

Global Solutions to the Initial Value Problem for the Nonlinear Boltzmann Equation

By

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Abstract

The nonlinear Boltzmann equation in the rarefied gas dynamics is investigated for the gas molecules with the cut-off hard potential in the sense of Grad. The solution to the initial value problem is proved to exist uniquely in the large in time and to have the decay order of $(1+t)^{-3/4}$ as $t \rightarrow +\infty$ for the small initial data in the space $H_{3,s} \cap L^1(x; L^2(v))$. The decay order is improved to $(1+t)^{-5/4}$ by the additional assumptions on the initial data.

§ 1. Introduction

The nonlinear Boltzmann equation in the rarefied gas dynamics for the gas molecules with the cut-off hard potential in the sense of Grad is given around the absolute Maxwellian state $M = (2\pi)^{-3/2} \exp(-|v|^2/2)$ by the following: (cf. [2], [3])

$$(1.1) \quad \partial f(t, x, v) / \partial t + \sum_{j=1}^3 v_j \partial f(t, x, v) / \partial x_j = Lf + \nu \Gamma(f, f)$$
$$\text{in } t \geq 0, x \in R^3, v \in R^3.$$

$$(1.2) \quad f(0, x, v) = f_0(x, v) \text{ in } x \in R^3, v \in R^3.$$

Here the operator L acting only on v is linear and nonpositive, i.e.,

$$(Lf, f) \leq 0 \text{ for } f \in D(L) \text{ and}$$

$$Lf = 0 \text{ iff } f \in \{\psi_j\}_{j=1}^5 = \{M^{1/2}, v_j M^{1/2}, |v|^2 M^{1/2}\}$$

and it is decomposed as follows:

$$(1.3) \quad Lf = -\nu(v)f + Kf \text{ in } L^2(v),$$

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where $\nu(v)$ is a monotone function in $|v|$ and

$$(1.4) \quad 0 < \nu_0 \leq \nu(v) \leq \nu_1(1 + |v|), \text{ and}$$

$$(1.5) \quad K \text{ is a compact selfadjoint operator on } L^2(v).$$

The operator $\nu\Gamma(f, g)$ depends on v and is bilinear in f, g , and the further properties are stated later.

The initial value problem (1.1) (1.2) is considered by Grad [3], [4] locally in time, and by Ukai [12] in the large in time for the small initial data which are periodic in x , whose solution decays exponentially in t . Here we consider the pure initial value problem (1.1) (1.2) and prove that the solution exists uniquely in the large in time for the initial data $f_0(x, v) \in H_{3,3} \cap L^1(x; L^2(v))$ with the small norm in these spaces and that it decays with the order of $(1+t)^{-3/4}$ as $t \rightarrow +\infty$, where

$$(1.6) \quad H_{l,m} = \{f = f(x, v); \|f\|_{l,m} \equiv \sup_v (1 + |v|)^m \|f(\cdot, v)\|_{H^l} < +\infty\}$$

$$(1.7) \quad H^l \text{ is the Sobolev space of } L^2\text{-functions in } x \text{ with the } l\text{-th } L^2\text{-derivatives, and}$$

$$(1.8) \quad L^p(x; L^2(v)) \text{ is the } L^p\text{-space in } x \text{ with the value in } L^2(v) \text{ and}$$

$$\|f\|_{L^p(x; L^2(v))} = \int \|f(x, \cdot)\|_{L^2(v)} dx.$$

The additional conditions on the initial data such that

$$xf_0(x, v) \in L^1(x; L^2(v)) \text{ and } \iint f_0(x, v) \psi_j(v) dx dv = 0, \quad j = 1, 2, \dots, 5$$

improve the decay order to $(1+t)^{-5/4}$ as $t \rightarrow +\infty$. In order to get the existence theorem and asymptotic decay of solutions, we use the arguments in [3], [12] cited above and those in [1] where the linearized Boltzmann equation is discussed about the fluid dynamical approximation, and those in [5], where the decay of solutions in the whole space as $t \rightarrow +\infty$ is considered for the Broadwell model of Boltzmann equation.

