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Scattering Theory for Wave Equations with Dissipative Terms

By

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§ 1. Introduction

We shall consider wave equations of the form

$$w_{tt}(x,t)+b(x,t)w_t(x,t)-\Delta w(x,t)=0\,, \tag{1·1}$$

where $x\in \boldsymbol{R}^n$ $(n\neq 2)$, $t\geq 0$, $w_t=\partial w/\partial t$, $w_{tt}=\partial^2 w/\partial t^2$ and Δ is the n-dimensional Laplacian. $b(x,t)$ is a non-negative function and is assumed to satisfy the following conditions:

(A1) There exist constants $C_1>0$ and $\delta>0$ such that

$$0\leq b(x,t)\leq C_1(1+|x|)^{-1-\delta} \quad \text{for any} \quad x\in \boldsymbol{R}^n,\ t\geq 0\,.$$

(A2) $b_t(x,t)$ is bounded continuous in $x\in \boldsymbol{R}^n$ and $t\geq 0$.

In the following we assume that $\delta\leq 1$ without any loss of genelarity. Since $b(x,t)\geq 0$, $b(x,t)w_t(x,t)$ represents the resistance of viscous type. Our aim of this note is to show that the solutions of (1·1) are asymptotically equal for $t\rightarrow\infty$ to those of the free wave equation

$$w^0_{tt}(x,t)-\Delta w^0(x,t)=0\,. \tag{1·2}$$

More precisely, we shall show the existence of the Møller wave operators.

We restrict ourselves to solutions with finite energy. For pairs $f=\{f_1,f_2\}$ of functions in $\boldsymbol{R}^n$ the energy is defined by

$$\|f\|_E^2=\int_{R^n}(|Df_1|^2+|f_2|^2)\,dx\,, \tag{1·3}$$

where $Df_1=(D_1f_1,\cdots,D_nf_1)$ $(D_j=\partial/\partial x_j)$ and $|Df_1|^2=\sum_{j=1}^n|D_jf_1|^2$. The Hilbert space $\mathscr{H}$ is defined as the completion in the energy norm of

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smooth data with bounded support in $\mathbf{R}^n$. Put $u=\{w, w_t\}$. Then (1·1) can be expressed in the matrix notation as

$$u_t = A(t)u = A_0 u - V(t)u\,, \tag{1·4}$$

where

$$A_0 = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} \quad \text{and} \quad V(t) = \begin{pmatrix} 0 & 0 \\ 0 & b(x,t) \end{pmatrix}.$$

Put $u^0 = \{w^0, w_t^0\}$. Then (1·2) is expressed as

$$u_t^0 = A_0 u^0\,. \tag{1·5}$$

A_0 determines a skew-selfadjoint operator in $\mathcal{H}$ with domain

$$\mathcal{D}(A_0) = \{f \in \mathcal{H};\, \Delta f_1, D_j f_2 \in L^2(\mathbf{R}^n) \quad (j=1, \cdots, n)\}, \tag{1·6}$$

where all the derivatives are considered in the distribution sense. Thus, A_0 generates a one-parameter group $\{U_0(t) = e^{A_0 t};\, t \in \mathbf{R}\}$ of unitary operators. Under the above conditions on $b(x,t)$, $A(t)$ determines for each $t \geq 0$ a closed operator in $\mathcal{H}$ with domain $\mathcal{D}(A(t)) = \mathcal{D}(A_0)$. Moreover, positive numbers belong to the resolvent set of each $A(t)$ and $A(t)(A(0)-I)^{-1}$, where I is the identity in $\mathcal{H}$, is continuously differentiable in t in operator norm. Thus applying results of Kato [2], we see that there exists a unique family $\{U(t,s);\, t \geq s \geq 0\}$ of contraction evolution operators which is defined as mapping solution data of (1·4) at time s into those at time t.

Now the main results can be stated as follows:

Theorem 1. (a) *The wave operator*

$$Z = \text{strong} \lim_{t\to\infty} U_0(-t)U(t,0) \tag{1·7}$$

exists. (b) *Z is a not identically vanishing contraction operator in $\mathcal{H}$.* (c) *If we denote by Z^* the adjoint of Z, then*

$$Z^* = \text{strong} \lim_{t\to\infty} U(t,0)^* U_0(t). \tag{1·8}$$

We also consider the special case where $b(x,t)$ is independent of t. Then the operator $A = A_0 - V$, where $V = \begin{pmatrix} 0 & 0 \\ 0 & b(x) \end{pmatrix}$, generates a semi-group $\{U(t);\, t \geq 0\}$ of contraction operators.

