Ann. Inst. H. Poincaré
Anal. Non Linéaire 39 (2022), 245–325
DOI 10.4171/AIHPC/8

Stability threshold of two-dimensional Couette flow in Sobolev spaces

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Abstract. We study the stability threshold of the two-dimensional Couette flow in Sobolev spaces at high Reynolds number Re. We prove that if the initial vorticity Ω_{in} satisfies $\|\Omega_{\text{in}} - (-1)\|_{H^\sigma} \le \varepsilon \text{Re}^{-1/3}$, then the solution of the two-dimensional Navier–Stokes equation approaches some shear flow which is also close to Couette flow for time $t \gg \text{Re}^{1/3}$ by a mixing-enhanced dissipation effect, and then converges back to Couette flow when $t \to +\infty$.

1. Introduction

In this paper, we consider the two-dimensional incompressible Navier–Stokes equations in $\mathbb{T} \times \mathbf{R}$:

$$\begin{cases} \partial_t V + V \cdot \nabla V + \nabla P - \nu \Delta V = 0, \\ \nabla \cdot V = 0, \\ V|_{t=0} = V_{\text{in}}(x, y), \end{cases} \tag{1.1}$$

where ν denotes the viscosity, which is the multiplicative inverse of the Reynolds number Re. We denote by $V = (V^1, V^2)$ and P the velocity and the pressure of the fluid respectively. Let $\Omega = \partial_x V^2 - \partial_y V^1$ be the vorticity, which satisfies

$$\Omega_t + V \cdot \nabla \Omega - \nu \Delta \Omega = 0. \tag{1.2}$$

The Couette flow $(y, 0)$ is a steady solution of (1.1).

Now we introduce the perturbation: let $\Omega = \omega - 1$ and $V = (y, 0) + (U^x, U^y)$; then $\omega = \partial_x U^y - \partial_y U^x$ satisfies

$$\begin{cases} \partial_t \omega + y \partial_x \omega - \nu \Delta \omega = -U \cdot \nabla \omega, \\ \omega|_{t=0} = \omega_{\text{in}}(x, y), \end{cases} \tag{1.3}$$

and $U = (U^x, U^y) = (-\partial_y \psi, \partial_x \psi)$ with $\Delta \psi = \omega$.

The study of (1.3) for small perturbations is an old problem in hydrodynamic stability, considered by both Rayleigh ([37]) and Kelvin ([28]), as well as by many modern authors

2020 Mathematics Subject Classification. 35B35, 35B40, 35Q30.
Keywords. Sobolev spaces, Couette flow.

with new perspectives (see e.g. the classical texts [19, 45] and the references therein). Rayleigh and Kelvin both studied the linearization of (1.3), which is simply

$$\begin{cases} \partial_t\omega + y\partial_x\omega - \nu\Delta\omega = 0, \\ \Delta\psi = \omega, \\ \omega|_{t=0} = \omega_{\mathrm{in}}(x,y). \end{cases} \tag{1.4}$$

Indeed, if we denote by $\widehat{\omega}(t,k,\eta)$ the Fourier transform of $\omega(t,x,y)$, then the solution of (1.4) can be written as

$$\begin{aligned} \widehat{\omega}(t,k,\eta) &= \widehat{\omega}_{\mathrm{in}}(k,\eta+kt)\exp\Big(-\nu\int_0^t |k|^2 + |\eta - ks + kt|^2\,ds\Big), \\ \widehat{\psi}(t,k,\eta) &= \frac{-\widehat{\omega}_{\mathrm{in}}(k,\eta+kt)}{k^2+\eta^2}\exp\Big(-\nu\int_0^t |k|^2 + |\eta - ks + kt|^2\,ds\Big), \end{aligned} \tag{1.5}$$

which gives

$$\begin{aligned} \|\partial_y P_{\neq}\psi\|_{L^2} + \langle t\rangle\|\partial_x P_{\neq}\psi\|_{L^2} &\le C\langle t\rangle^{-1}e^{-c\nu t^3}\|P_{\neq}\omega_{\mathrm{in}}\|_{H^2}, \\ \|P_{\neq}\omega\|_{L^2} &\le C\|P_{\neq}\omega_{\mathrm{in}}\|_{L^2}e^{-c\nu t^3}, \end{aligned} \tag{1.6}$$

where here we denote by $P_{\neq}f = f(x,y) - \frac{1}{2\pi}\int_{\mathbb{T}} f(x,y)\,dx$ the projection to nonzero mode of f. The first inequality in (1.6) is the inviscid damping and the second one is the enhanced dissipation. These two results are both related to the vorticity mixing effect.

In [36], Orr observed the important phenomenon that the velocity will tend to 0 as $t \to \infty$, even for a time reversible system such as the Euler equations ($\nu = 0$). This phenomenon is so-called inviscid damping, which is the analogue in hydrodynamics of Landau damping found by Landau ([29]), which predicted the rapid decay of the electric field of the linearized Vlasov equation around homogeneous equilibrium. Mouhot and Villani ([35]) made a breakthrough and proved nonlinear Landau damping for perturbation in the Gevrey class (see also [8]). In this case, the mechanism leading to the damping is the vorticity mixing driven by shear flow or the Orr mechanism ([36]). See [2, 22, 38, 39, 44] for similar phenomena in various systems. We point out that inviscid damping for general shear flow is a challenge problem even at the linear level due to the presence of the nonlocal operator for general shear flow. For linear inviscid damping we refer to [20, 26, 41, 46] for the results of general monotone flows. For nonmonotone flows such as Poiseuille flow and Kolmogorov flow, another dynamic phenomenon should be taken into consideration, which is the so-called vorticity depletion phenomenon, predicted by Bouchet and Morita ([12]) and later proved by Wei, Zhang and Zhao ([42, 43]). Due to possible nonlinear transient growth, it is a challenging task extending linear damping to nonlinear damping. Even for Couette flow there are only a few results. Moreover, nonlinear damping is sensitive to the topology of the perturbation. Indeed, Lin and Zeng ([32]) proved that nonlinear inviscid damping is not true for perturbation of Couette flow in H^s for $s < \frac{3}{2}$. Bedrossian and Masmoudi ([7]) proved nonlinear inviscid damping around Couette flow when the perturbation is in Gevrey class 2_-. Recently, Deng and Masmoudi ([17]) proved the instability for initial perturbations in Gevrey class 2_+. We refer to [23, 25] and references therein for other related interesting results. We also refer to the very recent papers [24, 33], which show that nonlinear inviscid damping holds for general linear stable monotone shear flows. Moreover, it is also observed by Orr that, if we rewrite the linearized system using the change of coordinates $f(t,z,y) = \omega(t, z+ty, y)$, then the Fourier transform of the stream function $\phi(t,z,y) = \psi(t, z+yt, y)$ is

$$\widehat{\phi}(t,k,\eta) = \frac{\hat{f}(t,k,\eta)}{(\eta - kt)^2 + k^2}. \tag{1.7}$$

The denominator of (1.7) is minimized at $t = \frac{\eta}{k}$ which is known as the Orr critical time.

The second phenomenon — enhanced dissipation — is sometimes referred to by modern authors as the "shear-diffusion mechanism". This decay rate is much faster than the diffusive decay of $e^{-\nu t}$. The mechanism leading to the enhanced dissipation is also due to vorticity mixing.

However, for the nonlinear system the Orr mechanism is known to interact poorly with the nonlinear term, creating a weakly nonlinear effect referred to as an echo. The basic mechanism is straightforward: a mode that is near its critical time is creating most of the velocity field and, at this point, it can interact with a part of enstrophy which is already mixed to transfer enstrophy to a mode which is un-mixing. When this third mode reaches its critical time, the result of the nonlinear interaction becomes very strong (the time delay explains the terminology "echo"). There are two necessary ways to control (compete against) the echo cascades. One is to assume enough smallness of the initial perturbations such that the rapid growth of the enstrophy may not happen before the enhanced dissipative time scale $\nu^{-\frac{1}{3}}$. The other is to assume enough regularity (Gevrey class) of the initial perturbations that one can pay enough regularity to control the growth caused by the echo cascade.

In this work, we are interested in the first method to stabilize the system and studying the long time behavior of (1.3) for small initial perturbations ω_{in}. We aim to find the largest perturbation (threshold) in Sobolev spaces below which Couette flow is stable. More precisely, we are studying the following classical question:

Given a norm $\|\cdot\|_X$, *find a* $\mu = \mu(X)$ *such that*

$$\|\omega_{\text{in}}\|_X \le \nu^{\mu} \Rightarrow \textit{stability},$$
$$\|\omega_{\text{in}}\|_X \gg \nu^{\mu} \Rightarrow \textit{instability}.$$

Another interesting question that is related to this problem is nonlinear enhanced dissipation and inviscid damping, which can be proposed in the following two ways:

1. Given a norm $\|\cdot\|_X$ $(X \subset L^2)$, determine a $\mu = \mu(X)$ such that for $\|\omega_{\text{in}}\|_X \ll \nu^{\mu}$ and for $t > 0$,

$$\|\omega_{\neq}\|_{L^2_{x,y}} \le C\|\omega_{\text{in}}\|_X e^{-c\nu^{\frac{1}{3}}t} \quad \text{and} \quad \|V_{\neq}\|_{L^2_{t,x,y}} \le C\|\omega_{\text{in}}\|_X, \tag{1.8}$$

or the weak enhanced-dissipation-type estimate

$$\|\omega_{\neq}\|_{L_t^2 L_{x,y}^2} \le C\nu^{-\frac{1}{6}}\|\omega_{\rm in}\|_X \tag{1.9}$$

holds for the Navier–Stokes equations (1.3).

2. Given μ, is there an optimal function space $X \subset L^2$ so that if the initial vorticity satisfies $\|\omega_{\rm in}\|_X \ll \nu^{\mu}$, then (1.8) or (1.9) holds for the Navier–Stokes equations (1.3)?

These two problems (find the smallest μ or find the largest function space X) are related to each other, since one can gain regularity in a short time by a standard time–weight argument if the initial perturbation is small enough.

We summarize the results as follows:

- For $\mu = 0$, Bedrossian, Masmoudi and Vicol ([9]) showed that if X is taken as Gevrey-m with $m < 2$, then Couette flow is stable and (1.9) holds.
- For $\mu = \frac{1}{2}$, Bedrossian, Vicol and Wang ([10]) proved that Couette flow is stable, as well as nonlinear enhanced dissipation and inviscid damping for perturbation of initial vorticity in H^s, $s > 1$.
- For $\mu = \frac{1}{2}$, recently in [34], we proved nonlinear enhanced dissipation and inviscid damping for perturbation of initial vorticity in the almost critical space $H_x^{\log} L_y^2 \subset L_{x,y}^2$.

Let us also mention some other recent progress on the stability problem of different types of shear flows in different domains:

- Three-dimensional Couette flow in $\mathbb{T} \times \mathbf{R} \times \mathbb{T}$: If X is taken as a Sobolev space, then $\mu = 1$ gives stability ([3–5, 40]). Also see the very recent paper [15] for three-dimensional Couette flow in a finite channel.
- Two-dimensional Couette flow in a finite channel: If X is taken as a Sobolev space, then $\mu = \frac{1}{2}$ gives stability ([6, 13, 14, 18]).
- Other shear flows: See [16, 20, 21, 30, 31, 43].

In this paper, we find a smaller μ $(= \frac{1}{3})$ such that Couette flow is stable, and nonlinear enhanced dissipation and inviscid damping hold, when X is taken as a Sobolev space. Our main result is stated as follows:

Theorem 1.1. *For $\sigma \ge 40$, $\nu > 0$, there exist $0 < \varepsilon_0, \nu_0 < 1$, such that for all $0 < \nu \le \nu_0$ and $0 < \varepsilon \le \varepsilon_0$, if $\omega_{\rm in}$ satisfies $\|\omega_{\rm in}\|_{H^\sigma} \le \varepsilon\nu^{\frac{1}{3}}$, then the solution $\omega(t)$ of (1.3) with initial data $\omega_{\rm in}$ satisfies the following properties:*

1. *Global stability in H^σ,*

$$\|\omega(t, x + ty + \Phi(t, y), y)\|_{H^\sigma} \le C\varepsilon\nu^{\frac{1}{3}}, \tag{1.10}$$

where $\Phi(t, y)$ is given explicitly by

$$\Phi(t, y) = \int_0^t e^{\nu(t-\tau)\partial_y^2}\left(\frac{1}{2\pi}\int_{\mathbb{T}} U^x(\tau, x, y)\, dx\right) d\tau.$$

2. *Inviscid damping,*

$$\|P_{\neq}U^x\|_2 + \langle t\rangle\|U^y t\|_2 \leq \frac{C\varepsilon\nu^{\frac{1}{3}}}{\langle\nu t^3\rangle}\langle t\rangle^{-1}. \tag{1.11}$$

3. *Weak enhanced dissipation,*

$$\|P_{\neq}\omega(t)\|_2 \leq \frac{C\varepsilon\nu^{\frac{1}{3}}}{\langle\nu t^3\rangle}. \tag{1.12}$$

The constant C is independent of ν and ε.

Remark 1.2. By replacing $D(t,\eta)$ by $D(t,\eta)^\alpha$ with $\alpha \geq 1$ in the proof and assuming σ large enough (depending on α), one can obtain stronger enhanced dissipation of the following form:

$$\|P_{\neq}\omega(t)\|_2 \leq \frac{C\varepsilon\nu^{\frac{1}{3}}}{\langle\nu t^3\rangle^\alpha}. \tag{1.13}$$

However, weak enhanced dissipation of the same decay rate as in Theorem 1.1 is enough for the proof of Sobolev stability. Both (1.12) and (1.13) are far from the exponential decay of the linear case.

It is natural to ask whether the exponent $\frac{1}{3}$ is sharp for stability with perturbations of finite regularity. In the proof, we show some evidence for the sharpness. There are two basic mechanisms leading to asymptotic stability: one is enhanced dissipation, the other is inviscid damping. Both will help us to estimate the accumulation of errors from the nonlinear interactions. We also face the following two facts: (1) Enhanced dissipation happens when $t \gg \nu^{-\frac{1}{3}}$. (2) We need to pay regularity to obtain inviscid damping. In order to prove stability with perturbation of finite regularity, we cannot use inviscid damping for free. So, roughly speaking, if we assume the perturbation is of size $\varepsilon\nu^\mu$, then the nonlinear term is of size $\varepsilon^2\nu^{2\mu}$ and this error will accumulate in the time interval. This is the time interval before the enhanced dissipation starts. Thus, if we want the perturbation to stay of the same size as the initial data $\varepsilon\nu^\mu$, we need $\varepsilon^2\nu^{2\mu-\frac{1}{3}} \leq \varepsilon\nu^\mu$, which gives us $\mu \geq \frac{1}{3}$. Of course, one can also allow the perturbation to grow to a different size. Indeed, from our proof (see (2.23)) one finds that the size of the coordinate system grows from $\nu^{\frac{1}{3}}$ to $\nu^{\frac{1}{6}}$.

For two-dimensional Couette flow in the finite channel $\mathbb{T}\times[-1,1]$, in [14] the authors proved that $\mu = \frac{1}{2}$ gives stability in Sobolev space, when the boundary effect is taken into consideration. It remains a very interesting problem whether the threshold can be improved to $\mu = \frac{1}{3}$ in the finite channel case.

Let us now outline the main ideas in the proof of Theorem 1.1. First, we provide a (well-chosen) change of variable that adapts to the solution as it evolves and yields a new "relative" velocity which is time integrable. Second, we will construct a new multiplier that can be regarded as a ghost weight in phase space and that will help us control the growth caused by echo cascades.

2. Proof of Theorem 1.1

In this section we will present several key propositions and complete the proof of Theorem 1.1 by admitting those propositions.

2.1. Notation and conventions

See Section A.1 for the Fourier analysis conventions we are taking. A convention we generally use is to denote the discrete x (or z) frequencies as subscripts. By convention we always use Greek letters such as η and ξ to denote frequencies in the y or v direction and lowercase Latin characters commonly used as indices such as k and l to denote frequencies in the x or z direction (which are discrete). Another convention we use is to denote M, N, K as dyadic integers $M, N, K \in \mathbb{D}$, where

$$\mathbb{D} = \left\{\tfrac{1}{2}, 1, 2, 4, 8, \dots, 2^j, \dots\right\}.$$

When a sum is written with indices K, M, M', N or N' it will always be over a subset of $\mathbb{D}$. We will use the same A for $Af = (A(\eta)\hat{f}(\eta))^\vee$ or $A\hat{f} = A(\eta)\hat{f}(\eta)$, where A is a Fourier multiplier.

We use the notation $f \lesssim g$ when there exists a constant $C > 0$ independent of the parameters of interest such that $f \le Cg$ (we define $g \gtrsim f$ analogously). Similarly, we use the notation $f \approx g$ when there exists $C > 0$ such that $C^{-1}g \le f \le Cg$.

We will denote the l^1 vector norm $|k, \eta| = |k| + |\eta|$, which by convention is the norm taken in our work. Similarly, given a scalar or vector in $\mathbf{R}^n$ we denote

$$\langle v \rangle = (1 + |v|^2)^{\frac{1}{2}}.$$

We use similar notation to denote the x or z average of a function:

$$\langle f \rangle = \frac{1}{2\pi} \int_{\mathbb{T}} f(x, y)\, dx = f_0.$$

We also frequently use the notation $f_{\neq} = P_{\neq} f = f - f_0$. We denote the standard L^2 norms by $\|\cdot\|_2$. The norm of the Sobolev space H^σ is given by

$$\|f\|_{H^\sigma} = \|(\langle \eta \rangle^\sigma \hat{f})^\vee\|_2$$

The norm space-time Sobolev space $L^p_T(H^\sigma)$ is given by

$$\|f\|_{L^p_T(H^\sigma)} = \begin{cases} \sup\limits_{t' \in [1,t]} \|f(t')\|_{H^\sigma}, & p = \infty, \\ \left(\displaystyle\int_1^t \|f(t')\|^p_{H^\sigma}\, dt' \right)^{\frac{1}{p}}, & 1 \le p < \infty. \end{cases}$$

For $|m| = 0, 1, 2, \dots$, and $m\eta \ge 0$, let

$$t_{m,\eta} = \frac{2\eta}{2m+1}. \tag{2.1}$$

We then use

$$I_{m,\eta} \stackrel{\text{def}}{=} [t_{m,\eta}, t_{m-1,\eta}] \tag{2.2}$$

for $m = 1, 2, \ldots,$ to denote any resonant interval and its left and right parts with $\eta \geq (2m+1)m$. For $|\eta| \geq 3$, we denote by $E(\sqrt{|\eta|})$ the largest integer satisfying $(2E(\sqrt{|\eta|})+1)E(\sqrt{|\eta|}) \leq |\eta|$ and then $E(\sqrt{|\eta|}) \approx \sqrt{|\eta|}$. Let $t(\eta) = \frac{2\eta}{2E(\sqrt{|\eta|})+1} \approx \sqrt{|\eta|}$ be the start of the resonant interval. Then we denote by

$$I_t(\eta) \stackrel{\text{def}}{=} [t(\eta), 2|\eta|] = \bigcup_{m=1}^{E(\sqrt{\eta})} I_{m,\eta} \tag{2.3}$$

the whole resonant interval.

For a statement Q, 1_Q or χ^Q will denote the function that equals 1 if Q is true and 0 otherwise.

2.2. Coordinate transform

We will use the same change of coordinates as in [9], which allows us to simultaneously "mod out" by the evolution of the time-dependent background shear flow and treat the mixing of this background shear as a perturbation of Couette flow (in particular, to understand the nonlinear effect of the Orr mechanism).

The change of coordinates used is $(t, x, y) \to (t, z, v)$, where $z(t, x, y) = x - tv(t, y)$ and $v(t, y)$ satisfies

$$(\partial_t - \nu\partial_{yy})(t(v(t, y) - y)) = \langle U^x\rangle(t, y),$$

with initial data $\lim_{t\to 0} t(v(t, y) - y) = 0$, and where $\langle U^x\rangle(t, y) = \frac{1}{2\pi}\int_{\mathbb{T}} U^x(t, x, y)\, dx$.

We define the following quantities:

$$C(t, v(t, y)) = v(t, y) - y, \tag{2.4}$$
$$v'(t, v(t, y)) = (\partial_y v)(t, y), \tag{2.5}$$
$$v''(t, v(t, y)) = (\partial_{yy} v)(t, y), \tag{2.6}$$
$$[\partial_t v](t, v(t, y)) = (\partial_t v)(t, y), \tag{2.7}$$
$$f(t, z(t, x, y), v(t, y)) = \omega(t, x, y), \tag{2.8}$$
$$\phi(t, z(t, x, y), v(t, y)) = \psi(t, x, y), \tag{2.9}$$
$$\tilde{u}(t, z(t, x, y), v(t, y)) = U^x(t, x, y). \tag{2.10}$$

Thus we get

$$\Delta_t\phi \stackrel{\text{def}}{=} \partial_{zz}\phi + (v')^2(\partial_v - t\partial_z)^2\phi + v''(\partial_v - t\partial_z)\phi = f, \tag{2.11}$$

and

$$\partial_t f + [\partial_t v]\partial_v f - \nu v'' t\partial_z f + v'\nabla^{\perp}_{z,v} P_{\neq}\phi \cdot \nabla_{z,v} f = \nu\Delta_t f, \tag{2.12}$$

where $\nabla^{\perp}_{z,v} = (-\partial_v, \partial_z)$, $\nabla_{z,v} = (\partial_z, \partial_v)$, $P_{\neq}\phi = \phi - \langle\phi\rangle$, $\tilde{u}_0(t, v) = \frac{1}{2\pi}\int_{\mathbb{T}_{2\pi}} \tilde{u}(t, z, v)\, dz$.

We also obtain

$$\partial_t \tilde{u}_0 + [\partial_t v]\partial_v \tilde{u}_0 + \langle v' \nabla_{z,v}^{\perp} P_{\neq 0}\phi \cdot \nabla \tilde{u}\rangle = \nu \Delta_t \tilde{u}_0. \tag{2.13}$$

Define the auxiliary function

$$g(t,v) = \frac{1}{t}(\tilde{u}_0(t,v) - C(t,v)),$$

which implies that

$$\begin{gathered}[\partial_t v] = g + \nu v'',\\ v'\partial_v C(t,v) = v'(t,v) - 1,\\ \partial_t C + [\partial_t v]\partial_v C = [\partial_t v],\\ v'\partial_v v' = v'' = \Delta_t C,\end{gathered}$$

and that g satisfies

$$\partial_t g + \frac{2g}{t} + g\partial_v g = -\frac{v'}{t}\langle \nabla_{z,v}^{\perp} P_{\neq}\phi \cdot \nabla_{z,v}\tilde{u}\rangle + \nu (v')^2 \partial_{vv} g. \tag{2.14}$$

If we denote $h = v' - 1$, we get

$$\partial_t h + g\partial_v h = \frac{-f_0 - h}{t} + \nu \widetilde{\Delta}_t h. \tag{2.15}$$

Let $\bar{h} = \frac{-f_0-h}{t}$; thus we obtain

$$\partial_t \bar{h} + g\partial_v \bar{h} = -\frac{2}{t}\bar{h} + \frac{v'}{t}\langle \nabla_{z,v}^{\perp} P_{\neq}\phi \cdot \nabla_{z,v} f\rangle + \nu \widetilde{\Delta}_t \bar{h}. \tag{2.16}$$

It gives

$$\partial_t f + u \cdot \nabla_{z,v} f = \nu \widetilde{\Delta}_t f, \tag{2.17}$$

where

$$u(t,z,v) = \begin{pmatrix} 0 \\ g \end{pmatrix} + v' \nabla_{z,v}^{\perp} P_{\neq}\phi$$

and $\widetilde{\Delta}_t f = \partial_{zz} f + (v')^2(\partial_v - t\partial_z)^2 f$.

By the change of the coordinates we reduce our problem to studying the following system:

$$\begin{cases} \partial_t f + u \cdot \nabla_{z,v} f = \nu \widetilde{\Delta}_t f, \\ u(t,z,v) = \begin{pmatrix} 0 \\ g \end{pmatrix} + v' \nabla_{z,v}^{\perp} P_{\neq}\phi, \\ \Delta_t \phi = f, \quad v'' = v'\partial_v v', \quad h = v' - 1, \end{cases} \tag{2.18}$$

$$\begin{cases} \partial_t g + \dfrac{2g}{t} + g\partial_v g = -\dfrac{v'}{t}\langle \nabla_{z,v}^{\perp} P_{\neq}\phi \cdot \nabla_{z,v}\tilde{u}\rangle + \nu(v')^2\partial_{vv} g, \\ \partial_t \bar{h} + \dfrac{2}{t}\bar{h} + g\partial_v \bar{h} = \dfrac{v'}{t}\langle \nabla_{z,v}^{\perp} P_{\neq}\phi \cdot \nabla_{z,v} f\rangle + \nu(v')^2\partial_{vv}\bar{h}, \\ \partial_t h + g\partial_v h = \bar{h} + \nu(v')^2\partial_{vv} h, \\ \tilde{u} = -v'(\partial_v - t\partial_z)\phi. \end{cases} \tag{2.19}$$

2.3. Main energy estimate

In light of the previous section, our goal is to control the solution to (2.18) and (2.19) uniformly in a suitable norm as $t \to \infty$. The key idea we use for this is the carefully designed time-dependent norm written as

$$\|A^{\sigma}(t,\nabla) f\|_2^2 = \sum_k \int_{\eta} |A_k^{\sigma}(t,\eta)\hat{f}_k(t,\eta)|^2\, d\eta,$$

where $A_k^{\sigma}(t,\eta)$ is defined in (3.9).

We also introduce another time-dependent norm for $8 \le s \le \sigma - 10$:

$$\|A_E^{s}(t,\partial_k,\partial_v) f\|_2^2 = \sum_{k\neq 0} \int_{\eta} |A_E^{s}(t,k,\eta)\hat{f}_k(t,\eta)|^2\, d\eta,$$

which quantifies the enhanced dissipation effect with

$$A_E^s(t,k,\eta) = \langle k,\eta\rangle^s D(t,\eta),$$

with

$$D(t,\eta) = \frac{1}{3}\nu|\eta|^3 + \frac{1}{24}\nu(t^3 - 8|\eta|^3)_+.$$

Here E stands for enhanced dissipation.

We define our higher Sobolev energy:

$$\mathcal{E}^{\sigma}(t) = \frac{1}{2}\|A^{\sigma}(t) f(t)\|_2^2 + \mathcal{E}_v(t), \tag{2.20}$$

where

$$\mathcal{E}_v(t) = \|g\|_{H^\sigma}^2 + \nu^{\frac{1}{3}}\|h\|_{H^\sigma}^2 + \nu^{\frac{1}{3}}\|\bar{h}\|_{H^\sigma}^2 + \|h\|_{H^{\sigma-1}}^2 + \|\bar{h}\|_{H^{\sigma-1}}^2. \tag{2.21}$$

By well-posedness theory for two-dimensional Navier–Stokes equations in Sobolev spaces we may safely ignore the time interval (say) [0, 1] by further restricting the size of the initial data. That is, we have the following lemma.

Lemma 2.1. *For $\varepsilon > 0$, $\nu > 0$ and $\sigma \ge 40$, there exists $\varepsilon' > 0$ independent of ν such that if $\|\omega_{\text{in}}\|_{H^\sigma} \le \varepsilon'\nu^{\frac{1}{3}}$, then*

$$\sup_{t\in[0,1]} \mathcal{E}^{\sigma}(t) \le (\varepsilon\nu^{\frac{1}{3}})^2.$$

We define the following controls referred to in the sequel as the bootstrap hypotheses for $t \geq 1$.

Higher regularity: main system.

$$\|A^\sigma f(t)\|_2^2 + \nu \int_1^t \left\| \sqrt{-\Delta_L} A^\sigma f(t')\right\|_2^2 dt' + \int_1^t \mathrm{CK}_w(t')\, dt' \leq (8\varepsilon\nu^{\frac{1}{3}})^2, \tag{2.22}$$

where the CK stands for "Cauchy–Kovalevskaya",

$$\mathrm{CK}_w(t) = \sum_k \int \frac{\partial_t w_k(t,\eta)}{w_k(t,\eta)} |A_k^\sigma(t,\eta)\hat{f}_k(t,\eta)|^2\, d\eta.$$

Higher regularity: coordinate system.

$$\begin{aligned}
&\langle t\rangle \|g\|_{H^\sigma} + \int_1^t \|g(t')\|_{H^\sigma}\, dt' \leq 8\varepsilon\nu^{\frac{1}{3}},\\
&t^3\|A^\sigma \bar{h}(t)\|_2^2 + \int_1^t t'^3 \left\| \sqrt{\frac{\partial_t w(\nabla)}{w(\nabla)}} A^\sigma \bar{h}\right\|_2^2 dt'\\
&\qquad + \frac{1}{4}\int_1^t t'^2 \|A^\sigma \bar{h}\|_2^2\, dt' + \frac{1}{4}\nu \int_1^t t'^3 \|\partial_v A^\sigma \bar{h}\|_2^2\, dt' \leq 8\varepsilon(\varepsilon\nu^{\frac{1}{6}})^2,\\
&\|h(t)\|_{H^\sigma}^2 + \nu \int_1^t \|\partial_v h(t')\|_{H^\sigma}^2\, dt' \leq 8(10\varepsilon\nu^{\frac{1}{6}})^2.
\end{aligned} \tag{2.23}$$

Lower regularity: enhanced dissipation.

$$\|A_E^s f(t)\|_2^2 + \frac{2}{5}\nu \int_1^t \left\| \sqrt{-\Delta_L} A_E^s f(t')\right\|_2^2 dt' \leq (8\varepsilon\nu^{\frac{1}{3}})^2. \tag{2.24}$$

Lower regularity: decay of the zero mode.

$$\begin{aligned}
&\langle t\rangle^4 \|g(t)\|_{H^{\sigma-6}}^2 + \nu \int_1^t t'^4 \|\partial_v g(t')\|_{H^{\sigma-6}}^2\, dt' \leq (8\varepsilon\nu^{\frac{1}{3}})^2,\\
&\langle t\rangle^4 \|\bar{h}(t)\|_{H^{\sigma-6}}^2 + \nu \int_1^t t'^4 \|\partial_v \bar{h}(t')\|_{H^{\sigma-6}}^2\, dt' \leq (8\varepsilon\nu^{\frac{1}{3}})^2,\\
&\|f_0\|_{H^s}^2 + \frac{t\nu}{2}\|\partial_v f_0\|_{H^s}^2 + \nu \int_1^t \left(\|\partial_v f_0(t')\|_{H^s}^2 + \frac{t'\nu}{2}\|\partial_v f_0(t')\|_{H^s}^2\right) dt' \leq (8\varepsilon\nu^{\frac{1}{3}})^2.
\end{aligned} \tag{2.25}$$

Assistant estimates.

$$\begin{aligned}
&\langle t\rangle \|\bar{h}\|_{H^{\sigma-1}} + \int_1^t \|\bar{h}(t')\|_{H^{\sigma-1}}\, dt' \leq 8\varepsilon\nu^{\frac{1}{3}},\\
&\|h(t)\|_{H^{\sigma-1}}^2 + \nu \int_1^t \|\partial_v h(t')\|_{H^{\sigma-1}}^2\, dt' \leq 8(10\varepsilon\nu^{\frac{1}{3}})^2.
\end{aligned} \tag{2.26}$$

The next proposition follows from the bootstrap hypotheses, elliptic estimates and the properties of the multipliers A^σ and A^s_E.

Proposition 2.2. *Under the bootstrap hypotheses, the following inequalities hold:*

$$\|f\|_{H^\sigma} + \nu^{\frac{1}{2}}\|\sqrt{-\Delta_L} f\|_{L^2_T(H^\sigma)} + \left\|\sqrt{\frac{\partial_t w(\nabla)}{w(\nabla)}} f\right\|_{L^2_T(H^\sigma)} \lesssim \varepsilon\nu^{\frac{1}{3}}, \tag{2.27}$$

$$\|f_{\neq}\|_{H^s} + \nu^{\frac{1}{2}}\|\sqrt{-\Delta_L} f_{\neq}\|_{L^2_T(H^s)} \lesssim \frac{\varepsilon\nu^{\frac{1}{3}}}{\langle \nu t^3\rangle}, \tag{2.28}$$

and the inviscid damping results

$$\|P_{\neq}\phi\|_{H^{\sigma-4}} \lesssim \frac{\varepsilon\nu^{\frac{1}{3}}}{\langle t^2\rangle}, \quad \|\tilde{u}_{\neq}\|_{H^{\sigma-3}} \lesssim \frac{\varepsilon\nu^{\frac{1}{3}}}{\langle t\rangle}. \tag{2.29}$$

This proposition together with Lemma A.2 implies Theorem 1.1.

Proof. By Lemma 3.3, we get $A^\sigma_k(t,\eta) \approx \langle k,\eta\rangle^\sigma$. Thus we have $\|A^\sigma f\|_2 \approx \|f\|_{H^\sigma}$ which implies (2.27).

By Lemma 3.4, we get $D(t,\eta) \gtrsim \nu t^3$, thus $\|A^s_E f\|_2 \gtrsim \nu t^3\|f\|_{H^s}$ which gives (2.28).

The inviscid damping result (2.29) follows from Lemmas 4.1 and 4.2. ∎

For enhanced dissipation and inviscid damping in the Sobolev norm, we also have the following remark.

Remark 2.3. Under the bootstrap hypotheses, it holds that

$$\|\omega_{\neq}(t, x+ty+\Phi(t,y), y)\|_{H^s} \lesssim \frac{\varepsilon\nu^{\frac{1}{3}}}{\langle \nu t^3\rangle}$$

and

$$\|U^y(t, x+ty+\Phi(t,y), y)\|_{H^{\sigma-4}} \lesssim \frac{\varepsilon\nu^{\frac{1}{3}}}{\langle t^2\rangle},$$

$$\|U^x_{\neq}(t, x+ty+\Phi(t,y), y)\|_{H^{\sigma-4}} \lesssim \frac{\varepsilon\nu^{\frac{1}{3}}}{\langle t\rangle}.$$

Recall that

$$\begin{aligned}
&f(t, z(t,x,y), v(t,y)) = \omega(t,x,y) \\
&\qquad \Rightarrow \omega(t, x+ty+\Phi(t,y), y) = f(t,x,v(t,y)), \\
&\tilde{u}(t, z(t,x,y), v(t,y)) = U^x(t,x,y) \\
&\qquad \Rightarrow U^x(t, x+ty+\Phi(t,y), y) = \tilde{u}(t,x,v(t,y)), \\
&\partial_z\phi(t, z(t,x,y), v(t,y)) = U^y(t,x,y) \\
&\qquad \Rightarrow U^y(t, x+ty+\Phi(t,y), y) = (\partial_z\phi)(t,x,v(t,y)).
\end{aligned}$$

The remark follows directly from (2.28), (2.29), the composition Lemma A.2 and the bootstrap hypotheses for the regularity of the coordinate system.

By Lemma 2.1, for the rest of the proof we may focus on times $t \geq 1$. Let I^* be the connected set of times $t \geq 1$ such that the bootstrap hypotheses (2.22)–(2.26) are all satisfied. We will work on regularized solutions for which we know $\mathcal{E}^\sigma(t)$ takes values continuously in time, and hence I^* is a closed interval $[1, T^*]$ with $T^* \geq 1$. The bootstrap is complete if we show that I^* is also open, which is the purpose of the following proposition, the proof of which constitutes the majority of this work.

Proposition 2.4. *For $\sigma \geq 40$, $\nu > 0$ and $8 \leq s \leq \sigma - 10$, there exist $0 < \varepsilon_0, \nu_0 < 1$, such that for all $0 < \nu \leq \nu_0$ and $0 < \varepsilon \leq \varepsilon_0$, such that if on $[1, T^*]$ the bootstrap hypotheses (2.22)–(2.26) hold, then for any $t \in [1, T^*]$, we have the following properties:*

1. *Vorticity boundedness:*
$$\|A^\sigma f(t)\|_2^2 + \nu \int_1^t \|\sqrt{-\Delta_L} A^\sigma f(t')\|_2^2 \, dt' + \int_1^t \mathrm{CK}_w(t') \, dt' \leq (6\varepsilon\nu^{\frac{1}{3}})^2.$$
2. *Control of coordinates system:*
$$\begin{aligned}
&\langle t \rangle \|g\|_{H^\sigma} + \int_1^t \|g(t')\|_{H^\sigma} \, dt' \leq 6\varepsilon\nu^{\frac{1}{3}}, \\
&t^3 \|A^\sigma \bar{h}(t)\|_2^2 + \int_1^t t'^3 \left\| \sqrt{\frac{\partial_t w}{w}} \bar{h} \right\|_{H^\sigma}^2 dt' \\
&\quad + \frac{1}{4} \int_1^t t'^2 \|A^\sigma \bar{h}\|_2^2 \, dt' + \frac{1}{4}\nu \int_1^t t'^3 \|\partial_v A^\sigma \bar{h}\|_2^2 \, dt' \leq 6\varepsilon(\varepsilon\nu^{\frac{1}{6}})^2, \\
&\|h(t)\|_{H^\sigma}^2 + \nu \int_1^t \|\partial_v h(t')\|_{H^\sigma}^2 \, dt' \leq 6(10\varepsilon\nu^{\frac{1}{6}})^2.
\end{aligned}$$
3. *Enhanced dissipation:*
$$\|A_E^s f(t)\|_2^2 + \frac{2}{5}\nu \int_1^t \left\| \sqrt{-\Delta_L} A_E^s f(t') \right\|_2^2 dt' \leq (6\varepsilon\nu^{\frac{1}{3}})^2,$$
4. *Decay of the zero mode:*
$$\begin{aligned}
&\langle t \rangle^4 \|g(t)\|_{H^{\sigma-6}}^2 + \nu \int_1^t t'^4 \|\partial_v g(t')\|_{H^{\sigma-6}}^2 \, dt \leq (6\varepsilon\nu^{\frac{1}{3}})^2, \\
&\langle t \rangle^4 \|\bar{h}(t)\|_{H^{\sigma-6}}^2 + \nu \int_1^t t'^4 \|\partial_v \bar{h}(t')\|_{H^{\sigma-6}}^2 \, dt \leq (6\varepsilon\nu^{\frac{1}{3}})^2, \\
&\|f_0(t)\|_{H^s}^2 + \frac{t\nu}{2} \|\partial_v f_0\|_{H^s}^2 + \nu \int_1^t \left(\|\partial_v f_0(t)\|_{H^s}^2 + \frac{t'\nu}{2} \|\partial_v f_0(t')\|_{H^s}^2 \right) dt' \leq (6\varepsilon\nu^{\frac{1}{3}})^2.
\end{aligned}$$
5. *Assistant estimate:*
$$\begin{aligned}
&\langle t \rangle \|\bar{h}\|_{H^{\sigma-1}} + \int_1^t \|\bar{h}(t')\|_{H^{\sigma-1}} \, dt' \leq 6\varepsilon\nu^{\frac{1}{3}}, \\
&\|h(t)\|_{H^{\sigma-1}}^2 + \nu \int_1^t \|\partial_v h(t')\|_{H^{\sigma-1}}^2 \, dt' \leq 6(10\varepsilon\nu^{\frac{1}{3}})^2,
\end{aligned}$$

from which it follows that $T^* = +\infty$.

The remainder of the paper is devoted to the proof of Proposition 2.4, the primary step being to show that on $[1, T^*]$, we have the following estimates:

$$\|A^\sigma f(t)\|_2^2 + \nu \int_1^t \left\|\sqrt{-\Delta_L} A^\sigma f(t')\right\|_2^2 dt' + \int_1^t \mathrm{CK}_w(t')\, dt' \\ \le 2\|A^\sigma f(1)\|_2^2 + C\varepsilon^3 \nu^{\frac{2}{3}}, \tag{2.30}$$

$$\langle t\rangle \|g\|_{H^\sigma} + \int_1^t \|g(t')\|_{H^\sigma}\, dt' \le 2\|g(1)\|_{H^\sigma} + C\varepsilon^2 \nu^{\frac{1}{3}}, \tag{2.31}$$

$$t^3 \|A^\sigma \bar{h}(t)\|_2^2 + \int_1^t t'^3 \left\| \sqrt{\frac{\partial_t w}{w}} \bar{h} \right\|_{H^\sigma}^2 dt' + \frac{1}{4} \int_1^t t'^2 \|A^\sigma \bar{h}\|_2^2\, dt' \\ + \frac{1}{4}\nu \int_1^t t'^3 \|\partial_v A^\sigma \bar{h}\|_2^2\, dt' \le 2\|\bar{h}(1)\|_{H^\sigma}^2 + C\varepsilon^4 \nu^{\frac{1}{3}}, \tag{2.32}$$

$$\|h(t)\|_{H^\sigma}^2 + \nu \int_1^t \|\partial_v h(t')\|_{H^\sigma}^2\, dt' \le 2\|h(1)\|_{H^\sigma}^2 + C\varepsilon^3 \nu^{\frac{1}{3}}, \tag{2.33}$$

$$\|A_E^s f(t)\|_2^2 + \frac{2}{5}\nu \int_1^t \left\|\sqrt{-\Delta_L} A_E^s f(t')\right\|_2^2 dt' \le 2\|A_E^s f(1)\|_2^2 + C\varepsilon^3 \nu^{\frac{2}{3}}, \tag{2.34}$$

$$\langle t\rangle^4 \|g(t)\|_{H^{\sigma-6}}^2 + \nu \int_1^t t'^4 \|\partial_v g(t')\|_{H^{\sigma-6}}^2\, dt' \le 2\|g(1)\|_{H^{\sigma-6}}^2 + C\varepsilon(\varepsilon\nu^{\frac{1}{3}})^2, \tag{2.35}$$

$$\langle t\rangle^4 \|\bar{h}(t)\|_{H^{\sigma-6}}^2 + \nu \int_1^t t'^4 \|\partial_v \bar{h}(t')\|_{H^{\sigma-6}}^2\, dt \le 2\|\bar{h}(1)\|_{H^{\sigma-6}}^2 + C\varepsilon(\varepsilon\nu^{\frac{1}{3}})^2, \tag{2.36}$$

$$\|f_0(t)\|_{H^s}^2 + \frac{t\nu}{2}\|\partial_v f_0\|_{H^s}^2 + \nu \int_1^t \left(\|\partial_v f_0(t)\|_{H^s}^2 + \frac{t'\nu}{2}\|\partial_v f_0(t')\|_{H^s}^2 \right) dt' \\ \le 2\|f_0(1)\|_{H^s}^2 + \nu\|\partial_v f_0(1)\|_{H^s}^2 + C\varepsilon^3 \nu^{\frac{2}{3}}, \tag{2.37}$$

$$\langle t\rangle \|\bar{h}\|_{H^{\sigma-1}} + \int_1^t \|\bar{h}(t')\|_{H^{\sigma-1}}\, dt' \le 2\|\bar{h}(1)\|_{H^{\sigma-1}} + C\varepsilon^2 \nu^{\frac{1}{3}}, \tag{2.38}$$

$$\|h(t)\|_{H^{\sigma-1}}^2 + \nu \int_1^t \|\partial_v h(t')\|_{H^{\sigma-1}}^2\, dt' \le 2\|h(1)\|_{H^{\sigma-1}}^2 + 8\|\bar{h}\|_{L_T^1(H^{\sigma-1})}^2 + C\varepsilon^3 \nu^{\frac{2}{3}}, \tag{2.39}$$

for some constant C independent of ε, ν and T^*. If ε is sufficiently small then (2.30)–(2.39) implies Proposition 2.4.

