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Cahn–Hilliard equations governed by weakly nonlocal conservation laws and weakly nonlocal particle interactions

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Abstract. We consider a doubly nonlocal nonlinear parabolic equation which describes phase segregation of a binary system subject to weak-to-weak interactions [Gal, Ann. Inst. H. Poincaré Anal. Non Linéaire 35 (2018)]. The proposed model reduces to the classical Cahn–Hilliard equation under certain conditions. We establish well-posedness results (based on regular and nonregular mild solutions) along with regularity and long-time results in terms of finite-dimensional attractors. Then we also establish the convergence of (certain) mild solutions to single steady states as time goes to infinity. These results are also supplemented by a handful of (two-dimensional) numerical experiments displaying phase-segregation phenomena with interesting interface morphologies, depending on various choices of the interaction kernels (i.e., Gaussian, logarithmic, Riesz and bimodal potentials). We develop a stable numerical scheme which is able to control the computations under the effect of the double nonlinear convolutions.

Contents

1. Introduction

The classical Cahn–Hilliard equation (cCHE) was proposed in the late 1950s as a fundamental model for (isothermal) phase-segregation phenomena in a binary alloy system.

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Since then, it has become equally important not only to material scientists, but also to many other areas of science, describing spinodal decomposition, microstructure formation in materials, image inpainting, multiphase fluid flows, biological aggregations and tumor growth, and the list goes on (see, for instance, [7]). Although the classical form can be formally derived as the conserved dynamics generated by the variational derivative of a (purely local) Ginzburg–Landau free energy, its range of physical applicability is quite *limited* to only a number of applications, in particular when the particle interactions are assumed to be only short ranged. Some recent proposals have further widened the application of these models to other (possibly, yet undiscovered) areas of science. We refer the reader to [17, 18] for a complete discussion of these issues. Among new models of phase segregation, allowing additional flexibility in the choice of particle interaction ([20, 26]) and (general) laws of mass conservation, one can mention both the nonlocal Cahn–Hilliard equation (nCHE) (see [1, 5, 14, 15, 18, 19] for the analytic theory and [23] for analysis of numerical schemes in periodic domains) and the doubly nonlocal Cahn–Hilliard equation (dnCHE) (see [2, 8, 16, 17] for analytic theory). In fact, according to the analysis in [16], the two nonlocal versions of the Cahn–Hilliard equation are very much related since both can be unified into one fundamental equation, in the form

$$\partial_t\phi = A\mu, \quad \mu = -B\phi + F'(\phi) \quad \text{in } (0,\infty)\times\Omega. \tag{1.1}$$

The set Ω is bounded and open in $\mathbb{R}^N$, $\phi \in [-1,1]$ represents the relative difference of the two (material) phases, with ± 1 denoting the pure phases and $\phi \in (-1,1)$ capturing the phase transition in the interfacial regions. Furthermore, A, B are self-adjoint operators in $L^2(\Omega)$, with $-B \geq 0$ describing the particle interaction (at the discrete level, either in the short range or long range, or even both), while the conservation law in (1.1) is found to be determined by either *classical transport* (e.g., $A = \Delta_{\Omega,N}$) or *anomalous transport* (e.g., $A = \mathcal{L}_J$). For the latter, this is better reflected in the choice of $\mathcal{L}_J$, defined as a nonlocal operator,

$$\begin{aligned} \mathcal{L}_J(\mu)(x) &= \text{P.V.} \int_\Omega J(x-y)(\mu(y)-\mu(x))\,dy \\ &= \lim_{\varepsilon\to 0^+} \int_{\Omega\setminus B_\varepsilon(x)} J(x-y)(\mu(y)-\mu(x))\,dy, \end{aligned} \tag{1.2}$$

provided that the limit exists,[1] whenever μ is a measurable function and the probability density $J\colon \mathbb{R}^N \to \mathbb{R}$ is measurable and *symmetric*. Also, F is the density of potential energy which features two local minima at the pure phases ± 1.

In fact, a complete classification of equation (1.1), depending on a proper abstraction of (A, B), is completely given in [16], allowing one to recover even the most popularized form (cCHE) when $A = B = \Delta_{\Omega,N}$. Although the study of (1.1) in [16] also recaptures

[1] The principal value is only necessary when $J \notin L^1(\mathbb{R}^N)$. If $J \in L^1(\mathbb{R}^N)$, it can be dropped since in that case $\mathcal{L}_J$ is a bounded mapping from $L^p(\Omega) \to L^p(\Omega)$.

the nonlocal equations investigated by [2, 8] when both A and B are related to a fractional version of $\Delta_{\Omega,N}$, it also extends beyond these cases when, more generally, either $A = \mathcal{L}_J$ or $B = \mathcal{L}_K$ (for a measurable symmetric function $K: \mathbb{R}^N \to \mathbb{R}$) is associated with definition (1.2).[2] Therefore, in the context of (1.1) when $A = \mathcal{L}_J$, $B = \mathcal{L}_K$, we observe the following cases of physical interest:

(a) The strong-to-weak interaction case when $J \notin L^1(\mathbb{R}^N)$ and $K \in L^1(\mathbb{R}^N)$. We refer to [17] for a complete analysis when F is a polynomial potential, where we establish well-posedness results along with some regularity and long-time results in terms of finite-dimensional attractors and convergence of solutions to (single) steady states.

(b) The weak-to-strong interaction case when $J \in L^1(\mathbb{R}^N)$ and $K \notin L^1(\mathbb{R}^N)$.

(c) The strong-to-strong interaction case when $J, K \notin L^1(\mathbb{R}^N)$. We refer to [16] for a complete analysis when F is a polynomial potential (in some special cases, see also [2, 8].[3]) We refer to the preceding references for precise statements of well-posedness along with some regularity and long-time results in terms of finite-dimensional attractors, and convergence of solutions to (single) steady states.

(d) The weak-to-weak interaction case when $J, K \in L^1(\mathbb{R}^N)$.

So far, both cases (b) and (d) appear to be completely open for study, whereas in this contribution we aim to close this gap in case (d). To this end, we consider the following *doubly nonlocal* system:

$$\partial_t \phi(t,x) = \mathcal{L}_J \mu(t,x), \tag{1.3}$$

$$\mu(t,x) = -\mathcal{L}_K \phi(t,x) + F'(\phi(t,x)), \tag{1.4}$$

for $(t,x) \in (0,\infty) \times \Omega$, with

$$\phi(0,x) = \phi_0(x), \quad x \in \Omega. \tag{1.5}$$

The operator $\mathcal{L}_H$ is bounded, as a mapping from $L^p(\Omega) \to L^p(\Omega)$, provided that $H \in L^1(\mathbb{R}^N)$, and is defined[4] by

$$\mathcal{L}_H(\mu)(x) := \int_\Omega H(x-y)(\mu(y) - \mu(x))\, dy.$$

For the sake of convenience, we also set

$$(H * v)(x) := \int_\Omega H(x-y) v(y)\, dy$$

[2]The (nCHE) is merely a special case of (1.1) when $A = \Delta_{\Omega,N}$ and $B = \mathcal{L}_K$.

[3]Briefly speaking, in that case $K(x) = C_K|x|^{-N-2s}$ and $J(x) = C_J|x|^{-N-2l}$ for $x \neq 0$, for some $s, l \in (0,1)$.

[4]H is either J or K.

and

$$a_H(x) = \int_\Omega H(x-y)\,dy = (H*1)(x).$$

The existence theory for (1.3)–(1.5) follows a different approach from the analytic theories of [16, 17] and no longer can rely on a Galerkin scheme for construction of the approximate solutions. For this reason, the above rigorous gradient flow theories are not directly applicable; however, certain aspects may be recovered, such as when F is generated by a quadratic perturbation of a convex function. This is further complicated by the presence of singular kernels $J, K \in L^1(\mathbb{R}^N)$ (as the fundamental solutions of a PDE) and a nonlinear polynomial function F (including the double well, $\theta s^4 - \theta_c s^2$), which prevent the regularization of L^p-solutions on any time frame. We develop a solution theory of L^∞-integral/mild solutions on any time frame, since it is naturally expected that $\phi_0 \in L^\infty(\Omega)$ from a physical point of view. This approach also allows us to prove the uniqueness of $L^\infty(\Omega)$-mild solutions, along with the validity of the energy identity, in the same class, without any further (essential) assumptions of regularity on J, K. We based it on [17], which provides for refined L^∞-estimates to handle the low spatial regularity of solutions, the presence of the "nonlinear" convolution $\mathcal{L}_J(F'(\phi))$ and the double interaction $\mathcal{L}_J(\mathcal{L}_K(\phi))$, in (1.3)–(1.4).

There are several notions of criticality associated with the general problem (1.1), (1.5). One is a natural *dissipation property*,[5] which roughly translates to whether any energy (nonregular, L^p-) solution of (1.3)–(1.5), in any of the cases (a)–(d), regularizes to a smooth solution on the time frame $(0, T]$, for any $T > 0$. The energy identity plays a major role in such schemes. For problem (1.3)–(1.5), in cases (a), (c), the refined analysis of [16, 17] (see also [2, 8]) suggests this happens naturally due to the smoothing property provided by either one of the diffusion operators A, B (associated with a strongly singular kernel), providing for the desired compactness of energy solutions. In this sense, case (d) appears supercritical with respect to cases (a), (c), since we cannot claim any compactness for the operators $A = \mathcal{L}_J$ and $B = \mathcal{L}_K$, as in the latter cases. This difficulty is further amplified by the growth (at infinity) of the function F, which can no longer be controlled by either one of the nonsmoothing operators A, B in case (d). This brings us to the second notion of criticality for our problem, as a function of the spatial dimension $N \geq 1$, for any arbitrary set $\Omega \subset \mathbb{R}^N$. It roughly corresponds to finding the correct balance between the (singular) diffusion and the nonlinear behavior of $F(s)$ as $|s| \to \infty$. If the initial datum $\phi_0 \in W^{1,p}(\Omega)$, with $p > N$, while the domain Ω satisfies the cone condition,[6] problem (1.3)–(1.5) in case (d) is subcritical in dimension $N = 1$, is critical in dimension $N = 2$, but the dissipation property holds naturally, providing for some Hölder continuity of energy solutions in $C^\alpha(\mathbb{R}_+ \times \overline{\Omega})$ for some $\alpha \in (0, 1)$ (and therefore the desired

[5]We say that a problem is dissipative in some subset $V \subset Y$, where Y is a topological space, endowed with a given metric, if trajectories corresponding to bounded sets of initial data in V will enter V after a certain time, and will stay there forever.

[6]See for instance, the paper by Adams and Fournier [J. Math. Anal. Appl. 61 (1977)].

compactness in L^∞ holds). It is worth emphasizing that such a property is ensured, for instance, by weakly singular kernels $J, K \in W^{1,1}(\mathbb{R}^N)$. On the other hand, the problem becomes supercritical in dimension $N \geq 3$ as the behavior of the nonlinearity $F(s)$, as $|s| \to \infty$, is dominant. We extend the above dissipative and smoothing property in dimension $N \geq 3$, by relying instead on refined energy estimates using *weak L^p-type spaces*, in order to *balance* the strength of the nonlinearity against that of the singularity (near the origin) of $J \in W^{1,1}(\mathbb{R}^N)$. Interestingly, our analysis uncovers that a natural maximal regularity theory in L^p-spaces holds for the aforementioned problem. It appears that such a property has not been observed before for our (nonlinear) nonlocal equation.

Our problem is also formally a gradient flow with respect to the L^2-distance for the free energy

$$\mathcal{E}(t) := \frac{1}{4}\int_\Omega \int_\Omega K(x-y)(\phi(t,x)-\phi(t,y))^2\, dx\, dy + \int_\Omega F\phi(t,x)\, dx.$$

As a general scheme to deal with the above issues, we use the free energy as an important dissipative quantity in our analytical arguments, leading to the existence of finite-dimensional attractors for the problem and the convergence to (single) steady states as time goes to infinity. The Hölder continuity of solutions plays a crucial role in these arguments. We also point out that the free energy has been used by many authors for the same purpose[7]. Our conclusion is that, under *suitable assumptions*[8] on the problem parameters, our case study (d) and the other nonlocal (nCHE) (see [1,5,14,15,18,19]) and (dnCHE) equations (see (a), (c)), are equivalent from a long-term perspective in that their corresponding *steady state* behavior is the same.[9] Thus, under those assumptions, these phase-segregation models appear only different in their transient (temporal) behaviors and corresponding interfacial morphologies in the binary system, as these systems evolve with time. In this contribution, we also only consider initial data in $L^\infty(\Omega)$, for which the free energy turns out to be finite on any time frame, although our arguments may hold in more generality. In the future, it may be possible to further relax the regularity assumptions on J, K, in order to produce the critical properties just described above. The problem of a singular potential F, which satisfies $F'(\pm 1) = \pm\infty$ and $F''(\pm 1) = \infty$, is clearly equally as important (see, for instance, [18] for the case of (nCHE)). It remains unclear to what extent the assumptions on F, and the interaction kernels J, K, single out the doubly nonlocal Cahn–Hilliard equation from other possible equations describing phase-separation phenomena (see also Remark 3.8).

Perhaps a complete study of the interface motions, obtained from these equations in the sharp interface limit, can shed light in further classifying these problems and establish their strong connection to other (yet undiscovered) theories of phase segregation. On that front, it then also becomes a simple fundamental question of whether the classical

[7] We refer the reader to [16] and [18] for additional discussions.

[8] Such as initial conditions and the horizons of the interaction kernels J, K.

[9] We also expect this to hold in case (b). Steady states satisfy $-\mathcal{L}_K(\phi_*) + F'(\phi_*) = \text{const}$, $\bar{\phi}_* = \bar{\phi}_0$.

Cahn–Hilliard equation (cCHE) arises as a suitable approximation of the doubly nonlocal Cahn–Hilliard equations under certain conditions on the problem parameters. This is relevant in particular for the (standard) sharp interface limit problem ([21]) associated[10] with the (cCHE) and (nCHE), which can then be seen as a limit of suitable sharp interface problems, associated with the late stages of the coarsening process in doubly nonlocal Cahn–Hilliard equations. In the future, it would be interesting to investigate the power laws which govern the evolution of the dominant length scale, for each separate model from (a)–(d). Namely, find appropriate upper bounds for coarsening rates and describe how these bounds depend on the parameters of the system, such as temperature, the mean concentration and the choice of interaction kernels J, K.

For a proper nonlinear (approximation, outside the interval $[-1,1]$) function F, possessing double-well features (i.e., $(\pm 1, F(\pm 1))$ are the two local minima), consider the nonlocal problem (1.1), (1.5), with $A = B = \mathcal{L}_{K_\delta}$, for a suitable smooth family of (radially symmetric) kernels

$$\{K_\delta\}_{\delta>0} \subset C(\mathbb{R}^N) \cap L^1_{\mathrm{loc}}(\mathbb{R}^N).$$

More precisely, we let

$$K_\delta(x) = C_K \delta^{-N} K(x\delta^{-1}) \quad \text{with } C_K^{-1} = \frac{1}{2}\int_{B(0,r)} K(x)|x_N|^2\,dx \text{ for some } r > 0.$$

Assuming sufficiently smooth initial data ϕ_0 and a domain Ω of class $C^{4+\beta}$, we establish the convergence of the corresponding (unique) mild solution, satisfying $\phi_\delta(0) = \psi(0) = \phi_0$,

$$\|\phi_\delta - \psi\|_{C([0,T];L^2(\Omega))} \to 0 \quad \text{as } \delta \to 0^+, \text{ for all } T > 0, \tag{1.6}$$

where ψ is a (unique) $C^{4+\beta}$-solution of the (classical) Cahn–Hilliard equation (cCHE) (i.e., (1.1) with $A = B = \Delta_{\Omega,N}$). We note that similar convergence results have also been established between the (cCHE) and the standard nonlocal Cahn–Hilliard equation (nCHE) (for $A = \Delta_{\Omega,N}$, $B = \mathcal{L}_{K_\delta}$, albeit with different assumptions on the kernel and the sequence of initial data[11]), for periodic Ω-domains ([25]) and for other general domains ([9, 10]). The same question of convergence remains open for the binary system in the remaining cases (a)–(c).

A study of a numerical representation of the doubly nonlocal problem is given for bounded two-dimensional domains. Our motivation for this is to illustrate case (d) and to extract some further interesting features about this doubly nonlocal problem. One numerical study, on which we partially base ours, is [4], whereby a difference scheme is presented to examine a weakly nonlocal variant of the Allen–Cahn equation. In [4] the convolution

[10]The sharp (interface) evolution laws for (nCHE) coincide with the ones which can be obtained in analogous limits from the classical CHE (i.e., (cCHE)).

[11]For instance, in [25], $\phi_\delta(0) \in H^1(\Omega)$, with (uniformly) bounded $\mathcal{E}$-energy, is such that $\phi_\delta(0) \to \phi_0 = \psi(0)$ weakly, whereas in our case $\phi_\delta(0) = \phi_0$, but the double-well potential F is truncated outside the interval $[-1, 1]$, such that F has at most quadratic growth at $\pm\infty$.

is expressed naturally as a literal double sum of products communicating the probability of interaction between various elements. It seems that our numerical study is novel to the literature in that case (d) of the doubly nonlocal CHE contains an iteration of (weakly nonlocal) convolution terms. One of the main difficulties with the numerical problem is how to control the computations under the effect of the double convolution. This is described in detail for various different interaction kernels. Indeed, we entertain the Gaussian kernel, a Newtonian (logarithmic) potential, and a Riesz potential, as well as (what we term) a bimodal kernel (a bimodal kernel is similar to a Gaussian, but with two peaks). We achieve stability over the numerical procedure by finding a control over the first iterate of the solution. This is then used to (heuristically) find control constants on each kernel. To not distance our numerical scheme from the assumptions of the principal existence result, we only work with initial data that are faithful to the existence result (cf. Theorems 2.3 and 2.5). Here we have generated initial data with the properties that at each mesh point we assume the value of 1 or -1, and the sum over all such points is zero. Obviously, we are using locally integrable L^1-kernels and L^∞-data. In the four different simulations, the kernel K is fixed as a Gaussian but J varies over the kernels mentioned above. When J and K are Gaussian we also observe how the phase morphology behaves as a function of the interaction (length) scale $0 \leq \delta \leq 1$. Each simulation is initiated using the same initial data so we can see the effect of the different possible long-range interactions being performed in various stages of the calculations. It should be mentioned that due to the very nature of the rough data and (weakly) singular kernels, no further approximations nor fast numerical solvers are employed. This means one convolution is computed with the operational order of $\mathcal{O}(M^2)$ for each iterate, M being the number of spatial one-dimensional subintervals. With N iterations in time, the double convolutions, and the simulations provided here, are on the order of $\mathcal{O}(NM^4)$. The final statement on the results from our numerical study concerns an experiment that represents a departure from the theoretical result in (1.6). Although the result in (1.6) holds for sufficiently smooth data, we devise a suitably rescaled problem (in the case when both kernels are the same Gaussian) and measure the L^2-norm between the solution of the classical CHE and the solutions of the various rescaled nonlocal CHE, all originating from the same rough data described above.

Outline of the paper. In Section 2 we state the relevant notation and the notion of mild/integral solutions which can be constructed by the Picard iteration scheme. Furthermore, we give a main summary of the main results and proofs, involving the existence and uniqueness of L^∞-mild solutions, along with the existence of a dissipative semigroup for our problem. In Section 3 we prove additional smoothing estimates for the aforementioned solutions, implying the desired Hölder continuity of the semigroup. Consequently, in Section 4 we give a complete characterization of the omega-limit sets associated with $W^{1,p}$-data and then, in Section 5, a theorem on the existence of exponential attractors is proved for the associated semigroup of solutions. In Section 6 we give the precise statement and a proof of the aforementioned convergence result in (1.6). Section 7 implements the (forward) Euler scheme for our case study in (d), confirming the analysis performed in the previous sections. The final section is an appendix that contains a number of technical results, assisting in the proofs of the main results.

2. Mild solution theory

Let Ω be a bounded open set in $\mathbb{R}^N$, $N \geq 1$. No further regularity assumptions on Ω are required at this point, but we will add any whenever it is necessary to do so by our proofs. Assume the following:

(H1) $J \in L^1_{\text{loc}}(\mathbb{R}^N)$ is nonnegative, $K \in L^1_{\text{loc}}(\mathbb{R}^N)$[12] and J, K are symmetric[13] over $\mathbb{R}^N$.

(H2) $F \in C^2(\mathbb{R}, \mathbb{R})$ satisfies $F(0) = F'(0) = 0$, and there exists a constant $c_0 > 0$ such that, for $a_K(x) := \int_\Omega K(x-y)\,dy$,

$$F''(r) + a_K(x) \geq c_0 \quad \text{for all } r \in \mathbb{R} \text{ and a.e. } x \in \Omega. \tag{2.1}$$

Moving forward, we assume (H1)–(H2) hold. We will first look for the existence of a globally defined mild solution to (1.3)–(1.5) using Picard's method of successive approximations for the unknown function ϕ in the variable t; i.e., we identify $\phi(t,x) = \phi(t) \in X$, where X denotes an appropriate Banach space. The iterates are defined as follows:

$$\phi_0(t,x) \equiv \phi_0(x), \tag{2.2}$$

$$\phi_{n+1}(t,x) = \phi_0(x) + \int_0^t ((J * \mu_n)(\tau,x) - a_J(x)\mu_n(\tau,x))\,d\tau \quad \text{for all } n \geq 0, \tag{2.3}$$

$$\mu_n(\tau,x) = -(K * \phi_n)(\tau,x) + a_K(x)\phi_n(\tau,x) + F'(\phi_n(\tau,x)) \quad \text{for all } n \geq 0. \tag{2.4}$$

More precisely, our notion of a solution to the system (1.3)–(1.5) is the following.

Definition 2.1. We say that ϕ is a mild solution on the time interval $(0,T)$ if $\phi(0,x) = \phi_0$ in the $L^\infty(\Omega)$-sense, and it satisfies

$$\phi(t,x) = \phi_0(x) + \int_0^t (J * \mu)(\tau,x) - a_J(x)\mu(\tau,x)\,d\tau, \tag{2.5}$$

$$\mu(\tau,x) = -(K * \phi)(\tau,x) + a_K(x)\phi(\tau,x) + F'(\phi(\tau,x)) \quad \text{a.e. in}(0,T) \times \Omega.$$

Moreover, the solution satisfies

$$\phi \in C([0,T]; L^\infty(\Omega)), \qquad F'(\phi) \in C([0,T]; L^\infty(\Omega)).$$

[12] Mathematically, the positivity of K is not required. However, in phase separation/aggregation phenomena one has $K \geq 0$. This is similar to other instances; refer also to [18, 19].

[13] A function H is symmetric if $H(x) = H(-x)$ for all $x \in \mathbb{R}^N$.

