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A fast regularisation of a Newtonian vortex equation

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Abstract. We consider equations of the form $u_t = \nabla \cdot (\gamma(u) \nabla \mathrm{N}(u))$, where N is the Newtonian potential (inverse of the Laplacian) posed in the whole space $\mathbb{R}^d$, and $\gamma(u)$ is the mobility. For linear mobility, $\gamma(u) = u$, the equation and some variations have been proposed as a model for superconductivity or superfluidity. In that case the theory leads to uniqueness of bounded weak solutions having the property of compact space support, and in particular there is a special solution in the form of a disk vortex of constant intensity in space $u = c_1 t^{-1}$ supported in a ball that spreads in time like $c_2 t^{1/d}$, thus showing a discontinuous leading front.

In this paper we propose the model with sublinear mobility $\gamma(u) = u^\alpha$, with $0 < \alpha < 1$, and prove that non-negative solutions recover positivity everywhere, and moreover display a fat tail at infinity. The model acts in many ways as a regularisation of the previous one. In particular, we find that the equivalent of the previous vortex is an explicit self-similar solution decaying in time like $u = O(t^{-1/\alpha})$ with a space tail with size $u = O(|x|^{-d/(1-\alpha)})$. We restrict the analysis to radial solutions and construct solutions by the method of characteristics. We introduce the mass function, which solves an unusual variation of Burgers' equation, and plays an important role in the analysis. We show well-posedness in the sense of viscosity solutions. We also construct numerical finite-difference convergent schemes.

1. Introduction

We will study equations of the form

$$u_t = \nabla \cdot (\gamma(u) \nabla \mathrm{N}(u)) \tag{1.1}$$

where N is the Newtonian potential

$$\mathrm{N}(u(t,\cdot)) = \int_{\mathbb{R}^d} \mathbb{G}(x,y) u(t,y)\, \mathrm{d}y$$

for $\mathbb{G}$ the Green kernel, and $\gamma(u)$ is called the mobility. For linear mobility $\gamma(u) = u$, the equation has been studied by a number of authors as a model for superconductivity or superfluidity, cf. Lin and Zhang ([20]), Ambrosio, Mainini, and Serfaty ([2, 3]), Bertozzi, Laurent, and Léger ([5]), Serfaty and Vázquez ([21]). The theory of the last paper leads to uniqueness of bounded weak solutions having the property of compact support, and in

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particular to a special solution in the form of a disk vortex of constant intensity in space that decays in time like $c_1 t^{-1}$ and is supported in a ball that spreads with radius $R = c_2 t^{1/d}$, thus showing a discontinuous leading front, i.e. $u = c_1 t^{-1} \chi_{B(0, c_2 t^{1/d})}$. This vortex solution is an asymptotic attractor for a large class of solutions. Moreover, in dimension 2, the equation is directly related to the Chapman–Rubinstein–Schatzman ([10]) mean field model of superconductivity and to E's model of superfluidity ([16]), which would correspond rather to the equation $u_t = \nabla \cdot (|u| \nabla p)$.

On the other hand, we can formally understand (1.1) as a gradient flow equation with the non-linear mobility $\gamma(u)$ by rewriting it as

$$u_t = \nabla \cdot \Big(\gamma(u) \nabla \frac{\delta \mathcal{F}}{\delta u} \Big) \quad \text{with } \frac{\delta \mathcal{F}}{\delta u} = \mathrm{N}(u),$$

and the associated energy functional

$$\mathcal{F}(u) = \frac{1}{2} \int_{\mathbb{R}^d} \mathrm{N}(u) u \, \mathrm{d}x.$$

The transport distance associated to this non-linear continuity equation was shown in [15] to be well defined for non-linear mobilities of the form $\gamma(u) = u^\alpha$, $0 < \alpha < 1$, and for general concave non-linear mobilities, while transport distances associated with convex non-linear mobilities are not well defined in general. Gradient flows associated to homogeneous concave mobilities were studied subsequently in [9]. This interesting line of research will not be pursued further in this paper.

Statement of the problem and outline of results. In this paper we study the problem with non-linear mobility $\gamma(u) = u^\alpha$, with $0 < \alpha < 1$. The presence of the sublinear mobility leads to a number of results that strongly depart from the linear mobility case, and at the current time implies the need for significant new tools to develop the theory. In particular, we show that the sublinear non-linearity eliminates the compact support effect of the typical vortex solutions, and leads to profiles with fat tails at infinity (of the space variable). They can be interpreted as a diffused vortex. Moreover, the tails depend in a very precise way on the exponent $\alpha < 1$. The case $\alpha \geq 1$ leads to completely different behaviour: compactly supported self-similar solutions (see [7]). We write the problem as

$$\begin{cases} u_t = \nabla \cdot (u^\alpha \nabla v), & (0, +\infty) \times \mathbb{R}^d, \\ -\Delta v = u, & (0, +\infty) \times \mathbb{R}^d, \\ u(t,x), v(t,x) \to 0, & |x| \to +\infty, \\ u = u_0, & t = 0, \end{cases} \tag{P}$$

in all space dimensions $d \geq 1$. We assume that $u_0 \geq 0$. We will show that this implies that $u \geq 0$.

In the first part of this paper, Sections 2–5, we will focus on constructing radial weak solutions by characteristics, introducing rarefaction fans and shocks as appropriate. This will sometimes lead to the existence of multiple weak solutions for certain initial data. The second part, Sections 6–8, deals with the selection of the stable solutions in the sense of vanishing viscosity and the notion of a viscosity solution of the mass equation, presented below. This allows for a well-posedness theory of equation (P) for radial solutions. We now explain in detail the main results of each section.

We begin our study in Section 2 by looking for relevant explicit solutions. Notably, we find a self-similar solution with finite mass that will be the equivalent in our model of the vortex solution mentioned above for linear mobility. This solution is explicit, radially symmetric, and it has power decay rate in space for every $t > 0$ while it decays like $O(t^{-1/\alpha})$ in sup norm. In particular, we will show that the self-similar solution of total mass M is given by

$$U_M(t,x) = t^{-\frac{1}{\alpha}}\left(\alpha + \left(\frac{\omega_d |x|^d t^{-\frac{1}{\alpha}}}{\alpha M}\right)^{\frac{\alpha}{1-\alpha}}\right)^{-1/\alpha}. \tag{1.2}$$

Letting $\alpha \to 1$ we get the compactly supported vortex created by the equation with linear mobility.

In the first part of the paper we are particularly interested in radial solutions for which a very detailed description can be obtained. For these solutions we can study the mass function, which is introduced in Section 3 as

$$m(t,r) = \int_{B_r} u(t,x)\,\mathrm{d}x \tag{1.3}$$

which is the solution of a Hamilton–Jacobi-type equation when written in the volume variable $\rho = \omega_d r^d$:

$$m_t + m(m_\rho)^\alpha = 0. \tag{1.4}$$

This equation is reminiscent of Burgers' equation. Indeed, it is a very unusual version of it that needs careful development. We remark that our study is dimension independent. We recall that $m_\rho = u \geq 0$.

Equation (1.4) will be studied by the method of characteristics, following [17]. This is done in Section 3.2 and we obtain solutions by gluing characteristic lines (see Theorem 3.1). In particular, we recover the self-similar solution (1.2) again. We devote Section 4 to showing that the method of characteristics works well when $u_0(r)$ is radially symmetric and decreasing. First, in Section 4.1 we discuss the case where u_0 is non-increasing and continuous, and the characteristics fill the space. Then, in Section 4.2, we study the case in which u_0 is non-increasing and discontinuous, where characteristics leave gaps. One way to fill these gaps is the introduction of a rarefaction fan, which is presented in Section 4.2.3. This important topic is treated in detail. Then we derive mass conservation (Proposition 4.4), a comparison principle (Theorem 4.5), and asymptotic behaviour for such solutions (as $t \to \infty$ in Theorem 4.6 and Lemma 4.8 and for fixed t as $|x| \to \infty$ in Theorem 4.10).

Next, in Section 5, we enlarge the class of initial data, still radially symmetric, but only piecewise decreasing. Then shocks may appear, and we need Rankine–Hugoniot conditions (given by (5.1)) to select the correct shock solutions. In fact, we give in Section 5.2 an example of non-uniqueness of weak solutions: the square functions.

We then address the issue of constructing solutions for a large class of initial data and selecting the physical ones. We devote two sections to constructing viscosity approximations, as it is customary to do for similar problems. In Section 6 we will consider a regularised problem with a viscous term $\varepsilon\Delta u$:

$$\begin{cases} u_t = \nabla\cdot((\varepsilon + u_+)^\alpha \nabla v) + \varepsilon \Delta u, & (0,+\infty)\times\mathbb{R}^d, \\ -\Delta v = u, & (0,+\infty)\times\mathbb{R}^d, \\ u(t,x), v(t,x) \to 0, & |x|\to+\infty, \\ u = u_0, & t = 0. \end{cases} \tag{P$_\varepsilon$}$$

The limit of this problem as $\varepsilon \to 0$ is called the vanishing viscosity limit. We prove that, for general (non-radial) initial data, (P$_\varepsilon$) is well posed (Theorem 6.1), has suitable L^p estimates (Proposition 6.2), its mass satisfies (6.3) similar to (1.4), and it converges in the sense of weak solutions (Theorem 6.3). Passing to the limit $\varepsilon \to 0$ thanks to suitable a priori estimates we get weak solutions for quite general, not necessarily radial data.

We still have the problem of uniqueness that we solve for radially symmetric data by passing to the limit in the above approximation, but now in the mass variable. In Section 7 we obtain a unique viscosity solution in the sense of Crandall–Lions ([13]). We prove that bounded and uniformly continuous viscosity solutions of (1.4) satisfy a comparison principle (Theorem 7.5) and can be recovered as the limit of the solutions of (6.3) (Theorem 7.12). This allows us to state the well-posedness in Theorem 7.14. We conclude the section by discussing the asymptotic behaviour of viscosity solutions in Theorem 7.17.

Finally, we devote Section 8 to constructing numerical finite-difference convergent schemes for the mass function using viscosity-solution techniques. Numerical calculations illustrate the main results of the paper at different stages. We close the paper with some comments on extensions and open problems in Section 9.

2. Explicit solutions

In this section we construct two families of explicit solutions for (P).

2.1. Constant-in-space solutions and the friendly giant

We look for ODE-type solutions for (P). Indeed, for initial constant data $u_0(x)$ we may look for supersolutions $u(t,x) = g(t)$. We write the equation

$$u_t = \nabla\cdot(u^\alpha\nabla v) = \nabla u^\alpha \nabla v + u^\alpha \Delta v = \nabla u^\alpha\nabla v - u^{\alpha+1}.$$

Hence,

$$g' = -g^{\alpha+1}.$$

Therefore, we have the friendly giant solution:

$$g(t) = (u_0^{-\alpha} + \alpha t)^{-1/\alpha}.$$

Assuming that a comparison principle works, this solution will allow us below to show that

$$\bar{u}(t,x) = (\|u_0\|_{L^\infty}^{-\alpha} + \alpha t)^{-1/\alpha} \tag{2.1}$$

is a supersolution.

Global supersolution. Even as $\|u_0\|_{L^\infty} \to +\infty$ we have the so-called friendly giant

$$\tilde{u}(t) = (\alpha t)^{-1/\alpha}. \tag{2.2}$$

Even if these solutions are not in L^1, comparison works for any viscosity solution or for any limit of approximate classical solutions like the ones obtained by the vanishing viscosity method.

2.2. Self-similar solutions

Next we establish the existence of the important class of self-similar solutions, which take the form

$$U(t,x) = t^{-\gamma} F(|x|t^{-\beta}). \tag{2.3}$$

In order to satisfy (P) and conserve mass we must take

$$\gamma = \frac{1}{\alpha}, \quad \beta = \frac{1}{\alpha d}.$$

A PDE in self-similar variables. Then the equation for the profile $U(t,x) = t^{-\gamma} F(|y|)$ where $y = xt^{-\beta}$ is

$$-\frac{1}{\alpha d} \nabla \cdot (yF) = \nabla \cdot (F^\alpha \nabla \mathrm{N}(F)).$$

Eliminating the nablas and rearranging, we get the fractional stationary equation

$$yF^{1-\alpha} = -\alpha d \nabla \mathrm{N}(F).$$

Applying the divergence operator to the latter equation, we get

$$\nabla \cdot (yF^{1-\alpha}) = -\alpha d \Delta \mathrm{N}(F) = \alpha d F \tag{2.4}$$

since N is the inverse of $-\Delta$ in $\mathbb{R}^d$.

An ODE for F in radial coordinates. In order to solve this equation we put $w(r) = F(r)^{1-\alpha}$ so that $F = w^p$ with $p = 1/(1-\alpha) > 1$. Notice that $p - 1 = \alpha/(1-\alpha)$. Also, $p \to \infty$ as $\alpha \to 1^-$ and for $\alpha = 1/2$ we get $p = 2$. We also assume that w is a radial function $w = w(r)$. We get

$$rw'(r) = \alpha d w(r)^p - d w(r) = d w(r)(\alpha w(r)^{p-1} - 1). \tag{2.5}$$

There is an equilibrium point $w_* = (1/\alpha)^{1/(p-1)} = \alpha^{-(1-\alpha)/\alpha}$ (for $\alpha = 1/2$ we get $w_* = 2$). This gives rise to the constant solution that is also found in the limit case of linear mobility. But in the case of linear mobility we have $\alpha = 1$, $p = 1$ and there is no preferred critical value for (2.5).

Actually, the existence of the critical value for $0 < \alpha < 1$ allows us to construct solutions in the region $D = \{(r, v) : r > 0,\ 0 < w < w_*\}$ of the ODE phase plane. It is clear that D is an invariant region; it is bounded by the solutions $w = 0$ and $w = w_*$ from below and above.

Quantitative analysis of (2.5). An asymptotic analysis as $r \to \infty$ gives for all possible solutions $rw'(r) \sim -dw(r)$ so that $w(r) \sim r^{-d}$ and the original profile v behaves as

$$F(r) \sim r^{-dp} \quad \text{as } r \to \infty.$$

Since $dp = d/(1-\alpha) > d$, this tail is integrable. As for the limit $r \to 0$, the only admissible option is to enter the corner point so that

$$F(0^+) = w_*^p = \alpha^{-\alpha}.$$

Hence, all the solutions in this region will have the same behaviour at $r = 0$ to zero order. They are all decreasing and positive for $r > 0$.

Explicit expression for F. An explicit computation is possible as follows. Since we have by (2.4) that

$$r^{1-d}(r^d w)' = \alpha d w^p,$$

if we define $z = r^d w$, then we get the ODE $z' = Ar^{-a-1} z^p$ with

$$A = \alpha d, \quad a = dp - (d-1) - 1 = d(p-1) = d\alpha/(1-\alpha).$$

Integration of $z^{-p} z' = Ar^{-a-1}$ gives

$$-\frac{1}{p-1}(z^{-(p-1)} - C_1) = -\frac{A}{a} r^{-a},$$
$$z^{-(p-1)} = C_1 + \frac{A(p-1)}{a} r^{-a}.$$

We have $A(p-1)/a = \alpha$ so that

$$w(r) = (C + \alpha r^{-d(p-1)})^{-1/(p-1)} r^{-d} = \frac{1}{(\alpha + C r^{d(p-1)})^{1/(p-1)}},$$

where $1/(p-1) = (1-\alpha)/\alpha$ ranges in $(0, \infty)$. Finally, the profile $F = w^p$ is given by

$$F(r) = F(0^+)(1 + C r^{d\alpha/(1-\alpha)})^{-1/\alpha}, \quad F(0^+) = \alpha^{-1/\alpha}, \tag{2.6}$$

where the exponent $1/\alpha = p/(p-1)$ ranges in $(1, \infty)$ and C is left to be determined.

