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Global well-posedness of a binary–ternary Boltzmann equation

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Abstract. In this paper we show global well-posedness near vacuum for the binary–ternary Boltzmann equation. The binary–ternary Boltzmann equation provides a correction term to the classical Boltzmann equation, taking into account both binary and ternary interactions of particles, and may serve as a more accurate description model for denser gases in non-equilibrium. Well-posedness of the classical Boltzmann equation and, independently, the purely ternary Boltzmann equation follow as special cases. To prove global well-posedness, we use a Kaniel–Shinbrot iteration and related work to approximate the solution of the non-linear equation by monotone sequences of supersolutions and subsolutions. This analysis required establishing new convolution-type estimates to control the contribution of the ternary collisional operator to the model. We show that the ternary operator allows consideration of softer potentials than the one binary operator, and consequently our solution to the ternary correction of the Boltzmann equation preserves all the properties of the binary interactions solution. These results are novel for collisional operators of monoatomic gases with either hard or soft potentials that model both binary and ternary interactions.

1. Introduction

We study global-in-time well-posedness near vacuum of the Cauchy problem for an extension of the classical Boltzmann transport equation (BTE) for monoatomic binary interactions gases that includes ternary interactions. This equation, which can be viewed as a model of a denser gas dynamics, has been recently introduced by two of the authors in [5], who rigorously derived, from finitely many particle dynamics, the purely ternary model for the case of hard potential interactions zone for short times. Moreover, it is seen in [3] that the ternary collisional operator derived in [5] has the same conservation laws and entropy production properties as the classical binary operator, which justifies that the introduced ternary term can serve as a higher-order correction to the Boltzmann equation. Such rigorous derivation of the full binary–ternary model is a work in progress ([4]). Let us also mention that Maxwell models with multiple particle interactions have been studied in [7, 8] for the space homogeneous case via Fourier transform methods.

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In this paper we provide the first rigorous analytical result that shows global-in-time existence and uniqueness of *mild solutions* near vacuum to the binary–ternary model and the purely ternary model on its own. By mild solutions we mean that the x-space dependence of the solution is evaluated along the characteristic curves given by the Hamiltonian evolution of the particle system in between collisions (denoted by $f^{\#}$, which is introduced in Section 2.2). The analytical techniques we use are inspired by the works [6, 13, 14, 16, 19, 20] and the more recent work of [1, 2]. These techniques rely on finding convergent supersolutions and subsolutions to the strong form of (1.1) in the associated strong topology of space–velocity Maxwellian weighted in L^∞-functions.

The binary–ternary Boltzmann transport equation we focus on is given by

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = Q_2(f, f) + Q_3(f, f, f), & (t, x, v) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d, \\ f(0) = f_0, & (x, v) \in \mathbb{R}^d \times \mathbb{R}^d, \end{cases} \tag{1.1}$$

and describes the evolution of the probability density f of a dilute gas in non-equilibrium in $\mathbb{R}^d$, $d \geq 2$, given an initial condition $f_0 \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, when both binary and ternary interactions among particles can occur. The operator $Q_2(f, f)$ is the classical binary collisional operator, which expresses binary elastic interactions between particles, and is of quadratic order, while the operator $Q_3(f, f, f)$ is the ternary collisional operator which expresses ternary interactions among particles, and is of cubic order. For the exact forms of the operators $Q_2(f, f)$, $Q_3(f, f, f)$ used in this paper, see (2.1), (2.14) respectively. We should mention that the purely ternary model, rigorously derived for short times in [5], is given by

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = Q_3(f, f, f), & (t, x, v) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d, \\ f(0) = f_0, & (x, v) \in \mathbb{R}^d \times \mathbb{R}^d. \end{cases} \tag{1.2}$$

We refer to (1.2) as the ternary Boltzmann transport equation.

For the classical Boltzmann transport equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = Q_2(f, f), & (t, x, v) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d, \\ f(0) = f_0, & (x, v) \in \mathbb{R}^d \times \mathbb{R}^d, \end{cases} \tag{1.3}$$

one way to obtain global well-posedness near vacuum is by utilizing an iterative scheme that constructs monotone sequences of supersolutions and subsolutions that converge to the global solution of (1.3). This has been carried out for the first time by Illner and Shinbrot ([13]), who were motivated by the work of Kaniel and Shinbrot ([14]), who in turn showed local-in-time well-posedness for (1.3) following this program. Later, this work was extended by Bellomo and Toscani ([6]) , Toscani ([19, 20]) and Palczewski and Toscani ([16]) to include a wider range of potentials and to relax assumptions on initial data. Alonso and Gamba ([2]) used Kaniel–Shinbrot iteration to derive distributional and classical solutions to (1.3) for soft potentials for large initial data near two sufficiently close Maxwellians in position and velocity space, while Alonso ([1]) used this technique to study the inelastic Boltzmann equation for hard spheres. Strain ([18]) remarks that the estimates he derives can be combined with the Kaniel–Shinbrot iteration to obtain existence of a unique mild solution for the relativistic Boltzmann equation.

Kaniel–Shinbrot iteration is also an important tool for proving non-negativity of solutions; see for example [9, 10, 17]. Also, when initial data has decay in the direction of $x - v$ as opposed to x and v separately, Kaniel–Shinbrot iteration can be used to construct solutions with infinite energy; see for example [15, 23, 24].

Certain problems have been solved by considering modifications of the Kaniel–Shinbrot iteration. For example, Bellomo and Toscani ([21]) adapted the iteration to the Boltzmann–Enskog equation. Ha, Noh and Yun ([12]) and Ha and Noh ([11]) also modified the iteration to prove global existence of mild solutions to the Boltzmann system for gas mixtures in the elastic and the inelastic cases respectively. Also, Wei and Zhang ([22]) used another modified iteration to obtain eternal solutions for the Boltzmann equation.

The goal of this paper is to establish global existence and uniqueness of a mild solution near vacuum to the binary–ternary Boltzmann equation (1.1) in spaces of non-negative functions bounded by a Maxwellian. Moreover, solution of (1.2) follows as a special case. Inspired by [2, 13, 14], we devise an iterative scheme that constructs monotone sequences of supersolutions and subsolutions to (1.1). For small enough initial data, the beginning condition of the iteration holds globally in time and the two sequences can be shown to converge to the same limit, namely a function f that solves equation (1.1) in a mild sense. This strategy requires new ideas given the fact that ternary interactions are also taken into account in (1.1).

In particular, due to the presence of the ternary correction term, one needs to properly adapt the iteration, so that the corresponding supersolutions and subsolutions remain monotone and convergent. One of the main tools is stated in Lemma 3.2, which provides important exponentially weighted convolution estimates. This lemma not only recovers the estimates developed in [2] for the binary interaction case, but also develops a new approach in order to treat the ternary interaction case. Lemma 3.2 is crucially used to obtain uniform-in-time, space–velocity L^1-bounds that control the ternary gain and loss terms ($L^\infty L^1$ estimates). In fact, using Lemma 3.2 one first obtains asymmetric estimates (see Lemma 3.3) because of the asymmetry introduced by the ternary collisional operator, which is not present in the binary case. However, to obtain convergence, it is essential to have symmetry with respect to the inputs of the gain and loss operators. We were able to achieve this symmetrization in Proposition 3.4. Finally, we also use Lemma 3.2 to prove a global estimate for the time average of the gain and loss operators along the characteristics of the Hamiltonian, see Proposition 3.7. With this, we were able to extend the argument for controlling the binary time integrals of both gain and loss terms (see [2]) to the ternary case by invoking properties of ternary interactions and a two-dimensional analog of the time integration bound for a traveling Maxwellian.

With these tools in hand, for small initial data, the constructed iteration scheme is proved to converge to the unique, global-in-time mild solution of (1.1). For more details see Sections 4 and 5.

Organization of the paper

In Section 2 we review the binary and ternary collisional operators and decompose them into gain and loss forms. We then introduce some necessary notation and state our main result (Theorem 2.10). In Section 3 we prove the convolution estimate and derive essential bounds for the gain and loss operators. In Section 4 we inductively construct monotone sequences of supersolutions and subsolutions which are shown to converge to a common limit that solves the binary–ternary Boltzmann equation (1.1), as long as a beginning condition is satisfied. Finally, in Section 5 we provide the proof of our main result (Theorem 2.10).

2. Towards the statement of the main result

The goal of this section is to present the precise statement of the main result of this paper. In order to do so, we first review the collisional operators and decompose them to gain and loss form in Section 2.1, introduce necessary notation and the notion of a solution in Section 2.2, and then state the main result in Section 2.3 (Theorem 2.10).

2.1. Collisional operators

2.1.1. Binary collisional operator. The binary collisional operator is given by

$$\begin{aligned} &Q_2(f,f)(t,x,v) \\ &\quad= \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} B_2(u,\omega)\big(f(t,x,v')f(t,x,v_1') - f(t,x,v)f(t,x,v_1)\big)\,d\omega\,dv_1, \end{aligned} \tag{2.1}$$

where

$$u := v_1 - v \tag{2.2}$$

is the relative velocity of a pair of interacting particles centered at $x, x_1 \in \mathbb{R}^d$, with velocities $v, v_1 \in \mathbb{R}^d$ before the binary interaction with respect to the impact direction

$$\omega := \frac{x_1 - x}{|x - x_1|} \in \mathbb{S}^{d-1}, \tag{2.3}$$

and

$$v' := v + (\omega\cdot u)\omega, \quad v_1' := v_1 - (\omega\cdot u)\omega \tag{2.4}$$

are the outgoing velocities after the binary interaction.