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§ 2. Linearized Boltzmann Equation

We summarize here some known results about the linearized Boltzmann equation,

$$(2.1) \quad \partial f(t, x, v) / \partial t + v \cdot \partial f / \partial x = Lf.$$

It is considered in the Fourier transform in x :

$$(2.2) \quad \begin{aligned} \partial \hat{f}(t, k, v) / \partial t = -ik \cdot v \hat{f} + L\hat{f} \equiv B\hat{f} \\ \text{in } t \geq 0, k \in R^3, v \in R^3. \end{aligned}$$

Let $k \in R^3$ be a fixed parameter and consider the following operators in $L^2(v)$.

$$(2.3) \quad B_k = -ik \cdot v - \nu(v) + K,$$

$$(2.4) \quad A_k = -ik \cdot v - \nu(v),$$

where $D(A_k) = D(B_k) = \{f \in L^2(v), A_k f \in L^2(v)\}$.

Since K is a compact operator, B_k is considered as the compact perturbation from A_k .

Proposition 1. B_k generates the strongly continuous contraction semigroup e^{tB_k} on $L^2(v)$.

Proof. See [1], [9].

Definition. $\sigma(B_k), \sigma_e(B_k)$ and $\sigma_d(B_k)$ denote the spectrum, essential spectrum in the sense of Browder [11] and the set of isolated eigenvalues with finite multiplicity, respectively.

Proposition 2. $\sigma_e(B_k) \subset \sigma_e(A_k) \cup \{\lambda = \beta + i\gamma, \beta \leq -\nu(v), \gamma = -k \cdot v \text{ for } \forall v \in R^3\}$, $\sigma(B_k) = \sigma_e(B_k) \cup \sigma_d(B_k)$.

Proof. See [1], [8].

It is wellknown (cf. [2]) that $0 \in \sigma(B_0) = \sigma(L)$ is an eigenvalue with multiplicity 5 and its eigenspace is spanned by $\{\psi_j\}_{j=1}^5$, and that

the other eigenvalue of B_0 (if exist) are on the negative real axis such that $0 > \lambda_1 \geq \lambda_2 \geq \dots \rightarrow -\nu_0$.

Now let us parametrize $k = \kappa\omega$ where $\kappa \in R^1, \omega \in S^2$.

Proposition 3. *There exists $\delta > 0, \beta_1 > 0$ such that for $|k| < \delta$*

$$\sigma_a(B_k) \cap \{\text{Re } \lambda > -\beta_1\} = \{\alpha_j(k)\}_{j=1}^5,$$

where the perturbed eigenvalues $\alpha_j(k)$ of 0 are smooth in $\kappa, |\kappa| < \delta$ and have the expansion

$$\alpha_j(k) = a_{j,1}(i\kappa) + a_{j,2}(i\kappa)^2 + a_{j,3}(i\kappa)^3 + O(\kappa^4),$$

where

$$a_{j,2} > 0, \quad j=1, 2, \dots, 5.$$

The corresponding eigenfunctions are smooth in κ and have the expansion

$$e_j(k) = e_{j,0}(\omega) + e_{j,1}(\omega)(i\kappa) + e_{j,2}(\omega)(i\kappa)^2 + \dots,$$

and

$$(e_j(-k), e_l(k))_{L^2(\omega)} = \delta_{j,l}, \quad j, l = 1, 2, \dots, 5.$$

Proof. In particular for the hard sphere molecule, the expansions for $\alpha_j(k)$ and $e_j(k)$ converge in $\kappa, |\kappa| < \delta$ and they are analytic in κ there. (cf. [7], [8]) For the general cut-off hardpotential, see [1].

Proposition 4.

(i) For any fixed $\beta < \nu_0$,

$$\sup_{\substack{\text{Re } \lambda \geq -\beta \\ |k| > r}} \|K(\lambda - A_k)^{-1}\| \rightarrow 0 \quad \text{as } r \rightarrow +\infty.$$

(ii) For any fixed $\beta < \nu_0$, any fixed $r > 0$,

$$\sup_{\substack{\text{Re } \lambda \geq -\beta \\ |k| \leq r \\ |\lambda| \geq l}} \|K(\lambda - A_k)^{-1}\| \rightarrow 0 \quad \text{as } l \rightarrow +\infty.$$

(iii) For any fixed $\beta < \nu_0$, there are no eigenvalues of B_k in $\{\text{Re } \lambda \geq -\beta\}$ for large $|k|$. And for $|k| \leq \forall r < +\infty$, there exists $c = c(r)$ such that

$$\sigma_a(B_k) \cap \{\operatorname{Re} \lambda \geq -\beta\} \subset \{|\operatorname{Im} \lambda| \leq c\}.$$

Proof. See [12]. (iii) follows from (i) and (ii).