We will later show, by a different method, that, under the additional condition

$$\int_{\mathbb{R}^d} U(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^d} F(|y|)\,\mathrm{d}y = 1,$$

we deduce that the self-similar profile is

$$F(|y|) = \Big(\Big(\frac{\omega_d |y|^d}{\alpha}\Big)^{\frac{\alpha}{1-\alpha}} + \alpha\Big)^{-\frac{1}{\alpha}}.$$

Hence, the self-similar solution (2.3) of mass 1 is given by

$$U(t,x) = t^{-\frac{1}{\alpha}}\Big(\alpha + \Big(\frac{\omega_d |x|^d t^{-\frac{1}{\alpha}}}{\alpha}\Big)^{\frac{\alpha}{1-\alpha}}\Big)^{-\frac{1}{\alpha}}. \tag{2.7}$$

Remark 2.1. (1) Self-similar solutions of mass M can be obtained by the rescaling

$$U_M(t,x) = MU(M^\alpha t, x).$$

Going back to (2.3), the profile of the solution of mass M is given by

$$F_M(|y|) = F\Big(\frac{|y|}{M^{\frac{1}{d}}}\Big)$$

which yields solutions of the form (1.2) The initial datum of such a solution is a Dirac delta. The whole class reminds us of the Barenblatt solutions of fast diffusion equations, cf. [24]. Notice that for large y we have

$$F_M(|y|) \sim \Big(\frac{\alpha M}{\omega_d |y|^d}\Big)^{\frac{1}{1-\alpha}} \quad \text{for } |y| \gg 1$$

so the tail depends on the total mass, unlike in the fast diffusion equation, where the constant for the tail is uniform (see [25]). On the other hand, $F_M(0) = F(0)$ for all M, i.e. near $y = 0$, the self-similar solution does not detect the mass. Notice that

$$F_M(y) \to F(0) \quad \text{as } M \to +\infty.$$

In particular, the constant value $F(0)$ is the self-similar profile of the global supersolution $\tilde{u}(t) = (\alpha t)^{-1/\alpha}$, which has infinite mass.

(2) The formula for $\alpha = 1/2$ is

$$F(|y|) = \frac{1}{4(1 + C|y|^d)^2}$$

and the self-similar solution in original variables is

$$U(t,x) = \frac{t^{-2}}{4(1 + C|x|^d t^{-2})^2} = \frac{t^2}{4(t^2 + C|x|^d)^2},$$

which is the Cauchy distribution in $d = 1$ and the stereographic projection to some sphere in dimension $d = 2$.

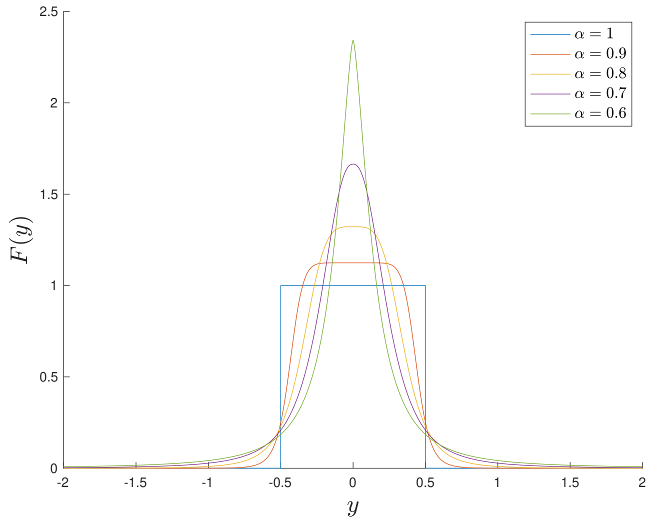

Figure 1. Self-similar profiles for $d = 1$ and different values of α.

(3) The self-similar solution is a C^∞ solution in space and time. This regularity will not be achieved by the general class of solutions we will describe below, where Lipschitz continuity will be the rule.

(4) In the limit $\alpha \to 1$ we obtain the expanding vortex solution described in [21], given by

$$U(t,x) = \frac{1}{t}\chi_{B(0,R_1 t)}, \quad \text{with } \omega_d R_1^d = M.$$

This limit is illustrated in Figure 1.

3. Mass function of radial solutions

In order to proceed with our mathematical analysis, we restrict consideration to radially symmetric solutions and introduce an important tool, the mass function.

3.1. A PDE for the mass

3.1.1. Radial coordinates. Let us consider $u = u(t, |x|)$ a radial function and let us define its mass in radial coordinates as

$$m(t,r) = d\omega_d \overline{m}(t,r), \quad \text{with } \overline{m}(t,r) = \int_0^r u(t,\tau)\tau^{d-1}\,d\tau.$$

We have $\overline{m}_r(t,r) = r^{d-1}u(t,r)$. Taking the derivative in t,

$$\overline{m}_t = \int_0^r \tau^{d-1} u_t \, \mathrm{d}\tau = \int_0^r \tau^{d-1} \frac{1}{\tau^{d-1}} \frac{\partial}{\partial r}\left(\tau^{d-1} u^\alpha \frac{\partial v}{\partial r}\right) \mathrm{d}\tau$$
$$= r^{d-1}(\overline{m}_r r^{-(d-1)})^\alpha \frac{\partial v}{\partial r} = r^{-\alpha(d-1)} \overline{m}_r^\alpha r^{d-1} \frac{\mathrm{d}v}{\mathrm{d}r}.$$

Since u is radial, then v is also radial and its equation can be written

$$-\frac{1}{r^{d-1}} \frac{\partial}{\partial r}\left(r^{d-1} \frac{\partial v}{\partial r}\right) = u.$$

Hence

$$-r^{d-1} \frac{\partial v}{\partial r} = \int_0^r \tau^{d-1} u = \overline{m}.$$

Therefore, we can write a first-order equation for m of the form

$$\overline{m}_t + r^{-\alpha(d-1)} \overline{m}\, \overline{m}_r^\alpha = 0, \tag{3.1}$$

which looks like a difficult variation of the classical Burgers' equation.

3.1.2. Volume coordinates. Equation (3.3) above includes an unwelcome $r^{-\alpha(d-1)}$. However, by choosing the volume-scaling coordinates

$$\rho = \omega_d r^d, \tag{3.2}$$

we can write $\overline{m}_r = d\omega_d r^{d-1} \overline{m}_\rho$ and hence

$$\overline{m}_t + r^{-\alpha(d-1)} \overline{m} (d\omega_d r^{d-1} \overline{m}_\rho)^\alpha = 0.$$

In particular, multiplying by $d\omega_d$ we have

$$(d\omega_d \overline{m})_t + (d\omega_d \overline{m})(d\omega_d \overline{m}_\rho)^\alpha = 0.$$

Changing back to the m variable,

$$m_t + m m_\rho^\alpha = 0. \tag{3.3}$$

In this variable, the equation for m does not depend on d any more. This is a surprising new version of Burgers' equation, which is not in divergence form. For $\alpha \neq 1$, to our knowledge there is no reference in the mathematical literature to this equation.

3.2. Method of generalised characteristics

The method of generalised characteristics (see [17, Section 3.2]) for a generic first-order equation

$$G(Dw, w, \mathbf{y}) = 0,$$

where $\mathbf{y} = (t, x)$, consists in constructing parametric characteristic $\mathbf{y}(s)$ that can be solved independently. Applying this theory to (3.3), with the notation $\mathbf{y} = (t, r)$, $w = m$, $p_1 = m_t$, $p_2 = m_\rho$, so that our equation becomes $G = 0$ with

$$G(p, z, \mathbf{y}) = p_1 + z p_2^\alpha, \tag{3.4}$$

we deduce the following theorem.

Theorem 3.1. *Let m be a classical solution of (3.3) with initial data m_0, and let the derivative be called $u_0 = (m_0)_\rho \geq 0$. As long as the characteristics*

$$\rho(t) = \rho_0 + \alpha m_0(\rho_0) u_0(\rho_0)^{\alpha-1} t \tag{3.5}$$

do not cross, the solution is given by

$$m(t, \rho(t)) = m_0(\rho_0)(1 + \alpha u_0(\rho_0)^\alpha t)^{1-\frac{1}{\alpha}} \tag{3.6}$$

and its derivative $u = m_\rho$ by

$$u(t, \rho(t)) = (u_0(\rho_0)^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}}. \tag{3.7}$$

Remark 3.2. (1) Notice that characteristics are always straight lines. Recall that ρ is a volume variable.

(2) Due to our choice of coordinates $\omega_d r^{d-1} u = m_r = d\omega_d r^{d-1} m_\rho$, so that $m(\rho) = \int_0^\rho u(s)\, ds$.

(3) The equation for the mass (3.3) has infinite speed of propagation for the derivative u. When u_0 is the triangle $u_0(\rho) = (1 - \rho)_+$ then the mass is given by

$$m_0(\rho) = \frac{1}{2}(1 - (1 - \rho_0)^2) \quad \text{for } 0 \leq \rho_0 \leq 1.$$

Hence the characteristics from $\rho_0 \in [0, 1]$ are written

$$\rho = \rho_0 + \frac{\alpha}{2}(1 - (1 - \rho_0)^2)(1 - \rho_0)^{\alpha-1} t \quad \text{for } 0 \leq \rho_0 \leq 1.$$

For any $t > 0$, these characteristics cover all $\rho \geq 0$, as shown in Figure 2.

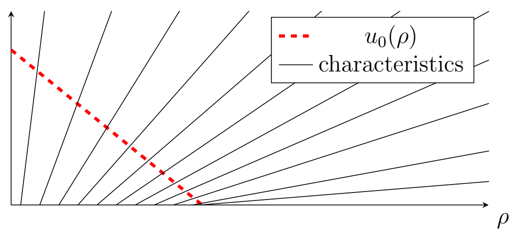

Figure 2. Characteristics corresponding to $u_0(\rho) = (1 - \rho)_+$ for $\rho_0 \in [0, 1]$.

Notice that characteristics from $\rho_0 \in (1, +\infty)$ are constant $\rho = \rho_0$, since $u_0(\rho_0)$ and $m(\rho_0) = 1/2$.

(4) A remarkable difference in (3.3) with respect to Burgers' equation is the fact that, even for Lipschitz initial data m_0, characteristics may cross for all $t > 0$ (see Figure 3). These intersections will lead to a shock, governed by a variant of the classical Rankine–Hugoniot conditions ([18, 22]), as we will see below.

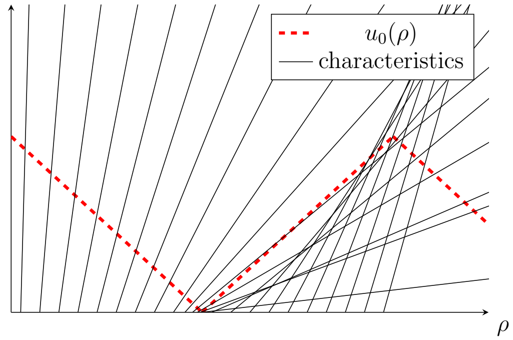

Figure 3. Characteristics corresponding to $u_0(\rho) = (1-\rho)_+ + (1-|\rho-2|)_+$ for $\rho_0 \in [0, 2]$.

(5) Notice that by a happy coincidence, $u = m_\rho = p_2$. Hence, by solving the system of ODEs, we already obtain the value of the original function u along the characteristic.

Proof of Theorem 3.1. As usual, we form a two-parametric family $\mathbf{y}(s, \rho)$ of characteristics and then identify the surface they build as the solution. Following [17, Section 3.2] we next construct the characteristics. Using the notation

$$z(s) = w(\mathbf{y}(s)), \quad \mathbf{p}(s) = Dw(\mathbf{y}(s)),$$

the equations for the characteristics are

$$\dot{\mathbf{p}}(s) = -D_y G(\mathbf{p}(s), z(s), \mathbf{y}(s)) - D_z G(\mathbf{p}(s), z(s), \mathbf{y}(s))\mathbf{p}(s), \tag{3.8a}$$
$$\dot{z}(s) = D_p G(\mathbf{p}(s), z(s), \mathbf{y}(s)) \cdot \mathbf{p}(s), \tag{3.8b}$$
$$\dot{\mathbf{y}}(s) = D_p G(\mathbf{p}(s), z(s), \mathbf{y}(s)). \tag{3.8c}$$

We write system (3.8) as

$$\dot{p}_1(s) = -p_2^\alpha p_1, \tag{3.9a}$$
$$\dot{p}_2(s) = -p_2^{\alpha+1}, \tag{3.9b}$$
$$\dot{z}(s) = (1, \alpha z p_2^{\alpha-1}) \cdot (p_1, p_2) = p_1 + \alpha z p_2^\alpha, \tag{3.9c}$$
$$\dot{t}(s) = 1, \tag{3.9d}$$
$$\dot{\rho}(s) = \alpha z p_2^{\alpha-1}. \tag{3.9e}$$

For the initial data we take $t(0) = 0$ and $\rho(0) = \rho_0 > 0$. We have the following initial conditions:

$$z(0) = z_0 = m(0, \rho_0) \int_0^{\rho_0} u_0(s)\, \mathrm{d}s, \tag{3.10a}$$

$$p_2(0) = p_{2,0} = m_r(0, \rho_0) = u(0, \rho_0) = u_0(\rho_0). \tag{3.10b}$$

The equation relates the values of $p_1(0)$, $p_{2,0}$ and z_0: $p_1(0) = -p_{2,0}^{\alpha} z_0$.

We first notice that $t(s) = s$. If $u_0(\rho_0) = 0$ then $p_1 = p_2 = 0$, and hence $z \equiv z_0$ and $\rho(s) \equiv \rho_0$. In other words, points outside the support of u_0 do not propagate in any direction. Furthermore, if $u_0(r) = 0$ then $m(r,t) = m(r,0)$. However, if $u_0(\rho_0) = p_{2,0} > 0$ then $p_2(t) > 0$, then $\dot{\rho} > 0$, and hence it is increasing.

Observe that the equation for p_2 is autonomous; then it can be solved explicitly to get

$$p_2(t) = (p_{2,0}^{-\alpha} + \alpha t)^{-1/\alpha}, \tag{3.11}$$

if $p_{2,0} > 0$. Notice that $p_1 \dot{p}_2 - \dot{p}_1 p_2 = 0$, therefore p_1/p_2 is constant and

$$p_1(t) = \frac{p_1(0)}{p_{2,0}} (p_{2,0}^{-\alpha} + \alpha t)^{-1/\alpha}.$$

Using the condition on the initial data we finally obtain

$$p_1(t) = -\frac{z_0}{p_{2,0}} (1 + \alpha p_{2,0} t)^{-1/\alpha}. \tag{3.12}$$

Once p_1 and p_2 are known, we can solve for z as a linear equation with variable coefficients. We have

$$\dot{z}(t) - \frac{\alpha}{p_{2,0}^{-\alpha} + \alpha t} z(t) = p_1(t).$$

Since, for any two functions $f'g - fg' = (f/g)'g^2$, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{z(t)}{(p_{2,0}^{-\alpha} + \alpha t)^{-1}} \right) = -\frac{z_0}{p_{2,0}} (1 + \alpha p_{2,0}^{\alpha} t)^{-1-1/\alpha}.$$

Integrating on $[0, t]$ and solving for z we deduce that

$$z(t) = z_0 (1 + \alpha p_{2,0}^{\alpha} t)^{1-1/\alpha}. \tag{3.13}$$

Thus we deduce

$$\dot{\rho}(t) = z(t) p_2(t)^{\alpha-1} = z_0 p_{2,0}^{\alpha-1}.$$

Hence

$$\rho(t) = \rho_0 + \alpha z_0 p_{2,0}^{\alpha-1} t. \tag{3.14}$$

To deduce (3.5) we substitute the values from the initial data in (3.10). ∎

Remark 3.3. Notice that the argument works for any $\alpha > 0$. The characteristics formula (3.5) shows us that the cases $\alpha \in (0, 1), \{1\}, (1, +\infty)$ behave quite differently. The case $\alpha = 1$ is the Burgers' equation. In the case $0 < \alpha < 1$, solutions with small positive initial value will disperse almost instantaneously (as in the fast diffusion equation; see [25]). Conversely, for $\alpha > 1$ the larger the initial data the slower it will diffuse (as in the porous medium equation; see [25]).