One can easily verify the binary energy–momentum conservation system is satisfied:

$$v' + v_1' = v + v_1, \tag{2.5}$$

$$|v'|^2 + |v_1'|^2 = |v|^2 + |v_1|^2. \tag{2.6}$$

Either (2.4) or (2.5)–(2.6) imply

$$|u'| = |u|, \quad \text{where } u' := v_1' - v'. \tag{2.7}$$

In addition, equation (2.4) yields the specular reflection with respect to the impact direction ω:

$$\omega \cdot u' = -\omega \cdot u. \tag{2.8}$$

In fact, it is not hard to show that, given $v, v_1 \in \mathbb{R}^d$, expression (2.4) gives the general solution of system (2.5)–(2.6), parametrized by $\omega \in \mathbb{S}^{d-1}$. The factor B_2 in the integrand of (2.1) is referred to as the binary interaction differential cross-section, which depends on relative velocity u and the impact direction ω. It expresses the transition probability of binary interactions, and we assume it is of the form

$$B_2(u, \omega) = |u|^{\gamma_2} b_2(\hat{u} \cdot \omega), \quad \gamma_2 \in (-d+1, 1], \tag{2.9}$$

where $\hat{u} = \frac{u}{|u|} \in \mathbb{S}^{d-1}$ is the relative velocity direction and $b_2\colon [-1,1] \to [0,\infty)$ is the binary angular transition probability density. It is worth mentioning that the case $\gamma_2 \in (0,1]$ corresponds to hard potentials, the case $\gamma_2 \in (-d+1, 0)$ corresponds to soft potentials and the case $\gamma_2 = 0$ corresponds to Maxwell molecules.

We assume that the binary angular transition probability density b_2 satisfies the following properties:

- $b_2\colon [-1,1] \to \mathbb{R}$ is a measurable, non-negative probability density.
- b_2 is even, i.e.

  $$b_2(-z) = b_2(z) \quad \forall z \in [-1,1], \tag{2.10}$$

  which, due to the property from (2.8), yields the binary micro-reversibility condition

  $$b_2(\hat{u}' \cdot \omega) = b_2(\hat{u} \cdot \omega) \quad \forall \omega \in \mathbb{S}^{d-1}, \quad \forall v, v_1 \in \mathbb{R}^d, \tag{2.11}$$

  where $\hat{u}' = \frac{u'}{|u|} \in \mathbb{S}^{d-1}$ is the scattering direction. In addition, relations (2.7), (2.9) and (2.11) yield

  $$B_2(u', \omega) = B_2(u, \omega) \quad \forall \omega \in \mathbb{S}^{d-1}, \quad \forall v, v_1 \in \mathbb{R}^d. \tag{2.12}$$

- The probability density is integrable on the sphere; i.e. for any fixed $\hat{u}$ we have $b_2(\hat{u} \cdot \omega) \in L^1(\mathbb{S}^{d-1})$ or, equivalently, $b_2(z)(1-z^2)^{\frac{d-3}{2}} \in L^1([-1,1])$, for $z = \hat{u} \cdot \omega$, and

  $$\|b_2\|_{L^1(\mathbb{S}^{d-1})} = |\mathbb{S}^{d-2}| \int_{-1}^{1} |b_2(z)|(1-z^2)^{\frac{d-3}{2}}\, dz < \infty, \tag{2.13}$$

  where $|\mathbb{S}^{d-2}|$ is the volume of the $(d-2)$-dimensional sphere.

Remark 2.1. The integrability condition on b_2 is weaker than the classical Grad cut-off assumption, which assumes b_2 is a bounded function of $z = \hat{u} \cdot \omega$. So our result is valid for a broader class of angular transition probability measures.

Remark 2.2. One can see that the usual hard sphere model is a special case of the form (2.9) for

$$\gamma_2 = 1, \quad b_2(z) = \frac{|z|}{2}.$$

2.1.2. Ternary collisional operator. The ternary collisional operator is given by (see [5] for details)

$$\begin{aligned} Q_3(f,f,f)(t,x,v) &= \int_{\mathbb{S}^{2d-1}\times\mathbb{R}^{2d}} B_3(\boldsymbol{u},\boldsymbol{\omega})\big(f(t,x,v,v^*)f(t,x,v,v_1^*)f(t,x,v,v_2^*) \\ &\qquad - f(t,x,v)f(t,x,v_1)f(t,x,v_2)\big)\,d\omega_1\,d\omega_2\,dv_1\,dv_2, \end{aligned} \tag{2.14}$$

where

$$\boldsymbol{u} := \begin{pmatrix} v_1 - v \\ v_2 - v \end{pmatrix} \in \mathbb{R}^{2d} \tag{2.15}$$

is the relative velocity of some colliding particles centered at $x, x_1, x_2 \in \mathbb{R}^d$, with velocities $v, v_1, v_2 \in \mathbb{R}^d$ before the ternary interaction with respect to the impact directions vector

$$\boldsymbol{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} := \frac{1}{\sqrt{|x-x_1|^2 + |x-x_2|^2}} \begin{pmatrix} x_1 - x \\ x_2 - x \end{pmatrix} \in \mathbb{S}^{2d-1}, \tag{2.16}$$

and

$$\begin{aligned} v^* &= v + \frac{\omega_1\cdot(v_1-v)+\omega_2\cdot(v_2-v)}{1+\omega_1\cdot\omega_2}(\omega_1+\omega_2), \\ v_1^* &= v_1 - \frac{\omega_1\cdot(v_1-v)+\omega_2\cdot(v_2-v)}{1+\omega_1\cdot\omega_2}\omega_1, \\ v_2^* &= v_2 - \frac{\omega_1\cdot(v_1-v)+\omega_2\cdot(v_2-v)}{1+\omega_1\cdot\omega_2}\omega_2 \end{aligned} \tag{2.17}$$

are the outgoing velocities of the particles after the ternary interaction. It can be easily seen that if v^*, v_1^*, v_2^* are given by (2.17), the ternary energy–momentum conservation system

$$v^* + v_1^* + v_2^* = v + v_1 + v_2, \tag{2.18}$$

$$|v^*|^2 + |v_1^*|^2 + |v_2^*|^2 = |v|^2 + |v_1|^2 + |v_2|^2 \tag{2.19}$$

is satisfied. Expressions (2.18)–(2.19) also imply the ternary velocities conservation law

$$|v^* - v_1^*|^2 + |v^* - v_2^*|^2 + |v_1^* - v_2^*|^2 = |v-v_1|^2 + |v-v_2|^2 + |v_1-v_2|^2. \tag{2.20}$$

For the postcollisional relative velocity, we will write

$$\boldsymbol{u}^* := \begin{pmatrix} v_1^* - v^* \\ v_2^* - v^* \end{pmatrix}, \tag{2.21}$$

and let us also define the quantities

$$|\tilde{\boldsymbol{u}}| := \sqrt{|v-v_1|^2 + |v-v_2|^2 + |v_1-v_2|^2}, \tag{2.22}$$

$$|\tilde{\boldsymbol{u}}^*| := \sqrt{|v^*-v_1^*|^2 + |v^*-v_2^*|^2 + |v_1^*-v_2^*|^2}. \tag{2.23}$$

Then (2.20) can be written as

$$|\tilde{\boldsymbol{u}}| = |\tilde{\boldsymbol{u}}^*|, \tag{2.24}$$

which is the ternary analog of the binary expression (2.7). Defining

$$\bar{\boldsymbol{u}} := \frac{\boldsymbol{u}}{|\tilde{\boldsymbol{u}}|}, \quad \bar{\boldsymbol{u}}^* := \frac{\boldsymbol{u}^*}{|\tilde{\boldsymbol{u}}|}, \tag{2.25}$$

equality (2.20) implies $\bar{\boldsymbol{u}}, \bar{\boldsymbol{u}}^* \in \mathbb{E}^{2d-1}$, where

$$\mathbb{E}^{2d-1} := \{(\nu_1, \nu_2) \in \mathbb{R}^{2d} : |\nu_1|^2 + |\nu_2|^2 + |\nu_1 - \nu_2|^2 = 1\} \tag{2.26}$$

is a $(2d-1)$-dimensional ellipsoid. The vectors $\bar{\boldsymbol{u}}$, $\bar{\boldsymbol{u}}^*$ are the ternary analogs of the relative velocity direction and the scattering direction of the binary interaction. Because of the asymmetry of the ternary interaction they are not unit vectors, but they lie on the ellipsoid $\mathbb{E}^{2d-1}$ instead. However, for convenience we will refer to $\bar{\boldsymbol{u}}$, $\bar{\boldsymbol{u}}^*$ as the relative velocity direction and scattering direction respectively.

The collisional formulas (2.17) also imply

$$\boldsymbol{\omega} \cdot \bar{\boldsymbol{u}}^* = -\boldsymbol{\omega} \cdot \bar{\boldsymbol{u}}, \tag{2.27}$$

which is the ternary analog to specular reflection with respect to the impact directions vector $\boldsymbol{\omega} = (\omega_1, \omega_2) \in \mathbb{S}^{2d-1}$. Indeed, one has

$$\begin{aligned}\boldsymbol{\omega} \cdot \boldsymbol{u}^* &= \omega_1 \cdot (v_1^* - v^*) + \omega_2 \cdot (v_2^* - v^*) \\ &= \boldsymbol{u} \cdot \boldsymbol{\omega} - \frac{2\boldsymbol{u} \cdot \boldsymbol{\omega}}{1 + \omega_1 \cdot \omega_2}(|\omega_1|^2 + \omega_1 \cdot \omega_2 + |\omega_2|^2) = -\boldsymbol{\omega} \cdot \boldsymbol{u},\end{aligned}$$

which is equivalent to (2.27) due to (2.25).

The term B_3 in the integrand of (2.14), depending on the relative velocity $\boldsymbol{u} \in \mathbb{R}^{2d}$ and the impact directions vector $\boldsymbol{\omega} = (\omega_1, \omega_2) \in \mathbb{S}^{2d-1}$, is the ternary interaction differential cross-section, which describes the transition probability of ternary interactions. Recalling $|\tilde{\boldsymbol{u}}|$ from (2.22) and $\bar{\boldsymbol{u}} \in \mathbb{E}^{2d-1}$ from (2.25), we assume B_3 takes the form

$$B_3(\boldsymbol{u}, \boldsymbol{\omega}) = |\tilde{\boldsymbol{u}}|^{\gamma_3} b_3(\bar{\boldsymbol{u}} \cdot \boldsymbol{\omega}, \omega_1 \cdot \omega_2), \quad \gamma_3 \in (-2d + 1, 1], \tag{2.28}$$

and $b_3\colon [-1, 1] \times [-\frac{1}{2}, \frac{1}{2}] \to [0, \infty)$ is the ternary angular transition probability density. Since $\boldsymbol{\omega} = (\omega_1, \omega_2) \in \mathbb{S}^{2d-1}$, the Cauchy–Schwarz inequality and (2.22) yield

$$|\bar{\boldsymbol{u}} \cdot \boldsymbol{\omega}| \leq |\bar{\boldsymbol{u}}| \cdot |\boldsymbol{\omega}| = \frac{|\boldsymbol{u}|}{|\tilde{\boldsymbol{u}}|} \leq 1. \tag{2.29}$$

Moreover, for any $\boldsymbol{\omega} = (\omega_1, \omega_2) \in \mathbb{S}^{2d-1}$, the Cauchy–Schwarz inequality followed by Young's inequality yields

$$|\omega_1 \cdot \omega_2| \leq |\omega_1| \cdot |\omega_2| \leq \frac{|\omega_1|^2 + |\omega_2|^2}{2} = \frac{1}{2}.$$

Therefore, for any $\boldsymbol{\omega} = (\omega_1, \omega_2) \in \mathbb{S}_1^{2d-1}$, the expression $b_3(\bar{\boldsymbol{u}} \cdot \boldsymbol{\omega}, \omega_1 \cdot \omega_2)$ is well defined.