If we note that the set $\sigma_a(B_k)$ in $\operatorname{Re} \lambda > -\beta (> -\nu_0)$ is continuous in $\kappa \in R^1$ and so it is in $k \in R^3$ by using the rotation in v for B_k , it follows from propositions 3 and 4 that for any $k, |k| > \delta > 0$ there exists $\beta_2 > 0$ such that

$$(2.5) \quad \sigma_a(B_k) \subset \{\operatorname{Re} \lambda < -\beta_2\}.$$

Theorem 1. *There exists $\delta > 0, \beta_1 > 0, \beta_2 > 0$ such that the following (i) and (ii) are valid for any $f \in D(B_k)$.*

(i) *For any $k, |k| \leq \delta,$*

$$(2.6) \quad e^{tB_k}f = \lim_{r \rightarrow +\infty} \frac{1}{2\pi i} \int_{-\beta_1 - ir}^{-\beta_1 + ir} e^{t\lambda} (\lambda - B_k)^{-1} f d\lambda \\ + \sum_{j=1}^5 e^{t\alpha_j(k)} (e_j(-k), f)_{L^2(v)} e_j(k),$$

where $\alpha_j(k)$ and $e_j(k)$ ($j=1, 2, \dots, 5$) are the eigenvalues and the corresponding eigenfunctions of B_k in proposition 3.

(ii) *For any $k, |k| \geq \delta,$*

$$(2.7) \quad e^{tB_k}f = \lim_{r \rightarrow +\infty} \frac{1}{2\pi i} \int_{-\beta_2 - ir}^{-\beta_2 + ir} e^{t\lambda} (\lambda - B_k)^{-1} f d\lambda.$$

Proof. If we note that proposition 4 implies $\|(\lambda - B_k)^{-1}f\| \rightarrow 0$ as $|\operatorname{Im} \lambda| \rightarrow \infty$ for $\operatorname{Re} \lambda \geq -\beta (> -\nu_0)$, [1] gives (i) and (2.5) gives (ii).

Remark The first terms on the right hand side in (2.6) and (2.7) can be rewritten in the form (cf. [12]);

$$\lim_{r \rightarrow +\infty} \frac{1}{2\pi i} \int_{-\beta_j - ir}^{-\beta_j + ir} e^{t\lambda} (\lambda - B_k)^{-1} f d\lambda \\ = e^{tA_k}f + e^{-\beta_j t} \lim_{r \rightarrow +\infty} \frac{1}{2\pi i} \int_{-ir}^{ir} e^{t\lambda} (\lambda - A_k)^{-1} \\ \cdot (I - K(\lambda - A_k)^{-1})^{-1} K(\lambda - A_k)^{-1} f d\lambda$$

$$= e^{tA}f + e^{-\beta t}Z(\beta, t)f,$$

where $\|Z(\beta, t)\| \leq c$ independent of t . ($j=1, 2$)

§ 3. Decay of Solutions for the Linearized Boltzmann Equation

Definition. $H = \{f(x, v) \in L^2(x, v), (x, v) \in R^6\} \subset L^2(x; L^2(v))$,

$$\begin{aligned} \hat{H} &= \left\{ \hat{f}(k, v) = (2\pi)^{-3/2} \int e^{-ik \cdot x} f(x, v) dx \text{ for } f \in H \right\} \\ &= L^2(k, v) \subset L^2(k; L^2(v)), \end{aligned}$$

$$H_l = \{f \in H, |k|^l \hat{f}(k, v) \in \hat{H}\},$$

$$\|f\|_l^2 = \int (1 + |k|)^{2l} \|\hat{f}(k, v)\|_{L^2(v)}^2 dk.$$

Definition. Boltzmann semigroup (cf. [1]).

$$e^{tB}f = (2\pi)^{-3/2} \int e^{ik \cdot x} e^{tB_k} \hat{f}(k, v) dk \text{ for } f(x, v) \in H.$$

Theorem 2. *The solution of the linearized Boltzmann equation (2.1) with the initial $f(x, v)$ has the following decay estimate for $l=0, 1, 2, \dots$.*

