale Euler characteristic of F^D is the positive real number $\chi_X(F^D)$ such that

$$\lambda\left(\Delta_X(F^D)\right) = \left(\chi_X(F^D)\right)^{-1} \mathbb{Z} \hookrightarrow \mathbb{R}.$$

Remark. If we let $\cup \theta$ be the map given by the composition of projections and inclusions

$$\cup \theta : R\Gamma_{\mathrm{ar},c}(X, F_{\mathbb{R}}^D) \to R\Gamma_{c,\mathcal{D}}(X, F_{\mathbb{R}}^D) \to R\Gamma_{\mathrm{ar},c}(X, F_{\mathbb{R}}^D)[1]$$

then $\cup\theta$ induces a long exact sequence

$$\cdots \to H_{\mathrm{ar},c}^{i-1}(X, F_{\mathbb{R}}^D) \xrightarrow{\cup\theta} H_{\mathrm{ar},c}^i(X, F_{\mathbb{R}}^D) \xrightarrow{\cup\theta} H_{\mathrm{ar},c}^{i+1}(X, F_{\mathbb{R}}^D) \to \cdots$$

which gives a trivialization

$$\lambda' : \bigotimes_{i\in\mathbb{Z}} \left(\det_{\mathbb{R}} H_{\mathrm{ar},c}^i(X, F_{\mathbb{R}}^D)\right)^{(-1)^i} \xrightarrow{\simeq} \mathbb{R}$$

which coincides with λ under the isomorphism

$$\det_{\mathbb{R}} R\Gamma_{\mathrm{ar},c}(X, F_{\mathbb{R}}^D) \simeq \bigotimes_{i\in\mathbb{Z}} \left(\det_{\mathbb{R}} H_{\mathrm{ar},c}^i(X, F_{\mathbb{R}}^D)\right)^{(-1)^i}.$$

We can thus define alternatively $\chi_X(F^D)$ such that

$$\lambda\left(\bigotimes_{i\in\mathbb{Z}} \left(\det_{\mathbb{Z}} H_{W,c}^i(X, F^D)\right)^{(-1)^i} \otimes \bigotimes_{i\in\mathbb{Z}} \left(\det_{\mathbb{Z}} H^i \operatorname{Lie}_X(F^D)\right)^{(-1)^{i+1}}\right)$$
$$= \left(\chi_X(F^D)\right)^{-1}\mathbb{Z} \hookrightarrow \mathbb{R}.$$

Theorem 7.3. *The Weil-étale Euler characteristic is multiplicative with respect to red-to-blue short exact sequences of tamely ramified sheaves.*

Proof. Let $0 \to F \to G \to H \to 0$ be a red-to-blue short exact sequence of tamely ramified sheaves. We let

$$T_1 : R\Gamma_{W,c}(X, H^D) \to R\Gamma_{W,c}(X, G^D) \to R\Gamma_{W,c}(X, F^D) \to$$
$$T_2 : \operatorname{Lie}_X(H^D) \to \operatorname{Lie}_X(G^D) \to \operatorname{Lie}_X(F^D) \to$$

be the two natural triangles; the first one is *not* distinguished. For $R = \mathbb{Z}, \mathbb{R}$, let g_R be a determinant functor on $\mathrm{Gr}^b(\mathrm{Mod}_R^{\mathrm{ft}})$ with values in graded R-lines extending the usual determinant functor on projective R-modules of finite type. The graded cohomology functor

$$H : D_{\mathrm{perf}}(R) \to \mathrm{Gr}^b(\mathrm{Mod}_R^{\mathrm{ft}})$$

together with the assignment that sends a distinguished triangle $T : X \xrightarrow{u} Y \to Z \to$ to the exact sequence

$$0 \to \ker\left(H(u)\right) \to H(X) \to H(Y) \to H(Z) \xrightarrow{\partial} \ker\left(H(u)\right)[1] \to 0$$

induced by the long exact sequence, induces a pullback functor H^* on Picard groupoids of determinants. We put $f_R = H^*g_R$. Notice that $H(T)$ makes sense for any triangle (not necessarily distinguished) that induces a long exact cohomology sequence. There is a base change isomorphism $\gamma : (g_\mathbb{Z}(-))_\mathbb{R} \xrightarrow{\simeq} g_\mathbb{R}((-)_\mathbb{R})$, which induces a base change isomorphism $H^*(\gamma) : (f_\mathbb{Z}(-))_\mathbb{R} \xrightarrow{\simeq} f_\mathbb{R}((-)_\mathbb{R})$. By Theorem 3.10, $H(T_1)$ is an exact sequence

in $\mathrm{Gr}^b(\mathrm{Mod}_\mathbb{Z}^{\mathrm{ft}})$, while $H(T_2)$ is because T_2 is distinguished. On the other hand, $T_{1,\mathbb{R}}$ and $T_{2,\mathbb{R}}$ are both distinguished triangles and the structure of exact sequence on $H(T_{1,\mathbb{R}})$ and $H(T_{2,\mathbb{R}})$ coming from the distinguished triangles coincides with the structure coming by base change from $H(T_1)$ and $H(T_2)$. We get a commutative diagram (see [31, Thm. 6.3 and its proof]):

$$
\begin{array}{ccc}
\begin{array}{c} f_\mathbb{Z}\big(R\Gamma_{W,c}(X, H^D)\big) \otimes f_\mathbb{Z}\big(R\Gamma_{W,c}(X, F^D)\big) \otimes \\ f_\mathbb{Z}\big(\mathrm{Lie}_X(H^D)\big)^{-1} \otimes f_\mathbb{Z}\big(\mathrm{Lie}_X(F^D)\big)^{-1} \end{array} & \xrightarrow{g_\mathbb{Z}(HT_1) \otimes g_\mathbb{Z}(HT_2)^{-1}} & \begin{array}{c} f_\mathbb{Z}\big(R\Gamma_{W,c}(X, G^D)\big) \otimes \\ f_\mathbb{Z}\big(\mathrm{Lie}_X(G^D)\big)^{-1} \end{array} \\
\downarrow & & \downarrow \\
\begin{array}{c} \big(f_\mathbb{Z}\big(R\Gamma_{W,c}(X, H^D)\big)\big)_\mathbb{R} \otimes \big(f_\mathbb{Z}\big(R\Gamma_{W,c}(X, F^D)\big)\big)_\mathbb{R} \otimes \\ \big(f_\mathbb{Z}\big(\mathrm{Lie}_X(H^D)\big)^{-1}\big)_\mathbb{R} \otimes \big(f_\mathbb{Z}\big(\mathrm{Lie}_X(F^D)\big)^{-1}\big)_\mathbb{R} \end{array} & \xrightarrow{(g_\mathbb{Z}(HT_1))_\mathbb{R} \otimes (g_\mathbb{Z}(HT_2)^{-1})_\mathbb{R}} & \begin{array}{c} \big(f_\mathbb{Z}\big(R\Gamma_{W,c}(X, G)\big)\big)_\mathbb{R} \otimes \\ \big(f_\mathbb{Z}\big(\mathrm{Lie}_X(G)\big)\big)_\mathbb{R} \end{array} \\
\Big\downarrow {\scriptstyle H^*(\gamma) \otimes H^*(\gamma) \otimes H^*(\gamma)^{-1} \otimes H^*(\gamma)^{-1}} & {\scriptstyle H^*(\gamma) \otimes H^*(\gamma)^{-1}} & \Big\downarrow \\
f_\mathbb{R}\big(\Delta_X(H^D)_\mathbb{R}\big) \otimes f_\mathbb{R}\big(\Delta_X(F^D)_\mathbb{R}\big) & \longrightarrow & f_\mathbb{R}\big(\Delta_X(F^D)_\mathbb{R}\big) \\
\downarrow & {\scriptstyle g_\mathbb{R}(H(T_{1,\mathbb{R}})) \otimes g_\mathbb{R}(H(T_{2,\mathbb{R}})^{-1}) = f_\mathbb{R}(T_{1,\mathbb{R}}) \otimes f_\mathbb{R}(T_{2,\mathbb{R}})^{-1}} & \downarrow \\
\mathbb{R} \otimes \mathbb{R} & \xrightarrow{\qquad \mathrm{mult} \qquad} & \mathbb{R}
\end{array}
$$

Under the multiplication map, the image of $x\mathbb{Z} \otimes y\mathbb{Z} \subset \mathbb{R} \otimes \mathbb{R}$ is $xy\mathbb{Z}$. ∎

Proposition 7.4. *Let $Y = \mathrm{Spec}(\mathcal{O}')$ be the spectrum of an order in a number field with a finite dominant morphism $\pi : Y \to X$. Suppose that $K(Y)/K$ is tamely ramified, and let F be a tamely ramified red or blue sheaf on Y. Then*

$$
\chi_X\big((\pi_* F)^D\big) = \chi_Y(F^D).
$$

Proof. This follows from the isomorphism

$$
R\Gamma_{W,c}(Y, F^D) \xrightarrow{\simeq} R\Gamma_{W,c}\big(X, (\pi_* F)^D\big)
$$
$$
\mathrm{Lie}_Y(F^D) \xrightarrow{\simeq} \mathrm{Lie}_X\big((\pi_* F)^D\big)
$$
$$
R\Gamma_{c,\mathscr{D}}(Y, F_\mathbb{R}^D) \xrightarrow{\simeq} R\Gamma_{c,\mathscr{D}}\big(X, (\pi_* F)_\mathbb{R}^D\big)
$$

and the isomorphism of distinguished triangles

$$
\begin{array}{ccccc}
R\Gamma_{\mathrm{ar},c}(Y, F_\mathbb{R}^D) & \longrightarrow & R\Gamma_{W,c}(Y, F^D)_\mathbb{R} & \longrightarrow & \mathrm{Lie}_Y(F^D)_\mathbb{R} \longrightarrow \\
{\scriptstyle \simeq}\Big\downarrow & & {\scriptstyle \simeq}\Big\downarrow & & {\scriptstyle \simeq}\Big\downarrow \\
R\Gamma_{\mathrm{ar},c}\big(X, (\pi_* F)_\mathbb{R}^D\big) & \longrightarrow & R\Gamma_{W,c}\big(X, (\pi_* F)^D\big)_\mathbb{R} & \longrightarrow & \mathrm{Lie}_X\big((\pi_* F)^D\big)_\mathbb{R} \longrightarrow
\end{array}
$$
∎

We now use the multiplicativity of the Weil-étale Euler characteristic to extend it to arbitrary tamely ramified $\mathbb{Z}$-constructible sheaves:

Definition 7.5. Let F be a tamely ramified $\mathbb{Z}$-constructible sheaf, let $j : U \to X$ be an open subscheme such that $F_{|U}$ is locally constant and let $i : Z \to X$ be the closed complement. The sheaves $j_! F_{|U}$ and $i_* i^* F$ are tamely ramified and respectively red and blue, and we define the Weil-étale Euler characteristic of F^D as

$$\chi_X(F^D) = \chi_X\big((j_! F_{|U})^D\big) \chi_X\big((i_* i^* F)^D\big).$$

The definition does not depend on the choice of U. The Weil-étale Euler characteristic is multiplicative with respect to short exact sequences of tamely ramified sheaves. Let $Y = \mathrm{Spec}(\mathcal{O}')$ be the spectrum of an order in a number field with a finite dominant morphism $\pi : Y \to X$ such that $K(Y)/K$ is tamely ramified, and let F be a tamely ramified $\mathbb{Z}$-constructible sheaf on Y. Then

$$\chi_X\big((\pi_* F)^D\big) = \chi_Y(F^D).$$

Proof. This follows formally from the previous results by using the open-closed decomposition lemma, see [31, §6.5]. ∎

7.2. Computations of the Euler characteristic, and the special value theorem

We first give an explicit expression of our Euler characteristic which does not involve Weil-étale cohomology anymore. This will allow us to give an expression valid for any tamely ramified $\mathbb{Z}$-constructible sheaf.