In addition, we assume that b_3 satisfies the following properties:

- $b_3\colon [-1,1]\times[-\frac{1}{2},\frac{1}{2}] \to \mathbb{R}$ is a measurable, non-negative probability density.
- b_3 is even with respect to the first argument, i.e.

$$b_3(-z,w) = b_3(z,w) \quad \forall (z,w) \in [-1,1]\times[-\tfrac{1}{2},\tfrac{1}{2}]. \tag{2.30}$$

In addition, due to (2.27), the ternary micro-reversibility condition holds:

$$b_3(\bar{\boldsymbol{u}}^* \cdot \boldsymbol{\omega}, \omega_1\cdot\omega_2) = b_3(\bar{\boldsymbol{u}}\cdot\boldsymbol{\omega}, \omega_1\cdot\omega_2) \quad \forall \boldsymbol{\omega}\in\mathbb{S}^{2d-1}, \quad \forall v, v_1, v_2 \in \mathbb{R}^d, \tag{2.31}$$

and relations (2.28), (2.25) and (2.31) imply the total ternary collision kernel satisfies

$$B_3(\boldsymbol{u}^*, \boldsymbol{\omega}) = B_3(\boldsymbol{u},\boldsymbol{\omega}) \quad \forall \boldsymbol{\omega}\in\mathbb{S}^{2d-1}, \quad \forall v, v_1, v_2\in\mathbb{R}^d. \tag{2.32}$$

- The probability density b_3 is integrable on $\mathbb{S}^{2d-1}$, i.e.

$$\|b_3\|_{L^1(\mathbb{S}^{2d-1})} := \sup_{\boldsymbol{\nu}\in\mathbb{E}_1^{2d-1}} \int_{\mathbb{S}^{2d-1}} b_3(\boldsymbol{\nu}\cdot\boldsymbol{\omega}, \omega_1\cdot\omega_2)\,d\boldsymbol{\omega} < \infty. \tag{2.33}$$

Remark 2.3. One can see that the ternary operator introduced in [5] is a special case of (2.28) for

$$\gamma_3 = 1, \quad b_3(z,w) = \frac{1}{2}\frac{|z|}{\sqrt{1+w}}.$$

Remark 2.4. Throughout the paper we assume that at least one of b_2, b_3 is not trivially zero; one of the two, however, could be zero. If $b_3 = 0$ we recover the classical Boltzmann equation (1.3), while if $b_2 = 0$ we recover the ternary Boltzmann equation (1.2). As will become clear, see for instance (2.77), the dependence on the sizes of b_2 and b_3 is additive, implying that the two collisional operators can be studied separately.

2.1.3. Gain and loss operators. It turns out to be more convenient to study the more general collisional operators

$$\begin{aligned} &Q_2(f,g)(t,x,v) \\ &\quad = \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} B_2(u,\omega)\big(f(t,x,v')g(t,x,v_1') - f(t,x,v)g(t,x,v_1)\big)\,d\omega\,dv_1, \end{aligned} \tag{2.34}$$

$$\begin{aligned} &Q_3(f,g,h)(t,x,v) \\ &\quad = \int_{\mathbb{S}^{2d-1}\times\mathbb{R}^{2d}} B_3(\boldsymbol{u},\boldsymbol{\omega})\big(f(t,x,v^*)g(t,x,v_1^*)h(t,x,v_2^*) \\ &\qquad\qquad - f(t,x,v)g(t,x,v_1)h(t,x,v_2)\big)\,d\omega_1\,d\omega_2\,dv_1\,dv_2. \end{aligned} \tag{2.35}$$

Due to assumptions (2.13), (2.33), the binary–ternary operator $Q_2(f,g) + Q_3(f,g,h)$ can be decomposed into a gain and a loss term as

$$Q_2(f,g) + Q_3(f,g,h) = G(f,g,h) - L(f,g,h), \tag{2.36}$$

where

$$L(f,g,h) = L_2(f,g) + L_3(f,g,h), \tag{2.37}$$

$$G(f,g,h) = G_2(f,g) + G_3(f,g,h). \tag{2.38}$$

The binary gain and loss operators G_2, L_2 are given by

$$G_2(f,g)(t,x,v) = \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} B_2(u,\omega) f(t,x,v')g(t,x,v_1')\,d\omega\,dv_1, \tag{2.39}$$

$$L_2(f,g)(t,x,v) = \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} B_2(u,\omega) f(t,x,v)g(t,x,v_1)\,d\omega\,dv_1, \tag{2.40}$$

and are clearly bilinear. The ternary gain and loss operators L_3, G_3 are given by

$$G_3(f,g,h)(t,x,v) = \int_{\mathbb{S}^{2d-1}\times\mathbb{R}^{2d}} B_3(\boldsymbol{u},\boldsymbol{\omega}) f(t,x,v^*)g(t,x,v_1^*) \times h(t,x,v_2^*)\,d\omega_1\,d\omega_2\,dv_1\,dv_2, \tag{2.41}$$

$$L_3(f,g,h)(t,x,v) = \int_{\mathbb{S}^{2d-1}\times\mathbb{R}^{2d}} B_3(\boldsymbol{u},\boldsymbol{\omega}) f(t,x,v)g(t,x,v_1) \times h(t,x,v_2)\,d\omega_1\,d\omega_2\,dv_1\,dv_2, \tag{2.42}$$

and are clearly trilinear. Notice the loss term can be factorized as

$$L(f,g,h) = fR(g,h), \tag{2.43}$$

where R is given by

$$R(g,h) := R_2(g) + R_3(g,h), \tag{2.44}$$

R_2 is the linear operator

$$R_2(g)(t,x,v) := \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} B_2(u,\omega)g(t,x,v_1)\,d\omega\,dv_1 \tag{2.45}$$

and R_3 is the bilinear operator

$$R_3(g,h)(t,x,v) := \int_{\mathbb{S}^{2d-1}\times\mathbb{R}^{2d}} B_3(\boldsymbol{u},\boldsymbol{\omega})g(t,x,v_1)h(t,x,v_2)\,d\omega_1\,d\omega_2\,dv_1\,dv_2. \tag{2.46}$$

2.2. Some notation and the notion of a solution

Throughout the paper, the dimension $d \geq 2$, the binary and ternary integrability assumptions (2.13), (2.33) respectively, and the cross-section exponents

$$\gamma_2 \in (-d+1, 1], \quad \gamma_3 \in (-2d+1, 1], \tag{2.47}$$

appearing respectively in (2.9), (2.28), will be fixed. Moreover, C_d denotes a general constant depending on the dimension d and can change value.

2.2.1. Functional spaces. Let us introduce the functional spaces used in this paper. First, in order to point out the dependence on positions and velocities, we will use the notation

$$L^1_{x,v} := L^1(\mathbb{R}^d \times \mathbb{R}^d), \tag{2.48}$$

$$L^\infty_{x,v} := L^\infty(\mathbb{R}^d \times \mathbb{R}^d). \tag{2.49}$$

We also define the sets of space–velocity functions

$$F_{x,v} := \big\{f : \mathbb{R}^d \times \mathbb{R}^d \to \overline{\mathbb{R}},\ \text{such that } f \text{ is measurable}\big\}, \tag{2.50}$$

$$F^+_{x,v} := \big\{f \in F_{x,v} : f(x,v) \geq 0,\ \text{for a.e. } (x,v) \in \mathbb{R}^d \times \mathbb{R}^d\big\}, \tag{2.51}$$

$$L^{1,+}_{x,v} := L^1_{x,v} \cap F^+_{x,v}. \tag{2.52}$$

In general, for $f, g \in F_{x,v}$, we write $f \geq g$ iff $f(x,v) \geq g(x,v)$ for a.e. $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$. The same notation will hold for equality as well.

Given $\alpha, \beta > 0$, we define the corresponding (non-normalized) Maxwellian $M_{\alpha,\beta}$: $\mathbb{R}^d \times \mathbb{R}^d \to (0,\infty)$ by

$$M_{\alpha,\beta}(x,v) := e^{-\alpha|x|^2-\beta|v|^2}. \tag{2.53}$$

We also define the corresponding Banach space of functions essentially bounded by $M_{\alpha,\beta}$ as

$$\mathcal{M}_{\alpha,\beta} := \big\{f \in F_{x,v} : \|f\|_{\mathcal{M}_{\alpha,\beta}} < \infty\big\}, \tag{2.54}$$

where

$$\|f\|_{\mathcal{M}_{\alpha,\beta}} := \|f M^{-1}_{\alpha,\beta}\|_{L^\infty_{x,v}}.$$

We will write $f_n \xrightarrow{\mathcal{M}_{\alpha,\beta}} f$ if

$$f_n \xrightarrow{\text{a.e.}} f \quad \text{and} \quad \sup_{n\in\mathbb{N}} \|f_n\|_{\mathcal{M}_{\alpha,\beta}} < \infty. \tag{2.55}$$

It is clear that if $f_n \xrightarrow{\mathcal{M}_{\alpha,\beta}} f$ then $f_n \in \mathcal{M}_{\alpha,\beta}$ for all $n \in \mathbb{N}$ and $f \in \mathcal{M}_{\alpha,\beta}$. If $k \in \mathbb{N}$ and $f_{1,n} \xrightarrow{\mathcal{M}_{\alpha,\beta}} f_1$, $f_{2,n} \xrightarrow{\mathcal{M}_{\alpha,\beta}} f_2, \dots, f_{k,n} \xrightarrow{\mathcal{M}_{\alpha,\beta}} f_k$, we will write

$$(f_{1,n}, \dots, f_{k,n}) \xrightarrow{\mathcal{M}_{\alpha,\beta}} (f_1, \dots, f_k).$$

We also define the set of a.e. non-negative functions essentially bounded by $M_{\alpha,\beta}$ as

$$\mathcal{M}^+_{\alpha,\beta} := \mathcal{M}_{\alpha,\beta} \cap F^+_{x,v}. \tag{2.56}$$

Given $0 < T \leq \infty$, we define the sets of time-dependent functions

$$\mathcal{F}_T := \{f : [0,T) \to F_{x,v}\}, \tag{2.57}$$

$$\mathcal{F}^+_T := \{f : [0,T) \to F^+_{x,v}\}, \tag{2.58}$$

and given $f, g \in \mathcal{F}_T$, we will write $f \geq g$ iff $f(t) \geq g(t)$ for all $t \in [0,T)$. The same notation will hold for equalities as well.