(i) *Let $f(x, v) \in H_l \cap L^1(x; L^2(v))$. Then*

$$(3.1) \quad \|e^{tB}f\|_l \leq c(\|f\|_l + \|f\|_{L^1(x; L^2(v))}) / (1+t)^{3/4}.$$

(ii) *In addition to the condition of (i) let $xf(x, v) \in L^1(x; L^2(v))$ and*

$$\iint \phi_j(v) f(x, v) dx dv = 0 \text{ for } j=1, 2, \dots, 5. \text{ Then}$$

$$(3.2) \quad \|e^{tB}f\|_l \leq c(\|f\|_l + \|f\|_{L^1(x; L^2(v))} + \|xf\|_{L^1(x; L^2(v))}) / (1+t)^{5/4}.$$

(iii) *Let $f \in H_l$, $\|\hat{f}(k, v)\|_{L^2(v)} \leq c$ for any $k \in R^3$ and*

$$\int \phi_j(v) f(x, v) dv = 0 \text{ for a.a. } x \in R^3, j=1, 2, \dots, 5. \text{ Then}$$

$$(3.3) \quad \|e^{tB}f\| \leq c(\|f\|_l + \sup_k \|\hat{f}(k, \cdot)\|_{L^2(v)}) / (1+t)^{5/4}.$$

It is shown in [10] that the decay of solution to (2.1) as $t \rightarrow +\infty$

without the decay order for the hard sphere molecule.

Proof. Let us estimate

$$\begin{aligned} \|e^{tB}f\|_l^2 &= \int (1 + |k|)^{2l} \|e^{tB_k}f(k, v)\|_{L^2(v)}^2 dk \\ &= \int_{|k| \leq \delta} + \int_{|k| > \delta} = I + I_1. \end{aligned}$$

By Theorem 1 (ii) and the remark to it,

$$I_1 \leq c^2 e^{-2\beta_2 t} \int_{|k| > \delta} (1 + |k|)^{2l} \|\hat{f}(k, v)\|_{L^2(v)}^2 dk.$$

For $|k| \leq \delta$ we decompose I as follows;

$$I = \int_{|k| \leq \delta} \leq I_2 + I_3$$

where I_2 and I_3 correspond to the first term and the second term in the right hand side of (2.6) for $e^{tB_k}\hat{f}$ respectively. Therefore using again the remark to Theorem 1 we have

$$\begin{aligned} I_2 &\leq \int_{|k| \leq \delta} c^2 e^{-\beta_1 t} (1 + |k|)^{2l} \|\hat{f}(k, v)\|_{L^2(v)}^2 dk, \\ I_3 &\leq \int_{|k| \leq \delta} c^2 (1 + |k|)^{2l} \left\| \sum_{j=1}^5 e^{t\alpha_j(k)} (e_j(-k), \hat{f}(k, \cdot)) \right\|_{L^2(v)}^2 c_j(k) \|_{L^2(v)}^2 dk \\ &\leq c^2 \sup_{|k| \leq \delta} \|\hat{f}(k, \cdot)\|_{L^2(v)}^2 \int_{|k| \leq \delta} e^{-a|k|^{2l}} dk \\ &\leq c^2 \|f\|_{L^1(x; L^2(v))}^2 / (1 + t)^{3/2}, \quad (a = \min_j \alpha_j, 2). \end{aligned}$$

Thus we have for $\beta_0 = \min\{\beta_1, \beta_2\}$

$$\begin{aligned} \|e^{tB}f\|_l^2 &\leq I_1 + I_2 + I_3 \\ &\leq c^2 e^{-2\beta_0 t} \|f\|_l^2 + c^2 \|f\|_{L^1(x; L^2(v))}^2 / (1 + t)^{3/2}. \end{aligned}$$

(ii) and (iii) are proved analogously. (cf. [5]). Q.E.D.