Remark 3.4. Notice that for points in the support of u_0, characteristics are increasing straight lines. If the support of u_0 is bounded, characteristics coming from the support of u_0 (with positive values of u), will intersect characteristics from outside the support. We will see later how solutions overcome this difficulty.

4. Radial non-increasing data u_0

In this section we will consider non-increasing radial data for (P), by the method of characteristics.

4.1. Continuous u_0

In this case we will show that the characteristics do not cross, and hence we can construct classical solutions of (3.3) using Theorem 3.1. Then u is determined in an implicit way by

$$u(t,x) = (u_0(\rho_0)^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}} \quad \text{where } \omega_d |x|^d = \rho_0 + \alpha m_0(\rho_0) u_0(\rho_0)^{\alpha-1} t. \tag{4.1}$$

We introduce the function

$$P_t(\rho_0) = \rho_0 + \alpha m_0(\rho_0) u_0(\rho_0)^{\alpha-1} t.$$

Let us distinguish two cases.

Positive u_0. Since u is positive, m is strictly increasing and, since u_0 is strictly decreasing, then $u_0(\rho_0)^{\alpha-1}$ is non-decreasing. Hence, for every $t > 0$, P_t is a strictly increasing function of ρ_0, and therefore invertible. It is null at zero and unbounded at infinity. Hence, $P_t\colon [0, +\infty) \to [0, +\infty)$ is invertible. Therefore, for every $t > 0$ and $x \in \mathbb{R}^d$ there exists a unique ρ_0 such $\omega_d |x|^d = P_t(\rho_0)$.

Compactly supported u_0. If the initial datum reaches zero, then $\operatorname{supp} u_0 = \overline{B_R}$ for some $R > 0$. Then P_t is still a strictly increasing function for $\rho < R$. Clearly $P_t(R^-) = +\infty$. Hence, for every $t > 0$ we have $P_t\colon [0, R) \to [0, +\infty)$ is invertible.

4.2. Discontinuous data: rarefaction fan solutions

4.2.1. Rarefaction fan solution for $u_0(\rho) = \chi_{[0,L]}(\rho)$. If one considers a regularised version of the square functions

$$u_0^{(\varepsilon)}(\rho) = \begin{cases} c_0, & \rho \le L, \\ \dfrac{c_0}{\varepsilon}(L+\varepsilon-\rho), & L \le \rho < L+\varepsilon, \\ 0, & \rho \ge L+\varepsilon. \end{cases} \tag{4.2}$$

The initial mass becomes

$$m_0^{(\varepsilon)}(\rho) = \begin{cases} c_0\rho, & \rho \le L, \\ c_0 L + \dfrac{c_0}{\varepsilon}\dfrac{(L+\varepsilon-\rho)^2}{2}, & L < \rho \le L+\varepsilon, \\ c_0 L + c_0\varepsilon, & \rho \ge L+\varepsilon. \end{cases}$$

We write the characteristics

$$\rho = \rho_0 + \alpha m_0^{(\varepsilon)}(\rho_0) u_0^{(\varepsilon)}(\rho_0)^{\alpha-1} t \quad \text{for } 0 < \rho < L+\varepsilon.$$

Since $0 < \alpha < 1$, these characteristics cover the whole space $(t,\rho) \in (0,+\infty)^2$. There is a function $\rho_0^{(\varepsilon)}(t,\rho)$, with no simple explicit formula. Then

$$u^{(\varepsilon)}(t,\rho) = \big(u_0^{(\varepsilon)}(\rho_0^{(\varepsilon)}(t,r))^{-\alpha} + \alpha t\big)^{-\frac{1}{\alpha}}. \tag{4.3}$$

The characteristic emanating from the end of the flat part is still

$$\rho = L + \alpha c_0 L (c_0)^{\alpha-1} t = L(1+\alpha c_0^\alpha t).$$

For $t > 0$ and $\rho > L(1+\alpha c_0^\alpha t)$, as $\varepsilon \to 0$ then $\rho_0^{(\varepsilon)}(\rho) \to L$, whereas $u_0^{(\varepsilon)}(\rho_0^{(\varepsilon)}(t,\rho))$ is a bounded sequence, so let $p_{2,0}(t,\rho) \in [0, c_0]$ be a pointwise limit. Then $p_{2,0}(t,\rho)$ is a solution of

$$\rho = L + \alpha m_0(L) p_{2,0}(t,\rho)^{\alpha-1} t.$$

Since $m_0(L) = c_0 L$ we have

$$\rho = L + \alpha c_0 L p_{2,0}(t,\rho)^{\alpha-1} t.$$

Therefore the limit is unique and

$$u_0^{(\varepsilon)}(\rho_0^{(\varepsilon)}(t,\rho)) \to p_{2,0}(t,\rho) = \Big(\frac{\rho - L}{\alpha c_0 L t}\Big)^{\frac{1}{\alpha-1}}.$$

In particular, the pointwise limit is unique. Hence, the whole sequence converges pointwise to the limit

$$u^{(\varepsilon)}(t,\rho) \to \bigg(\Big(\frac{\rho-L}{\alpha c_0 L t}\Big)^{\frac{\alpha}{1-\alpha}} + \alpha t\bigg)^{-\frac{1}{\alpha}} \quad \text{for } t > 0 \text{ and } \rho > L(1+\alpha c_0^\alpha t).$$

Since the pointwise limit elsewhere is given by $u^{(\varepsilon)}(t,\rho) \to u(t,\rho)$, where

$$u(t,\rho) = \begin{cases} (c_0^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}} & \text{if } \rho \le L(1+\alpha c_0^{\alpha} t), \\ \left(\left(\frac{\rho - L}{\alpha c_0 L t} \right)^{\frac{\alpha}{1-\alpha}} + \alpha t \right)^{-\frac{1}{\alpha}} & \text{if } \rho > L(1+\alpha c_0^{\alpha} t), \end{cases} \tag{4.4}$$

we have only proved pointwise convergence. Since the sequence $u^{(\varepsilon)} \in L^1 \cap L^\infty$ is bounded, the limit is L^p for $1 < p < +\infty$ and weak-$\star$ in L^∞.

Remark 4.1. This last function is continuous, and therefore its mass is a classical solution of (3.3) .

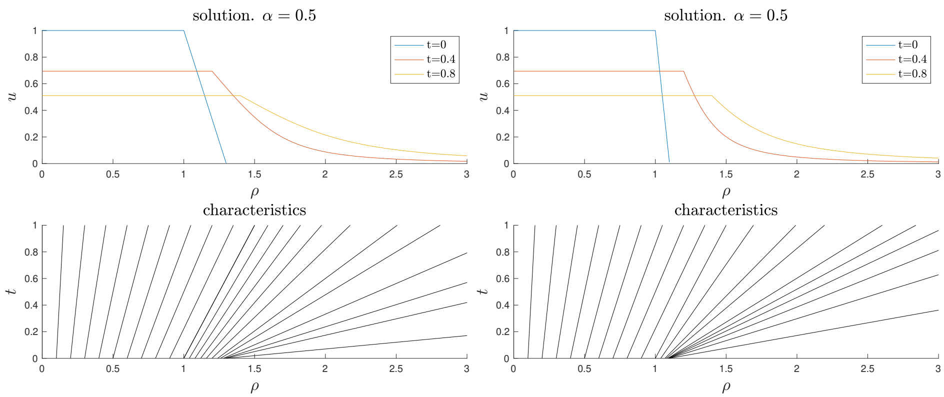

Figure 4. Solution by characteristics for the case $\alpha = 0.5$ and initial data $u_0^{(\varepsilon)}$ ($c_0 = 1$, $L = 1$ and $\varepsilon = 0.3$ left and $\varepsilon = 0.1$ right) a continuous version of the characteristic function, still with compact support. The characteristics guarantee that flat zones are preserved. For $t > 0$ there is no longer compact support.

We point out that the analogy to the fast diffusion equation is limited since Figure 4 shows that solutions are Lipschitz continuous and no more even if fat tails are produced.

4.2.2. Recovering the self-similar solution. Let us consider (4.4) and fix the total mass $M = c_0 L$. We get

$$u_L(t,x) = \begin{cases} (M^{-\alpha} L^{\alpha} + \alpha t)^{-\frac{1}{\alpha}}, & \omega_d |x|^d \le L + \alpha M^{\alpha} L^{1-\alpha} t, \\ \left(\left(\frac{\omega_d |x|^d - L}{\alpha M t} \right)^{\frac{\alpha}{1-\alpha}} + \alpha t \right)^{-\frac{1}{\alpha}}, & \omega_d |x|^d > L + \alpha M^{\alpha} L^{1-\alpha} t. \end{cases}$$

As $L \to 0$ we recover the self-similar solution of mass M,

$$\begin{aligned} u_L(t,x) &\to \left(\left(\frac{\omega_d |x|^d}{\alpha M t} \right)^{\frac{\alpha}{1-\alpha}} + \alpha t \right)^{-\frac{1}{\alpha}} = t^{-\frac{1}{\alpha}} \left(\left(\frac{t^{-\frac{1}{\alpha}} \omega_d |x|^d}{\alpha M} \right)^{\frac{\alpha}{1-\alpha}} + \alpha \right)^{-\frac{1}{\alpha}} \\ &= t^{-\frac{1}{\alpha}} F\left(\frac{t^{-\frac{1}{d\alpha}} |x|}{M^{\frac{1}{d}}} \right) = U_M(t,x). \end{aligned}$$

4.2.3. Rarefaction fan solution for discontinuous non-increasing data. Combining the formula of solutions by characteristics (4.1) with the rarefaction fan idea we construct

$$u(t,\rho) = (\eta_0^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}} \quad \text{where } \rho = \rho_0 + \alpha m_0(\rho_0)\eta_0^{\alpha-1} t, \tag{4.5a}$$

where η_0 is some value

$$\eta_0 \in [u_0(\rho_0^+), u_0(\rho_0^-)]. \tag{4.5b}$$

Let us now show that these solutions are well defined.

Proposition 4.2. *Let $u_0 \in L^\infty_+([0,+\infty)) \cap L^\infty_+([0,+\infty))$, $u_0 \not\equiv 0$, and radially non-increasing. For every $t > 0$, the map*

$$P_t \colon \bigcup_{\rho_0 : u_0(\rho_0)>0} \{\rho_0\} \times [u_0(\rho_0^+), u_0(\rho_0^-)] \to \mathbb{R}_+, \tag{4.6}$$

$$(\rho_0, \eta_0, t) \mapsto \rho_0 + \alpha m_0(\rho_0)\eta_0^{\alpha-1} t \tag{4.7}$$

is bijective. Therefore, the map (4.5) is well defined. Furthermore, it defines a function $u \in \mathcal{C}([0,+\infty)^2)$ such that

$$u(t,\cdot) \to u_0 \quad \text{in } L^1(\mathbb{R}^n). \tag{4.8}$$

The function m given by

$$m(t,\rho) = m_0(\rho_0)(1 + \alpha\eta_0^\alpha t)^{1-\frac{1}{\alpha}}, \quad \text{where } (\rho_0, \eta_0) = P_t^{-1}(\rho), \tag{4.9}$$

is a classical solution of (3.3) by characteristics.

Proof. We define an order over the domain of P_t,

$$(\rho_{0,1}, \eta_{0,1}) \prec (\rho_{0,2}, \eta_{0,2}) \equiv \begin{cases} \rho_{0,1} < \rho_{0,2} \\ \text{or} \\ \rho_{0,1} = \rho_{0,2} \text{ and } \eta_{0,1} > \eta_{0,2}. \end{cases}$$

This defines a strict total order in the domain of P_t. Furthermore, notice that

$$(\rho_{0,1}, \eta_{0,1}) \prec (\rho_{0,2}, \eta_{0,2}) \implies P_t(\rho_{0,1}, \eta_{0,1}) < P_t(\rho_{0,2}, \eta_{0,2}).$$

Hence, P_t is injective. Furthermore, it is continuous with the topology induced in the domain of P_t. It is immediate to check that

$$P_t(0, u(0^+)) = 0.$$

Notice that $\{\rho_0 : u_0(\rho_0) > 0\} = (0, R)$ where $R \le +\infty$. As $\rho_0 \nearrow R$ we have that $u_0(\rho_0^+) \to 0$ and $m_0(\rho_0) \nearrow M$, hence

$$P_t(\rho_0, u(\rho_0^+)) \nearrow +\infty.$$

Hence P_t is surjective. This completes the proof. ∎

4.2.4. Data with an initial gap. Notice that if u is given by (4.5), then

$$\tilde{u}(t,\rho) = \begin{cases} u(t,\rho-b), & \rho \geq b, \\ 0, & 0 \leq \rho < b, \end{cases}$$

is also a solution, and it corresponds to the initial datum

$$\tilde{u}_0(\rho) = \begin{cases} u_0(\rho-b), & \rho \geq b, \\ 0, & 0 \leq \rho < b. \end{cases}$$

Furthermore, this solution can be obtained by approximation by continuous initial data given by characteristics. Therefore $\operatorname{supp} \hat{u}(t,\cdot) = [b,+\infty)$. The conclusion is that this kind of gap is preserved in time. See Figure 5.

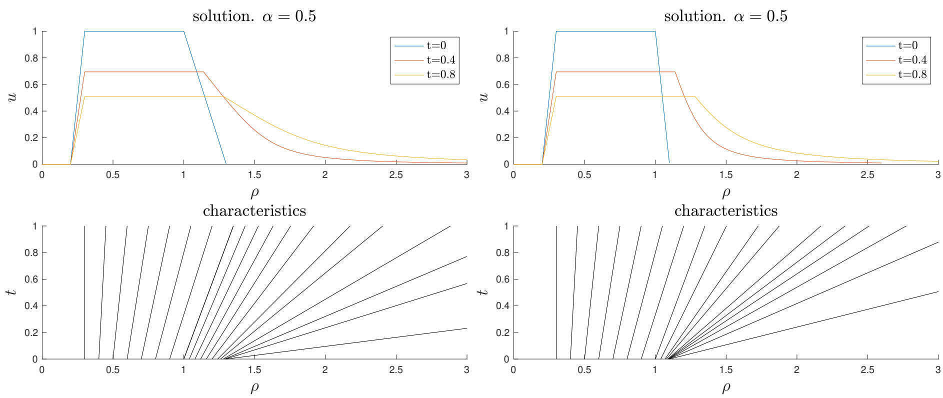

Figure 5. Solution by characteristics for the case $\alpha = 0.5$ where initial data has an initial gap.