Finally, we define the following subsets of functional spaces:

$$C^0([0,T),L^{1,+}_{x,v}) := C^0([0,T),L^1_{x,v}) \cap \mathcal{F}_T^+, \tag{2.59}$$
$$L^1_{\text{loc}}([0,T),L^{1,+}_{x,v}) := L^1_{\text{loc}}([0,T),L^1_{x,v}) \cap \mathcal{F}_T^+, \tag{2.60}$$
$$L^\infty([0,T),L^{1,+}_{x,v}) := L^\infty([0,T),L^1_{x,v}) \cap \mathcal{F}_T^+, \tag{2.61}$$

and given $\alpha,\beta > 0$, we define the Banach space of time-dependent functions uniformly essentially bounded by $M_{\alpha,\beta}$,

$$L^\infty([0,T),\mathcal{M}_{\alpha,\beta}) := \{f \in \mathcal{F}_T : |||f|||_\infty < \infty\}, \tag{2.62}$$

with norm

$$|||f|||_\infty = \sup_{t\in[0,T)} \|f(t)\|_{\mathcal{M}_{\alpha,\beta}}. \tag{2.63}$$

Notice that in definition (2.62), the supremum is taken with respect to all $t \in [0,T)$. We also write

$$L^\infty([0,T),\mathcal{M}^+_{\alpha,\beta}) := L^\infty([0,T),\mathcal{M}_{\alpha,\beta}) \cap \mathcal{F}_T^+. \tag{2.64}$$

2.2.2. Transport operator. We now introduce the transport operator, which will be crucial to define mild solutions to (1.1). Let us recall from (2.50)–(2.51) the sets of functions

$$F_{x,v} := \big\{f : \mathbb{R}^d \times \mathbb{R}^d \to \overline{\mathbb{R}},\ \text{such that } f \text{ is measurable}\big\},$$
$$F^+_{x,v} := \big\{f \in F_{x,v} : f(x,v) \geq 0,\ \text{for a.e. } (x,v) \in \mathbb{R}^d \times \mathbb{R}^d\big\}.$$

Consider a positive time $0 < T \leq \infty$ (we can have $T = \infty$) and recall from (2.57)–(2.58) the sets of time-dependent functions

$$\mathcal{F}_T := \big\{f : [0,T) \to F_{x,v}\big\},$$
$$\mathcal{F}_T^+ := \big\{f : [0,T) \to F^+_{x,v}\big\}.$$

Given $f \in \mathcal{F}_T$, we define $f^\# \in \mathcal{F}_T$ by

$$f^\#(t,x,v) := f(t,x+tv,v), \tag{2.65}$$

and $f^{-\#} \in \mathcal{F}_T$ by

$$f^{-\#}(t,x,v) := f(t,x-tv,v).$$

Clearly, the operators $\#: \mathcal{F}_T \to \mathcal{F}_T$ and $-\#: \mathcal{F}_T \to \mathcal{F}_T$ are linear and invertible and, in particular,

$$(\#)^{-1} = -\#.$$

Remark 2.5. Let $f,g \in \mathcal{F}_T$. Since the maps $(x,v) \to (x+tv,v)$ and $(x,v) \to (x-tv,v)$ are measure preserving, for all $t \in [0,T)$, we have

$$f \geq g \Leftrightarrow f^\# \geq g^\# \Leftrightarrow f^{-\#} \geq g^{-\#}.$$

In particular,

$$f \in \mathcal{F}_T^+ \Leftrightarrow f^\# \in \mathcal{F}_T^+ \Leftrightarrow f^{-\#} \in \mathcal{F}_T^+. \tag{2.66}$$

Remark 2.6. Let $f, g \in \mathcal{F}_T$. Since the maps $(x, v) \to (x + tv, v)$ and $(x, v) \to (x - tv, v)$ are measure preserving, for all $t \in [0, T)$, we have

$$\|f^{\#}(t)\|_{L^1_{x,v}} = \|f(t)\|_{L^1_{x,v}} = \|f^{-\#}(t)\|_{L^1_{x,v}} \quad \forall t \in [0, T). \tag{2.67}$$

Relations (2.66)–(2.67) and linearity of the transport operator imply

$$f \in C^0([0, T), L^{1,+}_{x,v}) \Leftrightarrow f^{\#} \in C^0([0, T), L^{1,+}_{x,v}) \Leftrightarrow f^{-\#} \in C^0([0, T), L^{1,+}_{x,v}). \tag{2.68}$$

Throughout the manuscript, we will often define $f^{\#} \in \mathcal{F}_T$ directly, implying that f is defined by $f := (f^{\#})^{-\#}$.

2.2.3. Transported gain and loss operators. In order to define mild solutions to (1.1), it is important to understand the action of the transport operator on the gain and loss operators. More specifically, given $f, g, h \in \mathcal{F}_T$, for the gain operators we write

$$\begin{aligned}
G_2^{\#}(f, g)(t, x, v) &:= (G_2(f, g))^{\#}(t, x, v) \\
&= \int_{\mathbb{S}^{d-1} \times \mathbb{R}^d} B_2(u, \omega) f(t, x + tv, v') g(t, x + tv, v_1') \, d\omega \, dv_1, \\
G_3^{\#}(f, g, h)(t, x, v) &:= (G_3(f, g, h))^{\#}(t, x, v) \\
&= \int_{\mathbb{S}^{2d-1} \times \mathbb{R}^{2d}} B_3(\boldsymbol{u}, \boldsymbol{\omega}) f(t, x + tv, v^*) g(t, x + tv, v_1^*) \\
&\qquad\qquad \times h(t, x + tv, v_2^*) \, d\omega_1 \, d\omega_2 \, dv_1 \, dv_2, \\
G^{\#}(f, g, h)(t, x, v) &:= G_2^{\#}(f, g)(t, x, v) + G_3^{\#}(f, g, h)(t, x, v),
\end{aligned} \tag{2.69}$$

and for the loss operators we write

$$\begin{aligned}
L_2^{\#}(f, g)(t, x, v) &:= (L_2(f, g))^{\#}(t, x, v) \\
&= \int_{\mathbb{S}^{d-1} \times \mathbb{R}^d} B_2(u, \omega) f(t, x + tv, v) g(t, x + tv, v_1) \, d\omega \, dv_1, \\
L_3^{\#}(f, g, h)(t, x, v) &:= (L_3(f, g, h))^{\#}(t, x, v) \\
&= \int_{\mathbb{S}^{2d-1} \times \mathbb{R}^{2d}} B_3(\boldsymbol{u}, \boldsymbol{\omega}) f(t, x + tv, v) g(t, x + tv, v_1) \\
&\qquad\qquad \times h(t, x + tv, v_2) \, d\omega_1 \, d\omega_2 \, dv_1 \, dv_2, \\
L^{\#}(f, g, h)(t, x, v) &:= L_2^{\#}(f, g)(t, x, v) + L_3^{\#}(f, g, h)(t, x, v).
\end{aligned} \tag{2.70}$$

Under this notation, it is straightforward to verify that

$$\begin{aligned}
L_2^{\#}(f, g, h)(t) &= f^{\#}(t) R_2^{\#}(g)(t), \\
L_3^{\#}(f, g, h)(t) &= f^{\#}(t) R_3^{\#}(g, h)(t), \\
L^{\#}(f, g, h)(t) &= f^{\#}(t) R^{\#}(g, h)(t),
\end{aligned} \tag{2.71}$$

where

$$R_2^{\#}(g)(t,x,v) := (R_2(g))^{\#}(t,x,v) = \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} B_2(u,\omega)g(t,x+tv,v_1)\,d\omega\,dv_1,$$

$$\begin{aligned} R_3^{\#}(g,h)(t,x,v) &:= (R_3(g,h))^{\#}(t,x,v) \\ &= \int_{\mathbb{S}^{2d-1}\times\mathbb{R}^{2d}} B_3(\boldsymbol{u},\boldsymbol{\omega})g(t,x+tv,v_1)h(t,x+tv,v_2)\,d\omega_1\,d\omega_2\,dv_1\,dv_2, \\ R^{\#}(g,h)(t,x,v) &:= R_2^{\#}(g)(t,x,v) + R_3^{\#}(g,h)(t,x,v). \end{aligned} \tag{2.72}$$

2.2.4. Notion of a mild solution. Using (2.36), the binary–ternary Boltzmann equation (1.1) is written as

$$\begin{cases} \partial_t f + v\cdot\nabla_x f = G(f,f,f) - L(f,f,f), & (t,x,v)\in(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d, \\ f(0) = f_0, & (x,v)\in\mathbb{R}^d\times\mathbb{R}^d, \end{cases} \tag{2.73}$$

where the gain term $G(f,f,f)$ and the loss term $L(f,f,f)$ are given by (2.38)–(2.37) respectively.