Proposition 7.6. *Let F be a tamely ramified red or blue sheaf. Let $R_1(F^D)$ be the absolute value of the determinant of the pairing*

$$H^{-1}_{c,\mathscr{D}}(X, F^D_{\mathbb{R}}) \times H^1(X, F)_{\mathbb{R}} \to \mathbb{R}$$

and $R_0(F^D)$ the absolute value of the determinant of the pairing

$$H^0_{c,\mathscr{D}}(X, F^D_{\mathbb{R}}) \times H^0(X, F)_{\mathbb{R}} \to \mathbb{R}$$

in the following bases : pick bases modulo torsion of $H^{-1}(\mathrm{Lie}_X(F^D))$, of $H^i(X, F)$ for $i = 0, 1$ and of $H^i_{c,B}(X, F^D)$ for $i = -1, 0$. Then pick any $\mathbb{R}$-bases of $H^i_{c,\mathscr{D}}(X, F^D_{\mathbb{R}})$ compatible[35] with the chosen bases in the following exact sequence

$$0 \to H^{-1}_{c,\mathscr{D}}(X, F^D_{\mathbb{R}}) \to H^{-1}_{c,B}(X, F^D)_{\mathbb{R}} \to H^{-1}\big(\mathrm{Lie}_X(F^D)\big)_{\mathbb{R}}$$
$$\to H^0_{c,\mathscr{D}}(X, F^D_{\mathbb{R}}) \to H^0_{c,B}(X, F^D)_{\mathbb{R}} \to 0.$$

We have

$$\chi_X(F^D) = \frac{\big[H^0(X, F)_{\mathrm{tor}}\big]\big[H^0_{c,B}(X, F^D)_{\mathrm{tor}}\big]}{\big[H^1(X, F)_{\mathrm{tor}}\big]\big[H^{-1}_{c,B}(X, F^D)_{\mathrm{tor}}\big]\big[H^0\big(\mathrm{Lie}_X(F^D)\big)\big]} \frac{R_1(F^D)}{R_0(F^D)}.$$

---

[35]In the following sense: denote $\mathcal{E}^\bullet$ the exact sequence seen as an acyclic complex, then the choice of bases gives an element of $\det_{\mathbb{R}} \mathcal{E}^\bullet$ which is required to be sent to 1 under $\det_{\mathbb{R}} \mathcal{E}^\bullet \xrightarrow[\simeq]{\det(0)} \mathbb{R}$.

Remark. We can reformulate the part about determinants of pairings. Consider an acyclic chain complex of $\mathbb{R}$-vector spaces $A^\bullet$ with a $\mathbb{Z}$-lattice $M^i \subset A^i$ given for each i. We define $\det_{\mathbb{Z},\mathbb{R}} A^\bullet$ to be the absolute value of the image of the vector corresponding to bases of the lattices under the canonical isomorphism $\det_{\mathbb{R}} A^\bullet \xrightarrow[\det_{\mathbb{R}}(0)]{\simeq} \mathbb{R}$; put another way, it is the positive real number such that

$$\bigotimes_{i \in \mathbb{Z}} (\det_{\mathbb{Z}} M^i)^{(-1)^i} \subset \Big(\bigotimes_{i \in \mathbb{Z}} (\det_{\mathbb{Z}} M^i)^{(-1)^i} \Big)_{\mathbb{R}} \simeq \det_{\mathbb{R}} A^\bullet$$

corresponds to $(\det_{\mathbb{Z},\mathbb{R}} A^\bullet)\mathbb{Z} \subset \mathbb{R}$ under the canonical isomorphism $\det_{\mathbb{R}} A^\bullet \xrightarrow[\det_{\mathbb{R}}(0)]{\simeq} \mathbb{R}$. Consider the following complex

$$A^\bullet : \quad 0 \to \big(H^1(X, F)_{\mathbb{R}} \big)^\vee \to H^{-1}_{c,B}(X, F^D)_{\mathbb{R}} \to H^{-1}\big(\mathrm{Lie}_X(F^D) \big)_{\mathbb{R}}$$
$$\to \big(H^0(X, F)_{\mathbb{R}} \big)^\vee \to H^0_{c,B}(X, F^D)_{\mathbb{R}} \to 0$$

with first non-zero term in degree -1 and endowed with the natural lattices. Then we have

$$\frac{R_1(F^D)}{R_0(F^D)} = \det_{\mathbb{Z},\mathbb{R}} A^\bullet.$$

Proof. Since determinants depend only on cohomology and are multiplicative in short exact sequences, we can write $\Delta_X(F^D)$ as an alternating tensor product of the determinants of the torsion and torsion-free parts of the cohomology groups:

$$\Delta_X(F^D) = \Delta_X(F^D)_{\mathrm{tor}} \Delta_X(F^D)/\mathrm{tor}.$$

We have $(\Delta_X(F^D)_{\mathrm{tor}})_{\mathbb{R}} = \det_{\mathbb{R}}(0) = \mathbb{R}$ canonically hence by [31, Lem. A.1] the contribution of $\Delta_X(F^D)_{\mathrm{tor}}$ inside $\Delta_X(F^D)_{\mathbb{R}}$ is sent under the trivialization to

$$1/\chi_X(F^D)_{\mathrm{tor}} := \frac{\prod \big[H^i \, \mathrm{Lie}_X(F^D)_{\mathrm{tor}} \big]^{(-1)^i}}{\prod \big[H^i_{W,c}(X, F^D)_{\mathrm{tor}} \big]^{(-1)^i}}.$$

We obtain the first part of the claimed expression by the computation of the involved cohomology groups in Proposition 3.5 and Corollary 4.6.

It remains to show that $\Delta_X(F^D)/\mathrm{tor} = \frac{R_1(F^D)}{R_0(F^D)}$. We will use [31, Lem. A.2]. Fix basis vectors of $H^i(X, F)/\mathrm{tor}$ for $i = 0, 1$, of $(H^{-1} \, \mathrm{Lie}_X(F^D))/\mathrm{tor}$ and of $H^i_{c,B}(X, F^D)$ for $i = -1, 0$. The image of $\Delta_X(F^D)/\mathrm{tor}$ inside

$$\Delta_X(F^D)_{\mathbb{R}} := \det_{\mathbb{R}} R\Gamma_{W,c}(X, F^D)_{\mathbb{R}} \otimes_{\mathbb{R}} \big(\det_{\mathbb{R}} \mathrm{Lie}_X(F^D)_{\mathbb{R}} \big)^{-1}$$
$$\simeq \det_{\mathbb{R}} \big(D_{F,\mathbb{R}}[1] \big) \otimes_{\mathbb{R}} \det_{\mathbb{R}} R\Gamma_{c,B}(X, F^D)_{\mathbb{R}} \otimes_{\mathbb{R}} \big(\det_{\mathbb{R}} \mathrm{Lie}_X(F^D)_{\mathbb{R}} \big)^{-1}$$
$$\simeq \det_{\mathbb{R}} \big(D_{F,\mathbb{R}}[1] \big) \otimes_{\mathbb{R}} \det_{\mathbb{R}} R\Gamma_{c,\mathcal{D}}(X, F^D_{\mathbb{R}})$$
$$\simeq \bigotimes_i \big(\det_{\mathbb{R}} H^{i+1}(D_{F,\mathbb{R}}) \big)^{(-1)^i} \otimes_{\mathbb{R}} \bigotimes_i \big(\det_{\mathbb{R}} H^i_{c,\mathcal{D}}(X, F^D_{\mathbb{R}}) \big)^{(-1)^i}$$

is a certain product of basis vectors and dual basis vectors, obtained from the basis vectors of $H^i_{c,B}(X, F^D)/\mathrm{tor}$, $H^i \mathrm{Lie}_X(F^D)/\mathrm{tor}$ and $H^i(X, F)/\mathrm{tor}$ through the identity

$$H^i(D_{F,\mathbb{R}}) = \mathrm{Hom}\left(H^{2-i}(X, F), \mathbb{R}\right)$$

and the long exact sequence associated to the fiber sequence

$$R\Gamma_{c,\mathcal{D}}(X, F^D_{\mathbb{R}}) \to R\Gamma_{c,B}(X, F^D)_{\mathbb{R}} \to \mathrm{Lie}_X(F^D)_{\mathbb{R}}.$$

Finally, the trivialization is obtained by combining together, for each $i \in \mathbb{Z}$, the terms related by the duality theorem Theorem 5.2:

$$\left(\det_{\mathbb{R}} H^i_{c,\mathcal{D}}(X, F^D_{\mathbb{R}})\right)^{(-1)^i} \otimes \left(\det_{\mathbb{R}} H^{i+2}(D_{F,\mathbb{R}})\right)^{(-1)^{i+1}}$$

$$\xrightarrow{\simeq} \left(\det_{\mathbb{R}} H^i_{c,\mathcal{D}}(X, F^D_{\mathbb{R}})\right)^{(-1)^i} \bigotimes \left(\det_{\mathbb{R}} H^i_{c,\mathcal{D}}(X, F^D_{\mathbb{R}})\right)^{(-1)^{i+1}}$$

$$\xrightarrow{\simeq} \mathbb{R}.$$

The isomorphism $H^i_{c,\mathcal{D}}(X, F^D_{\mathbb{R}}) \xrightarrow{\simeq} H^{i+2}(D_{F,\mathbb{R}})$ is non-trivial only for $i = -1, 0$ in which case its determinant in the bases obtained is exactly (up to sign) R_1, resp. R_0. By [31, Lem. A.2], the contribution of $\Delta_X(F^D)/\mathrm{tor}$ is thus sent under the trivialization to

$$\frac{1}{\chi_X(F^D)/\mathrm{tor}} := R_0/R_1. \qquad \blacksquare$$

Corollary 7.7. *Let F be a tamely ramified $\mathbb{Z}$-constructible sheaf. With the same notations, we have*

$$\chi_X(F^D) = \frac{\left[H^0(X, F)_{\mathrm{tor}}\right]\left[H^0_{c,B}(X, F^D)_{\mathrm{tor}}\right]}{\left[H^1(X, F)_{\mathrm{tor}}\right]\left[H^{-1}_{c,B}(X, F^D)_{\mathrm{tor}}\right]\left[H^0\left(\mathrm{Lie}_X(F^D)\right)\right]} \frac{R_1(F^D)}{R_0(F^D)}.$$

Proof. Let $j : U \to X$ be an open subscheme such that $F_{|U}$ is locally constant and let $i : Z \to X$ be the closed complement. Put $F_U = j_! F_{|U}$ and $F_Z = i_* i^* F$. Then F_U is red, F_Z is blue and there is a short exact sequence $0 \to F_U \to F \to F_Z \to 0$. Consider the two following acyclic complexes of abelian groups, with first non-zero term respectively in degree -1 and 0:

$$A^\bullet : 0 \to H^{-1}_{c,B}\left(X, (F_Z)^D\right) \to H^{-1}_{c,B}(X, F^D) \to H^{-1}_{c,B}\left(X, (F_U)^D\right)$$
$$\to H^0_{c,B}\left(X, (F_Z)^D\right) \to H^0_{c,B}(X, F^D) \to H^0_{c,B}\left(X, (F_U)^D\right) \to I \to 0,$$
$$B^\bullet : 0 \to H^0(X, F_U) \to H^0(X, F) \to H^0(X, F_Z)$$
$$\to H^1(X, F_U) \to H^1(X, F) \to H^1(X, F_Z) \to J \to 0$$

where I is the image of $H^0_{c,B}(X, F^D_U)$ in $H^1_{c,B}(X, F^D_Z)$ and J is the image of $H^1(X, F_Z)$ in $H^2(X, F_U)$. Since F_Z is blue, $H^1_{c,B}(X, F^D_Z)$ and $H^1(X, F_Z)$ are finite so I and J are

also finite. Moreover Artin–Verdier duality gives a commutative square

$$
\begin{array}{ccc}
H^0_{c,B}\big(X, (F_U)^D\big) & \xrightarrow{\ \partial\ } & H^1_{c,B}\big(X, (F_Z)^D\big) \\
\Big\downarrow{\simeq} & & \Big\downarrow{\simeq} \\
H^2(X, F_U)^* & \xrightarrow{\ \partial\ } & H^1(X, F_Z)^*
\end{array}
$$

whence $I \simeq J^*$.