It is natural to compute the time evolution of the following quantities:

$$\mathcal{E}_{H,f} = \|A^\sigma f(t)\|_2^2, \quad \mathcal{E}_{H,g} = t\|g\|_{H^\sigma}, \quad \mathcal{E}_{H,\bar{h}} = t^3 \|A^\sigma \bar{h}\|_{H^\sigma}^2, \quad \mathcal{E}_{H,h} = \|h(t)\|_{H^\sigma}^2$$

and

$$\mathcal{E}_{L,\neq} = \|A_E^s f(t)\|_2^2, \quad \mathcal{E}_{L,g} = t^4 \|g(t)\|_{H^{\sigma-6}}^2, \quad \mathcal{E}_{L,\bar{h}} = t^4 \|\bar{h}\|_{H^{\sigma-6}}^2, \\ \mathcal{E}_{L,0} = \|f_0(t)\|_{H^s}^2 + \frac{t\nu}{2}\|\partial_v f_0\|_{H^s}^2$$

and

$$\mathcal{E}_{\mathrm{as},\bar{h}} = t^2 \|\bar{h}\|_{H^{\sigma-1}}^2, \quad \mathcal{E}_{\mathrm{as},h} = \|h(t)\|_{H^{\sigma-1}}^2,$$

where H stands for the highest regularity, L stands for the lower regularity and "as" stands for assistant.

The most difficult part in the proof is to control the energy $\mathcal{E}_{H,f}$. Here we present the calculations of the time evolution of $\mathcal{E}_{H,f}$. The calculations of the time evolution of $\mathcal{E}_{H,g}$, $\mathcal{E}_{\mathrm{as},\bar{h}}$, $\mathcal{E}_{\mathrm{as},h}$, $\mathcal{E}_{H,\bar{h}}$ and $\mathcal{E}_{H,h}$ are in Section 9.1. The calculations of the time evolution of $\mathcal{E}_{L,g}$ and $\mathcal{E}_{L,\bar{h}}$ are in Section 9.3. The calculations of the time evolution of $\mathcal{E}_{L,\neq}$ and $\mathcal{E}_{L,0}$ are in Section 10.

The rest of this section will give an outline of the proof of (2.30).

The proof of (2.31) can be found in Section 9.1.1.

The proofs of (2.32) and (2.33) can be found in Section 9.2.

The proof of (2.34) can be found in Section 10.1.

The proof of (2.35) can be found in Section 9.3.1.

The proof of (2.36) can be found in Section 9.3.2.

The proof of (2.37) can be found in Section 10.2.

The proofs of (2.38) and (2.39) can be found in Section 9.1.2.

From the time evolution of $\mathcal{E}_{H,f}$ we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}\times\mathbf{R}}|A^\sigma f(t)\|^2\,dv\,dz$$
$$= -\mathrm{CK}_w - \int A^\sigma f A^\sigma(u\nabla f)\,dz\,dv + \nu\int A^\sigma f A^\sigma(\tilde{\Delta}_t f)\,dz\,dv, \tag{2.40}$$

where, as before, CK stands for "Cauchy–Kovalevskaya",

$$\mathrm{CK}_w = \sum_k \int \frac{\partial_t w_k(t,\eta)}{w_k(t,\eta)}|A_k^\sigma(t,\eta)\hat{f}_k(t,\eta)|^2\,d\eta. \tag{2.41}$$

To treat the second term in (2.40), we have

$$\int A^\sigma f A^\sigma(u\nabla_{z,v} f)\,dz\,dv = -\frac{1}{2}\int \nabla\cdot u|A^\sigma f|^2\,dv\,dz$$
$$+ \int A^\sigma f[A^\sigma(u\cdot\nabla f) - u\cdot\nabla A^\sigma f]\,dz\,dv.$$

Notice that the relative velocity is not divergence-free:

$$\nabla\cdot u = \partial_v g + \partial_z\phi\partial_v v' = \partial_v g + \partial_z P_{\neq}\phi\partial_v h.$$

The first term is controlled by the bootstrap hypothesis (2.25). For the second term we use the elliptic estimates, Lemma 4.1, which show that under the bootstrap hypotheses we have

$$\|P_{\neq}\phi\|_{H^{\sigma-4}} \lesssim \frac{\varepsilon\nu^{\frac{1}{3}}}{\langle t\rangle^2}. \tag{2.42}$$

Therefore, by the Sobolev embedding, $\sigma > 40$ and the bootstrap hypotheses,

$$\begin{aligned}\left|\int \nabla\cdot u|A^\sigma f|^2\,dv\,dz\right| &\lesssim \|\nabla u\|_{L^\infty}\|A^\sigma f\|_2^2\\ &\lesssim \big(\|g\|_{H^2}+(1+\|h\|_{H^2})\|P_{\neq}\phi\|_{H^3}\big)\|A^\sigma f\|_2^2\\ &\lesssim \frac{\varepsilon\nu^{\frac13}}{\langle t\rangle^2}\|A^\sigma f\|_2^2. \end{aligned}\tag{2.43}$$

To handle the commutator, $\int A^\sigma f[A^\sigma(u\cdot\nabla f)-u\cdot\nabla A^\sigma f]\,dz\,dv$, we use a paraproduct decomposition. Precisely, we define the three main contributors: transport, reaction and remainder:

$$\begin{aligned}&\int A^\sigma f[A^\sigma(u\cdot\nabla f)-u\cdot\nabla A^\sigma f]\,dz\,dv\\ &\quad=\frac{1}{2\pi}\sum_{N\geq 8}T_N+\frac{1}{2\pi}\sum_{N\geq 8}R_N+\frac{1}{2\pi}\mathcal{R},\end{aligned}\tag{2.44}$$

where

$$\begin{aligned}T_N&=2\pi\int A^\sigma f[A^\sigma(u_{<N/8}\cdot\nabla f_N)-u_{<N/8}\cdot\nabla A^\sigma f_N]\,dz\,dv,\\ R_N&=2\pi\int A^\sigma f[A^\sigma(u_N\cdot\nabla f_{<N/8})-u_N\cdot\nabla A^\sigma f_{<N/8}]\,dz\,dv,\\ \mathcal{R}&=2\pi\sum_{N\in\mathbb{D}}\sum_{\frac18 N\leq N'\leq 8N}\int A^\sigma f[A^\sigma(u_N\cdot\nabla f_{N'})-u_N\cdot\nabla A^\sigma f_{N'}]\,dz\,dv.\end{aligned}$$

Here $N\in\mathbb{D}=\{\frac12,1,2,4,\dots,2^j,\dots\}$, g_N denotes the Nth Littlewood–Paley projection and $g_{<N}$ means the Littlewood–Paley projection onto frequencies less than N.

For the last term, we get

$$\begin{aligned}\nu\int A^\sigma fA^\sigma(\widetilde{\Delta}_t f)\,dz\,dv &= \nu\int A^\sigma fA^\sigma(\Delta_L f)\,dz\,dv\\ &\quad-\nu\int A^\sigma fA^\sigma((1-(v')^2)(\partial_v-t\partial_z)^2 f)\,dz\,dv\\ &=-\nu\big\|\sqrt{-\Delta_L}A^\sigma f\big\|_2^2\\ &\quad-\nu\int A^\sigma f_{\neq}A^\sigma((1-(v')^2)(\partial_v-t\partial_z)^2 f_{\neq})\,dz\,dv\\ &\quad-\nu\int A^\sigma f_0A^\sigma((1-(v')^2)\partial_v^2 f_0)\,dv\\ &=-\nu\big\|\sqrt{-\Delta_L}A^\sigma f\big\|_2^2+E^{\neq}+E^0.\end{aligned}\tag{2.45}$$

The next four propositions, together with (2.43), imply (2.30). First, we deal with the dissipation term. In Section 5 we will prove the following proposition.

Proposition 2.5. *Under the bootstrap hypotheses,*

$$\nu \int_1^t \left(\int A^\sigma f A^\sigma (\widetilde{\Delta}_t f)\, dz\, dv \right) dt' \leq -\frac{7}{8}\nu \int_1^t \| \sqrt{-\Delta_L} A^\sigma f(t') \|_2^2\, dt' + C\varepsilon^3 \nu^{\frac{2}{3}}.$$

Next we control the transport part. In Section 6 we will prove the following proposition.

Proposition 2.6. *Under the bootstrap hypotheses,*

$$\int_1^t \sum_{N \geq 8} |T_N(t')|\, dt' \lesssim \varepsilon \sup_{t' \in [1,t]} \|A^\sigma f(t')\|_2^2.$$

Next we control the remainder. In Section 7 we will prove the following proposition.

Proposition 2.7. *Under the bootstrap hypotheses,*

$$|\mathcal{R}(t)| \lesssim \frac{\varepsilon \nu^{\frac{1}{3}}}{\langle t \rangle^2} \|A^\sigma f\|_2^2.$$

Finally, we control the reaction part. In Section 8 we will prove the following proposition.

Proposition 2.8. *Under the bootstrap hypotheses,*

$$\int_1^t \sum_{N \geq 8} |R_N(t')|\, dt' \lesssim \varepsilon \sup_{t' \in [1,t]} \|A^\sigma f(t')\|_2^2 + \varepsilon \int_1^t \mathrm{CK}_w(t')\, dt' + \varepsilon^3 \nu^{\frac{2}{3}}.$$

Let us admit the above propositions and finish the proof of (2.30).

Proof of (2.30). We then get by (2.40) that

$$\begin{aligned}
&\|A^\sigma f(t)\|_2^2 + 2\int_1^t \mathrm{CK}_w(t')\, dt' \\
&\quad = \|A^\sigma f(1)\|_2^2 - 2\int_1^t \int A^\sigma f A^\sigma (u \nabla f)\, dz\, dv\, dt' \\
&\quad\quad + \nu 2 \int_1^t \int A^\sigma f A^\sigma (\widetilde{\Delta}_t f)\, dz\, dv\, dt' \\
&\quad \leq \|A^\sigma f(1)\|_2^2 - \frac{7}{4}\nu \int_1^t \| \sqrt{-\Delta_L} A^\sigma f(t') \|_2^2\, dt' + C\varepsilon^3 \nu^{\frac{2}{3}} \\
&\quad\quad + C \int_1^t \left[\left| \int \nabla \cdot u |A^\sigma f|^2\, dv\, dz \right| + \sum_{N \geq 8} |T_N(t')| + |\mathcal{R}(t')| + \sum_{N \geq 8} |R_N(t')| \right] dt'.
\end{aligned}$$

Thus, by (2.43) and the above propositions we have

$$\begin{aligned}
&\|A^\sigma f(t)\|_2^2 + 2\int_1^t \mathrm{CK}_w(t')\, dt' + \frac{7}{4}\nu \int_1^t \| \sqrt{-\Delta_L} A^\sigma f(t') \|_2^2\, dt' \\
&\quad \leq \|A^\sigma f(1)\|_2^2 + C\varepsilon^3 \nu^{\frac{2}{3}} + C\varepsilon \sup_{t' \in [1,t]} \|A^\sigma f(t')\|_2^2 + C\varepsilon \int_1^t \mathrm{CK}_w(t')\, dt'.
\end{aligned}$$

Thus, by taking ε small enough we have proved (2.30). ■

3. Toy model and nonlinear growth

In this section we study nonlinear growth. For simplicity of notation, in this section we usually take $\eta, k > 0$ but the work applies equally well to $\eta, k < 0$. Note that modes where $\eta k < 0$ do not have resonance for positive times.

3.1. The toy model

According to the change of coordinates, the relative velocity now is time integrable. The growth may come from the reaction term. In each time interval $I_{m,\eta}$ which contains only one Orr critical time $t = \frac{\eta}{m}$, it is necessary to study the following toy model:

$$
\begin{aligned}
&\partial_t \hat{f}(t,m,\eta) + \nu(k^2 + (\eta - mt)^2)\hat{f}(t,m,\eta) \\
&\quad = \int_{|\eta-\xi|\le 1} \sum_{m-l=\pm 1} \frac{\pm\xi}{l^2 + (\xi - lt)^2} \hat{f}(t,l,\xi)\hat{f}(t,\pm 1,\eta-\xi)\, d\xi, \\
&\partial_t \hat{f}(t,m\pm 1,\eta) + \nu((m\pm 1)^2 + (\eta - (m\pm 1)t)^2)\hat{f}(t,m,\eta) \\
&\quad = \int_{|\eta-\xi|\le 1} \frac{\xi}{m^2 + (\xi - mt)^2} \hat{f}(t,m,\xi)\hat{f}(t,\pm 1,\eta-\xi)\, d\xi.
\end{aligned}
$$

Since the $\hat{f}(t,\pm 1,\xi-\eta)$ is restricted to the lower frequency $|\xi - \eta| \le 1$, we can regard it as a constant in the ξ variable. Moreover, $\hat{f}(t,\pm 1,\xi-\eta)$ also has enhanced dissipation. As $t \in I_{m,\eta}$, $(m\pm 1)^2 + (\eta - (m\pm 1)t)^2 \approx \frac{\eta^2}{m^2}$, thus we deduce the following simplified toy model:

$$
\partial_t \hat{f}(t,m,\eta) + \nu(m^2 + (\eta - mt)^2)\hat{f}(t,m,\eta) = \frac{\kappa e^{-c\nu^{\frac{1}{3}}t} m^2}{|\eta|} \hat{f}(t,m\pm 1,\eta), \tag{3.1}
$$

$$
\partial_t \hat{f}(t,m\pm 1,\eta) + \frac{\nu\eta^2}{m^2}\hat{f}(t,m\pm 1,\eta) = \frac{\kappa|\eta| e^{-c\nu^{\frac{1}{3}}t}}{m^2(1 + (\frac{\eta}{m} - t)^2)} \hat{f}(t,m,\eta), \tag{3.2}
$$

where κ stands for the smallness assumption of the initial data. Our goal is to find the largest κ such that we can control the total growth caused by the toy model. Thus we assume the enhanced dissipation is $e^{-c\nu^{\frac{1}{3}}t}$.

The next step of the simplification is based on the following observations:

- When $t \gg \nu^{-\frac{1}{3}}$, the enhanced dissipation will offer a small coefficient which makes the Orr mechanism weaker. So we focus on the region of time $t \lesssim \nu^{-\frac{1}{3}}$. The resonant time region is $I_t(\eta) \approx [\sqrt{|\eta|}, 2|\eta|]$. We are interested in the case $|\eta| \lesssim \nu^{-\frac{2}{3}}$ so that $I_t(\eta) \cap [1, C\nu^{-\frac{1}{3}}] \neq \emptyset$. During this region of time, $e^{-c\nu^{\frac{1}{3}}t} \approx 1$.
- The rapid growth of $\hat{f}(t,m\pm 1,\eta)$ happens when $|t - \frac{\eta}{m}| \approx 1$.
- The coefficient in front of $\hat{f}(t,m,\eta)$ on the right-hand side of (3.2) is much bigger than the coefficient in front of $\hat{f}(t,m\pm 1,\eta)$ on the right-hand side of (3.1). It means that $\hat{f}(t,m\pm 1,\eta)$ will grow faster than $\hat{f}(t,m,\eta)$. We may replace $\hat{f}(t,m,\eta)$ by $\hat{f}(t,m\pm 1,\eta)$ in the second equation.

- Since $|\eta| \lesssim \nu^{-\frac{2}{3}}$, when $|t - \frac{\eta}{m}| \approx 1$, the dissipation coefficient in (3.2), $\frac{\nu\eta^2}{m^2} \lesssim \frac{\nu^{\frac{1}{3}}\eta}{m^2}$, is not bigger than the coefficient of the right-hand side if $\kappa \approx \nu^{\frac{1}{3}}$. Thus we can remove the dissipation term.

  Thus we deduce to the following toy model:

$$\partial_t \hat{f}(t, m \pm 1, \eta) = \frac{\kappa|\eta|e^{-c\nu^{\frac{1}{3}}t}}{m^2(1 + (\frac{\eta}{m} - t)^2)} \hat{f}(t, m \pm 1, \eta).$$

For $t \in I_{m,\eta}$, let $\tau = t - \frac{\eta}{m}$; then $\tau \in [-D^-_{m,\eta}, D^+_{m,\eta}]$, where $D^-_{m,\eta} = \frac{\eta}{(2m+1)m} = \frac{\eta}{m} - t_{m,\eta}$ and $D^+_{m,\eta} = \frac{\eta}{(2m-1)m} = t_{m-1,\eta} - \frac{\eta}{m}$ for $m \geq 1$; then $D^\pm_{m,\eta} \approx \frac{\eta}{m^2}$.

Finally, we use the following model to control the entropy growth in each critical time region:

$$\begin{cases} \partial_\tau g_m = \langle \nu^{\frac{1}{3}} t_{m,\eta}\rangle^{-(1+\beta)} \dfrac{\nu^{\frac{1}{3}}\frac{\eta}{m^2}}{1 + \tau^2} g_m, \\ g_m(-D^-_{m,\eta}) = 1. \end{cases} \tag{3.3}$$

We need to point out that in the toy model, $e^{-c\nu^{\frac{1}{3}}t}$ is replaced by $\langle \nu^{\frac{1}{3}} t_{m,\eta}\rangle^{-(1+\beta)}$, with $0 < \beta \leq \frac{1}{2}$, owing to some technical reasons when we deal with the zero mode (see (8.6)). The condition $\beta > 0$ ensures the total growth is bounded (see Lemma 3.3).

For $m\eta > 0$ and $|m| \in [1, E(\sqrt{|\eta|})]$, with $|\eta| \geq 3$, we define for $0 < \beta \leq \frac{1}{2}$,

$$g_m(\tau, \eta) = \exp\Big(\langle \nu^{\frac{1}{3}} t_{m,\eta}\rangle^{-(1+\beta)} \frac{\nu^{\frac{1}{3}}\eta}{m^2}(\arctan(\tau) + \arctan(D^-_{m,\eta}))\Big); \tag{3.4}$$

then g_m solves (3.3).

Then we have

$$g_m(D^+_{m,\eta}, \eta) = G_m(\eta)\, \underbrace{g_m(-D^-_{m,\eta}, \eta)}_{=1},$$

with

$$G_m(\eta) = \exp\Bigg(\langle \nu^{\frac{1}{3}} t_{m,\eta}\rangle^{-(1+\beta)} \frac{\nu^{\frac{1}{3}}\eta}{m^2}(\arctan(D^+_{m,\eta}) + \arctan(D^-_{m,\eta}))\Bigg).$$

Otherwise, we let $g_m(\tau, \eta) = 1$.

3.2. Key multiplier

In this subsection we will define the key multiplier that governs the growth.

We define $w(t, \eta)$ in the following way:

- for $t \leq t(\eta)$, $w(t, \eta) = 1$;
- for $t \in I_{j,\eta}$ with $|j| \in [1, E(\sqrt{|\eta|})]$ and $j\eta > 0$, we have $w(t, \eta) = w(t_{j,\eta}, \eta)\times g_j(t - \frac{\eta}{j}, \eta)$;
- for $t \geq 2|\eta|$, we have $w(t, \eta) = w(2|\eta|, \eta)$.

According to the definition of g_m, we get

$$\frac{\partial_t w(t,\eta)}{w(t,\eta)} \approx \frac{\langle \nu^{\frac{1}{3}} t\rangle^{-(1+\beta)} \nu^{\frac{1}{3}} \frac{\eta}{m^2}}{1+(t-\frac{\eta}{m})^2} 1_{t\in I_{m,\eta}} \approx \frac{\langle \nu^{\frac{1}{3}} t\rangle^{-\beta} m^{-1}}{1+(t-\frac{\eta}{m})^2} 1_{t\in I_{m,\eta}}. \tag{3.5}$$

Next, for $m\eta > 0$ and $|m| \in [1, E(\sqrt{|\eta|})]$, with $|\eta| \geq 3$, we will construct a continuous function $\varrho(m,\eta) \approx \frac{m}{|m|}\max\{|m|,|\eta|\}$. First, let $\rho(x)$ be a bounded smooth function such that

$$\rho(x) = \begin{cases} 1, & x \geq \frac{1}{10}, \\ \text{smooth connected}, & x \in [\frac{1}{20}, \frac{1}{10}], \\ 0, & x \leq \frac{1}{20}. \end{cases} \tag{3.6}$$

We also let ρ satisfy

$$\int_{\frac{1}{20}}^{\frac{1}{10}} \rho(x)\,dx = \frac{1}{20}.$$

Let $\rho_k(x) = \rho(\frac{x}{|k|})$ and for $k\eta \geq 0$,

$$w_k(t,\eta) = w(t,\varrho(k,\eta)), \tag{3.7}$$

where

$$\varrho(k,\eta) = \begin{cases} \frac{|k|}{20} + \int_0^{|\eta|} \rho_k(x)\,dx, & k \neq 0, \\ |\eta|, & k = 0. \end{cases} \tag{3.8}$$

Then we get that for $|\eta| \leq \frac{|k|}{20}$, we have $\varrho(k,\eta) = \frac{|k|}{20}$ and $w_k(t,\eta) = w(t,\frac{k}{20})$, and for $|\eta| \geq \frac{|k|}{10}$, we have $\varrho(k,\eta) = |\eta|$ and $w_k(t,\eta) = w(t,\eta)$.

Lemma 3.1. *It holds that*

$$\varrho(k,\eta) \approx \langle k,\eta\rangle.$$

For $|k-l,\xi-\eta| \leq \frac{1}{100}|l,\xi|$, it holds that

$$|\varrho(k,\eta) - \varrho(l,\xi)| \lesssim |k-l,\xi-\eta|.$$

Proof. It is easy to obtain that $\varrho(k,\eta) \lesssim \langle k,\eta\rangle$. The lower bound follows from the fact that for $\frac{|k|}{20} \leq |\eta| \leq \frac{|k|}{10}$, we have $\varrho(k,\eta) \gtrsim \frac{|k|}{20} \geq \frac{|\eta|}{2}$.

If $|\xi| \geq |l|$, then

$$|k-l,\xi-\eta| \leq \frac{1}{50}|\xi|, \quad |\eta| \geq \frac{49}{50}|\xi| \quad \text{and} \quad |k| \leq |k-l| + |l| \leq \frac{51}{50}|\xi| \leq 2|\eta|.$$

Thus,

$$|\varrho(k,\eta) - \varrho(l,\xi)| = |\xi - \eta|.$$

If $|\xi| \leq \frac{|l|}{100}$, then

$$|k-l,\xi-\eta| \leq \frac{101}{10000}|l|, \quad |k| \geq \frac{9899}{10000}|l| \quad \text{and} \quad |\eta| \leq |\xi| + |\xi-\eta| \leq \frac{201}{10000}|l| \leq \frac{|k|}{20}.$$

Thus,

$$|\varrho(k,\eta)-\varrho(l,\xi)| = \frac{1}{20}|k-l|.$$

Then we only need to focus on $|\xi| \approx |l| \approx |\eta| \approx |k|$. Thus,

$$\begin{aligned}
|\varrho(k,\eta)-\varrho(l,\xi)| &= \left|\frac{|k|}{20}-\frac{|l|}{20}+|k|\int_0^{\frac{\eta}{k}}\rho(x)\,dx-|l|\int_0^{\frac{\xi}{l}}\rho(x)\,dx\right| \\
&\lesssim |k-l|+|k-l|\int_0^{\frac{\eta}{k}}\rho(x)\,dx+\left|l\int_{\frac{\eta}{k}}^{\frac{\xi}{l}}\rho(x)\,dx\right| \\
&\lesssim |k-l|+\frac{|\eta l-\xi k|}{|k|} \lesssim |k-l|+\frac{|\eta,k|\,|k-l,\eta-\xi|}{|k|} \\
&\lesssim |k-l,\xi-\eta|.
\end{aligned}$$

Thus we have proved the lemma. ∎

With $w_k(t,\eta)$, we can now define our key multiplier $A_k^\sigma(t,\eta)$,

$$A_k^\sigma(t,\eta) = \frac{\langle k,\eta\rangle^\sigma}{w_k(t,\eta)}. \tag{3.9}$$

3.3. Basic estimate for the multiplier

The following lemma expresses that the critical times are well separated.

Lemma 3.2 ([7]). *Let ξ, η be such that there exists some $\alpha \geq 1$ with $\alpha^{-1}|\xi| \leq |\eta| \leq \alpha|\xi|$ and let k, n be such that $t \in I_{k,\eta} \cap I_{n,\xi}$; then $k \approx n$ and, moreover, at least one of the following holds:*

(a) $k = n$;

(b) $|t-\frac{\eta}{k}| \geq \frac{1}{10\alpha}\frac{\eta}{k^2}$ *and* $|t-\frac{\xi}{n}| \geq \frac{1}{10\alpha}\frac{\xi}{n^2}$;

(c) $|\eta-\xi| \gtrsim_\alpha \frac{|\eta|}{|n|}$.

Now we will present a lemma about the upper and lower bound estimates for $w(t,\eta)$.

Lemma 3.3. *It holds that*

$$w(t,\eta) \approx 1.$$

As a consequence, $A_k^\sigma(t,\eta) \approx \langle k,\eta\rangle^\sigma$.

Proof. We have for any t,η,

$$\begin{aligned}
1 \leq w(t,\eta) &\leq \prod_{m=E(\sqrt{|\eta|})}^{1} G_m(\eta) \\
&\leq \exp\left(\sum_{m=E(\sqrt{|\eta|})}^{1} \frac{\pi\nu^{\frac{1}{3}}\eta}{m^2}(1+\nu^{\frac{1}{3}}t_{m,\eta})^{-(1+\beta)}\right)
\end{aligned}$$

$$\lesssim \begin{cases} \exp\Big(C \sum_{m=E(\sqrt{|\eta|})}^{1} \frac{\nu^{\frac{1}{3}}\eta}{m^2}\Big), & \nu^{\frac{1}{3}}\eta \le 1, \\ \exp\Big(C \sum_{m=\nu^{\frac{1}{3}}\eta}^{E(\sqrt{|\eta|})} \frac{\nu^{\frac{1}{3}}\eta}{m^2} + C \sum_{m=\nu^{\frac{1}{3}}\eta}^{1} \frac{m^{-1+\beta}}{(\nu^{\frac{1}{3}}\eta)^{\beta}}\Big), & 1 \le \nu^{\frac{1}{3}}\eta \le E(\sqrt{|\eta|}), \\ \exp\Big(C \sum_{m=E(\sqrt{|\eta|})}^{1} \frac{m^{-1+\beta}}{(\nu^{\frac{1}{3}}\eta)^{\beta}}\Big), & 1 \le E(\sqrt{|\eta|}) \le \nu^{\frac{1}{3}}\eta, \end{cases}$$

$$\lesssim \begin{cases} \nu^{\frac{1}{3}}\eta \lesssim 1, & |\eta| \lesssim \nu^{-\frac{1}{3}}, \\ 1, & \nu^{-\frac{1}{3}} \lesssim |\eta| \lesssim \nu^{-\frac{2}{3}}, \\ \frac{1}{(\nu^{\frac{1}{3}}\sqrt{|\eta|})^{\beta}} \lesssim 1, & \nu^{-\frac{2}{3}} \lesssim |\eta|. \end{cases}$$

Thus we have proved the lemma. ∎

The above lemma gives that for all t,

$$A_k^{\sigma}(t,\eta) \approx \langle k,\eta\rangle^{\sigma}. \tag{3.10}$$

Next we introduce several lemmas related to the properties of D. The first lemma can be found in [9], which will be useful in the proof of the commutator estimate in Section 10.

Lemma 3.4 ([9]). *Uniformly in ν, η, ξ and $t \ge 1$ we have*

$$D(t,\eta) \gtrsim \nu \max\{|\eta|^3, t^3\}$$

and

$$\frac{D(t,\xi)}{D(t,\eta)} \lesssim \langle \eta-\xi\rangle^3, \quad |D(t,\xi) - D(t,\eta)| \lesssim \frac{D(t,\xi)}{\langle\xi\rangle + \langle\eta\rangle}\langle \eta-\xi\rangle^3.$$

In the next lemma we will introduce the product lemma related to D which is a Sobolev-type estimate comparable to [9, Lemma 3.7].

Lemma 3.5. *The following holds for all q^1 and q^2 and $\gamma > 1$:*

$$\|D(q^1q^2)\|_{H^{\gamma}} \lesssim \|q^1\|_{H^{\gamma+3}}\|Dq^2\|_{H^{\gamma}}$$

and

$$\|D(\nabla^{\perp}q^1 \cdot \nabla q^2)\|_{H^{\gamma}} \lesssim \|q^1\|_{H^{\gamma+5}}\|Dq^2\|_{H^{\gamma}} + \|Dq_1\|_{H^{\gamma}}\|q_2\|_{H^{\gamma+5}}.$$

Proof. We use the dual method. By Lemma 3.4, we get

$$\begin{aligned} \|D(q^1q^2)\|_{H^{\gamma}} &= \|\langle\nabla\rangle^{\gamma} D(q^1q^2)\|_{L^2} \\ &= \sup_{\|\varphi\|_{L^2}=1} \Big|\sum_{k,l}\int_{\eta,\xi} \hat{\varphi}_k(\eta)\langle k,\eta\rangle^{\gamma} D(\eta)\hat{q}^1_{k-l}(\eta-\xi)\hat{q}^2_l(\xi)\,d\xi\,d\eta\Big| \end{aligned}$$

$$
\begin{aligned}
&\lesssim \sup_{\|\varphi\|_{L^2}=1} \sum_{k,l} \int_{\eta,\xi} |\widehat{\varphi}_k(\eta)| \langle k,\eta\rangle^{\gamma} \langle \xi-\eta\rangle^3 |\hat{q}^1_{k-l}(\eta-\xi)|\, |D(\xi)\hat{q}^2_l(\xi)|\, d\xi\, d\eta \\
&\lesssim \sup_{\|\varphi\|_{L^2}=1} \sum_{k,l} \int_{\eta,\xi} 1_{\substack{|k-l,\eta-\xi|\\ \le |l,\xi|}} |\widehat{\varphi}_k(\eta)| \langle \xi-\eta\rangle^3 |\hat{q}^1_{k-l}(\eta-\xi)|\, |D(\xi)\langle l,\xi\rangle^{\gamma}\hat{q}^2_l(\xi)|\, d\xi\, d\eta \\
&\quad + \sup_{\|\varphi\|_{L^2}=1} \sum_{k,l} \int_{\eta,\xi} 1_{\substack{|k-l,\eta-\xi|\\ > |l,\xi|}} |\widehat{\varphi}_k(\eta)| \langle k-l,\eta-\xi\rangle^{\gamma} \langle \xi-\eta\rangle^3 |\hat{q}^1_{k-l}(\eta-\xi)| \\
&\qquad\qquad \times |D(\xi)\hat{q}^2_l(\xi)|\, d\xi\, d\eta \\
&\lesssim \|\varphi\|_{L^2} \|Dq^2\|_{H^\gamma} \|q^1\|_{H^{\gamma+3}}.
\end{aligned}
$$

In the last inequality we use the fact that $\|\hat{q}\|_{L^1} \lesssim \|\langle k,\eta\rangle^{\gamma}\hat{q}\|_{L^2} \|\langle k,\eta\rangle^{-\gamma}\|_{L^2} \lesssim \|q\|_{H^\gamma}$ for $\gamma > 1$.

We also have

$$
\begin{aligned}
&\|D(\nabla^{\perp} q^1 \cdot \nabla q^2)\|_{H^\gamma} \\
&= \sup_{\|\varphi\|_{L^2}=1} \left| \sum_{k,l} \int_{\eta,\xi} \widehat{\varphi}_k(\eta) \langle k,\eta\rangle^{\gamma} D(\eta) \hat{q}^1_{k-l}(\eta-\xi) \hat{q}^2_l(\xi) (-\eta+\xi, k-l)\cdot(l,\xi)\, d\xi\, d\eta \right| \\
&\lesssim \sup_{\|\varphi\|_{L^2}=1} \sum_{k,l} \int_{\eta,\xi} 1_{\substack{|k-l,\eta-\xi|\\ \le |l,\xi|}} |\widehat{\varphi}_k(\eta)| |D(\eta-\xi)\hat{q}^1_{k-l}(\eta-\xi)|\, |\langle \xi\rangle^3 \langle l,\xi\rangle^{\gamma+2} \hat{q}^2_l(\xi)|\, d\xi\, d\eta \\
&\quad + \sup_{\|\varphi\|_{L^2}=1} \sum_{k,l} \int_{\eta,\xi} 1_{\substack{|k-l,\eta-\xi|\\ > |l,\xi|}} |\widehat{\varphi}_k(\eta)| \langle k-l,\eta-\xi\rangle^{\gamma+2} \langle \xi-\eta\rangle^3 |\hat{q}^1_{k-l}(\eta-\xi)| \\
&\qquad\qquad \times |D(\xi)\hat{q}^2_l(\xi)|\, d\xi\, d\eta \\
&\lesssim \|\varphi\|_{L^2} \|Dq^2\|_{H^\gamma} \|q^1\|_{H^{\gamma+5}} + \|\varphi\|_{L^2} \|Dq^1\|_{H^\gamma} \|q^2\|_{H^{\gamma+5}}.
\end{aligned}
$$

Thus we have proved the lemma. ∎

4. Elliptic estimate

The purpose of this section is to provide a thorough analysis of Δ_t.

Lemma 4.1. *Under the bootstrap hypotheses, for ν sufficiently small and $s' \in [0,2]$, it holds that for $2 \le \gamma \le \sigma - 1$,*

$$
\|P_{\neq}\phi\|_{H^{\gamma-s'}} \lesssim \frac{1}{\langle t\rangle^{s'}} \|\langle \partial_z\rangle^{-s'} f_{\neq}\|_{H^\gamma},
$$

and for $\gamma \le \sigma - 1$,

$$
\|\Delta_L \Delta_t^{-1} P_{\neq} f\|_{H^\gamma} = \|\Delta_L P_{\neq}\phi\|_{H^\gamma} \lesssim \|P_{\neq} f\|_{H^\gamma}.
$$

Proof. For $s' \in [0,2]$ and $s \geq 0$ we get

$$\begin{aligned}
\|P_{\neq}\phi\|_{H^s}^2 &= \sum_{k\neq 0}\int_\eta \langle k,\eta\rangle^{2s}|\widehat{\phi}(k,\eta)|^2\,d\eta \\
&\leq \sum_{k\neq 0}\int_\eta \frac{\langle k,\eta\rangle^{2s}\langle\frac{\eta}{k}\rangle^{2s'}}{\langle\frac{\eta}{k}\rangle^{2s'}(k^2+(\eta-kt)^2)^2}|\widehat{\Delta_L\phi}(k,\eta)|^2\,d\eta \\
&\lesssim \sum_{k\neq 0}\int_\eta \frac{\langle k,\eta\rangle^{2s+2s'}}{k^{2s'}(1+t^2)^{s'}}|\widehat{\Delta_L\phi}(k,\eta)|^2\,d\eta \\
&\lesssim \frac{1}{(1+t^2)^{s'}}\|\langle\partial_z\rangle^{-s'}\Delta_L P_{\neq}\phi\|_{H^{s+s'}}^2. \qquad (4.1)
\end{aligned}$$

We write Δ_t as a perturbation of Δ_L via

$$\Delta_L P_{\neq}\phi = P_{\neq}f + (1-(v')^2)(\partial_v - t\partial_z)^2 P_{\neq}\phi - v''(\partial_v - t\partial_z)P_{\neq}\phi.$$

Thus we get

$$\begin{aligned}
\|\Delta_L P_{\neq}\phi\|_{H^\gamma} \leq \|P_{\neq}f\|_{H^\gamma} &+ C\|(1-(v')^2)(\partial_v - t\partial_z)^2 P_{\neq}\phi\|_{H^\gamma} \\
&+ C\|v''(\partial_v - t\partial_z)P_{\neq}\phi\|_{H^\gamma};
\end{aligned}$$

then by using the fact that $v'' = (h+1)\partial_v h$, (A.3) and the bootstrap hypotheses, we get

$$\begin{aligned}
\|\Delta_L P_{\neq}\phi\|_{H^\gamma} &\leq \|P_{\neq}f\|_{H^\gamma} + C\|h\|_{H^{\sigma-1}}(1+\|h\|_{\sigma-1})\|\Delta_L P_{\neq}\phi\|_{H^\gamma} \\
&\quad + C(1+\|h\|_{\sigma-1})\|h\|_{H^\sigma}\|\Delta_L P_{\neq}\phi\|_{H^\gamma} \\
&\lesssim \|P_{\neq}f\|_{H^\gamma} + C\varepsilon\nu^{\frac{1}{6}}\|\Delta_L P_{\neq}\phi\|_{H^\gamma},
\end{aligned}$$

which implies $\|\Delta_L P_{\neq}\phi\|_{H^\gamma} \lesssim \|P_{\neq}f\|_{H^\gamma}$. The lemma follows from (4.1) with $s = \sigma - 2 - s'$. ∎

As $(1-(v')^2)$ and v'' are the zero mode, by the same argument as the proof we can easily get that for $\gamma \leq \sigma - 1$,

$$\|\langle\partial_z\rangle^{\sigma-\gamma}\langle\partial_v\rangle^\gamma \Delta_L\Delta_t^{-1}f_{\neq}\|_2 \lesssim \|f_{\neq}\|_{H^\sigma} \lesssim \|A^\sigma f\|_2. \qquad (4.2)$$

Lemma 4.2. *Under the bootstrap hypotheses, it holds that*

$$\|\nabla_L P_{\neq}\phi\|_{H^{\sigma-2}} + \|\tilde{u}_{\neq}\|_{H^{\sigma-2}} \lesssim \frac{1}{\langle t\rangle}\|f_{\neq}\|_{H^{\sigma-1}},$$

and $\gamma \leq \sigma - 1$,

$$\|\nabla_L\tilde{u}_{\neq}\|_{H^\gamma} \lesssim \|f_{\neq}\|_{H^\gamma}.$$

Proof. By the definition of $\tilde{u}$ we get

$$\tilde{u}_{\neq} = -(1+h)(\partial_v - t\partial_z)P_{\neq}\phi.$$

Here we use the same argument as (4.1) and we get

$$\begin{aligned}\|(\partial_v - t\partial_z)P_{\neq}\phi\|_{H^s}^2 &= \sum_{k\neq 0}\int_\eta \langle k,\eta\rangle^{2s}|\eta-kt|^2|\widehat{\phi}(k,\eta)|^2\,d\eta\\ &\leq \sum_{k\neq 0}\int_\eta \frac{\langle k,\eta\rangle^{2s}\langle\frac{\eta}{k}\rangle^2|\eta-kt|^2}{\langle\frac{\eta}{k}\rangle^2(k^2+(\eta-kt)^2)^2}|\widehat{\Delta_L\phi}(k,\eta)|^2\,d\eta\\ &\leq \sum_{k\neq 0}\int_\eta \frac{\langle k,\eta\rangle^{2s}\langle\frac{\eta}{k}\rangle^2}{\langle\frac{\eta}{k}\rangle^2(k^2+(\eta-kt)^2)}|\widehat{\Delta_L\phi}(k,\eta)|^2\,d\eta\\ &\lesssim \sum_{k\neq 0}\int_\eta \frac{\langle k,\eta\rangle^{2s+2}}{k^2(1+t^2)}|\widehat{\Delta_L\phi}(k,\eta)|^2\,d\eta\\ &\lesssim \frac{1}{1+t^2}\|\langle\partial_z\rangle^{-1}\Delta_L P_{\neq}\phi\|_{H^{s+1}}^2.\end{aligned} \tag{4.3}$$

Then by Lemma 4.1 and the bootstrap hypotheses, we have

$$\begin{aligned}\|\tilde{u}_{\neq}\|_{H^{\sigma-2}} &\lesssim (1+\|h\|_{H^{\sigma-2}})\|(\partial_v - t\partial_z)P_{\neq}\phi\|_{H^{\sigma-2}}\\ &\lesssim \frac{(1+\|h\|_{H^{\sigma-2}})}{\langle t\rangle}\|\Delta_L P_{\neq}\phi\|_{H^{\sigma-1}} \lesssim \frac{1}{\langle t\rangle}\|f_{\neq}\|_{H^{\sigma-1}}.\end{aligned}$$

The first inequality follows from (4.3) with $s=\sigma-2$.