Remark 2.2. Note that, due to Fubini's theorem, assumption (H1) and Definition 2.1, the total mass of ϕ over Ω is conserved, namely

$$\int_\Omega \phi(t,x)\,dx = \int_\Omega \phi(0,x)\,dx \quad \text{for all } t \geq 0.$$

Theorem 2.3. *Let the assumptions (H1)–(H2) hold, and let $\phi_0 \in L^\infty(\Omega)$.*

(i) *Then system (1.3)–(1.5) has a mild solution, in the sense of Definition 2.1, on the interval $(0,T)$, for any $T > 0$.*

(ii) *Each mild solution satisfies $\phi \in W^{2,\infty}(0,T;L^\infty(\Omega))$, and the following equations hold:*

$$\begin{aligned} \partial_t \phi &= \mathcal{L}_J(\mu(t,x)), && \text{a.e. } x \in \Omega, \text{ for all } t \in (0,T), \quad (2.6)\\ \mu(t,x) &= -\mathcal{L}_K \phi(t,x) + F'(\phi(t,x)), && \text{a.e. } x \in \Omega, \text{ for all } t \in (0,T). \end{aligned}$$

(iii) *Moreover, we have*

$$\mu \in W^{1,\infty}(0,T;L^\infty(\Omega)), \qquad F'(\phi) \in W^{1,\infty}(0,T;L^\infty(\Omega)). \tag{2.7}$$

Proof. For the sake of convenience, the reader can find the local-in-time result in the appendix (see Theorem A.1). Let $\delta \in (0, \frac{T}{2})$ be an arbitrarily small number and consider the right-difference

$$Z(t,h) := h^{-1}(\phi(t+h) - \phi(t)) \quad \text{for } h \in (0,\delta] \text{ and } 0 \leq t \leq T.$$

Here $T > 0$ is a fixed time, which is defined by how long the mild $C([0,T];L^\infty(\Omega))$-solution ϕ exists. Notice that $Z(t-h,h)$ coincides with the left-difference. For every mild solution ϕ, the continuous functions $Z(t,h)$ and $Z(t-h,h)$ then satisfy

$$Z(t,h) = h^{-1}\int_t^{t+h} \mathcal{L}_J(\mu(s))\,ds$$

and, respectively,

$$Z(t-h,h) = h^{-1}\int_{t-h}^{t} \mathcal{L}_J(\mu(s))\,ds.$$

As in the proof of Theorem A.1, we clearly have

$$\begin{cases} \|Z(t,h)\|_{L^\infty(\Omega)} \leq C\|\phi\|_{C([0,T]L^\infty(\Omega))} \leq C_T,\\ \|Z(t-h,h)\|_{L^\infty(\Omega)} \leq C\|\phi\|_{C([0,T]L^\infty(\Omega))} \leq C_T, \end{cases}$$

uniformly in $t \in [0,T]$. Passing now to the limit as $h \to 0^+$ in the limsup and liminf sense above, we deduce that both lower Dini derivatives $\partial_+\phi(t)$, $\partial_-\phi(t)$ and both upper Dini derivatives $\partial^+\phi(t)$, $\partial^-\phi(t)$ are bounded uniformly (as functions with values in $L^\infty(\Omega)$) for all $0 \leq t \leq T$. Thus, all four Dini derivatives are finite in the range for $0 \leq t \leq T$. By application of the celebrated theorem of Denjoy–Young–Saks ([6, Chapter IV, Theorem 4.4]), the continuous mild solution $\phi\colon [0,T] \to L^\infty(\Omega)$ is differentiable for almost all

$0 \le t \le T$, and all four Dini derivatives are equal to $\partial_t\phi(t)$ on the set $t \in [0,T] \setminus E$ (where E is a null set of Lebesgue measure; in fact, E is a set of first category; see [6, Chapter IV, Theorem 4.7]). In particular, this yields the fact that $\partial_t\phi \in L^\infty(0,T;L^\infty(\Omega))$, so that (2.6) follows. The last regularity (2.7) is an immediate consequence of the regularity for $\partial_t\phi$ and the (local) Lipschitz continuity of F'. In fact, on account of formula (2.5), a similar argument shows that the $C([0,T];L^\infty(\Omega))$-norm controls the norm of $\partial_t\mu \in L^\infty(0,T;L^\infty(\Omega))$ and $\partial_{tt}\phi \in L^\infty(0,T;L^\infty(\Omega))$. Thus, (iii) follows once again from the validity of (2.5), whence (ii) is satisfied due to the boundedness of the operators $\mathcal{L}_J, \mathcal{L}_K$.

It then suffices to show that $\phi \in C([0,T];L^\infty(\Omega))$ for any $T>0$; the local solution can then be continued on every interval by the usual ODE trick. Immediately after, (ii)–(iii) hold on any interval $(0,T)$, and not just locally in time. To show the global boundedness, we follow an argument from [17, Lemma 3.1] in order to deal with the double interaction in the Cahn–Hilliard equation. This argument requires mainly that $J \ge 0$ and the integrability of J and K. We multiply the first equation of (2.6) by $|\phi|^{p-1}\phi$, and then integrate over Ω. We obtain

$$\begin{aligned}\frac{1}{p+1}\frac{d}{dt}\int_\Omega |\phi|^{p+1}\,dx \\ &= -\int_\Omega\int_\Omega J(x-y)(\mu(x)-\mu(y))(|\phi(x)|^{p-1}\phi(x)-|\phi(y)|^{p-1}\phi(y))\,dy\,dx \\ &=: -(J_1+J_2+J_3), \end{aligned} \tag{2.8}$$

where we have set

$$q_F(\phi) := \frac{F'(\phi(x))-F'(\phi(y))}{\phi(x)-\phi(y)}$$

and

$$\begin{aligned} J_1 &:= \int_\Omega\int_\Omega J(x-y)(a_K(x)+q_F(\phi))(\phi(x)-\phi(y)) \\ &\qquad \times (|\phi(x)|^{p-1}\phi(x)-|\phi(y)|^{p-1}\phi(y))\,dy\,dx, \\ J_2 &:= \int_\Omega\int_\Omega J(x-y)(a_K(x)-a_K(y))\phi(y) \\ &\qquad \times (|\phi(x)|^{p-1}\phi(x)-|\phi(y)|^{p-1}\phi(y))\,dy\,dx \end{aligned}$$

and, finally,

$$\begin{aligned} J_3 &:= \int_\Omega\int_\Omega J(x-y)((K*\phi)(x)-(K*\phi)(y)) \\ &\qquad \times (|\phi(x)|^{p-1}\phi(x)-|\phi(y)|^{p-1}\phi(y))\,dy\,dx. \end{aligned}$$

We also recall that $a_K(x)+q_F(\phi) \ge c_0$ for all $\phi \in \mathbb{R}$, a.e. in Ω, on account of assumption (H2) and the mean value theorem for $F \in C^2$. Then, exploiting [17, Lemma 3.1, (3.5)], it follows that $-J_1 \le 0$ for all $t \ge 0$; henceforth (2.8) implies that

$$\|\phi(t)\|^p_{L^{p+1}(\Omega)}\frac{d}{dt}\|\phi(t)\|_{L^{p+1}(\Omega)} \le -J_2 - J_3$$

and the uniform estimate

$$\begin{aligned}|J_2| &\le \|a_K\|_{L^\infty(\Omega)}\|J*\phi\|_{L^{p+1}(\Omega)}\|\phi\|^p_{L^{p+1}(\Omega)}\\ &\le \|K\|_{L^1}\|J\|_{L^1}\|\phi\|^{p+1}_{L^{p+1}(\Omega)},\end{aligned}$$

as well as

$$|J_3| \le \|K\|_{L^1}\|J\|_{L^1}\|\phi\|^{p+1}_{L^{p+1}(\Omega)}.$$

Thus, for all $t \ge 0$, it follows that

$$\frac{d}{dt}\|\phi(t)\|_{L^{p+1}(\Omega)} \le 2\|K\|_{L^1}\|J\|_{L^1}\|\phi(t)\|_{L^{p+1}(\Omega)}.$$

The Grönwall inequality gives the desired uniform estimate in the L^{p+1}-norm, and the $L^\infty(\Omega)$-norm, as usual by passing to the limit as $p \to \infty$. Namely, we deduce

$$\|\phi(t)\|_{L^\infty(\Omega)} \le \|\phi_0\|_{L^\infty(\Omega)} e^{2\|K\|_{L^1}\|J\|_{L^1}T}$$

for all $t \in [0,T]$. The proof is now complete. ∎

Remark 2.4. The conclusions of Theorem 2.3 (and, in particular, the global boundedness) also hold if we replace (2.1) by the condition

$$F''(r) + a_K(x) \ge c_0|s|^{2q} - \bar{c}_0 \quad \text{for all } r \in \mathbb{R}, \text{ a.e. } x \in \Omega, \tag{2.9}$$

for some $q > 0$, $c_0 > 0$, $\bar{c}_0 \ge 0$. For comparison, we refer the reader to [13], where a nonlocal Cahn–Hilliard equation is coupled with the Navier–Stokes equation (and, where (2.9) plays some role in providing additional regularity properties for the velocity component).

Each mild solution satisfies an energy identity. To this end, let us also define the energy functional $\mathcal{E}\colon (0,\infty) \to \mathbb{R}$, along any given mild L^∞-solution, by

$$\mathcal{E}(t) := \frac{1}{4}\int_\Omega\int_\Omega K(x-y)(\phi(t,x)-\phi(t,y))^2\,dx\,dy + \int_\Omega F\phi(t,x)\,dx.$$

Theorem 2.5. *Let the assumptions of Theorem 2.3 hold.*

(i) $\mathcal{E} \in AC(0,T;\mathbb{R})$ *(i.e., $\mathcal{E}$ is absolutely continuous on $(0,T)$) and the energy identity*

$$\frac{d}{dt}\mathcal{E}(t) + \frac{1}{2}\int_\Omega\int_\Omega J(x-y)(\mu(t,x)-\mu(t,y))^2\,dx\,dy = 0 \tag{2.10}$$

holds for almost all $t \in (0,T)$.

(ii) *Moreover, if ϕ_1 and ϕ_2 are two mild solutions, subject to the initial conditions $\phi_1(0) = \phi_{10}$, $\phi_2(0) = \phi_{20}$, the following estimate also holds:*

$$\|\phi_1(t)-\phi_2(t)\|_{L^\infty(\Omega)} \le \|\phi_{10}-\phi_{20}\|_{L^\infty(\Omega)} e^{Ct} \quad \text{for all } t \in [0,T] \tag{2.11}$$

for some $C > 0$ independent of t and ϕ_i ($i = 1,2$).

Proof. The proof of (i) follows easily (on account of the regularity of ϕ), by multiplying the first equation of (2.6) by μ, the second by $\partial_t\phi$, and then combining the resulting equations to conclude with (2.10).

For the Lipschitz estimate (and uniqueness), we consider two mild solutions ϕ_1, ϕ_2 subject to the initial conditions $\phi_1(0) = \phi_{10}$, $\phi_2(0) = \phi_{20}$. We then set $v(t) := \phi_1(t) - \phi_2(t)$, $t \in [0, T]$, and observe that v satisfies

$$v(t) = v(0) + \int_0^t ((J * \bar{\mu})(\tau, x) - a_J(x)\bar{\mu}(\tau, x))\, d\tau,$$
$$\bar{\mu}(\tau, x) = -(K * v)(\tau, x) + a_K(x)v(\tau, x) + F'(\phi_1(\tau, x)) - F'(\phi_2(\tau, x)).$$

Following the standard existence argument, we then find

$$\|v(t)\|_{L^\infty(\Omega)} \le \|v(0)\|_{L^\infty(\Omega)} + C_T \int_0^t \|v(\tau)\|_{L^\infty(\Omega)}\, d\tau$$

for some $C > 0$ independent of v, t. The Grönwall inequality immediately yields the final claim (2.11) of the theorem. The proof is finished. ∎

In what follows, we also set

$$L^p_{(m)}(\Omega) = \left\{\phi \in L^p(\Omega) : \bar{\phi} := \tfrac{1}{|\Omega|}\textstyle\int_\Omega \phi(x)\, dx = m\right\}, \quad 1 \le p \le \infty,$$

which we endow with the metric L^p-topology. We remark that problem (1.3)–(1.5) generates a (strongly) continuous semigroup

$$S(t)\colon L^\infty_{(m)}(\Omega) \to L^\infty_{(m)}(\Omega)$$

given by

$$S(t)\phi_0 = \phi(t), \quad t \ge 0,$$

where ϕ is the unique mild solution in the sense of Definition 2.1.

For additional (physical) properties of the solution (and/or semigroup), we will also need the following assumptions.

(H3) Given J and Ω, there exists a constant $\lambda_1 = \lambda_{1,J}(\Omega) > 0$ such that, for all $v \in L^2_{(0)}(\Omega)$, the Poincaré inequality holds:

$$\lambda_1 \|v\|^2_{L^2(\Omega)} \le \frac{1}{2}\int_\Omega\int_\Omega J(x-y)(v(t,x) - v(t,y))^2\, dx\, dy.$$

(H4) There exist $c_1 > 0$, $c_2 \ge 0$ such that $F(r) \ge c_1|r|^{2q+2} - c_2$ for some $q > 0$, for all $r \in \mathbb{R}$.

Remark 2.6. • (H3) is satisfied for instance by any integrable kernel $J \ge 0$ whose support contains the ball $B(0, \rho)$ for some $\rho > 0$. In that case, $\mathcal{L}_J\colon L^2_{(0)}(\Omega) \to L^2_{(0)}(\Omega)$ is a positive (self-adjoint) bounded operator. We remark that

$$\lambda_{1,J} = \inf_{v \in L^2_{(0)}(\Omega)} \frac{\frac{1}{2}\int_\Omega\int_\Omega J(x-y)(v(t,x)-v(t,y))^2\, dx\, dy}{\|v\|^2_{L^2}}.$$

The Fredholm alternative implies that $\lambda_1 = \lambda_{1,J} > 0$.

- By (H3), $\mathcal{L}_J$ is a linear homeomorphism (or topological isomorphism) as a mapping from $L^2_{(0)}(\Omega) \to L^2_{(0)}(\Omega)$ (and so is $\mathcal{L}_J^{-1}$). Consequently, there exist constants $m_i, M_i > 0$ $(i = 1, 2)$ such that $m_i \|\phi\|_{L^2(\Omega)} \le \|T_i\phi\|_{L^2(\Omega)} \le M_i \|\phi\|_{L^2(\Omega)}$ for all $\phi \in L^2_{(0)}(\Omega)$, where $T_1 = \mathcal{L}_J$, $T_2 = \mathcal{L}_J^{-1}$.
- The double-well potential $F(r) = \theta r^4 - \theta_c r^2$, $0 < \theta < \theta_c$, satisfies (2.9) and (H4) with $q = 1$.

Theorem 2.7. *Let the assumptions of Theorem 2.5 be satisfied and further assume (H3)–(H4).*

(a) *Then the following dissipative estimate holds:*

$$\mathcal{E}(t) \le \mathcal{E}(0)e^{-t} + L_1 \quad \textit{for all } t \ge 0, \tag{2.12}$$

where $L_1 = L_1(m) > 0$ is a constant which is independent of the initial data, ϕ and time.

(b) *There is a bounded absorbing set (in the L^{2q+2}-topology) for the semigroup family $\{S(t)\}_{t\ge 0}$. Namely, for any $\phi_0 \in L^\infty_{(m)}(\Omega)$ such that $\|\phi_0\|_{L^\infty(\Omega)} \le R$, there exists a time $t_0 = t_0(R, m) > 0$ such that $\|\phi(t)\|_{L^{2q+2}(\Omega)} \le C_m$ for all $t \ge t_0$, with constant $C_m > 0$ independent of time, ϕ and the initial datum (depending only on m, and the structural assumptions of the theorem).*

Proof. To show (2.12), let us test $\mu = a_K\phi - K * \phi + F'(\phi)$ by ϕ in $L^2(\Omega)$. We obtain

$$(\mu, \phi)_{L^2} = \frac{1}{2}\int_\Omega\int_\Omega K(x-y)(\phi(x)-\phi(y))^2\,dy\,dx + (F'(\phi), \phi)_{L^2}. \tag{2.13}$$

By the convexity of $G(r) = F(r) + \frac{\|a_K\|_\infty}{2}r^2$ (and therefore $G''(r) \ge c_0$ a.e. in Ω, owing to (H2)), we have

$$F'(r)r \ge F(r) - \frac{\|a_K\|_\infty}{2}r^2 \quad \text{for any } r \in \mathbb{R}.$$

Therefore, from (2.13) we get

$$(\mu, \phi)_{L^2} \ge \frac{1}{2}\int_\Omega\int_\Omega K(x-y)(\phi(x)-\phi(y))^2\,dy\,dx + \int_\Omega F(\phi(t))\,dx - \frac{\|a_K\|_\infty}{2}\|\phi\|^2_{L^2(\Omega)}. \tag{2.14}$$

On the other hand, by (H3) we can exploit the Poincaré inequality

$$\lambda_1\|\mu - \bar{\mu}\|^2_{L^2(\Omega)} \le l(\mu, \mu) := \frac{1}{2}\int_\Omega\int_\Omega J(x-y)(\mu(x)-\mu(y))^2\,dy\,dx$$

and the conservation of mass $\bar{\phi} = \bar{\phi}_0 = m$, to observe that

$$(\mu, \phi)_{L^2} = (\mu - \bar{\mu}, \phi)_{L^2} \le \lambda_1^{-1/2}\sqrt{l(\mu, \mu)}\|\phi\|_{L^2(\Omega)},$$

assuming for simplicity (for now) that $m = 0$. Thus, by virtue of assumption (H4) (recall that $q > 2$) we rewrite (2.14) and estimate in a simple fashion, in order to get

$$\lambda_1^{-1/2}\sqrt{l(\mu,\mu)}\|\phi\|_{L^2(\Omega)} \geq \frac{1}{2}\mathcal{E}(t) + \frac{c_1}{2}\|\phi\|_{L^{2q+2}(\Omega)}^{2q+2} - \Big(\frac{c_2}{2}|\Omega| + \frac{\|a_K\|_\infty}{2}\|\phi\|_{L^2(\Omega)}^2\Big).$$

An application of the Young inequality yields

$$\begin{aligned}&\frac{1}{2}\mathcal{E}(t) + \frac{c_1}{2}\|\phi\|_{L^{2q+2}(\Omega)}^{2q+2} - \Big(\frac{c_2}{2}|\Omega| + \frac{\|a_K\|_\infty}{2}\|\phi\|_{L^2(\Omega)}^2\Big)\\ &\quad\leq \frac{1}{2}l(\mu,\mu) + \frac{\lambda_1^{-1}}{2}\|\phi\|_{L^2(\Omega)}^2.\end{aligned}$$

We thus easily deduce $\frac{1}{2}\mathcal{E}(t) \leq \frac{1}{2}l(\mu,\mu) + C_*$, with constant $C_* > 0$ depending only on q, c_1, λ_1, $\|K\|_{L^1}$, c_2 and $|\Omega|$. It follows by virtue of the foregoing inequality and the energy identity for ϕ that we have

$$\frac{d}{dt}\mathcal{E}(t) + \mathcal{E}(t) \leq 2C_* \quad \text{for all } t \geq 0. \tag{2.15}$$

Since $|\mathcal{E}(0)| \leq \mathcal{C}_R$ with $\|\phi_0\|_{L^\infty(\Omega)} \leq R$, by means of the Grönwall inequality we obtain

$$\mathcal{E}(t) \leq \mathcal{E}(0)e^{-t} + L \leq \mathcal{C}_R e^{-t} + L, \tag{2.16}$$

with $L = 2C_*$. If $m \neq 0$, observe that if ϕ is a mild solution with initial datum ϕ_0 for the problem with potential F, then $\tilde{\phi} = \phi - m$ is a mild solution with initial datum $\tilde{\phi}(0) = \phi_0 - m$ for the same problem with potential $\widetilde{F}(s) := F(r+m) - F(m)$. Since now $\bar{\tilde{\phi}} = 0$, we can employ the dissipative estimate (2.16) for the solution $\tilde{\phi}$ and easily arrive at the final inequality (2.12). The proof is complete. ∎

3. Regular mild solutions and uniform estimates

Our main goal of this section is to investigate whether the mild solution is eventually more regular, in a suitable class of Sobolev spaces. Furthermore, we aim to give dissipative estimates which are crucial for the long-term behavior (as time goes to infinity) of the mild solution in the following sections.

We introduce the ith difference quotient (of size h) in any open set $\Omega' \subset \Omega$ (where Ω satisfies the cone condition), for all $t \geq 0$,

$$D_{i,h}\phi(x,t) = h^{-1}(\phi(x + e_i h, t) - \phi(x,t)), i = 1, \ldots, N,$$

for $x \in \Omega'$ and $0 < |h| < (1/2)\,\mathrm{dist}(\Omega', \partial\Omega)$. As usual, $\nabla = (D_1, D_2, \ldots, D_N)$.

We impose additional assumptions on J, K and replace (H4) by a condition which implies that F has some rational/polynomial growth at infinity.

(H5) There exist $c_i > 0$, $\tilde{c}_i \geq 0$ $(i = 1, 2)$ such that

$$c_2|r|^{2l} + \tilde{c}_2 \geq F''(r) \geq c_1|r|^{2q} - \tilde{c}_1 \quad \text{for all } r \in \mathbb{R},$$

for some $l \geq q > 0$.

(H6) The interaction kernels $J, K \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$.

We remark that (H6) also implies that $a_J(x) = (J * 1)(x)$ is continuous in $x \in \overline{\Omega}$, and so

$$\min_{x \in \overline{\Omega}} a_J(x) \geq \lambda_{1,J} > 0$$

(see [3, Lemma 3.15]). Recall that $\lambda_{1,J} > 0$ is the first eigenvalue of the (self-adjoint) operator $\mathcal{L}_J \colon L^2_{(0)}(\Omega) \to L^2_{(0)}(\Omega)$ (see Remark 2.6).

Our first regularity result is the following.

Theorem 3.1. *Let (H1)–(H3) and (H5)–(H6) hold, and assume $\phi_0 \in L^\infty_{(m)}(\Omega) \cap H^1(\Omega)$ such that $\|\phi_0\|_{L^\infty(\Omega)} \leq R_1$ and $\|\phi_0\|_{H^1(\Omega)} \leq R_2$. Then for all $t \geq 0$, the following dissipative estimate holds:*

$$\|\phi(t)\|_{H^1(\Omega)} \leq \|\phi_0\|_{H^1(\Omega)} e^{-\lambda_{1,J} c_0 t} + L_2, \tag{3.1}$$

where the constant $L_2 = L_2(m, R_1) > 0$ is independent of the initial data, ϕ and time. Moreover, there exists a time $t_1 = t_1(R_1, R_2) > 0$ such that

$$\|\phi(t)\|_{H^1(\Omega)} \leq C_m \quad \text{for all } t \geq t_1. \tag{3.2}$$

Here the constant $C_m > 0$ is independent of time, R_i, ϕ and the initial datum (depending only on m and the structural assumptions of the theorem).