Since F_Z is supported on a finite closed subscheme, we have $\mathrm{Lie}_X\big((F_Z)^D\big) = 0$ hence an isomorphism $\mathrm{Lie}_X(F^D) \xrightarrow{\simeq} \mathrm{Lie}_X((F_U)^D)$ which gives a canonical isomorphism

$$
\det_{\mathbb{Z}} \mathrm{Lie}_X(F^D)\big(\det_{\mathbb{Z}} \mathrm{Lie}_X\big((F_U)^D\big)\big)^{-1} \xrightarrow[\tau]{\simeq} \mathbb{Z}.
$$

On the other hand, since $A^\bullet$ and $B^\bullet$ are acyclic, we obtain canonical trivializations

$$
\det_{\mathbb{Z}} A^\bullet \xrightarrow[\det 0]{\simeq} \mathbb{Z}, \quad \det_{\mathbb{Z}} B^\bullet \xrightarrow[\det 0]{\simeq} \mathbb{Z}.
$$

Consider now the line

$$
\Delta := \det_{\mathbb{Z}} A^\bullet \det_{\mathbb{Z}} B^\bullet \det_{\mathbb{Z}} \mathrm{Lie}_X(F^D)\big(\det_{\mathbb{Z}} \mathrm{Lie}_X\big((F_U)^D\big)\big)^{-1}.
$$

It obtains a canonical trivialization $\alpha : \Delta \xrightarrow{\det 0 \otimes \det 0 \otimes \tau} \mathbb{Z} \otimes \mathbb{Z} \otimes \mathbb{Z} \xrightarrow{\simeq} \mathbb{Z}$ through the previous remarks.

For a distinguished triangle $X \to Y \to Z \to X[1]$, denote $i : (\det Y)^{-1} \det X \det Z \to 1$ the structural isomorphism to the unit. From the octahedral diagram

$$
\begin{array}{ccc}
R\Gamma_{c,\mathcal{D}}\big(X, (F_Z)^D_{\mathbb{R}}\big) & \longrightarrow & R\Gamma_{c,\mathcal{D}}(X, F^D_{\mathbb{R}}) & \longrightarrow & R\Gamma_{c,\mathcal{D}}\big(X, (F_U)^D_{\mathbb{R}}\big) \\
\Big\downarrow{\simeq} & & \Big\downarrow & & \Big\downarrow \\
R\Gamma_{c,B}\big(X, (F_Z)^D\big)_{\mathbb{R}} & \longrightarrow & R\Gamma_{c,B}(X, F^D)_{\mathbb{R}} & \longrightarrow & R\Gamma_{c,B}\big(X, (F_U)^D\big)_{\mathbb{R}} \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
\mathrm{Lie}_X\big((F_Z)^D\big)_{\mathbb{R}} = 0 & \longrightarrow & \mathrm{Lie}_X(F)_{\mathbb{R}} & \xrightarrow{\simeq} & \mathrm{Lie}_X\big((F_U)^D\big)_{\mathbb{R}}
\end{array}
$$

we find using the associativity axiom of determinant functors [6, §3.1] a commutative diagram

$$
\begin{array}{ccc}
\big(\det_{\mathbb{R}} R\Gamma_{c,\mathcal{D}}(X, F^D_{\mathbb{R}})\big)^{-1} \det_{\mathbb{R}} R\Gamma_{c,\mathcal{D}}\big(X, (F_U)^D_{\mathbb{R}}\big) \det_{\mathbb{R}} R\Gamma_{c,\mathcal{D}}\big(X, (F_Z)^D_{\mathbb{R}}\big) & \xrightarrow{\ i\ } & \mathbb{R} \\
\Big\downarrow{\simeq} & & \Big\uparrow \\
\begin{array}{c}\big(\det_{\mathbb{R}} R\Gamma_{c,B}(X, F^D)_{\mathbb{R}}\big)^{-1} \det_{\mathbb{R}} R\Gamma_{c,B}\big(X, (F_U)^D\big)_{\mathbb{R}} \\ \det_{\mathbb{R}} R\Gamma_{c,B}\big(X, (F_Z)^D\big)_{\mathbb{R}} \det_{\mathbb{R}} \mathrm{Lie}_X(F^D)_{\mathbb{R}} \\ \big(\det_{\mathbb{R}} \mathrm{Lie}_X\big((F_U)^D\big)_{\mathbb{R}}\big)^{-1}\end{array} & \xrightarrow{i_{\mathbb{R}} \otimes \tau_{\mathbb{R}}} & \mathbb{R} \otimes_{\mathbb{R}} \mathbb{R}
\end{array}
$$

$$(7.1)$$

There are isomorphisms

$$\det_{\mathbb{R}} A_{\mathbb{R}}^{\bullet} \simeq \big(\det_{\mathbb{R}} R\Gamma_{c,B}(X, F^D)_{\mathbb{R}}\big)^{-1} \det_{\mathbb{R}} R\Gamma_{c,B}\big(X, (F_U)^D\big)_{\mathbb{R}} \det_{\mathbb{R}} R\Gamma_{c,B}\big(X, (F_Z)^D\big)_{\mathbb{R}}, \quad (7.2)$$

$$\det_{\mathbb{R}} B_{\mathbb{R}}^{\bullet} \simeq \big(\det_{\mathbb{R}} R\Gamma(X, F)_{\mathbb{R}}\big)^{(-1)} \det_{\mathbb{R}} R\Gamma(X, F_U)_{\mathbb{R}} \det_{\mathbb{R}} R\Gamma(X, F_Z)_{\mathbb{R}} \quad (7.3)$$

under which the trivialization $\det(0)_{\mathbb{R}} = \det_{\mathbb{R}}(0)$ on the left corresponds to $i_{\mathbb{R}}$ on the right. By combining (7.1) to (7.3), we find a commutative diagram of isomorphisms

$$\begin{array}{ccc} \Delta_{\mathbb{R}} & \xrightarrow{\simeq} & \dfrac{\big(\det_{\mathbb{R}} R\Gamma_{c,\mathcal{D}}(X, F_{\mathbb{R}}^D)\big)^{(-1)} \det_{\mathbb{R}} R\Gamma_{c,\mathcal{D}}\big(X, (F_U)_{\mathbb{R}}^D\big) \det_{\mathbb{R}} R\Gamma_{c,\mathcal{D}}\big(X, (F_Z)_{\mathbb{R}}^D\big)}{\big(\det_{\mathbb{R}} R\Gamma(X, F)_{\mathbb{R}}\big)^{(-1)} \det_{\mathbb{R}} R\Gamma(X, F_U)_{\mathbb{R}} \det_{\mathbb{R}} R\Gamma(X, F_Z)_{\mathbb{R}}} \\[2ex] {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle i \cdot i_{\mathbb{R}}} \\[1ex] \mathbb{R} & =\!=\!=\!=\!=\!=\!= & \mathbb{R} \end{array}$$

We now describe how to produce another trivialization of $\Delta_{\mathbb{R}}$. By Theorem 5.2, there is an isomorphism

$$\psi_{(-)} : R\Gamma_{c,\mathcal{D}}\big(X, (-)_{\mathbb{R}}^D\big) \xrightarrow{\simeq} R\Gamma\big(X, (-)\big)_{\mathbb{R}}^{\vee}.$$

If L is a graded $\mathbb{R}$-line, there is a canonical isomorphism $L \otimes L^{-1} \xrightarrow{\delta} \mathbb{R}$ and we take the trivialization

$$\dfrac{\big(\det_{\mathbb{R}} R\Gamma_{c,\mathcal{D}}(X, F_{\mathbb{R}}^D)\big)^{(-1)} \det_{\mathbb{R}} R\Gamma_{c,\mathcal{D}}\big(X, (F_U)_{\mathbb{R}}^D\big) \det_{\mathbb{R}} R\Gamma_{c,\mathcal{D}}\big(X, (F_Z)_{\mathbb{R}}^D\big)}{\big(\det_{\mathbb{R}} R\Gamma(X, F)_{\mathbb{R}}\big)^{(-1)} \det_{\mathbb{R}} R\Gamma(X, F_U)_{\mathbb{R}} \det_{\mathbb{R}} R\Gamma(X, F_Z)_{\mathbb{R}}}$$

$$\downarrow$$

$$\dfrac{\det_{\mathbb{R}} R\Gamma(X, F)_{\mathbb{R}} \big(\det_{\mathbb{R}} R\Gamma(X, F_U)_{\mathbb{R}}\big)^{(-1)} \big(\det_{\mathbb{R}} R\Gamma(X, F_Z)_{\mathbb{R}}\big)^{(-1)}}{\big(\det_{\mathbb{R}} R\Gamma(X, F)_{\mathbb{R}}\big)^{(-1)} \det_{\mathbb{R}} R\Gamma(X, F_U)_{\mathbb{R}} \det_{\mathbb{R}} R\Gamma(X, F_Z)_{\mathbb{R}}}$$

$$\downarrow{\scriptstyle \delta \otimes \delta \otimes \delta}$$

$$\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R} \otimes_{\mathbb{R}} \mathbb{R}$$

$$\downarrow{\scriptstyle \text{mult}}$$

$$\mathbb{R}$$

Consider Diagram 4. From the isomorphism of fiber sequences

$$\begin{array}{ccccc} R\Gamma_{c,\mathcal{D}}\big(X, (F_Z)_{\mathbb{R}}^D\big) & \longrightarrow & R\Gamma_{c,\mathcal{D}}(X, F_{\mathbb{R}}^D) & \longrightarrow & R\Gamma_{c,\mathcal{D}}\big(X, (F_U)_{\mathbb{R}}^D\big) \\[1ex] \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\downarrow \\[1ex] R\Gamma(X, F_Z)_{\mathbb{R}}^{\vee} & \longrightarrow & R\Gamma(X, F)_{\mathbb{R}}^{\vee} & \longrightarrow & R\Gamma(X, F_U)_{\mathbb{R}}^{\vee} \end{array}$$

we deduce that the upper triangle in Diagram 4 commutes, while the lower square commutes formally. Thus the trivialization by duality equals the trivialization $i \cdot i_{\mathbb{R}}$.

$(\det_{\mathbb{R}} R\Gamma_{c,\mathcal{D}}(X, F_{\mathbb{R}}^D))^{(-1)} \det_{\mathbb{R}} R\Gamma_{c,\mathcal{D}}(X, (F_U)_{\mathbb{R}}^D) \det_{\mathbb{R}} R\Gamma_{c,\mathcal{D}}(X, (F_Z)_{\mathbb{R}}^D)$

$(\det_{\mathbb{R}} R\Gamma(X, F)_{\mathbb{R}})^{(-1)} \det_{\mathbb{R}} R\Gamma(X, F_U)_{\mathbb{R}} \det_{\mathbb{R}} R\Gamma(X, F_Z)_{\mathbb{R}}$

$\xrightarrow{((\det \psi_F)^t)^{-1} \det \psi_{F_U} \det \psi_{F_Z}}$

$\det_{\mathbb{R}} R\Gamma(X, F)(\det_{\mathbb{R}} R\Gamma(X, F_U))^{(-1)}(\det_{\mathbb{R}} R\Gamma(X, F_Z))^{(-1)}$

$(\det_{\mathbb{R}} R\Gamma(X, F)_{\mathbb{R}})^{(-1)} \det_{\mathbb{R}} R\Gamma(X, F_U)_{\mathbb{R}} \det_{\mathbb{R}} R\Gamma(X, F_Z)_{\mathbb{R}}$

$i \otimes i_{\mathbb{R}}$

$i \cdot i_{\mathbb{R}}$

$((i_{\mathbb{R}})^t)^{-1} \otimes i_{\mathbb{R}}$

$\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R}$

$\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R} \otimes_{\mathbb{R}} \mathbb{R}$

mult

mult

$\mathbb{R}$

Diagram 4.