We also have

$$\begin{aligned}\partial_z\tilde{u}_{\neq} &= -(1+h)(\partial_v - t\partial_z)\partial_z P_{\neq}\phi,\\ (\partial_v - t\partial_z)\tilde{u}_{\neq} &= -(1+h)(\partial_v - t\partial_z)^2 P_{\neq}\phi - \partial_v h(\partial_v - t\partial_z)P_{\neq}\phi.\end{aligned}$$

Therefore, by Lemma 4.1 and the bootstrap hypotheses, we get

$$\|\nabla_L\tilde{u}_{\neq}\|_{H^\gamma} \lesssim (1+\|h\|_{H^{\sigma-1}})\|\Delta_L P_{\neq}\phi\|_{H^\gamma} \lesssim \|f_{\neq}\|_{H^\gamma}.$$

Thus we have proved the lemma. ∎

Lemma 4.3. *Under the bootstrap hypotheses, it holds that*

$$\|\nabla_L\tilde{u}_{\neq}\|_{H^\sigma} \lesssim \|\Delta_L\Delta_t^{-1}f_{\neq}\|_{H^\sigma} \lesssim \|f_{\neq}\|_{H^\sigma} + \frac{\varepsilon\nu^{\frac13}}{\langle t\rangle\langle \nu t^3\rangle}\|\partial_v h\|_{H^\sigma}$$

and

$$\left\|\sqrt{\frac{\partial_t w}{w}}\chi_R\Delta_L\Delta_t^{-1}f_{\neq}\right\|_{H^\sigma} \lesssim \left\|\sqrt{\frac{\partial_t w}{w}}f_{\neq}\right\|_{H^\sigma} + \frac{\varepsilon^2\nu^{\frac12}}{\langle\nu t^3\rangle}.$$

Proof. We have

$$\Delta_L\Delta_t^{-1}f_{\neq} = \Delta_L P_{\neq}\phi = P_{\neq}f + (1-(v')^2)(\partial_v - t\partial_z)^2 P_{\neq}\phi - v''(\partial_v - t\partial_z)P_{\neq}\phi.$$

Thus we get

$$\begin{aligned}
\|\Delta_L\Delta_t^{-1} f_{\neq}\|_{H^\sigma} &\lesssim \|f_{\neq}\|_{H^\sigma} + \|(1-(v')^2)\|_{H^3}\|(\partial_v - t\partial_z)^2 P_{\neq}\phi\|_{H^\sigma} \\
&\quad + \|(1-(v')^2)\|_{H^\sigma}\|(\partial_v - t\partial_z)^2 P_{\neq}\phi\|_{H^3} \\
&\quad + \|(1+h)\partial_v h\|_{H^3}\|\nabla_L P_{\neq}\phi\|_{H^\sigma} + \|(1+h)\partial_v h\|_{H^\sigma}\|\nabla_L P_{\neq}\phi\|_{H^3} \\
&\lesssim \|f_{\neq}\|_{H^\sigma} + \varepsilon\nu^{\frac{1}{3}}\|\Delta_L\Delta_t^{-1} f_{\neq}\|_{H^\sigma} + \varepsilon\nu^{\frac{1}{6}}\|f_{\neq}\|_{H^3} + \varepsilon\nu^{\frac{1}{6}}\|\nabla_L P_{\neq}\phi\|_{H^3} \\
&\quad + \|\partial_v h\|_{H^\sigma}\|\nabla_L P_{\neq}\phi\|_{H^3} \\
&\lesssim \|f_{\neq}\|_{H^\sigma} + \varepsilon\nu^{\frac{1}{6}}\|\Delta_L\Delta_t^{-1} f_{\neq}\|_{H^\sigma} + \langle t\rangle^{-1}\|\partial_v h\|_{H^\sigma}\|\Delta_L P_{\neq}\phi\|_{H^4} \\
&\lesssim \|f_{\neq}\|_{H^\sigma} + \varepsilon\nu^{\frac{1}{6}}\|\Delta_L\Delta_t^{-1} f_{\neq}\|_{H^\sigma} + \frac{\nu^{\frac{1}{3}}}{\langle t\rangle\langle \nu t^3\rangle}\|\partial_v h\|_{H^\sigma}.
\end{aligned}$$

We also have

$$\begin{aligned}
\partial_z\tilde{u}_{\neq} &= -(1+h)(\partial_v - t\partial_z)\partial_z P_{\neq}\phi, \\
(\partial_v - t\partial_z)\tilde{u}_{\neq} &= -(1+h)(\partial_v - t\partial_z)^2 P_{\neq}\phi - \partial_v h(\partial_v - t\partial_z)P_{\neq}\phi.
\end{aligned}$$

Therefore by Lemma 4.1 and the bootstrap hypotheses, we get

$$\|\nabla_L\tilde{u}_{\neq}\|_{H^\sigma} \lesssim (1+\|h\|_{H^\sigma})\|\Delta_L P_{\neq}\phi\|_{H^\sigma}.$$

By taking ε small enough, we get the first inequality.

In what follows we use the shorthand

$$G(\xi) = \widehat{1-(v')^2}(\xi),$$

and then

$$\begin{aligned}
&\sqrt{\frac{\partial_t w_k(t,\eta)}{w_k(t,\eta)}}1_{t\in I_{k,\eta}}1_{k\neq 0}(k^2+(\eta-kt)^2)\phi_k(t,\eta) \\
&\quad = \sqrt{\frac{\partial_t w_k(t,\eta)}{w_k(t,\eta)}}1_{t\in I_{k,\eta}}1_{k\neq 0} f_k(t,\eta) \\
&\qquad - \sqrt{\frac{\partial_t w_k(t,\eta)}{w_k(t,\eta)}}1_{t\in I_{k,\eta}}1_{k\neq 0}\int_{|\xi|\geq|\eta-\xi|} G(\xi)(\eta-\xi-kt)^2\widehat{\phi}_k(t,\eta-\xi)\,d\xi \\
&\qquad - \sqrt{\frac{\partial_t w_k(t,\eta)}{w_k(t,\eta)}}1_{t\in I_{k,\eta}}1_{k\neq 0}\int_{|\xi|\geq|\eta-\xi|} G(\eta-\xi)(\xi-kt)^2\widehat{\phi}_k(t,\xi)\,d\xi \\
&\qquad - i\sqrt{\frac{\partial_t w_k(t,\eta)}{w_k(t,\eta)}}1_{t\in I_{k,\eta}}1_{k\neq 0}\int_{|\xi|\geq|\eta-\xi|} \widehat{v''}(\xi)(\eta-\xi-kt)\widehat{\phi}_k(t,\eta-\xi)\,d\xi \\
&\qquad - i\sqrt{\frac{\partial_t w_k(t,\eta)}{w_k(t,\eta)}}1_{t\in I_{k,\eta}}1_{k\neq 0}\int_{|\xi|\geq|\eta-\xi|} \widehat{v''}(\eta-\xi)(\xi-kt)\widehat{\phi}_k(t,\xi)\,d\xi \\
&\quad = \sqrt{\frac{\partial_t w_k(t,\eta)}{w_k(t,\eta)}}1_{t\in I_{k,\eta}}1_{k\neq 0} f_k(t,\eta) + E^1_{HL} + E^1_{LH} + E^2_{HL} + E^2_{LH}.
\end{aligned}$$

We have $t \approx t_{k,\eta} \approx \frac{\eta}{k}$ and then

$$\frac{\partial_t w_k(t,\eta)}{w_k(t,\eta)} = \frac{\langle \nu^{\frac{1}{3}} t_{k,\eta}\rangle^{-(1+\beta)} \nu^{\frac{1}{3}} \frac{\eta}{k^2}}{1+(\frac{\eta}{k}-t)^2} \lesssim \frac{1}{k}\langle \nu^{\frac{1}{3}} t\rangle^{-(1+\beta)} \nu^{\frac{1}{3}} t \lesssim \frac{1}{k}.$$

Thus we get

$$\begin{aligned}\|E^1_{HL}\|_{H^\sigma} &\lesssim \|G\|_{H^\sigma} \|(\partial_v - t\partial_z)^2 P_{\neq}\phi\|_{H^4}\\ &\lesssim \|h\|_{H^\sigma}(\|h\|_{H^3}+1)\|f_{\neq}\|_{H^4}\\ &\lesssim \frac{\varepsilon^2\nu^{\frac{1}{2}}}{\langle \nu t^3\rangle}.\end{aligned}$$

For E^1_{LH} we get

$$\frac{\partial_t w_k(t,\eta)}{w_k(t,\eta)} \frac{w_k(t,\xi)}{\partial_t w_k(t,\xi)} \approx \frac{1+|\frac{\xi}{k}-t|^2}{1+|\frac{\eta}{k}-t|^2} \lesssim \langle \eta-\xi\rangle^2.$$

Then we get

$$\begin{aligned}&\|E^1_{LH}\|_{H^\sigma}\\ &\lesssim \left\| \langle k,\eta\rangle^\sigma \sqrt{\frac{\partial_t w_k(t,\eta)}{w_k(t,\eta)}} 1_{t\in I_{k,\eta}} 1_{k\neq 0} \int_{|\xi|\geq|\eta-\xi|} \widehat{G}(\eta-\xi)(\xi-kt)^2 \widehat{\phi}_k(t,\xi)\,d\xi \right\|_{L^2}\\ &\lesssim \left\| 1_{t\in I_{k,\eta}} 1_{k\neq 0} \int_{|\xi|\geq|\eta-\xi|} \langle \eta-\xi\rangle \widehat{G}(\eta-\xi)(\xi-kt)^2 \langle k,\xi\rangle^\sigma \sqrt{\frac{\partial_t w_k(t,\xi)}{w_k(t,\xi)}} \widehat{\phi}_k(t,\xi)\,d\xi \right\|_{L^2}\\ &\lesssim \|G\|_{H^6} \left\| \sqrt{\frac{\partial_t w}{w}} \chi_R \Delta_L \Delta_t^{-1} f_{\neq} \right\|_{H^\sigma} \lesssim \varepsilon\nu^{\frac{1}{3}} \left\| \sqrt{\frac{\partial_t w}{w}} \chi_R \Delta_L \Delta_t^{-1} f_{\neq} \right\|_{H^\sigma},\end{aligned}$$

and similarly we have

$$\begin{aligned}&\|E^2_{LH}\|_{H^\sigma}\\ &\lesssim \left\| \langle k,\eta\rangle^\sigma \sqrt{\frac{\partial_t w_k(t,\eta)}{w_k(t,\eta)}} 1_{t\in I_{k,\eta}} 1_{k\neq 0} \int_{|\xi|\geq|\eta-\xi|} \widehat{v''}(\eta-\xi)|\xi-kt| \widehat{\phi}_k(t,\xi)\,d\xi \right\|_{L^2}\\ &\lesssim \left\| 1_{t\in I_{k,\eta}} 1_{k\neq 0} \int_{|\xi|\geq|\eta-\xi|} \langle \eta-\xi\rangle \widehat{v''}(\eta-\xi)|\xi-kt| \langle k,\xi\rangle^\sigma \sqrt{\frac{\partial_t w_k(t,\xi)}{w_k(t,\xi)}} \widehat{\phi}_k(t,\xi)\,d\xi \right\|_{L^2}\\ &\lesssim \|v''\|_{H^6} \left\| \sqrt{\frac{\partial_t w}{w}} \chi_R \Delta_L \Delta_t^{-1} f_{\neq} \right\|_{H^\sigma} \lesssim \varepsilon\nu^{\frac{1}{3}} \left\| \sqrt{\frac{\partial_t w}{w}} \chi_R \Delta_L \Delta_t^{-1} f_{\neq} \right\|_{H^\sigma}.\end{aligned}$$

Finally, we deal with T^2_{HL}: we have $\sqrt{\frac{\partial_t w_k(t,\xi)}{w_k(t,\xi)}} \lesssim \frac{kt}{\eta}$ and then we get

$$\begin{aligned}\|E^2_{HL}\|_{H^\sigma} &\lesssim \left\| \langle k,\eta\rangle^\sigma \frac{kt}{\eta} 1_{t\in I_{k,\eta}} 1_{k\neq 0} \int_{|\xi|\geq|\eta-\xi|} \widehat{v''}(\xi)(\eta-\xi-kt)\widehat{\phi}_k(t,\eta-\xi)\,d\xi \right\|_{L^2}\\ &\lesssim \left\| 1_{t\in I_{k,\eta}} 1_{k\neq 0} \int_{|\xi|\geq|\eta-\xi|} \langle \xi\rangle^{\sigma-1} |\widehat{v''}(\xi)|\,|kt|\,|\eta-\xi-kt|\,|\widehat{\phi}_k(t,\eta-\xi)|\,d\xi \right\|_{L^2}\end{aligned}$$

$$\lesssim t\|v''\|_{H^{\sigma-1}}\|\nabla_L\phi_{\neq}\|_{H^4}$$
$$\lesssim \|h\|_{H^\sigma}\|f_{\neq}\|_{H^5} \lesssim \frac{\varepsilon^2\nu^{\frac{1}{2}}}{\langle \nu t^3\rangle}.$$

Thus we have proved the lemma. ■

By the fact that $u = (0, g)^T + (1+h)\nabla^{\perp}_{z,v}P_{\neq}\phi$, Lemma 4.1 and under the bootstrap hypotheses, it holds that

$$\|u\|_{H^s} \lesssim \|g\|_{H^s} + \|P_{\neq}\phi\|_{H^{s+1}} \lesssim \frac{\varepsilon\nu^{\frac{1}{3}}}{\langle t^2\rangle}. \tag{4.4}$$

Lemma 4.4. *Under the bootstrap hypotheses for ε sufficiently small, for $s \le \sigma - 7$ it holds that*

$$\|A^s_E(P_{\neq}\phi)\|_2 \lesssim \frac{1}{\langle t\rangle^2}(\|A^s_E f\|_2 + \|f\|_{H^\sigma}).$$

Proof. By Lemma 3.4, we have

$$\begin{aligned}
\|A^s_E(\nabla^{\perp}P_{\neq}\phi)\|_2^2 &\approx \|\nu\max\{t^3,\eta^3\}\widehat{\phi}_{\neq}\|^2_{H^s} \\
&\lesssim \nu^2\sum_{k\neq 0}\int_{2|\eta|\ge t}\langle k,\eta\rangle^{2s+6}|\widehat{\phi}_k(t,\eta)|^2\,d\eta \\
&\quad + \nu^2\sum_{k\neq 0}\int_{2|\eta|< t} t^6\langle k,\eta\rangle^{2s}|\widehat{\phi}_k(t,\eta)|^2\,d\eta \\
&= \Pi_1 + \Pi_2.
\end{aligned}$$

By Lemma 4.1 we get

$$|\Pi_1| \lesssim \|P_{\neq}\phi\|^2_{H^{s+3}} \lesssim \frac{1}{\langle t\rangle^4}\|f_{\neq}\|^2_{H^{s+5}} \lesssim \frac{1}{\langle t\rangle^4}\|f_{\neq}\|^2_{H^\sigma}$$

and

$$\begin{aligned}
|\Pi_2| \lesssim \nu^2 t^6\|P_{\neq}\phi\|^2_{H^s} &\lesssim \nu^2\sum_{k\neq 0}\int_{2|\eta|<t} t^6\frac{\langle k,\eta\rangle^{2s}}{(k^2+|\eta-kt|^2)^2}|\widehat{\Delta_L\phi}_k(t,\eta)|^2\,d\eta \\
&\lesssim \nu^2\sum_{k\neq 0}\int_{2|\eta|<t} t^6\frac{\langle k,\eta\rangle^{2s}}{(k^2+|\eta|^2+k^2t^2)^2}|\widehat{\Delta_L\phi}_k(t,\eta)|^2\,d\eta \\
&\lesssim \frac{1}{\langle t\rangle^4}\|\nu t^3\Delta_L\phi_{\neq}\|^2_{H^s}.
\end{aligned}$$

By Lemma 4.1, we then obtain

$$|\Pi_2| \lesssim \frac{1}{\langle t\rangle^4}\|\nu t^3 f_{\neq}\|^2_{H^s} \lesssim \frac{1}{\langle t\rangle^4}\|A^s_E f_{\neq}\|_2^2.$$

Thus we have proved the lemma. ■

5. Dissipation error term

In this section we will deal with the dissipation error term in (2.45).

5.1. Treatment of the zero mode

By the fact that

$$A_0^\sigma(\eta) \approx \langle\eta\rangle^\sigma \approx 1+|\eta|^\sigma \quad \text{and} \quad |\eta| \le |\xi|+|\eta-\xi| \lesssim \max\{|\xi|,|\eta-\xi|\},$$

we get

$$\begin{aligned}
|E^0| &\lesssim \int_{\xi,\eta} \langle\eta\rangle^{2\sigma}|\bar{\hat{f}}_0(\eta)|\left|\widehat{(1-(v')^2}(\eta-\xi))|\xi|^2\hat{f}_0(\xi)\right| d\xi\, d\eta \\
&\lesssim \int_{\xi,\eta} 1_{|\eta|\le 1}|\bar{\hat{f}}_0(\eta)|\left|\widehat{(1-(v')^2}(\eta-\xi))|\xi|(|\eta|+|\xi-\eta|)\hat{f}_0(\xi)\right| d\eta\, d\xi \\
&\quad + \int_{\xi,\eta} 1_{|\eta|\ge 1}1_{|\xi-\eta|\ge|\xi|}|\eta|^{2\sigma}|\bar{\hat{f}}_0(\eta)|\left|\widehat{(1-(v')^2}(\eta-\xi))|\xi|^2\hat{f}_0(\xi)\right| d\xi\, d\eta \\
&\quad + \int_{\xi,\eta} 1_{|\eta|\ge 1}1_{|\xi-\eta|<|\xi|}|\eta|^{2\sigma}|\bar{\hat{f}}_0(\eta)|\left|\widehat{(1-(v')^2}(\eta-\xi))|\xi|^2\hat{f}_0(\xi)\right| d\xi\, d\eta \\
&\lesssim \int_{\xi,\eta} 1_{|\eta|\le 1}\left|\overline{\widehat{\partial_v f}_0}(\eta)\right|\left|\widehat{(1-(v')^2}(\eta-\xi))|\xi|\hat{f}_0(\xi)\right| d\eta\, d\xi \\
&\quad + \int_{\xi,\eta} 1_{|\eta|\le 1}|\bar{\hat{f}}_0(\eta)|\left|\widehat{(\partial_v(1-(v')^2)}(\eta-\xi))|\xi|\hat{f}_0(\xi)\right| d\eta\, d\xi \\
&\quad + \int_{\xi,\eta} 1_{|\eta|\ge 1}1_{|\xi-\eta|<|\xi|}|\eta|^{\sigma+1}|\bar{\hat{f}}_0(\eta)|\left|\widehat{(1-(v')^2}(\eta-\xi))|\xi|^{\sigma+1}\hat{f}_0(\xi)\right| d\xi\, d\eta \\
&\quad + \int_{\xi,\eta} 1_{|\eta|\ge 1}1_{|\xi-\eta|\ge|\xi|}|\eta|^{\sigma+1}|\bar{\hat{f}}_0(\eta)|\left|\widehat{(1-(v')^2}(\eta-\xi))|\eta-\xi|^{\sigma-1}|\xi|^2\hat{f}_0(\xi)\right| d\xi\, d\eta \\
&\lesssim \nu\|\partial_v f_0\|_{H^2}^2\|1-(v')^2\|_2 + \nu\|\partial_v f_0\|_{H^2}\|\partial_v(1-(v')^2)\|_2\|f_0\|_2 \\
&\quad + \nu\|\partial_v f_0\|_{H^\sigma}^2\|1-(v')^2\|_{H^2} + \nu\|\partial_v f_0\|_{H^\sigma}\|\partial_v(1-(v')^2)\|_{H^{\sigma-2}}\|f_0\|_{H^4}.
\end{aligned}$$

The purpose of the above estimate is to obtain the homogeneous derivative. By the fact that $(v')^2-1=(1-(v'))^2+2(v'-1)=h^2+2h$ and

$$\|h^2\|_{H^s} \lesssim \|h\|_{H^1}\|h\|_{H^s}, \quad \|\partial_v h^2\|_{H^s} \lesssim \|h\|_{H^1}\|\partial_v h\|_{H^s}, \quad s\ge 1,$$

we obtain by the bootstrap hypotheses that

$$\begin{aligned}
|E^0| &\lesssim \nu(\|h\|_{H^2}+1)(\|\partial_v A^\sigma f_0\|_2^2\|h\|_{H^2} + \|\partial_v h\|_{H^{\sigma-2}}\|\partial_v f_0\|_{H^\sigma}\|f_0\|_{H^4}) \\
&\lesssim \varepsilon\nu^{\frac13}\nu\|\partial_v A^\sigma f_0\|_2^2 + \varepsilon\nu^{\frac13}\nu\|\partial_v h\|_{H^{\sigma-1}}^2.
\end{aligned} \tag{5.1}$$

5.2. Treatment of the nonzero mode

We use a paraproduct decomposition in v. Then we have

$$E^{\neq} = E^{\neq}_{LH} + E^{\neq}_{HL} + E^{\neq}_{HH},$$

where

$$E^{\neq}_{LH} = -\sum_{M\geq 8}\nu\int A^\sigma f_{\neq} A^\sigma\big((1-(v')^2)_{<M/8}(\partial_v - t\partial_z)^2(f_{\neq})_M\big)\,dz\,dv,$$

$$E^{\neq}_{HL} = -\sum_{M\geq 8}\nu\int A^\sigma f_{\neq} A^\sigma\big((1-(v')^2)_{M}(\partial_v - t\partial_z)^2(f_{\neq})_{<M/8}\big)\,dz\,dv,$$

$$E^{\neq}_{HH} = -\nu\sum_{M\in\mathbb{D}}\sum_{\frac{1}{8}M\leq M'\leq 8M}\int A^\sigma f_{\neq} A^\sigma\big((1-(v')^2)_{M}(\partial_v - t\partial_z)^2(f_{\neq})_{M'}\big)\,dz\,dv.$$

5.2.1. Treatment of $E^{\neq}_{LH}$. We have

$$E^{\neq}_{LH} \lesssim \nu\sum_{M\geq 8}\sum_{k\neq 0}\int_{\eta,\xi} A^\sigma|\bar{\hat{f}}_k(\eta)|A^\sigma_k(\eta)\big|\widehat{(1-(v')^2)}(\eta-\xi)_{<M/8}|\xi-kt|^2\hat{f}_k(\xi)_M\big|\,d\xi\,d\eta.$$

By the fact that $\xi\approx\eta\approx M$, $|k,\eta|\approx|k,\xi|$ and

$$|\xi-kt|\lesssim|\xi-\eta|+|\eta-kt|\lesssim\langle\xi-\eta\rangle\sqrt{k^2+|\eta-kt|^2},$$

we have

$$\begin{aligned}E^{\neq}_{LH} &\lesssim \nu\sum_{M\geq 8}\sum_{k\neq 0}\int_{\eta,\xi}\sqrt{k^2+|\eta-kt|^2}A^\sigma|\bar{\hat{f}}_k(\eta)|\,\big|\widehat{\langle\partial_v\rangle(1-(v')^2)}(\eta-\xi)_{<M/8}\\ &\qquad\qquad\times|\xi-kt|A^\sigma_k(\xi)\hat{f}_k(\xi)_M\big|\,d\xi\,d\eta\\ &\lesssim \nu\sum_{M\geq 8}\big\|\big(\sqrt{-\Delta_L}A^\sigma f_{\neq}\big)_{\sim M}\big\|_2\big\|\big(\sqrt{-\Delta_L}A^\sigma f_{\neq}\big)_M\big\|_2\|(1-(v')^2)\|_{H^4},\end{aligned}$$

which gives

$$E^{\neq}_{LH}\lesssim\nu\big\|\big(\sqrt{-\Delta_L}A^\sigma f_{\neq}\big)\big\|_2^2\|(1-(v')^2)\|_{H^4}.$$

5.2.2. Treatment of $E^{\neq}_{HL}$. We have

$$\begin{aligned}E^{\neq}_{HL} &\lesssim \nu\sum_{M\geq 8}\sum_{k\neq 0}\int_{\eta,\xi}[1_{|\eta|\leq 16|k|}+1_{|\eta|>16|k|}]A^\sigma|\bar{\hat{f}}_k(\eta)|A^\sigma_k(\eta)\\ &\qquad\qquad\times\big|\widehat{(1-(v')^2)}(\eta-\xi)_M|\xi-kt|^2\hat{f}_k(\xi)_{<M/8}\big|\,d\xi\,d\eta\\ &= E^{\neq,z}_{HL}+E^{\neq,v}_{HL}.\end{aligned}$$

For $E^{\neq,z}_{HL}$, we have $|k,\eta|\approx|k|\approx|k,\xi|$ and

$$|\xi-kt|\lesssim|\xi-\eta|+|\eta-kt|\lesssim\langle\xi-\eta\rangle\sqrt{k^2+|\eta-kt|^2},$$

which then implies

$$E_{HL}^{\neq,z} \lesssim \nu \sum_{M\geq 8}\sum_{k\neq 0}\int_{\eta,\xi} 1_{|\eta|\leq 16|k|} A^\sigma \sqrt{k^2+|\eta-kt|^2}|\bar{\hat{f}}_k(\eta)| \times \left|\langle\eta-\xi\rangle\widehat{(1-(v')^2)}(\eta-\xi)_M|\xi-kt|\,|k|^\sigma \hat{f}_k(\xi)_{<M/8}\right| d\xi\, d\eta$$
$$\lesssim \nu \sum_{M\geq 8} M^{-2}\|(1-(v')^2)_M\|_{H^5}\left\|\left(\sqrt{-\Delta_L}A^\sigma f_{\neq}\right)\right\|_2^2.$$

Thus we have

$$E_{HL}^{\neq,z} \lesssim \nu\|(1-(v')^2)\|_{H^5}\left\|\left(\sqrt{-\Delta_L}A^\sigma f_{\neq}\right)\right\|_2^2.$$

We turn to $E_{HL}^{\neq,v}$. In this case, $|k,\eta|\approx|\eta|\approx|\eta-\xi|\approx M$, and then we get

$$|E_{HL}^{\neq,v}| \lesssim \nu \sum_{M\geq 8}\sum_{k\neq 0}\int_{\eta,\xi} 1_{|\eta|>16|k|} A^\sigma |\bar{\hat{f}}_k(\eta)|\,|\eta-\xi|\langle\eta-\xi\rangle^{\sigma-1}\left|\widehat{(1-(v')^2)}(\eta-\xi)_M\right| \times |\xi-kt|^2|\hat{f}_k(\xi)_{<M/8}|\, d\xi\, d\eta$$
$$\lesssim \nu \sum_{M\geq 8}\|(f_{\neq})_{\sim M}\|_{H^\sigma}\|\partial_v(1-(v')^2)_M\|_{H^{\sigma-1}}\langle t\rangle^2\|f_{\neq}\|_{H^5}$$
$$\lesssim \nu\|f_{\neq}\|_{H^\sigma}\|\partial_v(1-(v')^2)\|_{H^{\sigma-1}}\langle t\rangle^2\|f_{\neq}\|_{H^5}.$$

5.2.3. Treatment of $E_{HH}^{\neq}$. In this case, it holds that $|\eta-\xi|\approx|\xi|\approx M'$. We divide the problem into two parts:

$$|E_{HH}^{\neq}| \lesssim \nu \sum_{M\in\mathbb{D}}\sum_{\frac{1}{8}M\leq M'\leq 8M}\sum_{k\neq 0}\int_{\eta,\xi}[1_{|k|\geq 16|\xi|}+1_{|k|<16|\xi|}] \times A^\sigma|\bar{\hat{f}}_k(\eta)|A_k^\sigma(\eta)\left|\widehat{(1-(v')^2)}(\eta-\xi)_{M'}|\xi-kt|^2\hat{f}_k(\xi)_M\right| d\xi\, d\eta$$
$$= E_{HH}^{\neq,z}+E_{HH}^{\neq,v}.$$

To treat $E_{HH}^{\neq,z}$, we have

$$|k|\lesssim|k,\eta|\lesssim|k|+|\eta-\xi|+|\xi|\lesssim|k|$$

and

$$|\xi-kt|\lesssim|\xi-\eta|+|\eta-kt|\lesssim\langle\xi-\eta\rangle\sqrt{k^2+|\eta-kt|^2}.$$

Therefore we get

$$E_{HH}^{\neq,z} \lesssim \nu \sum_{M\in\mathbb{D}}\sum_{\frac{1}{8}M\leq M'\leq 8M}\sum_{k\neq 0}\int_{\eta,\xi} 1_{|k|\geq 16|\xi|}A^\sigma\sqrt{k^2+|\eta-kt|^2}|\bar{\hat{f}}_k(\eta)| \times \langle\xi-\eta\rangle\left|\widehat{(1-(v')^2)}(\eta-\xi)_{M'}|\xi-kt|\,|k|^\sigma\hat{f}_k(\xi)_M\right| d\xi\, d\eta$$
$$\lesssim \nu \sum_{M\in\mathbb{D}}\left\|\sqrt{-\Delta_L}A^\sigma f\right\|_2\left\|\left(\sqrt{-\Delta_L}A^\sigma f\right)_M\right\|_2\|(1-(v')^2)_{\sim M}\|_{H^3}$$
$$\lesssim \nu\left\|\sqrt{-\Delta_L}A^\sigma f\right\|_2^2\|(1-(v')^2)\|_{H^3}.$$

Next we turn to $E^{\neq,v}_{HH}$, in which case we have

$$|k,\eta| \lesssim |k| + |\eta| \lesssim |k| + |\eta-\xi| + |\xi| \approx |k| + |\xi| \lesssim |\xi| \approx |\xi-\eta|$$

and

$$|\xi - kt| \lesssim |\xi-\eta| + |\eta-kt| \lesssim \langle \xi-\eta\rangle \sqrt{k^2+|\eta-kt|^2}.$$

Therefore we get

$$\begin{aligned}
E^{\neq,v}_{HH} &\lesssim \nu \sum_{M\in\mathbb{D}} \sum_{\frac{1}{8}M\le M'\le 8M} \sum_{k\neq 0} \int_{\eta,\xi} 1_{|k|<16|\xi|} A^\sigma \sqrt{k^2+|\eta-kt|^2}\,|\bar{\hat{f}}_k(\eta)| \\
&\qquad\qquad \times \langle \xi-\eta\rangle \big|\widehat{(1-(v')^2)}(\eta-\xi)_{M'}|\xi-kt|\,|\xi|^\sigma \hat{f}_k(\xi)_M\big|\,d\xi\,d\eta \\
&\lesssim \nu \sum_{M\in\mathbb{D}} \big\|\sqrt{-\Delta_L}A^\sigma f\big\|_2 \big\|\big(\sqrt{-\Delta_L}A^\sigma f\big)_M\big\|_2 \|(1-(v')^2)_{\sim M}\|_{H^3} \\
&\lesssim \nu \big\|\sqrt{-\Delta_L}A^\sigma f\big\|_2^2 \|(1-(v')^2)\|_{H^3}.
\end{aligned}$$

By the fact that $(v')^2-1 = h^2+2h$, the bootstrap hypotheses and (2.28), we obtain

$$\begin{aligned}
|E^{\neq}| &\lesssim \nu \|f_{\neq}\|_{H^\sigma} \|\partial_v(1-(v')^2)\|_{H^{\sigma-1}} \langle t\rangle^2 \|f_{\neq}\|_{H^5} \\
&\quad + \nu \big\|\sqrt{-\Delta_L}A^\sigma f\big\|_2^2 \|(1-(v')^2)\|_{H^4} \\
&\lesssim \nu(1+\|h\|_{H^2})\big(\|f_{\neq}\|_{H^\sigma}\|\partial_v h\|_{H^{\sigma-1}}\langle t\rangle^2\|f_{\neq}\|_{H^5} + \big\|\sqrt{-\Delta_L}A^\sigma f\big\|_2^2 \|h\|_{H^4}\big) \\
&\lesssim \nu\varepsilon\nu^{\frac{1}{3}} \big\|\sqrt{-\Delta_L}A^\sigma f\big\|_2^2 + (\varepsilon\nu^{\frac{1}{3}})^2 \nu \|\partial_v h\|_{H^{\sigma-1}} \frac{\langle t\rangle^2}{\langle \nu t^3\rangle}.
\end{aligned} \tag{5.2}$$

We end the section by proving Proposition 2.5.

Proof of Proposition 2.5. We get by (2.45) that

$$\begin{aligned}
&\int_1^t \Big(\nu \int A^\sigma f(t') A^\sigma(\widetilde{\Delta}_t f(t'))\,dz\,dv\Big)\,dt' \\
&\qquad \le -\int_1^t \nu \big\|\sqrt{-\Delta_L}A^\sigma f(t')\big\|_2^2\,dt' + \int_1^t |E^{\neq}(t')| + |E^0(t')|\,dt'.
\end{aligned}$$

Then by (5.1) and (5.2) we obtain

$$\begin{aligned}
&\int_1^t \Big(\nu \int A^\sigma f(t') A^\sigma(\widetilde{\Delta}_t f(t'))\,dz\,dv\Big)\,dt' \\
&\quad \le -\int_1^t \nu \big\|\sqrt{-\Delta_L}A^\sigma f(t')\big\|_2^2\,dt' + C\varepsilon\nu^{\frac{1}{3}} \int_1^t \nu \big\|\sqrt{-\Delta_L}A^\sigma f(t')\big\|_2^2\,dt' \\
&\qquad + C\varepsilon\nu^{\frac{1}{3}}\nu\|\partial_v h\|^2_{L^2_T(H^{\sigma-1})} + C\int_1^t (\varepsilon\nu^{\frac{1}{3}})^2 \nu \|\partial_v h(t')\|_{H^{\sigma-1}} \frac{\langle t\rangle^2}{\langle \nu t'^3\rangle}\,dt'.
\end{aligned}$$

Thus, by taking ε small enough and using Proposition 2.2 we get

$$\begin{aligned}
&\int_1^t \left(\nu \int A^\sigma f(t') A^\sigma (\widetilde{\Delta}_t f(t'))\,dz\,dv\right) dt' \\
&\quad \le -\frac{7}{8}\int_1^t \nu \big\|\sqrt{-\Delta_L} A^\sigma f(t')\big\|_2^2\,dt' + \varepsilon^2 \nu \|\partial_v h(t')\|_{L^2_T H^{\sigma-1}} \left(\int_1^t \frac{1}{\langle \nu^{\frac13} t'\rangle^2}\,dt'\right)^{\frac12} \\
&\quad \le -\frac{7}{8}\int_1^t \nu \big\|\sqrt{-\Delta_L} A^\sigma f(t')\big\|_2^2\,dt' + C\varepsilon^3 \nu^{\frac23}.
\end{aligned}$$

Thus we have proved Proposition 2.5. ∎

6. Transport

To treat the transport term, we need to consider the commutator. The following lemma gives the key commutator estimate.

Lemma 6.1. *Assume that* $|\xi-\eta| \le \frac{1}{10}|\eta|$*; then it holds that*

$$|w(t,\eta) - w(t,\xi)| \lesssim \frac{|\xi-\eta|}{\langle \eta\rangle} \times \begin{cases} \nu^{-\frac13}, & t \lesssim \nu^{-\frac13}, \\ \nu^{-\frac13\beta} t^{1-\beta}, & t \gtrsim \nu^{-\frac13}. \end{cases}$$

Let us admit the lemma and finish the estimate of transport term first. Then the proof of the lemma will be presented at the end of this section.

We write

$$\begin{aligned}
T_N &= i\sum_{k,l}\int_{\eta,\xi} A_k^\sigma(\eta)\bar{\hat{f}}_k(\eta)\hat{u}_{k-l}(\eta-\xi)_{<N/8}\cdot(l,\xi)A_l^\sigma(\xi)\hat{f}_l(\xi)_N\left(\frac{A_k^\sigma(\eta)}{A_l^\sigma(\xi)}-1\right)d\xi\,d\eta \\
&= i\sum_{k,l}\int_{\eta,\xi} A_k^\sigma(\eta)\bar{\hat{f}}_k(\eta)\hat{u}_{k-l}(\eta-\xi)_{<N/8}\cdot(l,\xi)A_l^\sigma(\xi)\hat{f}_l(\xi)_N \\
&\qquad\qquad \times\left(\frac{\langle k,\eta\rangle^\sigma}{\langle l,\xi\rangle^\sigma}-1\right)\frac{w_l(t,\xi)}{w_k(t,\eta)}\,d\xi\,d\eta \\
&\quad + i\sum_{k,l}\int_{\eta,\xi} A_k^\sigma(\eta)\bar{\hat{f}}_k(\eta)\hat{u}_{k-l}(\eta-\xi)_{<N/8}\cdot(l,\xi)A_l^\sigma(\xi)\hat{f}_l(\xi)_N \\
&\qquad\qquad \times\left(\frac{w_l(t,\xi)}{w_k(t,\eta)}-1\right)d\xi\,d\eta \\
&= T_N^1 + T_N^2.
\end{aligned}$$

For the first term we get

$$\left|\frac{\langle k,\eta\rangle^\sigma}{\langle l,\xi\rangle^\sigma}-1\right| \lesssim \frac{\langle k-l,\eta-\xi\rangle}{\langle l,\xi\rangle},$$

which gives

$$|T_N^1| \lesssim \|A^\sigma f_{\sim N}\|_2 \|A^\sigma f_N\|_2 \|u\|_{H^4}.$$

Next we will deal with T_N^2. By the support of the integrand we get

$$\frac{N}{16} \le |k-l, \xi-\eta| \le \frac{3N}{16}, \quad \frac{N}{2} \le |l, \xi| \le \frac{3N}{2}.$$

We then set more restrictions on the support of the integrand to make k, η and l, ξ closer. We get

$$\begin{aligned} T_N^2 &= i \sum_{k,l} \int_{\eta,\xi} (\chi^D + (1-\chi^D)) A_k(\eta) \bar{\hat{f}}_k(\eta) \hat{u}_{k-l}(\eta-\xi)_{<N/8} \cdot (l,\xi) A_l(\xi) \hat{f}_l(\xi)_N \\ &\qquad \times \Big(\frac{w_l(t,\xi)}{w_k(t,\eta)} - 1\Big)\, d\xi\, d\eta \\ &= T_{N,D}^2 + T_{N,*}^2, \end{aligned}$$

where χ^D is a characteristic function (the indicator function) of the set

$$D = \big\{(k,l,\xi,\eta) : |k-l, \xi-\eta| \le \tfrac{1}{1000}|l,\xi|\big\}.$$

Then, by $|\frac{w(t,\xi)}{w(t,\eta)}| \lesssim 1$, we get

$$|T_{N,*}^2| \lesssim \|A^\sigma f_{\sim N}\|_2 \|A^\sigma f_N\|_2 \|u\|_{H^4}.$$

We rewrite $T_{N,D}^2$ as

$$\begin{aligned} T_{N,D}^2 &= i \sum_{k\ne l} \int_{\eta,\xi} \chi^D A_k(\eta) \bar{\hat{f}}_k(\eta) \hat{u}_{k-l}(\eta-\xi)_{<N/8} \cdot (l,\xi) A_l(\xi) \hat{f}_l(\xi)_N \\ &\qquad \times \Big(\frac{w_l(t,\xi)}{w_k(t,\eta)} - 1\Big)\, d\xi\, d\eta \\ &\quad + i \sum_{l} \int_{\eta,\xi} \chi^D A_k(\eta) \bar{\hat{f}}_l(\eta) \hat{u}_0(\eta-\xi)_{<N/8} \cdot (l,\xi) A_l(\xi) \hat{f}_l(\xi)_N \\ &\qquad \times \Big(\frac{w_l(t,\xi)}{w_l(t,\eta)} - 1\Big)\, d\xi\, d\eta \\ &= T_{N,\ne}^2 + T_{N,=}^2. \end{aligned}$$

6.1. Treatment of $T_{N,=}^2$

By the fact that $u_0 = (0, g)$ and since for $|l| \ge 20\max\{|\xi|, |\eta|\}$, we have $w_l(t,\xi) = w_l(t,\eta) = w(t, \frac{l}{20})$, we get

$$\begin{aligned} T_{N,=}^2 = i \sum_{0\ne l\le 20\max\{|\xi|,|\eta|\}} \int_{\eta,\xi} \chi^D A_k(\eta) \bar{\hat{f}}_l(\eta) \hat{g}(\eta-\xi)_{<N/8} \cdot \xi A_l(\xi) \hat{f}_l(\xi)_N \\ \times \Big(\frac{w_l(t,\xi)}{w_l(t,\eta)} - 1\Big)\, d\xi\, d\eta. \end{aligned}$$

Due to the fact that $0 \neq l \leq 20\max\{|\xi|,|\eta|\} \approx |\xi|$, we get $\varrho(l,\eta) \approx |\eta|$. Thus, by Lemmas 6.1 and 3.1 we obtain

$$\begin{aligned}\left|\frac{w_l(t,\xi)}{w_l(t,\eta)}-1\right| &\lesssim |w(t,\varrho(l,\xi))-w(t,\varrho(l,\eta))|\\ &\lesssim \big(\nu^{-\frac13}\chi_{t\lesssim\nu^{-\frac13}}(t)+\nu^{-\frac13\beta}t^{1-\beta}\chi_{t\gtrsim\nu^{-\frac13}}(t)\big)\frac{|\varrho(l,\xi)-\varrho(l,\eta)|}{|\varrho(l,\eta)|}\\ &\lesssim \big(\nu^{-\frac13}\chi_{t\lesssim\nu^{-\frac13}}(t)+\nu^{-\frac13\beta}t^{1-\beta}\chi_{t\gtrsim\nu^{-\frac13}}(t)\big)\frac{|\eta-\xi|}{\eta}.\end{aligned}$$

Therefore we get

$$\begin{aligned}|T^2_{N,=}| \lesssim{}& \|A^\sigma f_{\sim N}\|_2\|A^\sigma f_N\|_2\|g\|_{H^4}\nu^{-\frac13}\chi_{t\lesssim\nu^{-\frac13}}(t)\\ &+\|A^\sigma f_{\sim N}\|_2\|A^\sigma f_N\|_2\|g\|_{H^4}\nu^{-\frac13\beta}t^{1-\beta}\chi_{t\gtrsim\nu^{-\frac13}}(t).\end{aligned}$$

6.2. Treatment of $T^2_{N,\neq}$

By the definition of $\varrho(k,\eta)$, we have for $(l,k,\xi,\eta)\in D$ that

$$|\varrho(k,\eta)| \approx |\varrho(l,\xi)| \approx |l,\xi|.$$

We get by Lemmas 6.1 and 3.1 that

$$\begin{aligned}\left|\frac{w_l(t,\xi)}{w_k(t,\eta)}-1\right| &\lesssim |w(t,\varrho(l,\xi))-w(t,\varrho(k,\eta))|\\ &\lesssim \frac{|\varrho(l,\xi)-\varrho(k,\eta)|}{|\varrho(k,\eta)|}\big(\nu^{-\frac13}\chi_{t\lesssim\nu^{-\frac13}}(t)+\nu^{-\frac13\beta}t^{1-\beta}\chi_{t\gtrsim\nu^{-\frac13}}(t)\big)\\ &\lesssim \frac{|l-k,\xi-\eta|}{|l,\xi|}\big(\nu^{-\frac13}\chi_{t\lesssim\nu^{-\frac13}}(t)+\nu^{-\frac13\beta}t^{1-\beta}\chi_{t\gtrsim\nu^{-\frac13}}(t)\big),\end{aligned}$$

which implies that

$$|T^2_{N,\neq}| \lesssim \|A^\sigma f_{\sim N}\|_2\|A^\sigma f_N\|_2\|u_{\neq}\|_{H^4}\big(\nu^{-\frac13}\chi_{t\lesssim\nu^{-\frac13}}(t)+\nu^{-\frac13\beta}t^{1-\beta}\chi_{t\gtrsim\nu^{-\frac13}}(t)\big).$$

By the fact that

$$u_{\neq} = h\nabla^{\perp}_{z,v}P_{\neq}\phi+\nabla^{\perp}_{z,v}P_{\neq}\phi,$$

we then get by Lemma 4.1 (by taking $s'=2$ in the lemma) that

$$\|u_{\neq}\|_{H^4} \lesssim (1+\|h\|_{H^4})\|\nabla^{\perp}_{z,v}P_{\neq}\phi\|_{H^4} \lesssim \frac{1}{\langle t\rangle^2}(1+\|h\|_{H^4})\|f_{\neq}\|_{H^7}.$$

Therefore we get

$$\begin{aligned}|T_N| \lesssim{}& \|A^\sigma f_{\sim N}\|_2\|A^\sigma f_N\|_2(\|g\|_{H^4}+\|u_{\neq}\|_{H^4})\\ &\times\big(\nu^{-\frac13}\chi_{t\lesssim\nu^{-\frac13}}(t)+\nu^{-\frac13\beta}t^{1-\beta}\chi_{t\gtrsim\nu^{-\frac13}}(t)\big)\\ \lesssim{}& \varepsilon\|A^\sigma f_{\sim N}\|_2\|A^\sigma f_N\|_2\left(\frac{\chi_{t\lesssim\nu^{-\frac13}}(t)}{\langle t\rangle^2}+\frac{\nu^{-\frac13\beta+\frac13}\chi_{t\gtrsim\nu^{-\frac13}}(t)}{\langle t\rangle^{1+\beta}}\right).\end{aligned}\tag{6.1}$$

Now we are able to prove Proposition 2.6.