Remark 4.3. Notice that if $\operatorname{supp} u_0 = [b, L]$ then for any $t > 0$, $\lim_{\rho\to+\infty} P_t^{-1}(\rho) = (L, 0)$. If $\operatorname{supp} u_0 = [0,+\infty)$ then $\lim_{\rho\to+\infty} P_t^{-1}(\rho) \to +\infty$.

4.3. Qualitative properties

4.3.1. Mass conservation.

Proposition 4.4. *For the classical solution m of* (3.3) *given by* (4.5) *under the hypothesis of Proposition* 4.2*, for every $t \geq 0$ we have*

$$\lim_{\rho\to+\infty} m(t,\rho) = \lim_{\rho\to+\infty} m_0(\rho). \tag{4.10}$$

Proof. Combining (4.9) and Remark 4.3 we conclude the result. ∎

4.3.2. Comparison principle for *u* by characteristics. Since we have classical solutions by characteristics, we prove a comparison principle for $u = m_\rho$ by direct computation. This immediately implies there exists some kind of comparison principle for m.

Theorem 4.5. *Assume that $u_{0,1}, u_{0,2} \in L^\infty(\mathbb{R}^d)$ is radially non-increasing and such that $u_{0,1} \ge u_{0,2}$ in $\mathbb{R}^d$. Let u_1, u_2 be given by (4.5). Then $u_1 \ge u_2$ in $[0,+\infty) \times \mathbb{R}^d$.*

Proof. Assume, towards a contradiction, that $u_1(t,\rho) < u_2(t,\rho)$ for some $t > 0$, ρ. Since the functions are given by (4.5) we write this inequality as

$$(\eta_{0,1}^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}} < (\eta_{0,2}^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}}, \tag{4.11}$$

where $\eta_{0,i} \in [u_{0,i}(\rho_{0,i}^+), u_{0,i}(\rho_{0,i}^-)]$. Therefore, we have $\eta_{0,1} < \eta_{0,2}$. Since $u_{0,1} \ge u_{0,2}$ we have $\rho_{0,2} \le \rho_{0,1}$. Since $m_{0,2} \le m_{0,1}$ we have $m_{0,2}(\rho_{0,2}) \le m_{0,1}(\rho_{0,1})$.

If $m_{0,2}(\rho_{0,2}) = 0$, then $\rho = \rho_{0,2} = 0$, but at $\rho = 0$ we have $u_2(t,\rho) = u_2(t,0) = u_{0,2}(0) \ge u_{0,1}(0) = u_1(t,0)$ and we reach a contradiction.

If $m_{0,2}(\rho_{0,2}) > 0$, we have

$$\rho = \rho_{0,1} + \alpha m_{0,1}(\rho_{0,1})\eta_{0,1}^{\alpha-1} t < \rho_{0,2} + \alpha m_{0,2}(\rho_{0,2})\eta_{0,2}^{\alpha-1} t = \rho. \tag{4.12}$$

This is a contradiction. ∎

4.4. Asymptotic behaviour

For the study of the asymptotic behaviour we consider a rescaled version of the solution u by considering scaling as the self-similar solution

$$w(t,y) = t^{\frac{1}{\alpha}} u(t, t^{\frac{1}{d\alpha}} y). \tag{4.13}$$

This is the natural candidate to converge to a stationary non-trivial profile. We will prove stabilisation of the rescaled flow in the strong form of uniform convergence in relative error to the self-similar profile of the solution of the same total mass M, i.e. F_M.

4.4.1. Asymptotic behaviour as $t \to \infty$ for non-increasing data u_0. We tackle the general case for solutions given by characteristics and state the convergence in relative error.

Theorem 4.6. *Let $u_0 \in L_c^\infty(\mathbb{R}^d)$ be radially non-increasing, $M = \|u_0\|_{L^1}$, and let u be given by (4.5). Then we have*

$$\sup_{y\in\mathbb{R}^d} \left| \frac{w(t,y) - F_M(|y|)}{F_M(|y|)} \right| \to 0 \quad \text{as } t \to +\infty, \tag{4.14}$$

where w is given by (4.13).

As as direct consequence of this theorem, we have L^∞ convergence with sharp rate:

Corollary 4.7. *Under the same assumptions we have*

$$\lim_{t\to\infty} t^{1/\alpha} |u(t,x) - U_M(t,x)| = 0 \tag{4.15}$$

uniformly in $x \in \mathbb{R}^d$.

We split the proof of Theorem 4.6 into several lemmas.

Lemma 4.8. *Let* u_0 *be radially non-increasing and let* $\omega_d|x|^d = \rho = \rho_0 + \alpha m_0(\rho_0)\eta_0^{\alpha-1}t$. *Then*

$$\frac{u(t,x)^\alpha - U_M(t,x)^\alpha}{U_M(t,x)^\alpha} = \Big[1 - \Big(\frac{\rho/(\rho-\rho_0)}{M/m_0(\rho_0)}\Big)^{\frac{\alpha}{1-\alpha}}\Big] \times \Big[1 + \alpha\Big(\frac{\rho-\rho_0}{\alpha m_0(\rho_0)t^{\frac{1}{\alpha}}}\Big)^{-\frac{\alpha}{1-\alpha}}\Big]^{-1}. \tag{4.16}$$

Proof. We have

$$u(x,t) = (\eta_0^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}}, \tag{4.17}$$

where η_0 is given by (3.5). Going back to (1.2) we have

$$u(t,x)^{-\alpha} - U_M(t,x)^{-\alpha} = \Big[\Big(\frac{\rho-\rho_0}{\alpha m_0(\rho_0)}\Big)^{\frac{\alpha}{1-\alpha}} - \Big(\frac{\rho}{\alpha M}\Big)^{\frac{\alpha}{1-\alpha}}\Big]t^{-\frac{\alpha}{1-\alpha}} \tag{4.18}$$

and hence

$$\begin{aligned}\frac{u(t,x)^\alpha - U_M(t,x)^\alpha}{U_M(t,x)^\alpha} &= \frac{u(x,t)^{-\alpha} - U_M(t,x)^{-\alpha}}{u(t,x)^{-\alpha}}\\ &= \Big[1 - \Big(\frac{\rho/(\rho-\rho_0)}{M/m_0(\rho_0)}\Big)^{\frac{\alpha}{1-\alpha}}\Big]\Big[1 + \alpha\Big(\frac{\rho-\rho_0}{\alpha m_0(\rho_0)}\Big)^{-\frac{\alpha}{1-\alpha}} t^{\frac{1}{1-\alpha}}\Big]^{-1}. \qquad \blacksquare\end{aligned}$$

Equation (4.16) looks better in rescaled variables:

$$\begin{aligned}\Big|\frac{w(t,y)^\alpha - F_M(|y|)^\alpha}{F_M(|y|)^\alpha}\Big| &= \Big|\Big[1 - \Big(\frac{|y|^d}{|y|^d - \rho_0(t,y)t^{-\frac{1}{\alpha}}}\frac{m_0(\rho_0(t,y))}{M}\Big)^{\frac{\alpha}{1-\alpha}}\Big]\Big|\\ &\quad\times\Big[1 + \alpha\Big(\frac{|y|^d - \rho_0(t,y)t^{-\frac{1}{\alpha}}}{\alpha m_0(\rho_0(t,y))}\Big)^{-\frac{\alpha}{1-\alpha}}\Big]^{-1}\\ &\le \Big|1 - \Big(\frac{|y|^d}{|y|^d - \rho_0(t,y)t^{-\frac{1}{\alpha}}}\frac{m_0(\rho_0(t,y))}{M}\Big)^{\frac{\alpha}{1-\alpha}}\Big|,\end{aligned} \tag{4.19}$$

where ρ_0 represents the foot of the rarefaction fan solution, i.e. in the notation of Section 4.2.3,

$$\rho_0(t,y) = P_t^{-1}(t^{\frac{1}{\alpha}}y).$$

For compactly supported u_0, we have that $\rho_0(t,y)$ is bounded and the first fraction tends to 1 uniform in t. For the second fraction we need to know whether $m_0(\rho_0(t,y)) \to M$ when $t \to +\infty$.

Lemma 4.9. *Let* u_0 *be radially non-increasing, bounded, and compactly supported. Let* $\operatorname{supp} u_0 = [0,\rho_*]$. *Then, for every* $t \ge 0$, *we have*

$$\rho_0(t,y) \to \rho_* \quad \text{as } t \to +\infty \text{ uniformly for } |y| \ge \delta.$$

Proof. We write

$$\rho = \rho_0 + \alpha \eta_0^{\alpha-1} m_0(\rho_0) t, \quad \eta_0 \in [u_0(\rho_0^+), u_0(\rho_0^-)].$$

In order to recover the scaling factor, we multiply by $t^{-\frac{1}{\alpha}}$ and bound some terms from below and some from above:

$$\omega_d \delta^d \le \omega_d |y|^d = t^{-\frac{1}{\alpha}} \rho_0 + \alpha \eta_0^{\alpha-1} m_0(\rho_0) t^{-\frac{1-\alpha}{\alpha}} \le t^{-\frac{1}{\alpha}} \rho_* + \alpha \eta_0^{\alpha-1} M t^{-\frac{1-\alpha}{\alpha}}.$$

Hence,

$$\eta_0 \le t^{-\frac{1}{\alpha}} \Big(\frac{\alpha M}{\omega_d \delta^d - t^{-\frac{1}{\alpha}} \rho_*} \Big)^{\frac{1}{1-\alpha}}$$

so long as t is large enough that $\omega_d y_0^d > t^{-\frac{1}{\alpha}} \rho_*$. Since $\eta_0 \to 0$ uniformly in $|y_0|$ and u_0 is radially decreasing, then $\rho_0(t, y) \to \rho_*$ uniformly in δ. ■

We can now prove the main theorem.

Proof of Theorem 4.6. Let $\varepsilon > 0$. Since F_M is continuous and $F_M > 0$, let us take $\delta > 0$ such that

$$\left| \frac{F_M(|y|) - F_M(0)}{F_M(0)} \right| \le \varepsilon \quad \forall |y| \le \delta. \tag{4.20}$$

Step 1. Close to $y = 0$. One-sided bound. We assume first that $|y| < \delta$. Since v is non-increasing in y, we have

$$w(t, y_0) \le w(t, y) \le w(t, 0) \quad \forall |y| \le |y_0| = \delta.$$

On the one hand we notice that

$$w(t, 0) = t^{\frac{1}{\alpha}} u(t, 0) = (u_0 t^{-1} + \alpha)^{-\frac{1}{\alpha}} \to \alpha^{-\frac{1}{\alpha}} = F_M(0)$$

as $t \to +\infty$. Hence, there exists $t_1 > 0$ such that

$$w(t, 0) \le F_M(0)(1 + \varepsilon) \quad \forall t \ge t_1.$$

Hence,

$$w(t, y) \le w(t, 0) \le F_M(0)(1 + \varepsilon) \le F_M(|y|)(1 + \varepsilon)^2.$$

Therefore,

$$\frac{w(t, y) - F_M(|y|)}{F_M(|y|)} \le 2\varepsilon + \varepsilon^2 \quad \forall t \ge t_1, |y| \le \delta. \tag{4.21}$$

Step 2. Away from 0. $|y| \ge \delta$. Through Lemma 4.8 in version (4.19) and Lemma 4.9 we have

$$\sup_{|y| \ge \delta} \left| \frac{w(t, y)^\alpha - F_M(|y|)^\alpha}{F_M(|y|)^\alpha} \right| \to 0 \quad \text{as } t \to +\infty.$$

Therefore, there exists $t_2 > 0$ dependent on δ such that

$$\frac{1}{2}F_M(|y|)^\alpha \le w(t,y)^\alpha \le \frac{3}{2}F_M(|y|)^\alpha \quad \forall t \ge t_2,\ |y| \ge \delta.$$

Taking roots,

$$\frac{1}{2^{\frac{1}{\alpha}}}F_M(|y|) \le w(t,y) \le \left(\frac{3}{2}\right)^\alpha F_M(|y|).$$

Since we want to compare v and F_M rather than their power, we use the intermediate value theorem which gives

$$w(t,y)^\alpha - F_M(|y|)^\alpha = \alpha \nu(t,y)^{\alpha-1}(w(t,y) - F_M(|y|)),$$

where $\nu(t,y)$ is between $w(t,y)$ and $F_M(|y|)$. Therefore,

$$\begin{aligned}
\left|\frac{w(t,y) - F_M(|y|)}{F_M(|y|)}\right| &= \frac{1}{\alpha\nu(t,y)^{\alpha-1}}\left|\frac{w(t,y)^\alpha - F_M(|y|)^\alpha}{F_M(|y|)}\right| \\
&= \frac{1}{\alpha}\frac{F_M(|y|)^{\alpha-1}}{\nu(t,y)^{\alpha-1}}\left|\frac{w(t,y)^\alpha - F_M(|y|)^\alpha}{F_M(|y|)^\alpha}\right| \\
&= \frac{1}{\alpha}\left(\frac{\nu(t,y)}{F_M(|y|)}\right)^{1-\alpha}\left|\frac{w(t,y)^\alpha - F_M(|y|)^\alpha}{F_M(|y|)^\alpha}\right| \\
&\le \frac{1}{\alpha}\left(\frac{3}{2}\right)^{\frac{\alpha}{1-\alpha}}\left|\frac{w(t,y)^\alpha - F_M(|y|)^\alpha}{F_M(|y|)^\alpha}\right|.
\end{aligned} \tag{4.22}$$

Hence, there exists $t_3 \ge t_2$ such that

$$\left|\frac{w(t,y) - F_M(|y|)}{F_M(|y|)}\right| \le \varepsilon \quad \forall t \ge t_3,\ |y| \ge \delta. \tag{4.23}$$

Step 3. Close to $y = 0$. Other bound. We write (4.23) as

$$w(t,y) \ge (1-\varepsilon)F_M(|y|) \quad \forall t \ge t_3,\ |y| \ge \delta.$$

Therefore, taking some $|y_0| = \delta$ and using that v is non-increasing in y we have

$$w(t,y) \ge w(t,y_0) \ge (1-\varepsilon)F_M(\delta) \quad \forall t \ge t_3,\ |y| \le |y_0|.$$

Going back to (4.20) we have

$$F_M(\delta) \ge (1-\varepsilon)F_M(0) \ge (1-\varepsilon)^2 F_M(|y|),$$

so

$$w(t,y) \ge F_M(|y|)(1-\varepsilon)^3 \quad \forall t \ge t_3,\ |y| \le \delta.$$

Therefore

$$\frac{w(t,y) - F_M(|y|)}{F_M(|y|)} \ge -3\varepsilon + 3\varepsilon^2 - \varepsilon^3. \tag{4.24}$$

Joining the information from (4.21) and (4.24) we have

$$\left|\frac{w(t,y) - F_M(|y|)}{F_M(|y|)}\right| \le 4\varepsilon + 4\varepsilon^2 + \varepsilon^3 \quad \forall t \ge t_2, t_3,\ |y| \le \delta.$$

Together with (4.23), this completes the proof. ∎

4.4.2. Asymptotic behaviour as $t \to +\infty$ for non-increasing data u_0 with an initial gap. Going back to what was said in Section 4.2.4, if we have an initial datum

$$u_0(\rho) = \begin{cases} 0, & \rho \le b, \\ \tilde u_0(\rho - b), & \rho > b, \end{cases}$$

where u_0 is non-increasing, then a solution of (P) by characteristics is given by

$$u(t,\rho) = \begin{cases} 0, & \rho \le b, \\ \tilde u(t, \rho - b), & \rho > b, \end{cases}$$

where u is the solution by characteristics with datum u_0. In particular, $u(t,0) = 0$ and, therefore, $v(t,0) = 0$. Hence, (4.14) cannot hold. Nevertheless, due to Theorem 4.6 we have the weaker form

$$\sup_{\omega_d |x|^d \ge b} \left| \frac{u(t,x) - U_M(t,x)}{U_M(t,x)} \right| \to 0 \quad \text{as } t \to +\infty. \tag{4.25}$$

In rescaled variable this reads

$$\sup_{\omega_d |y|^d \ge b t^{-\frac{1}{\alpha}}} \left| \frac{w(t,y) - F_M(|y|)}{F_M(|y|)} \right| \to 0 \quad \text{as } t \to +\infty.$$

Asymptotically, this covers every $|y| > 0$. In particular, it guarantees that

$$\sup_{|y| \ge \delta} \left| \frac{w(t,y) - F_M(|y|)}{F_M(|y|)} \right| \to 0 \quad \text{as } t \to +\infty, \ \forall \delta > 0. \tag{4.26}$$

This result is the most that can be expected in general.