Proof. In this proof (and everywhere else in this section), the constant $C_* > 0$ is independent of the initial data, ϕ, R_i and time (and may change from line to line). By Theorem 2.7 (a) and (H5), $\phi \in C_b(\mathbb{R}_+; L^{2q+2}(\Omega))$ with $F(\phi) \in C_b(\mathbb{R}_+; L^1(\Omega))$, and

$$\int_t^{t+1} \|\phi(s)\|^2_{L^2(\Omega)}\, ds \leq C_{R_1} e^{-t} + C_* \quad \text{for all } t \geq 0. \tag{3.3}$$

Here $C_{R_1} > 0$ is such that $|\mathcal{E}(0)| \leq C_{R_1}$ since $\|\phi_0\|_{L^\infty(\Omega)} \leq R_1$. Moreover,

$$\mu \in C_b(\mathbb{R}_+; L^1(\Omega))$$

with $|\bar{\mu}(t)| \leq C_{R_1} e^{-t} + C_*$, since $|F'(r)| \leq C_*(|F(r)| + 1)$, and

$$\int_t^{t+1} \|\mu(s)\|^2_{L^2(\Omega)}\, ds \leq C_{R_1} e^{-t} + C_* \quad \text{for all } t \geq 0. \tag{3.4}$$

Indeed, we infer from the energy identity (2.10) that

$$\begin{aligned} &\mathcal{E}(t+1) + \frac{1}{2} \int_t^{t+1} \int_\Omega \int_\Omega J(x-y)(\mu(t,x) - \mu(t,y))^2\, dx\, dy\, dt \\ &\quad = \mathcal{E}(t) \leq C_{R_1} e^{-t} + (L_1 + C_*), \end{aligned} \tag{3.5}$$

which in view of the Poincaré inequality (H3) yields

$$\int_t^{t+1} \|\mu(s) - \bar{\mu}(s)\|_{L^2(\Omega)}^2 \, ds \le C_{R_1} e^{-t} + C_*.$$

Thus, (3.4) holds.

Next, each bounded mild solution of Theorem 2.7 satisfies for a.e. $x \in \Omega' \subset \Omega$, all $t \ge 0$ and $i = 1, 2, \dots, N$,

$$\partial_t D_{i,h}\phi(x,t) + a_J(x) D_{i,h}\mu(x,t) = (D_{i,h} J * \mu)(x,t) - (D_{i,h} a_J)(x) \mu(x + e_i h, t) \tag{3.6}$$

where

$$D_{i,h}\mu(x,t) = -(D_{i,h} K * \phi)(x,t) + (D_{i,h} a_K)(x) \phi(x + e_i h, t) + (a_K(x) + F''(\xi(x,t))) D_{i,h}\phi(x,t), \tag{3.7}$$

and $\xi(x, \cdot) = \lambda \phi(x + e_i h, \cdot) + (1 - \lambda)\phi(x, \cdot)$ for some $\lambda \in (0, 1)$.

Our goal is to derive uniform (in $0 < |h| \le 2\,\mathrm{dist}(\Omega', \partial\Omega)$ and in time) estimates for $D_{i,h}\phi$. To this end, we multiply (3.6) by $2D_{i,h}\phi$ and integrate the resulting identity over Ω'. We deduce

$$\begin{aligned} &\frac{d}{dt} \|D_{i,h}\phi\|_{L^2(\Omega')}^2 + 2 \int_{\Omega'} a_J(x)(a_K(x) + F''(\xi(x,t))) |D_{i,h}\phi|^2 \, dx \\ &= 2(a_J D_{i,h} K * \phi, D_{i,h}\phi)_{L^2} - 2(a_J D_{i,h} a_K \phi(x + e_i h, t), D_{i,h}\phi)_{L^2} \\ &\quad + 2(D_{i,h} J * \mu, D_{i,h}\phi)_{L^2} - 2((D_{i,h} a_J)\mu(x + e_i h, t), D_{i,h}\phi)_{L^2}. \end{aligned} \tag{3.8}$$

The Young convolution theorem and (H2), (H6) then imply

$$\begin{aligned} &\frac{d}{dt} \|D_{i,h}\phi\|_{L^2(\Omega')}^2 + 2c_0\lambda_{1,J} \|D_{i,h}\phi\|_{L^2(\Omega')}^2 \\ &\le 4(\|J\|_{L^1} \|K\|_{W^{1,1}} \|\phi\|_{L^2(\Omega)} + \|J\|_{W^{1,1}} \|\mu\|_{L^2(\Omega)}) \|D_{i,h}\phi\|_{L^2(\Omega')} \\ &\le 2C_* g(t) \|D_{i,h}\phi\|_{L^2(\Omega')}, \end{aligned} \tag{3.9}$$

where we have set $g := \|\phi\|_{L^2(\Omega)} + \|\mu\|_{L^2(\Omega)}$. On account of (3.3)–(3.4), we have

$$\int_t^{t+1} g(s) \, ds \le C_{R_1}^{1/2} e^{-t/2} + C_*^{1/2} \quad \text{for all } t \ge 0. \tag{3.10}$$

There is also a time $t_0 = t_0(R_1) > 0$ such that

$$\int_t^{t+1} g(s) \, ds \le 2C_*^{1/2} \quad \text{for all } t \ge t_0. \tag{3.11}$$

Then, for all $t \ge 0$, (3.9) implies

$$\frac{d}{dt} \|D_{i,h}\phi(t)\|_{L^2(\Omega')} + c_0\lambda_{1,J} \|D_{i,h}\phi(t)\|_{L^2(\Omega')} \le C_* g(t). \tag{3.12}$$

The Grönwall inequality yields from (3.12) that

$$\|D_{i,h}\phi(t)\|_{L^2(\Omega')} \le e^{-c_0\lambda_1(t-t_0)}\|D_{i,h}\phi(t_0)\|_{L^2(\Omega')} + e^{-c_0\lambda_1(t-t_0)}\int_{t_0}^{t} g(s)e^{c_0\lambda_1 s}\,ds$$

for all $t \ge t_0 \ge 0$. Suppose now $k + t_0 \le t < t_0 + k + 1$ for some integer k. Then in view of (3.10) we have

$$\begin{aligned}
&\|D_{i,h}\phi(t)\|_{L^2(\Omega')} \\
&\le e^{-c_0\lambda_1(t-t_0)}\|D_{i,h}\phi(t_0)\|_{L^2(\Omega')} + e^{-c_0\lambda_1(t-t_0)}\sum_{j=0}^{k}\int_{j+t_0}^{1+j+t_0} g(s)e^{c_0\lambda_1 s}\,ds \\
&\le e^{-c_0\lambda_1(t-t_0)}\|D_{i,h}\phi(t_0)\|_{L^2(\Omega')} \\
&\quad + e^{-c_0\lambda_1(t-t_0)}\sum_{j=0}^{k} e^{c_0\lambda_1(t_0+j+1)}\left(\sup_{t\ge t_0}\int_t^{t+1} g(s)\,ds\right) \\
&\le e^{-c_0\lambda_1(t-t_0)}\|D_{i,h}\phi(t_0)\|_{L^2(\Omega')} + e^{c_0\lambda_1(k+1+t_0-t)}\frac{e^{c_0\lambda_1}}{e^{c_0\lambda_1}-1}\left(\sup_{t\ge t_0}\int_t^{t+1} g(s)\,ds\right) \\
&\le e^{-c_0\lambda_1(t-t_0)}\|D_{i,h}\phi(t_0)\|_{L^2(\Omega')} + \frac{e^{2c_0\lambda_1}}{e^{c_0\lambda_1}-1}\left(\sup_{t\ge t_0}\int_t^{t+1} g(s)\,ds\right). \qquad (3.13)
\end{aligned}$$

The above estimate easily yields claim (3.1) when $t_0 = 0$, on account of (3.10), since (3.13) is also uniform with respect to h. In the case when $t_0 > 0$, the existence of a bounded absorbing set in $H^1(\Omega)$ (see (3.2)) follows from (3.11) instead of (3.10), together with (3.1). The proof of the theorem is complete. ∎

Let us set $p_N = 2N/(N-2)$ if $N \ge 3$, and by the usual convention[14] we notice that $p_2 \in (1,\infty)$ is arbitrary in dimension $N = 2$, and $p_1 = \infty$ when $N = 1$. Due to the presence of a nonlinear term, the chemical potential μ suffers from loss of integrability, in any of its first derivatives, in any dimension $N \ge 2$ (cf. also Remark 3.3 below).

Corollary 3.2. *Let the assumptions of Theorem 3.1 hold. Then the following assertions hold for some constant $C_m > 0$ independent of time, R_i, ϕ and the initial datum.*

(1) *In dimension $N = 1$ we have*

$$\|\mu(t)\|_{H^1(\Omega)} + \|\partial_t\phi(t)\|_{H^1(\Omega)} \le C_m \quad \text{for all } t \ge t_1. \qquad (3.14)$$

(2) *In dimension $N \ge 2$, setting $q_N := \frac{2p_N}{4l+p_N} (< 2)$, we have*

$$\|\mu(t)\|_{W^{1,q_N}(\Omega)} + \|\partial_t\phi(t)\|_{W^{1,q_N}(\Omega)} \le C_m \quad \text{for all } t \ge t_1. \qquad (3.15)$$

[14] Due to the Sobolev embedding $H^1(\Omega) \subset L^{p_N}(\Omega)$.

Proof. First, we notice that by (3.2) we have

$$\|\phi(t)\|_{L^{p_N}(\Omega)} \le C_m \quad \text{for all } t \ge t_1. \tag{3.16}$$

From equation (3.7) for $D_{i,h}\mu$, it follows in view of (3.2) that

$$\begin{aligned}\|D_{i,h}\mu\|_{L^2(\Omega')} &\le 2\|K\|_{W^{1,1}}\|\phi\|_{L^2(\Omega)} + \|K\|_{L^1}\|\phi\|_{H^1(\Omega)} + \|F''(\xi)D_{i,h}\phi\|_{L^2(\Omega')} \\ &\le C_*(1 + \|\phi\|_{H^1(\Omega)} + \|\phi(x+e_ih)\|^{2l}_{L^\infty(\Omega')} + \|\phi(x)\|^{2l}_{L^\infty(\Omega')}).\end{aligned}$$

Using (3.6), for any $p > 1$ we also have

$$\|D_{i,h}\partial_t\phi\|_{L^p(\Omega')} \le \|J\|_{L^1}\|D_{i,h}\mu\|_{L^p(\Omega')} + 2\|J\|_{W^{1,1}}\|\mu\|_{L^p(\Omega)}. \tag{3.17}$$

In dimension $N = 1$, (3.14) is a consequence of (3.2), in view of the embedding $H^1(\Omega) \subset L^\infty(\Omega)$, (3.16)–(3.17) and the obvious inequality $\|\partial_t\phi(t)\|_{L^\infty} \le 2\|J\|_{L^1}\|\mu(t)\|_{L^\infty}$. In dimension $N \ge 2$, we have, in view of (3.2) and (3.16)–(3.17),

$$\begin{aligned}\|D_{i,h}\mu\|_{L^{q_N}(\Omega')} &\le 2\|K\|_{W^{1,1}}\|\phi\|_{L^2(\Omega)} + \|K\|_{L^1}\|\phi\|_{H^1(\Omega)} \\ &\quad + \|F''(\xi)\|_{L^{\frac{p_N}{2l}}(\Omega')}\|\phi\|_{H^1(\Omega)} \\ &\le C_*(\|\phi(x+e_ih)\|_{L^{p_N}(\Omega')} + \|\phi(x)\|_{L^{p_N}(\Omega')} + 1) \\ &\le C_*\end{aligned}$$

for all $t \ge t_1$. This gives the first part of the conclusion in (3.15). For the second part, we exploit the continuous embedding $W^{1,q_N}(\Omega) \subset L^{\frac{p_N}{2l+1}}(\Omega)$, the boundedness of the mapping $\mathcal{L}_J$, and once again (3.17) with $p = q_N < p_N/(2l+1)$. The theorem is proved. ∎

Remark 3.3. If $F(r) = \theta r^4 - \theta_c r^2$, $(0 < \theta < \theta_c)$ we have in dimension $N = 3$, that

$$\partial_t\phi, \mu \in L^\infty((t_1,\infty); W^{1,6/5}(\Omega)) \subset L^\infty((t_1,\infty); L^2(\Omega)).$$

We can improve the $W^{1,2}$-regularity of ϕ to $W^{1,p}$-regularity ($p > 2$), at least in dimension $N = 1, 2$, without any further assumptions on J and F.

Theorem 3.4. *Let the assumptions of Theorem 3.1 hold in dimension $N = 1, 2$. Assume $\phi_0 \in W^{1,p}(\Omega)$ for any $p \in (N,\infty)$, such that $\|\phi_0\|_{H^1(\Omega)} \le R_3$ and $\|\phi_0\|_{W^{1,p}(\Omega)} \le R_4$.*

(1) *For all $t \ge 0$, the following dissipative estimate holds:*

$$\|\phi(t)\|_{W^{1,p}(\Omega)} \le \|\phi_0\|_{W^{1,p}(\Omega)} e^{-\lambda_{1,J}c_0 t} + L_3, \tag{3.18}$$

where the constant $L_3 = L_2(m, R_3) > 0$ is independent of the initial data, ϕ and time.

(2) *There exists a time $t_2 = t_2(R_3, R_4) > 0$ such that*

$$\|\phi(t)\|_{W^{1,p}(\Omega)} \le C_m \quad \text{for all } t \ge t_2. \tag{3.19}$$

The constant $C_m > 0$ is independent of time, R_i, ϕ and the initial datum.

(3) *Finally,*

$$\|\mu(t)\|_{W^{1,p}(\Omega)} + \|\partial_t\phi(t)\|_{W^{1,p}(\Omega)} \leq C_m \quad \text{for all } t \geq t_2. \tag{3.20}$$

Proof. We multiply (3.6) by $|D_{i,h}\phi|^{p-2}D_{i,h}\phi$ (for any $p > 2$) and integrate the resulting identity over Ω'. We deduce

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|D_{i,h}\phi\|^p_{L^p(\Omega')} &+ \int_{\Omega'} a_J(x)\big(a_K(x) + F''(\xi(x,t))\big)|D_{i,h}\phi|^p\,dx \\ &= (a_J D_{i,h}K * \phi, |D_{i,h}\phi|^{p-2}D_{i,h}\phi)_{L^2} \\ &\quad - (a_J D_{i,h}a_K\phi(x+e_ih,t), |D_{i,h}\phi|^{p-2}D_{i,h}\phi)_{L^2} \\ &\quad + (D_{i,h}J * \mu, |D_{i,h}\phi|^{p-2}D_{i,h}\phi)_{L^2} \\ &\quad - ((D_{i,h}a_J)\mu(x+e_ih,t), |D_{i,h}\phi|^{p-2}D_{i,h}\phi)_{L^2}.\end{aligned} \tag{3.21}$$

The first two summands on the right-hand side are bounded collectively and uniformly by

$$2\|J\|_{L^1}\|K\|_{W^{1,1}}\|\phi\|_{L^p(\Omega)}\|D_{i,h}\phi\|^{p-1}_{L^p(\Omega')}, \tag{3.22}$$

while the last two can be bounded uniformly in terms of

$$2\|J\|_{W^{1,1}}\|\mu\|_{L^p(\Omega)}\|D_{i,h}\phi\|^{p-1}_{L^p(\Omega')}. \tag{3.23}$$

Thus, as in the proof of (3.9), we deduce

$$\frac{d}{dt}\|D_{i,h}\phi\|_{L^p(\Omega')} + c_0\lambda_1\|D_{i,h}\phi\|_{L^p(\Omega')} \leq C_*\theta(t), \tag{3.24}$$

where we have set $\theta := \|\phi\|_{L^p(\Omega)} + \|\mu\|_{L^p(\Omega)}$. In dimension $N = 1$ or 2, the embedding $W^{1,q_n}(\Omega) \subset L^{p_N/(2l+1)}(\Omega)$ holds, where we recall that p_N is arbitrary in (N,∞) (and so is $p_N/(2l+1)$ for any fixed $l > 0$). In fact, $N = 1$ is subcritical with respect to the energy estimate since $H^1(\Omega) \subset L^\infty(\Omega)$, while $N = 2$ is only critically so. Thus, we can set $p = \frac{p_N}{2l+1} < p_N$ in (3.24), and notice that $\theta \lesssim \|\phi\|_{L^{p_N}(\Omega)} + \|\mu\|_{W^{1,q_N}(\Omega)}$. On account of Theorem 3.1 and Corollary 3.2, this implies that

$$\int_t^{t+1}\theta(s)\,ds \leq C_{R_3}e^{-\lambda_{1,J}c_0t} + L_2 \quad \text{for all } t \geq 0. \tag{3.25}$$

Clearly, there is also a time $t_2 = t_2(t_1) > 0$ such that

$$\int_t^{t+1}\theta(s)\,ds \leq C_m \quad \text{for all } t \geq t_2. \tag{3.26}$$

The argument leading to (3.13) then yields from (3.24), for any $t_0 \geq 0$ and all $t \geq t_0$,

$$\begin{aligned}\|D_{i,h}\phi(t)\|_{L^p(\Omega')} &\leq e^{-c_0\lambda_1(t-t_0)}\|D_{i,h}\phi(t_0)\|_{L^p(\Omega')} \\ &\qquad + \frac{e^{2c_0\lambda_1}}{e^{c_0\lambda_1}-1}\left(\sup_{t\geq t_0}\int_t^{t+1}\theta(s)\,ds\right).\end{aligned}$$

The final conclusions (3.18)–(3.19) then follow because of (3.25)–(3.26). Finally, arguing exactly as in the proof of Corollary 3.2, while observing that $W^{1,p}(\Omega)$ is embedded continuously into $C^{0,1-N/p}(\overline{\Omega})$, one also gets (3.20). The proof is complete. ■

Suppose now that $N = 3$. In (3.24), we observe that the L^p-estimate for $D_{i,h}\phi$ requires that $\mu \in L^p(\Omega)$ (for $p > 2$), which is a gap that cannot be filled in the case of the double-well potential (see Remark 3.3). Indeed, from (3.15), $\mu \in W^{1,q_N}(\Omega) \subset L^{p_N/(2l+1)}(\Omega)$ is optimal. To close the "energy" gap, we will employ the notion of weak L^p-spaces, the Young inequality in weak L^p-spaces, and impose some (new, but) reasonable assumptions on the kernel J. These assumptions still include important cases of interest, such as when J is the fundamental solution to an elliptic PDE (see below).

To this end, we denote the weak L^p-space by $L^{p,\infty}(\Omega)$ and the associated quasi-norm

$$\|\phi\|_{L^{p,\infty}(\Omega)} = \left(\sup_{\beta>0} \beta^p \lambda_\phi(\beta)\right)^{1/p},$$

where $\lambda_\phi(\beta) = |\{|\phi| > \beta\}|$ is the distribution function of ϕ. In this case, the Young inequality for convolutions in weak L^p-spaces reads[15]

$$\|f * g\|_{L^r} \le C_{p,q,r}\|f\|_{L^{p,\infty}}\|g\|_{L^q},$$

for $1 < p, q, r < \infty$, $1/p + 1/q = 1/r + 1$ (cf. [22, Theorem 1.4.25]).

We consider the following assumptions on J, only in dimension[16] $N = 3$ (recall that $p_N = 6$).

(H7) We assume $J \in W^{1,1}_{\rm loc}(\mathbb{R}^3) \cap C^1(\mathbb{R}^3\backslash\{0\})$ satisfies the following conditions:

- $J(x) = j(|x|) = j(r)$ such that j is nonincreasing as a function $r \in (0, \delta)$, for some $\delta > 0$;
- $j'(r)$ is monotone for $r \in (0, \delta)$;
- $|\nabla J(x)| \le C|x|^{-3/(1+\varepsilon)}$ as $|x| \to 0^+$, for some $\varepsilon > 0$ and $C > 0$.

Regarding the potential F, we assume instead of (H5), the following:

(H8) There exist $c_i > 0$, $\tilde{c}_i \ge 0$ ($i = 1, 2$) such that

$$c_2|r|^{2l} + \tilde{c}_2 \ge F''(r) \ge c_1|r|^{2q} - \tilde{c}_1 \quad \text{for all } r \in \mathbb{R},$$

for some $5/2 > l \ge q > 0$, satisfying $0 < Q_{l,\varepsilon}(p_3) < 1/2$, where

$$Q_{l,\varepsilon}(p_3) := \frac{2l+1}{p_3} - \frac{\varepsilon}{1+\varepsilon},$$

and $\varepsilon > 0$ is the parameter in (H7).

The interplay between the singularity (at the origin) of J, in (H7), and the (growth) exponent l in (H8) implies the $W^{1,p}$-regularity of the order parameter for any $p > 2$,

[15] The endpoints $r, q = 1$ and $r, q = \infty$ fail in general, compared to the standard Young convolution theorem.

[16] We consider the case $N = 3$ only, in order to avoid additional (nonessential) technicalities. Our approach can be extended to the higher-dimensional case $N \ge 4$ with minor modifications.

in dimension $N = 3$. Our argument emphasizes how the (integrable) singularity of J at 0 controls the value of the exponent $l > 0$ in (H8), and vice versa. This property appears natural since it is known also to occur in the case when $\mathcal{L}_J$ is replaced by an operator (of strong type), such as the fractional (Neumann) Laplacian (see, for instance, [17, 19]). Our approach to show $W^{1,p}$-regularity is based on an iteration procedure that allows us to increase the regularity of $\phi \in W^{1,2}$, beyond that of W^{1,p_j}-regularity, for some $p_j > p_{j-1} > 2$, at each step, in order to reach $W^{1,3}$-regularity in finitely many steps. At that point, we have reached the critical point where either $p_j = N = 3$ (and the embedding $W^{1,3}(\Omega) \subset L^s(\Omega)$ holds for arbitrary $s \in (2,\infty)$) or $p_j > 3$ (with $W^{1,p_j}(\Omega) \subset L^\infty(\Omega)$). Therefore, the same argument (in dimension $N = 2$) developed in the proof of Theorem 3.4 applies (albeit with some nonessential modifications), due to the validity of the foregoing embeddings.

Theorem 3.5. *Let (H1)–(H3) and (H6)–(H8) hold. Let $\phi_0 \in W^{1,p}(\Omega)$ for any $p \in (2,\infty)$, such that $\|\phi_0\|_{H^1(\Omega)} \le R_3$ and $\|\phi_0\|_{W^{1,p}(\Omega)} \le R_4$ for some $R_3, R_4 > 0$. Next assume that, for given $\varepsilon > 0$, $l > 0$, there exists $\sigma \ge 0$ such that the condition*

$$0 < Q_{l,\varepsilon}(p_3 + \sigma) < \tfrac{1}{3} \tag{3.27}$$

holds. Then the same conclusions as in Theorem 3.4 also hold in dimension $N = 3$.