Proof of Proposition 2.6. We get by (6.1) and Proposition 2.2, (A.1) and (A.2) that

$$\int_1^t \sum_{N\geq 8} |T_N(t')|\,dt' \lesssim \varepsilon \int_1^t \sum_{N\geq 8} \|A^\sigma f_{\sim N}\|_2 \|A^\sigma f_N\|_2 \times \left(\frac{\chi_{t'\lesssim \nu^{-\frac13}}(t')}{\langle t'\rangle^2} + \frac{\nu^{-\frac13\beta+\frac13}\chi_{t'\gtrsim \nu^{-\frac13}}(t')}{\langle t'\rangle^{1+\beta}} \right) dt' \lesssim \varepsilon \sup_{t'\in[1,t]} \|A^\sigma f(t')\|_2^2.$$

Thus we have proved Proposition 2.6. ■

6.3. Proof of Lemma 6.1

We end this section by proving Lemma 6.1.

Proof of Lemma 6.1. Without loss of the generality, we assume $0<\eta<\xi$. Then according to the relation between t and ξ,η, we need to consider following five cases:

Case 1. For $t\leq t(\eta)$, we have $w(t,\eta)=w(t,\xi)=1$.

Case 2. For $t(\eta)\leq t\leq t(\xi)$, there exists $l\in[1,E(\sqrt{|\eta|})]$ such that $t\in I_{l,\eta}$; then $|l-E(\sqrt{|\eta|})|\lesssim \sqrt{\xi}-\sqrt{\eta}$ and

$$\begin{aligned}
&|w(t,\eta)-w(t,\xi)| = |w(t,\eta)-1| \\
&\leq \left| \prod_{m=E(\sqrt{|\eta|})}^{l+1} G_m(\eta)\exp\Big(\langle \nu^{\frac13} t_{l,\eta}\rangle^{-(1+\beta)} \frac{\nu^{\frac13}\eta}{l^2} \Big(\arctan\Big(t-\frac{\eta}{l}\Big)+\arctan(D^-_{l,\eta})\Big)\Big) - 1 \right| \\
&\leq \begin{cases} \exp\Big(\displaystyle\sum_{m=E(\sqrt{|\eta|})}^{l} \frac{C\nu^{\frac13}\eta}{m^2}\Big) - 1 \leq \nu^{\frac13}|\sqrt{\xi}-\sqrt{\eta}|, & \sqrt{\eta}\leq \nu^{-\frac13}, \\ \exp\Big(\displaystyle\sum_{m=E(\sqrt{|\eta|})}^{l} \frac{Cl^{-1+\beta}}{(\nu^{\frac13}\eta)^\beta}\Big) - 1 \leq \frac{|\sqrt{\xi}-\sqrt{\eta}|}{\sqrt{|\xi|}} \Big(\frac{\sqrt{|\xi|}}{\nu^{\frac13}\eta}\Big)^\beta, & \sqrt{\eta}>\nu^{-\frac13}, \end{cases} \\
&\lesssim |\eta-\xi|\langle\xi\rangle^{-1}.
\end{aligned}$$

Here we use the fact that $|e^x-1|\lesssim|x|$ for $|x|\lesssim 1$.

Case 3. For $t(\xi)\leq t\leq 2\eta$, there exist k,l such that $t\in I_{k,\eta}\cap I_{l,\xi}$. By Lemma 3.2, we need to consider the following three cases.

(3a) $k=l$. Let

$$F_1(m,\eta) = \nu^{\frac13}\langle \nu^{\frac13} t_{m,\eta}\rangle^{-(1+\beta)} \frac{\eta}{m^2},$$
$$F_2^\pm(m,\eta) = \nu^{\frac13}\langle \nu^{\frac13} t_{m,\eta}\rangle^{-(1+\beta)} \times \frac{\eta}{m^2}\arctan(D^\pm_{m,\eta});$$

then $G_m(\eta) = e^{F^+(m,\eta)+F^-(m,\eta)}$ and we get

$$\begin{aligned}
&|\partial_\eta F_1(m,\eta)| \lesssim \frac{F_1(m,\eta)}{\langle\eta\rangle}, \quad |\partial_\eta F_2^\pm(m,\eta)| \lesssim \frac{F_2^\pm(m,\eta)}{\langle\eta\rangle},\\
&\Big|\arctan\Big(t-\frac{\eta}{l}\Big) - \arctan\Big(t-\frac{\xi}{l}\Big)\Big| \lesssim \min\Big\{\frac{|\xi-\eta|}{l}, 1\Big\},\\
&e^x - 1 \le (e^x+1)|x| \lesssim |x|, \quad \text{for } |x| \lesssim 1.
\end{aligned} \tag{6.2}$$

Therefore, we obtain

$$\begin{aligned}
|w(t,\eta) - w(t,\xi)| &= w(t,\xi)\Big|\frac{w(t,\eta)}{w(t,\xi)} - 1\Big|\\
&\lesssim \Big|\prod_{m=E(\sqrt{|\xi|})}^{m=E(\sqrt{|\eta|})+1} \frac{1}{G_m(\xi)} \prod_{m=E(\sqrt{|\eta|})}^{l+1} \frac{G_m(\eta)}{G_m(\xi)}\\
&\qquad \times \frac{\exp\big(\nu^{\frac13}\langle\nu^{\frac13}t_{l,\eta}\rangle^{-(1+\beta)}\frac{\eta}{l^2}(\arctan(t-\frac{\eta}{l}) + \arctan(D^-_{l,\eta}))\big)}{\exp\big(\nu^{\frac13}\langle\nu^{\frac13}t_{l,\xi}\rangle^{-(1+\beta)}\frac{\xi}{l^2}(\arctan(t-\frac{\xi}{l}) + \arctan(D^-_{l,\xi}))\big)} - 1\Big|\\
&\lesssim \Big|\prod_{m=E(\sqrt{|\xi|})}^{m=E(\sqrt{|\eta|})+1} \frac{1}{G_m(\xi)} - 1\Big| + \Big|\prod_{m=E(\sqrt{|\eta|})}^{l+1} \frac{G_m(\eta)}{G_m(\xi)} - 1\Big|\\
&\quad + \Big|\frac{\exp\big(\nu^{\frac13}\langle\nu^{\frac13}t_{l,\eta}\rangle^{-(1+\beta)}\frac{\eta}{l^2}(\arctan(t-\frac{\eta}{l}) + \arctan(D^-_{l,\eta}))\big)}{\exp\big(\nu^{\frac13}\langle\nu^{\frac13}t_{l,\xi}\rangle^{-(1+\beta)}\frac{\xi}{l^2}(\arctan(t-\frac{\xi}{l}) + \arctan(D^-_{l,\xi}))\big)} - 1\Big|\\
&\lesssim \Big|\prod_{m=E(\sqrt{|\xi|})}^{m=E(\sqrt{|\eta|})+1} \frac{1}{G_m(\xi)} - 1\Big| + \Big|\prod_{m=E(\sqrt{|\eta|})}^{l+1} \frac{G_m(\eta)}{G_m(\xi)} - 1\Big|\\
&\quad + \Big|\frac{\exp\big(\nu^{\frac13}\langle\nu^{\frac13}t_{l,\eta}\rangle^{-(1+\beta)}\frac{\eta}{l^2}\arctan(D^-_{l,\eta})\big)}{\exp\big(\nu^{\frac13}\langle\nu^{\frac13}t_{l,\xi}\rangle^{-(1+\beta)}\frac{\xi}{l^2}\arctan(D^-_{l,\xi})\big)} - 1\Big| + \Big|\frac{\exp\big(\nu^{\frac13}\langle\nu^{\frac13}t_{l,\eta}\rangle^{-(1+\beta)}\frac{\eta}{l^2}\big)}{\exp\big(\nu^{\frac13}\langle\nu^{\frac13}t_{l,\xi}\rangle^{-(1+\beta)}\frac{\xi}{l^2}\big)} - 1\Big|\\
&\quad + \Big|\exp\Big(\nu^{\frac13}\langle\nu^{\frac13}t_{l,\eta}\rangle^{-(1+\beta)}\frac{\eta}{l^2}\Big(\arctan\Big(t-\frac{\eta}{l}\Big) - \arctan\Big(t-\frac{\xi}{l}\Big)\Big)\Big) - 1\Big|.
\end{aligned}$$

Here and in the rest of the proof we will always use the fact that for $x, y \lesssim 1$, we have $|xy-1| \lesssim |x|\,|y-1| + |x-1| \lesssim |x-1| + |y-1|$.

Then the lemma follows from the inequalities below, which follow from (6.2) and the fact that $|f(x) - f(y)| \lesssim \|f'(z)\|_{L^\infty}|x-y|$:

$$\Big|\prod_{m=E(\sqrt{|\xi|})}^{m=E(\sqrt{|\eta|})+1} \frac{1}{G_m(\xi)} - 1\Big| \lesssim |\sqrt{\xi} - \sqrt{\eta}|\frac{\nu^{\frac13}}{\langle\nu^{\frac13}\eta^{\frac12}\rangle^{1+\beta}} \lesssim |\eta-\xi|\langle\xi\rangle^{-1}, \tag{6.3}$$

$$\Big|\prod_{m=E(\sqrt{|\eta|})}^{l+1} \frac{G_m(\eta)}{G_m(\xi)} - 1\Big| \lesssim \exp\Big(C\frac{|\xi-\eta|}{\langle\eta\rangle}\sum_{m=E(\sqrt{|\eta|})}^{l+1}(F_2^+ + F_2^-)(m,\eta)\Big) - 1 \lesssim \frac{|\xi-\eta|}{\langle\eta\rangle}, \tag{6.4}$$

and

$$\left|\frac{\exp\big(\nu^{\frac{1}{3}}\langle\nu^{\frac{1}{3}}t_{l,\eta}\rangle^{-(1+\beta)}\frac{\eta}{l^2}\arctan(D^-_{l,\eta})\big)}{\exp\big(\nu^{\frac{1}{3}}\langle\nu^{\frac{1}{3}}t_{l,\xi}\rangle^{-(1+\beta)}\frac{\xi}{l^2}\arctan(D^-_{l,\xi})\big)}-1\right| \lesssim \exp\Big(C\frac{|\xi-\eta|}{\langle\eta\rangle}\Big)-1 \lesssim \frac{|\xi-\eta|}{\langle\eta\rangle},$$

$$\left|\frac{\exp\big(\nu^{\frac{1}{3}}\langle\nu^{\frac{1}{3}}t_{l,\eta}\rangle^{-(1+\beta)}\frac{\eta}{l^2}\big)}{\exp\big(\nu^{\frac{1}{3}}\langle\nu^{\frac{1}{3}}t_{l,\xi}\rangle^{-(1+\beta)}\frac{\xi}{l^2}\big)}-1\right| \lesssim \exp\Big(C\frac{|\xi-\eta|}{\langle\eta\rangle}\Big)-1 \lesssim \frac{|\xi-\eta|}{\langle\eta\rangle},$$

$$\begin{aligned}
&\left|\exp\bigg(\nu^{\frac{1}{3}}\langle\nu^{\frac{1}{3}}t_{l,\eta}\rangle^{-(1+\beta)}\frac{\eta}{l^2}\Big(\arctan\Big(t-\frac{\eta}{l}\Big)-\arctan\Big(t-\frac{\xi}{l}\Big)\Big)\bigg)-1\right| \\
&\qquad\lesssim \nu^{\frac{1}{3}}\langle\nu^{\frac{1}{3}}t_{l,\eta}\rangle^{-(1+\beta)}\frac{\eta}{l^2}\Big(\arctan\Big(t-\frac{\eta}{l}\Big)-\arctan\Big(t-\frac{\xi}{l}\Big)\Big) \\
&\qquad\lesssim \nu^{\frac{1}{3}}\langle\nu^{\frac{1}{3}}t_{l,\eta}\rangle^{-(1+\beta)}\frac{\eta}{l^2}\min\Big\{\frac{|\xi-\eta|}{l},1\Big\} \lesssim \begin{cases}\nu^{-\frac{1}{3}}\dfrac{|\xi-\eta|}{\eta}, & t\approx\dfrac{\eta}{l}\lesssim\nu^{-\frac{1}{3}},\\[2mm] \nu^{-\frac{1}{3}\beta}t^{1-\beta}\dfrac{|\xi-\eta|}{\eta}, & t\approx\dfrac{\eta}{l}\gtrsim\nu^{-\frac{1}{3}}.\end{cases}
\end{aligned}$$

(3b) $k\neq l$, $|t-\frac{\eta}{k}|\gtrsim\frac{\eta}{k^2}$ and $|t-\frac{\xi}{l}|\gtrsim\frac{\xi}{l^2}$ with $k<l$. We have

$$\begin{aligned}
|w(t,\eta)-w(t,\xi)| &= w(t,\xi)\left|\frac{w(t,\eta)}{w(t,\xi)}-1\right| \\
&\lesssim \left|\prod_{m=E(\sqrt{|\xi|})}^{m=E(\sqrt{|\eta|})+1}\frac{1}{G_m(\xi)}-1\right| + \left|\prod_{m=E(\sqrt{|\eta|})}^{\max\{l,k\}+1}\frac{G_m(\eta)}{G_m(\xi)}-1\right| + \left|\prod_{m=\max\{l,k\}}^{\min\{k,l\}+1}G_m(\eta)-1\right| \\
&\quad+\left|\frac{\exp\big(\nu^{\frac{1}{3}}\langle\nu^{\frac{1}{3}}t_{k,\eta}\rangle^{-(1+\beta)}\frac{\eta}{k^2}(\arctan(t-\frac{\eta}{k})+\arctan(D^-_{k,\eta}))\big)}{\exp\big(\nu^{\frac{1}{3}}\langle\nu^{\frac{1}{3}}t_{l,\xi}\rangle^{-(1+\beta)}\frac{\xi}{l^2}(\arctan(t-\frac{\xi}{l})+\arctan(D^-_{l,\xi}))\big)}-1\right|.
\end{aligned}\tag{6.5}$$

Let $F_3(t,k,\eta)=\nu^{\frac{1}{3}}\langle\nu^{\frac{1}{3}}t_{k,\eta}\rangle^{-(1+\beta)}\frac{\eta}{k^2}(\arctan(t-\frac{\eta}{k})+\arctan(D^-_{k,\eta}))$; we get

$$|F_3|\lesssim k^{-1},\quad |\partial_k F_3|\lesssim\frac{1}{\langle k\rangle},\quad |\partial_\eta F_3|\lesssim\frac{1}{\langle\eta\rangle},$$

where we used the fact that $\eta\gtrsim k^2$. Then the lemma follows from (6.3), (6.4) and the following two inequalities:

$$\begin{aligned}
&\left|\frac{\exp\big(\nu^{\frac{1}{3}}\langle\nu^{\frac{1}{3}}t_{k,\eta}\rangle^{-(1+\beta)}\frac{\eta}{k^2}(\arctan(t-\frac{\eta}{k})+\arctan(D^-_{k,\eta}))\big)}{\exp\big(\nu^{\frac{1}{3}}\langle\nu^{\frac{1}{3}}t_{l,\xi}\rangle^{-(1+\beta)}\frac{\xi}{l^2}(\arctan(t-\frac{\xi}{l})+\arctan(D^-_{l,\xi}))\big)}-1\right| \\
&\qquad\lesssim e^{F_3(t,k,\eta)-F_3(t,l,\xi)}-1\lesssim|F_3(t,k,\eta)-F_3(t,l,\xi)| \\
&\qquad\lesssim\frac{1}{\langle k\rangle}|k-l|+\frac{1}{\langle\eta\rangle}|\xi-\eta|\lesssim\frac{|\xi-\eta|}{\langle\eta\rangle}
\end{aligned}$$

and

$$\left|\prod_{m=\max\{k,l\}}^{\min\{k,l\}+1}G_m(\eta)-1\right|\lesssim|l-k|\frac{\frac{\nu^{\frac{1}{3}}\eta}{l^2}}{\langle\frac{\nu^{\frac{1}{3}}\eta}{l}\rangle^{1+\beta}}\lesssim\frac{|k-l|}{l}\lesssim\frac{|\xi-\eta|}{\langle\eta\rangle},$$

where we used the fact that $|k-l|\lesssim\langle\frac{\xi-\eta}{t}\rangle$, $l\approx\frac{\eta}{t}$.

(3c) $|\xi-\eta| \gtrsim \frac{\xi}{l} \approx \frac{\eta}{k}$. In this case, similarly we have

$$|w(t,\xi)-w(t,\eta)| \lesssim \frac{|\xi-\eta|}{\langle\eta\rangle}.$$

Case 4. For $2\eta \le t \le 2\xi$, then $t \in I_{1,\xi}$ and

$$\begin{aligned}
|w(t,\eta)-w(t,\xi)| &= w(t,\xi)\left|\frac{w(2\eta,\eta)}{w(t_{l,\xi},\xi)g_l(t-\frac{\xi}{l},\xi)}-1\right| \\
&\lesssim \left|\frac{w(2\eta,\eta)}{w(t_{l,\xi},\xi)g_l(t-\frac{\xi}{l},\xi)}-1\right| \\
&\lesssim \left|\prod_{m=E(\sqrt{|\xi|})}^{m=E(\sqrt{|\eta|})+1}\frac{1}{G_m(\xi)}-1\right| + \left|\prod_{m=E(\sqrt{|\eta|})}^{2}\frac{G_m(\eta)}{G_m(\xi)}-1\right| \\
&\quad + \left|\frac{\exp\big(\nu^{\frac13}\langle\nu^{\frac13}t_{1,\eta}\rangle^{-(1+\beta)}\eta(\arctan(\eta)+\arctan(\frac13\eta))\big)}{\exp\big(\nu^{\frac13}\langle\nu^{\frac13}t_{1,\xi}\rangle^{-(1+\beta)}\xi(\arctan(t-\xi)+\arctan(\frac13\xi))\big)}-1\right|.
\end{aligned}$$

Thus, the lemma follows from (6.3), (6.4) and the following inequalities:

$$\begin{aligned}
&\left|\exp\big(\nu^{\frac13}\langle\nu^{\frac13}t_{1,\xi}\rangle^{-(1+\beta)}\xi(\arctan(t-\xi)+\arctan(D^-_{1,\xi}))\big)-1\right| \\
&\lesssim \big|\nu^{\frac13}\langle\nu^{\frac13}t_{1,\xi}\rangle^{-(1+\beta)}\xi(\arctan(t-\xi)+\arctan(\tfrac13\xi)) \\
&\quad -\big(\nu^{\frac13}\langle\nu^{\frac13}t_{1,\eta}\rangle^{-(1+\beta)}\eta(\arctan(\eta)+\arctan(\tfrac13\eta))\big)\big| \\
&\lesssim |\arctan(t-\xi)-\arctan(\eta)| + |\xi-\eta|\langle\xi\rangle^{-1} \\
&\lesssim \max\big\{\arctan(\xi)-\arctan(\eta), \arctan(\eta)-\arctan(2\eta-\xi)\big\} + |\xi-\eta|\langle\xi\rangle^{-1} \\
&\lesssim |\xi-\eta|\langle\xi\rangle^{-1}.
\end{aligned}$$

Case 5. For $t \ge 2\xi$. We get by (6.3) and (6.4) that

$$\begin{aligned}
|w(2\eta,\eta)-w(2\xi,\xi)| &= w(2\xi,\xi)\left|\frac{w(2\eta,\eta)}{w(2\xi,\xi)}-1\right| \\
&\lesssim \left|\prod_{m=E(\sqrt{|\xi|})}^{m=E(\sqrt{|\eta|})+1}\frac{1}{G_m(\xi)}-1\right| + \left|\prod_{m=E(\sqrt{|\eta|})}^{1}\frac{G_m(\eta)}{G_m(\xi)}-1\right| \\
&\lesssim |\xi-\eta|\langle\xi\rangle^{-1}.
\end{aligned}$$

Thus we have proved Lemma 6.1. ∎

7. Remainder

In this section we deal with the remainder and prove Proposition 2.7. Now, the commutator cannot gain us anything so we may as well treat each term separately. We rewrite both terms on the Fourier side:

$$\mathcal{R} = \sum_{N\in\mathbb{D}}\sum_{N'\approx N}\sum_{k,l}\int_{\eta,\xi} A^\sigma \bar{\hat{f}}_k(\eta) A^\sigma_k(\eta)\hat{u}_l(\xi)_N \widehat{\nabla f}_{k-l}(\eta-\xi)_{N'}\,d\eta\,d\xi$$
$$+\sum_{N\in\mathbb{D}}\sum_{N'\approx N}\sum_{k,l}\int_{\eta,\xi} A^\sigma \bar{\hat{f}}_k(\eta)\hat{u}_l(\xi)_N A^\sigma_{k-l}(\eta-\xi)\widehat{\nabla f}_{k-l}(\eta-\xi)_{N'}\,d\eta\,d\xi.$$

On the support of the integrand, $|l,\xi| \approx |k-l,\eta-\xi|$; thus

$$A^\sigma_k(\eta) \approx \langle k,\eta\rangle^\sigma \lesssim \langle l,\xi\rangle^\sigma + \langle k-l,\eta-\xi\rangle^\sigma \approx \langle l,\xi\rangle\langle k-l,\eta-\xi\rangle^{\sigma-1} \approx A^\sigma_{k-l}(\eta-\xi),$$

which implies that

$$|\mathcal{R}| \lesssim \sum_{N\in\mathbb{D}} \|A^\sigma f\|_2 \|u_N\|_{H^3}\|f_{\sim N}\|_{H^\sigma}.$$

Therefore we get by (4.4),

$$|\mathcal{R}| \lesssim \|f\|^2_{H^\sigma}\|u\|_{H^3} \lesssim \frac{\varepsilon\nu^{\frac{1}{3}}}{\langle t^2\rangle}\|A^\sigma f\|_2^2, \tag{7.1}$$

which gives Proposition 2.7.

8. Reaction

In this section we deal with the reaction term and prove Proposition 2.8. We focus first on an individual frequency shell and divide each into several natural pieces:

$$R_N = R^1_N + R^{\varepsilon,1}_N + R^2_N + R^3_N,$$

where

$$R^1_N = \sum_{k,l\neq 0}\int_{\eta,\xi} A^\sigma \bar{\hat{f}}_k(\eta)A^\sigma_k(\eta)(\eta l - \xi k)\hat{\phi}_l(\xi)_N \hat{f}_{k-l}(\eta-\xi)_{<N/8}\,d\eta\,d\xi,$$
$$R^{\varepsilon,1}_N = -\sum_{k,l\neq 0}\int_{\eta,\xi} A^\sigma \bar{\hat{f}}_k(\eta)A^\sigma_k(\eta)\big[\widehat{(1-v')\nabla^\perp\phi_l}\big](\xi)_N \cdot \widehat{\nabla f}_{k-l}(\eta-\xi)_{<N/8}\,d\eta\,d\xi,$$
$$R^2_N = \sum_k \int_{\eta,\xi} A^\sigma \bar{\hat{f}}_k(\eta)A^\sigma_k(\eta)\hat{g}(\xi)_N \widehat{\partial_v f}_k(\eta-\xi)_{<N/8}\,d\eta\,d\xi,$$
$$R^3_N = -\sum_{k,l}\int_{\eta,\xi} A^\sigma \bar{\hat{f}}_k(\eta)A^\sigma_{k-l}(\eta-\xi)\hat{u}_l(\xi)_N \widehat{\nabla f}_{k-l}(\eta-\xi)_{<N/8}\,d\eta\,d\xi.$$

8.1. Main contribution

The main contribution comes from R^1_N. We subdivide this integral depending on whether (l,ξ) and/or (k,η) are resonant, as each combination requires slightly different treatment.

Define the partition

$$\begin{aligned}1 &= 1_{t\notin I_{k,\eta},t\notin I_{l,\xi}} + 1_{t\notin I_{k,\eta},t\in I_{l,\xi}} + 1_{t\in I_{k,\eta},t\notin I_{l,\xi}} + 1_{t\in I_{k,\eta},t\in I_{l,\xi}}\\ &= \chi^{\mathrm{NR,NR}} + \chi^{\mathrm{NR,R}} + \chi^{\mathrm{R,NR}} + \chi^{\mathrm{R,R}},\end{aligned}$$

where the NR and R denote "nonresonant" and "resonant" respectively, referring to (k,η) and (l,ξ). Correspondingly, denote

$$\begin{aligned}R_N^1 =& \underbrace{\sum_{l\neq 0}\int_{\eta,\xi} \chi^D A^\sigma \bar{\hat{f}}_l(\eta) A_l^\sigma(\eta)(\eta l - \xi l)\hat{\phi}_l(\xi)_N \hat{f}_0(\eta-\xi)_{<N/8}\, d\eta\, d\xi}_{R_{N,D}}\\ &+ \underbrace{\sum_{l\neq 0}\int_{\eta,\xi}[\chi^{\mathrm{NR,NR}} + \chi^{\mathrm{NR,R}} + \chi^{\mathrm{R,NR}} + \chi^{\mathrm{R,R}}](1-\chi^D) \times A^\sigma \bar{\hat{f}}_l(\eta) A_l^\sigma(\eta)(\eta l - \xi l)\hat{\phi}_l(\xi)_N \hat{f}_0(\eta-\xi)_{<N/8}\, d\eta\, d\xi}_{R_{N,=}^{\mathrm{NR,NR}} + R_{N,=}^{\mathrm{NR,R}} + R_{N,=}^{\mathrm{R,NR}} + R_{N,=}^{\mathrm{R,R}}}\\ &+ \underbrace{\sum_{k,l\neq 0, k\neq l}\int_{\eta,\xi}(1-\chi^{D_1}) A^\sigma \bar{\hat{f}}_k(\eta) A_k^\sigma(\eta)(\eta l - \xi k)\hat{\phi}_l(\xi)_N \hat{f}_{k-l}(\eta-\xi)_{<N/8}\, d\eta\, d\xi}_{R_{N,\neq,*}}\\ &+ \underbrace{\sum_{k,l\neq 0, k\neq l}\int_{\eta,\xi}[\chi^{\mathrm{NR,NR}} + \chi^{\mathrm{NR,R}} + \chi^{\mathrm{R,NR}} + \chi^{\mathrm{R,R}}]\chi^{D_1} \times A^\sigma \bar{\hat{f}}_k(\eta) A_k^\sigma(\eta)(\eta l - \xi k)\hat{\phi}_l(\xi)_N \hat{f}_{k-l}(\eta-\xi)_{<N/8}\, d\eta\, d\xi}_{R_N^{\mathrm{NR,NR}} + R_N^{\mathrm{NR,R}} + R_N^{\mathrm{R,NR}} + R_N^{\mathrm{R,R}}}\\ =& R_{N,D} + R_{N,=}^{\mathrm{NR,NR}} + R_{N,=}^{\mathrm{NR,R}} + R_{N,=}^{\mathrm{R,NR}} + R_{N,=}^{\mathrm{R,R}}\\ &+ R_{N,\neq,*} + R_N^{\mathrm{NR,NR}} + R_N^{\mathrm{NR,R}} + R_N^{\mathrm{R,NR}} + R_N^{\mathrm{R,R}},\end{aligned}$$

where χ^D is a characteristic function (the indicator function) of the set

$$D = \left\{(l,\xi) : |l| \geq \tfrac{5}{4}|\xi|\right\},$$

and χ^{D_1} is a characteristic function (the indicator function) of the set

$$D_1 = \left\{(l,k,\xi,\eta) : |l| \leq |\xi|,\ |l-k, \xi-\eta| \leq \tfrac{1}{1000}|l,\xi|\right\},$$

8.1.1. Treatment of $R_{N,D}$. For the case $|l| \geq \frac{5}{4}|\xi|$, we get for $t \geq 1$,

$$\frac{A_l^\sigma(\eta)|l|}{l^2 + |lt-\xi|^2} \lesssim \frac{A_l^\sigma(\xi)|l|}{l^2 + t^2 l^2} \lesssim \frac{|l|^{\sigma-1}}{1+t^2},$$

which implies that

$$\begin{aligned}|R_{N,D}| &\lesssim \frac{1}{\langle t^2\rangle}\|A^\sigma f_{\sim N}\|_{L^2}\|\langle\partial_z\rangle^{\sigma-1}\Delta_L \Delta_t^{-1} P_{\neq} f_N\|_{L^2}\|f_0\|_{H^3}\\ &\lesssim \frac{\varepsilon \nu^{\frac{1}{3}}}{\langle t^2\rangle}\|A^\sigma f\|_2^2. \end{aligned} \tag{8.1}$$

In the next four subsections we will use the fact that for $|l| \le \frac{5}{4}|\xi|$ and $|\xi - \eta| \le \frac{3}{8}|l,\xi| \le \frac{27}{40}|\xi|$, we have $|\eta| \ge \frac{13}{40}|\xi| \ge \frac{13}{50}|l|$, which gives that

$$w_l(t,\eta) = w(t,\eta), \quad w_l(t,\xi) = w(t,\xi). \tag{8.2}$$

8.1.2. Treatment of the zero mode $R_{N,=}^{\mathrm{NR,NR}}$. We have

$$R_{N,=}^{\mathrm{NR,NR}} \lesssim \sum_{l \neq 0} \int_{\eta,\xi} \chi^{\mathrm{NR,NR}}(1-\chi^D) \times \left| A^\sigma \bar{\hat{f}}_l(\eta) \frac{A_l^\sigma(\eta)|l|}{l^2 + |lt-\xi|^2} \widehat{\Delta_L \Delta_t^{-1} f}_l(\xi)_N \widehat{\partial_v f}_0(\eta-\xi)_{<N/8} \right| d\eta\, d\xi$$

According to the relation between t and ξ, we have the following three cases.

Case 1: $t \le \max\{t(\xi), t(\eta)\} \approx \sqrt{|\xi|} \approx \sqrt{N}$. Then in this case,

$$\frac{A_l^\sigma(\eta)|l|}{l^2+|lt-\xi|^2} \lesssim \begin{cases} \dfrac{A_l^\sigma(\xi)|l|}{l^2(1+\frac{|\xi|^2}{l^4})} \lesssim \dfrac{A_l^\sigma(\xi)}{\sqrt{\xi}} & \text{if } |l| \le E(\sqrt{\xi}), \\ \dfrac{A_l^\sigma(\xi)}{\sqrt{\xi}} & \text{if } |l| \ge E(\sqrt{\xi})+1, \end{cases} \tag{8.3}$$

which implies

$$\begin{aligned} |R_{N,=}^{\mathrm{NR,NR}}| &\lesssim \frac{1}{\langle t\rangle} \|A^\sigma f_{\sim N}\|_{L^2} \|A^\sigma \Delta_L \Delta_t^{-1} P_{\neq} f_N\|_{L^2} \|\partial_v f_0\|_{H^3} \\ &\lesssim \frac{\varepsilon \nu^{\frac{1}{3}}}{\langle \nu^{\frac{1}{2}} t^{\frac{1}{2}}\rangle \langle t\rangle} \|A^\sigma f_{\sim N}\|_{L^2} \|A^\sigma \Delta_L \Delta_t^{-1} P_{\neq} f_N\|_{L^2}. \end{aligned}$$

Case 2: $t \ge 2|\xi|$ or $t \ge 2|\eta|$. Then in this case,

$$\frac{|l|}{l^2+|lt-\xi|^2} \lesssim \frac{1}{1+t^2} \tag{8.4}$$

and

$$\begin{aligned} \frac{|l|}{l^2+|lt-\xi|^2} &\lesssim \frac{|l|}{l^2+|lt-\xi|^2} - \frac{|l|}{l^2+|lt-\eta|^2} + \frac{|l|}{l^2+|lt-\eta|^2} \\ &\lesssim \frac{|l|\,|\eta-\xi|(2|\xi-lt|+|\xi-\eta|)}{(l^2+|lt-\eta|^2)(l^2+|lt-\xi|^2)} + \frac{|l|}{l^2+|lt-\eta|^2} \\ &\lesssim \frac{|l|\langle \xi-\eta\rangle^2}{l^2+|lt-\eta|^2} \lesssim \frac{\langle \xi-\eta\rangle^2}{1+t^2}, \end{aligned} \tag{8.5}$$

which implies that

$$\begin{aligned} |R_{N,=}^{\mathrm{NR,NR}}| &\lesssim \frac{1}{\langle t^2\rangle} \|A^\sigma f_{\sim N}\|_{L^2} \|A^\sigma \Delta_L \Delta_t^{-1} P_{\neq} f_N\|_{L^2} \|f_0\|_{H^5} \\ &\lesssim \frac{\varepsilon \nu^{\frac{1}{3}}}{\langle t^2\rangle} \|A^\sigma f_{\sim N}\|_{L^2} \|A^\sigma \Delta_L \Delta_t^{-1} P_{\neq} f_N\|_{L^2}. \end{aligned}$$

Case 3: $t \in I_t(\xi) \cap I_t(\eta)$. In this case there exists k, l' so that $t \in I_{k,\eta} \cap I_{l',\xi}$. By Lemma 3.2, we get $k \approx l'$.

If $l \le \frac{1}{2}\min\{|l'|, |k|\}$, then

$$\begin{aligned}\frac{|l|}{l^2(1+|t-\frac{\xi}{l}|^2)} &\lesssim \sqrt{\frac{|l|}{l^2(1+|t-\frac{\eta}{l}|^2)}}\sqrt{\frac{|l|}{l^2(1+|t-\frac{\xi}{l}|^2)}}\langle \xi-\eta\rangle\\ &\lesssim \frac{\langle\xi-\eta\rangle}{l(1+\frac{\xi^2}{l^2})} \lesssim \frac{\langle\xi-\eta\rangle}{1+\frac{\xi^2}{l'^2}} \lesssim \frac{\langle\xi-\eta\rangle}{1+t^2},\end{aligned}$$

where here we use the fact that $t \approx \frac{\xi}{l'}$.