§ 4. Existence and Decay of Solutions for the Nonlinear Boltzmann Equation

In order to get the existence and the decay estimate of the solutions to the Cauchy problem (1.1) (1.2) we follow the line of [3] and

[12], when the space of functions $H_{l,m}$ for $l, m \geq 3$ is used. (cf. (1.6))

Theorem 3. *Let $l, m \geq 3$*

(i) *Let the initial data $f_0(x, v) \in H_{l,m} \cap L^1(x; L^2(v))$ and put*

$$E_1 = \|f_0\|_{l,m} + \int \|f_0(x, \cdot)\|_{L^2(v)} dx.$$

There exists a constant $E_0 > 0$ such that for any initial data with $E_1 < E_0$ the solution $f(t)$ of (1.1) (1.2) exists uniquely in the large in time and has the decay estimate,

$$\|f(t)\|_{l,m} \leq cE_1 / (1+t)^{3/4}.$$

(ii) *In addition let the initials satisfy $xf_0(x, v) \in L^1(x; L^2(v))$ and*

$$\iint \psi_j(v) f_0(x, v) dx dv = 0, \quad j=1, 2, \dots, 5.$$

Put $E_2 = E_1 + \int \|xf_0(x, v)\|_{L^2(v)} dx$.

Then the decay estimate of the solution of (i) improves to

$$\|f(t)\|_{l,m} \leq cE_2 / (1+t)^{5/4}.$$

Proof. In order to obtain the solution of (1.1), (1.2) we introduce the iteration

$$(4.1) \quad f^n(t) = e^{tB} f_0 + \int_0^t e^{(t-s)B} \nu \Gamma(f^{n-1}(s), f^{n-1}(s)) ds, \quad n=1, 2, \dots.$$

$$f^0(t) = 0$$

in $H_{l,m} \subset H_l$ where e^{tB} is the linear solution operator in Theorem 2. To prove the convergence of this iteration globally in time, we use the integral equation

$$(4.2) \quad f(t) = e^{tB} f_0 + \int_0^t e^{(t-s)B} \nu g(s) ds$$

where $\nu g(t) = \nu(v)g(t, x, v)$ is assumed to satisfy the conditions of Theorem 2, (iii), and follow Grad's and Ukai's arguments ([3], [12]). We use the operators $e^{tA} f = e^{-tv(v)} f(x-tv, v)$ and a version of (4.2)

$$(4.3) \quad f(t) = e^{tA}f_0 + \int_0^t e^{(t-s)A}\nu g(s) ds + \int_0^t e^{(t-s)A}Kf(s) ds$$

to estimate (4.2) in $H_{l,m}$. Using the property of K ([2], [3]) (4.3) has the next estimates for $\beta=3/4, 5/4$

$$(4.4) \quad \|f\|_{\beta}^{l,0} \leq c(\|f_0\|_{l,0} + \|g\|_{\beta}^{l,0} + \|f\|_{\beta}^l)$$

$$(4.5) \quad \|f\|_{\beta}^{l,j} \leq c(\|f_0\|_{l,j} + \|g\|_{\beta}^{l,j} + \|f\|_{\beta}^{l,j-1}), \quad j \geq 1,$$

where c means some positive constant and

$$\|f\|_{\beta}^{l,j} \equiv \sup_{t \geq 0} (1+t)^{\beta} \|f(t)\|_{l,j},$$

$$\|f\|_{\beta}^l \equiv \sup_{t \geq 0} (1+t)^{\beta} \|f(t)\|_l.$$

Iterating these inequalities $j=0, 1, 2 \dots m$, we obtain

$$(4.6) \quad \|f\|_{\beta}^{l,m} \leq c(\|f_0\|_{l,m} + \|g\|_{\beta}^{l,m} + \|f\|_{\beta}^l).$$

To estimate $\|f\|_{\beta}^l$ we consider (4.2) in H_l . By the aid of the inequality

$$(4.7) \quad (1+t)^{\beta} \int_0^t \frac{ds}{(1+s)^{2\beta} \{1+(t-s)\}^{5/4}} \leq c$$

c : independent of t ,

Theorem 2 gives

$$(4.8) \quad \|f\|_{\beta}^l \leq c(E + \|g\|_{2\beta}^{l,m} + \sup_{\substack{k \in \mathbb{R}^3 \\ t \geq 0}} (1+t)^{2\beta} \|(\nu g)^{\wedge}\|_{L^2(v)})$$

where $E=E_1, E_2$ for $\beta=3/4, 5/4$ respectively. From (4.6) and (4.8) it follows that

$$(4.9) \quad \|f\|_{\beta}^{l,m} \leq aE + b(\|g\|_{2\beta}^{l,m} + \sup_{\substack{k \in \mathbb{R}^3 \\ t \geq 0}} (1+t)^{2\beta} \|(\nu g)^{\wedge}\|_{L^2(v)}),$$

a, b : positive constants.