Proof. Without loss of generality, assume $0 \in \Omega$, and consider the ball $B_d := B(0,d) \supset \Omega$, where $d > 2\,\mathrm{diameter}(\Omega)$ and $d > \delta$. Let $\widehat{\phi}$ and $\widehat{\mu}$ be the trivial extensions of ϕ, μ to B_d such that $\widehat{\phi}_{|B_d\backslash\Omega} = 0$, $\widehat{\mu}_{|B_d\backslash\Omega} = 0$. We will drop the hats, for the sake of convenience, in what follows. By (H7), $|D_i J(x)| \le \eta(r) = |j'(r)|$ near the origin, for $x \in B_\delta$. Since η is monotone in $r \in (0,\delta)$ and $\eta = O(-r^{3/(1+\varepsilon)})$ as $r \to 0^+$, it follows that there exists $C > 0$ such that for all $0 < r < \delta$, $\eta(r) \le C r^{-3/(1+\varepsilon)}$ and, for any $\beta > 0$, there exists a unique $r_* = r(\beta) \in [0,\delta]$ such that $\eta(r) > \beta$ for $r < r_*$. We take $r_* = 0$ if $\eta(r) < \beta$ over the entire interval $[0,\delta]$. Then we get $r_* = r(\beta) \le C\beta^{-(1+\varepsilon)/3}$ for some $C > 0$, and

$$\beta^{1+\varepsilon}\lambda_\eta(\beta) = \beta^{1+\varepsilon}(\omega_3 r_*^3) \le C \tag{3.28}$$

for some (finite) constant $C > 0$. Here ω_3 is the volume of the unit ball in $\mathbb{R}^3$. By definition, (3.28) implies that $\eta 1_{B_\delta} \in L^{1+\varepsilon,\infty}(B_d)$. Since $D_i J$ is also continuous in $B_d\backslash B_\delta$, we obtain

$$\nabla J(x) \in L^{1+\varepsilon,\infty}(B_\delta) \quad \text{and} \quad \nabla J(x) 1_{B_d\backslash B_\delta} \in L^\infty(B_d).$$

Consider now the identity (3.21) for $p > 2$. We estimate the last summand in (3.21) since one argues in a similar fashion for the first three summands, observing also that $\|\phi\|_{L^{s/(2l+1)}} \lesssim \|\phi\|_{L^s}$ for all s. Indeed, for any $s_1, s_2, s_3 \in (1,\infty)$ with $1/s_1 + 1/s_2 + 1/s_3 = 1$, the Hölder inequality, together with Young inequalities in L^p- and weak L^p-spaces, gives

$$\begin{aligned}
&|((D_{i,h} a_J)\mu(x + e_i h, t), |D_{i,h}\phi|^{p-2} D_{i,h}\phi)_{L^1}| \\
&\quad \le \|D_{i,h} J * 1\|_{L^{s_1}} \|\mu\|_{L^{s_2}} \|D_{i,h}\phi\|^{p-1}_{L^{s_3(p-1)}}
\end{aligned}$$

$$\le \|\nabla J\|_{L^{1+\varepsilon,\infty}(B_\delta)}\|1_{B_\delta}\|_{L^{\tilde{t}}}\|\mu\|_{L^{s_2}}\|D_{i,h}\phi\|^{p-1}_{L^{s_3(p-1)}}$$
$$+ \|\nabla J 1_{B_d\backslash B_\rho}\|_{L^\infty}\|1_{B_d\backslash B_\rho}\|_{L^p}\|\mu\|_{L^1}\|D_{i,h}\phi\|^{p-1}_{L^p} \tag{3.29}$$

for any $\tilde{t} \in (1,\infty)$ provided that

$$\frac{1}{1+\varepsilon} + \frac{1}{\tilde{t}} + \frac{1}{s_2} + \frac{1}{s_3} = 2.$$

Pick $s_3 := \frac{p}{p-1}$ and $s_2 := \frac{p_3}{2l+1} > 1$ (since $l < 5/2$). The previous condition implies, for any $p > 2$, that

$$\frac{-\varepsilon}{1+\varepsilon} + \frac{2l+1}{p_3} = \frac{1}{q_*} < \frac{1}{1+\varepsilon} - 1 + \frac{1}{\tilde{t}} + \frac{2l+1}{p_3} = \frac{1}{p} < \frac{1}{2}. \tag{3.30}$$

This defines $q_* = q_*(\varepsilon, l) > 2$ for each fixed ε, l. Note that $1/q_* \in (0, 1/2)$, due to the given range for l in assumption (H8) (i.e., $q_* = Q^{-1}_{l,\varepsilon}(p_3)$). Moreover, due to the openness of this interval, we find that $p \in (2, q_*)$ and that, in view of (3.29),

$$|((D_{i,h}a_J)\mu(x+e_ih,t), |D_{i,h}\phi|^{p-2}D_{i,h}\phi)| \le C_{J,\rho,d,p}\|\mu\|_{L^{\frac{p_3}{2l+1}}}\|D_{i,h}\phi\|^{p-1}_{L^p}.$$

Thus, we infer from (3.21), estimating the remaining summands in a similar way, that

$$\frac{d}{dt}\|D_{i,h}\phi\|_{L^p(\Omega')} + \lambda_1 c_0\|D_{i,h}\phi\|_{L^p(\Omega')} \lesssim \|\mu\|_{L^{\frac{p_3}{2l+1}}(\Omega)} + \|\phi\|_{L^{p_3}(\Omega)}. \tag{3.31}$$

Since the right-hand side of (3.31) can be estimated in terms of the H^1-regularity of ϕ (see the proof of Theorem 3.4), we can argue once again as in the proof of Theorem 3.1, by means of the Grönwall inequality. We obtain

$$\|\phi(t)\|_{W^{1,p}(\Omega)} \le \|\phi_0\|_{W^{1,p}(\Omega)}e^{-\lambda_{1,J}c_0t} + L_4, \tag{3.32}$$

where the constant $L_4 = L_2(m, R_3) > 0$ is independent of the initial data, ϕ and time. Moreover, there exists a time $t_3 > 0$ such that

$$\|\phi(t)\|_{W^{1,p}(\Omega)} \le C_m \quad \text{for all } t \ge t_3 = t_3(R_3, R_4). \tag{3.33}$$

These arguments yield the conclusions of the theorem in the range for all $p \in (2, q_*)$. If $q_* > 3$ (or, equivalently, for given (ε, l), there holds $0 < Q_{l,\varepsilon}(p_3) < \frac{1}{3}$), then we are done since the next energy computations can be carried out as in dimension 2, whenever $p \in [q_*, \infty)$. If not (i.e., $q_* \le 3$, or simply, $Q_{l,\varepsilon}(p_3) \ge \frac{1}{3}$), we iterate the above argument finitely many times to find a (positive) increasing sequence $\{q_{*j}\}$ such that, at some (finite) $j \in \mathbb{N}_0$, $q_{*j} > 3$, owing to the fact that both ε and l are fixed at the beginning of the iteration. To set up the scheme, first set $q_* =: q_0^*$, $p_{*0} := p_3 = 6$, such that (3.31) holds for any $p \in (2, q_0^*)$ whenever q_* is defined by (3.30). Next we choose $q_1 \in (2, q_*)$ as close to $q_* =: q_0^*$ as possible, for which the dissipative estimates (3.32)–(3.33) hold with

$p = q_1$. Then consider $p_{*1} > p_{*0} = 6$, such that $W^{1,q_1}(\Omega)$ is embedded continuously into $L^{p_{*1}}(\Omega)$, and then define q_1^* by

$$\frac{1}{q_1^*} = \frac{-\varepsilon}{1+\varepsilon} + \frac{2l+1}{p_{*1}} =: Q_{l,\varepsilon}(p_{*1}) < \frac{1}{2}.$$

Then, arguing as above in (3.29), a version of (3.31) now holds for all $p \in (2, q_1^*)$ and with the exponent $p_3 = p_{*0}$ replaced by $p_{*1} > p_3$ on the right-hand side of (3.31). Since the right-hand side is also bounded uniformly in terms of (3.32)–(3.33), by the Grönwall inequality we may then conclude the same uniform estimates, in the wider range, for all $p \in (2, q_1^*) \supset (2, q_0^*)$. Thus, at once we can define an (increasing, finite) sequence $\{q_j\}$ by picking $q_{j+1} \in (2, q_j^*)$, as close as possible to q_j^*, such that the corresponding estimates (3.32)–(3.33) hold for $p = q_{j+1}$. If $q_j^* \geq 3$ (or, equivalently, $0 < Q_{l,\varepsilon}(p_{*j}) < \frac{1}{3}$), the iteration stops; if not, the (increasing) sequence p_{*j}, defined by the continuous embedding $W^{1,q_{j+1}}(\Omega) \subset L^{p_{*j}}(\Omega)$, allows one to define the sequence

$$\frac{1}{q_{j+1}^*} = \frac{-\varepsilon}{1+\varepsilon} + \frac{2l+1}{p_{*j}} =: Q_{l,\varepsilon}(p_{*j}) < \frac{1}{2}, \quad j \geq 1.$$

Finally, for all $p \in (2, q_{j+1}^*) \supset (2, q_j^*)$, one can show that (3.31) holds in this range as well (arguing by the Hölder inequality and by Young inequalities, as in (3.29)), with a right-hand side that contains the exponent p_{*j} $(> p_{*j-1})$, in place of $p_3 = p_{*0}$. Since ε, l are fixed and finite, there is ultimately a finite $j = j_\# \in \mathbb{N}$, such that $q_{j_\#}^* > 3$ (or, equivalently, $Q_{l,\varepsilon}(p_{*j_\#}) < \frac{1}{3}$) because of the stopping condition (3.27); at that point, the (uniform) estimates (3.32)–(3.33) close in light of our observations before the theorem. ■

Example 3.6. Let $F(r) = \theta r^4 - \theta_c r^2$ with $0 < \theta < \theta_c$. Notice that F satisfies (H8) with $l = q = 1$. Then, in dimension 3, a radially symmetric J satisfies (H7) for *any* $\varepsilon > 1/5$.

Example 3.7. Among radially symmetric potentials that satisfy (H6) and (H8) are the Newtonian, Bessel and Riesz-like potentials. Consider, for $x \in \mathbb{R}^N \backslash \{0\}$, the Bessel potential

$$b_s(|x|) = \frac{e^{-|x|}}{(2\pi)^{N-1} 2^{s/2} \Gamma(\frac{s}{2}) \Gamma(\frac{N-s+1}{2})} \int_0^\infty e^{-|x|t} \Big(t + \frac{t^2}{2}\Big)^{\frac{N-s-1}{2}} dt,$$

where Γ is the Gamma function and $0 < s$. Note that on $\mathbb{R}^N$, $(I - \Delta)^{-s/2} \upsilon = b_s * \upsilon$. In particular, b_s behaves as the Riesz potential, asymptotically as $|x| \to 0^+$, since

$$b_s(|x|) = \frac{\Gamma(N-s)}{2^s \pi^{s/2}} \frac{1}{|x|^{N-s}} (1 + o(1)) \quad \text{if } 0 < s < N.$$

Logarithmically behaving kernels are also included in this analysis, as

$$b_N(|x|) = -\frac{1}{2^{N-1} \pi^{N/2}} \log |x| (1 + o(1)) \quad \text{as } |x| \to 0^+.$$

Under the assumptions of the previous theorems, problem (1.3)–(1.5) generates a (strongly) continuous semigroup

$$S_p(t)\colon W^{1,p}_{(m)}(\Omega) \to W^{1,p}_{(m)}(\Omega), \quad p > N \text{ and } p \geq 2,$$

given by

$$S_p(t)\phi_0 = \phi(t), \quad t \geq 0,$$

where ϕ is the unique mild solution in the sense of Definition 2.1. Here, $W^{1,p}_{(m)}(\Omega) = W^{1,p}(\Omega) \cap L^\infty_{(m)}(\Omega)$.

Remark 3.8. The main results in this section determine the space $X = W^{1,p}_{(m)}(\Omega)$ as one possible candidate for the problem to possess the *dissipation property*.[17] However, we will see in Section 7 that certain equilibrium configurations can also be achieved for an initial datum

$$\phi_0(x) = \begin{cases} -1, & x \in \Omega_-, \\ +1, & x \in \Omega_+, \end{cases} \tag{3.34}$$

bounded in $L^\infty_{(m)}(\Omega)$, but which is not in $W^{1,p}_{(m)}(\Omega)$ for $p > N$ (here, $\Omega = \Omega_+ \cup \Omega_-$; see Example 7.1). Note that $\phi_0 \in W^{s,1}(\Omega) \cap L^\infty_{(m)}(\Omega)$ for any $s \in (0,1)$ since the characteristic function $1_{\Omega_\pm}$, of the set $\Omega_\pm$, belongs to $W^{s,1}(\Omega)$. It is an open question whether our problem also possesses a dissipation property in the space $W^{s,1}(\Omega)$, under suitable assumptions on the parameters of the problem.

4. Characterization of omega-limit sets

Let ϕ be the (unique) mild solution of (1.3)–(1.5), corresponding to some given initial datum $\phi_0 \in W^{1,p}_{(m)}(\Omega)$, $p > N$ and $p \geq 2$ (see Section 3). Our goal is to establish that once the solution ϕ enters a small L^2-neighborhood of a nonzero stationary state ϕ_*, then it must remain there for all time $t \geq t_*$, t_* large enough and, consequently, $\phi(t)$ fully converges to ϕ_* as $t \to \infty$ (and not just along subsequences!). But first we show that every mild solution $\phi = S_p(t)\phi_0$ has a nonempty ω-limit set $\omega(\phi_0)$, where $\omega(\phi_0)$ is defined by

$$\omega(\phi_0) = \left\{\phi_* : \exists\, t_n \to \infty \text{ such that } \phi(t_n) \to \phi_* \text{ strongly in } C(\overline{\Omega})\right\}.$$

To study the asymptotic behavior of solutions, we first need the following.

Lemma 4.1. *Consider the dynamical system $(W^{1,p}_{(m)}(\Omega), \{S_p(t)\}_{t\geq 0})$ under the assumptions of Theorems 3.4, 3.5. Then, any divergent sequence $\{t_n\} \subset [0,\infty)$ admits a subsequence, denoted by $\{t_{n_k}\}$, such that*

$$\lim_{t_{n_k}\to\infty} \phi(t_{n_k}) = \phi_* \textit{ strongly in } C(\overline{\Omega}), \tag{4.1}$$

[17] Namely, the trajectories corresponding to bounded sets of initial data in X, enter X after a certain time, and will stay there forever.

for some $\phi_* \in C^\alpha(\Omega)$, $0 < \alpha < 1$, *which is a solution of*

$$\begin{cases} a_K(x)\phi_* - K * \phi_* + F'(\phi_*) = \mu_* & \text{a.e. in } \Omega, \\ \mu_* = \text{constant}, \quad \bar{\phi}_* = \bar{\phi}_0 = m. \end{cases} \tag{4.2}$$

Proof. By Theorems 3.4, 3.5, $\phi \in C_b(\mathbb{R}_+; W^{1,p}_{(m)}(\Omega))$ with

$$\|\phi\|_{C^{1-N/p}(\bar{\Omega})} \lesssim \|\phi(t)\|_{W^{1,p}(\Omega)} \le C_m \quad \text{for all } t \ge t_3. \tag{4.3}$$

It follows that $\mu \in C_b(\mathbb{R}_+; L^\infty(\Omega))$, $\partial_t \phi \in L^\infty(\mathbb{R}_+; L^\infty(\Omega))$ with

$$\|\mu(t)\|_{L^\infty(\Omega)} \le C_{J,l_1,K}, \quad \|\partial_t \phi(t)\|_{L^\infty(\Omega)} \le C_{J,l_1,K}, \quad \text{for all } t \ge t_3. \tag{4.4}$$

We also infer from the energy identity (2.10) that

$$\mathcal{E}(t_2) - \mathcal{E}(t_1) = -\frac{1}{2}\int_{t_1}^{t_2}\int_\Omega\int_\Omega J(x-y)(\mu(t,x)-\mu(t,y))^2\,dx\,dy\,dt \le 0 \tag{4.5}$$

for all $0 \le t_1 < t_2 < \infty$. Furthermore, due to (4.3), $\mathcal{E}(t) \ge -C$ for all $t \ge 0$, for some positive constant $C = C(K, F)$, and so $\mathcal{E}(t)$ converges to a certain constant $\mathcal{E}_\infty$ as $t \to \infty$. Therefore, setting $t_1 = 0$ and letting $t_2 \to \infty$, we also deduce from (4.5) that

$$\int_0^\infty\int_\Omega\int_\Omega J(x-y)(\mu(t,x)-\mu(t,y))^2\,dx\,dy\,dt = \mathcal{E}_\infty - \mathcal{E}(0) \le C \tag{4.6}$$

for some constant $C < \infty$. By Theorem 2.5, each mild solution satisfies $\partial_t \phi(t) = \mathcal{L}_J(\mu(t) - \bar{\mu}(t))$ for a.e. $t > 0$. Thus, by (H3) we find

$$\begin{aligned} \|\partial_t \phi(t)\|^2_{L^2(\Omega)} &= \|\mathcal{L}_J(\mu(t) - \bar{\mu}(t))\|^2_{L^2(\Omega)} \le C_J \|\mu(t) - \bar{\mu}(t)\|^2_{L^2(\Omega)} \\ &\le \frac{C_J}{2\lambda_1}\int_\Omega J(x-y)(\mu(t,x)-\mu(t,y))^2\,dx\,dy. \end{aligned}$$

This, together with (4.6), also implies the uniform estimate

$$\int_0^\infty \|\partial_t \phi(t)\|^2_{L^2(\Omega)}\,dt \le C < \infty. \tag{4.7}$$

In particular, it follows from (4.4) and (4.7) that

$$\|\partial_t \phi(t)\|_{L^2(\Omega)} \to 0 \quad \text{as } t \to \infty. \tag{4.8}$$

Now let $\{t_n\} \subset [0, \infty)$ be a divergent sequence. Then, by (4.3)–(4.4), at least along a suitable subsequence $\{t_{n_k}\}$ of $\{t_n\}$, it follows that

$$\phi(t_{n_k}) \to \phi_* \quad \text{strongly in } C(\bar{\Omega}) \tag{4.9}$$

and

$$\mu(t_{n_k}) \rightharpoonup \mu_* \quad \text{weakly-* in } L^\infty(\Omega), \tag{4.10}$$

for some $\phi_* \in C^\alpha(\overline{\Omega})$ and $\mu_* \in L^\infty(\Omega)$. We next claim that

$$\mu(t_{n_k}) \to \mu_* \quad \text{strongly in } L^2(\Omega), \tag{4.11}$$

at least along a further subsequence (still denoted by $\{t_{n_k}\}$). Indeed, this is obvious due to the strong convergence (4.9) and the fact that $F'(\phi(t_{n_k})) \to F'(\phi_*)$ strongly in $L^2(\Omega)$. Also, (4.11) and (4.8) imply that $\mathcal{L}_J(\mu_*) = 0$, and so μ_* is constant (in Ω). We finally observe that $\phi_* \in C^\alpha(\overline{\Omega})$ is a solution of (4.2), and that the energy level $\mathcal{E}_\infty$ is the same for each stationary state ϕ_*, as claimed. The lemma is proved. ∎

We have the following convergence result which is the main result of this section.

Theorem 4.2. *Let the assumptions of Theorem 3.5 (if $N = 3$) or Theorem 3.4 (if $N = 1, 2$) hold. Assume F is real analytic*[18] *on $[-C_m, C_m]$. Then $\omega(\phi_0) = \{\phi_*\}$; namely, for any (given) $\phi_0 \in W^{1,p}_{(m)}(\Omega)$, the corresponding mild solution ϕ satisfies*

$$\|\phi(t) - \phi_*\|_{C(\overline{\Omega})} = O((1+t)^{-\frac{1}{\gamma}}) \quad \text{as } t \to \infty, \tag{4.12}$$

for some $\gamma > 0$, where ϕ_* is (some) solution of (4.2).

Proof. First, we observe that since the mapping $\phi \mapsto K * \phi \colon L^\infty(\Omega) \to C(\overline{\Omega})$ is compact, all stationary solutions $\phi_* \in \omega(\phi_0)$ are continuous in $\overline{\Omega}$ and bounded in $C^\alpha(\overline{\Omega})$. Secondly, setting

$$l(\mu(t), \mu(t)) := \frac{1}{2}\int_\Omega \int_\Omega J(x-y)(\mu(t,x) - \mu(t,y))^2 \, dx\, dy,$$

by the energy identity (2.10),

$$\frac{d\mathcal{E}(t)}{dt} = -l(\mu(t), \mu(t)) \quad \text{for } t \geq 0. \tag{4.13}$$

Thus, integrating (4.13) over (t, ∞), we get

$$\int_t^\infty l(\mu(s), \mu(s))\, ds = \mathcal{E}(t) - \mathcal{E}_\infty. \tag{4.14}$$

We can now apply Lemma A.3 and (H3) for the Poincaré inequality, to infer the existence of some constants $\theta \in (0, \frac{1}{2})$, $C > 0$, $\varepsilon > 0$, such that

$$\begin{aligned} |\mathcal{E}(t) - \mathcal{E}_\infty|^{1-\theta} &\leq C\|\mu(t) - \bar{\mu}(t)\|_{L^2(\Omega)} \\ &\leq \frac{C}{\lambda_1^{1/2}}\sqrt{l(\mu(t), \mu(t))} \end{aligned} \tag{4.15}$$

provided that

$$\|\phi - \phi_*\|_{L^2(\Omega)} \leq \varepsilon. \tag{4.16}$$

[18] $C_m > 0$ is the radius of the absorbing ball; see Theorems 3.4, 3.5.