Here is where the importance of the transport operator will become clear. Indeed, using the chain rule, the initial value problem (2.73) can be formally written as

$$\begin{cases} \partial_t f^{\#} + L^{\#}(f,f,f) = G^{\#}(f,f,f), & (t,x,v)\in(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d, \\ f^{\#}(0) = f_0, & (x,v)\in\mathbb{R}^d\times\mathbb{R}^d. \end{cases} \tag{2.74}$$

Motivated by (2.74), we aim to define solutions of (1.1) up to time $0<T\le\infty$, with respect to a given Maxwellian $M_{\alpha,\beta}$, where $\alpha,\beta>0$.

Definition 2.7. Let $0<T\le\infty$, $\alpha,\beta>0$ and $f_0\in\mathcal{M}^+_{\alpha,\beta}$. A mild solution to (1.1) in $[0,T)$, with initial data $f_0\in\mathcal{M}^+_{\alpha,\beta}$, is a function $f\in\mathcal{F}_T^+$ such that

(i) $f^{\#}\in C^0([0,T),L^{1,+}_{x,v})\cap L^\infty([0,T),\mathcal{M}^+_{\alpha,\beta})$,

(ii) $L^{\#}(f,f,f), G^{\#}(f,f,f)\in L^\infty([0,T),L^{1,+}_{x,v})$,

(iii) $f^{\#}$ is weakly differentiable and satisfies

$$\begin{cases} \dfrac{df^{\#}}{dt} + L^{\#}(f,f,f) = G^{\#}(f,f,f), \\ f^{\#}(0) = f_0. \end{cases} \tag{2.75}$$

Remark 2.8. The differential equation of (2.75) is interpreted as an equality of distributions since all terms involved belong to $L^1_{\text{loc}}([0,T),L^{1,+}_{x,v})$.

Remark 2.9. Remarks 2.5–2.6 imply that a mild solution f to (1.1) belongs to $C^0([0,T), L^{1,+}_{x,v})$.

2.3. Statement of the main result

Now we are ready to state the main result of the paper.

Theorem 2.10. *Let* $0 < T \le \infty$, $\alpha, \beta > 0$. *Then for any initial data* $f_0 \in \mathcal{M}^+_{\alpha,\beta}$ *with*

$$\|f_0\|_{\mathcal{M}_{\alpha,\beta}} < \frac{\alpha^{1/2}}{48K_\beta\big(1 + \frac{\alpha^{1/4}}{2\sqrt{6K_\beta}}\big)}, \tag{2.76}$$

where

$$\begin{aligned} K_\beta = C_d\Big[&\|b_2\|_{L^1(\mathbb{S}^{d-1})}\Big(\beta^{-d/2} + \frac{1}{d+\gamma_2-1}\Big) \\ &+ \|b_3\|_{L^1(\mathbb{S}^{2d-1})}\Big(\beta^{-d} + \frac{1}{2d+\gamma_3-1}\Big)\Big] > 0 \end{aligned} \tag{2.77}$$

and C_d *is an appropriate constant depending on the dimension* d, *the binary–ternary Boltzmann equation* (1.1) *has a unique mild solution* f *satisfying the bound*

$$|||f^\#|||_\infty \le \frac{1 - \sqrt{1 - 48K_\beta\alpha^{-1/2}\big(1 + \frac{\alpha^{1/4}}{2\sqrt{6K_\beta}}\big)\|f_0\|_{\mathcal{M}_{\alpha,\beta}}}}{24K_\beta\alpha^{-1/2}\big(1 + \frac{\alpha^{1/4}}{2\sqrt{6K_\beta}}\big)}. \tag{2.78}$$

Remark 2.11. As we will see, the uniqueness claimed above holds in the class of solutions of (1.1) satisfying (2.78).

Remark 2.12. According to the assumptions on b_2, b_3 made in Remark 2.4, Theorem 2.10 applies as well to the endpoint cases where either $b_2 = 0$ or $b_3 = 0$ (but not both). In the case $b_3 = 0$, one recovers the solution of the classical Boltzmann equation (1.3) constructed in [13], while in the case $b_2 = 0$, one obtains well-posedness of the ternary Boltzmann equation (1.2) introduced in [5].

3. Properties of the transported gain and loss operators

In this section we investigate properties of the transported gain and loss operators, which will be important for proving global well-posedness of (1.1).

3.1. Monotonicity and L^1-norms

As we will see, the transported gain and loss operators are monotone increasing when acting on non-negative functions. These monotonicity properties will allow us to construct monotone sequences of supersolutions and subsolutions to (1.1). Moreover, we show that the L^1-norm of the gain is equal to the L^1-norm of the loss. This equality will allow us to reduce estimates on the norm of the gain term to estimating the norm of the loss term. In the following, saying that an operator is bilinear/trilinear, we mean it is linear in each argument, and saying it is monotone increasing, we mean it is increasing in each argument.

Proposition 3.1. *Let* $0 < T \leq \infty$. *Then the following hold:*

(i) $R_2^\#\colon \mathcal{F}_T^+ \to \mathcal{F}_T^+$ *is linear and monotone increasing.*

(ii) $L_2^\#, G_2^\#, R_3^\#\colon \mathcal{F}_T^+ \times \mathcal{F}_T^+ \to \mathcal{F}_T^+$ *are bilinear and monotone increasing.*

(iii) $L_3^\#, G_3^\#\colon \mathcal{F}_T^+ \times \mathcal{F}_T^+ \times \mathcal{F}_T^+ \to \mathcal{F}_T^+$ *are trilinear and monotone increasing.*

(iv) $L^\#, G^\#\colon \mathcal{F}_T^+ \times \mathcal{F}_T^+ \times \mathcal{F}_T^+ \to \mathcal{F}_T^+$ *and* $R^\#\colon \mathcal{F}_T^+ \times \mathcal{F}_T^+ \to \mathcal{F}_T^+$ *are monotone increasing.*

(v) *For any* $f, g, h \in \mathcal{F}_T^+$, *the following identities hold:*

$$\begin{aligned}
\|G_2^\#(f,g)(t)\|_{L^1_{x,v}} &= \|L_2^\#(f,g)(t)\|_{L^1_{x,v}} && \forall t \in [0,T),\\
\|G_3^\#(f,g,h)(t)\|_{L^1_{x,v}} &= \|L_3^\#(f,g,h)(t)\|_{L^1_{x,v}} && \forall t \in [0,T),\\
\|G^\#(f,g,h)(t)\|_{L^1_{x,v}} &= \|L^\#(f,g,h)(t)\|_{L^1_{x,v}} && \forall t \in [0,T).
\end{aligned} \tag{3.1}$$

Proof. Parts (i)–(iv) are immediate by linearity of the integral, positivity of the functions considered and relation (2.66).

Let us now prove (v). We first prove (3.1) for the binary case. By (2.67) we have

$$\begin{aligned}
\|G_2^\#(f,g)(t)\|_{L^1_{x,v}} &= \|G_2(f,g)(t)\|_{L^1_{x,v}} \quad \forall t \in [0,T),\\
\|L_2^\#(f,g)(t)\|_{L^1_{x,v}} &= \|L_2(f,g)(t)\|_{L^1_{x,v}} \quad \forall t \in [0,T).
\end{aligned}$$

Therefore, for any $t \in [0,T)$, using (2.12) and the involutionary substitution $(v', v_1') \to (v, v_1)$, we obtain

$$\begin{aligned}
\|G_2^\#(f,g)(t)\|_{L^1_{x,v}} &= \|G_2(f,g)(t)\|_{L^1_{x,v}}\\
&= \int_{\mathbb{R}^{3d}\times\mathbb{S}^{d-1}} B_2(u,\omega) f(t,x,v') g(t,x,v_1')\, d\omega\, dv_1\, dv\, dx\\
&= \int_{\mathbb{R}^{3d}\times\mathbb{S}^{d-1}} B_2(u',\omega) f(t,x,v') g(t,x,v_1')\, d\omega\, dv_1\, dv\, dx\\
&= \int_{\mathbb{R}^{3d}\times\mathbb{S}^{d-1}} B_2(u,\omega) f(t,x,v) g(t,x,v_1)\, d\omega\, dv_1\, dv\, dx\\
&= \|L_2(f,g)(t)\|_{L^1_{x,v}} = \|L_2^\#(f,g)(t)\|_{L^1_{x,v}}.
\end{aligned}$$

We now prove (3.1) for the ternary case. By (2.67) we have

$$\begin{aligned}
\|G_3^\#(f,g,h)(t)\|_{L^1_{x,v}} &= \|G_3(f,g,h)(t)\|_{L^1_{x,v}} \quad \forall t \in [0,T),\\
\|L_3^\#(f,g,h)(t)\|_{L^1_{x,v}} &= \|L_3(f,g,h)(t)\|_{L^1_{x,v}} \quad \forall t \in [0,T).
\end{aligned}$$

Therefore, for any $t \in [0,T)$, using (2.32) and the involutionary substitution $(v^*, v_1^*, v_2^*) \to (v, v_1, v_2)$, we obtain

$$\begin{aligned}
&\|G_3^\#(f,g,h)(t)\|_{L^1_{x,v}} = \|G_3(f,g,h)(t)\|_{L^1_{x,v}}\\
&\quad = \int_{\mathbb{R}^{4d}\times\mathbb{S}^{2d-1}} B_3(\boldsymbol{u},\boldsymbol{\omega}) f(t,x,v^*) g(t,x,v_1^*) h(t,x,v_2^*)\, d\omega_1\, d\omega_2\, dv_1\, dv_2\, dv\, dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^{4d}\times\mathbb{S}^{2d-1}} B_3(\boldsymbol{u}^*,\boldsymbol{\omega}) f(t,x,v^*)g(t,x,v_1^*)h(t,x,v_2^*)\,d\omega_1\,d\omega_2\,dv_1\,dv_2\,dv\,dx \\
&= \int_{\mathbb{R}^{4d}\times\mathbb{S}^{2d-1}} B_3(\boldsymbol{u},\boldsymbol{\omega}) f(t,x,v)g(t,x,v_1)h(t,x,v_2)\,d\omega_1\,d\omega_2\,dv_1\,dv_2\,dv\,dx \\
&= \|L_3(f,g,h)(t)\|_{L^1_{x,v}} = \|L_3^\#(f,g,h)(t)\|_{L^1_{x,v}}.
\end{aligned}
$$