If $l > \frac{1}{2}\min\{|l'|, |k|\}$, then

$$\frac{1}{l^2} \lesssim \frac{1}{\min\{|l'|,|k|\}^2} \lesssim \frac{1}{|l'k|}$$

and by the fact that for $t \in I_{l',\xi} \cap I_{k,\eta}$ and $t \notin I_{l,\xi} \cup I_{l,\eta}$, it holds that $|t - \frac{\eta}{l}| \ge |t - \frac{\eta}{k}|$ and $|t - \frac{\xi}{l}| \ge |t - \frac{\xi}{l'}|$. Thus, by (8.2) we get

$$\begin{aligned}\frac{|l|\langle \nu^{\frac12}(\frac{\xi}{k})^{\frac12}\rangle^{-1}}{l^2(1+|t-\frac{\xi}{l}|^2)} &\lesssim \sqrt{\frac{|l|}{l^2(1+|t-\frac{\eta}{l}|^2)}}\sqrt{\frac{|l|}{l^2(1+|t-\frac{\xi}{l}|^2)}}\langle\xi-\eta\rangle\langle \nu^{\frac12}(\tfrac{\xi}{k})^{\frac12}\rangle^{-1}\\ &\lesssim \sqrt{\frac{1}{k(1+|t-\frac{\eta}{k}|^2)}}\sqrt{\frac{1}{l'(1+|t-\frac{\xi}{l'}|^2)}}\langle\xi-\eta\rangle\langle \nu^{\frac12}(\tfrac{\xi}{k})^{\frac12}\rangle^{-1}\\ &\lesssim \sqrt{\frac{\eta}{k^2(1+|t-\frac{\eta}{k}|^2)}}\sqrt{\frac{\xi}{l'^2(1+|t-\frac{\xi}{l'}|^2)}}\langle\xi-\eta\rangle\langle \nu^{\frac13}(\tfrac{\xi}{k})\rangle^{-(1+\beta)}\\ &\lesssim \sqrt{\frac{\partial_t w(t,\eta)}{w(t,\eta)}}\sqrt{\frac{\partial_t w(t,\xi)}{w(t,\xi)}}\langle\xi-\eta\rangle\nu^{-\frac13}.\end{aligned}$$

In the third inequality we use the fact that for $0 < \beta \le \frac{1}{2}$,

$$\langle \nu^{\frac12}(\tfrac{\xi}{k})^{\frac12}\rangle^{-1} \lesssim \frac{\xi}{k}\langle \nu^{\frac13}(\tfrac{\xi}{k})\rangle^{-(1+\beta)}. \tag{8.6}$$

Thus, by the fact that $t \approx \frac{\xi}{k}$ we get

$$\begin{aligned}|R_{N,=}^{\mathrm{NR,NR}}| &\lesssim \frac{1}{\langle t^2\rangle}\|A^\sigma f_{\sim N}\|_{L^2}\|A^\sigma \Delta_L\Delta_t^{-1} f_N\|_{L^2}\|f_0\|_{H^4}\\ &\quad + \left\|\sqrt{\frac{\partial_t w}{w}}A^\sigma f_{\sim N}\right\|_{L^2}\left\|\sqrt{\frac{\partial_t w}{w}}\chi_R A^\sigma\Delta_L\Delta_t^{-1} f_N\right\|_{L^2}\|\nu^{-\frac13}(\nu^{\frac12}t^{\frac12}\partial_v)f_0\|_{H^3}\\ &\lesssim \frac{\varepsilon\nu^{\frac13}}{\langle t^2\rangle}\|A^\sigma f_{\sim N}\|_{L^2}\|A^\sigma\Delta_L\Delta_t^{-1} f_N\|_{L^2}\\ &\quad + \varepsilon\left\|\sqrt{\frac{\partial_t w}{w}}A^\sigma f_{\sim N}\right\|_{L^2}\left\|\sqrt{\frac{\partial_t w}{w}}\chi_R A^\sigma\Delta_L\Delta_t^{-1} P_{\neq} f_N\right\|_{L^2}.\end{aligned}$$

Putting together all the above estimates, we conclude that

$$
\begin{aligned}
|R_{N,=}^{\mathrm{NR,NR}}| \lesssim{}& \frac{\varepsilon\nu^{\frac{1}{3}}}{\langle \nu^{\frac{1}{2}} t^{\frac{3}{2}}\rangle} \|A^\sigma f_{\sim N}\|_{L^2} \|A^\sigma \Delta_L \Delta_t^{-1} f_N\|_{L^2} \\
&+ \frac{\varepsilon\nu^{\frac{1}{3}}}{\langle t^2\rangle} \|A^\sigma f_{\sim N}\|_{L^2} \|A^\sigma \Delta_L \Delta_t^{-1} f_N\|_{L^2} \\
&+ \varepsilon \left\| \sqrt{\frac{\partial_t w}{w}} A^\sigma f_{\sim N} \right\|_{L^2} \left\| \sqrt{\frac{\partial_t w}{w}} \chi_R A^\sigma \Delta_L \Delta_t^{-1} P_{\neq} f_N \right\|_{L^2}.
\end{aligned} \tag{8.7}
$$

8.1.3. Treatment of the zero mode $R_{N,=}^{\mathrm{NR,R}}$. Since $t \in I_{l,\xi}$, if $t \le t(\eta) \approx \sqrt{|\xi|} \approx \sqrt{N}$, then $l \approx \sqrt{|\xi|}$; thus by the fact that

$$
\frac{A_l^\sigma(\eta)|l|}{l^2 + |lt - \xi|^2} \lesssim \frac{A_l^\sigma(\xi)}{|l|} \lesssim \frac{A_l^\sigma(\xi)}{\sqrt{N}},
$$

we obtain

$$
\begin{aligned}
|R_{N,=}^{\mathrm{NR,R}}| &\lesssim \frac{1}{\langle t\rangle} \|A^\sigma f_{\sim N}\|_{L^2} \|A^\sigma \Delta_L \Delta_t^{-1} f_N\|_{L^2} \|\partial_v f_0\|_{H^3} \\
&\lesssim \frac{\varepsilon\nu^{\frac{1}{3}}}{\langle \nu^{\frac{1}{2}} t^{\frac{3}{2}}\rangle} \|A^\sigma f_{\sim N}\|_{L^2} \|A^\sigma \Delta_L \Delta_t^{-1} f_N\|_{L^2}.
\end{aligned}
$$

If $t \ge 2|\eta|$, then we use the same argument as (8.4) and (8.5) and get

$$
\frac{|l| A_k^\sigma(\eta)}{l^2 + (\xi - lt)^2} \lesssim \frac{\langle \xi - \eta\rangle^2}{\langle t\rangle^2} A_l^\sigma(\xi).
$$

Therefore we get

$$
\begin{aligned}
|R_{N,=}^{\mathrm{NR,R}}| &\lesssim \frac{1}{\langle t^2\rangle} \|A^\sigma f_{\sim N}\|_{L^2} \|A^\sigma \Delta_L \Delta_t^{-1} f_N\|_{L^2} \|f_0\|_{H^5} \\
&\lesssim \frac{\varepsilon\nu^{\frac{1}{3}}}{\langle t^2\rangle} \|A^\sigma f_{\sim N}\|_{L^2} \|A^\sigma \Delta_L \Delta_t^{-1} f_N\|_{L^2}.
\end{aligned}
$$

If $t \in I_t(\eta)$, then there is $k \neq l$ such that $t \in I_{k,\eta} \cap I_{l,\xi}$. By the fact that for $t \in I_{k,\eta}$ and $t \notin I_{l,\eta}$, $|t - \frac{\eta}{l}| > |t - \frac{\eta}{k}|$, then we get by (8.6) that

$$
\begin{aligned}
\frac{|l|\langle \nu^{\frac{1}{2}} (\frac{\xi}{k})^{\frac{1}{2}}\rangle^{-1}}{l^2(1 + |t - \frac{\xi}{l}|^2)} &\lesssim \sqrt{\frac{|l|}{l^2(1 + |t - \frac{\eta}{l}|^2)}} \sqrt{\frac{|l|}{l^2(1 + |t - \frac{\xi}{l}|^2)}} \langle \xi - \eta\rangle \langle \nu^{\frac{1}{2}} (\tfrac{\xi}{k})^{\frac{1}{2}}\rangle^{-1} \\
&\lesssim \sqrt{\frac{1}{k(1 + |t - \frac{\eta}{k}|^2)}} \sqrt{\frac{1}{l(1 + |t - \frac{\xi}{l}|^2)}} \langle \xi - \eta\rangle \langle \nu^{\frac{1}{2}} (\tfrac{\xi}{k})^{\frac{1}{2}}\rangle^{-1} \\
&\lesssim \sqrt{\frac{\eta}{k^2(1 + |t - \frac{\eta}{k}|^2)}} \sqrt{\frac{\xi}{l^2(1 + |t - \frac{\xi}{l}|^2)}} \langle \xi - \eta\rangle \langle \nu^{\frac{1}{3}} (\tfrac{\xi}{k})\rangle^{-(1+\beta)} \\
&\lesssim \sqrt{\frac{\partial_t w(t,\eta)}{w(t,\eta)}} \sqrt{\frac{\partial_t w(t,\xi)}{w(t,\xi)}} \langle \xi - \eta\rangle \nu^{-\frac{1}{3}}.
\end{aligned}
$$

Therefore, we get

$$\begin{aligned}|R_{N,=}^{\mathrm{NR,R}}| &\lesssim \left\|\sqrt{\frac{\partial_t w}{w}}A^\sigma f_{\sim N}\right\|_{L^2}\left\|\sqrt{\frac{\partial_t w}{w}}A^\sigma \chi_R\Delta_L\Delta_t^{-1}f_N\right\|_{L^2}\|\nu^{-\frac13}(\nu^{\frac12}t^{\frac12}\partial_v)f_0\|_{H^3}\\ &\lesssim \varepsilon\left\|\sqrt{\frac{\partial_t w}{w}}A^\sigma f_{\sim N}\right\|_{L^2}\left\|\sqrt{\frac{\partial_t w}{w}}A^\sigma \chi_R\Delta_L\Delta_t^{-1}f_N\right\|_{L^2}.\end{aligned}$$

Putting together all the above estimates, we conclude that

$$\begin{aligned}|R_{N,=}^{\mathrm{NR,R}}| &\lesssim \frac{\varepsilon\nu^{\frac13}}{\langle \nu^{\frac12}t^{\frac32}\rangle}\|A^\sigma f_{\sim N}\|_{L^2}\|A^\sigma\Delta_L\Delta_t^{-1}f_N\|_{L^2}\\ &\quad+\frac{\varepsilon\nu^{\frac13}}{\langle t^2\rangle}\|A^\sigma f_{\sim N}\|_{L^2}\|A^\sigma\Delta_L\Delta_t^{-1}f_N\|_{L^2}\\ &\quad+\varepsilon\left\|\sqrt{\frac{\partial_t w}{w}}A^\sigma f_{\sim N}\right\|_{L^2}\left\|\sqrt{\frac{\partial_t w}{w}}\chi_R A^\sigma\Delta_L\Delta_t^{-1}f_N\right\|_{L^2}. \end{aligned}\tag{8.8}$$

8.1.4. Treatment of the zero mode $R_{N,=}^{\mathrm{R,NR}}$. Since $t\in I_{l,\eta}$, if $t\le t(\xi)\approx\sqrt{|\xi|}$, then $l\approx\sqrt{|\xi|}$. By using (8.3), we get

$$\begin{aligned}|R_{N,=}^{\mathrm{R,NR}}| &\lesssim \frac{1}{\langle t^2\rangle}\|A^\sigma f_{\sim N}\|_{L^2}\|A^\sigma\Delta_L\Delta_t^{-1}f_N\|_{L^2}\|f_0\|_{H^5}\\ &\lesssim \frac{\varepsilon\nu^{\frac13}}{\langle t^2\rangle}\|A^\sigma f_{\sim N}\|_{L^2}\|A^\sigma\Delta_L\Delta_t^{-1}f_N\|_{L^2}.\end{aligned}$$

Similarly if $t\ge 2|\xi|$, then by (8.5), we get

$$\begin{aligned}|R_{N,=}^{\mathrm{R,NR}}| &\lesssim \frac{1}{\langle t^2\rangle}\|A^\sigma f_{\sim N}\|_{L^2}\|A^\sigma\Delta_L\Delta_t^{-1}f_N\|_{L^2}\|f_0\|_{H^5}\\ &\lesssim \frac{\varepsilon\nu^{\frac13}}{\langle t^2\rangle}\|A^\sigma f_{\sim N}\|_{L^2}\|A^\sigma\Delta_L\Delta_t^{-1}f_N\|_{L^2}.\end{aligned}$$

If $t\in I_t(\xi)$, then there is $k\neq l$ such that $t\in I_{k,\xi}\cap I_{l,\eta}$. By the fact that for $t\in I_{k,\xi}$ and $t\notin I_{l,\xi}$, $|t-\frac{\xi}{l}|>|t-\frac{\xi}{k}|$, then we get by (8.6) that

$$\begin{aligned}\frac{|l|\langle\nu^{\frac12}(\frac{\xi}{k})^{\frac12}\rangle^{-1}}{l^2(1+|t-\frac{\xi}{l}|^2)} &\lesssim \sqrt{\frac{|l|}{l^2(1+|t-\frac{\xi}{l}|^2)}}\sqrt{\frac{|l|}{l^2(1+|t-\frac{\eta}{l}|^2)}}\langle\xi-\eta\rangle\langle\nu^{\frac12}(\frac{\xi}{k})^{\frac12}\rangle^{-1}\\ &\lesssim \sqrt{\frac{1}{k(1+|t-\frac{\xi}{k}|^2)}}\sqrt{\frac{1}{l(1+|t-\frac{\eta}{l}|^2)}}\langle\xi-\eta\rangle\langle\nu^{\frac12}(\frac{\xi}{k})^{\frac12}\rangle^{-1}\\ &\lesssim \sqrt{\frac{\xi}{k^2(1+|t-\frac{\xi}{k}|^2)}}\sqrt{\frac{\eta}{l^2(1+|t-\frac{\eta}{l}|^2)}}\langle\xi-\eta\rangle\langle\nu^{\frac13}(\frac{\xi}{k})\rangle^{-(1+\beta)}\\ &\lesssim \sqrt{\frac{\partial_t w(t,\eta)}{w(t,\eta)}}\sqrt{\frac{\partial_t w(t,\xi)}{w(t,\xi)}}\langle\xi-\eta\rangle\nu^{-\frac13}.\end{aligned}$$

Therefore, we get

$$\begin{aligned}|R^{\mathrm{R,NR}}_{N,=}| &\lesssim \Big\|\sqrt{\frac{\partial_t w}{w}}A^\sigma f_{\sim N}\Big\|_{L^2}\Big\|\sqrt{\frac{\partial_t w}{w}}\chi_R A^\sigma \Delta_L\Delta_t^{-1} f_N\Big\|_{L^2}\|\nu^{-\frac13}(\nu^{\frac12}t^{\frac12}\partial_v)f_0\|_{H^3}\\ &\lesssim \varepsilon\Big\|\sqrt{\frac{\partial_t w}{w}}A^\sigma f_{\sim N}\Big\|_{L^2}\Big\|\sqrt{\frac{\partial_t w}{w}}\chi_R A^\sigma \Delta_L\Delta_t^{-1} f_N\Big\|_{L^2}.\end{aligned}$$

Putting together all the above estimates, we conclude that

$$\begin{aligned}|R^{\mathrm{R,NR}}_{N,=}| &\lesssim \frac{\varepsilon\nu^{\frac13}}{\langle t^2\rangle}\|A^\sigma f_{\sim N}\|_{L^2}\|A^\sigma \Delta_L\Delta_t^{-1} f_N\|_{L^2}\\ &\quad+\varepsilon\Big\|\sqrt{\frac{\partial_t w}{w}}A^\sigma f_{\sim N}\Big\|_{L^2}\Big\|\sqrt{\frac{\partial_t w}{w}}\chi_R A^\sigma \Delta_L\Delta_t^{-1} f_N\Big\|_{L^2}. \end{aligned}\tag{8.9}$$

8.1.5. Treatment of the zero mode $R^{\mathrm{R,R}}_{N,=}$. We get by (8.6) that

$$\begin{aligned}\frac{|l|\langle\nu^{\frac12}(\frac{\xi}{l})^{\frac12}\rangle^{-1}}{l^2(1+|t-\frac{\xi}{l}|^2)} &\lesssim \sqrt{\frac{|l|}{l^2(1+|t-\frac{\xi}{l}|^2)}}\sqrt{\frac{|l|}{l^2(1+|t-\frac{\eta}{l}|^2)}}\langle\xi-\eta\rangle\langle\nu^{\frac12}(\tfrac{\xi}{l})^{\frac12}\rangle^{-1}\\ &\lesssim \sqrt{\frac{\xi}{l^2(1+|t-\frac{\xi}{l}|^2)}}\sqrt{\frac{\eta}{l^2(1+|t-\frac{\eta}{l}|^2)}}\langle\xi-\eta\rangle\langle\nu^{\frac13}(\tfrac{\xi}{l})\rangle^{-(1+\beta)}\\ &\lesssim \sqrt{\frac{\partial_t w(t,\eta)}{w(t,\eta)}}\sqrt{\frac{\partial_t w(t,\xi)}{w(t,\xi)}}\langle\xi-\eta\rangle\nu^{-\frac13},\end{aligned}$$

which gives

$$\begin{aligned}|R^{\mathrm{R,R}}_{N,=}| &\lesssim \Big\|\sqrt{\frac{\partial_t w}{w}}A^\sigma f_{\sim N}\Big\|_{L^2}\Big\|\sqrt{\frac{\partial_t w}{w}}\chi_R A^\sigma \Delta_L\Delta_t^{-1} f_N\Big\|_{L^2}\|\nu^{-\frac13}(\nu^{\frac12}t^{\frac12}\partial_v)f_0\|_{H^3}\\ &\lesssim \varepsilon\Big\|\sqrt{\frac{\partial_t w}{w}}A^\sigma f_{\sim N}\Big\|_{L^2}\Big\|\sqrt{\frac{\partial_t w}{w}}\chi_R A^\sigma \Delta_L\Delta_t^{-1} f_N\Big\|_{L^2}.\end{aligned}\tag{8.10}$$

8.1.6. Treatment of $R_{N,\neq,*}$. In this case we get $(l,k,\xi,\eta)\notin D_1$ which means that at least one of the two inequalities $|l|\geq|\xi|$, $|l-k,\xi-\eta|\geq\frac{1}{1000}|l,\xi|$ holds. Thus we get

$$\frac{A^\sigma_k(\eta)|l,\xi|}{l^2+|lt-\xi|^2}\lesssim A^\sigma_l(\xi)\langle l-k,\xi-\eta\rangle,$$

which implies

$$\begin{aligned}|R_{N,\neq,*}| &\lesssim \|A^\sigma f_{\sim N}\|_2\|A^\sigma\Delta_L\Delta_t^{-1}f_N\|_2\|f_{\neq}\|_{H^4}\\ &\lesssim \frac{\varepsilon\nu^{\frac13}}{\langle\nu t^3\rangle}\|A^\sigma f_{\sim N}\|_2\|A^\sigma\Delta_L\Delta_t^{-1}f_N\|_2.\end{aligned}\tag{8.11}$$

In the next four subsections, we restrict $(l,k,\xi,\eta)\in D_1$, which gives $|\eta|\geq\frac{1}{20}|k|$ and $|\xi|\geq\frac{1}{20}|l|$; thus

$$w_k(t,\eta)=w(t,\eta),\quad w_l(t,\xi)=w(t,\xi).\tag{8.12}$$

8.1.7. Treatment of $R_N^{\mathrm{NR,NR}}$. Since we restrict the integrand to D_1, it holds that

$$\big||l,\xi|-|k,\eta|\big| \leq |k-l,\eta-\xi| \leq \frac{1}{1000}|l,\xi|. \tag{8.13}$$

It follows from the fact that

$$(\eta l-\xi k)=(\eta-\xi)l+(l-k)\xi$$

and Lemma 3.3 that

$$\begin{aligned}
|R_N^{\mathrm{NR,NR}}| &\lesssim \sum_{k,l\neq 0,\,k\neq l}\int_{\eta,\xi} 1_{t\notin I_{k,\eta},t\notin I_{l,\xi}}|A^\sigma \bar{\hat{f}}_k(\eta)| \\
&\qquad\times\Big|\frac{A_l^\sigma(\xi)|l,\xi|}{l^2+(\xi-lt)^2}\widehat{\Delta_L\Delta_t^{-1}f}_l(\xi)_N\widehat{\nabla f}_{k-l}(\eta-\xi)_{<N/8}\Big|\,d\eta\,d\xi \\
&\lesssim \sum_{k,l\neq 0,\,k\neq l}\int_{\eta,\xi} 1_{t\notin I_{k,\eta},t\notin I_{l,\xi}}A^\sigma|\bar{\hat{f}}_k(\eta)|\frac{A_l^\sigma(\xi)|l,\xi|}{l^2(1+\frac{\xi^2}{l^4})} \\
&\qquad\times\big|\widehat{\Delta_L\Delta_t^{-1}f}_l(\xi)_N\widehat{\nabla f}_{k-l}(\eta-\xi)_{<N/8}\big|\,d\eta\,d\xi.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
|R_N^{\mathrm{NR,NR}}| &\lesssim \|A^\sigma f_{\sim N}\|_2\|A^\sigma\Delta_L\Delta_t^{-1}f_N\|_2\|f_{\neq}\|_{H^3} \\
&\lesssim \frac{\varepsilon\nu^{\frac{1}{3}}}{\langle\nu t^3\rangle}\|A^\sigma f_{\sim N}\|_2\|A^\sigma\Delta_L\Delta_t^{-1}f_N\|_2.
\end{aligned} \tag{8.14}$$

8.1.8. Treatment of $R_N^{\mathrm{NR,R}}$. By (8.13) we have $\langle k,\eta\rangle^\sigma\approx\langle l,\xi\rangle^\sigma$ and $A_l(\xi)\approx A_k(\eta)$, which gives

$$\begin{aligned}
|R_N^{\mathrm{NR,R}}| &\lesssim \sum_{k,l\neq 0,\,k\neq l}\int_{|\eta-\xi|\leq\frac{|\eta|}{100}}\chi^{\mathrm{NR,R}}|A^\sigma\bar{\hat{f}}_k(\eta)|\frac{A_l^\sigma(\xi)|l,\xi|}{l^2+(\xi-lt)^2} \\
&\qquad\times\big|\widehat{\Delta_L\Delta_t^{-1}f}_l(\xi)_N\widehat{\nabla f}_{k-l}(\eta-\xi)_{<N/8}\big|\,d\eta\,d\xi \\
&\quad+\sum_{k,l\neq 0,\,k\neq l}\int_{|\eta-\xi|>\frac{|\eta|}{100}}\chi^{\mathrm{NR,R}}|A^\sigma\bar{\hat{f}}_k(\eta)|\frac{A_l^\sigma(\xi)|l,\xi|}{l^2+(\xi-lt)^2} \\
&\qquad\times\big|\widehat{\Delta_L\Delta_t^{-1}f}_l(\xi)_N\widehat{\nabla f}_{k-l}(\eta-\xi)_{<N/8}\big|\,d\eta\,d\xi \\
&= R_{N,1}^{\mathrm{NR,R}}+R_{N,2}^{\mathrm{NR,R}}.
\end{aligned}$$

Let us first deal with $R_{N,1}^{\mathrm{NR,R}}$, so that $|\eta-\xi|\leq\frac{|\eta|}{100}$.

If $t\leq t(\eta)\approx\sqrt{|\eta|}$, then $w_k(\eta)=1$ and by the fact that $t\geq t(\xi)\approx\sqrt{|\eta|}$, we get that $t\approx\sqrt{|\xi|}$, $|l|\approx\sqrt{|\xi|}$ and then

$$\begin{aligned}
|R_{N,1}^{\mathrm{NR,R}}| &\lesssim \sum_{k,l\neq 0,\,k\neq l}\int_{\eta,\xi} 1_{t\notin I_{k,\eta},t\in I_{l,\xi}}|A^\sigma\bar{\hat{f}}_k(\eta)|A^\sigma \\
&\qquad\times\big|\widehat{\Delta_L\Delta_t^{-1}f}_l(\xi)_N\widehat{\nabla f}_{k-l}(\eta-\xi)_{<N/8}\big|\,d\eta\,d\xi \\
&\lesssim \|A^\sigma f_{\sim N}\|_2\|A^\sigma\Delta_L\Delta_t^{-1}f_N\|_2\|f_{\neq}\|_{H^3}.
\end{aligned}$$

If $t \geq 2|\eta|$, and for $|\eta - \xi| \leq \frac{1}{100}|\eta|$ we have $|l|t - |\xi| \geq 2|\eta| - |\xi| \geq |\eta| - \big||\eta| - |\xi|\big| \gtrsim |\xi|$, this implies

$$|R_{N,1}^{\mathrm{NR,R}}| \lesssim \|A^\sigma f_{\sim N}\|_{L^2} \|A^\sigma \Delta_L \Delta_t^{-1} f_N\|_{H^\sigma} \|f_{\neq}\|_{H^3}.$$

If $t \in I_t(\eta)$, there exists $k' \in [1, E(\sqrt{|\eta|})]$ such that $t \in I_{k',\eta} \cap I_{l,\xi}$. Then by Lemma 3.2, we need to consider the following three cases:

(a) $k' = l$. In this case, by using the fact that

$$\begin{aligned}\frac{|\xi|}{l^2 + (\xi - lt)^2} &\lesssim \sqrt{\frac{|\xi|}{l^2 + (\xi - lt)^2}} \sqrt{\frac{|\eta|}{l^2 + (\eta - lt)^2}} \langle \xi - \eta \rangle \\ &\lesssim \langle \nu^{\frac{1}{3}} t \rangle^{1+\beta} \nu^{-\frac{1}{3}} \langle \xi - \eta \rangle \sqrt{\frac{\partial_t w(t,\xi)}{w(t,\xi)}} \sqrt{\frac{\partial_t w(t,\eta)}{w(t,\eta)}},\end{aligned}$$

and $|l| \lesssim \sqrt{|\xi|} \lesssim |\xi|$, we get

$$\begin{aligned}|R_{N,1}^{\mathrm{NR,R}}| \lesssim &\sum_{k,l \neq 0, k \neq l} \int_{|\eta - \xi| \leq \frac{|\eta|}{100}} 1_{t \in I_{k',\eta} \cap I_{l,\xi}} |A^\sigma \bar{\hat{f}}_k(\eta)| \frac{A_l^\sigma(t,\xi)|\xi|}{l^2 + (\xi - lt)^2} \\ &\qquad \times \left|\widehat{\Delta_L \Delta_t^{-1} f}_l(\xi)_N \widehat{\nabla f}_{k-l}(\eta - \xi)_{<N/8}\right| d\eta\, d\xi \\ \lesssim &\sum_{k,l \neq 0, k \neq l} \int_{|\eta - \xi| \leq \frac{1}{100}|\eta|} 1_{t \in I_{l,\eta} \cap I_{l,\xi}} \left|\sqrt{\frac{\partial_t w(\eta)}{w(\eta)}} A^\sigma \bar{\hat{f}}_k(\eta)\right| \\ &\qquad \times \Bigg|\sqrt{\frac{\partial_t w(\eta)}{w(\eta)}} A^\sigma \widehat{\Delta_L \Delta_t^{-1} f}_l(\xi)_N (1 + \nu^{\frac{1}{3}} t)^{1+\beta} \nu^{-\frac{1}{3}} \\ &\qquad \times \widehat{\langle \nabla \partial_v \rangle f}_{k-l}(\eta - \xi)_{<N/8}\Bigg| d\eta\, d\xi \\ \lesssim &\left\|\sqrt{\frac{\partial_t w(\eta)}{w(\eta)}} A^\sigma f_{\sim N}\right\|_2 \left\|\sqrt{\frac{\partial_t w(\xi)}{w(\xi)}} \chi_R A^\sigma \Delta_L \Delta_t^{-1} f_N\right\|_2 \|\langle \nu^{\frac{1}{3}} t \rangle^{1+\beta} \nu^{-\frac{1}{3}} f_{\neq}\|_{H^4}.\end{aligned}$$

(b) $|t - \frac{\eta}{k'}| \gtrsim \frac{\eta}{k'^2}$ and $|t - \frac{\xi}{l}| \gtrsim \frac{\xi}{l^2}$. In this case, by using the fact that

$$\frac{|l,\xi|}{l^2 + (\xi - lt)^2} \lesssim \frac{|\xi|/l^2}{1 + (\xi/l - t)^2} \lesssim \frac{|\xi|/l^2}{1 + \frac{\xi^2}{l^4}} \lesssim 1, \tag{8.15}$$

we obtain

$$|R_{N,1}^{\mathrm{NR,R}}| \lesssim \|A^\sigma f_{\sim N}\|_{L^2} \|A^\sigma \Delta_L \Delta_t^{-1} P_{\neq} f_N\|_{L^2} \|f_{\neq}\|_{H^3}.$$

(c) $|\eta - \xi| \gtrsim \frac{|\eta|}{|l|} \approx \frac{|\xi|}{|l|}$. In this case we get

$$\frac{|\xi|}{l^2 + (\xi - lt)^2} \lesssim \frac{|\xi|}{l^2} \lesssim |\eta - \xi|,$$

which gives

$$R_{N,1}^{\mathrm{NR,R}} \lesssim \|A^\sigma f_{\sim N}\|_{L^2}\|A^\sigma \Delta_L \Delta_t^{-1} P_{\neq} f_N\|_{L^2}\|f_{\neq}\|_{H^4}.$$

Next we deal with $R_{N,2}^{\mathrm{NR,R}}$, for which we will use the fact that

$$\frac{|l,\xi|}{l^2+(\xi-lt)^2} \lesssim 1+\frac{|\xi|}{l^2} \lesssim \langle \xi-\eta\rangle.$$

Thus we get

$$\begin{aligned} R_{N,2}^{\mathrm{NR,R}} \lesssim & \sum_{k,l\neq 0,\,k\neq l} \int_{|\eta-\xi|>\frac{|\eta|}{100}} \chi^{\mathrm{NR,R}} |A^\sigma \bar{\hat{f}}_k(\eta)| \frac{A_l^\sigma(\xi)|l,\xi|}{l^2+(\xi-lt)^2} \\ & \qquad \times \left|\widehat{\Delta_L\Delta_t^{-1} f}_l(\xi)_N \widehat{\nabla f}_{k-l}(\eta-\xi)_{<N/8}\right| d\eta\, d\xi \\ \lesssim & \|A^\sigma f_{\sim N}\|_2 \|A^\sigma \Delta_L \Delta_t^{-1} P_{\neq} f_N\|_{L^2}\|f_{\neq}\|_{H^4}. \end{aligned}$$

Therefore we conclude that

$$\begin{aligned} |R_N^{\mathrm{NR,R}}| \lesssim & \|A^\sigma f_{\sim N}\|_{L^2}\|A^\sigma \Delta_L\Delta_t^{-1} f_N\|_{L^2}\|f_{\neq}\|_{H^4} \\ & + \left\|\sqrt{\frac{\partial_t w}{w}} A^\sigma f_{\sim N}\right\|_2 \left\|\sqrt{\frac{\partial_t w}{w}} \chi_R A^\sigma \Delta_L \Delta_t^{-1} f_N\right\|_2 \|\langle \nu^{\frac13} t\rangle^{1+\beta} \nu^{-\frac13} f_{\neq}\|_{H^4} \\ \lesssim & \frac{\varepsilon \nu^{\frac13}}{\langle \nu t^3\rangle} \|A^\sigma f_{\sim N}\|_{L^2}\|A^\sigma \Delta_L\Delta_t^{-1} f_N\|_{L^2} \\ & + \varepsilon \left\|\sqrt{\frac{\partial_t w}{w}} A^\sigma f_{\sim N}\right\|_2 \left\|\sqrt{\frac{\partial_t w}{w}} \chi_R A^\sigma \Delta_L \Delta_t^{-1} f_N\right\|_2. \end{aligned} \tag{8.16}$$

8.1.9. Treatment of $R_N^{\mathrm{R,NR}}$. In this case we have

$$\frac{|l,\xi|}{l^2+(\xi-lt)^2} \lesssim \begin{cases} \dfrac{|l|}{l^2(1+\frac{|\xi|^2}{l^4})} \lesssim 1 & \text{if } |l| \le E(\sqrt{\xi}), \\ \dfrac{1}{\sqrt{\xi}} \lesssim 1 & \text{if } |l| \ge E(\sqrt{\xi})+1, \end{cases}$$

which implies

$$\begin{aligned} R_N^{\mathrm{R,NR}} \lesssim & \sum_{k,l\neq 0,\,k\neq l} \int_{\eta,\xi} \chi^{\mathrm{R,NR}} |A^\sigma \bar{\hat{f}}_k(\eta)| \frac{A_l^\sigma(\xi)|l,\xi|}{l^2+(\xi-lt)^2} \\ & \qquad \times \left|\widehat{\Delta_L\Delta_t^{-1} f}_l(\xi)_N \widehat{\nabla f}_{k-l}(\eta-\xi)_{<N/8}\right| d\eta\, d\xi \\ \lesssim & \|A^\sigma f_{\sim N}\|_2 \|A^\sigma \Delta_L \Delta_t^{-1} P_{\neq} f_N\|_{L^2}\|f_{\neq}\|_{H^3} \\ \lesssim & \frac{\varepsilon \nu^{\frac13}}{\langle \nu t^3\rangle} \|A^\sigma f_{\sim N}\|_2 \|A^\sigma \Delta_L \Delta_t^{-1} P_{\neq} f_N\|_{L^2}. \end{aligned} \tag{8.17}$$

8.1.10. Treatment of $R_N^{\mathrm{R,R}}$. In this case $t \in I_{k,\eta} \cap I_{l,\xi}$ with $k \neq l$. By Lemma 3.2, we only need to deal with the following two cases:

(b) $|t - \frac{\eta}{k}| \gtrsim \frac{\eta}{k^2}$ and $|t - \frac{\xi}{l}| \gtrsim \frac{\xi}{l^2}$. In this case, by the fact that $A_l^\sigma(\xi) \approx A_k^\sigma(\eta)$ and

$$\frac{|l,\xi|}{l^2 + (\xi - lt)^2} \lesssim 1 + \frac{|\xi|}{l^2(1 + \frac{\xi^2}{l^4})} \lesssim 1,$$

we get

$$\begin{aligned}|R_N^{\mathrm{R,R}}| \lesssim & \sum_{k,l\neq 0,\, k\neq l} \int_{\eta,\xi} \chi^{\mathrm{R,R}} A^\sigma |\bar{\hat{f}}_k(\eta)| \frac{A_l^\sigma(\xi)|l,\xi|}{l^2 + (\xi - lt)^2} \\ & \times \left|\widehat{\Delta_L \Delta_t^{-1} f}_l(\xi)_N \widehat{\nabla f}_{k-l}(\eta - \xi)_{<N/8}\right| d\eta\, d\xi \\ \lesssim & \|A^\sigma f_{\sim N}\|_{L^2} \|A^\sigma \Delta_L \Delta_t^{-1} f_N\|_{L^2} \|f_{\neq}\|_{H^3}.\end{aligned}$$

(c) $|\eta - \xi| \gtrsim \frac{|\eta|}{|l|} \approx \frac{|\xi|}{|l|}$. In this case, by using the fact that $A_l^\sigma(\xi) \approx A_k^\sigma(\eta)$ and

$$\frac{|l,\xi|}{l^2 + (\xi - lt)^2} \lesssim 1 + \frac{|\xi|}{l^2} \lesssim |\eta - \xi|,$$

we get

$$\begin{aligned}|R_N^{\mathrm{R,R}}| \lesssim & \sum_{k,l\neq 0,\, k\neq l} \int_{\eta,\xi} \chi^{\mathrm{R,R}} A^\sigma |\bar{\hat{f}}_k(\eta)| \frac{A_l(\xi)|l,\xi|}{l^2 + (\xi - lt)^2} \\ & \times \left|\widehat{\Delta_L \Delta_t^{-1} f}_l(\xi)_N \widehat{\nabla f}_{k-l}(\eta - \xi)_{<N/8}\right| d\eta\, d\xi \\ \lesssim & \|A^\sigma f_{\sim N}\|_2 \|A^\sigma \Delta_L \Delta_t^{-1} f_N\|_2 \|f_{\neq}\|_{H^4}.\end{aligned}$$

Therefore we conclude that

$$\begin{aligned}|R_N^{\mathrm{R,R}}| &\lesssim \|A^\sigma f_{\sim N}\|_2 \|A^\sigma \Delta_L \Delta_t^{-1} P_{\neq} f_N\|_2 \|f_{\neq}\|_{H^4} \\ &\lesssim \frac{\varepsilon \nu^{\frac{1}{3}}}{\langle \nu t^3 \rangle} \|A^\sigma f_{\sim N}\|_2 \|A^\sigma \Delta_L \Delta_t^{-1} P_{\neq} f_N\|_2. \end{aligned} \tag{8.18}$$

Combining (8.1), (8.7), (8.8), (8.9), (8.10), (8.11), (8.14), (8.16), (8.17) and (8.18), we deduce that

$$\begin{aligned}|R_N^1| \lesssim & \left(\frac{\varepsilon \nu^{\frac{1}{3}}}{\langle \nu^{\frac{1}{2}} t^{\frac{3}{2}} \rangle} + \frac{\varepsilon \nu^{\frac{1}{3}}}{\langle t^2 \rangle} + \frac{\varepsilon \nu^{\frac{1}{3}}}{\langle \nu t^3 \rangle}\right) \|A^\sigma f_{\sim N}\|_{L^2} \|A^\sigma \Delta_L \Delta_t^{-1} P_{\neq} f_N\|_{L^2} \\ & + \varepsilon \left\| \sqrt{\frac{\partial_t w}{w}} A^\sigma f_{\sim N} \right\|_2 \left\| \sqrt{\frac{\partial_t w}{w}} \chi_R A^\sigma \Delta_L \Delta_t^{-1} P_{\neq} f_N \right\|_2. \end{aligned} \tag{8.19}$$

8.2. Treatment of R_N^2

We recall that

$$\begin{aligned} R_N^2 &= \sum_{k\neq 0}\int_{\eta,\xi} A^\sigma \bar{\hat{f}}_k(\eta) A_k^\sigma(\eta)\hat{g}(\xi)_N \widehat{\partial_v f}_k(\eta-\xi)_{<N/8}\, d\eta\, d\xi \\ &\quad + \int_{\eta,\xi} A^\sigma \bar{\hat{f}}_0(\eta) A_0^\sigma(\eta)\hat{g}(\xi)_N \widehat{\partial_v f}_0(\eta-\xi)_{<N/8}\, d\eta\, d\xi \\ &= R_{N,\neq}^2 + R_{N,0}^2. \end{aligned}$$

By the fact that $|k,\eta-\xi| \leq \frac{3}{16}N \leq \frac{3}{8}|\xi| \approx |k,\eta|$, we have

$$|R_{N,\neq}^2| \lesssim \|A^\sigma (f_{\neq})_{\sim N}\|_2 \|\langle \partial_v\rangle^\sigma g_N\|_2 \|f_{\neq}\|_{H^3}$$

and

$$|R_{N,0}^2| \lesssim \|A^\sigma (f_0)_{\sim N}\|_2 \|\langle \partial_v\rangle^\sigma g_N\|_2 \|f_0\|_{H^3}.$$

Thus we obtain

$$|R_N^2| \lesssim \|A^\sigma f_{\sim N}\|_2 \|g_N\|_{H^\sigma} \|f\|_{H^3}. \tag{8.20}$$

8.3. Treatment of R_N^3

The term R_N^3 is easy to deal with, because the derivatives land on the low frequency. We then get

$$\begin{aligned} |R_N^3| &\leq \left| \sum_{k,l}\int_{\eta,\xi} A^\sigma \bar{\hat{f}}_k(\eta) A_{k-l}^\sigma(\eta-\xi)\hat{u}_l(\xi)_N \widehat{\nabla f}_{k-l}(\eta-\xi)_{<N/8}\, d\eta\, d\xi \right| \\ &\leq \sum_{k,l}\int_{\eta,\xi} A^\sigma |\bar{\hat{f}}_k(\eta)|\, |l,\xi|\, |\hat{u}_l(\xi)_N| A_{k-l}^\sigma(\eta-\xi) |\hat{f}_{k-l}(\eta-\xi)_{<N/8}|\, d\eta\, d\xi, \end{aligned}$$

which gives

$$|R_N^3| \lesssim \|A^\sigma f_{\sim N}\|_2 \|u_N\|_{H^3} \|f\|_{H^\sigma}. \tag{8.21}$$

8.4. Corrections

In this section we treat $R_N^{\varepsilon,1}$, which is higher order in $\nu^{\frac{1}{3}}$ than R_N^1. We expand $(1-v')\phi_l$ with a paraproduct only in v:

$$\begin{aligned} R_N^{\varepsilon,1} &= -\frac{1}{2\pi}\sum_{M\geq 8}\sum_{k,l\neq 0}\int_{\eta,\xi,\xi'} A^\sigma \bar{\hat{f}}_k(\eta) A_k^\sigma(\eta)\big((\xi-\eta)l - \xi'(k-l)\big)\chi_N(l,\xi) \\ &\qquad\qquad \times \big[\widehat{(1-v')}(\xi'-\xi)\big]_{<M/8} \hat{\phi}_l(\xi')_M \hat{f}_{k-l}(\eta-\xi)_{<N/8}\, d\eta\, d\xi\, d\xi' \\ &\quad -\frac{1}{2\pi}\sum_{M\geq 8}\sum_{k,l\neq 0}\int_{\eta,\xi,\xi'} A^\sigma \bar{\hat{f}}_k(\eta) A_k^\sigma(\eta)\big((\xi-\eta)l - \xi'(k-l)\big)\chi_N(l,\xi) \\ &\qquad\qquad \times \big[\widehat{(1-v')}(\xi'-\xi)\big]_{M} \hat{\phi}_l(\xi')_{<M/8} \hat{f}_{k-l}(\eta-\xi)_{<N/8}\, d\eta\, d\xi\, d\xi' \end{aligned}$$

$$-\frac{1}{2\pi}\sum_{M\in\mathbb{D}}\sum_{\frac{1}{8}M\le M'\le M}\sum_{k,l\ne 0}\int_{\eta,\xi,\xi'} A^\sigma \bar{\hat{f}}_k(\eta) A_k^\sigma(\eta)\big((\xi-\eta)l-\xi'(k-l)\big)$$
$$\times \chi_N(l,\xi)\big[\widehat{(1-v')}(\xi'-\xi)\big]_{M'}\widehat{\phi}_l(\xi')_M \hat{f}_{k-l}(\eta-\xi)_{<N/8}\, d\eta\, d\xi\, d\xi'$$
$$= R^{\varepsilon,1}_{N,LH} + R^{\varepsilon,1}_{N,HL} + R^{\varepsilon,1}_{N,HH}.$$

We recall that χ_N denotes the Littlewood–Paley cutoff to the Nth dyadic shell in $\mathbb{Z}\times\mathbf{R}$; see Section A.1.

We begin with $R^{\varepsilon,1}_{N,LH}$. On the support of the integrand,

$$\big||k,\eta|-|l,\xi|\big| \le |k-l,\eta-\xi| \le \frac{3}{8}|l,\xi|,$$
$$\big||l,\xi'|-|l,\xi|\big| \le |\xi-\xi'| \le \frac{3}{8}|l,\xi'|.$$

Thus $A_k^\sigma(\eta) \approx A_l^\sigma(\xi')$ and

$$|R^{\varepsilon,1}_{N,LH}| \lesssim \sum_{M\ge 8}\sum_{k,l\ne 0}\int_{\eta,\xi,\xi'} |A^\sigma \bar{\hat{f}}_k(\eta)||(\xi-\eta)l-\xi'(k-l)|\chi_N(l,\xi)$$
$$\times \big|\big[\widehat{(1-v')}(\xi'-\xi)\big]_{<M/8} A_l^\sigma(\xi')\widehat{\phi}_l(\xi')_M \hat{f}_{k-l}(\eta-\xi)_{<N/8}\big|\, d\eta\, d\xi\, d\xi'.$$

From here we may proceed analogously to the treatment of R^1_N with (l,ξ') playing the role of (l,ξ). We omit the details and simply conclude the result is

$$|R^{\varepsilon,1}_{N,LH}| \lesssim \|h\|_{H^3}\Bigg[\bigg(\frac{\varepsilon\nu^{\frac{1}{3}}}{\langle t^{\frac{3}{2}}\nu^{\frac{1}{2}}\rangle}+\frac{\varepsilon\nu^{\frac{1}{3}}}{\langle t^2\rangle}+\frac{\varepsilon\nu^{\frac{1}{3}}}{\langle t^3\nu\rangle}\bigg)\|A^\sigma f_{\sim N}\|_{L^2}\|A^\sigma\Delta_L\Delta_t^{-1}P_{\ne} f_N\|_{L^2}$$
$$+\varepsilon\bigg\|\sqrt{\frac{\partial_t w}{w}}A^\sigma f_{\sim N}\bigg\|_2\bigg\|\sqrt{\frac{\partial_t w}{w}}A^\sigma\chi_R\Delta_L\Delta_t^{-1}P_{\ne} f_N\bigg\|_2\Bigg]. \tag{8.22}$$

Turn now to $R^{\varepsilon,1}_{N,HL}$. On the support of the integrand, it holds that

$$\langle k,\eta\rangle^\sigma \approx \langle l,\xi\rangle^\sigma \approx \langle l,\xi'-\xi\rangle^\sigma.$$

Thus we get

$$|R^{\varepsilon,1}_{N,HL}| \lesssim \sum_{M\ge 8}\sum_{k,l\ne 0}\int_{\eta,\xi,\xi'} \chi_{|l|\ge\frac{|\xi|}{16}}|A^\sigma \bar{\hat{f}}_k(\eta)|\chi_N(l,\xi)\big|\big[\widehat{(1-v')}(\xi'-\xi)\big]_M$$
$$\times \langle l\rangle^\sigma|l,\xi'|\widehat{\phi}_l(\xi')_{<M/8}\widehat{\nabla f}_{k-l}(\eta-\xi)_{<N/8}\big|\, d\eta\, d\xi\, d\xi'$$
$$+\sum_{M\ge 8}\sum_{k,l\ne 0}\int_{\eta,\xi,\xi'} \chi_{|l|<\frac{|\xi|}{16}}|A^\sigma \bar{\hat{f}}_k(\eta)|\chi_N(l,\xi)\langle\xi'-\xi\rangle^\sigma\big|\big[\widehat{(1-v')}(\xi'-\xi)\big]_M$$
$$\times |l,\xi'|\widehat{\phi}_l(\xi')_{<M/8}\widehat{\nabla f}_{k-l}(\eta-\xi)_{<N/8}\big|\, d\eta\, d\xi\, d\xi'$$
$$= R^{\varepsilon,1,z}_{N,HL} + R^{\varepsilon,1,v}_{N,HL}.$$

First, consider $R^{\varepsilon,1,z}_{N,HL}$, where on the support of the integrand, $16|l| \geq |\xi|$:

$$\big||k,\eta| - |l,\xi|\big| \leq |k-l,\xi-\eta| \leq \frac{3}{16}|l,\xi|,$$
$$\big||l,\xi| - |l,\xi'|\big| \leq |\xi-\xi'| \leq \frac{38|\xi|}{32} \lesssim |l|.$$

Thus we divide $R^{\varepsilon,1,z}_{N,HL}$ into two parts,

$$\begin{aligned}
R^{\varepsilon,1,z}_{N,HL} \lesssim &\sum_{M\geq 8}\sum_{k,l\neq 0}\int_{\eta,\xi,\xi'} \chi_{|l|\geq 16|\xi|}|A^\sigma \bar{\hat{f}}_k(\eta)|\chi_N(l,\xi)\big|\big[\widehat{(1-v')}(\xi'-\xi)\big]_M\\
&\qquad\times\big[\langle l\rangle^{\sigma+1}\hat{\phi}_l(\xi')_{<M/8}\big]_{\sim N}\widehat{\nabla f}_{k-l}(\eta-\xi)_{<N/8}\big|\,d\eta\,d\xi\,d\xi'\\
&+\sum_{M\geq 8}\sum_{k,l\neq 0}\int_{\eta,\xi,\xi'} \chi_{16|\xi|\geq|l|\geq\frac{|\xi|}{16}}|A^\sigma \bar{\hat{f}}_k(\eta)|\chi_N(l,\xi)\big|\big[\widehat{(1-v')}(\xi'-\xi)\big]_{M\sim N}\\
&\qquad\times\langle l\rangle^{\sigma}|l,\xi'|\hat{\phi}_l(\xi')_{<M/8}\widehat{\nabla f}_{k-l}(\eta-\xi)_{<N/8}\big|\,d\eta\,d\xi\,d\xi'\\
=\ & R^{\varepsilon,1,z}_{N,HL,1} + R^{\varepsilon,1,z}_{N,HL,2}
\end{aligned}$$

To make it summable in M we need more "derivatives" at higher (in M) frequency; luckily, all of the "derivatives" land on the lower (in M) frequency.