4.4.3. Asymptotic behaviour as $|x| \to +\infty$ for $t > 0$ fixed and general non-increasing data u_0. We can repeat the argument to check that the tails of the self-similar solution are maintained as $|x| \to +\infty$ for any $t > 0$, if u_0 is compactly supported

Theorem 4.10. *Let $u_0 \in L^\infty_c(\mathbb{R}^d)$ be radially non-increasing, $M = \|u_0\|_{L^1}$, and let u be given by* (4.5)*. Then, for every $t > 0$ fixed,*

$$\frac{u(t,x) - U_M(t,x)}{U_M(t,x)} \to 0, \quad |x| \to +\infty. \tag{4.27}$$

Proof. We repeat the same argument as before. First, from Lemma 4.8 and the fact that $\rho_0 = P_t^{-1}(\rho)$ we have

$$\frac{u(t,x)^\alpha - U_M(t,x)^\alpha}{U_M(t,x)^\alpha} \to 0, \quad |x| \to +\infty. \tag{4.28}$$

Therefore, $u/U_M \in (c_1, c_2)$ at least for $|x|$ large. Then, arguing as in (4.22), it holds that

$$\left| \frac{u(t,x) - U_M(t,x)}{U_M(t,x)} \right| \le \frac{c_2^{\frac{\alpha}{1-\alpha}}}{\alpha} \left| \frac{u(t,x)^\alpha - U_M(t,x)^\alpha}{U_M(t,x)^\alpha} \right| \to 0. \qquad \blacksquare$$

5. General radial data

As we pointed out in item (4) of Remark 3.2 if the data u_0 is not monotone, then characteristics can cross instantaneously. This leads to the appearance of a shock using the conservation law theory, as we will discuss next.

5.1. Shocks. The Rankine–Hugoniot equation

The choice of the free boundary is not trivial in principle. As usual in conservation laws, let us assume that the solution is classical at either side of a shock wave $(t, S(t))$, and let us denote these solutions by u^+ and u^-.

If we consider the mass at either side S, continuity of the mass means

$$m(t, S(t)^+) = m(t, S(t)^-).$$

Taking derivatives,

$$m(t, S(t)^+)_t + m_\rho(t, S(t)^+)S' = m_t(t, S(t)^-) + m_\rho(t, S(t)^-)S'.$$

Due to (3.3) and the continuity of m,

$$\begin{aligned} &- m_\rho(t, S(t)^+)^\alpha m(t, S(t)) + m_\rho(t, S(t)^+)S' \\ &\quad = -m_\rho(t, S(t)^-)^\alpha m(t, S(t)) + m_\rho(t, S(t)^-)S'. \end{aligned}$$

Taking into account that $m_\rho = u$, we deduce that

$$-(u^+)^\alpha m + u^+ S' = -(u^-)^\alpha m + u^- S'$$

and solving for S' we deduce the equation

$$S'(t) = m(t, S(t)) \frac{u^+(t, S(t))^\alpha - u^-(t, S(t))^\alpha}{u^+(t, S(t)) - u^-(t, S(t))}. \tag{5.1}$$

We can call this the generalised Rankine–Hugoniot condition for (3.3). We leave it to the reader to check that if u^+ and u^- are weak local solutions satisfying (5.1) then the solution constructed by pasting them,

$$u(t, x) = \begin{cases} u^+(t, x), & x < S(t), \\ u^-(t, x), & x > S(t), \end{cases}$$

is a weak local solution.

Remark 5.1. For solutions jumping at the boundary of the support, formula (5.1) simplifies into

$$S'(t) = m(t, S(t))u^+(t, S(t))^{\alpha-1}. \tag{5.2}$$

5.2. Example of non-uniqueness of weak solutions: the square functions

Let $u_0(\rho) = c_0\chi_{[0,L]}(\rho)$. Then the mass in volumetric coordinates is

$$m(0,\rho) = \begin{cases} c_0\rho, & \rho < L, \\ c_0 L, & \rho \geq L. \end{cases}$$

Let us find weak local solutions by characteristics. For $0 < \rho_0 < L$ we have

$$\rho(t) = \rho_0 + \alpha\rho_0 c_0^\alpha t.$$

We can therefore invert:

$$\rho_0 = \frac{r}{1+\alpha c_0^\alpha t}.$$

Assume $0 < \rho/(1+\alpha c_0^\alpha t) < L$; then we can follow a characteristic back to $\rho_0 < L$. As a consequence we get

$$u(t,\rho) = c_0(1+\alpha c_0^\alpha t)^{-\frac{1}{\alpha}}.$$

Using the generalised Rankine–Hugoniot condition (5.1), we select two weak local solutions $u^+(t,\rho) = (c_0^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}}$ and $u^-(t,\rho) = 0$. The mass $m(t,r)$ is clearly

$$m(t,\rho) = (c_0^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}}(\rho \wedge S(t)).$$

Therefore, we substitute in (5.2) to deduce that

$$S'(t) = S(t)(c_0^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}}(c_0^{-\alpha} + \alpha t)^{-\frac{\alpha-1}{\alpha}}.$$

Hence

$$S'(t) = S(t)(c_0^{-\alpha} + \alpha t)^{-1}.$$

By definition the jump must begin at the jump of u_0, hence $S(0) = L$. We integrate the separable equation to deduce that

$$S(t) = L(c_0^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}}.$$

Therefore, the solutions obtained by elementary mass conservation arguments are precisely the weak solutions. We have therefore constructed the following function:

$$u(t,\rho) = \begin{cases} c_0(1+\alpha c_0^\alpha t)^{-\frac{1}{\alpha}}, & \rho < L(c_0^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}}, \\ 0, & \rho > L(c_0^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}}. \end{cases}$$

It is an additional weak local solution to equation (P) with initial data $u_0 = c_0\chi_{[0,L]}$. Notice that we have already constructed a rarefaction fan solution in Section 4.2.1. We leave it to the reader to check that this solution does not satisfy the Lax–Oleinik condition of incoming characteristics.

5.3. Initial data with two bumps

We now consider the initial data

$$u_0 = c_1 \chi_{[0,1]} + c_2 \chi_{[a,b]};$$

then physically meaningful solutions must develop a free boundary $S(t)$. The solution can be computed classically on the left ($u_1(t,x)$) and right ($u_2(t,x)$) of $S(t)$. We presume that $S(0) = a$. Since the characteristics emanating from the second bump have increasing slope, the Lax–Oleinik condition guarantees that S is increasing, so that they are incoming characteristics. An ODE for $S(t)$ can be written via the Rankine–Hugoniot equation (5.1).

As u_1 we can take the solution by characteristics constructed in Section 4.2.1, i.e.

$$u_1(t,\rho) = \begin{cases} (c_1^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}}, & \rho \le 1 + \alpha c_1^\alpha t, \\ \left(\left(\dfrac{\rho - 1}{\alpha c_1 t}\right)^{\frac{\alpha}{1-\alpha}} + \alpha t\right)^{-1/\alpha}, & \rho > 1 + \alpha c_1^\alpha t. \end{cases}$$

For u_2 we have to do some additional computations. We will construct a solution by characteristics which is, as we have seen, a weak local solution. We look at the equation for the characteristics coming from $\rho_0 \in [a,b)$,

$$\rho = \rho_0 + \alpha u_{2,0}(\rho_0)^{\alpha-1} m_{0,2}(\rho_0) t = \rho_0 + \alpha c_2^{\alpha-1}(c_1 + c_2(\rho_0 - a))t,$$

where $m_{0,2}(a) = c_1$ is the mass accumulated from the first bump. These characteristics correspond to the flat zone $u_2(t,\rho) = (c_2^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}}$. On the other hand, the characteristics of the rarefaction fan tail come from $\rho_0 = b$ and have as equations

$$\rho = b + \alpha \eta_{0,2}^{\alpha-1}(c_1 + c_2(b-a))t,$$

where each characteristic is determined by a unique value $\eta_{0,2} \in [0, c_2]$, and over this characteristic we have $u_2(t,\rho) = (\eta_{0,2}^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}}$. Solving for $\eta_{0,2}$ we have

$$\eta_{0,2} = \left(\frac{\rho - b}{\alpha(c_1 + c_2(b-a))t}\right)^{\frac{1}{\alpha-1}}.$$

Therefore, we can construct the solution right of the shock as

$$u_2(t,\rho) = \begin{cases} (c_2^{-\alpha} + \alpha t)^{-1/\alpha}, & \rho \le b + \alpha(c_1 + c_2(b-a))c_2^{\alpha-1} t, \\ \left(\left(\dfrac{\rho - b}{\alpha(c_1 + c_2(b-a))t}\right)^{\frac{\alpha}{1-\alpha}} + \alpha t\right)^{-1/\alpha}, & \rho > b + \alpha(c_1 + c_2(b-a))c_2^{\alpha-1} t. \end{cases}$$

We can compute the mass at the shock as that coming from the left. The shock will move faster than the characteristic coming from 1, and going back to (3.6),

$$\begin{aligned} m(t,S(t)) &= m_1(t,S(t)) = m_{0,1}(\rho_0)(1 + \alpha u_0(\rho_0)^\alpha t)^{1-\frac{1}{\alpha}} \\ &= \begin{cases} c_1 S(t)(1 + \alpha c_1^\alpha t)^{1-\frac{1}{\alpha}}, & S(t) \le 1 + \alpha c_1^\alpha t, \\ c_1\left(1 + \alpha\left(\dfrac{S(t) - 1}{\alpha c_1 t}\right)^{\frac{\alpha}{\alpha-1}} t\right)^{1-1/\alpha}, & S(t) > 1 + \alpha c_1^\alpha t, \end{cases} \end{aligned}$$

and we end up with a piecewise-defined ODE

$$S'(t) = m(t, S(t)) \frac{u_1(t, S(t))^\alpha - u_2(t, S(t))^\alpha}{u_1(t, S(t)) - u_2(t, S(t))}.$$

The right-hand side of the ODE for $S(t)$ is continuous and locally Lipschitz, so a unique solution exists. Solving this ODE numerically, we obtain the characteristics given in Figure 6.

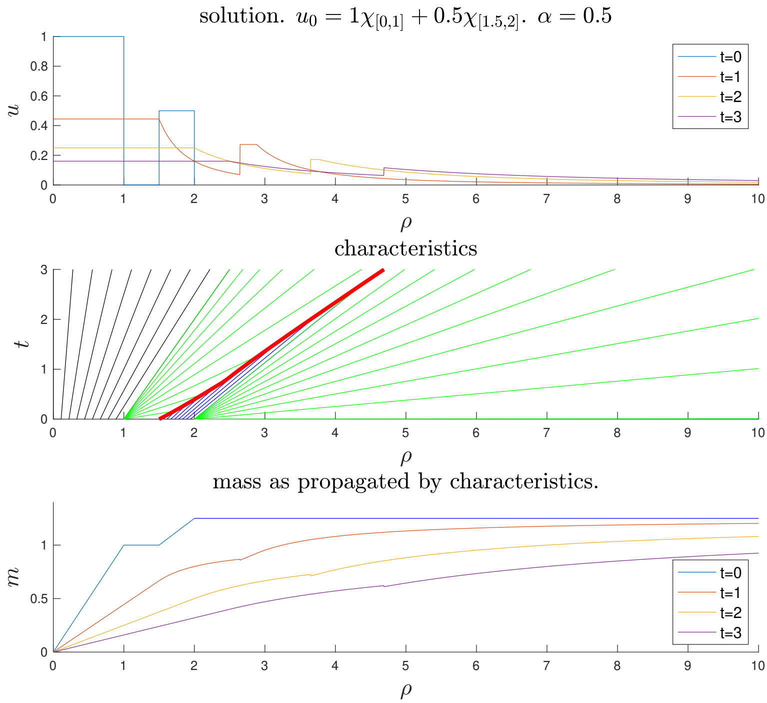

Figure 6. Solution using explicit solutions on either side of the shock, and a Runge–Kutta scheme to solve the Rankine–Hugoniot condition. We show only a few characteristics. The characteristics in black and blue correspond to the flat part of the solution, characteristics in green to the rarefaction fan tail, and the red line represents the shocks.

6. Existence theory: vanishing viscosity

Up to now we have constructed solutions by the method of characteristics. The problem is then to show that for this class of solutions the initial value problem is well posed. As is frequently done, we proceed further by constructing first a well-posed theory for the approximate regularised problem (P_ε), and then we pass to the limit to construct solutions of (P) that coincide with our previous constructions. Finally, we prove uniqueness of the limit by yet another theory.

6.1. Classical solutions of the viscous problem (P_ε)

We construct a classical solution via a fixed point of the heat equation.

Theorem 6.1. *Let $u_0 \in L^1(\mathbb{R}^d) \cap \mathcal{C}_b(\mathbb{R}^d)$. Then there exists a classical solution of* (P_ε).

Proof. We define the map

$$G\colon u \mapsto \nabla \cdot ((\varepsilon + u_+)^\alpha \nabla \mathrm{N}(u)).$$

One can write the equation as $u_t - \varepsilon \Delta u = G(u)$. Hence, it is natural to look for solutions as fixed points of Duhamel's formula

$$u(t) = S_\varepsilon(t) u_0 + \int_0^t S_\varepsilon(t-s) G(u(s))\, \mathrm{d}s, \tag{6.1}$$

where S_ε is the semigroup for $-\varepsilon\Delta$. We recall the regularisation properties of the Newtonian potential $\mathrm{N}\colon \mathcal{C}_b(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \to \mathcal{C}_b^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$. By writing

$$G(u) = (\varepsilon + u_+)^\alpha \nabla u \cdot \nabla \mathrm{N}(u) - (\varepsilon + u_+)^\alpha u,$$

we deduce that $G\colon H^1(\mathbb{R}^d) \cap \mathcal{C}^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d) \cap \mathcal{C}_b(\mathbb{R}^d)$. Furthermore, it is a Lipschitz operator. The operator from (6.1) is given by

$$K_t(u) = S_\varepsilon(t) u_0 + \int_0^t S_\varepsilon(t-s) G(u(s))\, \mathrm{d}s.$$

Due to the standard decay properties of the heat semigroup we have

$$K_t\colon X \to X, \quad \text{where } X = L^2(0,T; L^1(\mathbb{R}^d)) \cap \mathcal{C}_b(\mathbb{R}^d),$$

is Lipschitz with a constant depending on t. For t small enough, the operator is contractive and we can use Banach's fixed point theorem to show there exists a unique solution of (6.1). Since G is Lipschitz, this constant can be taken uniformly, and hence the solution is global. By a simple bootstrap argument, we show that the solution is classical. ■

Proposition 6.2. *Let $u_0 \in L^1(\mathbb{R}^d) \cap \mathcal{C}_b(\mathbb{R}^d)$. Then any classical solutions of* (P_ε) *satisfy*

$$\|u(t,\cdot)_+\|_{L^p} \le \|(u_0)_+\|_{L^p}. \tag{6.2}$$

The same holds for u_-, so if $u_0 \ge 0$, then $u \ge 0$.