We finally prove (3.1) for the mixed case. By positivity, for any $t \in [0,T)$, we have

$$
\begin{aligned}
\|G^\#(f,g,h)(t)\|_{L^1_{x,v}} &= \|G_2^\#(f,g)(t) + G_3^\#(f,g,h)(t)\|_{L^1_{x,v}} \\
&= \|G_2^\#(f,g)(t)\|_{L^1_{x,v}} + \|G_3^\#(f,g,h)(t)\|_{L^1_{x,v}}, \\
\|L^\#(f,g,h)(t)\|_{L^1_{x,v}} &= \|L_2^\#(f,g)(t) + L_3^\#(f,g,h)(t)\|_{L^1_{x,v}} \\
&= \|L_2^\#(f,g)(t)\|_{L^1_{x,v}} + \|L_3^\#(f,g,h)(t)\|_{L^1_{x,v}}.
\end{aligned}
$$

Equality (3.1) for the mixed case immediately follows from the corresponding binary and ternary equalities. ∎

3.2. Convolution estimates

We now present a general convolution-type result, which will be essential for control of the binary and the ternary collisional operators. These estimates will be of fundamental importance in the proof of the $L^\infty L^1$ estimates (see Section 3.3) and the global estimate on the time average of the transported gain and loss operators appearing in Proposition 3.7, which in turn will be crucial for the proof of global well-posedness of (1.1). For the binary case one can find similar convolution estimates in [2, 13, 14]. Here, our contribution is the derivation of these estimates for the ternary case, since this is the first time global well-posedness has been studied for such a ternary correction of the Boltzmann equation. The estimates of the ternary term illustrate that consideration of softer potentials is allowed for the ternary collisional operator.

Lemma 3.2. *Let $\beta > 0$, $q_2 \in (-d,1]$ and $q_3 \in (-2d,1]$. Then the following hold:*

(i) *For any $v \in \mathbb{R}^d$, we have*

$$
\int_{\mathbb{R}^d} |u|^{q_2} e^{-\beta|v_1|^2}\,dv_1 \le \widetilde{K}^2_{\beta,q_2}(1+|v|^{q_2^+}), \tag{3.2}
$$

where $u = v_1 - v$, $q_2^+ := \max\{0,q_2\}$, $\widetilde{K}^2_{\beta,q_2}$ is given by

$$
\widetilde{K}^2_{\beta,q_2} = C_d\Big[(1+\beta^{-d/2}+\beta^{-\frac{d+1}{2}})\mathbb{1}_{q_2>0}(q_2) + \Big(\beta^{-d/2}+\frac{1}{d+q_2}\Big)\mathbb{1}_{q_2\le 0}(q_2)\Big] \tag{3.3}
$$

and C_d is an appropriate constant depending on the dimension d.

(ii) *For any $v \in \mathbb{R}^d$, we have*

$$
\int_{\mathbb{R}^{2d}} |\tilde{\boldsymbol{u}}|^{q_3} e^{-\beta(|v_1|^2+|v_2|^2)}\,dv_1\,dv_2 \le \widetilde{K}^3_{\beta,q_3}(1+|v|^{q_3^+}), \tag{3.4}
$$

where $|\tilde{\boldsymbol{u}}|$ *is given by* (2.22), $q_3^+ := \max\{0, q_3\}$, $\widetilde{K}^3_{\beta,q_3}$ *is given by*

$$\widetilde{K}^3_{\beta,q_3} = C_d\Big[(1+\beta^{-d}+\beta^{-\frac{2d+1}{2}})\mathbb{1}_{q_3>0}(q_3) + \Big(\beta^{-d}+\frac{1}{2d+q_3}\Big)\mathbb{1}_{q_3\le 0}(q_3)\Big] \tag{3.5}$$

and C_d *is an appropriate constant depending on the dimension* d.

Proof. We will rely on the elementary estimate

$$\int_{\mathbb{R}^d} e^{-\beta|v_1|^2}\,dv_1 \le C_d\beta^{-d/2} \tag{3.6}$$

and, given $q \in (0,1]$, on the estimate

$$\begin{aligned}\int_{\mathbb{R}^d} |v_1|^q e^{-\beta|v_1|^2}\,dv_1 &\le |B_1^d| + \int_{|v_1|>1} |v_1|^q e^{-\beta|v_1|^2}\,dv_1\\ &\le |B_1^d| + \int_{|v_1|>1} |v_1| e^{-\beta|v_1|^2}\,dv_1\\ &\le C_d(1+\beta^{-\frac{d+1}{2}}),\end{aligned} \tag{3.7}$$

where $|B_1^d|$ denotes the volume of the d-dimensional unit ball.

(i) We take separate cases for $q_2 \in (-d, 1]$.

- $q_2 \in (0,1]$: Since $q_2 \in (0,1]$ we have

$$|u|^{q_2} = |v-v_1|^{q_2} \le (|v|+|v_1|)^{q_2} \le |v|^{q_2} + |v_1|^{q_2}.$$

Therefore

$$\begin{aligned}\int_{\mathbb{R}^d} |u|^{q_2} e^{-\beta|v_1|^2}\,dv_1 &\le \int_{\mathbb{R}^d} (|v|^{q_2}+|v_1|^{q_2})e^{-\beta|v_1|^2}\,dv_1\\ &\le C_d(1+\beta^{-d/2}+\beta^{-\frac{d+1}{2}})(1+|v|^{q_2}),\end{aligned} \tag{3.8}$$

where to obtain (3.8), we use estimates (3.6)–(3.7) for $q = q_2$.

- $q_2 \in (-d, 0]$: Since $q_2 \le 0$, estimate (3.6) implies

$$\begin{aligned}\int_{\mathbb{R}^d} |v-v_1|^{q_2} e^{-\beta|v_1|^2}\,dv_1 &\le \int_{|v-v_1|>1} e^{-\beta|v_1|^2}\,dv_1 + \int_{|v-v_1|<1} |v-v_1|^{q_2}\,dv_1\\ &= C_d\beta^{-d/2} + \int_{|y|<1} |y|^{q_2}\,dy\\ &= C_d\beta^{-d/2} + C_d\int_0^1 r^{d-1+q_2}\,dr\\ &= C_d\Big(\beta^{-d/2}+\frac{1}{d+q_2}\Big),\end{aligned} \tag{3.9}$$

since we have assumed $q_2 > -d$.

(ii) We take separate cases for $q_3 \in (-2d, 1]$

• $q_3 \in (0, 1]$: Since $q_3 \in (0, 1]$, we have

$$\begin{aligned} |\tilde{\boldsymbol{u}}|^{q_3} &= (|v - v_1|^2 + |v - v_2|^2 + |v_1 - v_2|^2)^{q_3/2} \\ &\le 2^{q_3}(|v|^2 + |v_1|^2 + |v_2|^2)^{q_3/2} \\ &\le 2(|v|^{q_3} + |v_1|^{q_3} + |v_2|^{q_3}). \end{aligned}$$

Therefore, Fubini's theorem and estimates (3.6)–(3.7) applied for $q = q_3$ imply

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} |\tilde{\boldsymbol{u}}|^{q_3} e^{-\beta(|v_1|^2+|v_2|^2)}\, dv_1\, dv_2 \\ &\quad\le 2\int_{\mathbb{R}^{2d}} (|v|^{q_3} + |v_1|^{q_3} + |v_2|^{q_3}) e^{-\beta(|v_1|^2+|v_2|^2)}\, dv_1\, dv_2 \\ &\quad\le C_d(1 + \beta^{-d} + \beta^{-\frac{2d+1}{2}})(1 + |v|^{q_3}). \end{aligned} \tag{3.10}$$

• $q_3 \in (-2d, 0]$: Recalling (2.22) and using the fact that $q_3 \le 0$, Fubini's theorem and estimates (3.6)–(3.7) imply

$$\begin{aligned} \int_{\mathbb{R}^{2d}} |\tilde{\boldsymbol{u}}|^{q_3} e^{-\beta(|v_1|^2+|v_2|^2)}\, dv_1\, dv_2 &\le \int_{\mathbb{R}^{2d}} |\boldsymbol{u}|^{q_3} e^{-\beta(|v_1|^2+|v_2|^2)}\, dv_1\, dv_2 \\ &\le \int_{|\boldsymbol{u}|>1} e^{-\beta(|v_1|^2+|v_2|^2)}\, dv_1\, dv_2 + \int_{|\boldsymbol{u}|<1} |\boldsymbol{u}|^{q_3}\, dv_1\, dv_2 \\ &\le C_d\beta^{-d} + \int_{|\boldsymbol{u}|<1} |\boldsymbol{u}|^{q_3}\, dv_1\, dv_2 \\ &= C_d\beta^{-d} + \int_{|\boldsymbol{y}|<1} |\boldsymbol{y}|^{q_3}\, d\boldsymbol{y} \\ &\le C_d\beta^{-d} + C_d\int_0^1 r^{2d-1+q_3}\, dr \\ &= C_d\Big(\beta^{-d} + \frac{1}{2d + q_3}\Big), \end{aligned} \tag{3.11}$$

since we have assumed $q_3 > -2d$.

Combining (3.8)–(3.10) and (3.11), we obtain (3.2)–(3.4). ■

3.3. $L^\infty L^1$ estimates

Here we prove uniform-in-time, space–velocity L^1 estimates on the transported gain and loss operators. These estimates will be of fundamental importance for the convergence of the iteration to the global solution. As we will see, the ternary collisional operator introduces some asymmetry which is not present in the binary case. For this reason, when we use Lemma 3.2 we first obtain estimates in asymmetric form (see Lemma 3.3). However, we will need a symmetric version of this estimate, which we derive in Proposition 3.4. To achieve that, we crucially rely on properties of the ternary interactions.

Recall from (2.47) the fixed cross-section exponents $\gamma_2 \in (-d+1, 1]$ and $\gamma_3 \in (-2d+1, 1]$. For convenience, we define the function

$$p_{\gamma_2,\gamma_3}(v) = 1 + |v|^{\gamma_2^+} + |v|^{\gamma_3^+}. \tag{3.12}$$

Notice that, given $\alpha > 0$, $\beta > 0$, we have

$$p_{\gamma_2,\gamma_3} M_{\alpha,\beta} \in L^1_{x,v}. \tag{3.13}$$

Using Lemma 3.2 for $q_2 = \gamma_2$ and $q_3 = \gamma_3$, we obtain some of the asymmetric estimates mentioned above.