Under those equal trivializations, the image of Δ inside $\mathbb{R}$ is $\mathbb{Z}$ on the one hand and is computed with [31, Prop. A.3] on the other hand. We get an identity

$$
\frac{\left[H^0(X, F)_{\text{tor}}\right]\left[H^0_{c,B}(X, F^D)_{\text{tor}}\right]}{\left[H^1(X, F)_{\text{tor}}\right]\left[H^{-1}_{c,B}(X, F^D)_{\text{tor}}\right]\left[H^0\left(\text{Lie}_X(F^D)\right)\right]} \frac{R_1(F^D)}{R_0(F^D)} \frac{[J]}{[I]} \frac{1}{\chi_X\left((F_U)^D\right)\chi_X\left((F_Z)^D\right)}
$$
$$
= 1
$$

and thus, because $[I] = [J]$:

$$
\chi_X(F^D) := \chi_X\left((F_U)^D\right)\chi_X\left((F_Z)^D\right)
$$
$$
= \frac{\left[H^0(X, F)_{\text{tor}}\right]\left[H^0_{c,B}(X, F^D)_{\text{tor}}\right]}{\left[H^1(X, F)_{\text{tor}}\right]\left[H^{-1}_{c,B}(X, F^D)_{\text{tor}}\right]\left[H^0\left(\text{Lie}_X(F^D)\right)\right]} \frac{R_1(F^D)}{R_0(F^D)}. \qquad \blacksquare
$$

Definition 7.8. Let F be a tamely ramified $\mathbb{Z}$-constructible sheaf. We put

$$
r_1(F) = \log_2\left(\left[\text{Ext}^1_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^* F, 2i\pi\mathbb{Z})\right]\right),
$$
$$
r_2(F) = \text{rank}_{\mathbb{Z}}\,\text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^* F, 2i\pi\mathbb{Z}).
$$

For each archimedean place v of K, fix a corresponding embedding σ_v. There is an iso-morphism[36]

$$
\text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^* F, \mathbb{R}) \xrightarrow{\widetilde{}} \prod_v \text{Hom}_{G_v}(F_v, \mathbb{Z})_{\mathbb{R}}
$$

given by $(\phi_\sigma) \mapsto (\phi_{\sigma_v})_{v\ \text{real}}, (2\phi_{\sigma_v})_{v\ \text{complex}}$. Using Proposition 4.8, the short exact sequence $0 \to 2i\pi\mathbb{R} \to \mathbb{C} \xrightarrow{\Re} \mathbb{R} \to 0$ and the above isomorphism, we can consider the following acyclic complex of $\mathbb{R}$-vector spaces with lattices:

$$
D^\bullet : 0 \to \text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^* F, 2i\pi\mathbb{Z})_{\mathbb{R}} \to H^{-1}\text{Lie}_X(F^D)_{\mathbb{R}} \to \prod_v \text{Hom}_{G_v}(F_v, \mathbb{Z})_{\mathbb{R}} \to 0
$$

(with first non-zero term in degree -1). We denote

$$
\text{Disc}(F) = (2\pi)^{r_2(F)} \det_{\mathbb{Z}, \mathbb{R}} D^\bullet.
$$

Remark. • If $\widetilde{D}^\bullet$ is the complex

$$
0 \to \text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^* F, i\mathbb{Z})_{\mathbb{R}} \to H^{-1}\text{Lie}_X(F^D)_{\mathbb{R}} \to \prod_v \text{Hom}_{G_v}(F_v, \mathbb{Z})_{\mathbb{R}} \to 0
$$

with similar degree conventions, then $\text{Disc}(F) = \det_{\mathbb{Z}, \mathbb{R}} \widetilde{D}^\bullet$.

• The quantity $\text{Disc}(F)$ only depends on F_η.

---

[36]Non-canonical; the particular choice here is justified by the later computation of $\text{Disc}(F)$ and $R(F^D)$ for the case $F = \mathbb{Z}$.

We now reformulate the previous expression obtained to make it closer to a formula looking like the analytic class number formula.

Proposition 7.9. *Let F be a tamely ramified $\mathbb{Z}$-constructible sheaf. Denote*

$$N_2 := \mathrm{Ker}\left(\mathrm{Ext}^2_{G_\mathbb{R}, X(\mathbb{C})}(\alpha^* F, 2i\pi\mathbb{Z}) \to H^1_{c,B}(X, F^D)\right)$$

and let $\mathbb{G}_X = [g_* \mathbb{G}_m \to \oplus_{x \in X_0} i_{x,*} \mathbb{Z}]$ denote Deninger's dualizing complex. Let $R(F^D) := \det_{\mathbb{Z}, \mathbb{R}} B^\bullet$ with $B^\bullet$ the exact sequence of $\mathbb{R}$-vector spaces with lattices

$$B^\bullet: \quad 0 \to \left(H^1(X, F)_\mathbb{R}\right)^\vee \to \mathrm{Hom}_X(F, \mathbb{G}_X)_\mathbb{R} \to \prod_v \mathrm{Hom}_{G_v}(F_v, \mathbb{Z})_\mathbb{R}$$

$$\to \left(H^0(X, F)_\mathbb{R}\right)^\vee \to \mathrm{Ext}^1_X(F, \mathbb{G}_X)_\mathbb{R} \to 0$$

with first non-zero term in degree -1. We have

$$\chi_X(F^D) = \frac{(2\pi)^{r_2(F)} 2^{r_1(F)} \left[H^0(X, F)_{\mathrm{tor}}\right] \left[\mathrm{Ext}^1_X(F, \mathbb{G}_X)_{\mathrm{tor}}\right]}{\left[H^1(X, F)_{\mathrm{tor}}\right] \left[\mathrm{Hom}_X(F, \mathbb{G}_X)_{\mathrm{tor}}\right] \left[\mathrm{Ext}^1_{G^t_K}(F_\eta, \mathcal{O}_{K^t})\right] [N_2] \mathrm{Disc}(F)} R(F^D).$$

Proof. We have $\mathbb{Z}^c_X \simeq \mathbb{G}_X[1]$ [34]. Consider the biacyclic double complex of Diagram 5 (numbers on the side indicate indexing, zeros are omitted). If $C^{\bullet,\bullet}$ is any biacyclic double complex of $\mathbb{R}$-vector spaces with lattices, we find by applying Lemma E.1 to both the rows and the columns that the following diagram is commutative:

Fix bases $(u_k^{(i,j)})$ of the lattices inside the $C^{i,j}$, and denote $(u_k^{i,j})^{-1}$ the dual basis vectors. By looking at the image under the composite morphism $\bigotimes_{i,j}(\det_\mathbb{R} C^{i,j})^{(-1)^{i+j}} \to \mathbb{R}$ of $\bigotimes_{i,j} \bigwedge_k((u_k^{(i,j)})^{(-1)^{i+j}})$, we find the relation

$$\prod_i \left(\det_{\mathbb{Z},\mathbb{R}} C^{i,\bullet}\right)^{(-1)^i} = \prod_j \left(\det_{\mathbb{Z},\mathbb{R}} C^{\bullet,j}\right)^{(-1)^j}.$$

Hence in our case we obtain

$$\det_{\mathbb{Z},\mathbb{R}} A^\bullet \left(\det_{\mathbb{Z},\mathbb{R}} B^\bullet\right)^{-1} = \det_{\mathbb{Z},\mathbb{R}} C^\bullet \left(\det_{\mathbb{Z},\mathbb{R}} D^\bullet\right)^{-1} \left(\det_{\mathbb{Z},\mathbb{R}} E^\bullet\right)^{-1}.$$

$$C^\bullet \qquad\qquad D^\bullet \qquad\qquad E^\bullet \qquad -1$$

$$\mathrm{Hom}_{G_{\mathbb{R}},X(\mathbb{C})}(\alpha^*F,2i\pi\mathbb{Z})_{\mathbb{R}} = \mathrm{Hom}_{G_{\mathbb{R}},X(\mathbb{C})}(\alpha^*F,2i\pi\mathbb{Z})_{\mathbb{R}}$$

$$A^\bullet: \quad \left(H^1(X,F)_{\mathbb{R}}\right)^\vee \longrightarrow H^{-1}_{c,B}(X,F^D)_{\mathbb{R}} \longrightarrow H^1\,\mathrm{Lie}_X(F^D)_{\mathbb{R}} \longrightarrow \left(H^0(X,F)_{\mathbb{R}}\right)^\vee \longrightarrow H^0_{c,B}(X,F^D)_{\mathbb{R}} \qquad 0$$

$$\| \qquad\qquad \| \qquad\qquad \| \qquad\qquad \simeq$$

$$B^\bullet: \quad \left(H^1(X,F)_{\mathbb{R}}\right)^\vee \longrightarrow H^{-1}(X,F^D)_{\mathbb{R}} \longrightarrow \prod_v \mathrm{Hom}_{G_v}(F_v,\mathbb{Z})_{\mathbb{R}} \longrightarrow \left(H^0(X,F)_{\mathbb{R}}\right)^\vee \longrightarrow H^0(X,F^D)_{\mathbb{R}} \qquad 1$$

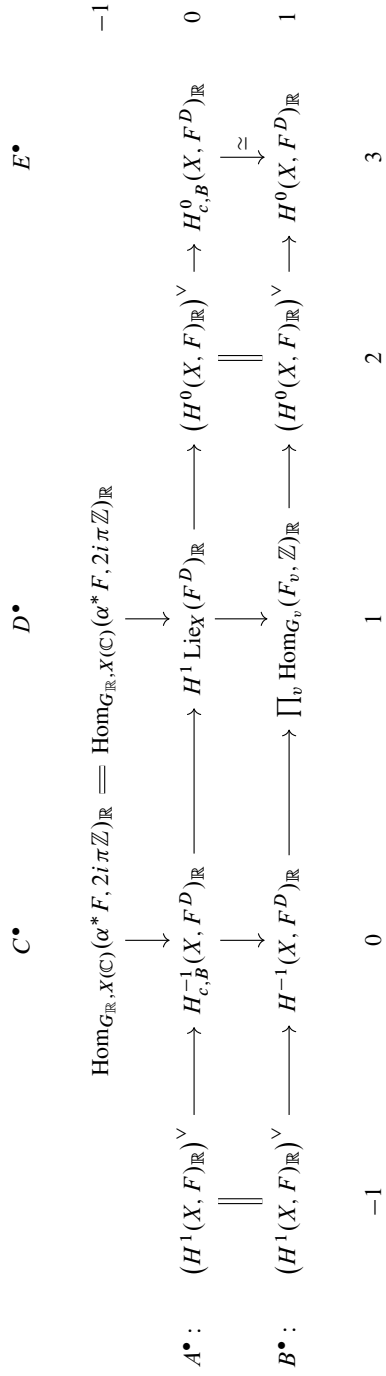$$-1 \qquad\qquad 0 \qquad\qquad 1 \qquad\qquad 2 \qquad\qquad 3$$

Diagram 5.