Combining (4.15) with (4.14) yields

$$\left(\int_t^\infty l(\mu(s),\mu(s))\,ds\right)^{2(1-\theta)} \le C\,l(\mu(t),\mu(t)) \tag{4.17}$$

for all $t > 0$, for as long as (4.16) holds. Note that, in general, the quantities θ, C and ε above may depend on ϕ_* and $\lambda_1 = \lambda_{1,J}$. Let us set

$$M = \cup\{\mathcal{I} : \mathcal{I} \text{ is an open interval on which (4.16) holds}\}.$$

Clearly, M is nonempty since $\phi_* \in \omega(\phi_0)$. As usual, we can then use (4.17), the fact that $Z(t) := \sqrt{l(\mu(t),\mu(t))} \in L^2(0,\infty)$ (cf. (4.13)), and exploit [12, Lemma 7.1] with $\alpha = 2(1-\theta)$ to deduce that $Z(\cdot) \in L^1(M)$ and

$$\int_M Z(s)\,ds = \int_M \sqrt{l(\mu(s),\mu(s))}\,ds \le C(\phi_*) < \infty. \tag{4.18}$$

Consequently, using (4.18) and the fact that $\partial_t\phi = \mathcal{L}_J(\mu-\bar{\mu})$, we also obtain

$$\int_M \|\partial_t\phi(s)\|_{L^2(\Omega)}\,ds < \infty. \tag{4.19}$$

Using (4.19), we can derive the integrability of $\partial_t\phi$ in $L^1(\tau,\infty;L^2(\Omega))$ for some $\tau > 0$. Indeed, we claim that we can find a sufficiently large time $\tau > 0$ such that $(\tau,\infty) \subset M$. To this end, recalling (4.14) and the above bounds, we also have that $\partial_t\phi \in L^2(0,\infty;L^2(\Omega))$, $Z \in L^2(0,\infty)$ and, furthermore, for any $\eta > 0$ there exists a time $t_* = t_*(\eta) > 0$ such that

$$\begin{gathered} \|\partial_t\phi\|_{L^1(M\cap(t_*,\infty);L^2(\Omega))} \le \eta, \quad \|\partial_t\phi\|_{L^2((t_*,\infty);L^2(\Omega))} \le \eta, \\ \|Z\|_{L^2((t_*,\infty))} \le \eta. \end{gathered} \tag{4.20}$$

Next, observe that by the uniform bounds provided in Section 3, there is a time $t_m > 0$ such that

$$\sup_{t\ge t_m} \|\phi(t)\|_{W^{1,p}(\Omega)} \le C_m. \tag{4.21}$$

Now, let $(t_0,t_2) \subset M$, for some $t_2 > t_0 \ge t_*(\eta)$, $|t_0 - t_2| \ge 1$ such that (4.21) holds (without loss of generality, we shall assume that $t_* \ge t_m$). Exploiting (4.20) and (4.21), we obtain

$$\begin{aligned} \|\phi(t_0)-\phi(t_2)\|_{L^2(\Omega)}^2 &= 2\int_{t_0}^{t_2} \langle \partial_t\phi(s), \phi(s)-\phi(t_0)\rangle\,ds \\ &\le 2\int_{t_0}^{t_2} \|\partial_t\phi(s)\|_{L^2(\Omega)}(\|\phi(s)\|_{L^2(\Omega)} + \|\phi(t_0)\|_{L^2(\Omega)})\,ds \\ &\le 2C\|\partial_t\phi\|_{L^1(t_0,t_2;L^2(\Omega))}(\|\phi\|_{L^\infty(t_*,\infty;L^2(\Omega))} + 1) \\ &\le 2C(1+C_m)\eta. \end{aligned}$$

Therefore we can choose a time $t_*(\eta) = \tau < t_0 < t_2$, such that

$$\|\phi(t_0)-\phi(t_2)\|_{L^2(\Omega)} < \varepsilon/3, \tag{4.22}$$

provided that (4.16) holds for all $t \in (t_0, t_2)$. Since $\phi_* \in \omega[\phi]$, a large (redefined) τ can be chosen such that

$$\|\phi(\tau) - \phi_*\|_{L^2(\Omega)} < \varepsilon/3. \tag{4.23}$$

Hence, (4.22) yields $(\tau, \infty) \subset M$. Indeed, taking $\bar{t} = \inf\{t > \tau : \|\phi(t) - \phi_*\|_{L^2(\Omega)} \geq \varepsilon\}$, we have $\bar{t} > \tau$ and $\|\phi(\bar{t}) - \phi_*\|_{L^2(\Omega)} \geq \varepsilon$ if $\bar{t}$ is finite. On the other hand, in view of (4.22) and (4.23), we have

$$\|\phi(t) - \phi_*\|_{L^2(\Omega)} \leq \|\phi(t) - \phi(\tau)\|_{L^2(\Omega)} + \|\phi(\tau) - \phi_*\|_{L^2(\Omega)} < 2\varepsilon/3$$

for all $\bar{t} > t \geq \tau$, and this leads to a contradiction. Therefore, $\bar{t} = \infty$ and by (4.20) the integrability of $\partial_t \phi$ in $L^1(\tau, \infty; L^2(\Omega))$ follows. Hence, $\omega[\phi] = \{\phi_*\}$ in the strong topology of $L^2(\Omega)$, as well as in the $C(\overline{\Omega})$-topology, owing to the compactness of $W^{1,p}(\Omega) \subset C(\overline{\Omega})$ and interpolation. As usual, when applying the Łojasiewicz inequality, the following rate of convergence holds, due to (4.15):

$$\|\phi(t) - \phi_*\|_{C(\overline{\Omega})} \leq C(1 + t)^{-\frac{1}{\gamma}} \quad \text{as } t \to \infty,$$

for some constants $C > 0$, $\gamma = \gamma(\theta, \phi_*) > 0$. The proof is finished. ∎

5. Finite-dimensional attractors

We show that problem (1.3)–(1.5) possesses finite-dimensional exponential attractors, provided the assumptions of the previous sections hold. The existence of the (finite-dimensional) global attractor, with similar properties, follows immediately as a corollary of the subsequent result (see, e.g., [19]). Let $p > N$ and $p \geq 2$ in what follows.

Theorem 5.1. *Let the assumptions of Theorem 3.5 (if $N = 3$) or Theorem 3.4 (if $N = 1, 2$) hold for some $F \in C^3(\mathbb{R})$. For every fixed $M \geq 0$ such that $m \in [-M, M]$, there exists an exponential attractor $\mathcal{G} = \mathcal{G}_M$ bounded in $W^{1,p}(\Omega)$, for the dynamical system $(W^{1,p}_{(m)}(\Omega), S_p(t))$, which satisfies the following properties:*

(1) *semi-invariance:*

$$S_p(t)\mathcal{G} \subseteq \mathcal{G} \quad \text{for every } t \geq 0;$$

(2) *exponential attraction:*

$$\operatorname{dist}_{C(\overline{\Omega})}(S_p(t)B, \mathcal{G}) \leq C_M e^{-\kappa t} \quad \text{for all } t \geq 0,$$

for any bounded $B \subset W^{1,p}(\Omega)$, for some positive constants C_M and κ (which are independent of B);

(3) *finite-dimensionality:*

$$\dim_F(\mathcal{G}, C(\overline{\Omega})) \leq C_M < \infty.$$

Proof. Let B_0 be a bounded absorbing set with respect to the $W^{1,p}$-topology. This set clearly exists by Theorem 3.5 or Theorem 3.4, respectively. We next define $B_1 = [\bigcup_{t\geq 0} S_p(t)B_0]_{H^1}$, where $[\cdot]_{H^1}$ denotes closure in the space $H^1(\Omega)$, and then set $\mathbb{B} = S(1)B_1$. Thus, $\mathbb{B}$ is a semi-invariant and closed (for the H^1-metric) subset of the phase space $W^{1,p}_{(m)}(\Omega)$. On the other hand, for $m \in [-M, M]$ we have

$$\sup_{t\geq 0}(\|\phi(t)\|_{W^{1,p}(\Omega)} + \|\mu(t)\|_{W^{1,p}(\Omega)} + \|\partial_t\phi(t)\|_{W^{1,p}(\Omega)}) \leq C_M \tag{5.1}$$

for every trajectory ϕ originating from $\phi_0 = \phi(0) \in \mathbb{B}$, for some positive constant C_M, which is independent of the choice of $\phi_0 \in \mathbb{B}$.

We can now define the map $\mathbb{S} = S_p(T)\colon \mathbb{B} \to \mathbb{B}$ and $\mathcal{H} = H^1(\Omega)$ for a fixed $T > 0$ such that $e^{-c_0\lambda_1 T} < \frac{1}{2}$. Then set

$$\begin{aligned} \mathcal{V}_1 &:= L^2([0,T]; H^1(\Omega)) \cap H^1([0,T]; H^1(\Omega)^*), \\ \mathcal{V} &:= L^2([0,T]; L^q(\Omega)). \end{aligned} \tag{5.2}$$

Here $q := 2p/(p-2)$ if $N = 2, 3$, and $q = \infty$ if $N = 1$. Define the operator $\mathbb{T}\colon \mathbb{B} \to \mathcal{V}_1$ by $\mathbb{T}\phi_0 := \phi \in \mathcal{V}_1$, where ϕ solves the nonlocal problem (1.3)–(1.5), with $\phi(0) = \phi_0 \in \mathbb{B}$. Notice that $\mathcal{V}_1$ is compactly embedded into $\mathcal{V}$. Also set $b_i := \phi_i(0) \in \mathbb{B}$. With this choice of spaces and operators, it follows from (A.13)–(A.14) of Proposition A.2 that

$$\begin{aligned} &\|\mathbb{T}b_1 - Tb_2\|_{\mathcal{V}_1} \leq L\|b_1 - b_2\|_{\mathcal{H}}, \\ &\|\mathbb{S}b_1 - \mathbb{S}b_2\|_{\mathcal{H}} \leq \gamma\|b_1 - b_2\|_{\mathcal{H}} + K\|\mathbb{T}b_1 - Tb_2\|_{\mathcal{V}} \end{aligned}$$

for some $K := C > 0$, independent of time and b_i, with $L := Ce^{CT}$ and $\gamma := e^{-c_0\lambda_1 T}$. Therefore, all the hypotheses of Proposition A.4 hold for the maps $\mathbb{S}$ and $\mathbb{T}$, respectively. Subsequently, we can infer the existence of a (discrete) exponential attractor $\mathcal{G}_d \subset \mathbb{B}$ associated with the semigroup $\mathbb{S}(n) = S_p(nT)$, $n \in \mathbb{N}$. Finally, the map $(t, \phi_0) \mapsto S_p(t)\phi_0$ is also uniformly Hölder continuous on $[0,T] \times \mathbb{B}$, when $\mathbb{B}$ is endowed with the H^1-topology. Next, one has that

$$\mathcal{G} := \bigcup_{t\in[0,T]} S_p(t)\mathcal{G}_d$$

is the desired exponential attractor for the continuous dynamical system $(W^{1,p}_{(m)}(\Omega), S_p(t))$. In particular, the exponential attraction and finite-dimensionality of $\mathcal{G}$, in the statement of Theorem 5.1, are satisfied with respect to the H^1-topology. However, owing to the fact that $\mathbb{B}$ is bounded in $W^{1,p}_{(m)}(\Omega) \subset C^{1-N/p}(\overline{\Omega})$, and interpolation (i.e., $\|\phi\|_{C(\overline{\Omega})} \leq C_\lambda\|\phi\|_{W^{1,\lambda}} \leq C_{p,q}\|\phi\|^s_{W^{1,p}}\|\phi\|^{1-s}_{H^1}$ for some $s = s(p,\lambda) \in (0,1)$ and $N < \lambda < p$), $\mathcal{G}$ is also an exponential attractor for $S_p(t)$ restricted to $\mathbb{B}$ with respect to the stronger metric on $C(\overline{\Omega})$ (and in fact, in the stronger metric of $W^{1,\lambda}(\Omega)$). Therefore, all the conclusions of Theorem 5.1 hold. ∎

6. Convergence to the classical Cahn–Hilliard equation

Consider the classical problem for the Cahn–Hilliard equation

$$\partial_t \psi(t,x) = \Delta v(t,x), \quad v(t,x) = -\Delta\psi(t,x) + F'(\psi(t,x)), \tag{6.1}$$

for $(t,x) \in (0,\infty)\times\Omega$, with

$$\nabla v(t,x)\cdot n = \nabla\psi(t,x)\cdot n = 0 \quad \text{for } (t,x)\in(0,\infty)\times\partial\Omega, \tag{6.2}$$

and subject to the initial condition

$$\psi(0,x) = \phi_0(x), \quad x\in\Omega. \tag{6.3}$$

Here, n stands for the outer *normal* to the boundary $\partial\Omega$ of the domain Ω.

In this section we show that problem (6.1)–(6.3) can be approximated by suitable doubly nonlocal problems of the form (1.3)–(1.5) when the symmetric kernel $K = J$ is rescaled appropriately, the domain Ω is *smooth* enough, and the second derivative of the potential F is bounded away from *any interval* containing $(-1,1)$. In what follows we will assume the following hypotheses:

(H9) $J \in C(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)$ is radially symmetric over $\mathbb{R}^N$ with supp$(J)= B(0,r)$ (for some $r>0$).

(H10) Let Ω be a bounded $C^{4+\alpha}$-domain, $0<\alpha<1$, and let $\phi_0\in C^{4+\alpha}(\overline{\Omega})$ satisfy $\nabla\phi_0\cdot n = \nabla(\Delta\phi_0)\cdot n = 0$ on $\partial\Omega$.[19]

(H11) $F\in C^2(\mathbb{R})$ satisfies $F(0)=F'(0)=0$, and for some $\varepsilon\ge 0$, $\ell_1>0$,

$$|F''(r)|\le \ell_1 \quad \text{for all } r\in(-\infty,-1-\varepsilon)\cup(1+\varepsilon,\infty). \tag{6.4}$$

Clearly, $\ell_1>0$ may depend on ε; in particular, (6.4) implies that F has at most quadratic growth[20] at $\pm\infty$. The reasoning behind assumption (H11) is as follows. The physically relevant example is the logarithmic potential defined by

$$\overline{F}(s) = \theta F_0(s) - \theta_c s^2, \quad F_0(s) := (1+s)\log(1+s) + (1-s)\log(1-s), \tag{6.5}$$

for all $s\in[-1,1]$, with $0<\theta<\theta_c$, where θ is the temperature of the system and θ_c the critical temperature, both assumed to be constant. In this context, F_0 refers to the entropy of the binary mixture (see, e.g., [17]). However, F_0 is also quite often replaced[21] by a polynomial approximation over the interval $[-1,1]$ (typically, by s^4), which then leads to an approximation of $\overline{F}$ (over the interval $[-1,1]$) by the double-well potential

[19] In particular, this is also equivalent to $\nabla\phi_0\cdot n = \nabla(-\Delta\phi_0 + F'(\phi_0))\cdot n = 0$ on $\partial\Omega$.

[20] (H11) implies that the mapping $F'\colon L^p(\Omega)\to L^p(\Omega)$ is (globally) Lipschitz continuous.

[21] Since any higher-order derivatives $F_0^{(j)}$, $j=1,2$ end up being singular at the endpoints ±1. In particular, one has $F_0{}'(\pm1)=\pm\infty$ and $F_0''(\pm1)=\infty$.

$F(s) = \theta s^4 - \theta_c s^2$. Finally, since concentration values outside the interval $\mathbb{I}_\varepsilon := [-1-\varepsilon, 1+\varepsilon]$, for some $\varepsilon \geq 0$, are not physically relevant, we can modify the double-well potential F outside the interval $\mathbb{I}_\varepsilon$ in such a way that the resulting potential F is of class C^2 over the whole interval $\mathbb{R}$, satisfying (6.4). Although the study of the singular potential case in (6.5) is not the goal of the present contribution, we point out that the subsequent result immediately applies once one knows that the *separation property* holds for the order parameter satisfying both the doubly nonlocal problem as well as the classical problem, with a singular potential. We will return to this question elsewhere.

Given $\delta > 0$, consider the rescaled kernel

$$J_\delta(x) = \frac{C_J}{\delta^N} J\left(\frac{x}{\delta}\right) \quad \text{with } C_J^{-1} = \frac{1}{2}\int_{B(0,r)} J(x)|x_N|^2\,dx. \tag{6.6}$$

For $(t,x) \in (0,\infty)\times\Omega$, let $\phi_\delta = \phi_\delta(t,x)$ be the unique solution of the doubly nonlocal problem

$$\begin{cases} \partial_t\phi_\delta(t,x) = \delta^{-2}\displaystyle\int_\Omega J_\delta(x-y)(\mu_\delta(y,t)-\mu_\delta(x,t))\,dy, \\ \mu_\delta(t,x) = -\delta^{-2}\displaystyle\int_\Omega J_\delta(x-y)(\phi_\delta(y,t)-\phi_\delta(x,t))\,dy + F'(\phi_\delta(t,x)), \\ \phi_\delta(0,x) = \phi_0(x), \quad x\in\Omega. \end{cases} \tag{6.7}$$

By Section 2, such a solution ϕ_δ exists for all $t\in[0,T]$, for any $T>0$, even upon imposing the (slightly more restrictive) conditions (H1), (H9)–(H11). Also, assumptions (H10)–(H11) imply the existence of a (unique) classical solution $\psi \in C^{4+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])$ to (6.1)–(6.3), such that $v\in C^{2+\alpha,\alpha/2}(\overline{\Omega}\times[0,T])$. See, for instance, [24].

The main result of this section reads as follows.

Theorem 6.1. *Let* (H1) *and* (H9)–(H11) *hold. Let* ψ *be the solution of* (6.1)–(6.3) *and* ϕ_δ *be the solution of* (6.7) *with* J_δ *as in* (6.6)*. Then, for any* $T>0$*,*

$$\lim_{\delta\to 0^+} \|\phi_\delta - \psi\|_{C([0,T];L^2(\Omega))} = 0. \tag{6.8}$$

Proof. The proof of (6.8) is based on an energy estimate for the difference $w_\delta = \phi_\delta - \psi$ since no maximum principle holds for our problem. Let $\tilde\psi$ be a $C^{4+\alpha,1+\alpha/2}(\mathbb{R}^N\times[0,T])$-smooth extension of ψ, and $\tilde v$ be a $C^{2+\alpha,\alpha/2}$-smooth extension v to $\mathbb{R}^N\times[0,T]$ (recall also that $\nabla\tilde\psi\cdot n = \nabla\tilde v\cdot n = 0$ on $\partial\Omega$). Following [3], we also set $m_\delta = \mu_\delta - v$ and consider the following operators:

$$L_\delta(w)(x,t) = \delta^{-2}\int_\Omega J_\delta(x-y)(w(y,t)-w(x,t))\,dy$$

and

$$\widetilde{L}_\delta(w)(x,t) = \delta^{-2}\int_{\mathbb{R}^N} J_\delta(x-y)(w(y,t)-w(x,t))\,dy.$$

Since $\Delta\psi = \Delta\tilde{\psi}$ and $\Delta v = \Delta\tilde{v}$ in Ω, the difference w_δ then satisfies for $(t,x) \in [0,T] \times \Omega$,

$$\begin{cases} \partial_t w_\delta(t,x) = L_\delta(m_\delta)(t,x) + F_\delta(\tilde{v})(x,t), \\ m_\delta(t,x) = -L_\delta(w_\delta)(t,x) + F'(\phi_\delta(t,x)) - F'(\psi(t,x)) - F_\delta(\tilde{\psi})(x,t), \\ w_\delta(0,x) = 0, \quad x \in \Omega. \end{cases} \tag{6.9}$$

Here, for any smooth function $\omega \in C^{2+\alpha,\alpha/2}(\mathbb{R}^N \times [0,T])$, we have set

$$F_\delta(\omega)(x,t) = \widetilde{L}_\delta(\omega)(x,t) - \Delta\omega - \delta^{-2} \int_{\mathbb{R}^N \setminus \Omega} J_\delta(x-y)(\omega(y,t) - \omega(x,t))\,dy. \tag{6.10}$$

Since J is radially symmetric and $\tilde{v} \in C^{2+\alpha,\alpha/2}(\mathbb{R}^N \times [0,T])$, $\tilde{\psi} \in C^{4+\alpha,1+\alpha/2}(\mathbb{R}^N \times [0,T])$, we have by a simple Taylor expansion (see [3, p. 50]) that

$$\sup_{t\in[0,T]} \|\widetilde{L}_\delta(\tilde{v}) - \Delta\tilde{v}\|_{L^\infty(\Omega)} = \sup_{t\in[0,T]} \|\widetilde{L}_\delta(\tilde{\psi}) - \Delta\tilde{\psi}\|_{L^\infty(\Omega)} = O(\delta^a).$$

For the last integral term in (6.10), we apply [3, Lemma 3.14] to find that $|F_\delta(\omega)| \le C\delta^\alpha + C\int_{\mathbb{R}^N\setminus\Omega} J_\delta(x-y)\,dy$, with $\omega \in \{\tilde{v}, \tilde{\psi}\}$, for some constant $C > 0$ independent of δ. In particular, this implies that $F_\delta(\omega)(x,t)$ is bounded for all $x \in \Omega_\delta = \{x \in \Omega : \mathrm{dist}(x,\partial\Omega) < \delta\}$ and $0 \le t \le T$. Since $|\Omega_\delta| = O(\delta)$ and $F_\delta(\omega)(x,t) = O(\delta^\alpha)$ for $x \in \Omega\setminus\Omega_\delta$, $t \in [0,T]$, we then get

$$\lim_{\delta\to0^+} \|F_\delta(\tilde{v})\|_{C([0,T];L^2(\Omega))} = \lim_{\delta\to0^+} \|F_\delta(\tilde{\psi})\|_{C([0,T];L^2(\Omega))} = 0. \tag{6.11}$$

Next we multiply the first equation of (6.9) by w_δ, the second equation of (6.9) by m_δ, and then integrate the resulting identities over Ω. We obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w_\delta(t)\|^2_{L^2(\Omega)} &= \big(L_\delta(m_\delta(t)), w_\delta(t)\big)_{L^2} + (F_\delta(\tilde{v}), w_\delta(t))_{L^2}, \\ \|m_\delta(t)\|^2_{L^2(\Omega)} &= \big(-L_\delta(w_\delta(t)), m_\delta(t)\big)_{L^2} + \big(F'(\phi_\delta(t,x)) - F'(\psi(t,x)), m_\delta(t)\big)_{L^2} \\ &\quad - (F_\delta(\tilde{\psi}), m_\delta(t))_{L^2}. \end{aligned}$$

Using the fact that $L_\delta\colon L^2(\Omega) \to L^2(\Omega)$ is a self-adjoint (bounded) operator, we derive

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|w_\delta(t)\|^2_{L^2(\Omega)} + \|m_\delta(t)\|^2_{L^2(\Omega)} \\ &\quad = (F_\delta(\tilde{v}), w_\delta(t))_{L^2} + \big(F'(\phi_\delta(t)) - F'(\psi(t)), m_\delta(t)\big)_{L^2} - (F_\delta(\tilde{\psi}), m_\delta(t))_{L^2}. \end{aligned}$$

The Young inequality combined with the fact that F' is Lipschitz continuous (without loss of generality, we assume that the (uniform) Lipschitz constant of F' is $\ell_1 = \ell_1(\varepsilon)$; cf. (6.4)) as a mapping from $L^2(\Omega) \to L^2(\Omega)$ then yields the estimate

$$\begin{aligned} &\frac{d}{dt}\|w_\delta(t)\|^2_{L^2(\Omega)} + \|m_\delta(t)\|^2_{L^2(\Omega)} \\ &\quad \le C_\varepsilon\|w_\delta(t)\|^2_{L^2(\Omega)} + 2(\|F_\delta(\tilde{\psi}(t))\|^2_{L^2(\Omega)} + \|F_\delta(\tilde{v}(t))\|^2_{L^2(\Omega)}). \end{aligned} \tag{6.12}$$

Here, $C_\varepsilon := 2(\ell_1^2(\varepsilon)+1) > 0$ is independent of δ and w_δ. Since $w_\delta(0) = 0$, we infer from (6.12) and the application of the Grönwall inequality that

$$\|w_\delta(t)\|^2_{L^2(\Omega)} \leq 2(\|F_\delta(\tilde{\psi})\|^2_{C([0,T];L^2(\Omega))} + \|F_\delta(\tilde{v})\|^2_{C([0,T];L^2(\Omega))})e^{C_\varepsilon T}$$

for all $t \in [0,T]$. Claim (6.8) now follows in view of (6.11), and the theorem is proved. ∎

7. Numerical analysis and implementation of a forward Euler scheme

In this section we present four examples of the numerical doubly nonlocal Cahn–Hilliard equation posed on a two-dimensional domain. In each example we treat a different interaction kernel J: Gaussian, logarithmic, Riesz and bimodal (see Figures 1 and 2 for one-dimensional plots of each kernel). The second kernel K is kept as a Gaussian throughout.