In this case we have the following

Theorem 2. (a) *The wave operators*

$$W=\text{strong}\lim_{t\to\infty} U(t)U_0(-t), \tag{1·9}$$

$$Z=\text{strong}\lim_{t\to\infty} U_0(-t)U(t) \tag{1·10}$$

exist. (b) *They both are not identically vanishing contraction operators in* $\mathcal{H}$. (c) $U_0(t)$ *and* $U(t)$ *are intertwined by both* W *and* Z, *i.e.,*

$$WU_0(t)=U(t)W,\ ZU(t)=U_0(t)Z \quad \textit{for any } t\geq 0. \tag{1·11}$$

(d) *The scattering operator, defined by* $S=ZW$, *commutes with* $U_0(t)$:

$$SU_0(t)=U_0(t)S \quad \textit{for any } t\in\boldsymbol{R}. \tag{1·12}$$

The proof of these theorems will be based on the "smooth perturbation theory" developed by Kato [3].

The above theorems generalize some results already announced in Mochizuki [7], where the main concern was in the local energy decay for wave equations with non-linear dissipative terms. The scattering theory has been developed by Lax-Phillips [4] for wave equation: $w_{tt}=\Delta w$ in an exterior domain of $\boldsymbol{R}^n$ ($n\geq 2$) with lossy boundary conditions: $w_n+\alpha(x)w_t=0$, $\alpha(x)\geq 0$. Some related problems has been studied in [1] and [5].

§ 2. Preliminaries

First we shall show an inequality for L^2-solutions of the Helmholtz equation

$$-\Delta u-\kappa^2 u=f(x) \text{ in } \boldsymbol{R}^n, \tag{2·1}$$

where κ is a complex number such that Im $\kappa\neq 0$ and $f(x)$ is a function such that $(1+|x|)^{(1+\delta)/2}f(x)\in L^2(\boldsymbol{R}^n)$.

Lemma 2. 1. *Let* Im $\kappa\gtrless 0$. *Then we have for any* $\rho>0$

$$(2\cdot 2)\qquad \frac{1}{2}\int_{S_\rho}\left(\left|\frac{\partial u}{\partial r}+\frac{n-1}{2r}u\right|^2+|\kappa|^2|u|^2\right)dS$$

$$+|\mathrm{Im}\,\kappa|\int_{K_\rho}\left(|Du|^2+\frac{n-1}{2r}|u|^2+|\kappa|^2|u|^2\right)dx$$

$$=\frac{1}{2}\int_{S_\rho}|\theta_\pm|^2dS\mp\int_{K_\rho}\mathrm{Re}[f\,\overline{i\kappa u}]dx,$$

where $r=|x|$, $S_\rho=\{x;|x|=\rho\}$, $K_\rho=\{x;|x|<\rho\}$ *and*

$$(2\cdot 3)\qquad \theta_\pm=\frac{\partial u}{\partial r}+\frac{n-1}{2r}u\mp i\kappa u.$$

Proof. Note the identity

$$-\mathrm{Re}\left[\frac{\partial u}{\partial r}\overline{i\kappa u}\right]=-\mathrm{Im}\,\kappa\frac{n-1}{2r}|u|^2\pm\frac{1}{2}|\theta_\pm|^2\mp\frac{1}{2}\left(\left|\frac{\partial u}{\partial r}+\frac{n-1}{2r}u\right|^2+|\kappa|^2|u|^2\right).$$

Then $(2\cdot 2)$ follows from the integration by parts of $(2\cdot 1)$ multiplied by $\overline{i\kappa u}$.

Lemma 2.2. *Let* Im $\kappa\gtrless 0$. *Then we have*

$$(2\cdot 4)\qquad |\mathrm{Im}\,\kappa|\int_{\boldsymbol{R}^n}r^\delta\left\{|\zeta_\pm|^2+\frac{(n-1)(n-3)}{4r^2}|u|^2\right\}dx$$

$$+\int_{\boldsymbol{R}^n}r^{-1+\delta}\left\{\left(1-\frac{\delta}{2}\right)(|\zeta_\pm|^2-|\theta_\pm|^2)+\frac{\delta}{2}|\theta_\pm|^2\right\}dx$$

$$+\frac{(n-1)(n-3)(2-\delta)}{8}\int_{\boldsymbol{R}^n}r^{-3+\delta}|u|^2dx=\int_{\boldsymbol{R}^n}r^\delta\mathrm{Re}[f\overline{\theta_\pm}]dx,$$

where

$$(2\cdot 5)\qquad \zeta_\pm=Du+\frac{n-1}{2r}\frac{x}{r}u\mp i\kappa\frac{x}{r}u.$$

Proof (cf., Mochizuki [6]). Put $v=e^{\mp i\kappa r}r^{(n-1)/2}u$. Then

$$(2\cdot 6)\qquad -\Delta v+\left(\frac{n-1}{2}\mp 2i\kappa\right)\frac{\partial v}{\partial r}+\frac{(n-1)(n-3)}{4r^2}v=e^{\mp i\kappa r}r^{(n-1)/2}f.$$