Now we can set $g(t) = \Gamma(h(t), h(t))$, $h \in H_{l,m}$. By virtue of [3] it has the estimates

$$(4.10) \quad \|\Gamma(h, h)\|_{2\beta}^{l,m} \leq c(\|h\|_{\beta}^{l,m})^2,$$

$$(4.11) \quad \sup_{\substack{k \in \mathbb{R}^3 \\ t \geq 0}} (1+t)^{2\beta} \|(\nu \Gamma(h, h))^{\wedge}\|_{L^2(v)} \leq c \sup_{t \geq 0} (1+t)^{2\beta} \|\nu \Gamma(h, h)\|_{L^2(v; L^1(x))}$$

$$\leq c(\|h\|_{\beta}^{0,m})^2.$$

Hence we obtain

$$(4.12) \quad \|f\|_{\beta}^{l,m} \leq aE + bc(\|h\|_{\beta}^{l,m})^2.$$

This inequality shows that there exists a constant E_0 such that for any $E < E_0$ the iteration (4.1) converges to the global solution in $H_{l,m}$.

Q.E.D.

Remark 1. While typing this work we received an abstract of Prof. Ukai that contains the existence theorem of the global solutions for the Cauchy problem of the nonlinear Boltzmann equation for the hard sphere molecule with the small initial data having no fluid part.

Remark 2. We can get the global solution of (1.1), (1.2) without the restriction $f_0 \in L^1(x; L^2(v))$. To prove this remark, we have only to note the facts

$$\|e^{tB}f_0\|_l \leq c\|f_0\|_l \leq c\|f_0\|_{l,m}$$

and (4.7) for $\beta=0$. Then the inequality (4.12) will become

$$\|f\|_0^{l,m} \leq a\|f_0\|_{l,m} + b(\|h\|_0^{l,m})^2.$$

Hence the iteration (4.1) gives the bounded (in t) solution in $H_{l,m}$, $l, m \geq 3$.

Moreover if we note that in this case the x -derivatives of the solution have the decay order as

$$\left\| \frac{\partial f(t)}{\partial x_i} \right\|_{2,3} \leq \frac{c\|f_0\|_{3,3}}{(1+t)^{1/2}}, \quad \left\| \frac{\partial^2 f(t)}{\partial x_i \partial x_j} \right\|_{1,3} \leq \frac{c\|f_0\|_{3,3}}{(1+t)},$$

we can get the decay of the solution $f(t)$ in the norm of $H_{0,3}$ by the aid of the Nirenberg's inequality^(*);

$$\|u(x)\|_{L^\infty} \leq c \left(\sum_{i,j} \|\partial^2 u / \partial x_i \partial x_j\|_{L^2} \right)^{3/4} (\|u\|_{L^2})^{1/4}.$$

In fact we know that by the Lebesgue's theorem

$$\|e^{tB}f_0\|_{0,3} \rightarrow 0 \quad \text{as } t \rightarrow +\infty$$

and we have for the nonlinear part

^(*) L. Nirenberg, On elliptic partial differential equations, *Annali Scuola Norm. Sup. Pisa*, 13 (1959) 115-162.

$$\begin{aligned}
\int_0^t \|e^{(t-s)B} \nu \Gamma(f(s), f(s))\|_{H_0} ds &\leq \int_0^t \frac{c \|\nu \Gamma(s)\|_{H_0}}{(1+t-s)^{1/2}} ds \\
&\leq \int_0^t \frac{c \|f(s)\|_{3,3} \cdot \sup_v (1+|v|)^3 \|f(s)\|_{L^\infty}}{(1+t-s)^{1/2}} ds \\
&\leq \int_0^t \frac{c \|f(s)\|_{3,3}^{5/4} \cdot (\sum \|\partial^2 f(s) / \partial x_i \partial x_j\|_{0,3})^{3/4}}{(1+t-s)^{1/2}} ds \\
&\leq \int_0^t c \|f_0\|_{3,3}^2 ds / (1+t-s)^{1/2} (1+s)^{3/4} \\
&\leq c \|f_0\|_{3,3}^2 / (1+t)^{1/4}.
\end{aligned}$$

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