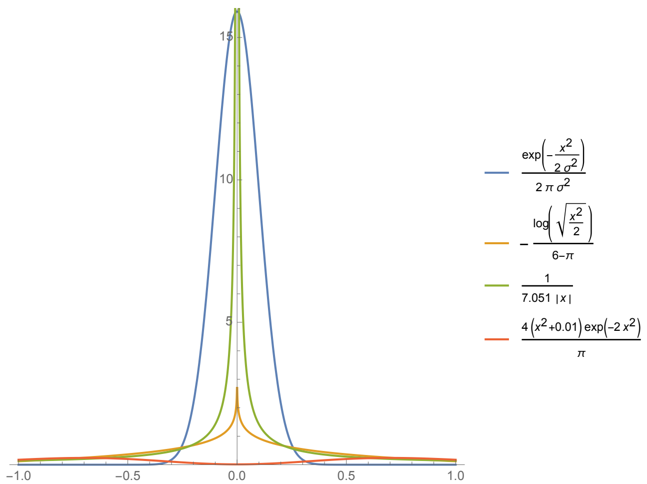


Figure 1. One-dimensional profiles of the four different kernels J used in the two-dimensional simulations.

This means, in the first instance when both kernels are the same, we may, and do, illustrate the convergence guaranteed in Theorem 6.1. In these studies we discretize the equation over time $[0,T]$ and space Ω in a conventional and literal sense. We treat the time derivative with the divided difference

$$\partial_t \phi(t,x) \approx \frac{\phi(n+1,m,l) - \phi(n,m,l)}{\Delta t}$$

and convolutions as simply (compare [4, p. 35])

$$(J * \mu)(t,x) = \int_\Omega J(x-y)\mu(y)\,dy \approx \Delta x^2 \sum_{k=1}^{M}\sum_{j=1}^{M} J(\Delta x(m-k), \Delta x(l-j))\mu(k,j)$$

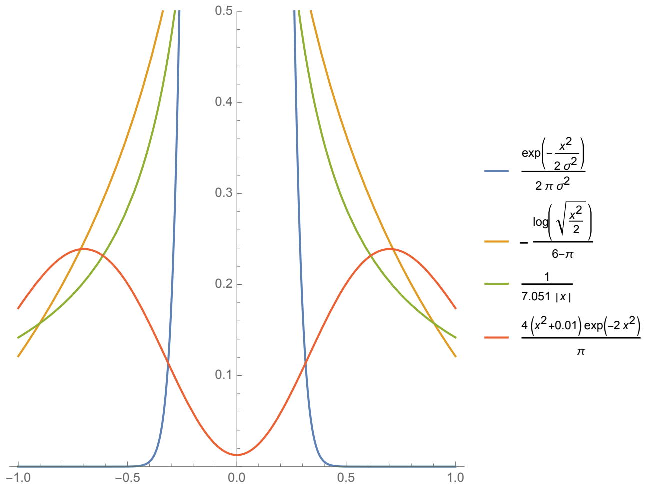


Figure 2. One-dimensional profiles of the four different kernels J used in the two-dimensional simulations. Here the range is restricted in order to express the "bimodal" kernel.

and

$$(K * \phi)(t, x) = \int_\Omega K(x-y)\phi(y)\,dy \approx \Delta x^2 \sum_{k=1}^{M} \sum_{j=1}^{M} K(\Delta x(m-k), \Delta x(l-j))\phi(k, j).$$

Initial data ϕ_0 is assumed to be rough/binary, taking only the values $+1$ or -1. In each example two control constants, α and β, will be introduced as multipliers on the discretized convolution. Bounds on the control constants are then sought to ensure that the absolute value of the first iterate $|\phi(1, m, l)|$ lies (approximately) within 1 over all m and l; that is, for any random initial data described above. With that in place we are able to (heuristically) determine suitable proportions, called α and β, so that the numerical model remains stable for all n.

Some global assumptions.

(1) $T > 0$ and $t \in [0, T]$.

(2) $N \in \mathbb{N}$ and $\Delta t = \frac{T}{N}$. The time interval $[0, T]$ is partitioned into N subintervals $[(n-1)\Delta t, n\Delta t]$ for $n = 1, 2, \ldots, N$.

(3) $L > 0$ and $x = (x_1, x_2) \in \Omega = (-L, L)^2$.

(4) M is a positive integer and $\Delta x = \frac{2L}{M}$. The domain Ω is partitioned into M^2 subsquares determined by $(-L + (m-1)\Delta x, -L + m\Delta x) \times (-L + (l-1)\Delta x, -L + l\Delta x)$ for $m, l = 1, 2, \ldots, M$.

(5) Initial data $\phi_0(x_1, x_2) \in L^\infty_{(0)}(\Omega)$ is discretized on the $(M \times M)$-mesh so that

$$\phi(0, m, l) \approx \phi_0(x_1, x_2).$$

(6) In accordance with Remark 2.6, the double-well potential is approximated by $F(r) = \theta r^4 - \theta_c r^2$ with $\theta = \frac{1}{4}$ and $\theta_c = \frac{1}{2}$.

Example 7.1. First we examine the discretized doubly nonlocal Cahn–Hilliard equation when both kernels are the same Gaussian.

(1) Here we set $T = 1$, $N = 2^8$, $L = 1$ and $M = 2^7$. Our choice of (discretized) initial condition $\phi(0, m, l)$ is the bounded function that consists of a uniformly randomly chosen number $\{-1, 1\}$ at each $(m, l) \in (M \times M)$-mesh. See Figure 3. Note that we choose $\phi(0, m, l)$ with the property $\sum_{m,l=1}^{M} \phi(0, m, l) = 0$ in order to emulate $\bar{\phi}_0 = 0$.

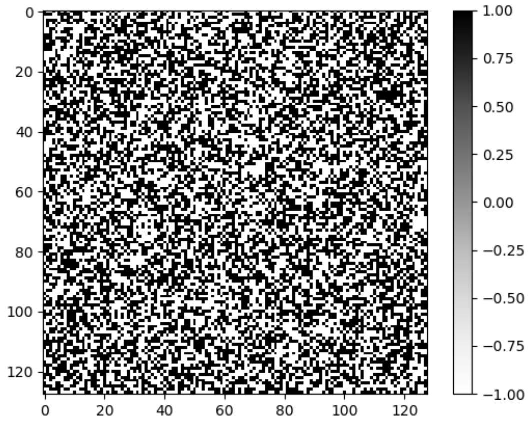


Figure 3. Discrete initial data $\phi(0, m, l)$ in the $(2^7 \times 2^7)$-mesh with values chosen randomly from $\{-1, 1\}$ with $\sum_{m,l=1}^{2^7} \phi(0, m, l) = 0$.

(2) $J(x_1, x_2) = K(x_1, x_2) = \frac{1}{2\pi\sigma^2} \exp(-\frac{x_1^2 + x_2^2}{2\sigma^2})$ with $\sigma = 0.1$ fixed. In this example we will employ the rescaled kernels appearing in (6.6) where δ is allowed to range in $\{0.03125, 0.0625, 0.125, 0.25, 0.5, 1.0\}$ and where the constant $C_J = 15.9577$.

Under the above assumptions, equations (6.7) are discretized in time with a forward Euler scheme and the convolution integrals are discretized with direct Riemann sums. For each $n = 1, 2, \dots, N$ and for each $m, l = 1, 2, \dots, M$, we consider the following approximations of $\phi(t, x)$ and $\mu(t, x)$:

$$\begin{aligned} \phi(n, m, l) &= \phi(n-1, m, l) \\ &\quad + \alpha \Delta t \Delta x^2 \sum_{k,j=1}^{M} \frac{C_J}{2\pi\sigma^2\delta^4} \exp\Big(-\frac{((m-k)\Delta x)^2 + ((l-j)\Delta x)^2}{2\sigma^2\delta^2}\Big) \\ &\qquad \times (\mu(n-1, k, j) - \mu(n-1, m, l)) \end{aligned}$$

and

$$\mu(n-1,m,l) = -\beta\Delta x^2 \sum_{k,j=1}^{M} \frac{C_J}{2\pi\sigma^2\delta^4} \exp\Big(-\frac{((m-k)\Delta x)^2 + ((l-j)\Delta x)^2}{2\sigma^2\delta^2}\Big) \times (\phi(n-1,k,j) - \phi(n-1,m,l)) + \phi(n-1,m,l)^3 - \phi(n-1,m,l),$$

where M, Δx, Δt, σ, δ and C_J are known values, and ranges of the "control constants" α and β will be sought to provide numerical stability. Indeed, it is worth analyzing the first step $n = 1$ to find suitable choices of the constants α and β that might aid in the convergence of the two discretized relations above. At $t = 1 \cdot \Delta t$ we find

$$\phi(1,m,l) = \phi(0,m,l) + \alpha\Delta t\,\Delta x^2 \sum_{k,j=1}^{M} \frac{C_J}{2\pi\sigma^2\delta^4} \exp\Big(-\frac{((m-k)\Delta x)^2 + ((l-j)\Delta x)^2}{2\sigma^2\delta^2}\Big) \times (\mu(0,k,j) - \mu(0,m,l)),$$

where now

$$\mu(0,m,l) = -\beta\Delta x^2 \sum_{k,j=1}^{M} \frac{C_J}{2\pi\sigma^2\delta^4} \exp\Big(-\frac{((m-k)\Delta x)^2 + ((l-j)\Delta x)^2}{2\sigma^2\delta^2}\Big) \times (\phi(0,k,j) - \phi(0,m,l)) + \phi(0,m,l)^3 - \phi(0,m,l).$$

Recall that the initial condition $\phi(0, m, l)$ takes only the values $\{-1, +1\}$. Hence, $\phi(0, m, l)$, $\phi(0, k, j) \in \{-1, +1\}$ for each choice of $m, l = 0, 1, 2, \ldots, M$ and $k, j = 1, 2, \ldots, M$, and we further find

$$\phi(0,k,j) - \phi(0,m,l) \in \{-2,0,2\},$$

and

$$\phi(0,m,l)^3 - \phi(0,m,l) = 0,$$

so that

$$\mu(0,m,l) = -\frac{\beta\Delta x^2 C_J}{2\pi\sigma^2\delta^4} \sum_{k,j=1}^{M} \exp\Big(-\frac{((m-k)\Delta x)^2 + ((l-j)\Delta x)^2}{2\sigma^2\delta^2}\Big)\{-2,0,2\}.$$

Illustration 1. To further motivate these formulas and to show how they are important to the general scheme, we provide a simple illustration of the first iterate $\phi(1, m, l)$ on the square $(-1, 1)^2$ with $M = 2$. So $\Delta x = 1$ and there are four subsquares determined by $(-1 + (m - 1), -1 + m) \times (-1 + (l - 1), -1 + l)$ where $m, l = 1, 2$. In this 2×2 case we show an example discretized initial condition $\phi(0, m, l)$ in Table 1.

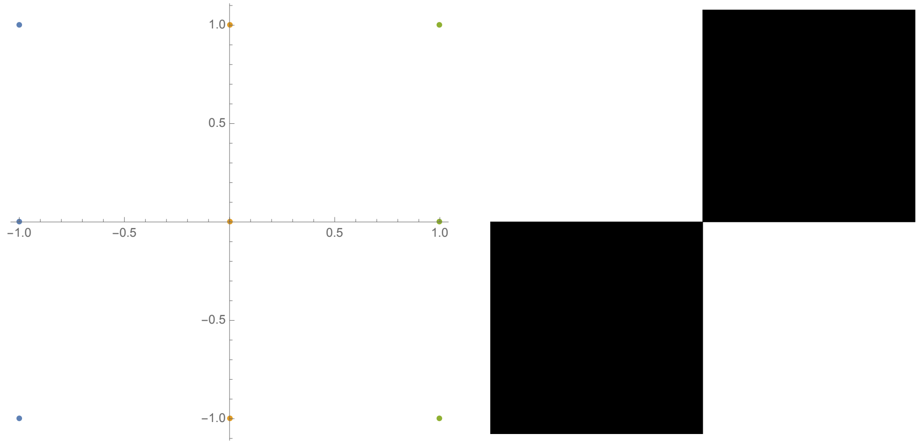


Table 1. Left: The nine mesh points on the domain $\Omega = (-1, 1)^2$. Right: A discretized initial condition $\phi(0, m, l)$.

We first obtain $\mu(0, m, l)$ on each of the four subsquares from simple calculations:

$$\mu(0,1,1) = \mu(0,2,2) = \frac{2\beta\Delta x^2 C_J}{\pi\sigma^2\delta^4 e^{\Delta x^2/2\sigma^2\delta^2}},$$
$$\mu(0,2,1) = \mu(0,1,2) = -\frac{2\beta\Delta x^2 C_J}{\pi\sigma^2\delta^4 e^{\Delta x^2/2\sigma^2\delta^2}}.$$

With these we are able to also find the four $\phi(1, m, l)$:

$$\phi(1,1,1) = \phi(1,2,2) = 1 - \frac{2\alpha\Delta t\,\Delta x^2 C_J}{\pi\sigma^2\delta^4 e^{\Delta x^2/2\sigma^2\delta^2}} \cdot \frac{2\beta\Delta x^2 C_J}{\pi\sigma^2\delta^4 e^{\Delta x^2/2\sigma^2\delta^2}},$$
$$\phi(1,2,1) = \phi(1,1,2) = -1 + \frac{2\alpha\Delta t\,\Delta x^2 C_J}{\pi\sigma^2\delta^4 e^{\Delta x^2/2\sigma^2\delta^2}} \cdot \frac{2\beta\Delta x^2 C_J}{\pi\sigma^2\delta^4 e^{\Delta x^2/2\sigma^2\delta^2}}.$$

It is worth noting that, based on the above calculations, a generic term in the 2×2 case could contain exponentials of the form

$$e^{-\Delta x^2/\sigma^2\delta^2}$$

which potentially arise from terms with, for example,

$$e^{-\Delta x^2 \frac{(2-1)^2+(2-1)^2}{2\sigma^2\delta^2}} \quad \text{or} \quad e^{-\Delta x^2 \frac{(1-2)^2+(1-2)^2}{2\sigma^2\delta^2}}.$$

We need to account for such terms when we make estimates or bounds. Based on the illustration so far we can determine the largest values of the initial chemical potential $\mu(0, 1, 1)$. The largest is, for example, when $\phi(0, 1, 1) = 1$ and $\phi(0, 2, 1) = \phi(0, 1, 2) = \phi(0, 2, 2) = -1$,

$$\mu(0,1,1) = \frac{\beta\Delta x^2 C_J}{\pi\sigma^2\delta^4}\left(2e^{-\Delta x^2/2\sigma^2\delta^2} + e^{-\Delta x^2/\sigma^2\delta^2}\right).$$

In that case the remaining μ are

$$\mu(0,1,2)=\mu(0,2,1)=-\frac{\beta\Delta x^2 C_J}{\pi\sigma^2\delta^4}e^{-\Delta x^2/2\sigma^2\delta^2}$$

and

$$\mu(0,2,2)=-\frac{\beta\Delta x^2 C_J}{\pi\sigma^2\delta^4}e^{-\Delta x^2/\sigma^2\delta^2}.$$

We conclude this illustration with the proposed control criteria obtained from the first iterate. Hence, the largest admissible α and β guaranteed to keep $|\phi(1,m,l)|\lessapprox 1$ for all possible $2^4=16$ possible different discretized initial conditions is

$$\beta=\frac{\pi\sigma^2\delta^4}{\Delta x^2 C_J}(2e^{-\Delta x^2/2\sigma^2\delta^2}+e^{-\Delta x^2/\sigma^2\delta^2})^{-1}\quad\text{and}\quad\alpha=\frac{\beta}{\Delta t}.$$

With the illustration concluded we look at the general $M\times M$ case. The largest admissible α and β guaranteed to keep $|\phi(1,m,l)|\lessapprox 1$ for all possible 2^{M^2} initial conditions is

$$\beta^*:=\frac{\pi\sigma^2\delta^4}{\Delta x^2 C_J}\Bigg(\underbrace{\sum_{d=2}^{M}e^{-\Delta x^2\frac{(1-d)^2}{\sigma^2\delta^2}}}_{\text{diagonal terms}}+\underbrace{2\sum_{k>j}^{M}\sum_{j=2}^{M}e^{-\Delta x^2\frac{(1-k)^2+(1-j)^2}{2\sigma^2\delta^2}}}_{\text{off-diagonal terms}}\Bigg)^{-1}$$

and

$$\alpha^*:=\frac{\beta^*}{\Delta t}.$$

Equivalently,

$$\beta^*:=\frac{\pi\sigma^2\delta^4}{\Delta x^2 C_J}\left(\sum_{k,j=1}^{M}e^{-\Delta x^2\frac{(1-k)^2+(1-j)^2}{2\sigma^2\delta^2}}-1\right)^{-1}\quad\text{and}\quad\alpha^*:=\frac{\beta^*}{\Delta t}.$$

We proceed with the numerical approximation for Example 7.1. A sequence of discretized solutions corresponding to the nonlocal Cahn–Hilliard equation given in (6.7) for each of the choices of $\delta\in\{0.03125,0.0625,0.125,0.25,0.5,1.0\}$ with $\alpha=(0.4)\frac{\beta^*}{\Delta t}$ and $\beta=(0.4)\beta^*$ are pictured in Tables 2–7. Each sequence is initiated with the same initial condition (with the zero sum property) given in Figure 3. Following these simulations is a table (Table 8) of the approximate solution, $\psi(n,m,l)$, corresponding to the classical problem also initiated with the data given in Figure 3. We investigate the L^2-convergence of the nonlocal problems to the classical problem in Figure 4. This is done by plotting the root of the sum of the squared differences between the numerical solution for the iterates $t=1,32,64,96,128,160,192,224,256$ of the classical Cahn–Hilliard equation (6.1)–(6.3) and the corresponding numerical solution to a rescaled nonlocal problem in (6.7). Hence, Figure 4 shows a plot of

$$\text{diff}(\delta):=\left(\sum_{m,l=1}^{M}|\phi_\delta(n,m,l)-\psi(n,m,l)|^2\right)^{1/2}.$$

In the interests of numerically validating Theorem 6.1 and in comparing the resulting Figure 4 when the numerical procedure is initialized with smooth data, we also obtain the result of this case (when both kernels are the same Gaussian) when the problems are given smooth initial data (as opposed to Figure 3). The results of this smooth data test appear in Appendix B.

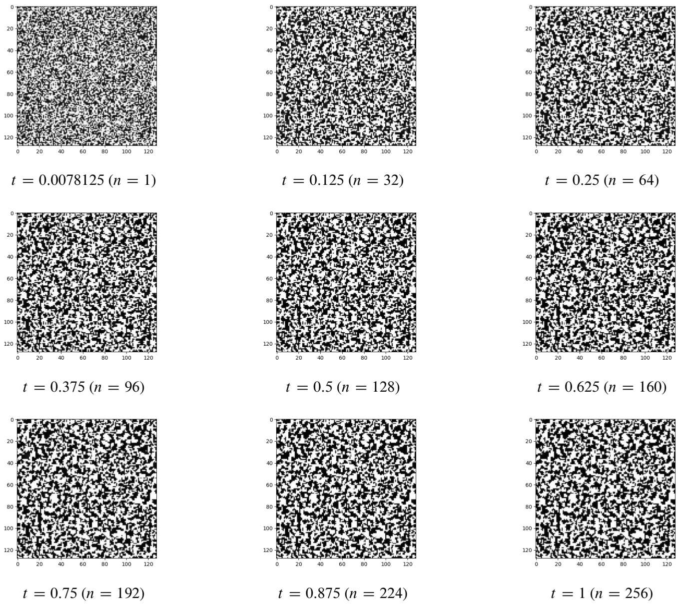


Table 2. Example 7.1: Evolution of the discretized solution of the Gauss–Gauss nonlocal CHE with $\delta = 0.03125$.

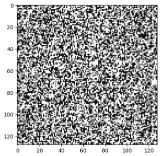
Example 7.2. We investigate the doubly nonlocal Cahn–Hilliard equation where the inner kernel K is the Gaussian from Example 7.1 and the outer kernel J is a logarithmic Newtonian potential.

(1) Again we take $T = 1$, $N = 2^8$, $L = 1$ and $M = 2^7$ and the (discretized) initial condition $\phi(0, m, l)$ is the same initial condition from Example 7.1.

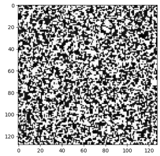
(2) Here

$$J(x_1, x_2) = \begin{cases} -c \log \sqrt{\dfrac{x_1^2 + x_2^2}{2}} & \text{when } (x_1, x_2) \neq (0,0), \\ J^* & \text{when } (x_1, x_2) = (0,0), \end{cases}$$

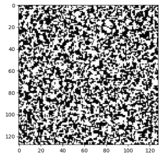
where c is a normalization constant; here, $c = \frac{1}{6-\pi}$.



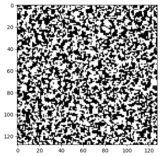
$t = 0.0078125$ ($n = 1$)



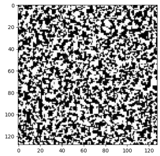
$t = 0.125$ ($n = 32$)



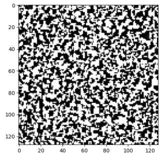
$t = 0.25$ ($n = 64$)



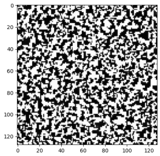
$t = 0.375$ ($n = 96$)



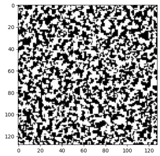
$t = 0.5$ ($n = 128$)



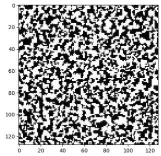
$t = 0.625$ ($n = 160$)



$t = 0.75$ ($n = 192$)

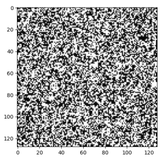


$t = 0.875$ ($n = 224$)

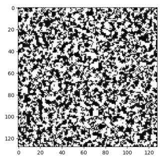


$t = 1$ ($n = 256$)

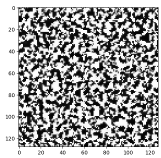
Table 3. Example 7.1: Evolution of the discretized solution of the Gauss–Gauss nonlocal CHE with $\delta = 0.0625$.