Multiply by $e^{\mp 2\,\mathrm{Im}\,\kappa r}r^{-n+1+\delta}(\partial\bar{v}/\partial r)$ on both sides and take the real parts. Then the repeated use of integration by parts gives $(2\cdot 4)$ if we note

$$(2\cdot7)\qquad \zeta_{\pm}=e^{\mp i\kappa r}r^{-(n-1)/2}Dv \quad\text{and}\quad \theta_{\pm}=\sum_{j=1}^{n}\frac{x_j}{r}[\zeta_{\pm}]_j,$$

where $[\zeta_{\pm}]_j$ is the j-th component of $\zeta_{\pm}$.

Proposition 2.1. *Let u be a L^2-solution of* $(2\cdot1)$. *Then there exists a constant $C_2>0$ such that for any $\kappa\in\boldsymbol{C}-\boldsymbol{R}$*

$$(2\cdot8)\qquad |\kappa|^2\int_{\boldsymbol{R}^n}(1+r)^{-1-\delta}|u|^2dx\leq C_2\int_{\boldsymbol{R}^n}(1+r)^{1+\delta}|f|^2dx.$$

Proof. Multiply by $(1+\rho)^{-2\delta}\rho^{-1+\delta}$ on both sides of $(2\cdot2)$ and integrate over $[0,\infty)$. Then we have

$$(2\cdot9)\qquad \frac{1}{2}|\kappa|^2\int_{\boldsymbol{R}^n}(1+r)^{-2\delta}r^{-1+\delta}|u|^2dx$$
$$\leq\frac{1}{2}\int_{\boldsymbol{R}^n}r^{-1+\delta}|\theta_{\pm}|^2dx+C(\delta)\int_{\boldsymbol{R}^n}|f\bar{i\kappa u}|dx.$$

On the other hand, noting that $n\neq2$, $0<\delta\leq1$ and $|\zeta_{\pm}|\geq|\theta_{\pm}|$, we have from $(2\cdot4)$

$$(2\cdot10)\qquad \int_{\boldsymbol{R}^n}r^{-1+\delta}|\theta_{\pm}|^2dx\leq\left(\frac{2}{\delta}\right)^2\int_{\boldsymbol{R}^n}r^{1+\delta}|f|^2dx.$$

Inequality $(2\cdot8)$ then follows if we note $(1+r)^{-1-\delta}\leq(1+r)^{-2\delta}r^{-1+\delta}$.

§ 3. Proof of Theorem 1

(a) Let $f=\{f_1,f_2\}\in\mathcal{H}$. Then $u(t)=U(t,0)f$ satisfies $(1\cdot4)$ and the initial condition $u(0)=f$. Since A_0 is skew-selfadjoint, we have from $(1\cdot4)$

$$(3\cdot1)\qquad U_0(-t)U(t,0)f=f-\int_0^t U_0(-\tau)V(\tau)U(\tau,0)fd\tau$$

and

$$(3\cdot2)\qquad \|U(t,0)f\|_E^2+2\int_0^t\|\sqrt{V(\tau)}U(\tau,0)f\|_E^2d\tau=\|f\|_E^2.$$

We put

$$(3\cdot3)\qquad A=\begin{pmatrix}0&0\\0&a(x)\end{pmatrix},\quad a(x)=\sqrt{C_1}(1+|x|)^{-(1+\delta)/2}.$$

Note that $A \geq \sqrt{V(t)}$. Then for any $g \in \mathcal{H}$

$$(3\cdot4) \qquad \int_s^t |(U_0(-\tau)V(\tau)U(\tau, 0)f, g)_E| d\tau$$

$$\leq \Big(\int_s^t \|\sqrt{V(\tau)}U(\tau, 0)f\|_E^2 d\tau\Big)^{1/2}\Big(\int_s^t \|AU_0(\tau)g\|_E^2 d\tau\Big)^{1/2},$$

where $(\ ,\ )_E$ denotes the inner product in $\mathcal{H}$. Thus, to see the existence of the strong limit of $(3\cdot1)$ as $t\to\infty$, it is sufficient to prove that there exists a constant $C_3>0$ such that

$$(3\cdot5) \qquad \int_0^\infty \|AU_0(t)g\|_E^2 dt \leq C_3\|g\|_E^2 \quad \text{for any} \quad g\in\mathcal{H}.$$

The following result is due to Kato [3].