Thus it remains to compute $\det_{\mathbb{Z},\mathbb{R}} C^\bullet (\det_{\mathbb{Z},\mathbb{R}} E^\bullet)^{-1}$. Consider the following acyclic complex of abelian groups:

$$0 \longrightarrow \mathrm{Hom}_{G_\mathbb{R},X(\mathbb{C})}(\alpha^* F, 2i\pi\mathbb{Z}) \longrightarrow H^{-1}_{c,B}(X, F^D) \longrightarrow H^{-1}(X, F^D)$$

$$G^\bullet : \qquad \longrightarrow \mathrm{Ext}^1_{G_\mathbb{R},X(\mathbb{C})}(\alpha^* F, 2i\pi\mathbb{Z}) \longrightarrow H^0_{c,B}(X, F^D) \longrightarrow H^0(X, F^D)$$

$$\xrightarrow{\hspace{3cm}} N_2 \xrightarrow{\hspace{3cm}} 0$$

where we have placed the first non-zero term in degree -1. The last term is a subgroup of $\mathrm{Ext}^2_{G_\mathbb{R},X(\mathbb{C})}(\alpha^* F, 2i\pi\mathbb{Z})$ and hence is finite 2-torsion. Then $\det_\mathbb{Z} G^\bullet \xrightarrow[\det(0)]{\simeq} \mathbb{Z}$ and $G^\bullet_\mathbb{R} = C^\bullet \oplus E^\bullet[-3]$, thus we find

$$\det_{\mathbb{Z},\mathbb{R}} C^\bullet \big(\det_{\mathbb{Z},\mathbb{R}} E^\bullet \big)^{-1} = \det_{\mathbb{Z},\mathbb{R}} G_\mathbb{R} = \prod_i [G^i_{\mathrm{tor}}]. \qquad \blacksquare$$

Proposition 7.10. *Suppose that X is regular. We have*

$$\sum (-1)^i i \cdot \dim_\mathbb{R} H^i_{\mathrm{ar},c}(X, \mathbb{Z}^D_\mathbb{R}) = -1, \quad \chi_X(\mathbb{Z}^D) = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{\omega \sqrt{|\Delta_K|}}$$

and thus the special value formula

$$\mathrm{ord}_{s=0} L_X(\mathbb{Z}^D) = \mathrm{ord}_{s=1} \zeta_X = \sum (-1)^i i \cdot \dim_\mathbb{R} H^i_{\mathrm{ar},c}(X, \mathbb{Z}^D_\mathbb{R}),$$
$$L^*_X(\mathbb{Z}^D, 0) = \zeta^*_X(1) = \chi_X(\mathbb{Z}^D).$$

Proof. The first formula is immediate from the previous computations.

We use Proposition 7.9. We easily find $r_2(\mathbb{Z}) = r_2$, $r_1(\mathbb{Z}) = r_1$, $N_2 = 0$, and $\mathrm{Lie}_X(\mathbb{Z}^D) = \mathcal{O}_K[1]$ so in particular $H^0 \mathrm{Lie}_X(\mathbb{Z}^D) = 0$. Thus using Propositions 2.10 and 3.6 we obtain

$$\chi_X(\mathbb{Z}^D) = \frac{2^{r_1}(2\pi)^{r_2} h_K R(\mathbb{Z}^D)}{\omega \, \mathrm{Disc}(\mathbb{Z}^D)}.$$

By the choices made in Definition 7.8, $\mathrm{Disc}(\mathbb{Z}^D)$ is computed as follows: fix a basis (α_i) of $\mathcal{O}_K$. Choose an ordering of the archimedean places, real places first and complex places second, and denote σ_i the choice of an embedding for each place v_i and $\sigma_{r_1+r_2+i} = \overline{\sigma_{r_1+i}}$. Thus $\sigma_1, \ldots, \sigma_{r_1}$ are the real embeddings and $\sigma_{r_1+1}, \ldots, \sigma_{r_1+r_2}$ the chosen complex embeddings for each complex place. By unravelling the definitions, we see that we are asked to compute the determinant of the matrix $\left(\begin{smallmatrix} (\Re(\sigma_i(\alpha_j)))_{1\leq i \leq r_1+r_2, 1\leq j \leq n} \\ (\Im(\sigma_{r_1+i'}(\alpha_j)))_{1\leq i' \leq r_2, 1\leq j \leq n} \end{smallmatrix} \right)$. Observe that it is obtained from the matrix $(\sigma_j(\alpha_i))_{i,j=1,\ldots,n}$ by some elementary transformations on the rows which will make a factor $(\frac{1}{i})^{r_2}$ appear. Since $(\det(\sigma_j(\alpha_i))_{i,j})^2 = \Delta_K$ and the sign of Δ_K is $(-1)^{r_2}$ (Brill's theorem [46, Lem. 2.2]), we obtain

$$\mathrm{Disc}(\mathbb{Z}^D) = \left| \left(\frac{1}{i} \right)^{r_2} \det (\sigma_j(\alpha_i))_{i,j} \right| = \sqrt{|\Delta_K|}.$$

It remains to determine $R(\mathbb{Z}^D)$. By definition, this is $\det_{\mathbb{Z},\mathbb{R}} B^\bullet$ where $B^\bullet$ is the acyclic complex with first non-zero term in degree 0

$$0 \to (\mathcal{O}_K^\times)_\mathbb{R} \to \left(\prod_v \mathbb{Z}\right)_\mathbb{R} \to (\mathbb{Z}_\mathbb{R})^\vee \to 0.$$

Let us make the maps explicit. Denote $|\cdot|_v$ the normalized absolute value for the archimedean place v: $|\cdot|_v = |\cdot|$ if v is real and $|\cdot|_v = |\cdot|^2$ if v is complex. Denote also $L := \prod_v \log|\sigma_v(\cdot)|_v$ the logarithmic embedding. We have a commutative diagram

$$
\begin{array}{ccc}
 & (\prod_\sigma \mathbb{R})^{G_\mathbb{R}} & \\
\overset{\prod_\sigma \log|\sigma(\cdot)|}{\nearrow} \quad \downarrow{\simeq} \quad \overset{\Sigma}{\searrow} & & \\
(\mathcal{O}_K^\times)_\mathbb{R} \xrightarrow{\quad L \quad} \prod_v \mathbb{R} \xrightarrow{\quad \Sigma \quad} \mathbb{R}
\end{array}
$$

where the middle isomorphism comes from Definition 7.8 and Σ is the sum of components map (see the discussion after Definition 5.1). Here $\mathbb{R} = H^0_{c,\mathcal{D}}(X,\mathbb{Z}^D_\mathbb{R}) = H^0(X,\mathbb{Z})^\vee_\mathbb{R}$ obtains the canonical basis because the perfect pairing $H^0_{c,\mathcal{D}}(X,\mathbb{Z}^D_\mathbb{R}) \times H^0(X,\mathbb{Z})_\mathbb{R} = \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by the multiplication map. The ordering on the places gives an ordered basis on $(\prod_v \mathbb{Z})_\mathbb{R}$. Let x be the vector $(1,0,\ldots,0) \in (\prod_v \mathbb{Z})_\mathbb{R}$. We have $\Sigma(x) = 1$. Denote (ε_i) a system of fundamental units of $\mathcal{O}_K^\times$. Then $\det_{\mathbb{Z},\mathbb{R}} B^\bullet$ is the absolute value of the determinant of the matrix P expressing $(L(\varepsilon_1), L(\varepsilon_2), \ldots, L(\varepsilon_{r_1+r_2-1}), x)$ in the basis of $(\prod_v \mathbb{Z})_\mathbb{R}$. Put $N_i = 1$ if σ_i is real and $N_i = 2$ if σ_i is complex. Then we have

$$
P = \begin{pmatrix}
N_1 \log|\sigma_1(\varepsilon_1)| & \cdots & N_1 \log|\sigma_1(\varepsilon_{s-1})| & 1 \\
N_2 \log|\sigma_2(\varepsilon_1)| & \cdots & N_2 \log|\sigma_2(\varepsilon_{s-1})| & 0 \\
\vdots & \ddots & \vdots & \vdots \\
N_s \log|\sigma_s(\varepsilon_1)| & \cdots & N_s \log|\sigma_s(\varepsilon_{s-1})| & 0
\end{pmatrix}.
$$

By definition, the regulator R_K is the absolute value of the determinant of any $(s-1) \times (s-1)$ minor of the matrix $(N_i \log|\sigma_i(\varepsilon_j)|)_{i=1,\ldots,s; j=1,\ldots,s-1}$; thus by developing the determinant of P with respect to the last column we find

$$R(\mathbb{Z}^D) = |\det P| = R_K.$$

We obtain finally

$$\chi_X(\mathbb{Z}^D) = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{\omega \sqrt{|\Delta_K|}}. \qquad \blacksquare$$

Remark. For $F = \mathbb{Z}$, the complexes

$$\mathrm{Lie}_X(\mathbb{Z}^D), \quad R\Gamma_{W,c}(X,\mathbb{Z}^D), \quad R\Gamma_{c,\mathcal{D}}(X,\mathbb{Z}^D_\mathbb{R}), \quad \text{and} \quad R\Gamma_{\mathrm{ar},c}(X,\mathbb{Z}^D_\mathbb{R})$$

are equal respectively (up to a shift) to the complexes

$$R\Gamma_{dR}(X/\mathbb{Z})/F^1, \quad R\Gamma_{W,c}(X,\mathbb{Z}(1)), \quad R\Gamma_c(X,\mathbb{R}(1)), \quad \text{and} \quad R\Gamma_{\mathrm{ar},c}(X,\widetilde{\mathbb{R}}(1))$$

of [15], in which the above formula was already obtained.

Proposition 7.11. *Let $i : x \to X$ be the inclusion of a closed point, M a discrete G_x-module of finite type. Let $\chi_{W,c}$ denote the Weil-étale Euler characteristic of a $\mathbb{Z}$-constructible sheaf on X from* [31] *and let $R(M)$ be the absolute value of the determinant of the pairing*

$$H^0(G_x, M)_{\mathbb{R}} \times H^0(G_x, M^{\vee})_{\mathbb{R}} \to \mathbb{R}$$

in bases modulo torsion. Then

$$\sum (-1)^i i \cdot \dim_{\mathbb{R}} H^i_{\mathrm{ar},c}\big(X, (i_* M)^D_{\mathbb{R}}\big) = -\mathrm{rank}_{\mathbb{Z}} H^0(G_x, M^{\vee}),$$

$$\chi_X\big((i_* M)^D\big) = \frac{\big[H^0(G_x, M)\big]_{\mathrm{tor}}}{\big[H^1(G_x, M)\big] R(M) \log\big(N(v)\big)^{\mathrm{rank}\, H^0(G_x, M)}}$$

$$= \chi_{W,c}(i_* M) = \chi_{W,c}(i_* M^{\vee}).$$

We have the special value formula

$$\mathrm{ord}_{s=0} L_X\big((i_* M)^D\big) = \sum (-1)^i i \cdot \dim_{\mathbb{R}} H^i_{\mathrm{ar},c}\big(X(, i_* M)^D_{\mathbb{R}}\big),$$

$$L_X^*\big((i_* M)^D, 0\big) = \chi_X\big((i_* M)^D\big).$$

Proof. The computation of $\sum (-1)^i i \cdot \dim_{\mathbb{R}} H^i_{\mathrm{ar},c}(X(, i_* M)^D_{\mathbb{R}})$ is straightforward. Let us compute $\chi_X((i_* M)^D)$. Using Proposition 7.9 we find

$$\chi_X\big((i_* M)^D\big) = \frac{\big[H^0(G_x, M)_{\mathrm{tor}}\big] R\big((i_* M)^D\big)}{\big[H^1(G_x, M)\big]}.$$