Lemma 3.3. *Let $0 < T \leq \infty$ and $\alpha, \beta > 0$. Then there is a constant $C_\beta > 0$ such that the following hold:*

(i) *For any $g, h \in \mathcal{F}_T^+$, with $g^\#, h^\# \in L^\infty([0,T), \mathcal{M}^+_{\alpha,\beta})$, and any $t \in [0,T)$, we have*

$$0 \leq R_2^\#(g)(t) \leq C_\beta |||g^\#|||_\infty p_{\gamma_2,\gamma_3}, \tag{3.14}$$

$$0 \leq R_3^\#(g,h)(t) \leq C_\beta |||g^\#|||_\infty |||h^\#|||_\infty p_{\gamma_2,\gamma_3}, \tag{3.15}$$

$$0 \leq R^\#(g,h)(t) \leq C_\beta |||g^\#|||_\infty (1 + |||h^\#|||_\infty) p_{\gamma_2,\gamma_3}. \tag{3.16}$$

(ii) *For any $f, g, h \in \mathcal{F}_T^+$, with $f^\#, g^\#, h^\# \in L^\infty([0,T), \mathcal{M}^+_{\alpha,\beta})$, and $t \in [0,T)$, we have*

$$\|L_2^\#(f,g)(t)\|_{L^1_{x,v}}, \ \|G_2^\#(f,g)(t)\|_{L^1_{x,v}} \leq C_\beta |||g^\#|||_\infty \|f^\#(t) p_{\gamma_2,\gamma_3}\|_{L^1_{x,v}}, \tag{3.17}$$

$$\begin{aligned} \|L_3^\#(f,g,h)(t)\|_{L^1_{x,v}}, \ \|G_3^\#(f,g,h)(t)\|_{L^1_{x,v}} &\leq C_\beta |||g^\#|||_\infty |||h^\#|||_\infty \\ &\quad \times \|f^\#(t) p_{\gamma_2,\gamma_3}\|_{L^1_{x,v}}, \end{aligned} \tag{3.18}$$

$$\begin{aligned} \|L^\#(f,g,h)(t)\|_{L^1_{x,v}}, \ \|G^\#(f,g,h)(t)\|_{L^1_{x,v}} &\leq C_\beta |||g^\#|||_\infty (1 + |||h^\#|||_\infty) \\ &\quad \times \|f^\#(t) p_{\gamma_2,\gamma_3}\|_{L^1_{x,v}}. \end{aligned} \tag{3.19}$$

Moreover,

$$L^\#(f,g,h), \ G^\#(f,g,h) \in L^\infty([0,T), L^{1,+}_{x,v}). \tag{3.20}$$

Proof. We prove each claim separately.

• Proof of (i): Positivity follows immediately by the monotonicity of $R_2^\#$, $R_3^\#$, $R^\#$ on $\mathcal{F}_T^+$ (see Proposition 3.1). Since $g^\#, h^\# \in L^\infty([0,T), \mathcal{M}^+_{\alpha,\beta})$, for any $t \in [0,T)$, we have

$$\begin{aligned} 0 \leq g(t,x,v) &\leq |||g^\#|||_\infty e^{-\alpha|x-tv|^2 - \beta|v|^2} \quad \text{for a.e. } (x,v) \in \mathbb{R}^d \times \mathbb{R}^d, \\ 0 \leq h(t,x,v) &\leq |||h^\#|||_\infty e^{-\alpha|x-tv|^2 - \beta|v|^2} \quad \text{for a.e. } (x,v) \in \mathbb{R}^d \times \mathbb{R}^d. \end{aligned} \tag{3.21}$$

Recalling the fact that $R^\#(g,h) = R_2^\#(g) + R_3^\#(g,h)$, it suffices to prove estimates (3.14)–(3.15).

Let us first prove (3.14). For a.e. $(x, v) \in \mathbb{R}^{2d}$, estimate (3.21) and Lemma 3.2 (i), applied for $q_2 = \gamma_2$ and $q_3 = \gamma_3$, imply

$$\begin{aligned} R_2(g)(t,x,v) &\leq \|b_2\|_{L^1(\mathbb{S}^{d-1})} \int_{\mathbb{R}^d} |u|^{\gamma_2} g(t,x,v_1)\,dv_1 \\ &\leq \|b_2\|_{L^1(\mathbb{S}^{d-1})} |||g^{\#}|||_\infty \int_{\mathbb{R}^d} |u|^{\gamma_2} e^{-\beta|v_1|^2}\,dv_1 \\ &\leq C_\beta |||g^{\#}|||_\infty (1+|v|^{\gamma_2^+}) \\ &\leq C_\beta |||g^{\#}|||_\infty p_{\gamma_2,\gamma_3}(v). \end{aligned} \tag{3.22}$$

Since the right-hand side of (3.22) does not depend on x, we obtain (3.14).

Let us now prove (3.15). For a.e. $(x, v) \in \mathbb{R}^{2d}$, estimate (3.21) and Lemma 3.2 (ii), applied for $q_2 = \gamma_2$ and $q_3 = \gamma_3$, imply

$$\begin{aligned} R_3(g,h)(t,x,v) &\leq \|b_3\|_{L^1(\mathbb{S}^{2d-1})} \int_{\mathbb{R}^{2d}} |\tilde{\boldsymbol{u}}|^{\gamma_3} g(t,x,v_1) h(t,x,v_2)\,dv_1\,dv_2 \\ &\leq \|b_3\|_{L^1(\mathbb{S}^{2d-1})} |||g^{\#}|||_\infty |||h^{\#}|||_\infty \int_{\mathbb{R}^{2d}} |\tilde{\boldsymbol{u}}|^{\gamma_3} e^{-\beta(|v_1|^2+|v_2|^2)}\,dv_1\,dv_2 \\ &\leq C_\beta |||g^{\#}|||_\infty |||h^{\#}|||_\infty (1+|v|^{\gamma_3^+}) \\ &\leq C_\beta |||g^{\#}|||_\infty |||h^{\#}|||_\infty p_{\gamma_2,\gamma_3}(v). \end{aligned} \tag{3.23}$$

Since the right-hand side of (3.23) does not depend on x, we obtain (3.15).

Estimate (3.16) follows by the fact that $R^{\#}(g,h) = R_2^{\#}(g) + R_3^{\#}(g,h)$.

• Proof of (ii): We first prove the claim for the loss operators. Positivity follows immediately from the monotonicity of $L_2^{\#}$, $L_3^{\#}$, $L^{\#}$ on $\mathcal{F}_T^+$. Estimates (3.17)–(3.19) follow directly from (2.71) and part (i). Moreover, estimate (3.19) implies (3.20) since $f^{\#}, g^{\#}, h^{\#} \in L^\infty([0,T), \mathcal{M}_{\alpha,\beta}^+)$ and $p_{\gamma_2,\gamma_3} M_{\alpha,\beta} \in L_{x,v}^1$ by (3.13).

For the gain operators, positivity follows immediately from the monotonicity of $G_2^{\#}$, $G_3^{\#}$, $G^{\#}$ on $\mathcal{F}_T^+$. Estimates (3.17)–(3.19) and (3.20) come from (3.1) and the estimates for the loss operators. ■

Notice that bounds (3.17)–(3.19) are only with respect to the first argument f. Although this is not an issue in the binary case, where the gain and loss collisional operators are symmetric with respect to the inputs in the L^1-norm, this is not the case for the ternary operators. In order to treat this asymmetry, we need to derive estimates with respect to all three inputs of the ternary gain and loss collisional operators. This is achieved in the following result

Proposition 3.4. *Let $0 < T \leq \infty$ and $\alpha, \beta > 0$. Consider $f_1, f_2, f_3 \in \mathcal{F}_T^+$ with $f_1^{\#}, f_2^{\#}, f_3^{\#} \in L^\infty([0,T), \mathcal{M}_{\alpha,\beta}^+)$. Then there is a constant $C_\beta > 0$ such that, for any permutation $\pi: \{1,2,3\} \to \{1,2,3\}$, the following estimates hold for any $t \in [0,T)$:*

$$\|L_2^{\#}(f_1,f_2)(t)\|_{L_{x,v}^1}, \quad \|G_2^{\#}(f_1,f_2)(t)\|_{L_{x,v}^1} \leq C_\beta |||f_{\pi_1}^{\#}|||_\infty \|f_{\pi_2}^{\#}(t) p_{\gamma_2,\gamma_3}\|_{L_{x,v}^1}, \tag{3.24}$$

$$\|L_3^\#(f_1,f_2,f_3)(t)\|_{L^1_{x,v}},\quad \|G_3^\#(f_1,f_2,f_3)(t)\|_{L^1_{x,v}} \le C_\beta |||f_{\pi_1}^\#|||_\infty |||f_{\pi_2}^\#|||_\infty \times \|f_{\pi_3}^\#(t)p_{\gamma_2,\gamma_3}\|_{L^1_{x,v}}, \tag{3.25}$$

$$\|L^\#(f_1,f_2,f_3)(t)\|_{L^1_{x,v}},\quad \|G^\#(f_1,f_2,f_3)(t)\|_{L^1_{x,v}} \le C_\beta |||f_{\pi_1}^\#|||_\infty (1+|||f_{\pi_2}^\#|||_\infty) \times \|f_{\pi_3}^\#(t)p_{\gamma_2,\gamma_3}\|_{L^1_{x,v}}. \tag{3.26}$$

Proof. By (3.1), the triangle inequality and Lemma 3.3 (ii), the proof of (3.24)–(3.26) for the loss term reduces to showing the following estimates:

$$\|L_2^\#(f_1,f_2)(t)\|_{L^1_{x,v}} \le C_\beta |||f_1^\#|||_\infty \|f_2^\# p_{\gamma_2,\gamma_3}\|_{L^1_{x,v}}, \tag{3.27}$$

$$\|L_3^\#(f_1,f_2,f_3)(t)\|_{L^1_{x,v}} \le C_\beta |||f_1^\#|||_\infty |||f_3^\#|||_\infty \|f_2^\# p_{\gamma_2,\gamma_3}\|_{L^1_{x,v}}, \tag{3.28}$$

$$\|L_3^\#(f_1,f_2,f_3)(t)\|_{L^1_{x,v}} \le C_\beta |||f_1^\#|||_\infty |||f_2^\#|||_\infty \|f_3^\# p_{\gamma_2,\gamma_3}\|_{L^1_{x,v}}. \tag{3.29}$$

- Proof of (3.27): Performing the involutionary change of variables $(v,v_1)\to(v_1,v)$ and using (2.10), for any $t\in[0,T)$, we have

$$\|L_2(f_1,f_2)(t)\|_{L^1_{x,v}} = \|L_2(f_2,f_1)(t)\|_{L^1_{x,v}} \Rightarrow \|L_2^\#(f_1,f_2)(t)\|_{L^1_{x,v}} = \|L_2^\#(f_2,f_1)(t)\|_{L^1_{x,v}}.$$

The claim comes from Lemma 3.3 (ii).