Proof. We study the positive and negative parts separately. Studying the negative part of u is very simple. For $1 < p < +\infty$, multiplying by $-(u_-)^{p-1}$ and integrating,

$$\int_{\mathbb{R}^d} u_t u_-^{p-1} - \int_{\mathbb{R}^d} (\varepsilon + u_+)^\alpha \nabla \mathrm{N}(u) \nabla u_-^{p-1} + \varepsilon \int_{\mathbb{R}^d} |\nabla u_-|^2 = 0.$$

Taking into account that u is a classical solution, that $(\varepsilon + u_+)^\alpha \nabla u_- = \varepsilon^\alpha \nabla u_-$, and the equation satisfied by $\mathrm{N}(u)$ we have

$$\begin{aligned}\int_{\mathbb{R}^d} u_t u_-^{p-1} &= \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |u_-|^p,\\ -\int_{\mathbb{R}^d} (\varepsilon + u_+)^\alpha \nabla \mathrm{N}(u) \nabla u_-^{p-1} &= -\varepsilon^\alpha \int_{\mathbb{R}^d} \nabla u_- \nabla\,\mathrm{N}(u),\\ &= -\varepsilon^\alpha \int_{\mathbb{R}^d} u u_- \geq 0.\end{aligned}$$

Hence, we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |u_-|^p \leq 0.$$

From this, we deduce that $\|u_-\|_{L^p} \leq \|(u_0)_-\|_{L^p}$.

For $1 < p < +\infty$, we repeat a similar argument for u_+, multiplying by u_+^{p-1} and integrating,

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u_+^p + \int_{\mathbb{R}^d} (\varepsilon + u_+)^\alpha \nabla \mathrm{N}(u) \nabla u_+^{p-1} + \varepsilon \int_{\mathbb{R}^d} |\nabla u_+|^2 = 0.$$

We have $(\varepsilon + u_+)^\alpha \nabla u_+^{p-1} = (p-1)(\varepsilon + u_+)^\alpha u_+^{p-2} \nabla u_+ = g(u_+)\nabla u_+ = \nabla G(u_+)$, where G is a primitive of g such that $G(0) = 0$. Since $\mathrm{N}(u)$ is the solution $-\Delta\,\mathrm{N}(u) = u$ we have

$$\int_{\mathbb{R}^d} \nabla \mathrm{N}(u) \nabla G(u_+) = \int_{\mathbb{R}^d} u G(u_+) = \int_{\mathbb{R}^d} u_+ G(u_+) \geq 0.$$

Finally,

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u_+^p \leq 0.$$

For $p = 1, +\infty$, the estimates hold by passing to the limit. ∎

6.2. An equation for the mass of ($\mathbf{P}_\varepsilon$)

Let us now compute

$$m_\varepsilon(t, r) = \int_{B_r} u_\varepsilon(t, x)\, \mathrm{d}x.$$

As above, we write the equation in radial coordinates as

$$\frac{\partial u_\varepsilon}{\partial t} = r^{-(d-1)}\frac{\partial}{\partial r}\Big(r^{d-1}((u_\varepsilon)_+ + \varepsilon)^\alpha \frac{\partial v}{\partial r}\Big) + \varepsilon r^{-(d-1)}\frac{\partial}{\partial r}\Big(r^{d-1}\frac{\partial u_\varepsilon}{\partial r}\Big).$$

Integrating over B_r we have

$$\frac{\partial m_\varepsilon}{\partial t} = d\omega_d r^{d-1}((u_\varepsilon)_+ + \varepsilon)^\alpha \frac{\partial v}{\partial r} + \varepsilon d\omega_d r^{d-1}\frac{\partial u_\varepsilon}{\partial r}.$$

Again $-d\omega_d r^{d-1}\frac{\partial v}{\partial r} = m_\varepsilon$ and

$$u_\varepsilon = \frac{1}{d\omega_d r^{d-1}}\frac{\partial m_\varepsilon}{\partial r}$$

so we have

$$\frac{\partial m_\varepsilon}{\partial t} = -\Big(\Big(\frac{1}{d\omega_d r^{d-1}}\frac{\partial m_\varepsilon}{\partial r}\Big)_+ + \varepsilon\Big)^\alpha m_\varepsilon + \varepsilon d\omega_d r^{d-1}\frac{\partial}{\partial r}\Big(\frac{1}{d\omega_d r^{d-1}}\frac{\partial m_\varepsilon}{\partial r}\Big).$$

Considering the change in variable $\rho = \omega_d r^d$ we have $\frac{\partial}{\partial r} = d\omega_d r^{d-1}\frac{\partial}{\partial \rho}$ and hence

$$\frac{\partial m_\varepsilon}{\partial t} = -\Big(\Big(\frac{\partial m_\varepsilon}{\partial \rho}\Big)_+ + \varepsilon\Big)^\alpha m_\varepsilon + \varepsilon(d\omega_d r^{d-1})^2\frac{\partial}{\partial \rho}\Big(\frac{\partial m_\varepsilon}{\partial \rho}\Big).$$

Replacing the last r by ρ

$$\frac{\partial m_\varepsilon}{\partial t} = -\Big(\Big(\frac{\partial m_\varepsilon}{\partial \rho}\Big)_+ + \varepsilon\Big)^\alpha m_\varepsilon + \varepsilon\big(d\omega_d^{\frac{1}{d}}\rho^{\frac{d-1}{d}}\big)^2\frac{\partial^2 m_\varepsilon}{\partial \rho^2}. \tag{6.3}$$

Therefore, the equation contains a term with degenerate viscosity for $d \geq 2$.

6.3. Weak solutions of (P) via vanishing viscosity

We can show existence of a weak solution of (P) by letting $\varepsilon \to 0$.

Theorem 6.3. *Let $u^{(n)}$ be a sequence of solutions of (P_ε) with $\varepsilon = \frac{1}{n}$. Then there is a subsequence converging weakly in $L^1_{\mathrm{loc}} \cap L^\infty_+$, and the limit is in $L^1 \cap L^\infty_+$.*

Proof. Since $\|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^d)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}$, there exists $u \in L^\infty(\mathbb{R}^d)$ such that, up to a subsequence, $u_\varepsilon \rightharpoonup u$ weak-$\star$ in $L^\infty(Q_T)$. Hence $u_n \rightharpoonup u$ weakly in $L^p_{\mathrm{loc}}(\mathbb{R}^d)$. Furthermore, by the lower semicontinuity of the norm, we have $u \in L^1 \cap L^\infty(\mathbb{R}^d)$. We finally use the compactness properties of the singular potentials, and we deduce the local strong convergence of $\nabla \mathrm{N}(u_\varepsilon)$ in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$. ∎

Instead of trying to prove uniqueness of weak solutions under additional conditions, we integrate and consider the mass function for radial initial data. In the next section we show uniqueness of solutions in the sense of viscosity solutions for the mass variable.

7. A theory of viscosity solutions for the mass equation

As shown above, weak solutions are not in general unique. This is a common problem of conservation laws. In some cases, this difficulty is overcome by introducing the notion of entropy solutions (see e.g. [4, 6, 19]). Such solutions are stable under passage to the limit and regularisation. They are understood as the "physically meaningful solutions". This notion works well for scalar laws, but authors have failed to extend it to systems, as is our case.

In one dimension, the primitive of entropy solutions of conservation laws (or of radial solution) is a solution of a Hamilton–Jacobi equation. The corresponding notion with uniqueness is that on *viscosity solutions* introduced by Crandall and Lions in [13] (a nice explanation of the link between entropy and viscosity solutions can be found in [1]). The nice properties are now well understood (see [11–13]; for a nice introduction to this theory we point the reader to [23]). Furthermore, viscosity solutions are approximated by finite-difference schemes (see [14]).

For the sake of clarity, let us recall here that vanishing viscosity solutions and viscosity solutions in the sense of Crandall–Lions are quite different concepts, though they often give the same class of solutions in practice. The latter concept will be used here below.

7.1. Viscosity solutions

The equation for the mass is written

$$m_t + (m_\rho)^\alpha m = 0$$

which is not problematic since we know that $m_\rho = u \ge 0$. To make a general theory it is better to write

$$m_t + (m_\rho)_+^\alpha m = 0. \tag{7.1}$$

Then the Hamiltonian $H(z, p_1, p_2) = (p_2)^\alpha z$ is defined and non-decreasing everywhere.

Letting $G(z, p_1, p_2, q) = p_1 + H(z, p_2)$ we have a monotone function. This recalls the theory in [8]. We are not exactly in their setting, since our function is not *weakly increasing*. The authors prove that viscosity solutions of this equation are non-decreasing in ρ (ρ-m in their notation). We could apply their existence theory, but not the uniqueness one. Still, the solutions for the general case are continuous, but not necessarily uniformly continuous.

We introduce the definition of a viscosity solution for our problem and some notation.

Definition 7.1. Let $f : \Omega \subset \mathbb{R}^m \to \mathbb{R}$. We define the Fréchet subdifferential and superdifferential,

$$D^- u(x) = \left\{ p \in \mathbb{R}^m : \liminf_{y \to x} \tfrac{u(y) - u(x) - p(y-x)}{|y-x|} \ge 0 \right\},$$
$$D^+ u(x) = \left\{ p \in \mathbb{R}^m : \limsup_{y \to x} \tfrac{u(y) - u(x) - p(y-x)}{|y-x|} \le 0 \right\}.$$

We recall the following result.

Theorem 7.2. *Let $\Omega \subset \mathbb{R}^m$ be an open set and $f : \Omega \to \mathbb{R}$ be a continuous function. Then $p \in D^+ f(x)$ if and only if there exists a function $\varphi \in C^1(I)$ such that $D\varphi(x) = p$ and $f - \varphi$ is a local maximum at x.*

With the initial condition $(m_0)_\rho = u_0$ and the fact that $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ (so that the mass over the point $\{0\} = B_0$ is null), we consider the Cauchy problem

$$\begin{cases} m_t + (m_\rho)_+^\alpha m = 0, & t > 0,\ \rho > 0, \\ m(0, \rho) = m_0(\rho), \\ m(t, 0) = 0. \end{cases} \tag{7.2}$$

The natural setting is with m_0 Lipschitz (i.e. $m_\rho = u \in L^\infty$) and bounded (i.e. $u \in L^1$).

Definition 7.3. We say that a continuous function $m \in \mathcal{C}([0,+\infty)^2)$ is

- a viscosity subsolution of (7.2) if
$$p_1 + (p_2)_+^\alpha m(t,\rho) \le 0 \quad \forall (t,\rho) \in \mathbb{R}_+^2 \text{ and } (p_1,p_2) \in D^+ m(t,\rho),$$
and $m(0,\rho) \le m_0(\rho)$ and $m(t,0) \le 0$;
- a viscosity supersolution of (7.2) if
$$p_1 + (p_2)_+^\alpha m(t,\rho) \ge 0 \quad \forall (t,\rho) \in \mathbb{R}_+^2 \text{ and } (p_1,p_2) \in D^- m(t,\rho),$$
and $m(0,\rho) \ge m_0(\rho)$ and $m(t,0) \ge 0$;
- a viscosity solution of (7.2) if it is both a sub- and a supersolution.

Remark 7.4. The more general theory in [8] allows for discontinuous sub- and super-solutions provided they are respectively lower and upper semicontinuous.

7.2. Comparison principle for the mass

Theorem 7.5. *Let m and M be uniformly continuous sub- and supersolutions of* (7.2) *in the sense of Definition* 7.3. *Then $m \le M$.*

We apply an old idea by Crandall and Lions ([14]) of variable doubling. For its application we follow the scheme as presented in [23, Theorem 1.18], there written for $u_t + H(D_x u) = 0$ with suitable modifications.

Proof of Theorem 7.5. Assume, towards a contradiction, that
$$\sup_{(t,\rho)\in[0,+\infty)^2} (m(t,\rho) - M(t,\rho)) = \sigma > 0.$$
Since both functions are continuous, there exists (t_1,ρ_1) such that $m(t_1,\rho_1) - M(t_1,\rho_1) > \frac{3\sigma}{4}$. Clearly, $t_1, \rho_1 > 0$. Let us take ε and λ positive such that
$$\varepsilon < \frac{\sigma}{16(\rho_1+1)}, \qquad \lambda < \frac{\sigma}{16(t_1+1)}.$$
With this choice we have
$$2\varepsilon\rho_1^2 + 2\lambda t_1 < \frac{\sigma}{4}.$$
For this ε and λ fixed, let us construct the variable doubling function:
$$\Phi(t,s,\rho,\xi) = m(t,\rho) - M(s,\xi) - \frac{|\rho-\xi|^2 + |s-t|^2}{\varepsilon^2} - \varepsilon(\rho^2+\xi^2) - \lambda(s+t).$$
This function is continuous and bounded above, so it achieves a maximum at some point. Let us name this maximum depending on ε, but not on λ:
$$\Phi(t_\varepsilon, s_\varepsilon, \rho_\varepsilon, s_\varepsilon) \ge \Phi(t_1,t_1,\rho_1,\rho_1) > \frac{3\sigma}{4} - 2\varepsilon\rho_1^2 - 2\lambda t_1 > \frac{\sigma}{2}.$$
In particular, it holds that
$$m(t_\varepsilon,\rho_\varepsilon) - M(s_\varepsilon,\xi_\varepsilon) \ge \Phi(t_\varepsilon,s_\varepsilon,\rho_\varepsilon,\xi_\varepsilon) > \frac{\sigma}{2}. \tag{7.3}$$

Step 1. Variables collapse. As $\Phi(t_\varepsilon, s_\varepsilon, \rho_\varepsilon, \xi_\varepsilon) \geq \Phi(0,0,0,0) = 0$, we have

$$\frac{|\rho_\varepsilon - \xi_\varepsilon|^2 + |s_\varepsilon - t_\varepsilon|^2}{\varepsilon^2} + \varepsilon(\rho_\varepsilon^2 + \xi_\varepsilon^2) + \lambda(s_\varepsilon + t_\varepsilon) \leq m(t_\varepsilon, \rho_\varepsilon) - M(s_\varepsilon, \xi_\varepsilon) \leq C.$$

Therefore, we obtain

$$|\rho_\varepsilon - \xi_\varepsilon| + |t_\varepsilon - s_\varepsilon| \leq C\varepsilon \quad \text{and} \quad \rho_\varepsilon + \xi_\varepsilon \leq \frac{C}{\sqrt{\varepsilon}}.$$

This implies that, as $\varepsilon \to 0$, the variable doubling collapses to a single point.