Proposition 3.1. *There exists a* $C_3>0$ *satisfying* $(3\cdot5)$ *if the operator* A *satisfies the condition*

$$(3\cdot6) \qquad \sup_{\kappa\in \boldsymbol{C}-\boldsymbol{R}} \|A(A_0-i\kappa I)^{-1}A\|_E < \infty .$$

For $g=\{g_1, g_2\}\in\mathcal{H}$ put

$$(3\cdot7) \qquad u=\{u_1, u_2\}=(A_0-i\kappa I)^{-1}Ag .$$

Then, as is easily seen, the second component u_2 satisfies equation $(2\cdot1)$ with $f=-i\kappa a(x)g_2$. Thus, by Proposition 2.1 we have

$$(3\cdot8) \qquad |\kappa|^2\int_{\boldsymbol{R}^n}(1+r)^{-1-\delta}|u_2|^2dx \leq C_2\int_{\boldsymbol{R}^n}(1+r)^{1+\delta}|i\kappa a(x)g_2|^2dx$$

$$\leq C_1C_2|\kappa|^2\int_{\boldsymbol{R}^n}|g_2|^2dx .$$

Since $A(A_0-i\kappa I)^{-1}Ag=\{0, a(x)u_2\}$, it follows from $(3\cdot8)$ that

$$(3\cdot9) \qquad \|A(A_0-i\kappa I)^{-1}Ag\|_E^2=\int_{\boldsymbol{R}^n}|a(x)u_2|^2dx$$

$$\leq C_1^2C_2\int_{\boldsymbol{R}^n}|g_2|^2dx \leq C_1\, C_2\|g\|_E^2 .$$

This proves that A satisfies condition $(3\cdot6)$. Hence, $(3\cdot5)$ holds and the wave operator Z exists.

(b) To show the existence of $f\in\mathcal{H}$ such that $Zf\neq0$, we assume contrary, i.e., for any $f\in\mathcal{H}$ $\|U(t,0)f\|_E\to 0$ as $t\to\infty$. Then we have from (3·2)

$$\|f\|_E^2=2\int_0^\infty\|\sqrt{V(t)}\,U(t,0)f\|_E^2dt. \tag{3·10}$$

Further, by (3·1) and (3·4)

$$\|f\|_E^2\leq\left(\int_0^\infty\|\sqrt{V(t)}\,U(t,0)f\|_E^2dt\right)^{1/2}\left(\int_0^\infty\|AU_0(t)f\|_E^2dt\right)^{1/2}. \tag{3·11}$$

Hence, it follows that

$$\|f\|_E^2\leq\frac{1}{2}\int_0^\infty\|AU_0(t)f\|_E^2dt. \tag{3·12}$$

Put $f=U_0(s)g$, where $\|g\|_E=1$. Then by (3·12)

$$\|U_0(s)g\|_E^2=1\leq\frac{1}{2}\int_s^\infty\|AU_0(t)g\|_E^2dt\to 0,\quad\text{as}\quad s\to\infty \tag{3·13}$$

(cf., (3·5)). This is a contradiction and (b) is proved.

(c) It follows from (3·5) that in (3·4)

$$\int_s^t\|AU_0(\tau)g\|_E^2d\tau\to 0\quad\text{as}\quad s,t\to\infty. \tag{3·14}$$

On the other hand, we have from (3·2)

$$\int_0^\infty\|\sqrt{V(t)}U(t,0)f\|_E^2dt\leq\frac{1}{2}\|f\|_E^2\quad\text{for any}\quad f\in\mathcal{H}. \tag{3·15}$$

Thus, $U(t,0)^*U_0(t)g$ converges in $\mathcal{H}$ as $t\to\infty$ and (c) is proved.

§ 4. Proof of Theorem 2

The assertions (a) and (b) for the operator W can be proved by the same argument as in the proof of Theorem 1 if we note that the adjoint semigroup $U(t)^*$ has generator

$$\Lambda^*=-\Lambda_0-V\quad\text{with domain}\quad\mathcal{D}(\Lambda^*)=\mathcal{D}(\Lambda_0). \tag{3·16}$$

(c) and (d) are obvious from the definition of W and Z.

References

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Added in Proof. Recently, Mr. A. Matsumura (Dept. Appl. Math. Phys., Fac. Engi., Kyoto U.) obtained the following result: If $b(x,t)$ in $(1\cdot 1)$ satisfies $t\gtrsim 0$

$$b_t(x,t)\leq 0 \quad \text{and} \quad \min_{|x|\leq R+t} b(x,t)\geq \frac{1}{K+\varepsilon t},$$

where R, K, ε are positive constants, and if the initial data $f=\{f_1, f_2\}$ has support contained in $\{x; |x|\leq R\}$, then the total energy of solution of $(1\cdot 1)$ decays like

$$\|U(t,0)f\|_E = O(t^{-1/(2+3\varepsilon)}) \quad \text{as} \quad t\to\infty.$$

By this result we can say that our assumption (A1) is settled in a sense.