If $|l| \geq 16|\xi|$, then in fact $\frac{38|\xi|}{32} \leq \frac{|l|}{4}$, and therefore $\big||k,\eta| - |l,\xi'|\big| \leq \frac{|l|}{2} \leq \frac{|l,\xi'|}{2}$, which gives

$$|k,\eta| \approx |l,\xi'| \approx |l| \approx N.$$

For $|l| \geq 16|\xi| \geq 16|\xi'|$, we get

$$\frac{|l|}{l^2(1+(\frac{\xi'}{l}-t)^2)} \lesssim \frac{|l|^{-1}\langle\frac{|\xi'|}{|l|}\rangle^2}{\langle\frac{|\xi'|}{|l|}\rangle^2(1+(\frac{\xi'}{l}-t)^2)} \lesssim \frac{|l|^{-1}}{\langle t\rangle^2}.$$

We then get

$$\begin{aligned}
R^{\varepsilon,1,z}_{N,HL,1} \lesssim &\sum_{M\geq 8}\sum_{k,l\neq 0}\int_{\eta,\xi,\xi'} \chi_{|l|\geq 16|\xi|}|A^\sigma \bar{\hat{f}}_k(\eta)|\chi_N(l,\xi)\Big|\big[\widehat{(1-v')}(\xi'-\xi)\big]_M\\
&\times\Big[\frac{\langle l\rangle^{\sigma+1}}{l^2+(\xi'-lt)^2}\widehat{\Delta_L\Delta_t^{-1}f}_l(\xi')_{<M/8}\Big]_{\sim N}\widehat{\nabla f}_{k-l}(\eta-\xi)_{<N/8}\Big|\,d\eta\,d\xi\,d\xi'\\
\lesssim &\langle t\rangle^{-2}\sum_{M\geq 8}\|A^\sigma f_{\sim N}\|_2 M^{-2}\|(v'-1)_M\|_{H^5}\|(\Delta_L\Delta_t^{-1}P_{\neq}f)_{\sim N}\|_{H^\sigma}\|f\|_{H^3},
\end{aligned}$$

which gives

$$\begin{aligned}
R^{\varepsilon,1,z}_{N,HL,1} &\lesssim \langle t\rangle^{-2}\|A^\sigma f_{\sim N}\|_2\|h\|_{H^5}\|A^\sigma(\Delta_L\Delta_t^{-1}P_{\neq}f)_{\sim N}\|_2\|f\|_{H^3}\\
&\lesssim \frac{\varepsilon^2\nu^{\frac{2}{3}}}{\langle t\rangle^2}\|A^\sigma f_{\sim N}\|_2\|A^\sigma(\Delta_L\Delta_t^{-1}P_{\neq}f)_{\sim N}\|_2.
\end{aligned}$$

Next we turn to $R^{\varepsilon,1,z}_{N,HL,2}$. If $\frac{1}{16}|\xi| \le |l| \le 16|\xi|$, then $|l| \approx |l,\xi| \approx |\xi-\xi'| \approx M \approx N$,

$$\begin{aligned}
R^{\varepsilon,1,z}_{N,HL,2} &\lesssim \sum_{M\ge 8} \|A^\sigma f_{\sim N}\|_2 M^{-1} \|(1-v')_{M\sim N}\|_{H^{\sigma-1}} \|P_{\neq}\phi\|_{H^4} \|f\|_{H^3} \\
&\lesssim \|A^\sigma f_{\sim N}\|_2 \|h_{\sim N}\|_{H^{\sigma-1}} \|P_{\neq}\phi\|_{H^4} \|f\|_{H^3} \\
&\lesssim \frac{\varepsilon^2 \nu^{\frac{2}{3}}}{\langle t\rangle^2} \|A^\sigma f_{\sim N}\|_2 \|h_{\sim N}\|_{H^{\sigma-1}}.
\end{aligned}$$

Next we turn to $R^{\varepsilon,1,v}_{N,HL}$, in which case we can consider all of the "derivatives" to be landing on $1-v'$. On the support of the integrand,

$$\begin{aligned}
\big||k,\eta| - |l,\xi|\big| &\le |k-l,\xi-\eta| \le \frac{3}{16}|l,\xi|, \\
\big||\xi-\xi'| - |l,\xi|\big| &\le |l,\xi'| \le \frac{|\xi|}{16} + |\xi'| \le \frac{67}{100}|\xi'-\xi|.
\end{aligned}$$

Since $|l,\xi| \approx |\xi-\xi'|$, the sum only includes boundedly many terms. Therefore,

$$\begin{aligned}
R^{\varepsilon,1,v}_{N,HL} &\lesssim \|A^\sigma f_{\sim N}\|_2 \|\partial_v h_{\sim N}\|_{H^{\sigma-1}} \|\Delta_L \Delta_t^{-1} f_{\neq}\|_{H^4} \|f\|_{H^3} \\
&\lesssim \frac{\varepsilon^2 \nu^{\frac{2}{3}}}{\langle \nu t^3\rangle} \|A^\sigma f_{\sim N}\|_2 \|\partial_v h_{\sim N}\|_{H^{\sigma-1}}.
\end{aligned}$$

We turn to the remainder term $R^{\varepsilon,1}_{N,HH}$. In this case we have

$$|\xi-\xi'| \approx |\xi'| \approx M \approx M'.$$

Thus we divide into two cases according to the relationship between l and ξ':

$$\begin{aligned}
|R^{\varepsilon,1}_{N,HH}| &\lesssim \sum_{M\in\mathbb{D}} \sum_{\frac{1}{8}M\le M'\le M} \sum_{k,l\neq 0} \int_{\eta,\xi,\xi'} 1_{|l|\ge 3|\xi'|} A^\sigma \bar{\hat{f}}_k(\eta) \chi_N(l,\xi) \big[\widehat{(1-v')}(\xi'-\xi)\big]_{M'} \\
&\qquad \times \frac{|l| A^\sigma_l(\xi)}{l^2+(\xi'-lt)^2} \big|\widehat{\Delta_L\Delta_t^{-1} f}_l(\xi')_M\big| \big|\widehat{\nabla f}_{k-l}(\eta-\xi)_{<N/8}\big| \, d\eta\, d\xi\, d\xi' \\
&\quad + \sum_{M\in\mathbb{D}} \sum_{\frac{1}{8}M\le M'\le M} \sum_{k,l\neq 0} \int_{\eta,\xi,\xi'} 1_{|l|<3|\xi'|} A^\sigma \bar{\hat{f}}_k(\eta) \chi_N(l,\xi) \big[\widehat{(1-v')}(\xi'-\xi)\big]_{M'} \\
&\qquad \times \frac{|l| A^\sigma_l(\xi)}{l^2+(\xi'-lt)^2} \big|\widehat{\Delta_L\Delta_t^{-1} f}_l(\xi')_M \widehat{\nabla f}_{k-l}(\eta-\xi)_{<N/8}\big| \, d\eta\, d\xi\, d\xi' \\
&= R^{\varepsilon,1,z}_{N,HH} + R^{\varepsilon,1,v}_{N,HH}.
\end{aligned}$$

First, we consider $R^{\varepsilon,1,z}_{N,HH}$. In this case we have $\langle \xi'-\xi\rangle \approx \langle \xi'\rangle$,

$$A^\sigma_k(\eta) \approx A^\sigma_l(\xi) \approx \langle l,\xi\rangle^\sigma \approx \langle l\rangle^\sigma + \langle \xi\rangle^\sigma \lesssim \langle l\rangle^\sigma + \langle \xi'\rangle^\sigma + \langle \xi'-\xi\rangle^\sigma \lesssim \langle l\rangle^\sigma,$$

and $N \approx |k,\eta| \approx \langle l\rangle$; then

$$\frac{|l| A^\sigma_l(\xi)}{l^2+(\xi'-lt)^2} \lesssim \frac{\langle l\rangle^{\sigma-1}}{\langle t\rangle^2}.$$

Therefore, we get

$$\begin{aligned}
R^{\varepsilon,1,z}_{N,HH} &\lesssim \langle t\rangle^{-2}\sum_{M\in\mathbb{D}}\|A^\sigma f_{\sim N}\|_2 M^{-2}\|h_{\sim M}\|_{H^4}\|\Delta_L\Delta_t^{-1} f_{\sim N}\|_{H^\sigma}\|f\|_{H^3}\\
&\lesssim \langle t\rangle^{-2}\|A^\sigma f_{\sim N}\|_2\|h\|_{H^4}\|\Delta_L\Delta_t^{-1}P_{\neq} f_{\sim N}\|_{H^\sigma}\|f\|_{H^3}\\
&\lesssim \frac{\varepsilon^2\nu^{\frac23}}{\langle t\rangle^2}\|A^\sigma f_{\sim N}\|_2\|\Delta_L\Delta_t^{-1}P_{\neq} f_{\sim N}\|_{H^\sigma}.
\end{aligned}$$

For $|l|<3|\xi'|$, we have $\langle \xi'-\xi\rangle\approx\langle\xi'\rangle$,

$$A^\sigma_k(\eta)\approx A^\sigma_l(\xi)\approx\langle l,\xi\rangle^\sigma\approx\langle l\rangle^\sigma+\langle\xi\rangle^\sigma\lesssim\langle l\rangle^\sigma+\langle\xi'\rangle^\sigma+\langle\xi'-\xi\rangle^\sigma\lesssim\langle\xi'\rangle^\sigma,$$

and $N\approx\langle l,\xi\rangle\approx\langle\xi'-\xi\rangle$. Thus,

$$\begin{aligned}
R^{\varepsilon,1,v}_{N,HH} &\lesssim \sum_{M\in\mathbb{D}}\sum_{\frac18 M\le M'\le M}\sum_{k,l\neq 0}\int_{\eta,\xi,\xi'} 1_{|l|<3|\xi'|}A^\sigma\bar{\hat{f}}_k(\eta)\chi_N(l,\xi)\\
&\quad\times\big[\langle\xi'-\xi\rangle^{\sigma-1}\widehat{(1-v')}(\xi'-\xi)\big]_{M'}\langle\xi'\rangle^2|\hat\phi_l(\xi')_M\widehat{\nabla f}_{k-l}(\eta-\xi)_{<N/8}|\,d\eta\,d\xi\,d\xi'\\
&\lesssim\sum_{M\in\mathbb{D}}\|A^\sigma f_{\sim N}\|_2 M^{-2}\|h_{\sim N}\|_{H^{\sigma-1}}\|(\Delta_L\Delta_t^{-1} f_{\neq})_{\sim M}\|_{H^4}\|f\|_{H^3}\\
&\lesssim \|A^\sigma f_{\sim N}\|_2\|h_{\sim N}\|_{H^{\sigma-1}}\|P_{\neq}\phi\|_{H^4}\|f\|_{H^3}\\
&\lesssim\frac{\varepsilon^2\nu^{\frac23}}{\langle t\rangle^2}\|A^\sigma f_{\sim N}\|_2\|h_{\sim N}\|_{H^{\sigma-1}}.
\end{aligned}$$

Therefore, we conclude by the bootstrap hypotheses and Lemma 4.1 that

$$\begin{aligned}
|R^{1,\varepsilon}_N| &\lesssim \varepsilon\nu^{\frac13}\Bigg[\Big(\frac{\varepsilon\nu^{\frac13}}{\langle t^{\frac32}\nu^{\frac12}\rangle}+\frac{\varepsilon\nu^{\frac13}}{\langle t^2\rangle}+\frac{\varepsilon\nu^{\frac13}}{\langle t^3\nu\rangle}\Big)\|A^\sigma f_{\sim N}\|_{L^2}\|A^\sigma\Delta_L\Delta_t^{-1}P_{\neq} f_N\|_{L^2}\\
&\qquad+\varepsilon\Big\|\sqrt{\frac{\partial_t w}{w}}A^\sigma f_{\sim N}\Big\|_2\Big\|\sqrt{\frac{\partial_t w}{w}}A^\sigma\chi_R\Delta_L\Delta_t^{-1}P_{\neq} f_N\Big\|_2\Bigg]\\
&\quad+\frac{\varepsilon^2\nu^{\frac23}}{\langle t\rangle^2}\|A^\sigma f_{\sim N}\|_2\big[\|\partial_v h_{\sim N}\|_{H^{\sigma-1}}+\|A^\sigma(\Delta_L\Delta_t^{-1}P_{\neq} f_{\neq})_{\sim N}\|_2\\
&\qquad+\|h_{\sim N}\|_{H^{\sigma-1}}\big].
\end{aligned}\tag{8.23}$$

We end this section by proving Proposition 2.8.

Proof of Proposition 2.8. By (8.19), (8.20), (8.21), (8.23), Lemma 4.1, Proposition 2.3, (A.2) and the bootstrap hypotheses, we get

$$\begin{aligned}
\sum_{N\ge 8}|R_N| &\lesssim \Big(\frac{\varepsilon\nu^{\frac13}}{\langle\nu^{\frac12}t^{\frac32}\rangle}+\frac{\varepsilon\nu^{\frac13}}{\langle t^2\rangle}+\frac{\varepsilon\nu^{\frac13}}{\langle\nu t^3\rangle}\Big)\|f\|_{H^\sigma}\|A^\sigma(\Delta_L\Delta_t^{-1}P_{\neq} f)\|_2\\
&\quad+\varepsilon\Big\|\sqrt{\frac{\partial_t w}{w}}A^\sigma f\Big\|_2\Big\|\sqrt{\frac{\partial_t w}{w}}A^\sigma\chi_R\Delta_L\Delta_t^{-1}P_{\neq} f_N\Big\|_2+\varepsilon\nu^{\frac13}\|A^\sigma f\|_2\|g\|_{H^\sigma}\\
&\quad+\frac{\varepsilon^2\nu^{\frac23}}{\langle t\rangle^2}\|A^\sigma f\|_2\big[\|\partial_v h\|_{H^{\sigma-1}}+\|A^\sigma(\Delta_L\Delta_t^{-1}P_{\neq} f)\|_2+\|h\|_{H^{\sigma-1}}\big].
\end{aligned}$$

By Lemma 4.3 we have

$$\sum_{N\ge 8}|R_N| \lesssim \Big(\frac{\varepsilon\nu^{\frac13}}{\langle \nu^{\frac12}t^{\frac32}\rangle}+\frac{\varepsilon\nu^{\frac13}}{\langle t^2\rangle}+\frac{\varepsilon\nu^{\frac13}}{\langle \nu t^3\rangle}\Big)\|f\|_{H^\sigma}\Big(\|f_{\neq}\|_{H^\sigma}+\frac{\varepsilon\nu^{\frac13}}{\langle t\rangle\langle \nu t^3\rangle}\|\partial_v h\|_{H^\sigma}\Big)$$
$$+\varepsilon\Big\|\sqrt{\frac{\partial_t w}{w}}A^\sigma f\Big\|_2\Big(\Big\|\sqrt{\frac{\partial_t w}{w}}f_{\neq}\Big\|_{H^\sigma}+\frac{\varepsilon^2\nu^{\frac12}}{\langle \nu t^3\rangle}\Big)$$
$$+\varepsilon\nu^{\frac13}\|A^\sigma f\|_2\|g\|_{H^\sigma}+\frac{\varepsilon^2\nu^{\frac23}}{\langle t\rangle^2}\|A^\sigma f\|_2[\|\partial_v h\|_{H^{\sigma-1}}+\|h\|_{H^{\sigma-1}}],$$

and then by the bootstrap hypotheses and the Young's inequality, it holds that

$$\int_1^t\sum_{N\ge 8}|R_N(t')|\,dt' \lesssim \varepsilon\sup_{t'\in[1,t]}\|A^\sigma f(t')\|_2^2+\varepsilon\int_1^t \mathrm{CK}_w(t')\,dt'$$
$$+\varepsilon\nu^{\frac13}\sup_{t'\in[1,t]}\|A^\sigma f(t')\|_2\|\partial_v h\|_{L^2_T(H^\sigma)}\Big(\int_1^t\Big(\frac{\varepsilon\nu^{\frac13}}{\langle t'\rangle\langle \nu t'^3\rangle}\Big)^2dt'\Big)^{\frac12}$$
$$+\varepsilon\Big(\int_1^t \mathrm{CK}_w(t')\,dt'\Big)^{\frac12}\Big(\int_1^t\Big(\frac{\varepsilon^2\nu^{\frac12}}{\langle \nu t'^3\rangle}\Big)^2dt'\Big)^{\frac12}$$
$$+\varepsilon\nu^{\frac13}\sup_{t'\in[1,t]}\|A^\sigma f(t')\|_2\|g\|_{L^1_T(H^\sigma)}+\varepsilon^3\nu\sup_{t'\in[1,t]}\|A^\sigma f(t')\|_2$$
$$+\varepsilon^2\nu^{\frac23}\sup_{t'\in[1,t]}\|A^\sigma f(t')\|_2\|\partial_v h\|_{L^2_T(H^{\sigma-1})}$$
$$\lesssim \varepsilon\sup_{t'\in[1,t]}\|A^\sigma f(t')\|_2^2+\varepsilon\int_1^t \mathrm{CK}_w(t')\,dt'$$
$$+\varepsilon^3\nu^{\frac13}\sup_{t'\in[1,t]}\|A^\sigma f(t')\|_2+\varepsilon^3\nu^{\frac13}\Big(\int_1^t \mathrm{CK}_w(t')\,dt'\Big)^{\frac12}$$
$$\lesssim \varepsilon\sup_{t'\in[1,t]}\|A^\sigma f(t')\|_2^2+\varepsilon\int_1^t \mathrm{CK}_w(t')\,dt'+\varepsilon^5\nu^{\frac23}.$$

Thus we have proved Proposition 2.8. ∎

9. Coordinate system

9.1. Higher regular controls

In this subsection we will study the energy estimate for g in H^σ and $h,\bar h$ in $H^{\sigma-1}$ and H^σ.

9.1.1. Energy estimate of g. In this section we will prove (2.31). We need to mention that the result (2.31) is not optimal; however it is enough. It is natural to compute the time evolution of $\|g\|_{H^\sigma}^2$. We get

$$\begin{aligned}\frac{d}{dt}\|\langle\partial_v\rangle^\sigma g\|_2^2 &= 2\int \langle\partial_v\rangle^\sigma g\langle\partial_v\rangle^\sigma \partial_t g\, dv\\ &= -\frac{4}{t}\|\langle\partial_v\rangle^\sigma g\|_2^2 - 2\int \langle\partial_v\rangle^\sigma g\langle\partial_v\rangle^\sigma (g\partial_v g)\, dv\\ &\quad -\frac{2}{t}\int \langle\partial_v\rangle^\sigma g\langle\partial_v\rangle^\sigma (v'\langle\nabla_{z,v}^\perp P_{\neq}\phi\cdot\nabla_{z,v}\tilde{u}\rangle)\, dv\\ &\quad + 2\nu\int \langle\partial_v\rangle^\sigma g\langle\partial_v\rangle^\sigma ((v')^2\partial_v^2 g)\, dv\\ &= -\frac{4}{t}\|\langle\partial_v\rangle^\sigma g\|_2^2 + V_1^{H,g} + V_2^{H,g} + V_3^{H,g}.\end{aligned}$$

To treat $V_1^{H,g}$, we get by using integration by parts,

$$|V_1^{H,g}| \lesssim \left|\int |\partial_v g|\,|\langle\partial_v\rangle^\sigma g|^2\, dv\right| + \|g\|_{H^\sigma}\|[\langle\partial_v\rangle^\sigma, g]\partial_v g\|_2.$$

By Lemma A.1 and the Sobolev embedding theory, we get

$$|V_1^{H,g}| \lesssim \|g\|_{H^2}\|\langle\partial_v\rangle^\sigma g\|_2^2.$$

Next we treat $V_2^{H,g}$. We now use the fact that

$$\langle\nabla_{z,v}^\perp P_{\neq}\phi\cdot\nabla_{z,v}\tilde{u}\rangle = \langle\nabla_L^\perp P_{\neq}\phi\cdot\nabla_L\tilde{u}_{\neq}\rangle.$$

Then by the bootstrap hypotheses, we get

$$\begin{aligned}|V_2^{H,g}| &\lesssim \frac{2}{t}\int \langle\partial_v\rangle^\sigma g\langle\partial_v\rangle^\sigma [(v'-1)\langle\nabla^\perp P_{\neq}\phi\cdot\nabla\tilde{u}_{\neq}\rangle]\, dv\\ &\quad + \frac{2}{t}\int \langle\partial_v\rangle^\sigma g\langle\partial_v\rangle^\sigma [\langle\nabla_L^\perp P_{\neq}\phi\cdot\nabla_L\tilde{u}_{\neq}\rangle]\, dv\\ &\lesssim \frac{1}{t}\|\langle\partial_v\rangle^\sigma g\|_2\|\langle\partial_v\rangle^\sigma (v'-1)\|_2\|\langle\nabla^\perp P_{\neq}\phi\cdot\nabla\tilde{u}_{\neq}\rangle\|_{H^\sigma}\\ &\quad + \frac{1}{t}\|\langle\partial_v\rangle^\sigma g\|_2\|\langle\partial_v\rangle^\sigma \langle\nabla_L^\perp P_{\neq}\phi\cdot\nabla_L\tilde{u}_{\neq}\rangle\|_2\\ &\lesssim \frac{1}{t}\|\langle\partial_v\rangle^\sigma g\|_2\|h\|_{H^\sigma}\|\nabla_L^\perp P_{\neq}\phi\|_{H^\sigma}\|\nabla_L\tilde{u}_{\neq}\|_{H^2}\\ &\quad + \frac{1}{t}\|\langle\partial_v\rangle^\sigma g\|_2\|h\|_{H^\sigma}\|\nabla_L^\perp P_{\neq}\phi\|_{H^2}\|\nabla_L\tilde{u}_{\neq}\|_{H^\sigma}\\ &\quad + \frac{1}{t}\|\langle\partial_v\rangle^\sigma g\|_2\|\nabla_L^\perp P_{\neq}\phi\|_{H^\sigma}\|\nabla_L\tilde{u}_{\neq}\|_{H^2}\\ &\quad + \frac{1}{t}\|\langle\partial_v\rangle^\sigma g\|_2\|\nabla_L^\perp P_{\neq}\phi\|_{H^2}\|\nabla_L\tilde{u}_{\neq}\|_{H^\sigma}\\ &\lesssim \frac{1}{t}\|\langle\partial_v\rangle^\sigma g\|_2\|\nabla_L^\perp P_{\neq}\phi\|_{H^\sigma}\|\nabla_L\tilde{u}_{\neq}\|_{H^2} + \frac{1}{t}\|\langle\partial_v\rangle^\sigma g\|_2\|\nabla_L^\perp P_{\neq}\phi\|_{H^2}\|\nabla_L\tilde{u}_{\neq}\|_{H^\sigma}.\end{aligned}$$

Finally, we treat the dissipation term $V_3^{H,g}$. We have

$$\begin{aligned}V_3^{H,g} &= 2\nu \int \langle\partial_v\rangle^\sigma g \langle\partial_v\rangle^\sigma (\partial_v^2 g)\, dv + 2\nu \int \langle\partial_v\rangle^\sigma g \langle\partial_v\rangle^\sigma \big(((v')^2-1)\partial_v^2 g\big)\, dv \\ &= -2\nu\|\partial_v\langle\partial_v\rangle^\sigma g\|_2^2 + 2\nu\int \langle\partial_v\rangle^\sigma g \langle\partial_v\rangle^\sigma \big(((v')^2-1)\partial_v^2 g\big)\, dv \\ &= -2\nu\|\partial_v\langle\partial_v\rangle^\sigma g\|_2^2 + V_{3,\varepsilon}^{H,g}.\end{aligned}$$

The term $V_{3,\varepsilon}^{H,g}$ is similar to E^0. We then obtain by Young's inequality that

$$\begin{aligned}|V_{3,\varepsilon}^{H,g}| &\lesssim \nu(1+\|h\|_{H^2})(\|\partial_v g\|_{H^\sigma}^2\|h\|_{H^2} + \|\partial_v h\|_{H^{\sigma-2}}\|\partial_v g\|_{H^\sigma}\|g\|_{H^4}) \qquad (9.1)\\ &\lesssim \nu(1+\|h\|_{H^2})(\|\partial_v g\|_{H^\sigma}^2\|h\|_{H^2} + \|h\|_{H^{\sigma-1}}\|\partial_v g\|_{H^\sigma}^2 + \|\partial_v h\|_{H^{\sigma-2}}\|g\|_{H^4}^2).\end{aligned}$$

By the bootstrap assumption, we get

$$\begin{aligned}\frac{d}{dt}\|g\|_{H^\sigma}^2 \le& -\frac{4}{t}\|g\|_{H^\sigma}^2 - \nu\|\partial_v g\|_{H^\sigma}^2 + C\Big(\|g\|_{H^2}\|g\|_{H^\sigma}^2 \\ &+ \frac{1}{t}\|g\|_{H^\sigma}(\|\nabla_L^\perp P_{\neq}\phi\|_{H^\sigma}\|\nabla_L\tilde{u}_{\neq}\|_{H^2} + \|\nabla_L^\perp P_{\neq}\phi\|_{H^2}\|\nabla_L\tilde{u}_{\neq}\|_{H^\sigma}) \\ &+ \nu\|\partial_v h\|_{H^{\sigma-2}}\|g\|_{H^4}^2\Big) \\ \le& -\frac{4}{t}\|g\|_{H^\sigma}^2 + C\Big[\|g\|_{H^2}\|g\|_{H^\sigma}^2 + \nu\|\partial_v h\|_{H^{\sigma-2}}\|g\|_{H^\sigma}\|g\|_{H^4} \\ &+ \frac{1}{t}\|g\|_{H^\sigma}(\|\Delta_L P_{\neq}\phi\|_{H^\sigma}\|\nabla_L\tilde{u}_{\neq}\|_{H^2} + \|\nabla_L^\perp P_{\neq}\phi\|_{H^2}\|\nabla_L\tilde{u}_{\neq}\|_{H^\sigma})\Big],\end{aligned}$$

which gives that for $t \ge 1$,

$$\begin{aligned}&\sup_{t'\in[1,t]}(t'\|g(t')\|_{H^\sigma}) + \int_1^t \|g(t')\|_{H^\sigma}\, dt' \\ &\le \|g(1)\|_{H^\sigma} + C\Big(\|g\|_{L_T^1H^2}\sup_{t'\in[1,T]} t'\|g(t')\|_{H^\sigma} + \nu^{\frac12}\|\partial_v h\|_{L_T^2(H^{\sigma-2})}\nu^{\frac12}\|tg\|_{L_T^2(H^4)} \\ &\quad + \|f_{\neq}\|_{L_T^\infty(H^\sigma)}\|\nabla_L\tilde{u}_{\neq}\|_{L_T^1H^2} + \Big\|\frac{\varepsilon\nu^{\frac13}}{\langle t\rangle\langle\nu t^3\rangle}\|\partial_v h\|_{H^\sigma}\Big\|_{L_T^1}\|\nabla_L\tilde{u}_{\neq}\|_{L_T^\infty(H^2)} \\ &\quad + \|\nabla_L^\perp P_{\neq}\phi\|_{L_T^1(H^2)}\|f_{\neq}\|_{L_T^\infty H^\sigma} + \|\nabla_L^\perp P_{\neq}\phi\|_{L_T^\infty(H^2)}\Big\|\frac{\varepsilon\nu^{\frac13}}{\langle t\rangle\langle\nu t^3\rangle}\|\partial_v h\|_{H^\sigma}\Big\|_{L_T^1}\Big) \\ &\le \|g(1)\|_{H^\sigma} + C\Big(\|g\|_{L_T^1H^2}\sup_{t'\in[1,T]} t'\|g(t')\|_{H^\sigma} + \nu^{\frac12}\|\partial_v h\|_{L_T^2(H^{\sigma-2})}\nu^{\frac12}\|tg\|_{L_T^2(H^4)} \\ &\quad + \varepsilon\nu^{\frac13}\|\nabla_L\tilde{u}_{\neq}\|_{L_T^1H^2} + \varepsilon\nu^{\frac13}\|\nabla_L^\perp P_{\neq}\phi\|_{L_T^1(H^2)} + \varepsilon^2\nu^{\frac23}\|\partial_v h\|_{L_T^2(H^\sigma)}\Big).\end{aligned}$$

Thus, by the bootstrap hypotheses we get

$$\sup_{t'\in[1,t]}(t'\|g(t')\|_{H^\sigma})+\int_1^t\|g(t')\|_{H^\sigma}\,dt' \\ \le \|g(1)\|_{H^\sigma}+C\Big(\varepsilon\nu^{\frac13}\sup_{t'\in[1,T]}t'\|g(t')\|_{H^\sigma}+\varepsilon^2\nu^{\frac13}\Big).$$

By taking ε small enough, we have proved (2.31).

9.1.2. Energy estimate of $\bar{h}$ and h. We get

$$\begin{aligned}\frac12\frac{d}{dt}\|\bar h\|^2_{H^{\sigma-1}} &= -\int\langle\partial_v\rangle^{\sigma-1}\bar h\langle\partial_v\rangle^{\sigma-1}(g\partial_v\bar h)\,dv-\frac{2}{t}\|\bar h\|^2_{H^{\sigma-1}}\\ &\quad+\frac1t\int\langle\partial_v\rangle^{\sigma-1}\bar h\langle\partial_v\rangle^{\sigma-1}(v'\langle\nabla^\perp_{z,v}P_{\neq}\phi\cdot\nabla_{z,v}f_{\neq}\rangle)\,dv\\ &\quad+\nu\int\langle\partial_v\rangle^{\sigma-1}\bar h\langle\partial_v\rangle^{\sigma-1}\big(((v')^2-1)\partial_{vv}\bar h\big)\,dv-\nu\|\partial_v\bar h\|_{H^{\sigma-1}}\\ &= -\frac2t\|\bar h\|^2_{H^{\sigma-1}}-\nu\|\partial_v\bar h\|_{H^{\sigma-1}}+V_1^{H,\bar h}+V_2^{H,\bar h}+V_3^{H,\bar h}\end{aligned}$$

and

$$\begin{aligned}\frac12\frac{d}{dt}\|h\|^2_{H^{\sigma-1}} &= -\int\langle\partial_v\rangle^{\sigma-1}h\langle\partial_v\rangle^{\sigma-1}(g\partial_v h-\bar h-\nu(v')^2\partial_v^2h)\,dv\\ &= -\int\langle\partial_v\rangle^{\sigma-1}h\langle\partial_v\rangle^{\sigma-1}(g\partial_v h)\,dv\\ &\quad+\int\langle\partial_v\rangle^{\sigma-1}h\langle\partial_v\rangle^{\sigma-1}\bar h\,dv\\ &\quad+\nu\int\langle\partial_v\rangle^{\sigma-1}h\langle\partial_v\rangle^{\sigma-1}\big(((v')^2-1)\partial_v^2h\big)\,dv-\nu\|\partial_v h\|^2_{H^{\sigma-1}}\\ &= -\nu\|\partial_v h\|^2_{H^{\sigma-1}}+V_1^{H,h}+V_2^{H,h}+V_3^{H,h}.\end{aligned}$$

We use the same argument as in the treatment of $V_1^{H,g}$ and get

$$|V_1^{H,\bar h}|\lesssim\|g\|_{H^{\sigma-1}}\|\bar h\|^2_{H^{\sigma-1}}$$

and

$$|V_1^{H,h}|\lesssim\|g\|_{H^{\sigma-1}}\|\bar h\|^2_{H^{\sigma-1}}.$$

We also have

$$|V_2^{H,h}|\le\|h\|_{H^{\sigma-1}}\|\bar h\|_{H^{\sigma-1}}.$$

To treat $V_2^{H,\bar h}$ we have

$$\begin{aligned}V_2^{H,\bar h} &= \frac1t\int\langle\partial_v\rangle^{\sigma-1}\bar h\langle\partial_v\rangle^{\sigma-1}(\langle\nabla^\perp_{z,v}P_{\neq}\phi\cdot\nabla_{z,v}f_{\neq}\rangle)\,dv\\ &\quad+\frac1t\int\langle\partial_v\rangle^{\sigma-1}\bar h\langle\partial_v\rangle^{\sigma-1}(h\langle\nabla^\perp_{z,v}P_{\neq}\phi\cdot\nabla_{z,v}f_{\neq}\rangle)\,dv.\end{aligned}$$

Thus, by the fact that

$$\langle -\partial_v F \partial_z G + \partial_z F \partial_v G \rangle = \partial_v \langle G \partial_z F \rangle,$$

we get

$$\begin{aligned}
|V_2^{H,\bar{h}}| &\lesssim \frac{1}{t}\|\bar{h}\|_{H^{\sigma-1}}(1+\|h\|_{H^2})\|\langle \nabla_{z,v}^{\perp} P_{\neq}\phi \cdot \nabla_{z,v} f_{\neq}\rangle\|_{H^{\sigma-1}} \\
&\quad + \frac{1}{t}\|\bar{h}\|_{H^{\sigma-1}}\|h\|_{H^{\sigma-1}}\|\langle \nabla_{z,v}^{\perp} P_{\neq}\phi \cdot \nabla_{z,v} f_{\neq}\rangle\|_{H^1} \\
&\lesssim \frac{1}{t}\|\bar{h}\|_{H^{\sigma-1}}(1+\|h\|_{H^2})\|\langle \partial_z P_{\neq}\phi f_{\neq}\rangle\|_{H^{\sigma}} \\
&\quad + \frac{1}{t}\|\bar{h}\|_{H^{\sigma-1}}\|h\|_{H^{\sigma-1}}\|\langle \partial_z P_{\neq}\phi f_{\neq}\rangle\|_{H^2} \\
&\lesssim \frac{1}{t}\|\bar{h}\|_{H^{\sigma-1}}(1+\|h\|_{H^2})\|P_{\neq}\phi\|_{H^{\sigma}}\|\partial_z f_{\neq}\|_{H^1} \\
&\quad + \frac{1}{t}\|\bar{h}\|_{H^{\sigma-1}}(1+\|h\|_{H^2})\|\partial_z P_{\neq}\phi\|_{H^1}\|f_{\neq}\|_{H^{\sigma}} \\
&\quad + \frac{1}{t}\|\bar{h}\|_{H^{\sigma-1}}\|h\|_{H^{\sigma-1}}\|\partial_z P_{\neq}\phi\|_{H^2}\|f_{\neq}\|_{H^2}.
\end{aligned}$$

Next we turn to $V_3^{H,\bar{h}}$ and $V_3^{H,h}$:

$$\begin{aligned}
|V_{3,\varepsilon}^{H,\bar{h}}| &\lesssim \int_{\xi,\eta} \langle\eta\rangle^{2\sigma-2}|\hat{\bar{h}}(\eta)|\big|\widehat{(1-(v')^2}(\eta-\xi))|\xi|^2\hat{\bar{h}}(\xi)\big|\,d\xi\,d\eta \\
&\lesssim \int_{\xi,\eta} 1_{|\eta|\le 1}|\hat{\bar{h}}(\eta)|\big|\widehat{(1-(v')^2}(\eta-\xi))|\xi|(|\eta|+|\xi-\eta|)\hat{\bar{h}}(\xi)\big|\,d\eta\,d\xi \\
&\quad + \int_{\xi,\eta} 1_{|\eta|\ge 1}1_{|\xi-\eta|\ge|\xi|}|\eta|^{2\sigma-2}|\hat{\bar{h}}(\eta)|\big|\widehat{(1-(v')^2}(\eta-\xi))|\xi|^2\hat{\bar{h}}(\xi)\big|\,d\xi\,d\eta \\
&\quad + \int_{\xi,\eta} 1_{|\eta|\ge 1}1_{|\xi-\eta|<|\xi|}|\eta|^{2\sigma-2}|\hat{\bar{h}}(\eta)|\big|\widehat{(1-(v')^2}(\eta-\xi))|\xi|^2\hat{\bar{h}}(\xi)\big|\,d\xi\,d\eta \\
&\lesssim \int_{\xi,\eta} 1_{|\eta|\le 1}|\eta|\,|\hat{\bar{h}}(\eta)|\big|\widehat{((1-(v')^2)}(\eta-\xi))|\xi|\hat{f}_0(\xi)\big|\,d\eta\,d\xi \\
&\quad + \int_{\xi,\eta} 1_{|\eta|\le 1}|\hat{\bar{h}}(\eta)|\big|\widehat{(\partial_v(1-(v')^2)}(\eta-\xi))|\xi|\hat{\bar{h}}(\xi)\big|\,d\eta\,d\xi \\
&\quad + \int_{\xi,\eta} 1_{|\eta|\ge 1}1_{|\xi-\eta|<|\xi|}|\eta|^{\sigma}|\hat{\bar{h}}(\eta)|\big|\widehat{(1-(v')^2}(\eta-\xi))|\xi|^{\sigma}\hat{\bar{h}}(\xi)\big|\,d\xi\,d\eta \\
&\quad + \int_{\xi,\eta} 1_{|\eta|\ge 1}1_{|\xi-\eta|\ge|\xi|}|\eta|^{\sigma-1}|\hat{\bar{h}}(\eta)|\big|\widehat{(1-(v')^2}(\eta-\xi))|\eta-\xi|^{\sigma-1}|\xi|^2\hat{\bar{h}}(\xi)\big|\,d\xi\,d\eta \\
&\lesssim \|(1-(v')^2)\|_{H^2}\|\partial_v\bar{h}\|_{H^{\sigma-1}} + \|\bar{h}\|_{H^{\sigma-1}}\|(1-(v')^2)\|_{H^{\sigma-1}}\|\bar{h}\|_{H^4} \\
&\lesssim (1+\|h\|_{H^2})\big(\|h\|_{H^2}\|\partial_v\bar{h}\|_{H^{\sigma-1}} + \|\bar{h}\|_{H^{\sigma-1}}\|\partial_v h\|_{H^{\sigma-2}}\|\bar{h}\|_{H^4}\big)
\end{aligned}$$

and

$$|V_{3,\varepsilon}^{H,h}| \lesssim \nu(1+\|h\|_{H^2})\|\partial_v h\|_{H^{\sigma-1}}^2\|h\|_{H^4}.$$

Putting these together and using the bootstrap assumption, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\bar{h}\|_{H^{\sigma-1}}^2 \leq -\frac{2}{t}\|\bar{h}\|_{H^{\sigma-1}}^2 + C\Big(&\|g\|_{H^2}\|\bar{h}\|_{H^{\sigma-1}}^2 \\ &+ \frac{1}{t}\|\bar{h}\|_{H^{\sigma-1}}(1+\|h\|_{H^2})\|P_{\neq}\phi\|_{H^\sigma}\|\partial_z f_{\neq}\|_{H^1} \\ &+ \frac{1}{t}\|\bar{h}\|_{H^{\sigma-1}}(1+\|h\|_{H^2})\|\partial_z P_{\neq}\phi\|_{H^1}\|f_{\neq}\|_{H^\sigma} \\ &+ \frac{1}{t}\|\bar{h}\|_{H^{\sigma-1}}\|h\|_{H^{\sigma-1}}\|\partial_z P_{\neq}\phi\|_{H^2}\|f_{\neq}\|_{H^2} \\ &+ \nu\|\partial_v h\|_{H^{\sigma-2}}\|\bar{h}\|_{H^{\sigma-1}}\|\bar{h}\|_{H^4}\Big).\end{aligned}$$

We also get

$$\begin{aligned}\frac{d}{dt}(t\|\bar{h}\|_{H^{\sigma-1}}) \leq -\|\bar{h}\|_{H^{\sigma-1}} + C\Big(&t\|g\|_{H^2}\|\bar{h}\|_{H^{\sigma-1}} + \|P_{\neq}\phi\|_{H^\sigma}\|\partial_z f_{\neq}\|_{H^1} \\ &+ \|\partial_z P_{\neq}\phi\|_{H^2}\|f_{\neq}\|_{H^\sigma} + \nu t\|\partial_v h\|_{H^{\sigma-2}}\|\bar{h}\|_{H^4}\Big),\end{aligned}$$

which then implies

$$\begin{aligned}&\sup_{t\in[1,T]}(t\|\bar{h}(t)\|_{H^{\sigma-1}}) + \int_0^T \|\bar{h}(t')\|_{H^{\sigma-1}}\,dt' \\ &\quad\leq \|\bar{h}(1)\|_{H^{\sigma-1}} + C\Big(\|g\|_{L^1_T(H^2)}\sup_{t\in[1,T]}\big(t\|\bar{h}(t)\|_{H^{\sigma-1}}\big) + \|P_{\neq}\phi\|_{L^\infty_T H^\sigma}\|\partial_z f_{\neq}\|_{L^1_T H^1} \\ &\qquad + \|\partial_z P_{\neq}\phi\|_{L^1_T H^2}\|f_{\neq}\|_{L^\infty(H^\sigma)} \\ &\qquad + \nu\|\partial_v h\|_{L^2_T H^{\sigma-2}}\big\|\langle t\rangle\|\bar{h}\|_{H^4}\big\|_{L^2_T}\Big) \\ &\quad\leq \|\bar{h}(1)\|_{H^{\sigma-1}} + C\Big(\varepsilon\nu^{\frac{1}{3}}\sup_{t\in[1,T]}\big(t\|\bar{h}(t)\|_{H^{\sigma-1}}\big) + \varepsilon^2\nu^{\frac{1}{3}} + \varepsilon^2\nu^{\frac{7}{6}}\Big).\end{aligned}$$

Then by taking ε small enough, we get (2.32).