We can improve the first estimate using that $\Phi(t_\varepsilon, s_\varepsilon, \rho_\varepsilon, \xi_\varepsilon) \geq \Phi(t_\varepsilon, t_\varepsilon, \rho_\varepsilon, \rho_\varepsilon)$. This gives us

$$\begin{aligned}\frac{|\rho_\varepsilon - \xi_\varepsilon|^2 + |s_\varepsilon - t_\varepsilon|^2}{\varepsilon^2} &\leq M(t_\varepsilon, \rho_\varepsilon) - M(s_\varepsilon, \xi_\varepsilon) + \varepsilon(\rho_\varepsilon^2 - \xi_\varepsilon^2) + \lambda(t_\varepsilon - s_\varepsilon) \\ &\leq M(t_\varepsilon, \rho_\varepsilon) - M(s_\varepsilon, \xi_\varepsilon) + \varepsilon\frac{C}{\sqrt{\varepsilon}}C\varepsilon + C\varepsilon.\end{aligned}$$

Since M is uniformly continuous, we have

$$\lim_{\varepsilon\to 0} \frac{|\rho_\varepsilon - \xi_\varepsilon|^2 + |s_\varepsilon - t_\varepsilon|^2}{\varepsilon^2} = 0.$$

Step 2. For $\varepsilon > 0$ sufficiently small, the points are interior. We show that there exists μ such that $t_\varepsilon, s_\varepsilon, \rho_\varepsilon, \xi_\varepsilon \geq \mu > 0$ for $\varepsilon > 0$ small enough. For this, since m and M are uniformly continuous,

$$\begin{aligned}\frac{\sigma}{2} &< m(t_\varepsilon, \rho_\varepsilon) - M(s_\varepsilon, \xi_\varepsilon) \\ &= m(t_\varepsilon, \rho_\varepsilon) - m(0, \rho_\varepsilon) + m(0, \rho_\varepsilon) - M(0, \rho_\varepsilon) \\ &\quad + M(0, \rho_\varepsilon) - M(t_\varepsilon, \rho_\varepsilon) + M(t_\varepsilon, \rho_\varepsilon) - M(s_\varepsilon, \xi_\varepsilon) \\ &\leq \omega(t_\varepsilon) + \omega(|\rho_\varepsilon - \xi_\varepsilon| + |t_\varepsilon - s_\varepsilon|),\end{aligned}$$

where $\omega \geq 0$ is a modulus of continuity (the minimum of the moduli of continuity of m and M), i.e. a continuous non-decreasing function such that $\lim_{r\to 0} \omega(r) = 0$. For $\varepsilon > 0$ such that

$$\omega(|\rho_\varepsilon - \xi_\varepsilon| + |t_\varepsilon - s_\varepsilon|) < \frac{\sigma}{4},$$

we have $\omega(t_\varepsilon) > \frac{\sigma}{4}$. The reasoning is analogous for s_ε. For ρ_ε we can proceed much in the same manner:

$$\begin{aligned}\frac{\sigma}{2} &< m(t_\varepsilon, \rho_\varepsilon) - M(s_\varepsilon, \xi_\varepsilon) \\ &= m(t_\varepsilon, \rho_\varepsilon) - m(t_\varepsilon, 0) + M(t_\varepsilon, 0) - M(t_\varepsilon, \rho_\varepsilon) \\ &\quad + M(t_\varepsilon, \rho_\varepsilon) - M(s_\varepsilon, \xi_\varepsilon) \\ &\leq \omega(\rho_\varepsilon) + \omega(|\rho_\varepsilon - \xi_\varepsilon| + |t_\varepsilon - s_\varepsilon|),\end{aligned}$$

and analogously for ξ_ε.

Step 3. Choosing viscosity test functions. With the construction we have made, the function $(t,\rho)\mapsto\Phi(t,s_\varepsilon,\rho,\xi_\varepsilon)$ has a maximum at $(t_\varepsilon,\rho_\varepsilon)$. Thus, so does the function

$$(t,\rho)\mapsto m(t,\rho)-\Big(\frac{|\rho-\xi_\varepsilon|^2+|t-s_\varepsilon|^2}{\varepsilon^2}+\varepsilon\rho^2+\lambda t\Big)=m(t,\rho)-\bar\varphi_\varepsilon(t,\rho).$$

We must be careful to ensure that the test function has contact with m at the right point $(t_\varepsilon,\rho_\varepsilon)$:

$$\varphi_\varepsilon(t,\rho)=\bar\varphi_\varepsilon(t,\rho)+m(t_\varepsilon,\rho_\varepsilon)-\bar\varphi_\varepsilon(t_\varepsilon,\rho_\varepsilon).$$

In fact, this is equivalent to

$$D\varphi_\varepsilon(t_\varepsilon,\rho_\varepsilon)=D\bar\varphi_\varepsilon(t_\varepsilon,\rho_\varepsilon)\in D^+m(t_\varepsilon,\rho_\varepsilon).$$

Since m is a viscosity subsolution, we recover

$$\frac{2(t_\varepsilon-s_\varepsilon)}{\varepsilon^2}+\lambda+\Big(\frac{2(\rho_\varepsilon-\xi_\varepsilon)}{\varepsilon^2}+2\varepsilon\rho_\varepsilon\Big)_+^\alpha m(t_\varepsilon,\rho_\varepsilon)\le 0. \tag{7.4}$$

Analogously, the following function has a minimum at $(s_\varepsilon,\xi_\varepsilon)$:

$$(s,\xi)\mapsto M(s,\xi)-\Big(-\frac{|\rho_\varepsilon-\xi|^2+|s-t_\varepsilon|^2}{\varepsilon^2}-\varepsilon\xi^2-\lambda s\Big)=M(s,\xi)-\bar\psi_\varepsilon(s,\xi).$$

Again, the correct test function is

$$\psi_\varepsilon(s,\xi)=\bar\psi_\varepsilon(s,\xi)+M(s_\varepsilon,\xi_\varepsilon)-\bar\psi_\varepsilon(s_\varepsilon,\xi_\varepsilon).$$

Since M is a viscosity supersolution, we recover

$$\frac{2(t_\varepsilon-s_\varepsilon)}{\varepsilon^2}-\lambda+\Big(\frac{2(\rho_\varepsilon-\xi_\varepsilon)}{\varepsilon^2}+2\varepsilon\xi_\varepsilon\Big)_+^\alpha M(s_\varepsilon,\xi_\varepsilon)\ge 0. \tag{7.5}$$

Step 4. A contradiction. Taking the difference between (7.4) and (7.5), we have

$$\begin{aligned}
2\lambda &\le \Big(\frac{2(\rho_\varepsilon-\xi_\varepsilon)}{\varepsilon^2}+2\varepsilon\xi_\varepsilon\Big)_+^\alpha M_\varepsilon(s_\varepsilon,\xi_\varepsilon)-\Big(\frac{2(\rho_\varepsilon-\xi_\varepsilon)}{\varepsilon^2}+2\varepsilon\rho_\varepsilon\Big)_+^\alpha m_\varepsilon(t_\varepsilon,\rho_\varepsilon)\\
&=\Big[\Big(\frac{2(\rho_\varepsilon-\xi_\varepsilon)}{\varepsilon^2}+2\varepsilon\xi_\varepsilon\Big)_+^\alpha-\Big(\frac{2(\rho_\varepsilon-\xi_\varepsilon)}{\varepsilon^2}+2\varepsilon\rho_\varepsilon\Big)_+^\alpha\Big]M(s_\varepsilon,\xi_\varepsilon)\\
&\quad+\Big(\frac{2(\rho_\varepsilon-\xi_\varepsilon)}{\varepsilon^2}+2\varepsilon\rho_\varepsilon\Big)_+^\alpha(M(s_\varepsilon,\xi_\varepsilon)-m(t_\varepsilon,\rho_\varepsilon))\\
&\le C\Big|2\varepsilon(\rho_\varepsilon-\xi_\varepsilon)\Big|^\alpha-\frac{\sigma}{2}\Big(\frac{2(\rho_\varepsilon-\xi_\varepsilon)}{\varepsilon^2}+2\varepsilon\rho_\varepsilon\Big)_+^\alpha\\
&\le C\Big|2\varepsilon(\rho_\varepsilon-\xi_\varepsilon)\Big|^\alpha,
\end{aligned}$$

due to the Hölder continuity of $\tau\mapsto\tau_+^\alpha$, the fact that M is bounded, and (7.3). As $\varepsilon\to 0$ we recover the contradiction $0<2\lambda\le 0$, and the proof is complete. ∎

Remark 7.6. In the argument above, the uniform continuity condition plays a key role. Notice that it is possible that $\rho_\varepsilon, \xi_\varepsilon \to +\infty$, and hence the continuity must be uniform to obtain the comparison estimate.

It was pointed out in [11, Remark 4.2] that the assumption of uniform continuity can be weakened (with minor modifications to the proof) to uniform continuity of u_0, v_0 and uniform convergence of $u(x,t) \to u_0(x)$ and $v(x,t) \to v_0(x)$ as $t \to 0$.

Remark 7.7. Notice that this proof can be extended to equations of the form $m_t + H(m_\rho)m = 0$ where $H \geq 0$ and uniformly continuous.

Remark 7.8. Notice that [13, Theorem V.3] covers the case $\alpha \geq 1$, and furthermore gives information on the cone of dependence. Naturally, in our setting there is no cone of dependence.

As a simple consequence of the comparison principle, we can take advantage of our explicit solutions for u in Section 2. The mass corresponding to the friendly giant should be a global supersolution. We compute the corresponding mass, which gives

$$M(t,\rho) = (\alpha t)^{-\frac{1}{\alpha}}\rho.$$

This is a classical solution of the equation

$$M_t + (M_\rho)_+^\alpha M = -\frac{1}{\alpha}\rho(\alpha t)^{-\frac{1}{\alpha}-1} + ((\alpha t)^{-\frac{1}{\alpha}-1})^\alpha(\rho(\alpha t)^{-\frac{1}{\alpha}}) = 0.$$

It is uniformly continuous for $t > \mu$. We can apply the proof of the comparison principle to this very nice classical solution, and hence we deduce that

$$m(t,\rho) \leq (\alpha t)^{-\frac{1}{\alpha}}\rho \tag{7.6}$$

holds for all uniformly continuous viscosity solutions.

Remark 7.9. This is known for Burgers' equation as the universal or absolute supersolution.

Remark 7.10. Notice that this implies $m(t,0) \leq 0$ for all $t \geq 0$. Since $m \equiv 0$ is also a solution, we check that, for all $m_0 \geq 0$, then the viscosity solution satisfies $m(t,0) = 0$ and $m(t,\rho) \geq 0$.

Remark 7.11. Formula (7.6) shows us that, for initial data $m_0 \geq 0$ and for all ρ fixed, $m(t,\rho) \to 0$ as $t \to +\infty$, i.e. eventually all mass travels to infinity.

7.3. The mass of (P_ε) converges to the viscosity solution of (7.2)

Theorem 7.12. *Let $d \geq 1$, and $u_0 \in L^1(\mathbb{R}^d) \cap \mathcal{C}_b(\mathbb{R}^d)$ be radially symmetric. Let u_ε be the solution of (P_ε) and m_ε its mass. Then $m_\varepsilon \to m$ in $\mathcal{C}_{\mathrm{loc}}([0,+\infty) \times [0,+\infty))$ where m is the viscosity solution of (7.2). Furthermore, m is Lipschitz continuous (in variable ρ).*

Proof. We first point out that $0 \le m_\varepsilon \le \|u_0\|_{L^1(\mathbb{R}^d)}$ and

$$(m_\varepsilon)_\rho = u_\varepsilon \in [0, \|u_0\|_{L^\infty(\mathbb{R}^d)}]. \tag{7.7}$$

Furthermore, we know that the solution is classical and that $(m_\varepsilon)_{\rho\rho}$ is also continuous. Hence, by the Arzelà–Ascoli theorem there is a convergent subsequence $m_\varepsilon \to m$ in $\mathcal{C}_{\mathrm{loc}}([0,+\infty) \times [0,+\infty))$. Let us now check that m is a viscosity solution. We begin by showing it is a viscosity subsolution and, likewise, one proves it is a supersolution. Fix $t, \rho > 0$, and let $(p_1, p_2) \in D^+ m(t,\rho)$. Again, we can construct a function $\varphi \in \mathcal{C}^2$ such that $m - \varphi$ has a local maximum at (t,ρ) and $p_1 = \varphi_t(t,\rho)$, $p_2 = \varphi_\rho(t,\rho)$. Now we need to prove that the quantity

$$\varphi_t(t,\rho) + (\varphi_\rho(t,\rho))_+^\alpha m(t,\rho)$$

is non-positive. For this, we go back to the viscosity equation. Since $m^\varepsilon \to m$ and $m - \varphi$ has a maximum at (t,ρ), by [23, Lemma 1.8] there exists a subsequence ε and $(t_\varepsilon, \rho_\varepsilon)$ of values such that $m^\varepsilon - \varphi$ has a maximum at $(t_\varepsilon, \rho_\varepsilon)$ and, furthermore, $(t_\varepsilon, \rho_\varepsilon) \to (t,\rho)$ as $\varepsilon \to 0$. We go back to (6.3):

$$\frac{\partial m_\varepsilon}{\partial t} + \Big(\Big(\frac{\partial m_\varepsilon}{\partial \rho}\Big)_+ + \varepsilon\Big)^\alpha m_\varepsilon = \varepsilon C_d(\rho) \frac{\partial^2 m_\varepsilon}{\partial \rho^2} n, \tag{7.8}$$

where $C_d(\rho)$ is defined and positive outside $\rho = 0$. For ε small enough, $\rho_\varepsilon > 0$ and, hence,

$$\frac{\partial m_\varepsilon}{\partial t}(t_\varepsilon, \rho_\varepsilon) = \frac{\partial \varphi}{\partial t}(t_\varepsilon, \rho_\varepsilon),$$
$$\frac{\partial m_\varepsilon}{\partial \rho}(t_\varepsilon, \rho_\varepsilon) = \frac{\partial \varphi}{\partial \rho}(t_\varepsilon, \rho_\varepsilon),$$

and

$$\frac{\partial^2 m_\varepsilon}{\partial \rho^2}(t_\varepsilon, \rho_\varepsilon) \le \frac{\partial^2 \varphi}{\partial \rho^2}(t_\varepsilon, \rho_\varepsilon).$$

Hence,

$$\varphi_t(t_\varepsilon, \rho_\varepsilon) + (\varepsilon + (\varphi_\rho(t_\varepsilon, \rho_\varepsilon))_+)^\alpha m_\varepsilon(t_\varepsilon, \rho_\varepsilon) \le \varepsilon C_d(\rho_\varepsilon) \frac{\partial^2 \varphi}{\partial \rho^2}(t_\varepsilon, \rho_\varepsilon).$$

As $\varepsilon \to 0$ we have

$$\varphi_t(t,\rho) + (\varphi_\rho(t,\rho))_+^\alpha m(t,\rho) \le 0.$$

Hence, we recover the sign

$$p_1 + (p_2)_+^\alpha m(t,\rho) \le 0.$$

Since this viscosity solution m is unique by the comparison principle, the whole sequence m_ε converges. ∎

7.4. Stability

Theorem 7.13. *Let m_j be viscosity subsolutions (resp. supersolutions) of* (7.2)*, and assume that $m_j \to m$ uniformly over compacts as $j \to +\infty$. Then m is a viscosity subsolution (resp. supersolution) of* (7.2)*.*

Proof. Fix $t, \rho > 0$, and let $(p_1, p_2) \in D^+ m(t, \rho)$. We can construct a function $\varphi \in \mathcal{C}^1$ (see [23, Theorem 1.4]) such that $m - \varphi$ has a local maximum at (t, ρ) and $p_1 = \varphi_t(t, \rho)$, $p_2 = \varphi_\rho(t, \rho)$. Now we need to check the sign of the quantity

$$\varphi_t(t, \rho) + (\varphi_\rho(t, \rho))_+^\alpha m(t, \rho).$$

Since $m_j \to m$ and $m - \varphi$ has a maximum at (t_j, ρ), by [23, Lemma 1.8] there exists a subsequence ε and (t_j, ρ_j) of values such that $m^\varepsilon - \varphi$ has a maximum at (t_j, ρ_j) and, furthermore, $(t_j, \rho_j) \to (t, \rho)$ as $j \to +\infty$. Then we have

$$\varphi_t(t_j, \rho_j) + (\varphi_\rho(t_j, \rho_j))_+^\alpha m_j(t_j, \rho_j) \le 0.$$

As $j \to +\infty$ we recover the definition of a viscosity subsolution for m. ∎

We can also prove continuous dependence on the data, using this result. If $m_{0,j} \to m_0$, then the solutions converge uniformly.