Here, the complex $B^{\bullet}$ is

$$\big(H^0(X, i_* M)_{\mathbb{R}}\big)^{\vee} \xrightarrow{\cong} H^0(X, i * M^{\vee})_{\mathbb{R}}$$

with first term in degree 2, so $R((i_* M)^D) = \det_{\mathbb{Z}, \mathbb{R}} B^{\bullet}$ is the inverse of the absolute value of the determinant of the pairing

$$H^0(X, i_* M)_{\mathbb{R}} \times H^0(X, i_* M^{\vee}) \to \mathbb{R}.$$

Hence from the proof of Theorem 5.2 we see that

$$R\big((i_* M)^D\big) = \frac{1}{R(M) \log\big(N(v)\big)^{\mathrm{rank}\, H^0(G_x, M)}}$$

which shows the first equality. The second equality is [31, Cor. 6.8]. Let us show the last equality. If M is torsion, then $M^{\vee} = M^*[-1]$ so by [31, Prop. 6.23] both terms equal 1. We can thus suppose that M is torsion-free. Let $n = \mathrm{rank}_{\mathbb{Z}} H^0(G_x, M)$. Then we have $\mathrm{rank}_{\mathbb{Z}} \mathrm{Hom}_{G_x}(M, \mathbb{Z}) = n$ by the perfectness of the above pairing, $M^{\vee} = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, and

$$\chi_{W,c}(i_* M) = \frac{1}{\big[H^1(G_x, M)\big] R(M) \log\big(N(v)\big)^n},$$

$$\chi_{W,c}(i_* M^{\vee}) = \frac{1}{\big[H^1(G_x, M^{\vee})\big] R(M^{\vee}) \log\big(N(v)\big)^n}$$

whence the result from the duality isomorphism

$$H^1(G_x, M) \simeq \mathrm{Ext}^1_{G_x}(M, \mathbb{Z})^* = H^1(G_x, M^\vee)^*$$

and the identification $R(M) = R(M^\vee)$.

Finally, since $L_X((i_*M)^D, s) = L_X(i_*M^\vee, s)$ is the L-function of a (complex of) $\mathbb{Z}$-constructible sheaves, from [31, Thm. 6.24] we find

$$\mathrm{ord}_{s=0} L_X\big((i_*M)^D, s\big) = -\mathrm{rank}_{\mathbb{Z}} H^0(G_x, M^*),$$
$$L_X^*\big((i_*M)^D, 0\big) = \chi_{W,c}(i_*M^\vee). \qquad \blacksquare$$

Theorem 7.12. *Let F be a constructible sheaf. Then*

$$\sum (-1)^i i \cdot \dim_{\mathbb{R}} H^i_{\mathrm{ar},c}(X, F^D_{\mathbb{R}}) = 0 = \mathrm{ord}_{s=0} L_X(F^D, s).$$

If F is moreover tamely ramified, then

$$\chi_X(F^D) = 1.$$

In particular, we have the special value formula $L_X^*(F^D, 0) = \pm \chi_X(F^D)$.

Proof. The vanishing order formula is immediate; let us show the special value formula. If F is supported on a finite closed subscheme, we reduce to the case $F = i_*M$ where $i : x \to X$ is a closed point and M is a finite discrete G_x-module. In that case, Proposition 7.11 gives $\chi_X((i_*M)^D) = 1$. The kernel and cokernel of the canonical map $F \to g_*g^*F$ are constructible supported on a finite closed subscheme. Therefore we are reduced to checking that $\chi_X((g_*M)^D) = 1$ for M a tamely ramified finite discrete G_K-module. Let L be a tamely ramified extension of K such that G_L acts trivially on M, and let $G_{L/K} := \mathrm{Gal}(L/K)$. For R a ring, denote $K_0'(R)$ the Grothendieck group of finitely generated left R-modules. The map $\theta : M \mapsto \chi_X((g_*M)^D)$ is multiplicative: indeed, if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of $G_{L/K}$-modules then we have an exact sequence $0 \to g_*M' \to g_*M \to g_*M'' \to N \to 0$. Then N is $\mathbb{Z}$-constructible as a quotient of g_*M''. We have $(R^1g_*M')_\eta = 0$ and the stalk $(R^1g_*M')_v = H^1(I_v, M')$ is Galois cohomology hence torsion. As $N \subset R^1g_*M'$, N is torsion and skyscraper, which shows that N is constructible supported on a finite closed subset. Then $\chi_X(N) = 1$ so $\chi_X(g_*M) = \chi_X(g_*M')\chi_X(g_*M'')$. We find that θ factors through $K_0'(\mathbb{Z}[G_{L/K}])$ and takes values in $\mathbb{R}^\times_{>0}$ which is torsion-free. By [41, Cor. 1], the kernel of $K_0'(\mathbb{Z}[G_{L/K}]) \to K_0'(\mathbb{Q}[G_{L/K}])$ is finite; since $M \otimes \mathbb{Q} = 0$, this implies that the class of M is a torsion element in $K_0'(\mathbb{Z}[G_{L/K}])$ so $\theta(M) = \chi_X(g_*M) = 1$. $\qquad \blacksquare$

Remark. • In [31], a similar formula for the Weil-étale Euler characteristic $\chi_X(F)$ of a constructible sheaf F is proven by reduction to Tate's formula for the Euler characteristic of a global field (see [30, Thm. I.5.1, Thm. II.2.13] and [31, Prop. 6.23] for the reduction). The method of the above proof applies *mutatis mutandis* to $\chi_X(F)$ and thus gives an alternative proof of Tate's formula (which still depends on Tate's formula in the local case).

• If F is a tamely ramified constructible sheaf, we have

$$R\Gamma_{\mathrm{ar},c}(X, F_{\mathbb{R}}^D) = 0,$$
$$R\Gamma_{W,c}(X, F^D) \simeq R\Gamma_{c,B}(X, F^D),$$
$$\mathrm{Lie}_X(F^D) = \mathrm{Ext}^1_{G_K^t}(F_\eta, \mathcal{O}_{K^t})[0]$$

and the two latter complexes are bounded with finite cohomology groups. Thus their Euler characteristics

$$\chi_{W,X}(F^D) := \prod_{i \in \mathbb{Z}} \big[H^i_{W,c}(X, F^D) \big]^{(-1)^i}$$

and

$$\chi_{L,X}(F^D) := \big[\mathrm{Ext}^1_{G_K^t}(F_\eta, \mathcal{O}_{K^t}) \big]$$

are well defined and we have $\chi_X(F^D) = \chi_{W,X}(F^D)/\chi_{L,X}(F^D)$. On the other hand, using Artin–Verdier duality, the morphism of fiber sequences (2.1) and [31, Prop. 6.23], we find

$$\chi_{W,X}(F^D) = [F_\eta]^{[K:\mathbb{Q}]}.$$

Thus the above proposition implies that

$$\big[\mathrm{Ext}^1_{G_K^t}(F_\eta, \mathcal{O}_{K^t}) \big] = [F_\eta]^{[K:\mathbb{Q}]}.$$

Theorem 7.13 (Special values theorem). *Let F be a $\mathbb{Z}$-constructible sheaf. We have the vanishing order formula*

$$\mathrm{ord}_{s=0} L_X(F^D, s) = \sum (-1)^i i \cdot \dim_{\mathbb{R}} H^i_{\mathrm{ar},c}(X, F_{\mathbb{R}}^D).$$

If F is tamely ramified, we have the special value formula

$$L_X^*(F^D, 0) = \pm \chi_X(F^D).$$

Proof. This follows from Propositions 7.10 and 7.11 and Theorem 7.12 by Artin induction. ∎

A. Duality for the Tate cohomology of finite groups

Let G be a finite group. The functor $R\widehat{\Gamma}(G, -)$ is lax monoidal, so for any complex of G-modules M we can construct the pairing

$$R\widehat{\Gamma}(G, M^\vee) \otimes^L R\widehat{\Gamma}(G, M) \to R\widehat{\Gamma}(G, \mathbb{Z}) \to \widehat{H}^0(G, \mathbb{Z})[0] = \mathbb{Z}/|G|\mathbb{Z}[0] \hookrightarrow \mathbb{Q}/\mathbb{Z}[0]$$

where $M^\vee = R\mathcal{H}om_G(M, \mathbb{Z})$. In the following, we will also denote

$$M^* = \mathcal{H}om_G(M, \mathbb{Q}/\mathbb{Z})$$

the Pontryagin dual.

Theorem A.1. *Let G be a finite group. The natural pairing*

$$R\widehat{\Gamma}(G, M^\vee) \otimes^L R\widehat{\Gamma}(G, M) \to \mathbb{Q}/\mathbb{Z}[0]$$

is perfect for any bounded complex M of G-modules with finite type cohomology groups.

Proof. By definition, showing that the pairing is perfect is showing that the adjoint map $R\widehat{\Gamma}(G, M^\vee) \to R\widehat{\Gamma}(G, M)^*$ is an isomorphism. Passing to cohomology, this is equivalent (using [42, Tag 0FP2] and the injectivity of $\mathbb{Q}/\mathbb{Z}$) to showing that the cup product pairing

$$\widehat{H}^i(G, M) \times \widehat{H}^{-i}(G, M^\vee) \to \widehat{H}^0(G, \mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

is perfect for each $i \in \mathbb{Z}$. Notice that if M is induced, then $M^\vee$ is also induced; moreover, if M_0 is a finite type abelian group, $\mathrm{ind}^G M_0$ also is. Thus the usual dimension shifting argument reduces to checking that the pairing is perfect in just one degree. We proceed by Artin induction:

- In an exact sequence $0 \to M \to P \to Q \to 0$ or more generally a fiber sequence $M \to P \to Q$, if the theorem is true for two out of the three terms then it is true for the third.

- We can filter a bounded complex with finite type cohomology groups by its truncations, which reduces to the case of a G-module M of finite type.

- Let H be a proper subgroup of G, let M be a torsion-free finite type discrete H-module and let us consider the induced G-module $\mathrm{ind}_H^G M$. We want to show compatibility of the pairing in degree 0 for $\mathrm{ind}_H^G M$ and the pairing in degree 0 for M.
  We have $M^\vee = \mathrm{Hom}_\mathbb{Z}(M, \mathbb{Z})$. Denote π_* for the induction functor ind_H^G and π^* for restriction to H. The functor π_* is a right adjoint of π^* and the finiteness of G also makes π_* into a left adjoint of π^*. Let us denote $\eta : \pi_*\pi^* \to \mathrm{id}$ the counit of the latter adjunction; the counit $\eta_\mathbb{Z} : \pi_*\mathbb{Z} \simeq \oplus_{G/H}\mathbb{Z} \to \mathbb{Z}$ is the sum map. As a right adjoint to a strict monoidal functor, π_* has a lax monoidal structure

$$\pi_*(-) \otimes \pi_*(-) \xrightarrow{\mathrm{lax}} \pi_*(- \otimes -).$$

Consider the following commutative diagram:

$$
\begin{array}{ccc}
\pi_* M \otimes (\pi_* M)^\vee & \xrightarrow{\quad\quad\quad\mathrm{ev}\quad\quad\quad} & \mathbb{Z} \\
{\scriptstyle\simeq}\big\uparrow & & \big\uparrow{\scriptstyle\eta=\Sigma} \\
\pi_* M \otimes \pi_*(M^\vee) \xrightarrow[\mathrm{lax}]{} \pi_*(M \otimes M^\vee) & \xrightarrow[\pi_*(\mathrm{ev})]{} & \pi_*\mathbb{Z}
\end{array}
$$

We can apply the lax-monoidal functor $\widehat{H}^0$ to the above to obtain Diagram 6 (using Shapiro's lemma's identifications). Square (1) commutes by functoriality of the cup product, square (2) commutes by functoriality of $\widehat{H}^0(G, -)$, square (3) commutes by inspection, square (4) commutes by functoriality in Shapiro's lemma and square

$$\hat{H}^0(G, \pi_* M) \otimes \hat{H}^0(G, (\pi_* M)^\vee) \overset{\cup}{\to} \hat{H}^0(G, \pi_* M \otimes (\pi_* M)^\vee) \xrightarrow{\quad \mathrm{ev}_* \quad} \hat{H}^0(G, \mathbb{Z}) = \mathbb{Z}/|G|\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z}$$

$$(1) \qquad (2)$$

$$\hat{H}^0(G, \pi_* M) \otimes \hat{H}^0(G, \pi_*(M^\vee)) \overset{\cup}{\to} \hat{H}^0(G, \pi_* M \otimes \pi_*(M^\vee)) \xrightarrow[\mathrm{lax}_*]{} \hat{H}^0(G, \pi_*(M \otimes M^\vee)) \xrightarrow[(\pi_*(\mathrm{ev}))_*]{} \hat{H}^0(G, \pi_* \mathbb{Z})$$

$$(3) \qquad (4) \qquad (5)$$

$$\hat{H}^0(H, M) \otimes \hat{H}^0(H, M^\vee) \overset{\cup}{\to} \hat{H}^0(H, M \otimes M^\vee) \xrightarrow{\quad \mathrm{ev}_* \quad} \hat{H}^0(H, \mathbb{Z}) = \mathbb{Z}/|H|\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z}$$

Diagram 6.