We also get by Young's inequality and the bootstrap hypotheses that

$$\begin{aligned}&\sup_{t\in[1,T]}(\|h(t)\|_{H^{\sigma-1}}^2) + \nu\|\partial_v h\|_{L^2_T H^{\sigma-1}}^2 \\ &\quad\leq \|h(1)\|_{H^{\sigma-1}}^2 + \|\bar{h}\|_{L^1_T(H^{\sigma-1})}\sup_{t\in[1,T]}(\|h(t)\|_{H^{\sigma-1}}) \\ &\qquad + C\|g\|_{L^1_T(H^2)}\sup_{t\in[1,T]}(\|h(t)\|_{H^{\sigma-1}}^2) \\ &\quad\leq \|h(1)\|_{H^{\sigma-1}}^2 + 4\|\bar{h}\|_{L^1_T(H^{\sigma-1})}^2 + \Big(\frac{1}{4} + C\|g\|_{L^1_T(H^2)}\Big)\sup_{t\in[1,T]}(\|h(t)\|_{H^{\sigma-1}}^2) \\ &\quad\leq \|h(1)\|_{H^{\sigma-1}}^2 + 4\|\bar{h}\|_{L^1_T(H^{\sigma-1})}^2 + \frac{3}{8}\sup_{t\in[1,T]}(\|h(t)\|_{H^{\sigma-1}}^2),\end{aligned}$$

which implies (2.33).

9.2. Energy estimates of $\bar{h}$ and h in H^σ

We have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|A^\sigma \bar{h}\|_2^2 \\
&= -\int \frac{\partial_t w(t,\eta)}{w(t,\eta)}\left|\frac{\langle\eta\rangle^\sigma \hat{\bar{h}}(t,\eta)}{w(t,\eta)}\right|^2 d\eta - \frac{2}{t}\|A^\sigma \bar{h}\|_2^2 \\
&\quad \underbrace{-\int A^\sigma \bar{h}[A^\sigma(g\partial_v \bar{h}) - g\partial_v A^\sigma \bar{h}]\,dv}_{V^{\bar{h}}_{1,\sigma}} + \frac{1}{2}\int g'|A^\sigma \bar{h}|^2\,dv \\
&\quad \underbrace{+\frac{1}{t}\sum_{l\neq 0}\int \frac{\langle\eta\rangle^\sigma \hat{\bar{h}}(t,\eta)}{w(t,\eta)}\,\frac{\langle\eta\rangle^\sigma \eta l}{w(t,\eta)}\overline{\hat{\phi}_{-l}(\eta-\xi)\,\hat{f}_l(\xi)}\,d\xi\,d\eta}_{V^{\bar{h}}_{2,\sigma}} \\
&\quad \underbrace{+\frac{1}{t}\sum_{l\neq 0}\int \frac{\langle\eta\rangle^\sigma \overline{\hat{\bar{h}}(t,\eta)}}{w(t,\eta)}\,\frac{\langle\eta\rangle^\sigma (\eta-\xi')l}{w(t,\eta)}\hat{h}(\xi')\hat{\phi}_{-l}(\eta-\xi)\,\hat{f}_l(\xi-\xi')\,d\xi'\,d\xi\,d\eta}_{V^{\bar{h},\varepsilon}_{2,\sigma}} \\
&\quad -\nu\|\partial_v A^\sigma \bar{h}\|_2^2 + \underbrace{\nu\int A^\sigma \bar{h} A^\sigma\big(((v')^2-1)\partial_{vv}\bar{h}\big)\,dv}_{V^{\bar{h}}_{3,\sigma}} \\
&= -\mathrm{CK}^{\bar{h}}_w - \frac{2}{t}\|A^\sigma \bar{h}\|_2^2 - \nu\|\partial_v A^\sigma \bar{h}\|_2^2 + V^{\bar{h}}_{1,\sigma} + V^{\bar{h}}_{2,\sigma} + V^{\bar{h},\varepsilon}_{2,\sigma} + V^{\bar{h}}_{3,\sigma}.
\end{aligned}$$

We then get

$$\begin{aligned}
V^{\bar{h}}_{1,\sigma} &= -\sum_{M\geq 8}\int A^\sigma \bar{h}[A^\sigma(g_M\partial_v \bar{h}_{<M/8}) - g_M\partial_v A^\sigma \bar{h}_{<M/8}]\,dv \\
&\quad -\sum_{M\geq 8}\int A^\sigma \bar{h}[A^\sigma(g_{<M/8}\partial_v \bar{h}_M) - g_{<M/8}\partial_v A^\sigma \bar{h}_M]\,dv \\
&\quad -\sum_{M\in\mathbb{D}}\sum_{\frac{1}{8}M\leq M'\leq 8M}\int A^\sigma \bar{h}[A^\sigma(g_M\partial_v \bar{h}_{M'}) - g_M\partial_v A^\sigma \bar{h}_{M'}]\,dv \\
&= V^{\bar{h},HL}_{1,\sigma} + V^{\bar{h},LH}_{1,\sigma} + V^{\bar{h},HH}_{1,\sigma}.
\end{aligned}$$

We have

$$\begin{aligned}
|V^{\bar{h},HL}_{1,\sigma}| \lesssim \sum_{M\geq 8}\int A^\sigma \hat{\bar{h}}(\eta)\big[&A^\sigma(\hat{g}(\xi)_M\widehat{\partial_v \bar{h}}(\eta-\xi)_{<M/8}) \\
&- \hat{g}(\xi)_M A^\sigma\widehat{\partial_v \bar{h}}(\eta-\xi)_{<M/8}\big]\,d\xi\,d\eta
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{M\geq 8}\int A^\sigma \hat{\bar{h}}(\eta)\langle\xi\rangle^\sigma \hat{g}(\xi)_M \widehat{\partial_v \bar{h}}(\eta-\xi)_{<M/8}\,d\xi\,d\eta\\
&\lesssim \sum_{M\geq 8}\|A^\sigma \bar{h}_{\sim M}\|_2\|g_M\|_{H^\sigma}\|\partial_v\bar{h}\|_{H^1}\\
&\lesssim \|A^\sigma\bar{h}\|_2\|g\|_{H^\sigma}\|\partial_v\bar{h}\|_{H^1}.
\end{aligned}
$$

Next we treat $V_{1,\sigma}^{\bar{h},LH}+V_{1,\sigma}$ and we get

$$
\begin{aligned}
|V_{1,\sigma}^{\bar{h},HL}| \lesssim &\sum_{M\geq 8}\int_{|\xi-\eta|\leq\frac{1}{10}|\eta|} A^\sigma\hat{\bar{h}}(\eta)\big[A^\sigma(\hat{g}(\eta-\xi)_{<M/8}\widehat{\partial_v\bar{h}}(\xi)_M)\\
&\qquad\qquad -\hat{g}(\eta-\xi)_{<M/8}A^\sigma\widehat{\partial_v\bar{h}}(\xi)_M\big]\,d\xi\,d\eta\\
&+\sum_{M\geq 8}\int_{|\xi-\eta|\geq\frac{1}{10}|\eta|} A^\sigma\hat{\bar{h}}(\eta)\big[A^\sigma(\hat{g}(\eta-\xi)_{<M/8}\widehat{\partial_v\bar{h}}(\xi)_M)\\
&\qquad\qquad -\hat{g}(\eta-\xi)_{<M/8}A^\sigma\widehat{\partial_v\bar{h}}(\xi)_M\big]\,d\xi\,d\eta.
\end{aligned}
$$

For the first term, by Lemma 6.1 we have $|\xi|\approx|\eta|$, and then

$$
\begin{aligned}
&\Big|\sum_{M\geq 8}\int_{|\xi-\eta|\leq\frac{1}{10}|\eta|} A^\sigma\hat{\bar{h}}(\eta)\big[A^\sigma(\hat{g}(\eta-\xi)_{<M/8}\widehat{\partial_v\bar{h}}(\xi)_M)\\
&\qquad\qquad -\hat{g}(\eta-\xi)_{<M/8}A^\sigma\widehat{\partial_v\bar{h}}(\xi)_M\big]\,d\xi\,d\eta\Big|\\
&\lesssim \sum_{M\geq 8}\int_{|\xi-\eta|\leq\frac{1}{10}|\eta|} A^\sigma|\hat{\bar{h}}(\eta)|\Big|\frac{\langle\eta\rangle\eta}{w(t,\eta)}-\frac{\langle\xi\rangle\xi}{w(t,\xi)}\Big||\hat{g}(\eta-\xi)_{<M/8}\hat{\bar{h}}(\xi)_M|\,d\xi\,d\eta\\
&\lesssim \sum_{M\geq 8}\int_{|\xi-\eta|\leq\frac{1}{10}|\eta|} A^\sigma|\hat{\bar{h}}(\eta)|\,|\langle\eta\rangle^\sigma\eta-\langle\xi\rangle^\sigma\xi|\,|\hat{g}(\eta-\xi)_{<M/8}\hat{\bar{h}}(\xi)_M|\,d\xi\,d\eta\\
&\quad+\sum_{M\geq 8}\int_{|\xi-\eta|\leq\frac{1}{10}|\eta|} A^\sigma|\hat{\bar{h}}(\eta)|\Big|\frac{1}{w(t,\eta)}-\frac{1}{w(t,\xi)}\Big|\langle\xi\rangle^{\sigma+1}|\hat{g}(\eta-\xi)_{<M/8}\hat{\bar{h}}(\xi)_M|\,d\xi\,d\eta\\
&\lesssim \sum_{M\geq 8}\int_{|\xi-\eta|\leq\frac{1}{10}|\eta|} A^\sigma|\hat{\bar{h}}(\eta)|\langle\eta-\xi\rangle\langle\xi\rangle^\sigma|\hat{g}(\eta-\xi)_{<M/8}\hat{\bar{h}}(\xi)_M|\,d\xi\,d\eta\\
&\quad+\sum_{M\geq 8}\int_{|\xi-\eta|\leq\frac{1}{10}|\eta|} A^\sigma|\hat{\bar{h}}(\eta)|\Big|\frac{1}{w(t,\eta)}-\frac{1}{w(t,\xi)}\Big|\langle\xi\rangle^{\sigma+1}|\hat{g}(\eta-\xi)_{<M/8}\hat{\bar{h}}(\xi)_M|\,d\xi\,d\eta\\
&\lesssim \sum_{M\geq 8}\int_{|\xi-\eta|\leq\frac{1}{10}|\eta|} A^\sigma|\hat{\bar{h}}(\eta)|\langle\eta-\xi\rangle\langle\xi\rangle^\sigma|\hat{g}(\eta-\xi)_{<M/8}\hat{\bar{h}}(\xi)_M|\,d\xi\,d\eta\\
&\quad+\sum_{M\geq 8}\int_{|\xi-\eta|\leq\frac{1}{10}|\eta|} A^\sigma|\hat{\bar{h}}(\eta)|\frac{\langle\eta-\xi\rangle}{\eta}\big(\nu^{-\frac{1}{3}}\chi_{t'\lesssim\nu^{-\frac{1}{3}}}(t')+\nu^{\frac{1}{3}\beta}t'^{1-\beta}\chi_{t'\gtrsim\nu^{-\frac{1}{3}}}(t')\big)\\
&\qquad\qquad\times\langle\xi\rangle^{\sigma+1}|\hat{g}(\eta-\xi)_{<M/8}\hat{\bar{h}}(\xi)_M|\,d\xi\,d\eta
\end{aligned}
$$

$$\lesssim \sum_{M\geq 8} \|A^\sigma \bar{h}_{\sim M}\|_2 \|g_M\|_{H^3} \|\bar{h}\|_{H^\sigma} \big(\nu^{-\frac{1}{3}} \chi_{t'\lesssim \nu^{-\frac{1}{3}}}(t') + \nu^{\frac{1}{3}\beta} t'^{1-\beta} \chi_{t'\gtrsim \nu^{-\frac{1}{3}}}(t')\big)$$
$$\lesssim \|A^\sigma \bar{h}\|_2 \|g\|_{H^3} \|\bar{h}\|_{H^\sigma} \big(\nu^{-\frac{1}{3}} \chi_{t'\lesssim \nu^{-\frac{1}{3}}}(t') + \nu^{\frac{1}{3}\beta} t'^{1-\beta} \chi_{t'\gtrsim \nu^{-\frac{1}{3}}}(t')\big).$$

For the second term, we have $|\xi-\eta| \geq \frac{1}{10}|\eta| \approx |\xi|$, and thus we get

$$\begin{aligned}
&\sum_{M\geq 8}\bigg|\int_{|\xi-\eta|\geq\frac{1}{10}|\eta|} A^\sigma \hat{\bar{h}}(\eta)\big[A^\sigma(\hat{g}(\eta-\xi)_{<M/8}\widehat{\partial_v \bar{h}}(\xi)_M)\\
&\qquad\qquad - \hat{g}(\eta-\xi)_{<M/8} A^\sigma \widehat{\partial_v \bar{h}}(\xi)_M\big]\, d\xi\, d\eta\bigg|\\
&\lesssim \sum_{M\geq 8}\int A^\sigma \hat{\bar{h}}(\eta)\langle\xi\rangle^\sigma \hat{g}(\xi)_M \widehat{\partial_v \bar{h}}(\eta-\xi)_{<M/8}\, d\xi\, d\eta\\
&\lesssim \sum_{M\geq 8} \|A^\sigma \bar{h}_{\sim M}\|_2 \|g_M\|_{H^\sigma} \|\partial_v \bar{h}\|_{H^1} \lesssim \|A^\sigma \bar{h}\|_2 \|g\|_{H^\sigma} \|\partial_v \bar{h}\|_{H^1}.
\end{aligned}$$

Now we turn to $V_{1,\sigma}^{\bar{h},HH}$. We have

$$|\eta| \lesssim |\eta-\xi| + |\xi| \approx |\xi| \approx |\eta-\xi|$$

and then

$$\begin{aligned}
|V_{1,\sigma}^{\bar{h},HH}| &\lesssim \sum_{M\in\mathbb{D}} \|A^\sigma \bar{h}\|_2 \|g_M\|_{H^\sigma} \|\bar{h}_{\sim M}\|_{H^3}\\
&\lesssim \|A^\sigma \bar{h}\|_2 \|g\|_{H^\sigma} \|\bar{h}\|_{H^3}.
\end{aligned}$$

As suggested by Section 8.4, $V_{2,\sigma}^{\bar{h},\varepsilon}$ is not significantly harder than $V_{2,\sigma}^{\bar{h}}$. In fact the primary complications that arise in the treatment of $R_N^{\varepsilon,1}$ do not arise in the treatment of $V_{2,\sigma}^{\bar{h},\varepsilon}$. Hence we focus only on $V_{2,\sigma}^{\bar{h}}$; the control of $V_{2,\sigma}^{\bar{h},\varepsilon}$ results in, at worst, similar contributions with an additional power of ε. Now we turn to $V_{2,\sigma}^{\bar{h}}$. We use the Littlewood–Paley decomposition in v and get

$$\begin{aligned}
V_{2,\sigma}^{\bar{h}} &= \frac{1}{t}\sum_{l\neq 0}\int \chi_{|\eta|\leq 100|l|} \frac{\langle\eta\rangle^\sigma \hat{\bar{h}}(t,\eta)}{w(t,\eta)} \frac{\langle\eta\rangle^\sigma \eta l}{w(t,\eta)} \overline{\hat{\phi}_{-l}(\eta-\xi)\hat{f}_l(\xi)}\, d\xi\, d\eta\\
&\quad + \sum_{M\geq 8}\frac{1}{t}\sum_{l\neq 0}\int \chi_{|\eta|\geq 100|l|} \frac{\langle\eta\rangle^\sigma \hat{\bar{h}}(t,\eta)}{w(t,\eta)} \frac{\langle\eta\rangle^\sigma \eta l}{w(t,\eta)} \overline{\hat{\phi}_{-l}(\eta-\xi)_{<M/8}\hat{f}_l(\xi)_M}\, d\xi\, d\eta\\
&\quad + \sum_{M\geq 8}\frac{1}{t}\sum_{l\neq 0}\int \chi_{|\eta|\geq 100|l|} \frac{\langle\eta\rangle^\sigma \hat{\bar{h}}(t,\eta)}{w(t,\eta)} \frac{\langle\eta\rangle^\sigma \eta l}{w(t,\eta)} \overline{\hat{\phi}_l(\xi)_M \hat{f}_{-l}(\eta-\xi)_{<M/8}}\, d\xi\, d\eta
\end{aligned}$$

$$+\sum_{M\in\mathbb{D}}\sum_{\frac{1}{8}M\le M'\le 8M}\frac{1}{t}\sum_{l\neq 0}\int \chi_{|\eta|\ge 100|l|}\frac{\langle\eta\rangle^\sigma\hat{\bar{h}}(t,\eta)}{w(t,\eta)}\frac{\langle\eta\rangle^\sigma\eta l}{w(t,\eta)}$$
$$\times\overline{\hat{\phi}_{-l}(\eta-\xi)_{M'}\hat{f}_l(\xi)_M}\,d\xi\,d\eta$$
$$=V_{2,\sigma}^{\bar{h},z}+\sum_{M\ge 8}T_M^{\bar{h}}+\sum_{M\ge 8}R_M^{\bar{h}}+\mathcal{R}^{\bar{h}}.$$

It is easy to obtain

$$|V_{2,\sigma}^{\bar{h},z}|\lesssim\frac{1}{t}\|\bar{h}\|_{H^\sigma}\|\phi_{\neq}\|_{H^4}\|f_{\neq}\|_{H^\sigma}\lesssim\frac{1}{t^3}\|\bar{h}\|_{H^\sigma}\|f_{\neq}\|_{H^\sigma}\|f_{\neq}\|_{H^6}.$$

There is a loss of derivative in $T_M^{\bar{h}}$. By using the fact that

$$|\eta|=|(\eta-\xi+lt)|+|(\xi-lt)|\lesssim\langle\eta-\xi+lt\rangle\langle\xi-lt\rangle,$$

we get

$$|T_M^{\bar{h}}|\lesssim\frac{1}{t}\sum_{l\neq 0}\int 1_{|\eta|\ge 100|l|}\frac{\langle\eta\rangle^\sigma|\hat{\bar{h}}(t,\eta)|}{w(t,\eta)}\frac{\langle\xi\rangle^\sigma|l|}{w(t,\xi)}|\langle\eta-\xi+lt\rangle\widehat{\phi}_l(\eta-\xi)_{<M/8}|$$
$$\times|\langle\xi-lt\rangle\hat{f}_{-l}(\xi)_M|\,d\xi\,d\eta$$
$$\lesssim\frac{1}{t}\|A^\sigma\bar{h}_{\sim M}\|_2\left\|\left(\sqrt{-\Delta_L}A^\sigma f_{\neq}\right)_M\right\|_2\|\partial_z\nabla_L\phi_{\neq}\|_{H^2}$$
$$\lesssim\frac{1}{t^2}\|A^\sigma\bar{h}_{\sim M}\|_2\left\|\left(\sqrt{-\Delta_L}A^\sigma f_{\neq}\right)_M\right\|_2\|f_{\neq}\|_{H^3}.$$

For $R_M^{\bar{h}}$ we have

$$|R_M^{\bar{h}}|\lesssim\frac{1}{t}\sum_{l\neq 0}\int 1_{|\eta|\ge 100|l|}|A^\sigma\hat{\bar{h}}(t,\eta)|A^\sigma(t,\xi)\frac{\xi/l^2}{\langle\frac{\xi}{l}-t\rangle^2}\widehat{\Delta_L\Delta_t^{-1}f}_l(\xi)_M$$
$$\times\widehat{\partial_z f}_{-l}(\eta-\xi)_{<M/8}\,d\xi\,d\eta$$
$$\lesssim\frac{1}{t}\sum_{l\neq 0}\int 1_{|\eta|\ge 100|l|}\chi_{|l|\ge\frac{1}{10}\sqrt{|\xi|}}|A^\sigma\hat{\bar{h}}|A^\sigma(t,\xi)\frac{\xi/l^2}{\langle\frac{\xi}{l}-t\rangle^2}\widehat{\Delta_L\Delta_t^{-1}f}_l(\xi)_M$$
$$\times\widehat{\partial_z f}_{-l}(\eta-\xi)_{<M/8}\,d\xi\,d\eta$$
$$+\frac{1}{t}\sum_{l\neq 0}\int 1_{|\eta|\ge 100|l|}1_{|l|\le\frac{1}{10}\sqrt{|\xi|}}[1_{t\in I_{l,\xi}}+1_{t\notin I_{l,\xi}}]|A^\sigma\hat{\bar{h}}|$$
$$\times A^\sigma(t,\xi)\frac{\xi/l^2}{\langle\frac{\xi}{l}-t\rangle^2}\widehat{\Delta_L\Delta_t^{-1}f}_l(\xi)_M\widehat{\partial_z f}_{-l}(\eta-\xi)_{<M/8}\,d\xi\,d\eta$$
$$=R_M^{\bar{h},z}+R_{M,R}^{\bar{h}}+R_{M,NR}^{\bar{h}}.$$

We have

$$|R_M^{\bar{h},z}| \lesssim \frac{1}{t}\|A^\sigma \bar{h}_{\sim M}\|_2 \|A^\sigma(\Delta_L \Delta_t^{-1} f_{\neq})_M\|_2 \|f_{\neq}\|_{H^3}.$$

Next we treat $R^{\bar{h}}_{M,NR}$. We have

$$|R^{\bar{h}}_{M,NR}| \lesssim \frac{1}{t}\|A^\sigma \bar{h}_{\sim M} t\|_2 \|A^\sigma(\Delta_L \Delta_t^{-1} f_{\neq})_M\|_2 \|f_{\neq}\|_{H^3}.$$

If $t \in I_{l,\xi}$ with $|l| \le \frac{1}{10}\sqrt{|\xi|}$, then according to the integrand we get $|\xi - \eta| \le \frac{3}{16}|\xi|$, and thus $t \in I_{k,\eta}$. By Lemma 3.2, we have the following three cases:

(a) $k = l$. We get

$$|R^{\bar{h}}_{M,R}| \lesssim \frac{1}{t}\left\|\sqrt{\frac{\partial_t w}{w}} A^\sigma \bar{h}_{\sim M}\right\|_2 \left\|\sqrt{\frac{\partial_t w}{w}} \chi_R A^\sigma(\Delta_L \Delta_t^{-1} f_{\neq})_M\right\|_2 \|\nu^{-\frac{1}{3}}\langle \nu^{\frac{1}{3}} t\rangle^{1+\beta} f_{\neq}\|_{H^6}.$$

(b) $|\frac{\eta}{k} - t| \gtrsim \frac{\eta}{k^2}$ and $|\frac{\xi}{k} - t| \gtrsim \frac{\xi}{k^2}$. Then the estimate is similar to $R^{\bar{h}}_{M,NR}$ and we get

$$|R^{\bar{h}}_{M,R}| \lesssim \frac{1}{t}\|A^\sigma \bar{h}_{\sim M}\|_2 \|A^\sigma(\Delta_L \Delta_t^{-1} f_{\neq})_M\|_2 \|f_{\neq}\|_{H^3}.$$

(c) $|\eta - \xi| \gtrsim \frac{\xi}{l^2}$. Then we get

$$|R^{\bar{h}}_{M,R}| \lesssim \frac{1}{t}\|A^\sigma \bar{h}_{\sim M} t\|_2 \|A^\sigma(\Delta_L \Delta_t^{-1} f_{\neq})_M\|_2 \|f_{\neq}\|_{H^4}.$$

The remainder term is easy to deal with. We use the fact that $|\eta| \lesssim |\eta - \xi| + |\xi|$ and $|\xi| \approx |\eta - \xi|$ and get

$$\begin{aligned}
|\mathcal{R}^{\bar{h}}| &\lesssim \sum_{M\in\mathbb{D}} \sum_{\frac{1}{8}M \le M' \le 8M} \frac{1}{t}\sum_{l\neq 0}\int \chi_{|\eta|\ge 100|l|} |A^\sigma \hat{\bar{h}}(t,\eta)| \langle\eta\rangle^{\sigma-2} |\hat{\phi}_{-l}(\eta-\xi)_{M'}|\,|\xi|^3|l| \\
&\qquad \times |\hat{f}_l(\xi)_M|\, d\xi\, d\eta \\
&\lesssim \sum_{M\in\mathbb{D}} \frac{1}{t}\|A^\sigma \bar{h}\|_2 \|P_{\neq}\phi_{\sim M}\|_{H^{\sigma-2}} \|(f_{\neq})_M\|_{H^7} \\
&\lesssim \frac{1}{t}\|A^\sigma \bar{h}\|_2 \|P_{\neq}\phi\|_{H^{\sigma-2}} \|f_{\neq}\|_{H^7} \lesssim \frac{1}{t}\|A^\sigma \bar{h}\|_2 \|f_{\neq}\|_{H^{\sigma-2}} \|f_{\neq}\|_{H^7}.
\end{aligned}$$

To treat the dissipation error term $V^{\bar{h}}_{3,\sigma}$, we have

$$\begin{aligned}
|V^{\bar{h}}_{3,\sigma}| &\lesssim \int \langle\eta\rangle^\sigma |\hat{\bar{h}}(\eta)| \langle\eta\rangle^\sigma |G(\eta-\xi)|\,|\xi|^2 |\hat{\bar{h}}(\xi)|\, d\xi\, d\eta \\
&\lesssim \int_{|\eta-\xi|\le|\xi|} 1_{|\eta|\le 1} |\hat{\bar{h}}(\eta)|\,|G(\eta-\xi)|\,|\xi|^2 |\hat{\bar{h}}(\xi)|\, d\xi\, d\eta \\
&\quad + \int_{|\eta-\xi|\le|\xi|} 1_{|\eta|\ge 1} \langle\eta\rangle^\sigma |\hat{\bar{h}}(\eta)| \langle\eta\rangle^\sigma |G(\eta-\xi)|\,|\xi|^2 |\hat{\bar{h}}(\xi)|\, d\xi\, d\eta \\
&\quad + \int_{|\eta-\xi|\ge|\xi|} 1_{|\eta|\ge 1} \langle\eta\rangle^\sigma |\hat{\bar{h}}(\eta)| \langle\eta\rangle^\sigma |G(\eta-\xi)|\,|\xi|^2 |\hat{\bar{h}}(\xi)|\, d\xi\, d\eta \\
&= V^{\bar{h},<1}_{3,\sigma} + V^{\bar{h},LH}_{3,\sigma} + V^{\bar{h},HL}_{3,\sigma}.
\end{aligned}$$

Now we treat $V_{3,\sigma}^{\bar{h},<1}$:

$$|V_{3,\sigma}^{\bar{h},<1}| \lesssim \nu \|\bar{h}\|_{L^2}\|\bar{h}\|_{H^2}\|G\|_{H^2} \lesssim \nu \|\bar{h}\|_{L^2}\|\bar{h}\|_{H^2}\|h\|_{H^2}.$$

Next we turn to $V_{3,\sigma}^{\bar{h},LH}$, in which case it holds that

$$|\eta| \leq |\eta-\xi| + |\xi| \lesssim |\xi|.$$

Then we have

$$\begin{aligned}V_{3,\sigma}^{\bar{h},LH} &\lesssim \nu \int_{|\eta-\xi|\leq|\xi|} |\eta|\langle\eta\rangle^{\sigma}|\hat{\bar{h}}(\eta)|\langle\xi\rangle^{\sigma-1}|G(\eta-\xi)|\,|\xi|^2|\hat{\bar{h}}(\xi)|\,d\xi\,d\eta \\ &\lesssim \nu\|h\|_{H^3}\|\partial_v A^{\sigma}\bar{h}\|_2^2 \lesssim \varepsilon\nu^{\frac{1}{3}}\nu\|\partial_v A^{\sigma}\bar{h}\|_2^2.\end{aligned}$$

Next we treat $V_{3,\sigma}^{\bar{h},HL}$ and we get

$$\begin{aligned}|V_{3,\sigma}^{\bar{h},HL}| &\lesssim \nu \int_{|\eta-\xi|\geq|\xi|} |\eta|\langle\eta\rangle^{\sigma}|\hat{\bar{h}}(\eta)|\langle\eta-\xi\rangle^{\sigma-1}|G(\eta-\xi)|\,|\xi|^2|\hat{\bar{h}}(\xi)|\,d\xi\,d\eta \\ &\lesssim \nu\|\partial_v\bar{h}\|_{H^{\sigma}}\|h\|_{H^{\sigma-1}}\|\partial_v\bar{h}\|_{H^3}.\end{aligned}$$

Thus we conclude that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|A^{\sigma}\bar{h}\|_2^2 + \mathrm{CK}_w^{\bar{h}} + \frac{2}{t}\|A^{\sigma}\bar{h}\|_2^2 + \frac{1}{2}\nu\|\partial_v A^{\sigma}\bar{h}\|_2^2 \\ &\quad\lesssim \|A^{\sigma}\bar{h}\|_2^2\|g\|_{H^{\sigma}} + \|A^{\sigma}\bar{h}\|_2^2\|g\|_{H^3}\big(\nu^{-\frac{1}{3}}\chi_{t'\lesssim\nu^{-\frac{1}{3}}}(t') + \nu^{\frac{1}{3}\beta}t'^{1-\beta}\chi_{t'\gtrsim\nu^{-\frac{1}{3}}}(t')\big) \\ &\qquad + \frac{1}{t}\|\bar{h}\|_{H^{\sigma}}\|f_{\neq}\|_{H^{\sigma}}\|f_{\neq}\|_{H^7} + \frac{1}{t^2}\|A^{\sigma}\bar{h}\|_2\big\|\big(\sqrt{-\Delta_L}A^{\sigma}f_{\neq}\big)\big\|_2\|f_{\neq}\|_{H^3} \\ &\qquad + \frac{1}{t}\|A^{\sigma}\bar{h}\|_2\|A^{\sigma}(\Delta_L\Delta_t^{-1}f_{\neq})\|_2\|f_{\neq}\|_{H^3} \\ &\qquad + \frac{1}{t}\left\|\sqrt{\frac{\partial_t w}{w}}A^{\sigma}\bar{h}\right\|_2\left\|\sqrt{\frac{\partial_t w}{w}}\chi_R A^{\sigma}(\Delta_L\Delta_t^{-1}f_{\neq})\right\|_2\|\nu^{-\frac{1}{3}}\langle\nu^{\frac{1}{3}}t\rangle^{1+\beta}f_{\neq}\|_{H^6} \\ &\qquad + \nu\|\bar{h}\|_{L^2}\|\bar{h}\|_{H^2}\|h\|_{H^2},\end{aligned}$$

which gives by the bootstrap hypotheses that

$$\begin{aligned}&(t^{\frac{3}{2}}\|A^{\sigma}\bar{h}(t)\|_2)^2 + \int_1^t t'^3\mathrm{CK}_w^{\bar{h}}\,dt' + \frac{1}{2}\int_1^t t'^2\|A^{\sigma}\bar{h}(t')\|_2^2\,dt' \\ &\qquad + \frac{1}{2}\nu\int_1^t t'^3\|\partial_v A^{\sigma}\bar{h}(t')\|_2^2\,dt' \\ &\quad\leq \|A^{\sigma}\bar{h}(1)\|_2^2 \\ &\qquad + C\left[\|g\|_{L_T^1(H^{\sigma})} + \int_1^t \frac{\varepsilon\nu^{\frac{1}{3}}}{t'^2}\big(\nu^{-\frac{1}{3}}\chi_{t'\lesssim\nu^{-\frac{1}{3}}}(t') + \nu^{\frac{1}{3}\beta}t'^{1-\beta}\chi_{t'\gtrsim\nu^{-\frac{1}{3}}}(t')\big)\,dt'\right] \\ &\qquad \times \|t^{\frac{3}{2}}\|A^{\sigma}\bar{h}(t)\|_2\|_{L_T^{\infty}}^2\end{aligned}$$

$$
\begin{aligned}
&+ C\int_1^t \frac{\varepsilon^2 \nu^{\frac{2}{3}} t^{\frac{1}{2}}}{\langle \nu t'^3\rangle}\, dt' \|t^{\frac{3}{2}}\|A^\sigma \bar{h}(t)\|_2\|_{L^\infty_T} \\
&+ C\left\|t^{\frac{3}{2}}\|A^\sigma \bar{h}(t)\|_2\right\|_{L^\infty_T}\left(\int_1^t \left[\frac{\varepsilon t'^{\frac{1}{2}}}{t'}\frac{\nu^{-\frac{1}{6}}}{\langle \nu t'^3\rangle}\right]^2 dt'\right)^{\frac{1}{2}} \nu^{\frac{1}{2}}\|\sqrt{-\Delta_L} f_{\neq}\|_{L^2_T(H^\sigma)} \\
&+ C\left\|t^{\frac{3}{2}}\|A^\sigma \bar{h}(t)\|_2\right\|_{L^\infty_T}\left(\int_1^t \left[t'^{\frac{1}{2}}\frac{\varepsilon \nu^{-\frac{1}{6}}}{\langle t\rangle\langle \nu t^3\rangle}\right]^2 dt'\right)^{\frac{1}{2}} \nu^{\frac{1}{2}}\|\partial_v h(t')\|_{L^2_T H^\sigma} \\
&+ C\left\|t'^{\frac{3}{2}}\sqrt{\frac{\partial_t w}{w}} A^\sigma \bar{h}(t')\right\|_{L^2_T(L^2)}\left\|\sqrt{\frac{\partial_t w}{w}} f_{\neq}\right\|_{L^2_T(H^\sigma)}\left\|\frac{\varepsilon^2 t^{\frac{1}{2}}\langle \nu^{\frac{1}{3}} t\rangle^{1+\beta}}{\langle \nu t^3\rangle}\right\|_{L^\infty} \\
&+ C\left\|t'^{\frac{3}{2}}\sqrt{\frac{\partial_t w}{w}} A^\sigma \bar{h}(t')\right\|_{L^2_T(L^2)}\left\|\frac{\varepsilon^3 \nu^{\frac{1}{2}} t^{\frac{1}{2}}}{\langle \nu t^3\rangle}\right\|_{L^2_T} \\
\le{}& \|A^\sigma \bar{h}(1)\|_2^2 + \frac{1}{100}\left\|t^{\frac{3}{2}}\|A^\sigma \bar{h}(t)\|_2\right\|^2_{L^\infty_T} + \frac{1}{100}\int_1^t t'^3 \mathrm{CK}_w^{\bar{h}}\, dt' + C\varepsilon^4 \nu^{\frac{1}{3}}.
\end{aligned}
$$

The energy for h in H^σ is similar to the estimate of $H^{\sigma-1}$. We have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|h\|^2_{H^\sigma} &= -\int \langle\partial_v\rangle^\sigma h \langle\partial_v\rangle^\sigma \big(g\partial_v h - \bar{h} - \nu (v')^2 \partial_v^2 h\big)\, dv \\
&= -\int \langle\partial_v\rangle^\sigma h \langle\partial_v\rangle^\sigma (g\partial_v h)\, dv \\
&\quad + \int \langle\partial_v\rangle^\sigma h \langle\partial_v\rangle^\sigma \bar{h}\, dv \\
&\quad + \nu \int \langle\partial_v\rangle^\sigma h \langle\partial_v\rangle^\sigma \big(((v')^2-1)\partial_v^2 h\big)\, dv - \nu\|\partial_v h\|^2_{H^\sigma} \\
&= -\nu\|\partial_v h\|^2_{H^\sigma} + V^{H,h}_{1,\sigma} + V^{H,h}_{2,\sigma} + V^{H,h}_{3,\sigma}.
\end{aligned}
$$

We get

$$
|V^{H,h}_{1,\sigma}| \lesssim \|g\|_{H^\sigma}\|h\|^2_{H^\sigma}, \quad |V^{H,h}_{2,\sigma}| \lesssim \|h\|_{H^\sigma}\|\bar{h}\|_{H^\sigma}, \quad |V^{H,h}_{3,\sigma}| \lesssim \varepsilon\nu^{\frac{1}{3}}\nu\|\partial_v h\|^2_{H^\sigma}.
$$

Thus we conclude that

$$
\begin{aligned}
\|h(t)\|^2_{H^\sigma} + \nu\|\partial_v h\|^2_{L^2_T(H^\sigma)} &\le \|h(1)\|^2_{H^\sigma} + C\|g\|_{L^1_T H^\sigma}\|h\|^2_{L^\infty_T H^\sigma} \\
&\quad + C\|h(1)\|_{H^\sigma}\|\bar{h}\|_{L^1_T(H^\sigma)} \\
&\le \|h(1)\|^2_{H^\sigma} + \frac{1}{100}\|h\|^2_{L^\infty_T H^\sigma} + C\varepsilon^3\nu^{\frac{1}{3}}.
\end{aligned}
$$

9.3. Lower energy estimate

9.3.1. Energy estimate of g in $H^{\sigma-6}$. We have

$$
\begin{aligned}
\frac{d}{dt}(t^4\|\langle\partial_v\rangle^{\sigma-6} g\|_2^2) &= (4)t^3\|\langle\partial_v\rangle^{\sigma-6} g\|_2^2 + t^4\frac{d}{dt}\|\langle\partial_v\rangle^{\sigma-6} g\|_2^2 \\
&= (4)t^3\|\langle\partial_v\rangle^{\sigma-6} g\|_2^2 + 2t^4\int \langle\partial_v\rangle^{\sigma-6} g \langle\partial_v\rangle^{\sigma-6}\partial_t g\, dv
\end{aligned}
$$

$$
\begin{aligned}
&= -2t^4 \int \langle \partial_v \rangle^{\sigma-6} g \langle \partial_v \rangle^{\sigma-6} (g \partial_v g)\, dv \\
&\quad - 2t^3 \int \langle \partial_v \rangle^{\sigma-6} g \langle \partial_v \rangle^{\sigma-6} (v' \langle \nabla_{z,v}^{\perp} P_{\neq} \phi \cdot \nabla_{z,v} \tilde{u} \rangle)\, dv \\
&\quad + 2t^4 \nu \int \langle \partial_v \rangle^{\sigma-6} g \langle \partial_v \rangle^{\sigma-6} \big(((v')^2 - 1) \partial_v^2 g \big)\, dv - 2t^4 \nu \|\partial_v g\|_{H^{\sigma-6}}^2 \\
&= -2t^4 \nu \|\partial_v g\|_{H^{\sigma-6}}^2 + V_1^{L,g} + V_2^{L,g} + V_3^{L,g}.
\end{aligned}
$$

As before we deal with $V_1^{L,g}$ by commutator estimate and integration by parts:

$$
\begin{aligned}
|V_1^{L,g}| &\lesssim \left| t^4 \int \partial_v g |\langle \partial_v \rangle^{\sigma-6} g|^2\, dv \right| + \left| t^4 \int \langle \partial_v \rangle^{\sigma-6} g [\langle \partial_v \rangle^{\sigma-6}, g] \partial_v g\, dv \right| \\
&\lesssim t^4 \|g\|_{H^3} \|g\|_{H^{\sigma-6}}^2.
\end{aligned}
$$

For $V_2^{L,g}$, by the fact that $\langle \nabla_{z,v}^{\perp} P_{\neq} \phi \cdot \nabla_{z,v} \tilde{u} \rangle = \partial_v \langle \partial_z P_{\neq} \phi \tilde{u} \rangle$, we get

$$
\begin{aligned}
|V_2^{L,g}| &\lesssim t^3 \|g\|_{H^{\sigma-6}} \big(\|((v'-1) \langle \nabla_{z,v}^{\perp} P_{\neq} \phi \cdot \nabla_{z,v} \tilde{u} \rangle)\|_{H^{\sigma-6}}^2 \\
&\qquad + \|\langle \nabla_{z,v}^{\perp} P_{\neq} \phi \cdot \nabla_{z,v} \tilde{u} \rangle\|_{H^{\sigma-6}}^2 \big) \\
&\lesssim t^3 \|g\|_{H^{\sigma-6}} (1 + \|h\|_{H^{\sigma-6}}) \|\langle \partial_z P_{\neq} \phi \tilde{u} \rangle\|_{H^{\sigma-5}} \\
&\lesssim t^3 \|g\|_{H^{\sigma-6}} (1 + \|h\|_{H^{\sigma-6}}) \|P_{\neq} \phi\|_{H^{\sigma-4}} \|\tilde{u}\|_{H^{\sigma-5}}.
\end{aligned}
$$

Then by the bootstrap assumption and Lemma 4.1, we get

$$
|V_2^{L,g}| \lesssim \|g\|_{H^{\sigma-6}} \|f_{\neq}\|_{H^{\sigma-2}} \|f_{\neq}\|_{H^{\sigma-4}}.
$$

For the dissipation error term we have

$$
\begin{aligned}
|V_3^{L,g}| &\lesssim \left| t^4 \nu \int \langle \partial_v \rangle^{\sigma-6} g \langle \partial_v \rangle^{\sigma-6} \big(((v')^2 - 1) \partial_v^2 g \big)\, dv \right| \\
&\lesssim t^4 \nu \|((v')^2 - 1)\|_{H^{\sigma-6}} \|\partial_v g\|_{H^{\sigma-6}}^2 \\
&\quad + t^4 \nu \|\partial_v ((v')^2 - 1)\|_2 \|\partial_v g\|_2 \|g\|_{H^1} + t^4 \nu \|((v')^2 - 1)\|_2 \|\partial_v g\|_2^2 \\
&\lesssim t^4 \nu (1 + \|h\|_{H^2}) \|h\|_{H^{\sigma-6}} \|\partial_v g\|_{H^{\sigma-6}}^2 \\
&\quad + t^4 \nu (1 + \|h\|_{H^2}) \|\partial_v h\|_{H^2} \|\partial_v g\|_{H^2} \|g\|_{H^1}.
\end{aligned}
$$

Thus, by the bootstrap assumption we get

$$
\begin{aligned}
&\sup_{t \in [1,T]} t^4 \|g(t)\|_{H^{\sigma-6}}^2 + \nu \int_1^T t'^4 \|\partial_v g(t')\|_{H^{\sigma-6}}^2\, dt' \\
&\quad \le \|g(1)\|_{H^{\sigma-6}}^2 + C \Bigg(\|g\|_{L_T^1(H^{\sigma-6})} \sup_{t \in [1,T]} t^4 \|g(t)\|_{H^{\sigma-6}}^2 \\
&\qquad\qquad + \sup_{t' \in [1,t]} t'^2 \|g(t')\|_{H^{\sigma-6}} \int_1^T \frac{\|f_{\neq}\|_{L_T^\infty(H^{\sigma-2})}^2}{t^2}\, dt \\
&\qquad\qquad + \nu \|\partial_v h\|_{L_T^2(H^2)} \|t^2 \partial_v g\|_{L_T^2(H^2)} \sup_{t' \in [1,t]} t'^2 \|g(t')\|_{H^{\sigma-6}} \Bigg)
\end{aligned}
$$

$$\le \|g(1)\|_{H^{\sigma-6}}^2 + C\varepsilon\nu^{\frac{1}{3}} \sup_{t\in[1,T]} t^4\|g(t)\|_{H^{\sigma-6}}^2 + \varepsilon^2\nu^{\frac{2}{3}} \sup_{t'\in[1,t]} t'^2\|g(t')\|_{H^{\sigma-6}}.$$

Thus, by taking ε small enough we have proved (2.35).