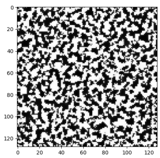
$t = 0.0078125$ ($n = 1$)



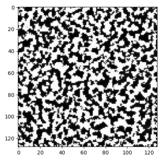
$t = 0.125$ ($n = 32$)



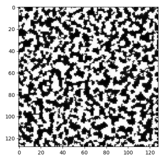
$t = 0.25$ ($n = 64$)



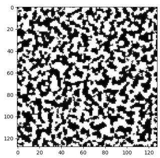
$t = 0.375$ ($n = 96$)



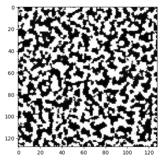
$t = 0.5$ ($n = 128$)



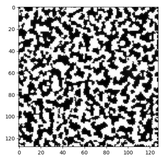
$t = 0.625$ ($n = 160$)



$t = 0.75$ ($n = 192$)

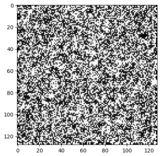
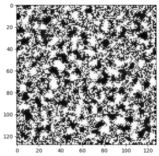
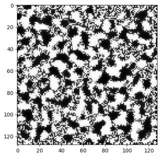


$t = 0.875$ ($n = 224$)

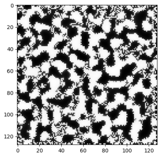
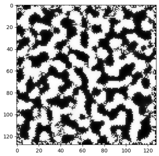
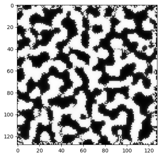


$t = 1$ ($n = 256$)

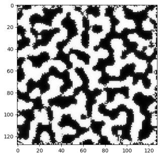
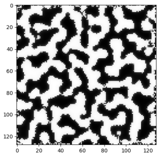
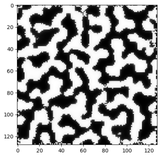
Table 4. Example 7.1: Evolution of the discretized solution of the Gauss–Gauss nonlocal CHE with $\delta = 0.125$.

$t = 0.0078125$ ($n = 1$) $t = 0.125$ ($n = 32$) $t = 0.25$ ($n = 64$)

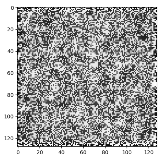
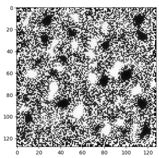
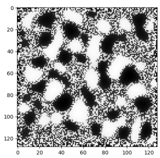




$t = 0.375$ ($n = 96$) $t = 0.5$ ($n = 128$) $t = 0.625$ ($n = 160$)

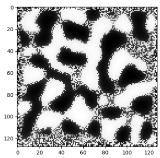
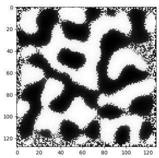
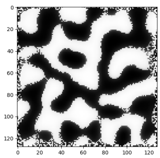




$t = 0.75$ ($n = 192$) $t = 0.875$ ($n = 224$) $t = 1$ ($n = 256$)

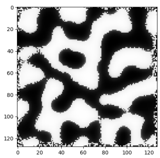
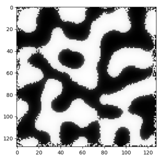
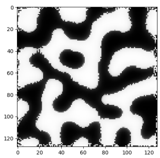
Table 5. Example 7.1: Evolution of the discretized solution of the Gauss–Gauss nonlocal CHE with $\delta = 0.25$.

$t = 0.0078125$ ($n = 1$) $t = 0.125$ ($n = 32$) $t = 0.25$ ($n = 64$)

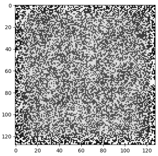




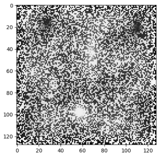
$t = 0.375$ ($n = 96$) $t = 0.5$ ($n = 128$) $t = 0.625$ ($n = 160$)

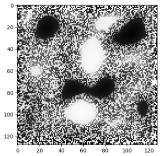




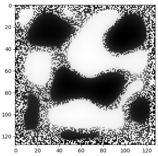
$t = 0.75$ ($n = 192$) $t = 0.875$ ($n = 224$) $t = 1$ ($n = 256$)

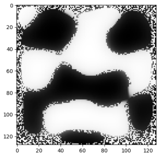
Table 6. Example 7.1: Evolution of the discretized solution of the Gauss–Gauss nonlocal CHE with $\delta = 0.5$.

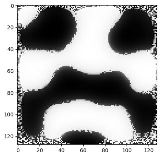

$t = 0.0078125\ (n = 1)$

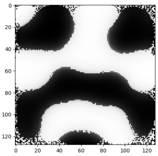

$t = 0.125\ (n = 32)$

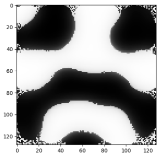

$t = 0.25\ (n = 64)$


$t = 0.375\ (n = 96)$


$t = 0.5\ (n = 128)$


$t = 0.625\ (n = 160)$


$t = 0.75\ (n = 192)$


$t = 0.875\ (n = 224)$

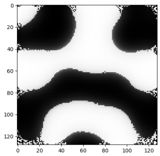
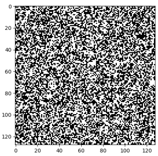
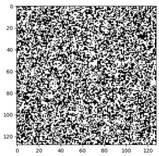
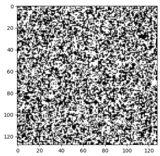
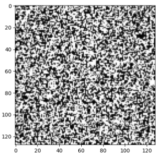

$t = 1\ (n = 256)$

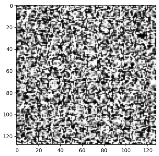
Table 7. Example 7.1: Evolution of the discretized solution of the Gauss–Gauss nonlocal CHE with $\delta = 1.0$.

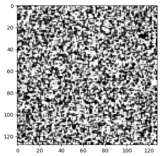

$t = 0.0078125\ (n = 1)$

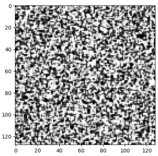

$t = 0.125\ (n = 32)$

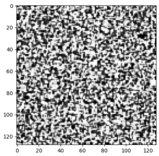

$t = 0.25\ (n = 64)$


$t = 0.375\ (n = 96)$


$t = 0.5\ (n = 128)$


$t = 0.625\ (n = 160)$


$t = 0.75\ (n = 192)$


$t = 0.875\ (n = 224)$

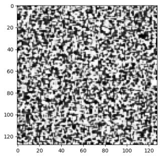

$t = 1\ (n = 256)$

Table 8. Evolution of the discretized solution of the classical Cahn–Hilliard equation ($\varepsilon = 0.007$).

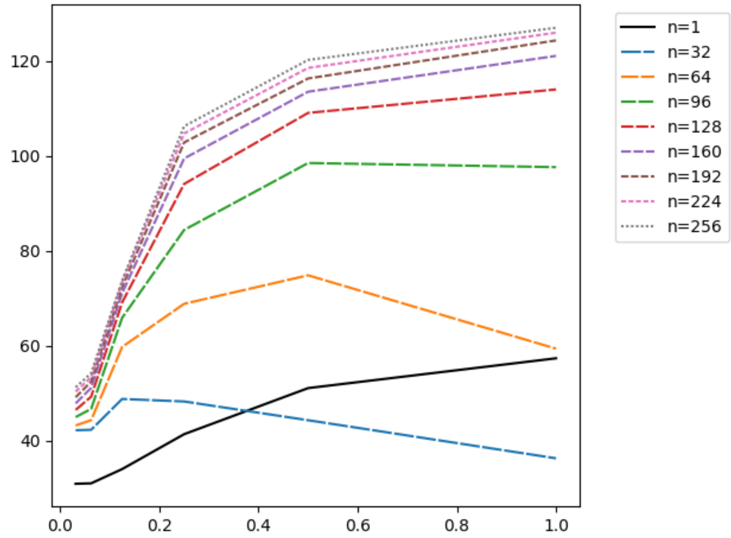


Figure 4. Example 7.1: The root of the sum of squared differences between various iterates ($n = 1, 32, 64, 96, 128, 160, 192, 224, 256$) of the solution of the classical CHE ($\varepsilon = 0.007$) and the corresponding doubly nonlocal CHE ($\sum_{m,l=1}^{2^7} |\phi_\delta(n,m,l) - \psi(n,m,l)|^2)^{1/2}$ over the scaling parameter $\delta \in \{0.03125, 0.0625, 0.125, 0.25, 0.5, 1.0\}$.

The constant $J^* > 0$ is meant to simulate the singularity at the origin.[22] The kernel K is the same Gaussian from Example 7.1, save that we now take $\sigma = 0.02$. In this example, we do not entertain the rescaled kernels, so there is no such constant C_K to compute. Also, for the Gaussian kernel K, we refer to the previous example with the scaling parameter now fixed at $\delta = 1$. It remains to compute α^* and β^* that guarantee a suitable bound on the first iterate. It is important to note that in Example 7.1 the kernels are the same Gaussian. Because of this we sought "stability constants" α and β, separately, so that $|\phi(1,m,l)| \lessapprox 1$. In this example, due to the presence of the singular kernel J and the regular Gaussian K, we will here seek a suitable bound on the product $\alpha\beta$ rather than separately. Imitating the simple calculations as in Example 7.1 we now find, for each $m, l = 1, 2, \ldots, M$,

$$\begin{aligned}\phi(1,m,l) = \phi(0,m,l) & \\ & - \alpha c \Delta t \Delta x^2 \sum_{k,j=1}^{M} \log\left(\Delta x \sqrt{\frac{(m-k)^2 + (l-j)^2}{2}}\right) \\ & \times (\mu(0,k,j) - \mu(0,m,l))\end{aligned}$$

[22] The number 9,223,372,036,854,775,807 is the maximum positive value for a 64-bit signed binary integer in computing.

where again (see above; but here $\delta = C_J = 1$)

$$\mu(0,m,l) = -\beta\Delta x^2 \sum_{k,j=1}^{M} \frac{1}{2\pi\sigma^2} \exp\Big(-\frac{\Delta x^2}{2\sigma^2}((m-k)^2 + (l-j)^2)\Big) \times (\phi(0,k,j) - \phi(0,m,l)) + \phi(0,m,l)^3 - \phi(0,m,l).$$

Illustration 2. Again, we visit the 2×2 case to gain insight into the largest value of the initial chemical potential. Supposing as before that $\phi(0,1,1) = 1$ and -1 elsewhere, then we now find

$$\begin{aligned}\phi(1,1,1) &= \phi(0,1,1) \\ &\quad - \alpha c \Delta t \Delta x^2 \sum_{k,j=1}^{2} \log\Big(\Delta x \sqrt{\frac{(1-k)^2 + (1-j)^2}{2}}\Big)(\mu(0,k,j) - \mu(0,1,1)) \\ &= 1 - \alpha c \Delta t \Delta x^2 \\ &\quad \times \Big(J^*(\mu(0,1,1) - \mu(0,1,1)) + \log\Big(\frac{\Delta x}{\sqrt{2}}\Big)(\mu(0,1,2) - \mu(0,1,1)) \\ &\quad + \log\Big(\frac{\Delta x}{\sqrt{2}}\Big)(\mu(0,2,1) - \mu(0,1,1)) + \log(\Delta x)(\mu(0,2,2) - \mu(0,1,1))\Big) \\ &= 1 - \frac{2\alpha\beta c \Delta t \Delta x^4}{\pi\sigma^2}\Big(\log\Big(\frac{\Delta x}{\sqrt{2}}\Big)(3e^{-\Delta x^2/2\sigma^2} + e^{-\Delta x^2/\sigma^2}) \\ &\quad + \log(\Delta x)(e^{-\Delta x^2/2\sigma^2} + e^{-\Delta x^2/\sigma^2})\Big).\end{aligned}$$

Hence, to bound the iterate $|\phi(1,m,l)| \lessapprox 1$ we require

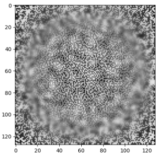
$$\alpha\beta \lessapprox -\frac{\pi\sigma^2}{2c\Delta t \Delta x^4}\Big(\log\Big(\frac{\Delta x}{\sqrt{2}}\Big)(3e^{-\Delta x^2/2\sigma^2} + e^{-\Delta x^2/\sigma^2}) + \log(\Delta x)(e^{-\Delta x^2/2\sigma^2} + e^{-\Delta x^2/\sigma^2})\Big)^{-1}.$$

Observe that the arbitrary (positive) value of J^* does not appear in the calculations. (This is also true of the general case.) The singularity in the logarithm is effectively limited out by the vanishing of the $\mu(0,k,j) - \mu(0,m,l)$ terms when both $k = m$ and $j = l$. The illustration is concluded.

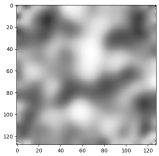
Moving on to the general case, we seek α and β so that $|\phi(1,m,l)| \lessapprox 1$ for all $m, l = 1, 2, \ldots, M$ and over any possible discretized initial data $\phi(0,m,n)$. Thus,

$$\alpha^*\beta^* = -\frac{\pi\sigma^2}{2c\Delta t \Delta x^4}\Big(\sum_{k,j=1}^{M} \log\Big(\Delta x \sqrt{\frac{(1-k)^2 + (1-j)^2}{2}}\Big) - J^*\Big)^{-1} \times \Big(\sum_{k,j=1}^{M} e^{-\Delta x^2 \frac{(1-k)^2 + (1-j)^2}{2\sigma^2}} - 1\Big)^{-1}.$$

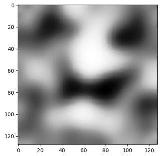
We illustrate the log-Gauss case in Table 9 with $\alpha = (0.3)\sqrt{\alpha^*\beta^*}$, $\beta = (0.3)\sqrt{\alpha^*\beta^*}$.



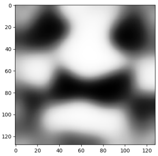
$t = 0.0078125$ ($n = 1$)



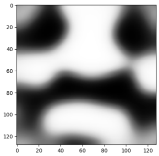
$t = 0.125$ ($n = 32$)



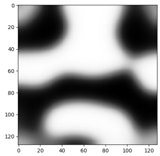
$t = 0.25$ ($n = 64$)



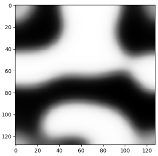
$t = 0.375$ ($n = 96$)



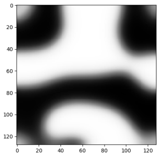
$t = 0.5$ ($n = 128$)



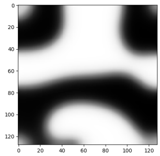
$t = 0.625$ ($n = 160$)



$t = 0.75$ ($n = 192$)



$t = 0.875$ ($n = 224$)



$t = 1$ ($n = 256$)

Table 9. Example 7.2: Evolution of the discretized solution of the log–Gauss nonlocal CHE.

In the next part of this section we study the Riesz potential.

Example 7.3. We investigate the doubly nonlocal Cahn–Hilliard equation where again the inner kernel K is the Gaussian from Example 7.1 and the outer kernel J is a Riesz potential.

(1) Again we take $T = 1$, $N = 2^8$, $L = 1$ and $M = 2^7$ and the (discretized) initial condition $\phi(0, m, l)$ is the same initial condition from Example 7.1.

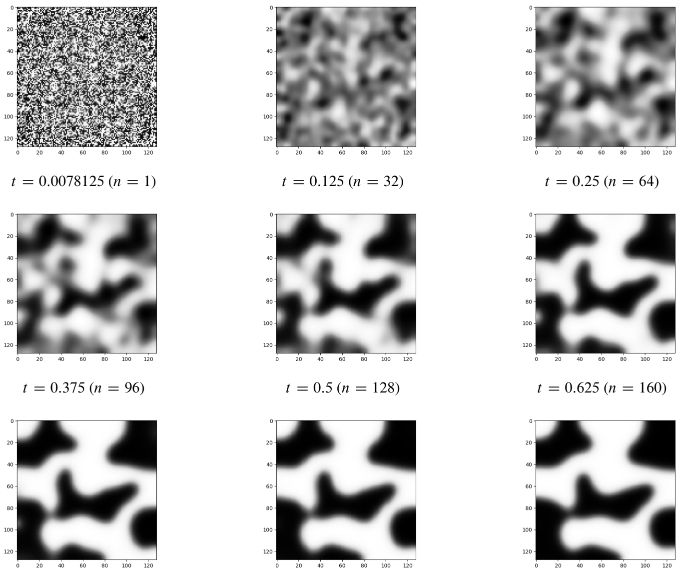
(2) Here,

$$J(x_1, x_2) = \begin{cases} \dfrac{c}{\sqrt{x_1^2 + x_2^2}} & \text{when } (x_1, x_2) \neq (0,0), \\ J^* & \text{when } (x_1, x_2) = (0,0), \end{cases}$$

with normalization constant $c = 7.051$.

As in the previous example, the kernel K is the same Gaussian from Example 7.1 but with $\sigma = 0.02$, and we do not apply the rescaled kernels. Concerning α^* and β^* we first have the calculations, for each $m, l = 1, 2, \dots, M$,

$$\phi(1, m, l) = \phi(0, m, l) + \alpha c \Delta t \Delta x \sum_{k,j=1}^{M} \frac{1}{\sqrt{(m-k)^2 + (l-j)^2}} (\mu(0, k, j) - \mu(0, m, l))$$



$t = 0.0078125$ ($n = 1$) | $t = 0.125$ ($n = 32$) | $t = 0.25$ ($n = 64$)

$t = 0.375$ ($n = 96$) | $t = 0.5$ ($n = 128$) | $t = 0.625$ ($n = 160$)

$t = 0.75$ ($n = 192$) | $t = 0.875$ ($n = 224$) | $t = 1$ ($n = 256$)

Table 10. Example 7.3: Evolution of the discretized solution of the Riesz–Gauss nonlocal CHE.

and again

$$\begin{aligned}\mu(0,m,l) = -\beta\Delta x^2 \sum_{k,j=1}^{M} \frac{1}{2\pi\sigma^2} \exp\Big(-\frac{\Delta x^2}{2\sigma^2}((m-k)^2+(l-j)^2)\Big) \\ \times (\phi(0,k,j)-\phi(0,m,l)) \\ +\phi(0,m,l)^3-\phi(0,m,l).\end{aligned}$$

In the general case we find the bound on the first iterate $|\phi(1,m,l)| \lessapprox 1$ for all $m,l = 1,2,\dots,M$ and over any possible discretized initial data $\phi(0,m,n)$ when

$$\begin{aligned}\alpha^*\beta^* = \frac{\pi\sigma^2}{2c\Delta t\Delta x^3}\Bigg(\sum_{k,j=1}^{M} \frac{1}{\sqrt{(1-k)^2+(1-j)^2}} - J^*\Bigg)^{-1} \\ \times \Bigg(\sum_{k,j=1}^{M} e^{-\Delta x^2 \frac{(1-k)^2+(1-j)^2}{2\sigma^2}} - 1\Bigg)^{-1}.\end{aligned}$$

We illustrate the Riesz–Gauss case with $\alpha = (0.6)\sqrt{\alpha^*\beta^*}$, $\beta = (0.5)\sqrt{\alpha^*\beta^*}$ in Table 10.

We conclude this section with a brief study of the so-called bimodal potential.

Example 7.4. Again we look at the doubly nonlocal Cahn–Hilliard equation where the inner kernel K is the Gaussian from Example 7.1, but now the outer kernel J is a "bimodal" potential.

(1) Again we take $T = 1$, $N = 2^8$, $L = 1$ and $M = 2^7$ and the (discretized) initial condition $\phi(0, m, l)$ is the same initial condition from Example 7.1.

(2) Here,

$$J(x_1, x_2) = c(x_1^2 + x_2^2 + 0.01)e^{-2(x_1^2+x_2^2)},$$

with normalization constant $c = \frac{4}{\pi}$.

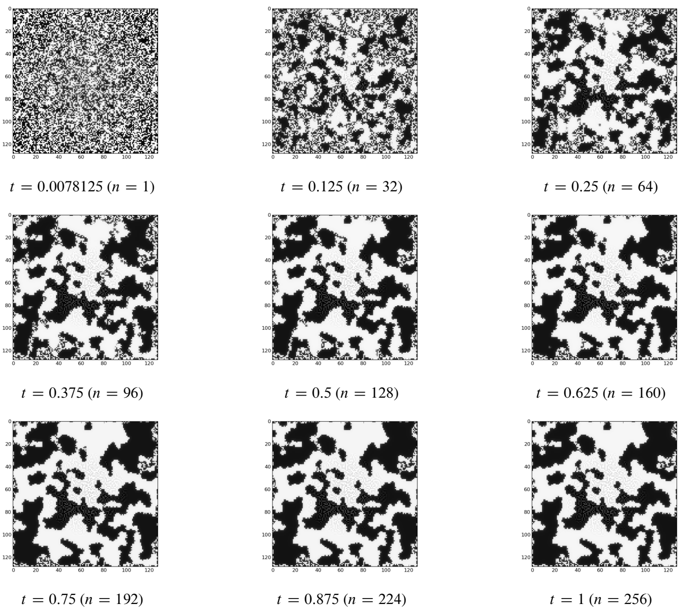
The kernel K is the same Gaussian from Example 7.3 (i.e., $\sigma = 0.02$). By following the calculations made through the previous three examples, we find

$$\beta^* = \frac{\pi\sigma^2}{\Delta x^2}\left(\sum_{k,j=1}^{M} e^{-\Delta x^2 \frac{(1-k)^2+(1-j)^2}{2\sigma^2}} - 1\right)^{-1}$$

and

$$\alpha^* = \frac{\pi}{4\Delta t \Delta x^4} \times \left(\sum_{k,j=1}^{M} ((1-k)^2 + (1-j)^2 + 0.01)e^{-2\Delta x^2((1-k)^2+(1-j)^2)}\right)^{-1}.$$

The illustration in this bimodal–Gauss case with $\alpha = (0.35)\sqrt{\alpha^*\beta^*}$, $\beta = (0.35)\sqrt{\alpha^*\beta^*}$ is in Table 11.



$t = 0.0078125$ ($n = 1$) | $t = 0.125$ ($n = 32$) | $t = 0.25$ ($n = 64$)

$t = 0.375$ ($n = 96$) | $t = 0.5$ ($n = 128$) | $t = 0.625$ ($n = 160$)

$t = 0.75$ ($n = 192$) | $t = 0.875$ ($n = 224$) | $t = 1$ ($n = 256$)

Table 11. Example 7.4: Evolution of the discretized solution of the bimodal–Gauss nonlocal CHE.

A. Some technical tools

For the sake of completeness, we include a simple proof of the local (in time) existence result of bounded mild solutions.