(5) commutes because $(e^{\frac{2i\pi}{|G|}})^{[G:H]} = e^{\frac{2i\pi}{|H|}}$. As the bottom left square commutes, the whole diagram commutes and the pairing for M is perfect if and only if the pairing for $\operatorname{ind}_H^G M$ is perfect.

- For $M = \mathbb{Z}$ we have $M^\vee = \mathbb{Z}$. The map $R\widehat{\Gamma}(G, M^\vee) \to R\widehat{\Gamma}(G, M)^*$ corresponds in degree 0 to the adjoint map to the cup product pairing

$$\mathbb{Z}/|G|\mathbb{Z} \otimes \mathbb{Z}/|G|\mathbb{Z} \to \mathbb{Z}/|G|\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$$

induced by the multiplication $\mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Z}$. Thus it is perfect.

- For M finite as an abelian group, we have $R\mathcal{H}om_G(M, \mathbb{Q}) = 0$ so the fiber sequence $\mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ gives $M^\vee \simeq M^*[-1]$. For any bounded complex M, the complex $M \otimes \mathbb{Q}$ is a complex of $\mathbb{Q}$-vector spaces hence cohomologically trivial. The map $\mathbb{Q}/\mathbb{Z}[-1] \to \mathbb{Z}$ thus induces an isomorphism of pairings

$$R\widehat{\Gamma}(G, M^*)[-1] \otimes^L R\widehat{\Gamma}(G, M) \to R\widehat{\Gamma}(G, \mathbb{Q}/\mathbb{Z})[-1] \to \widehat{H}^{-1}(G, \mathbb{Q}/\mathbb{Z})[0] \to \mathbb{Q}/\mathbb{Z}[0]$$
$$\simeq\downarrow \qquad\qquad\qquad \simeq\downarrow \qquad\qquad \simeq\downarrow \qquad\qquad \|$$
$$R\widehat{\Gamma}(G, M^\vee) \otimes^L R\widehat{\Gamma}(G, M) \longrightarrow R\widehat{\Gamma}(G, \mathbb{Z}) \longrightarrow \widehat{H}^0(G, \mathbb{Z})[0] \longrightarrow \mathbb{Q}/\mathbb{Z}[0]$$

It follows that it is enough to prove that the map $R\widehat{\Gamma}(G, M^*)[-1] \to R\widehat{\Gamma}(G, M)^*$ is an isomorphism in degree 0, i.e., that the cup-product pairing

$$\widehat{H}^{-1}(G, M^*) \times \widehat{H}^0(G, M) \to \widehat{H}^{-1}(G, \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

induced by the Pontryagin duality pairing $M^* \otimes M \to \mathbb{Q}/\mathbb{Z}$ is perfect. For $g \in G$, g acts on M^* as the transpose of g^{-1}. Thus, on M^*, N acts as N^t and the family $(1 - g)_{g \in G}$ is a permutation of the family $((1 - g)^t)_{g \in G}$. In the perfect pairing

$$M^* \otimes M \to \mathbb{Q}/\mathbb{Z},$$

we have $^\perp(\bigcap_{g \in G} \operatorname{Ker}(1 - g)) = \sum_{g \in G} \operatorname{Im}((1 - g)^t)$ and $\operatorname{Ker}(N)^\perp = \operatorname{Im}(N^t)$, so this pairing induces a perfect pairing between the subquotients:

$$\operatorname{Ker}(N^t)/\sum_{g \in G} \operatorname{Im}\big((1 - g)^t\big) \times \bigcap_{g \in G} \operatorname{Ker}(1 - g)/\operatorname{Im}(N) \to \mathbb{Q}/\mathbb{Z}.$$

This is exactly what we had to prove. ∎

B. The maximal tamely ramified extension of a number field

In this section, we discuss the tame Galois group of a number field. The results are certainly known but we did not find a convenient reference.

Let K be the field of fractions of a henselian DVR with finite residue field. A finite extension L/K is called tamely ramified if the ramification index $e_{L/K}$ is prime to the residual characteristic. We have the following properties:

Proposition B.1 ([35, §II.7, Cor. 7.8, Cor. 7.9, Def. 7.10]). (1) *If $M/L/K$ is a tower of finite extensions, then M/K is tamely ramified if and only if M/L and L/K are.*

(2) *The composite of two tamely ramified finite extensions of K is tamely ramified.*

(3) *The maximal tamely ramified extension K^t is defined as the composite of all finite tamely ramified extensions of K inside K^{sep}. Its finite subextensions are tamely ramified.*

Let K be a number field. For each finite place v of K, which corresponds to a closed point of $X = \mathrm{Spec}(\mathcal{O}_K)$, denote K_v the henselian local field at v. Let K^{sep} be a separable closure of K and for each finite place v, choose an embedding $K_v \hookrightarrow K^{\mathrm{sep}}$; this determines a place $\bar{v}$ of K^{sep} above v and gives K^{sep} the structure of a separable closure of K_v. Denote G_K, resp. G_{K_v} the absolute Galois group of K resp. K_v. The previous proposition is adapted to the global case by working simultaneously at all places, as following: a finite extension L/K is called tamely ramified if for all finite places v of K and w of L above v, the finite extension L_w/K_v is tamely ramified. We then have:

Corollary B.2. (1) *If $M/L/K$ is a tower of finite extensions, then M/K is tamely ramified if and only if M/L and L/K are.*

(2) *The composite of two tamely ramified finite extensions of K is tamely ramified.*

(3) *The maximal tamely ramified extension K^t is defined as the composite of all finite tamely ramified extensions of K inside K^{sep}. Its finite subextensions are tamely ramified.*

Proof. This follows from the previous propositions by taking henselizations, using the two following observations:

- For any choice of finite place in any of the three fields in a tower $M/L/K$, there exists a compatible system of places above and below it.

- If L/K and M/K are two finite extensions inside K^{sep} and z is a finite place of the composite extension LM mapping to places w, w' of L and M then $(LM)_z = L_w M_{w'}$; indeed, $(LM)_z$ is the smallest henselian field inside K^{sep} containing LM so $(LM)_z \subset L_w M_{w'}$, and the other inclusion is immediate. ∎

Definition B.3. The tame Galois group of K, denoted G_K^t, is the Galois group of the Galois extension K^t/K:
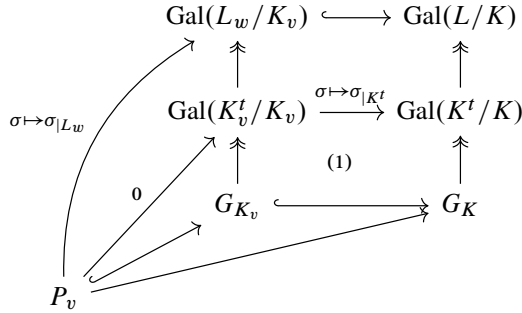$$G_K^t := \mathrm{Gal}(K^t/K).$$

For a finite place v of K, denote $P_v \subset D_v$ the wild ramification and decomposition subgroup of the place $\bar{v}$ of K^{sep}. There is an identification $D_v = G_{K_v}$ under which we have $P_v = \mathrm{Gal}(K^{\mathrm{sep}}/K_v^t)$. We now characterise the tame Galois group in terms of the wild inertia subgroups P_v.

Proposition B.4. *Let N be the smallest closed normal subgroup of G_K containing P_v for all finite places v. Then $N = \mathrm{Gal}(K^{\mathrm{sep}}/K^t)$, and consequently $G_K^t = G_K/N$.*

Proof. We first show $N \subset \mathrm{Gal}(K^{\mathrm{sep}}/K^t)$; since the latter is normal and closed it suffices to show that the elements of P_v, for any finite place v, fix K^t. Let L/K be a finite tamely ramified extension. The place $\bar{v}$ of K^{sep} determines a unique place w of L above v. The

following diagram commutes

$$
\begin{array}{ccc}
\mathrm{Gal}(L_w/K_v) & \hookrightarrow & \mathrm{Gal}(L/K) \\
\uparrow & & \uparrow \\
\mathrm{Gal}(K_v^t/K_v) & \xrightarrow{\sigma \mapsto \sigma_{|K^t}} & \mathrm{Gal}(K^t/K) \\
\uparrow & \mathrm{(1)} & \uparrow \\
G_{K_v} & \hookrightarrow & G_K \\
\end{array}
$$

with $\sigma \mapsto \sigma_{|L_w}$, 0, and P_v.

Indeed, the only nontrivial part is square (1). If L/K is a finite tamely ramified extension and w is the place of L induced by $\bar{v}$ then $L \subset L_w \subset K_v^t$; therefore there is an inclusion $K^t \subset K_v^t$ and the commutativity of (1) follows. Since the diagram commutes, the elements of P_v are sent to 0 in $\mathrm{Gal}(L/K)$ so they fix any finite tamely ramified extension of K, hence also K^t.

We now show $\mathrm{Gal}(K^{\mathrm{sep}}/K^t) \subset N$. Since N is normal, it suffices to show that for $N \subset U$ with U an open normal subgroup, we have $\mathrm{Gal}(K^{\mathrm{sep}}/K^t) \subset U$. This amounts to showing that the finite extension L/K corresponding to an open normal subgroup U containing N is tamely ramified. Let v be a finite place of K and denote w the place of L induced by $\bar{v}$. Then $L_w = LK_v$ is the fixed field of $U \cap G_{K_v}$, which contains P_v, so $L_w \subset K_v^t$ is tamely ramified. ∎

C. Miscellaneous results on proétale cohomology and condensed mathematics

In this section, we collect some results on condensed mathematics; these are certainly already known to experts.

Definition C.1 ([4, Def. 3.2.1]). An object F of a topos $\mathcal{X}$ is called weakly contractible if every surjection $G \to F$ has a section. We say that $\mathcal{X}$ is locally weakly contractible if it has enough weakly contractible coherent objects, i.e., each $X \in \mathcal{X}$ admits a surjection $\cup_i Y_i \to X$ with Y_i a coherent weakly contractible object.

The proétale topos of a scheme is locally weakly contractible [4, Prop. 4.2.8]; in particular the condensed topos (i.e., the proétale topos of a geometric point) is locally weakly contractible, and (the sheaves represented by) extremally disconnected profinite sets are a suitable family of weakly contractible objects.