9.3.2. Energy estimate of $\bar{h}$ in $H^{\sigma-6}$. The estimate is the same as the estimates of g in lower Sobolev spaces. We have

$$\begin{aligned}
\frac{d}{dt}(t^4\|\langle\partial_v\rangle^{\sigma-6}\bar{h}\|_2^2) &= 4t^3\|\langle\partial_v\rangle^{\sigma-6}\bar{h}\|_2^2 + t^4\frac{d}{dt}\|\langle\partial_v\rangle^{\sigma-6}\bar{h}\|_2^2 \\
&= (4)t^3\|\langle\partial_v\rangle^{\sigma-6}\bar{h}\|_2^2 + 2t^4\int\langle\partial_v\rangle^{\sigma-6}\bar{h}\langle\partial_v\rangle^{\sigma-6}\partial_t\bar{h}\,dv \\
&= -2t^4\int\langle\partial_v\rangle^{\sigma-6}\bar{h}\langle\partial_v\rangle^{\sigma-6}(g\partial_v\bar{h})\,dv \\
&\quad - 2t^3\int\langle\partial_v\rangle^{\sigma-6}\bar{h}\langle\partial_v\rangle^{\sigma-6}(v'\langle\nabla_{z,v}^{\perp}P_{\neq}\phi\cdot\nabla_{z,v}f\rangle)\,dv \\
&\quad + 2t^4\nu\int\langle\partial_v\rangle^{\sigma-6}\bar{h}\langle\partial_v\rangle^{\sigma-6}\big(((v')^2-1)\partial_v^2\bar{h}\big)\,dv - 2t^4\nu\|\partial_v\bar{h}\|_{H^{\sigma-6}}^2 \\
&= -2t^4\nu\|\partial_v\bar{h}\|_{H^{\sigma-6}}^2 + V_1^{L,\bar{h}} + V_2^{L,\bar{h}} + V_3^{L,\bar{h}}.
\end{aligned}$$

Then we have

$$|V_1^{L,\bar{h}}| \lesssim \|g\|_{H^{\sigma-6}}\|t^2\bar{h}\|_{H^{\sigma-6}}^2.$$

For $V_2^{L,\bar{h}}$, we get by Lemma 4.1 that

$$\begin{aligned}
|V_2^{L,\bar{h}}| &\lesssim t\|t^2\bar{h}\|_{H^{\sigma-6}}(1+\|h\|_{H^{\sigma-6}})\|\langle\partial_z P_{\neq}\phi f_{\neq}\rangle\|_{H^{\sigma-5}} \\
&\lesssim t\|t^2\bar{h}\|_{H^{\sigma-6}}(1+\|h\|_{H^{\sigma-6}})\|P_{\neq}\phi\|_{H^{\sigma-6}}\|f_{\neq}\|_{H^2} \\
&\quad + t\|t^2\bar{h}\|_{H^{\sigma-6}}(1+\|h\|_{H^{\sigma-6}})\|P_{\neq}\phi\|_{H^3}\|f_{\neq}\|_{H^{\sigma-5}} \\
&\lesssim t^{-1}\|t^2\bar{h}\|_{H^{\sigma-6}}(1+\|h\|_{H^{\sigma-6}})\|P_{\neq}f\|_{H^{\sigma-4}}\|f_{\neq}\|_{H^2} \\
&\quad + t^{-1}\|t^2\bar{h}\|_{H^{\sigma-6}}(1+\|h\|_{H^{\sigma-6}})\|P_{\neq}f\|_{H^5}\|f_{\neq}\|_{H^{\sigma-5}}.
\end{aligned}$$

Finally, for the dissipation error term we have

$$\begin{aligned}
|V_3^{L,\bar{h}}| &\lesssim t^4\nu(1+\|h\|_{H^2})\|h\|_{H^{\sigma-6}}\|\partial_v\bar{h}\|_{H^{\sigma-6}}^2 \\
&\quad + t^4\nu(1+\|h\|_{H^2})\|\partial_v h\|_{H^2}\|\partial_v\bar{h}\|_{H^2}\|\bar{h}\|_{H^1}.
\end{aligned}$$

Thus, by the bootstrap hypotheses we obtain

$$\begin{aligned}
&\sup_{t\in[1,T]} t^4\|\bar{h}(t)\|_{H^{\sigma-6}}^2 + \nu\int_1^T t'^4\|\partial_v\bar{h}(t')\|_{H^{\sigma-6}}^2\,dt' \\
&\quad\le \|\bar{h}(1)\|_{H^{\sigma-6}}^2 + C\Big(\varepsilon\nu^{\frac{1}{3}}\sup_{t\in[1,T]} t^4\|\bar{h}(t)\|_{H^{\sigma-6}}^2 + \varepsilon^2\nu^{\frac{1}{3}}\sup_{t\in[1,T]} t^2\|\bar{h}(t)\|_{H^{\sigma-6}}\Big) \\
&\quad\le \|\bar{h}(1)\|_{H^{\sigma-6}}^2 + \frac{1}{100}\sup_{t\in[1,T]} t^4\|\bar{h}(t)\|_{H^{\sigma-6}}^2 + C\varepsilon^4\nu^{\frac{2}{3}}.
\end{aligned}$$

Therefore, by taking ε small enough we have proved (2.36).

10. Decay estimate of vorticity

10.1. Decay estimate of nonzero mode: enhanced dissipation

Up to an adjustment of the constants in the bootstrap argument, it suffices to consider only t such that $\nu t^3 \gtrsim 1$ (say), as otherwise the decay estimate follows trivially from the higher regularity energy estimate.

Recall that

$$\|A_E^s f\|_2^2 = \sum_{k\neq 0}\int_\eta \langle k,\eta\rangle^{2s}|D(t,\eta)\hat{f}_k(t,\eta)|^2\,d\eta.$$

Computing the time evolution of $\|A_E^s f\|_2$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|A_E^s f\|_2^2 &= \sum_{k\neq 0}\int_\eta \frac{\partial_t D(t,\eta)}{D(t,\eta)}|A_E^s\hat{f}_k(t,\eta)|^2\,d\eta\\ &\quad -\int A_E^s f A_E^s(u\cdot\nabla f)\,dv\,dz + \nu\int A_E^s f A_E^s(\widetilde{\Delta}_t f)\,dv\,dz\\ &\le \frac{1}{8}\nu t^2\|1_{t\ge 2|\eta|}A_E^s\hat{f}_k(t,\eta)\|_2^2\\ &\quad -\int A_E^s f A_E^s(u\cdot\nabla f)\,dv\,dz + \nu\int A_E^s f A_E^s(\widetilde{\Delta}_t f)\,dv\,dz.\end{aligned}$$

We write the dissipation term as

$$\begin{aligned}\nu\int A_E^s f A_E^s(\widetilde{\Delta}_t f)\,dv\,dz &= -\nu\big\|\sqrt{-\Delta_L}A_E^s f\big\|_2^2\\ &\quad -\nu\int A_E^s f A_E^s\big(((v')^2-1)(\partial_v - t\partial_z)^2 f\big)\,dv\,dz\\ &= -\nu\big\|\sqrt{-\Delta_L}A_E^s f\big\|_2^2 + E^\nu.\end{aligned}$$

First, we need to cancel the growing term cause by $D(t,\eta)$. Indeed, we have

$$\begin{aligned}&\frac{1}{8}\nu t^2\|1_{t\ge 2|\eta|}A_E^s\hat{f}_k(t,\eta)\|_2^2 - \nu\big\|\sqrt{-\Delta_L}A_E^s f\big\|_2^2\\ &\quad= \sum_{k\neq 0}\int \nu\Big(\frac{1}{8}t^2 1_{t\ge 2\eta} - k^2 - (\eta - kt)^2\Big)|A_E^s\hat{f}_k(\eta)|^2\,d\eta\\ &\quad\le -\frac{1}{8}\nu\big\|\sqrt{-\Delta_L}A_E^s f\big\|_2^2,\end{aligned}$$

which gives that

$$\frac{1}{2}\frac{d}{dt}\|A_E^s f\|_2^2 \le -\int A_E^s f A_E^s(u\cdot\nabla f)\,dv\,dz - \frac{1}{8}\nu\big\|\sqrt{-\Delta_L}A_E^s f\big\|_2^2 + E^\nu. \tag{10.1}$$

10.1.1. Euler nonlinearity. We first divide into zero and nonzero frequency contributions, as they will be treated differently:

$$\begin{aligned}-\int A_E^s f A_E^s(u\cdot\nabla f)\,dv\,dz &= -\int A_E^s f A_E^s(g\partial_v f)\,dv\,dz\\ &\quad -\int A_E^s f A_E^s(v'\nabla^\perp P_{\neq}\phi\cdot\nabla f)\,dv\,dz\\ &= E_1+E_2.\end{aligned}$$

For E_1 we use the commutator trick and the paraproduct (in both z and v),

$$\begin{aligned}E_1 &= \frac{1}{2}\int \partial_v g|A_E^s f|^2\,dv\,dz+\int A_E^s f[g\partial_v A_E^s f - A_E^s(g\partial_v f)]\,dv\,dz\\ &= \frac{1}{2}\int \partial_v g|A_E^s f|^2\,dv\,dz+\sum_{N\geq 8} T_N^0+\sum_{N\geq 8} R_N^0+\mathcal{R}^0,\end{aligned}$$

where

$$\begin{aligned}T_N^0 &= \int A_E^s f[g_{<N/8}\partial_v A_E^s f_N - A_E^s(g_{<N/8}\partial_v f_N)]\,dv\,dz,\\ R_N^0 &= \int A_E^s f[g_N\partial_v A_E^s f_{<N/8} - A_E^s(g_N\partial_v f_{<N/8})]\,dv\,dz,\\ \mathcal{R}^0 &= \sum_{N\in\mathbb{D}}\sum_{N/8\leq N'\leq 8N}\int A_E^s f[g_{N'}\partial_v A_E^s f_N - A_E^s(g_{N'}\partial_v f_N)]\,dv\,dz.\end{aligned}$$

Treatment of T_N^0. We get

$$\begin{aligned}T_N^0 &= -i\sum_{k\neq 0}\int_{\eta,\xi} A_E^s\bar{\hat{f}}_k(\eta)D(\eta)(\langle k,\eta\rangle^s-\langle k,\xi\rangle^s)\hat{g}(\eta-\xi)_{<N/8}\xi\hat{f}_k(\xi)_N\,d\eta\,d\xi\\ &\quad -i\sum_{k\neq 0}\int_{\eta,\xi} A_E^s\bar{\hat{f}}_k(\eta)\langle k,\xi\rangle^s(D(\eta)-D(\xi))\hat{g}(\eta-\xi)_{<N/8}\xi\hat{f}_k(\xi)_N\,d\eta\,d\xi\\ &= T_N^{0,1}+T_N^{0,2}.\end{aligned}$$

For the term $T_N^{0,1}$, by Lemma 3.4 and the fact that $|k,\eta|\approx|k,\xi|$ and

$$|\langle k,\eta\rangle^s-\langle k,\xi\rangle^s|\lesssim\frac{|\xi-\eta|}{\langle\eta\rangle+\langle\xi\rangle}\langle k,\xi\rangle^s,$$

we have

$$\begin{aligned}|T_N^{0,1}| &\lesssim\sum_{k\neq 0}\int_{\eta,\xi}|A_E^s\bar{\hat{f}}_k(\eta)|\langle\eta-\xi\rangle^4|\hat{g}(\eta-\xi)_{<N/8}|\frac{|\xi|}{\langle\xi\rangle}A_E^s\hat{f}_k(\xi)_N\,d\eta\,d\xi\\ &\lesssim\|A_E^s f_{\sim N}\|_2\|A_E^s f_N\|_2\|g\|_{H^5}.\end{aligned}$$

We turn to $T_N^{0,2}$. By Lemma 3.4 we get

$$|T_N^{0,2}| \lesssim \sum_{k\neq 0}\int_{\eta,\xi} |A_E^s \bar{\hat{f}}_k(\eta)|\langle\eta-\xi\rangle^3|\hat{g}(\eta-\xi)_{<N/8}|\frac{|\xi|}{\langle\xi\rangle}A_E^s\hat{f}_k(\xi)_N\,d\eta\,d\xi$$
$$\lesssim \|A_E^s f_{\sim N}\|_2\|A_E^s f_N\|_2\|g\|_{H^5}.$$

Thus we get by (A.1) and (A.2) that

$$\sum_{N\geq 8}|T_N^0| \leq \|g\|_{H^5}\|A_E^s f\|_2^2. \tag{10.2}$$

Treatment of R_N^0. The "reaction" term R_N^0 is dealt with easily by "moving" the derivative to g. We have

$$\left|\int A_E^s f g_N\partial_v A_E^s f_{<N/8}\,dv\,dz\right| \lesssim \|A_E^s f_{\sim N}\|_2\|g_N\|_{H^2}\|A_E^s f\|_2,$$

and by Lemma 3.4 and the fact that $A_E^s(g_N\partial_v(f)_{<N/8}) = A_E^s(g_N\partial_v(f_{\neq})_{<N/8})$, we get

$$\left|\int A_E^s f A_E^s(g_N\partial_v(f_{\neq})_{<N/8})\,dv\,dz\right|$$
$$\lesssim \left|\sum_k\int_{\eta,\xi}A_E^s|\hat{f}_k(\eta)|\langle\eta-\xi\rangle^3\langle k,\eta\rangle^s|\hat{g}_N(\xi-\eta)|\,|\xi|\,|\hat{f}_k(\xi)_{<N/8}|\,d\eta\,d\xi\right|.$$

On the support of the integrand we have $|k,\eta|\approx|\xi-\eta|\gtrsim|k,\xi|$, and thus

$$\left|\int A_E^s f A_E^s(g_N\partial_v(f_{\neq})_{<N/8})\,dv\,dz\right| \lesssim \|A_E^s f_{\sim N}\|_2\|A_E^s f\|_2\|g_N\|_{H^{s+4}}.$$

The treatment of the remainder terms is similar to the reaction term. We have

$$\mathcal{R}^0 \lesssim \sum_{N\in\mathbb{D}}\|A_E^s f\|_2\|A_E^s f_N\|_2\|g_{\sim N}\|_{H^{s+4}}$$
$$\lesssim \|g\|_{H^{s+4}}\|A_E^s f\|_2^2.$$

Therefore, by the bootstrap hypotheses we conclude that

$$|E_1| \lesssim \|g\|_{H^{s+4}}\|A_E^s f\|_2^2 \leq \frac{\varepsilon\nu^{\frac{1}{3}}}{\langle t\rangle^2}\|A_E^s f\|_2^2. \tag{10.3}$$

Treatment of E_2. Next we turn to E_2. Now we need to use the inviscid damping to obtain decay in time. Roughly speaking, if f is of zero mode, we will land the operator A_E^s on $P_{\neq}\phi$ and use the lossy elliptic estimate for A_E^s.

Thus we get by Lemma 3.5 that

$$|E_2| \lesssim \|A_E^s f\|_2\|A_E^s((v'-1)\nabla^{\perp}P_{\neq}\phi\cdot\nabla f)\|_2 + \|A_E^s f\|_2\|A_E^s(\nabla^{\perp}P_{\neq}\phi\cdot\nabla f)\|_2$$
$$\lesssim \|A_E^s f\|_2(1+\|h\|_{H^{s+3}})\|A_E^s(\nabla^{\perp}P_{\neq}\phi\cdot\nabla f_{\neq})\|_2$$
$$\lesssim \|A_E^s f\|_2(1+\|h\|_{H^{s+3}})\|A_E^s P_{\neq}\phi\|_2\|f\|_{H^{s+5}}$$
$$+\|A_E^s f\|_2^2(1+\|h\|_{H^{s+3}})\|P_{\neq}\phi\|_{H^{s+5}}.$$

Thus, by the bootstrap hypotheses and Lemma 4.4 we have

$$|E_2| \lesssim \frac{\varepsilon\nu^{\frac{1}{3}}}{\langle t\rangle^2}(\|A_E^s f\|_2^2 + \|A^\sigma f\|_2 \|A_E^s f\|_2). \tag{10.4}$$

10.1.2. Dissipation error term. By Lemma 3.4 and the fact that

$$|\xi - kt| \le |\xi-\eta| + |\eta - kt| \le \langle \xi-\eta\rangle\sqrt{k^2+|\eta-kt|^2},$$

we have

$$\begin{aligned}
|E^\nu| &\lesssim \nu \sum_{k\neq 0}\int_{\eta,\xi} \big|A_E^s \hat{f}_k(\eta) A_E^s(k,\eta)\widehat{(1-(v')^2)}(\eta-\xi)|\xi-kt|^2 \hat{f}_k(\xi)\big|\, d\eta\, d\xi \\
&\lesssim \nu \sum_{k\neq 0}\int_{\eta,\xi} \big|A_E^s \sqrt{k^2+|\eta-kt|^2}\hat{f}_k(\eta) A_E^s(k,\eta)\langle\xi-\eta\rangle\widehat{(1-(v')^2)}(\eta-\xi) \\
&\qquad\qquad \times |\xi-kt|\hat{f}_k(\xi)\big|\, d\eta\, d\xi \\
&\lesssim \nu \sum_{k\neq 0}\int_{\eta,\xi} \big|A_E^s \sqrt{k^2+|\eta-kt|^2}\hat{f}_k(\eta)\langle\xi-\eta\rangle^4\widehat{(1-(v')^2)}(\eta-\xi)|\xi-kt| \\
&\qquad\qquad \times A_E^s(k,\xi)\hat{f}_k(\xi)\big|\, d\eta\, d\xi \\
&\lesssim \nu \big\|\sqrt{-\Delta_L}A_E^s f\big\|_2^2 \|(1-(v')^2)\|_{H^6} \lesssim \nu(1+\|h\|_{H^2})\|h\|_{H^6}\big\|\sqrt{-\Delta_L}A_E^s f\big\|_2^2.
\end{aligned}$$

Thus we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|A_E^s f\|_2^2 &\le E_1 + E_2 - \frac{1}{8}\nu\big\|\sqrt{-\Delta_L}A_E^s f\big\|_2^2 + E^\nu \\
&\le \frac{C\varepsilon\nu^{\frac{1}{3}}}{\langle t\rangle^2}\|A_E^s f\|_2^2 + \frac{C\varepsilon\nu^{\frac{1}{3}}}{\langle t\rangle^2}\|A^\sigma f\|_2^2 \\
&\quad - \frac{1}{8}\nu\big\|\sqrt{-\Delta_L}A_E^s f\big\|_2^2 + C\nu\varepsilon\nu^{\frac{1}{3}}\big\|\sqrt{-\Delta_L}A_E^s f\big\|_2^2,
\end{aligned}$$

which gives that

$$\begin{aligned}
&\|A_E^s f(t)\|_2^2 + \int_1^t \frac{1}{5}\nu\big\|\sqrt{-\Delta_L}A_E^s f(t')\big\|_2^2\, dt' \\
&\qquad \le \|A_E^s f(1)\|_2^2 + C\varepsilon\nu^{\frac{1}{3}}\|A_E^s f(t)\|_2^2 + C\varepsilon^3\nu.
\end{aligned}$$

Thus, by taking ε small enough we have proved (2.34).

10.2. Decay of the zero mode

Here we start the proof of (2.37). The zero mode f_0 satisfies

$$\partial_t f_0 + g\partial_v f_0 + v'\langle\nabla_{z,v}^\perp P_{\neq}\phi\cdot\nabla_{z,v} f\rangle - \nu(v')^2\partial_v^2 f_0 = 0. \tag{10.5}$$

We want to prove that the zero mode slightly decays. It is natural to study the time evolution of

$$\mathcal{E}_{L,0}(t) = \|\langle\partial_v\rangle^s f_0\|_2^2 + \frac{t\nu}{2}\|\langle\partial_v\rangle^s\partial_v f_0\|_2^2.$$

We get

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_{L,0}(t) &= \frac{1}{2}\nu\|\langle\partial_v\rangle^s\partial_v f_0\|_2^2 + t\nu\frac{1}{2}\frac{d}{dt}(\|\langle\partial_v\rangle^s\partial_v f_0\|_2^2) + \frac{d}{dt}(\|\langle\partial_v\rangle^s f_0\|_2^2)\\
&= -\frac{3}{2}\nu\|\langle\partial_v\rangle^s\partial_v f_0\|_2^2 - \nu^2 t\|\partial_v^2 f_0\|_{H^s}^2\\
&\quad - \nu t\int\langle\partial_v\rangle^s\partial_v f_0\langle\partial_v\rangle^s\partial_v(g\partial_v f_0)\,dv - 2\int\langle\partial_v\rangle^s f_0\langle\partial_v\rangle^s(g\partial_v f_0)\,dv\\
&\quad - \nu t\int\langle\partial_v\rangle^s\partial_v f_0\langle\partial_v\rangle^s\partial_v(v'\langle\nabla_{z,v}^\perp P_{\neq}\phi\cdot\nabla_{z,v}f\rangle)\,dv\\
&\quad - 2\int\langle\partial_v\rangle^s f_0\langle\partial_v\rangle^s(v'\langle\nabla_{z,v}^\perp P_{\neq}\phi\cdot\nabla_{z,v}f\rangle)\,dv\\
&\quad + \nu^2 t\int\langle\partial_v\rangle^s\partial_v f_0\langle\partial_v\rangle^s\partial_v\big(((v')^2-1)\partial_v^2 f_0\big)\,dv\\
&\quad + 2\nu\int\langle\partial_v\rangle^s f_0\langle\partial_v\rangle^s\big(((v')^2-1)\partial_v^2 f_0\big)\,dv\\
&= -\frac{3}{2}\nu\|\langle\partial_v\rangle^s\partial_v f_0\|_2^2 - \nu^2 t\|\partial_v^2 f_0\|_{H^s}^2\\
&\quad + V_{1,1} + V_{1,2} + V_{2,1} + V_{2,2} + V_{3,1} + V_{3,2}.
\end{aligned}$$

To treat $V_{1,1}$ and $V_{1,2}$, we use the commutator estimate and integration by parts:

$$\begin{aligned}
|V_{1,1}| + |V_{1,2}| &\lesssim \|\partial_v g\|_{L^\infty}(\|f_0\|_{H^s}^2 + \nu t\|\partial_v f_0\|_{H^s}^2)\\
&\quad + \|f_0\|_{H^s}\|[\langle\partial_v\rangle^s, g]\partial_v f_0\|_{L^2} + \nu t\|\partial_v f_0\|_{H^s}\|[\langle\partial_v\rangle^s\partial_v, g]\partial_v f_0\|_{L^2}\\
&\lesssim \|g\|_{H^s}\Big(\|f_0\|_{H^s}^2 + \frac{1}{2}\nu t\|\partial_v f_0\|_{H^s}^2\Big).
\end{aligned}$$

Next we turn to $V_{2,1}, V_{2,2}$: by using the fact that

$$\langle\nabla_{z,v}^\perp P_{\neq}\phi\cdot\nabla_{z,v}f\rangle = \partial_v\langle\partial_z P_{\neq}\phi f_{\neq}\rangle,$$

we get

$$\begin{aligned}
|V_{2,1}| &\lesssim \nu t\|\partial_v f_0\|_{H^s}(\|h\partial_v\langle\partial_z P_{\neq}\phi f_{\neq}\rangle\|_{H^{s+1}} + \|\partial_v\langle\partial_z P_{\neq}\phi f_{\neq}\rangle\|_{H^{s+1}})\\
&\lesssim \nu t(1 + \|h\|_{H^{s+1}})\|\partial_v f_0\|_{H^s}\|P_{\neq}\phi\|_{H^{s+3}}\|f_{\neq}\|_{H^{s+2}}
\end{aligned}$$

and similarly

$$\begin{aligned}
|V_{2,2}| &\lesssim \|f_0\|_{H^s}(\|h\partial_v\langle\partial_z P_{\neq}\phi f_{\neq}\rangle\|_{H^s} + \|\partial_v\langle\partial_z P_{\neq}\phi f_{\neq}\rangle\|_{H^{s+1}})\\
&\lesssim (1 + \|h\|_{H^s})\|f_0\|_{H^s}\|P_{\neq}\phi\|_{H^{s+2}}\|f_{\neq}\|_{H^{s+1}}.
\end{aligned}$$

Finally, we turn to $V_{3,1}, V_{3,2}$: as before we have

$$|V_{3,1}| \lesssim \nu^2 t \|\partial_v^2 f_0\|_{H^s}^2 \|(v')^2 - 1\|_{H^{s+1}} + \nu^2 t \|\partial_v((v')^2 - 1)\|_2 \|\partial_v^2 f_0\|_2 \|\partial_v f_0\|_{H^1}$$

and

$$|V_{3,2}| \lesssim \nu \|\partial_v^2 f_0\|_{H^s}^2 \|(v')^2 - 1\|_{H^s} + \nu \|\partial_v((v')^2 - 1)\|_2 \|\partial_v f_0\|_2 \|f_0\|_{H^1}.$$

Thus, by the bootstrap assumption we get

$$\begin{aligned}
\sup_{t'\in[1,t]} \mathcal{E}_{L,0}(t') &+ \nu \int_1^T \|\partial_v f_0(t)\|_{H^s}^2 \, dt \\
&\le \Big(\|\langle\partial_v\rangle^s f_0(1)\|_2^2 + \frac{\nu}{2}\|\langle\partial_v\rangle^s \partial_v f_0(1)\|_2^2\Big) \\
&\quad + C\Bigg[\|g\|_{L^1_T(H^s)} \sup_{t'\in[1,t]} \mathcal{E}_{L,0}(t') \\
&\qquad + \Big[\sup_{t'\in[1,t]} \mathcal{E}_{L,0}(t')\Big]^{\frac{1}{2}} \|f_{\neq}\|_{L^\infty_T(H^{s+5})}^2 \int_1^T \frac{\sqrt{\nu t} + 1}{t^2} \, dt \\
&\qquad + \nu \|\partial_v h\|_{L^2_T H^1} \big\|\sqrt{\nu t}\,\partial_v^2 f_0\big\|_{L^2_T(L^2)} \big\|\sqrt{\nu t}\,\partial_v f_0\big\|_{L^\infty_T(H^1)} \\
&\qquad + \nu \|\partial_v h\|_{L^2_T(L^2)} \|\partial_v f_0\|_{L^2_T(L^2)} \|f_0\|_{L^\infty_T(H^1)}\Bigg] \\
&\le \Big(\|\langle\partial_v\rangle^s f_0(1)\|_2^2 + \frac{\nu}{2}\|\langle\partial_v\rangle^s \partial_v f_0(1)\|_2^2\Big) + C\varepsilon\nu^{\frac{1}{3}} \sup_{t'\in[1,t]} \mathcal{E}_{L,0}(t') \\
&\quad + C\varepsilon^2 \nu^{\frac{2}{3}} \Big[\sup_{t'\in[1,t]} \mathcal{E}_{L,0}(t')\Big]^{\frac{1}{2}} + C\varepsilon^3 \nu \\
&\le \Big(\|\langle\partial_v\rangle^s f_0(1)\|_2^2 + \frac{\nu}{2}\|\langle\partial_v\rangle^s \partial_v f_0(1)\|_2^2\Big) + C\varepsilon \sup_{t'\in[1,t]} \mathcal{E}_{L,0}(t') + C\varepsilon^3 \nu.
\end{aligned}$$

Thus, by taking ε small enough we have proved (2.37).

A. Functional analysis tools

A.1. Littlewood–Paley decomposition and paraproducts

In this section we fix conventions and notation regarding Fourier analysis, Littlewood–Paley and paraproduct decompositions. See e.g. [1, 11] for more details.

For $f(z, v)$ in the Schwartz space, we define the Fourier transform $\hat{f}_k(\eta)$, where $(k, \eta) \in \mathbb{Z} \times \mathbf{R}$, as

$$\hat{f}_k(\eta) = \frac{1}{2\pi} \int_{\mathbb{T}\times R} f(z, v) e^{-ikz - iv\eta} \, dz \, dv$$

and the Fourier inversion formula is

$$f(z, v) = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \int_{\mathbf{R}} \hat{f}_k(\eta) e^{ikz + iv\eta} \, d\eta.$$

With these definitions we have

$$\int f(z,v)g(z,v)\,dz\,dv = \sum_k \int \hat{f}_k(\eta)\hat{g}_k(\eta)\,d\eta,$$
$$\hat{fg} = \hat{f} * \hat{g}.$$

This work makes heavy use of the Littlewood–Paley dyadic decomposition. Here we fix conventions and review the basic properties of this classical theory; see e.g. [1] for more details. First, we define the Littlewood–Paley decomposition only in the v variable. Let $\psi \in C_0^\infty(\mathbf{R};\mathbf{R})$ be such that $\psi(\xi) = 1$ for $|\xi| \le \frac{1}{2}$ and $\psi(\xi) = 0$ for $|\xi| \ge \frac{3}{4}$, and define $\chi(\xi) = \psi(\frac{\xi}{2}) - \psi(\xi)$ supported in the range $\xi \in (\frac{1}{2}, \frac{3}{2})$. Then we have the partition of unity

$$1 = \psi(\xi) + \sum_{M\in 2^{\mathbb{N}}} \chi_M(\xi),$$

where we mean that the sum runs over the dyadic numbers $M = 1, 2, 4, 8, \ldots, 2^j, \ldots$, and $\chi_M(\xi) = \chi(M^{-1}\xi)$, which has the compact support $\frac{M}{2} \le |\xi| \le \frac{3M}{2}$. For $f \in L^2(\mathbf{R})$, we define

$$f_M = \big(\chi_M(\xi)\hat{f}(\xi)\big)^\vee,$$
$$f_{\frac{1}{2}} = \big(\psi(\xi)\hat{f}(\xi)\big)^\vee,$$
$$f_{<M} = f_{\frac{1}{2}} + \sum_{K\in 2^{\mathbb{N}}, K<M} f_K,$$

which defines the decomposition

$$f = f_{\frac{1}{2}} + \sum_{K\in 2^{\mathbb{N}}} f_K.$$

There holds the almost orthogonality and the approximate projection property

$$\|f\|_2^2 \approx \sum_{K\in\mathbb{D}} \|f_K\|_2^2, \qquad \|f_M\|_2^2 \approx \|(f_M)_M\|_2^2. \tag{A.1}$$

The following is also clear for $M \ge 1$:

$$\|\,|\partial_v| f_M\|_2^2 \approx M\|f_M\|_2^2.$$

We make use of the notation

$$f_{\sim M} = \sum_{K\in\mathbb{D}:\frac{1}{C}M\le K\le CM} f_K,$$

for some constant C that is independent of M. Generally, the exact value of C being used is not important; what is important is that it is finite and independent of M. With this notation we also have

$$\|f\|_2^2 \approx_C \sum_{K\in\mathbb{D}} \|f_{\sim K}\|_2^2. \tag{A.2}$$

During much of the proof we are also working with Littlewood–Paley decompositions defined in the (z, v) variables, with the notation conventions being analogous. Our convention is to use N to denote Littlewood–Paley projections in (z, v) and M to denote projections only in the v direction.

Another key Fourier analysis tool employed in this work is the paraproduct decomposition, introduced by Bony ([11]) (see also [1]). Given suitable functions f, g, we may define the paraproduct decomposition (in either (z, v) or just v),

$$\begin{aligned} fg &= T_f g + T_g f + \mathcal{R}(f, g) \\ &= \sum_{N \geq 8} f_{<N/8} g_N + \sum_{N \geq 8} f_N g_{<N/8} + \sum_{N \in \mathbb{D}} \sum_{N/8 \leq N' \leq 8N} g_{N'} f_N, \end{aligned}$$

where all the sums are understood to run over $\mathbb{D}$. In our work we do not employ the notation in the first line since at most steps in the proof we are forced to explicitly write the sums and treat them term by term anyway. This is due to the fact that we are working in nonstandard regularity spaces and, more crucially, are usually applying multipliers that do not satisfy any version of $AT_f g \approx T_f Ag$. Hence, we have to prove almost everything "from scratch" and can only rely on standard para-differential calculus as a guide.

We also show some product estimates (or Young's inequality) based on Sobolev embedding. It holds for $s > 1$ that

$$\begin{aligned} \|fg\|_{H^s(\mathbb{T}\times R)} &\lesssim \|f\|_{H^s(\mathbb{T}\times R)} \|g\|_{H^s(\mathbb{T}\times R)}, \\ \|\hat{f} * \hat{g}\|_2 &\lesssim \|f\|_2 \|g\|_{H^s(\mathbb{T}\times R)}, \\ \|\hat{f} * \hat{g} * \hat{h}\|_2 &\lesssim \|f\|_2 \|g\|_2 \|h\|_{H^s(\mathbb{T}\times R)}. \end{aligned} \tag{A.3}$$

We end this subsection by introducing the commutator estimate which can be found in [27].

Lemma A.1 ([27]). *Let $J = (1 - \Delta)^{\frac{1}{2}}$; for $1 < p < \infty$ and $s \geq 0$ it holds that*

$$\|J^s(fg) - f(J^s g)\|_p \lesssim_{p,s} \|\nabla f\|_\infty \|J^{s-1} g\|_p + \|J^s f\|_p \|g\|_\infty.$$

A.2. Composition lemma

According to the coordinate transform, we need the following composition lemma.

Lemma A.2. *Suppose that $\gamma > 1$; let $F \in H^\gamma : \mathbb{T} \times \mathbf{R} \to \mathbf{R}$, $G : \mathbb{T} \times \mathbf{R} \to \mathbb{T} \times \mathbf{R}$ be such that $\|\nabla G - I_{2\times 2}\|_{L^\infty} \leq \frac{1}{4}$ and $\nabla G - I_{2\times 2} \in H^\gamma : \mathbb{T} \times \mathbf{R} \to \mathcal{M}_{2\times 2}$. Then there exists $C = C(\|\nabla G - I_{2\times 2}\|_{H^\gamma}, \gamma)$ such that*

$$\|F \circ G\|_{H^\gamma} \leq C \|F\|_{H^\gamma}.$$

Proof. First, we have $\|F \circ G\|_2^2 \approx \|F\|_2^2$. Then by the fact that $\nabla(F \circ G) = [(\nabla F) \circ G] (\nabla G - I_{2\times 2}) + (\nabla F) \circ G$, we have

$$\begin{aligned} \|F \circ G\|_{H^\gamma} &\lesssim \|(\nabla F) \circ G\|_{H^{\gamma-1}} \|(\nabla G - I_{2\times 2})\|_{H^\gamma} + \|(\nabla F) \circ G\|_{H^{\gamma-1}} \\ &\lesssim \|(\nabla F) \circ G\|_{H^{\gamma-1}}. \end{aligned}$$

Let $\gamma = [\gamma] + \{\gamma\}$ with $\{\gamma\} \in [0, 1)$; then by the equivalent definition of the fractional-order Sobolev spaces we get

$$\|F\|_{H^\gamma} \approx \|F\|_{H^{[\gamma]}} + \sum_{\gamma_1+\gamma_2=[\gamma]} \Big(\int_{(\mathbb{T}\times\mathbf{R})^2} \frac{|\partial_x^{\gamma_1}\partial_y^{\gamma_2} F(x_1, y_1) - \partial_x^{\gamma_1}\partial_y^{\gamma_2} F(x_2, y_2)|^2}{((x_1 - x_2)^2 + (y_1 - y_2)^2)^{1+\{\gamma\}}} \, dx_1\, dy_1\, dx_2\, dy_2 \Big)^{\frac{1}{2}}.$$

Therefore, we only need to prove

$$\|F \circ G\|_{H^{\{\gamma\}}} \le C\|F\|_{H^{\{\gamma\}}}.$$

Indeed, we have

$$\begin{aligned}
\|F \circ G\|_{H^{\{\gamma\}}}^2 &\approx \int_{\mathbb{T}\times\mathbf{R}} \int_{\mathbb{T}\times\mathbf{R}} \frac{|F(G(x_1, y_1)) - F(G(x_2, y_2))|^2}{((x_1 - x_2)^2 + (y_1 - y_2)^2)^{1+\{\gamma\}}} \, dx_1\, dy_1\, dx_2\, dy_2 \\
&\lesssim \int_{\mathbb{T}\times\mathbf{R}} \int_{\mathbb{T}\times\mathbf{R}} \frac{|F(G(x_1, y_1)) - F(G(x_2, y_2))|^2}{|G(x_1, y_1) - G(x_1, y_1)|^{2+2\{\gamma\}}} \\
&\qquad \times \frac{|G(x_1, y_1) - G(x_1, y_1)|^{2+2\{\gamma\}}}{((x_1 - x_2)^2 + (y_1 - y_2)^2)^{1+\{\gamma\}}} \, dx_1\, dy_1\, dx_2\, dy_2 \\
&\lesssim \|\nabla G\|_{L^\infty} \int_{\mathbb{T}\times\mathbf{R}} \int_{\mathbb{T}\times\mathbf{R}} \frac{|F(G(x_1, y_1)) - F(G(x_2, y_2))|^2}{|G(x_1, y_1) - G(x_1, y_1)|^{2+2\{\gamma\}}} \, dx_1\, dy_1\, dx_2\, dy_2.
\end{aligned}$$

By the assumption $\|\nabla G - I_{2\times 2}\|_{L^\infty} \le \frac{1}{4}$, we have that $(x, y) \to (z, v) = G(x, y)$ is invertible and thus

$$\begin{aligned}
\|F \circ G\|_{H^{\{\gamma\}}}^2 &\lesssim \|\nabla G\|_{L^\infty} \int_{\mathbb{T}\times\mathbf{R}} \int_{\mathbb{T}\times\mathbf{R}} \frac{|F(z_1, v_1) - F(z_1, v_1)|^2}{((z_1 - z_2)^2 + (v_1 - v_2)^2)^{1+\{\gamma\}}} \, dz_1\, dv_1\, dz_2\, dv_2 \\
&\lesssim (\|\nabla G - I_{2\times 2}\|_{H^2} + 1)\|F\|_{H^{\{\gamma\}}}^2.
\end{aligned}$$

Thus we have proved the lemma. ■

Acknowledgments. The work of N.M. is supported by NSF grant DMS-1716466 and by Tamkeen under the NYU Abu Dhabi Research Institute grant of the center SITE.

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Received 3 September 2019; revised 15 February 2021; accepted 3 May 2021.

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