7.5. Well-posedness

The mass associated to solutions of problem (P_ε) are always Lipschitz continuous, but the comparison principle holds true for uniformly continuous. Let us show that, for $m_0 \in \mathrm{BUC}$ (i.e. bounded and uniformly continuous), there is a viscosity solution $u \in \mathrm{BUC}$.

Theorem 7.14. *Let $m_0 \in \mathrm{BUC}(\mathbb{R}^d)$ be non-decreasing such that $m_0(0) = 0$. Then there exists a unique bounded and uniformly continuous viscosity solution. If m_0 is Lipschitz, then so is m.*

Proof. We first prove the case of $m_0 \in \mathcal{C}^1$ and Lipschitz, and then the general case.

Step 1. m_0 of class C^1 and Lipschitz. We can construct initial data in one dimension by taking $u_0(\rho) = (m_0)_\rho(|\rho|) \in L^1(\mathbb{R}) \cap \mathcal{C}_b(\mathbb{R})$ and we apply Theorem 7.12.

Step 2. Approximation arguments.

Step 2a. $m_0 \in \mathrm{BUC}(\mathbb{R}^d)$. Then it can be approximated from above by a decreasing sequence of Lipschitz functions $\overline{m}_{0,k}$ and from below by an increasing sequence of functions $\underline{m}_{0,k}$. We construct $\underline{m}_{0,k}$ as follows: it can be taken as a piecewise approximation of $m_0 - \frac{1}{k}$ by a piecewise constant function so that the uniform distance is less than $1/2k$. The procedure is analogous for $\overline{m}_{0,k}$.

By the comparison principle for m, the corresponding solutions are ordered $\underline{m}_k \le \overline{m}_k$, $\underline{m}_k$ is increasing, and $\overline{m}_k$ decreasing. Due to Dini's theorem, the pointwise limits exist and the convergence is uniform over compacts

$$\underline{m} = \lim_k \underline{m}_k \le \lim_k \overline{m}_k = \overline{m}.$$

By Theorem 7.13, these two functions are viscosity solutions, both corresponding to initial data m_0. Since they are the uniform limits of uniformly continuous functions, $\overline{m}, \underline{m}$ are uniformly continuous. By the comparison principle, $\overline{m} = \underline{m}$ and we can simply call this function m.

Step 2b. m_0 is Lipschitz continuous. If m_0 is not only BUC but also Lipschitz continuous, then $m_{0,k}$ are uniformly Lipschitz continuous, and due to (7.7) the functions m_k are also uniformly Lipschitz continuous. Hence, m is Lipschitz continuous. ■

Remark 7.15. If we repeat the approximation argument for m_0 continuous, we can repeat the argument and construct two continuous viscosity solutions $\overline{m}$ and $\underline{m}$. Since the comparison principle Theorem 7.5 cannot be applied, we do not know whether they are the same.

Remark 7.16. Notice that the rarefaction fan solution for $u_0 = c_0 \chi_{[0,L]}$, studied in Section 4.2.1, is continuous $(0,+\infty) \times (0,+\infty)$, therefore m is differentiable and thus a classical solution of (7.2), so the unique viscosity solution. The other solution constructed in Section 5.2 is therefore shown as the spurious solution.

7.6. Asymptotics for the mass

Theorem 7.17. *Let $m_0 \in \mathrm{BUC}([0,+\infty))$ be non-decreasing with $m_0(0) = 0$ and such that $(m_0)_\rho$ is compactly supported. Let us denote by*

$$G_M(\kappa) = \int_{\omega_d |y|^d \le \kappa} F_M(|y|)\, \mathrm{d}y$$

the mass function of the self-similar solution with total mass M. Then the unique viscosity solution satisfies for all $\kappa_0 \ge 0$ that

$$\sup_{\kappa \ge \kappa_0} \left| \frac{m(t, t^{\frac{1}{\alpha}} \kappa) - G_M(\kappa)}{G_M(\kappa)} \right| \to 0 \quad \text{as } t \to +\infty. \tag{7.9}$$

Proof. Let us define $M = m_0(+\infty)$. Consider the supersolution with initial datum

$$\overline{m}_0(\rho) = \begin{cases} \|(m_0)_\rho\|_\infty \rho, & \rho < M / \|(m_0)_\rho\|_\infty, \\ M, & \rho > M / \|(m_0)_\rho\|_\infty, \end{cases}$$

and the subsolution with initial datum, for δ such that $m_0(\delta) = M$,

$$\underline{m}_0(\rho) = \begin{cases} 0, & \rho < \delta, \\ M(\rho - \delta), & \delta < \rho < \delta + 1, \\ M, & \rho \ge \delta + 1. \end{cases} \tag{7.10}$$

Since $\bar{u}_0 = (\bar{m}_0)_\rho$ and $\underline{u}_0 = (\underline{m}_0)_\rho$ are square functions (and therefore non-increasing up to an initial gap), we know from Sections 4.4.1 and 4.4.2 that

$$\sup_{\omega_d |x|^d \geq \delta} \left| \frac{\underline{u}(t,x) - U_M(t,x)}{U_M(t,x)} \right| \to 0, \qquad \sup_{x \in \mathbb{R}^d} \left| \frac{\bar{u}(t,x) - U_M(t,x)}{U_M(t,x)} \right| \to 0, \qquad \text{as } t \to +\infty.$$

Furthermore, since we know they are given by characteristics, we can apply Theorem 4.5 to deduce that

$$\underline{u} \leq \bar{u}.$$

Let $\varepsilon > 0$ be fixed. From the previous estimate we have, for some $t_\varepsilon > 0$,

$$(1-\varepsilon) U_M(t,x) \leq \underline{u}(t,x) \leq \bar{u}(t,x) \leq (1+\varepsilon) U_M(t,x) \quad \forall t \geq 0,\ \omega_d |x|^d \geq \delta.$$

We will write in detail only upper bounds for $\underline{u}$, which are more delicate due to the presence of δ:

$$(1-\varepsilon) U_M(t,x) \leq \underline{u}(t,x) \leq (1+\varepsilon) U_M(t,x) \quad \forall t \geq t_\varepsilon,\ |x| \geq \delta.$$

We define the mass of the self-similar solution

$$m_M(t,\rho) = \int_{\omega_d |x|^d \leq \rho} U_M(t,x)\, \mathrm{d}x.$$

Integrating from ρ to $+\infty$,

$$(1-\varepsilon)(M - m_M(t,\rho)) \leq M - \underline{m}(t,\rho) \leq (1+\varepsilon)(M - m_M(t,\rho)).$$

From this, we deduce that

$$-\varepsilon M + (1+\varepsilon) m_M(t,\rho) \leq \underline{m}(t,\rho) \leq \varepsilon M + (1-\varepsilon) m_M(t,\rho).$$

We have

$$-\varepsilon \Big(\frac{M}{m_M(t,\rho)} - 1 \Big) \leq \frac{\underline{m}(t,\rho) - m_M(t,\rho)}{m_M(t,\rho)} \leq \varepsilon \Big(\frac{M}{m_M(t,\rho)} - 1 \Big) \quad \forall t \geq t_\varepsilon,\ \rho \geq \delta.$$

Notice that $M \geq m_M$ so $M/m_M - 1 \geq 0$. From here on, we write the estimate in absolute value. For ρ fixed, $m_M(t,\rho) \to 0$, so this estimate is not very nice on its own. However, we can pass to the scaling $y = x t^{-\frac{1}{\alpha d}}$ so in the rescaled volume variable,

$$\kappa = \rho t^{-\frac{1}{\alpha}}.$$

Let us look at the profile of m_M. Going back to the definition of U_M, we recover by integration that

$$m_M(t, t^{\frac{1}{\alpha}} \kappa) = \int_{\omega_d |y|^d \leq \kappa} F_M(|y|)\, \mathrm{d}y = G_M(\kappa).$$

Hence, in this rescaled variable we have

$$\left|\frac{\underline{m}(t,t^{\frac{1}{\alpha}}\kappa)-G_M(\kappa)}{G_M(\kappa)}\right| \leq \varepsilon\Big(\frac{M}{G_M(\kappa)}-1\Big) \quad \forall t \geq t_\varepsilon,\ \kappa \geq \delta t^{-\frac{1}{\alpha}}.$$

If we want a uniform bound, we cannot run all the range of κ; nevertheless we can fix a single $\kappa_0 > 0$ and, since $G_M(\kappa)$ is increasing in κ, get

$$\left|\frac{\underline{m}(t,t^{\frac{1}{\alpha}}\kappa)-G_M(\kappa)}{G_M(\kappa)}\right| \leq \varepsilon\Big(\frac{M}{G_M(\kappa_0)}-1\Big) \quad \forall t \geq t_\varepsilon,\ \kappa \geq \kappa_0.$$

Proceeding analogously for $\overline{m}$ we recover the same bounds and hence we recover (7.9). ∎

8. A finite-difference scheme for the mass

In this section we return to the consideration of non-negative solutions with positive non-decreasing mass function. Since we know that the characteristics arrive from the left due to m_ρ being positive, we can construct an upwind explicit scheme. We discretise the space and time variable by $t_n = nh_t$, $\rho_j = jh_\rho$, and propose the scheme

$$\frac{M_i^{k+1}-M_i^k}{h_t} + \Big(\frac{M_i^k - M_{i-1}^k}{h_\rho}\Big)^\alpha M_i^{k+1} = 0. \tag{8.1}$$

Factoring out M_i^{k+1}, we get

$$\Big(1 + h_t\Big(\frac{M_i^k - M_{i-1}^k}{h_\rho}\Big)^\alpha\Big)M_i^{k+1} = M_i^k, \tag{8.2}$$

hence, we deduce

$$M_i^{k+1} = \frac{M_i^k}{1 + h_t(\frac{M_i^k - M_{i-1}^k}{h_\rho})^\alpha}. \tag{8.3}$$

Unfortunately, this scheme is not a monotone scheme for $\alpha < 1$, see [14], due to the presence of the power, and hence it cannot be studied in the natural fashion proposed in [14].

Nevertheless, we can still propose a monotone scheme, given by regularising the power α. Including the initial and boundary conditions, one can write

$$\begin{cases} M_j^{n+1} = \dfrac{M_j^n}{1 + h_t H_\delta(\frac{M_j^n - M_{j-1}^n}{h_\rho})} & \text{if } j > 0,\ n \geq 0, \\ M_j^0 = m_0(h_\rho j) & \text{if } j \geq 0, \\ M_0^n = 0 & \text{if } n > 0, \end{cases} \tag{M$_\delta$}$$

where

$$H_\delta(s) = (s_+ + \delta)^\alpha - \delta^\alpha.$$

Now, we can set a CFL condition such that the method is monotone. Indeed, if we assume that

$$\frac{h_t}{h_\rho} < \frac{\delta^{1-\alpha}}{\alpha \overline{M}}, \tag{CFL$_\delta$}$$

then the method is monotone if $M_j^n \in [0, \overline{M}]$. Naturally, δ must go to zero with h_t and h_ρ. It is easy to see that if m_0 is non-decreasing in ρ then so is M_j^n in j.

Theorem 8.1. *Let m_0 be non-negative, non-decreasing, Lipschitz continuous, and bounded, M_j^n be the solution of* (M_δ)*, and m the viscosity solution of* (7.2)*. Let, for $\delta > 0$,*

$$h_\rho = \delta^{1+2\alpha}, \quad h_t = \frac{\delta^{2+\alpha}}{2\alpha \|m_0\|_\infty}.$$

Then, for any $T > 0$,

$$\sup_{\substack{j \ge 0 \\ 0 \le n \le T/h_t}} |m(t_n, \rho_j) - M_j^n| \le C\delta^\alpha,$$

where $C = C(\alpha, T, \|m_0\|_\infty, \|(m_0)_\rho\|_\infty)$. Hence, as $\delta \to 0$, the scheme (M_δ) *converges to the viscosity solution of* (7.2).

The lengthy details of the proof of this result are left to the interested reader following the blueprint of [14]. They crucially used the stability property of viscosity solutions proved in Section 7.4.

Remark 8.2. We have performed a numerical simulation of the case with two bumps, in order to see whether the results in Figure 6 are indeed the viscosity solution. The results can be seen in Figure 7.

9. Extensions and open problems

(1) The study of qualitative properties for non-radial solutions in several dimensions is completely open.

(2) The formal gradient flow structure of the equation may be relevant to discuss regularity and asymptotic properties of the equation for general initial data.

(3) Another interesting question arises by looking at the attractive case or equivalently the backward evolution of our model. In the case $\alpha = 1$, this was analysed in [5] and is known in the literature as the skeleton problem.

(4) Is there uniqueness of solutions for the mass equation with only continuous initial data?

(5) Can one construct convergent higher-order numerical schemes for the mass equation?

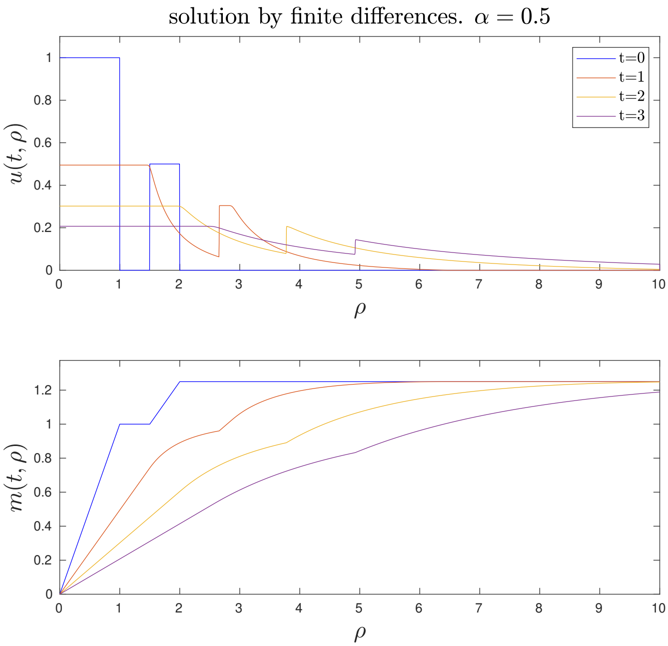

Figure 7. Solution by finite differences of the case with two bumps. Compare with Figure 6.

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