Lemma C.2. *Let $\mathcal{X}$ be a locally contractible topos. If (A_i) is a projective system of abelian group objects satisfying the Mittag-Leffler condition, then $R \lim A_i = \lim A_i$.*

Proof. Let Y be a weakly contractible object. As $\Gamma(Y, -)$ is exact and preserves injectives, we find $\Gamma(Y, -) = R\Gamma(Y, -)$ and thus $(R \lim A_i)(Y) = R \lim(A_i(Y))$. Moreover, $(A_i(Y))$ is a projective system of abelian groups satisfying the Mittag-Leffler condition, again by exactness and commutation with arbitrary limits of $\Gamma(Y, -)$; so we are done. ∎

Lemma C.3. *Let (A_i) be a family of discrete abelian groups satisfying the Mittag-Leffler condition, and let $\nu^* : Ab \to \mathrm{Cond}(Ab)$ be the constant sheaf functor. Then $(\nu^* A_i)$ satisfies the Mittag-Leffler condition.*

Proof. The functor ν^* is exact and agrees with the functor $(-) : Ab(\mathrm{Top}) \to \mathrm{Cond}(Ab)$ on discrete abelian groups [4, Lem. 4.2.12]; the latter functor is limit-preserving. An intersection of a decreasing family of abelian groups is a limit of the associated (discrete) topological abelian groups, so we are done. ∎

Proposition C.4 ([4, Lem. 6.1.17]). *Let $\pi : Y \to X$ be a finite morphism of finite presentation. Then $\pi_{\mathrm{proet},*} : D(Y_{\mathrm{proet}}) \to D(X_{\mathrm{proet}})$ has a right adjoint.*

Proof. Since both source and target triangulated categories are compactly generated, it suffices to show that $\pi_{\mathrm{proet},*}$ commutes with direct sums. This can be checked on w-contractible affines. Since evaluation of sheaves of abelian groups on the proétale site at w-contractible affines commutes with colimits, this is easily shown using [4, Lem. 2.4.10]. ∎

In the following we follow [1] and call a topology a family of sieves satisfying the usual axioms, and a pretopology a family of covers satisfying the usual axioms.

Proposition C.5. *The canonical topology on condensed sets induces the proétale topology (i.e., generated by finite jointly surjective families) on profinite sets under the Yoneda embedding.*

Proof. Denote J_{proet}, resp. J_{ind} the proétale topology resp. the topology induced from the canonical topology on condensed sets. Since condensed sets are covered under the canonical topology by profinite sets (under the fully faithful, limit-preserving Yoneda embedding), we conclude by [1, Exp. IV, Cor. 1.2.1] that $\mathrm{Sh}(\mathrm{Top}^{pf}, J_{\mathrm{proet}}) = \mathrm{Cond}(\mathrm{Set}) \simeq \mathrm{Sh}(\mathrm{Top}^{pf}, J_{\mathrm{ind}})$. Since the topology is characterized by its category of sheaves [1, Exp. II, Cor. 4.4.4] we find $J_{\mathrm{proet}} = J_{\mathrm{ind}}$. ∎

Definition C.6 ([4, Def. 4.3.1]). Let G be a compactly generated topological group. The pro-étale site BG_{proet} of G is defined as the site of profinite continuous G-sets with covers given by finite jointly surjective families.

Proposition C.7 ([13, Cor. 2]). *Let G be a compactly generated topological group. We have*

$$\mathrm{Sh}(BG_{\mathrm{proet}}) = \underline{G}\text{-}\mathrm{Cond}(Set).$$

If moreover $G = \mathrm{Gal}(k^{\mathrm{sep}}/k)$ for a field k, then

$$\mathrm{Sh}\left(\mathrm{Spec}(k)_{\mathrm{proet}}\right) = \underline{G}\text{-}\mathrm{Cond}(Set).$$

Proof. The Yoneda embedding from profinite sets to condensed sets if fully faithful, and it can be extended to a right adjoint (hence limit-preserving) faithful functor $\underline{(-)}$, which is moreover fully faithful when restricted to compactly generated topological space [11, Prop. 1.7]. The action of G on a profinite set X is encoded by a map $G \times X \to X$, where the product is computed in topological spaces. The product of a profinite set and a compactly generated space is compactly generated [39, Appendix A, Prop. A.2 (vi)][37], so this is the same as a map $\underline{G} \times \underline{X} \to \underline{X}$. Thus we get a fully faithful embedding of profinite continuous G-sets in $\underline{G}$-Cond(Set). The canonical topology on $\underline{G}$-condensed sets is obtained by forgetting the $\underline{G}$-structure, and similarly for the proétale topology on profinite continuous G-sets, so that the canonical topology on the former induces the proétale topology on the latter by the previous proposition. By [1, Exp. IV, Cor. 1.2.1] it remains only to show that a $\underline{G}$-condensed set is covered by profinite G-sets; for any cover of the underlying condensed set by profinite sets S_i, $G \times S_i$ (where G acts on the first factor) is a cover by profinite G-sets by an adjunction argument.   ∎

D. Strictly henselian local rings of singular schemes

Let X be an integral arithmetic scheme with finite normalization $\pi : Y \to X$, and let $x \in X$ be a closed point. We want to understand the fiber above the generic point of X of the strictly henselian local ring at x in terms of the fibers of strictly henselian local rings of points of Y above x. We can localize at x and we are thus in the following setting: let $(A, \mathfrak{m})$ be a Noetherian integral local ring with fraction field K, perfect residue field k and integral closure B. Can we describe $A^{\mathrm{sh}} \otimes_A K$ in terms of B and its strict henselizations at various maximal ideals?

We first have

Lemma D.1 ([42, Tag 07QQ]).

$$A^{\mathrm{sh}} \otimes_A K = \prod_{i=1}^{n} \kappa(\mathfrak{p}_i)$$

where $\mathfrak{p}_i$ are the prime ideals of $A^{\mathrm{sh}} \otimes_A K$ above (0)*; moreover each $k(\mathfrak{p}_i)/k$ is separable algebraic.*

Proof. A^{sh} is Noetherian and flat over A so it has finitely many minimal primes, which are exactly the primes lying above (0). Moreover it is reduced as A is reduced. Therefore $A^{\mathrm{sh}} \otimes_A K$ is Noetherian and has finitely many prime ideals, all minimals, so $A^{\mathrm{sh}} \otimes_A K$ is an Artinina ring; since it is reduced, its local rings are fields.   ∎

The lemma says that $A^{\mathrm{sh}} \otimes_A K$ is the total rings of fraction of A^{sh}. The total ring of fractions of the normalization of A^{sh} is the same, so we want to determine the normalization of A^{sh}:

---

[37]In the above reference, compactly generated spaces are called k-spaces.

Lemma D.2 ([42, Tag 0CBM]). *$B' := B \otimes_A A^{\mathrm{sh}}$ is the normalization of A^{sh}*

Proof. Normalization commutes with étale maps[38] and filtered colimits, and A^{sh} is a filtered colimit of étale A-algebras. ∎

By the previous lemma, the minimal primes of A^{sh} and B' are in bijection. Moreover, as B' is finite over A^{sh}, B' is a finite product of strictly henselian local rings each finite over A^{sh}. Since B' is normal, each local ring is moreover normal, hence integral. We now see that each minimal prime ideal is contained in a unique maximal ideal of B', and vice-versa. Denote $\mathfrak{q}_i$ the minimal prime of B' corresponding to $\mathfrak{p}_i$ and $\mathfrak{M}_i$ the maximal ideal containing $\mathfrak{q}_i$. We deduce

$$\kappa(\mathfrak{p}_i) = \kappa(\mathfrak{q}_i) = \mathrm{Frac}\left((B')_{\mathfrak{M}_i}\right).$$

Lemma D.3 ([42, Tag 08HV]). *Denote $\mathfrak{n} = \mathfrak{M}_i \cap B$. We have*

$$(B')_{\mathfrak{M}_i} = B_{\mathfrak{n}}^{\mathrm{sh}}.$$

Proof. B' is a filtered colimit of étale B-algebras and so is $B_{\mathfrak{n}}^{\mathrm{sh}}$. Since there is a natural morphism of B-algebras $B' \to B_{\mathfrak{n}}^{\mathrm{sh}}$, $B_{\mathfrak{n}}^{\mathrm{sh}}$ is also a filtered colimit of étale B'-algebras. Since $B_{\mathfrak{n}}^{\mathrm{sh}}$ is a strictly henselian local ring, it must be the strict henselian local ring of B' at any prime above $\mathfrak{n}$. ∎

Thus $\kappa(\mathfrak{q}_i) = \mathrm{Frac}((B')_{\mathfrak{M}_i}) = \mathrm{Frac}(B_{\mathfrak{n}}^{\mathrm{sh}})$. Because B' is a colimit of étale B-algebras, the maximal ideal $\mathfrak{n}$ of B is above $\mathfrak{m}$. Moreover we have:

Lemma D.4 ([42, Tag 0C25]). *The fiber above $\mathfrak{n}$ in B' is isomorphic to $\mathrm{Hom}_k(\kappa(\mathfrak{n}), k^{\mathrm{sep}})$*

Proof. We have

$$B' \otimes_B \kappa(\mathfrak{n}) = A^{\mathrm{sh}} \otimes_A \kappa(\mathfrak{n}) = A^{\mathrm{sh}} \otimes_A k \otimes_k \kappa(\mathfrak{n}) = k^{\mathrm{sep}} \otimes_k \kappa(\mathfrak{n})$$

hence the result. ∎

Combining everything, we obtain

Proposition D.5. *Let $(A, \mathfrak{m})$ be a Noetherian integral local ring with fraction field K, perfect residue field k and integral closure B finite over A. The total ring of fractions of A^{sh} is:*

$$A^{\mathrm{sh}} \otimes_A K = \prod_{\mathfrak{n}} \prod_{\mathrm{Hom}_k(\kappa(\mathfrak{n}), k^{\mathrm{sep}})} \mathrm{Frac}(B_{\mathfrak{n}}^{\mathrm{sh}})$$

where $\mathfrak{n}$ goes through the finitely many maximal ideals of B.

---

[38] It is easy to see the commutation with localization, so this reduces to commutation with standard étale maps.

E. A lemma on determinants of total complexes

Lemma E.1. *Let $\mathcal{A}$ be an abelian category, let C be a double complex of objects of $\mathcal{A}$ with uniformly bounded and acyclic rows and columns and let* det *be a determinant functor* $\mathrm{Ch}^b(\mathcal{A})_{qis} \to \mathcal{P}$. *We denote by $\mathbf{1}$ the unit of the Picard category $\mathcal{P}$. Define the filtration by rows on* $\mathrm{Tot}\, C$ *by* $F_r^n \,\mathrm{Tot}\, C = \mathrm{Tot}(\sigma_r^{\leq n} C)$,[39] *where* $(\sigma_r^{\leq n} C)^{p,q} = C^{p,q}$ *if $q \leq n$ and 0 otherwise. The filtration by rows induces a commutative diagram*

$$
\begin{array}{ccc}
\det \mathrm{Tot}\, C & \xrightarrow{\;\det \mathbf{0}\;} & \mathbf{1} \\
\downarrow & & \downarrow \\
\bigotimes_j (\det C^{\bullet,j})^{(-1)^j} & \xrightarrow[\otimes_j (\det 0)^{(-1)^j}]{} & \bigotimes_j (\mathbf{1})^{(-1)^j}
\end{array}
$$

Proof. The n-th graded piece of the filtration by rows of $\mathrm{Tot}\, C$ is $C^{\bullet,n}[-n]$. Since the rows are exact, the map $0 : \mathrm{Tot}\, C \to 0$ is a quasi-isomorphism; moreover it induces quasi-isomorphisms between the graded pieces of the filtrations by rows on $\mathrm{Tot}\, C$ and the trivial filtration on 0 because columns are exact. We conclude with [24, Prop. 1.7] and [7, Lem. 2.3]. ∎

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[39]Here *r* stands for *rows*.

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