Theorem A.1. *Let assumption (H1) hold and assume that $F \in C^2(\mathbb{R}, \mathbb{R})$ satisfies $F(0) = F'(0) = 0$. Then system (1.3)–(1.5) has at least one mild solution, in the sense of Definition 2.1, on the interval $(0, T_*)$, for some $T_* > 0$.*

Proof. We want to show that the sequence $\{\phi_n(t,x)\}_{n=0}^{\infty}$ defined by the iterates (2.2)–(2.4) converges to a solution $\phi(t,x)$ of (1.3)–(1.5). We know the solution $\phi(t,x)$ has the form

$$\phi(t,x) = \phi(0,x) + \int_0^t ((J * \mu)(\tau,x) - a_J(x)\mu(\tau,x))\, d\tau$$

where

$$\mu(\tau,x) = -(K * \phi)(\tau,x) + a_K(x)\phi(\tau,x) + F'(\phi(\tau,x)).$$

Given $\phi_0 \in L^\infty(\Omega)$, we claim the sequence $\phi_n(t,x)$ converges in $L^\infty(\Omega)$, uniformly in $t \in [0, T_*]$, for a suitable T_*. We also claim the limit $\phi(t,x)$ is a solution to problem (1.3)–(1.5). Indeed, using (2.3) and (2.4), for almost all $t \in [0, T_*]$ and for almost every $x \in \Omega$, there holds

$$\begin{aligned}\lim_{n\to\infty} \phi_n(t,x) &= \phi_0(x) + \lim_{n\to\infty} \int_0^t \left(\int_\Omega J(x-y)\mu_{n-1}(\tau,y)\, dy - a_J(x)\mu_{n-1}(\tau,x) \right) d\tau \\ &= \phi_0(x) + \int_0^t \left(\int_\Omega J(x-y) \lim_{n\to\infty} \mu_{n-1}(\tau,y)\, dy \right. \\ &\qquad \left. - a_J(x) \lim_{n\to\infty} \mu_{n-1}(\tau,x) \right) d\tau, \end{aligned} \tag{A.1}$$

and

$$\begin{aligned}\lim_{n\to\infty} \mu_{n-1}(\tau,x) &= \lim_{n\to\infty} \int_\Omega -K(x-y)\phi_{n-1}(\tau,y)\, dy + \lim_{n\to\infty} a_K(x)\phi_{n-1}(\tau,x) \\ &\quad + \lim_{n\to\infty} F'(\phi_{n-1}(\tau,x)) \\ &= \int_\Omega -K(x-y) \lim_{n\to\infty} \phi_{n-1}(\tau,y)\, dy + a_K(x) \lim_{n\to\infty} \phi_{n-1}(\tau,x) \\ &\quad + F'(\lim_{n\to\infty} \phi_{n-1}(\tau,x)) \\ &= \int_\Omega -K(x-y)\phi(\tau,y)\, dy + a_K(x)\phi(\tau,x) + F'(\phi(\tau,x)) \\ &= \mu(\tau,x). \end{aligned} \tag{A.2}$$

Hence, it follows that

$$\begin{aligned}\lim_{n\to\infty} \phi_n(t,x) = \phi_0(x) + \int_0^t \left(\int_\Omega J(x-y) \lim_{n\to\infty} \mu_{n-1}(\tau,y)\, dy \right. \\ \left. - a_J(x) \lim_{n\to\infty} \mu_{n-1}(\tau,x) \right) d\tau \end{aligned}$$

$$= \phi_0(x) + \int_0^t \left(\int_\Omega J(x-y)\mu(\tau,y)\,dy - a_J(x)\mu(\tau,x) \right) d\tau$$
$$= \phi(t,x).$$

We should note that the termwise integration is justified since it does not hold only on a set of measure zero. From now on, we denote by $C > 0$, a constant that depends on the structural parameters, and may even change from line to line. Such constant used below is independent of t and n, k. To this end, we define the closed subset Y_{M,T_*} of $C([0,T_*]; L^\infty(\Omega))$ as the set

$$Y_{M,T_*} := \left\{ \phi \in C([0,T_*]; L^\infty(\Omega)) : \|\phi\|_{C([0,T_*];L^\infty(\Omega))} \le M \right\}$$

for suitable constants $M, T_* > 0$, such that the sequence of iterates $\{\phi_n(t)\} \subset Y_{M,T_*}$ for all $n \in N_0$. This follows by an induction argument assuming that $M = 2\|\phi_0\|_{L^\infty(\Omega)}$ and $T_* > 0$ is small enough such that

$$T_* \le \frac{\|\phi_0\|_{L^\infty(\Omega)}}{\|J\|_{L^1}(\|K\|_{L^1} + |F''(\lambda M)|M)}$$

for some $\lambda \in (0,1)$. Indeed, by definition,

$$\begin{aligned}
\|\phi_{n+1}(t)\|_{L^\infty(\Omega)} &\le \|\phi_0\|_{L^\infty(\Omega)} + \int_0^t \|J\|_{L^1} \|\mu(s)\|_{L^\infty(\Omega)}\,ds \\
&\le \|\phi_0\|_{L^\infty(\Omega)} \\
&\quad + \int_0^t \|J\|_{L^1}(\|K\|_{L^1} + \|F''(\lambda\phi_n(s))\|_{L^\infty(\Omega)}\|\phi_n(s)\|_{L^\infty(\Omega)})\,ds \\
&\le \|\phi_0\|_{L^\infty(\Omega)} + t\|J\|_{L^1}(\|K\|_{L^1} + |F''(\lambda M)|M) \\
&\le 2\|\phi_0\|_{L^\infty(\Omega)} = M
\end{aligned}$$

for all $t \in [0, T_*]$.

The case $n = 0$. To begin, recall the Young inequality for convolutions: $\|f * g\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q}$ for $1 \le p,q,r \le \infty$, $1/p + 1/q = 1/r + 1$. Using (H1), we first estimate

$$\begin{aligned}
\|\phi_1(t) - \phi_0\|_{L^\infty(\Omega)} &= \left\| \int_0^t (J * \mu_0 - a\mu_0)\,d\tau \right\|_{L^\infty(\Omega)} \\
&\le (\|J * \mu_0\|_{L^\infty(\Omega)} + \|a\mu_0\|_{L^\infty(\Omega)})t \\
&= (\|J\|_{L^1(\Omega)} + \|a_J\|_{L^\infty(\Omega)})\|\mu_0\|_{L^\infty(\Omega)}t \\
&\le C\|\mu_0\|_{L^\infty(\Omega)}t,
\end{aligned} \tag{A.3}$$

where, using (2.4),

$$\begin{aligned}
\|\mu_0\|_{L^\infty(\Omega)} &\le \|K * \phi_0\|_{L^\infty(\Omega)} + \|b\phi_0\|_{L^\infty(\Omega)} + \|F'(\phi_0)\|_{L^\infty(\Omega)} \\
&\le (\|K\|_{L^1(\Omega)} + \|a_K\|_{L^\infty(\Omega)})\|\phi_0\|_{L^\infty(\Omega)} + C\|\phi_0\|_{L^\infty(\Omega)} \\
&\le C.
\end{aligned} \tag{A.4}$$

Hence, combining (A.3) and (A.4) yields

$$\|\phi_1(t) - \phi_0\|_{L^\infty(\Omega)} \leq Ct \quad \text{for all } t \in [0, T_*].$$

The $n = k$ case, for $k \geq 0$. We now assume that the following holds. For each $n = 0, 1, 2, \ldots, k$, for some positive constant C, independent of t and n (this is the weak inductive hypothesis),

$$\|\phi_{k+1}(t) - \phi_k(t)\|_{L^\infty(\Omega)} \leq C\frac{t^{k+1}}{(k+1)!} \quad \text{for all } t \in [0, T_*]. \tag{A.5}$$

We claim that there is a positive constant C such that

$$\begin{aligned}\|\phi_{k+2}(t) - \phi_{k+1}(t)\|_{L^\infty(\Omega)} &\leq C\int_0^t \|\mu_{k+1}(\tau) - \mu_k(\tau)\|_{L^\infty(\Omega)}\, d\tau \\ &\leq C\frac{t^{k+1}}{(k+1)!}.\end{aligned} \tag{A.6}$$

The first inequality in (A.6) follows naturally in view of the Young convolution theorem and the assumptions on J in (H1). It suffices to show the second inequality in (A.6). Namely,

$$\begin{aligned}&\int_0^t \|\mu_{k+1}(\tau) - \mu_k(\tau)\|_{L^\infty(\Omega)}\, d\tau \\ &\quad\leq C\int_0^t \big(\|K * (\phi_{k+1} - \phi_k)(\tau)\|_{L^\infty(\Omega)} + \|a_K(\phi_{k+1}(\tau) - \phi_k(\tau))\|_{L^\infty(\Omega)} \\ &\qquad + \|F'(\phi_{k+1}(\tau)) - F'(\phi_k(\tau))\|_{L^\infty(\Omega)}\big)\, d\tau.\end{aligned} \tag{A.7}$$

Furthermore, using (A.5) we find the bounds

$$\begin{aligned}\|K * (\phi_{k+1} - \phi_k)(\tau)\|_{L^p(\Omega)} &\leq \|K\|_{L^1(\Omega)}\|\phi_{k+1}(\tau) - \phi_k(\tau)\|_{L^p(\Omega)} \\ &\leq \frac{C\tau^{k+1}}{(k+1)!}\end{aligned} \tag{A.8}$$

and

$$\begin{aligned}\|a_K(\phi_{k+1}(\tau) - \phi_k(\tau))\|_{L^p(\Omega)} &\leq \|a_K\|_{L^\infty(\Omega)}\|\phi_{k+1}(\tau) - \phi_k(\tau)\|_{L^p(\Omega)} \\ &\leq \frac{C\tau^{k+1}}{(k+1)!}.\end{aligned} \tag{A.9}$$

Of course, the Lipschitz continuity of $\varphi \mapsto F'(\varphi) \colon L^\infty(\Omega) \to L^\infty(\Omega)$ implies

$$\|F'(\phi_{k+1}(\tau)) - F'(\phi_k(\tau))\|_{L^p(\Omega)} \leq C_M\frac{\tau^{k+1}}{(k+1)!}. \tag{A.10}$$

Putting together (A.7)–(A.10) we observe that, upon integrating, (A.6) holds. The above induction argument proves that, for all $t \in [0, T_*]$,

$$\left\| \phi_0 + \sum_{n=1}^{\infty} (\phi_n(t) - \phi_{n-1}(t)) \right\|_{L^\infty(\Omega)} \le C \sum_{n=1}^{\infty} \frac{t^{n+1}}{(n+1)!} \le C_M e^{T_*}.$$

By the Weierstrass M-test, the series converges uniformly in $t \in [0, T_*]$, in the space $L^\infty(\Omega)$. Thus, for each $t \in [0, T_*]$ there exists a limit function $\phi(t) \in L^\infty(\Omega)$ in which $\lim_{n\to\infty} \phi_n(t) = \phi(t)$ in $L^\infty(\Omega)$ and (A.1)–(A.2) hold in $L^\infty(\Omega)$. By construction, each term $\phi_n(t) \in L^\infty(\Omega)$ is continuous in t, so by the uniform convergence for $t \in [0, T_*]$, it follows with an application of the uniform limit theorem that the limit $\phi(t) \in L^\infty(\Omega)$ is also continuous in t; i.e.,

$$\phi_n \in C([0, T]; L^\infty(\Omega)) \quad \text{and} \quad \mu \in C([0, T]; L^\infty(\Omega)).$$

Since the $L^\infty(\Omega)$-normed convergence implies convergence pointwise a.e., the function $\phi(t, x)$ satisfies problem (1.3)–(1.5) pointwise a.e., and is a mild solution. As a further consequence, we also have the (local-in-time) bound

$$\begin{aligned} \|\phi(t)\|_{L^\infty(\Omega)} &= \lim_{n\to} \|\phi_n(t)\|_{L^\infty(\Omega)} \\ &= \|\phi_0 + \sum_{n=1}^{\infty} (\phi_n(t) - \phi_{n-1}(t))\|_{L^\infty(\Omega)} \le C e^{T_*}. \end{aligned} \tag{A.11}$$

Of course, we now examine the bound on $\mu(t)$; indeed, thanks to the Lipschitz continuity of the map $\varphi \mapsto F'(\varphi)$,

$$\begin{aligned} \|\mu(t)\|_{L^\infty(\Omega)} &= \| - K * \phi(t) + a_K \phi(t) + F'(\phi(t))\|_{L^\infty(\Omega)} \\ &\le \|K * \phi(t)\|_{L^\infty(\Omega)} + \|a_K \phi(t)\|_{L^\infty(\Omega)} + \|F'(\phi(t))\|_{L^\infty(\Omega)} \\ &\le \|K\|_{L^1(\Omega)} \|\phi(t)\|_{L^\infty(\Omega)} + \|a_K\|_{L^\infty(\Omega)} \|\phi(t)\|_{L^\infty(\Omega)} + C\|\phi(t)\|_{L^\infty(\Omega)} \\ &\le C\|\phi(t)\|_{L^\infty(\Omega)}. \end{aligned} \tag{A.12}$$

This concludes the existence argument for a locally defined mild solution. ∎

The following statement is required in the proof of Theorem 5.1. It allows one to find some compactness along differences of any two trajectories for the problem.

Proposition A.2. *Let ϕ_i, $i = 1, 2$ be a pair of mild solutions corresponding to $\phi_i(0) \in B$. Then the following estimates*[23] *hold:*

$$\begin{aligned} &\|\phi_1(t) - \phi_2(t)\|^2_{H^1(\Omega)} \\ &\quad \le \|\phi_1(0) - \phi_2(0)\|^2_{H^1(\Omega)} e^{-c_0\lambda_1 t} + C \int_0^t \|\phi_1(s) - \phi_2(s)\|^2_{L^q(\Omega)}\, ds \end{aligned} \tag{A.13}$$

[23] Here $q > 2$ is the same exponent as in (5.2). Namely, $q = \frac{2p}{p-2}$ if $p > N = 2, 3$, and $q = \infty$ if $N = 1$.

and

$$\int_0^t (\|\partial_t\phi_1(s) - \partial_t\phi_2(s)\|^2_{(H^1(\Omega))^*} + c_0\lambda_1\|\phi_1(s) - \phi_2(s)\|^2_{H^1(\Omega)})\, ds \\ \le Ce^{Ct}\|\phi_1(0) - \phi_2(0)\|^2_{H^1(\Omega)} \tag{A.14}$$

for all $t \ge 0$, for some positive constant C which depends on c_0, λ_1 and J, K, but is independent of $\phi_i(0)$ and time.

Proof. We have that $\phi := \phi_1 - \phi_2$ satisfies the problem

$$\partial_t\phi = \mathcal{L}_J\tilde{\mu}, \quad \tilde{\mu} = -K * \phi + a_K(x)\phi + F'(\phi_1) - F'(\phi_2), \tag{A.15}$$

subject to the initial condition

$$\phi_{|t=0} = \phi_1(0) - \phi_2(0) \quad \text{in } \Omega. \tag{A.16}$$

Also, observe that $D_i\phi$ satisfies the problem

$$\partial_t D_i\phi + a_J(x)(a_K(x) + F''(\xi))D_i\phi = \varrho(\phi), \tag{A.17}$$

where

$$\varrho(\phi) := -(D_i a_J(x))\tilde{\mu} + D_i J * \tilde{\mu} + a_J(x)D_i K * \phi - a_J(x)(D_i a_K(x))\phi \\ - (J * 1)(x)(\phi F'''(\xi)D_i\xi)$$

and $\xi := \lambda\phi_1 + (1-\lambda)\phi_2$ for some $\lambda \in (0,1)$. Multiply (A.15) and (A.17) by 2ϕ and $2D_i\phi$, respectively, and integrate over Ω. We obtain, thanks to the assumptions of Theorem 5.1, the following identities:

$$\frac{d}{dt}\|\phi(t)\|^2_{L^2(\Omega)} + 2c_0\lambda_1\|\phi(t)\|^2_{L^2(\Omega)} \\ = 2(J * \tilde{\mu}, \phi)_{L^2} + 2(a_J(x)K * \phi, \phi)_{L^2} \le C_M\|\phi\|^2_{L^2(\Omega)} \tag{A.18}$$

and

$$\frac{d}{dt}\|D_i\phi(t)\|^2_{L^2(\Omega)} + 2c_0\lambda_1\|D_i\phi(t)\|^2_{L^2(\Omega)} \\ = 2(\varrho(\phi), D_i\phi)_{L^2} \le C_M\|\varrho(\phi)\|^2_{L^2(\Omega)} + c_0\lambda_1\|D_i\phi\|^2_{L^2(\Omega)} \tag{A.19}$$

for all $t \ge 0$. Adding together (A.18)–(A.19), owing to (5.1) (which holds for every trajectory $\phi_i(t)$) and arguing as in the proofs of Theorems 3.5, 3.4 (on account of the Young convolution theorem), we arrive at the estimate

$$\frac{d}{dt}\|\phi(t)\|^2_{H^1(\Omega)} + c_0\lambda_1\|\phi(t)\|^2_{H^1(\Omega)} \\ \le C(\|\varrho(\phi)\|^2_{L^2(\Omega)} + \|\phi\|^2_{L^2(\Omega)}) \le C_M\|\phi\|^2_{L^q(\Omega)} \tag{A.20}$$

for all $t \geq 0$. Then the Grönwall inequality yields (A.13) from (A.20). We also note that, upon integrating (A.18) over $(0,t)$, we find

$$\|\phi(t)\|^2_{L^2(\Omega)} + c_0\lambda_1 \int_0^t \|\phi(s)\|^2_{L^2(\Omega)}\, ds \leq \|\phi(0)\|^2_{L^2(\Omega)} e^{2c_0\lambda_1 t}. \tag{A.21}$$

Then, upon integrating (A.20) once again, owing to $H^1(\Omega) \subset L^q(\Omega)$ we deduce

$$\int_0^t \|\phi(s)\|^2_{H^1(\Omega)}\, ds \leq C_M \|\phi(0)\|^2_{H^1(\Omega)} e^{c_0\lambda_1 t} \quad \text{for all } t \geq 0. \tag{A.22}$$

Moreover, by (A.15), we have

$$\int_0^t \|\partial_t\phi(s)\|^2_{H^1(\Omega)^*}\, ds \leq C_M \int_0^t \|\tilde{\mu}(s)\|^2_{L^2(\Omega)}\, ds \leq C_M \int_0^t \|\phi(s)\|^2_{L^2(\Omega)}\, ds. \tag{A.23}$$

Finally, combining (A.21) with (A.23) and then recalling (A.22), we immediately arrive at (A.14). The proof is finished. ∎

The main tool, used in Section 4, is the Łojasiewicz–Simon theorem for the energy functional $\mathcal{E} = \mathcal{E}(t)$ (see, e.g., [19, Lemma 2.20] for a proof).

Lemma A.3. *There exist constants $\theta \in (0, \frac{1}{2}]$, $C > 0$, $\delta > 0$ such that the following inequality holds:*

$$|\mathcal{E} - \mathcal{E}_\infty|^{1-\theta} \leq C\|\mu - \bar{\mu}\|_{L^2(\Omega)} \tag{A.24}$$

for all $\phi \in L^\infty_{(m)}(\Omega)$, provided that $\|\phi - \phi_*\|_{L^2(\Omega)} \leq \delta$.

We report for the reader's convenience the following abstract result on the existence of exponential attractors ([11, Proposition 4.1]), used in Section 5.

Proposition A.4. *Let H, V, V_1 be Banach spaces such that the embedding $V_1 \subset V$ is compact. Let B be a closed bounded subset of H and let $S\colon B \to B$ be a map. Assume also that there exists a uniformly Lipschitz continuous map $T\colon B \to V_1$, i.e.,*

$$\|\mathbb{T}b_1 - \mathbb{T}b_2\|_{\mathcal{V}_1} \leq L\|b_1 - b_2\|_{\mathcal{H}} \quad \textit{for all } b_1, b_2 \in B, \tag{A.25}$$

for some $L \geq 0$, such that

$$\|\mathbb{S}b_1 - \mathbb{S}b_2\|_{\mathcal{H}} \leq \gamma\|b_1 - b_2\|_{\mathcal{H}} + K\|\mathbb{T}b_1 - \mathbb{T}b_2\|_{\mathcal{V}} \quad \textit{for all } b_1, b_2 \in B, \tag{A.26}$$

for some $\gamma < \frac{1}{2}$ and $K \geq 0$. Then there exists a (discrete) exponential attractor $M_d \subset B$ of the semigroup $\{S(n) := S^n, n \in \mathbb{Z}_+\}$ with discrete time in the phase space H.

B. Supplement to Example 7.1

In this short section we present another plot similar to Figure 4 in Example 7.1. For validation purposes we here choose (smooth) initial data that satisfy the assumptions of

Theorem 6.1. Hence, we expect to see the convergence indicated by (6.8), at least an approximation at the numerical level. It is important to keep in mind that this convergence was not expected in Example 7.1. Indeed, the initial data used throughout Examples 7.1–7.4 were nonsmooth and meant to emulate a substance containing two components that are thoroughly mixed and randomly placed. We are again interested in seeing the L^2-distance between the numerical solution to the classical CHE and the numerical solution of the rescaled doubly nonlocal CHE, but here each problem is initialized with $\phi_0(\mathbf{x}) = \cos(2\pi x)\cos(2\pi y)$. The result appears in Figure 5.

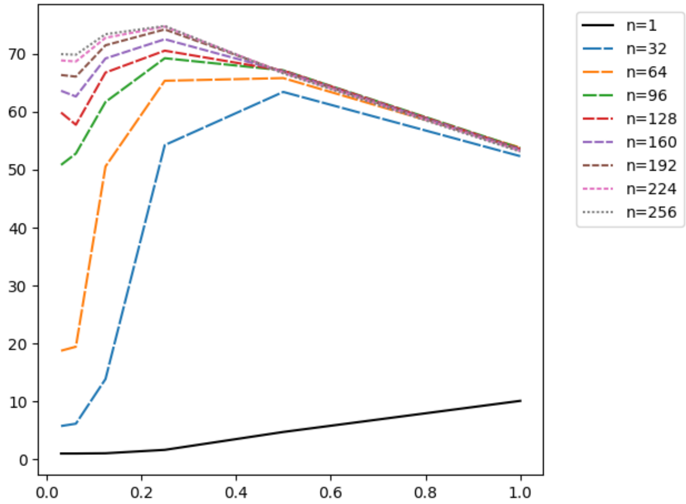


Figure 5. Smooth initial data. The root of the sum of squared differences between various iterates ($n = 1, 16, 32, 48, 64, 80, 96, 112, 128$) of the solution of the classical CHE ($\varepsilon = 0.007$) and the corresponding doubly nonlocal CHE $(\sum_{m,l=1}^{2^7} |\phi_\delta(n,m,l) - \psi(n,m,l)|^2)^{1/2}$ over the scaling parameter $\delta \in \{0.03125, 0.0625, 0.125, 0.25, 0.5, 1.0\}$.

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