- Proof of (3.28): Here the proof is subtler because the inner product $\bar{\boldsymbol{u}}\cdot\boldsymbol{\omega}$ is not symmetric upon renaming the velocities. However, we will strongly rely on the fact that the expression

$$|\tilde{\boldsymbol{u}}|^2 = |v-v_1|^2 + |v-v_2|^2 + |v_1-v_2|^2,$$

given in (2.22), is symmetric with respect to the inputs v, v_1, v_2.

Since $f_1^\#, f_3^\# \in L^\infty([0,T),\mathcal{M}^+_{\alpha,\beta})$, for any $t\in[0,T)$ and a.e. $(x,v)\in\mathbb{R}^d\times\mathbb{R}^d$, we have

$$0 \le f_i(t,x,v) \le |||f_i^\#|||_\infty e^{-\alpha|x-tv|^2-\beta|v|^2} \le |||f_i^\#|||_\infty e^{-\beta|v|^2} \quad \forall i\in\{1,3\}. \tag{3.30}$$

Using (2.67), the change of variables $(v,v_1)\to(v_1,v)$, bound (3.30), Lemma 3.2 (ii) and the fact that p_{γ_2,γ_3} is invariant in space, we obtain

$$\begin{aligned}
\|L_3^\#(f_1,f_2,f_3)(t)\|_{L^1_{x,v}} &= \|L_3(f_1,f_2,f_3)\|_{L^1_{x,v}} \\
&\le \|b_3\|_{L^1(\mathbb{S}^{2d-1})} \int_{\mathbb{R}^{4d}} |\tilde{\boldsymbol{u}}|^{\gamma_3} |f_1(t,x,v)|\,|f_2(t,x,v_1)|\,|f_3(t,x,v_2)|\,dv_1\,dv_2\,dv\,dx \\
&= \|b_3\|_{L^1(\mathbb{S}^{2d-1})} \int_{\mathbb{R}^{4d}} (|v-v_1|^2+|v-v_2|^2+|v_1-v_2|^2)^{\gamma_3/2} |f_1(t,x,v)| \\
&\qquad \times |f_2(t,x,v_1)|\,|f_3(t,x,v_2)|\,dv_1\,dv_2\,dv\,dx \\
&= \|b_3\|_{L^1(\mathbb{S}^{2d-1})} \int_{\mathbb{R}^{4d}} (|v-v_1|^2+|v-v_2|^2+|v_1-v_2|^2)^{\gamma_3/2} |f_2(t,x,v)| \\
&\qquad \times |f_1(t,x,v_1)|\,|f_3(t,x,v_2)|\,dv_1\,dv_2\,dv\,dx \\
&\le \|b_3\|_{L^1(\mathbb{S}^{2d-1})} \int_{\mathbb{R}^{4d}} |\tilde{\boldsymbol{u}}|^{\gamma_3} |f_2(t,x,v)|\,|f_1(t,x,v_1)|\,|f_3(t,x,v_2)|\,dv_1\,dv_2\,dv\,dx
\end{aligned}$$

$$= \|b_3\|_{L^1(\mathbb{S}^{2d-1})} \int_{\mathbb{R}^d\times\mathbb{R}^d} |f_2(t,x,v)| \int_{\mathbb{R}^{2d}} |\tilde{\boldsymbol{u}}|^{\gamma_3} |f_1(t,x,v_1)| \times |f_3(t,x,v_2)|\, dv_1\, dv_2\, dv\, dx$$
$$\leq \|b_3\|_{L^1(\mathbb{S}^{2d-1})} |||f_1^{\#}|||_\infty |||f_3^{\#}|||_\infty \int_{\mathbb{R}^d\times\mathbb{R}^d} |f_2(t,x,v)| \times \int_{\mathbb{R}^{2d}} |\tilde{\boldsymbol{u}}|^{\gamma_3} e^{-\beta(|v_1|^2+|v_2|^2)}\, dv_1\, dv_2\, dv\, dx$$
$$\leq C_\beta |||f_1^{\#}|||_\infty |||f_3^{\#}|||_\infty \|f_2(t) p_{\gamma_2,\gamma_3}\|_{L^1_{x,v}}$$
$$= C_\beta |||f_1^{\#}|||_\infty |||f_3^{\#}|||_\infty \|(f_2(t) p_{\gamma_2,\gamma_3})^{\#}\|_{L^1_{x,v}}$$
$$= C_\beta |||f_1^{\#}|||_\infty |||f_3^{\#}|||_\infty \|f_2^{\#}(t) p_{\gamma_2,\gamma_3}\|_{L^1_{x,v}}.$$

- Proof of (3.29): This follows in a similar way to the proof of (3.28).

Estimates (3.24)–(3.26) for the loss operators follow. Estimates for the gain operators follow from (3.1) and the estimates for the loss operators. The proof is complete. ■

Proposition 3.4 also implies an L^1-continuity result for the transported gain and loss operators.

Corollary 3.5. *Let $0 < T \leq \infty$ and $\alpha, \beta > 0$. For $i \in \{1,2,3\}$, consider some sequences $(f_{i,n})_n \subseteq \mathcal{F}_T^+$ and $f_i \in \mathcal{F}_T^+$ such that $f_{i,n}^{\#}(t) \xrightarrow{\mathcal{M}_{\alpha,\beta}} f_i^{\#}(t)$ for all $t \in [0,T)$. Then, for all $t \in [0,T)$, the following convergence holds:*

$$\big(L^{\#}(f_{1,n}, f_{2,n}, f_{3,n})(t), G^{\#}(f_{1,n}, f_{2,n}, f_{3,n})(t)\big) \xrightarrow{L^1_{x,v}} \big(L^{\#}(f_1, f_2, f_3)(t), G^{\#}(f_1, f_2, f_3)(t)\big) \text{ as } n \to \infty. \tag{3.31}$$

Proof. Fix $t \in [0,T)$. Since $f_{i,n}^{\#}(t) \xrightarrow{\mathcal{M}_{\alpha,\beta}} f_i^{\#}(t)$, for any $i \in \{1,2,3\}$, we have

$$f_{i,n}^{\#}(t) \xrightarrow{\text{a.e.}} f_i^{\#}(t), \qquad \sup_{n\in\mathbb{N}} \{|f_{i,n}^{\#}(t)|, |f_i^{\#}(t)|\} \leq C M_{\alpha,\beta}, \tag{3.32}$$

for some constant $C > 0$. Thus

$$f_{i,n}(t) \xrightarrow{\text{a.e.}} f_i(t), \qquad \sup_{n\in\mathbb{N}} \{|f_{i,n}(t)|, |f_i(t)|\} \leq C M_{\alpha,\beta}^{-\#}(t). \tag{3.33}$$

Let us first prove (3.31) for the loss case. By (2.37) and the triangle inequality, it suffices to prove

$$\|L_2^{\#}(f_{1,n}, f_{2,n})(t) - L_2^{\#}(f_1, f_2)(t)\|_{L^1_{x,v}} \xrightarrow{n\to\infty} 0, \tag{3.34}$$
$$\|L_3^{\#}(f_{1,n}, f_{2,n}, f_{3,n})(t) - L_3^{\#}(f_1, f_2, f_3)(t)\|_{L^1_{x,v}} \xrightarrow{n\to\infty} 0. \tag{3.35}$$

- Proof of (3.34): Using bilinearity of $L_2^{\#}$, bound (3.33) and monotonicity of $L_2^{\#}$, we have

$$\|L_2^{\#}(f_{1,n}, f_{2,n})(t) - L_2^{\#}(f_1, f_2)(t)\|_{L^1_{x,v}}$$
$$\leq \|L_2^{\#}(f_{1,n} - f_1, f_{2,n})(t)\|_{L^1_{x,v}} + \|L_2^{\#}(f_1, f_{2,n} - f_2)(t)\|_{L^1_{x,v}}$$

$$\leq C\|L_2^\#(f_{1,n}-f_1, M_{\alpha,\beta}^{-\#})(t)\|_{L^1_{x,v}} + C\|L_2^\#(M_{\alpha,\beta}^{-\#}, f_{2,n}-f_2)(t)\|_{L^1_{x,v}}$$
$$\leq C_\beta(\|(f_{1,n}^\#(t)-f_1^\#(t))p_{\gamma_2,\gamma_3}\|_{L^1_{x,v}} + C_\beta\|(f_{2,n}^\#(t)-f_2^\#(t))p_{\gamma_2,\gamma_3}\|_{L^1_{x,v}}), \quad (3.36)$$

where to obtain the last inequality we use (3.24) from Proposition 3.4 and $\|M_{\alpha,\beta}\|_{\mathcal{M}_{\alpha,\beta}}=1$.

By (3.32) and the dominated convergence theorem, each of the terms in (3.36) goes to zero as $n\to\infty$ and (3.34) is proved.

- Proof of (3.35): Using trilinearity of $L_3^\#$, bound (3.33) and monotonicity of $L_3^\#$, we have

$$\begin{aligned}
&\|L_3^\#(f_{1,n}, f_{2,n}, f_{3,n})(t) - L_3^\#(f_1, f_2, f_3)(t)\|_{L^1_{x,v}} \\
&\quad \leq \|L_3^\#(f_{1,n}-f_1, f_{2,n}, f_{3,n})(t)\|_{L^1_{x,v}} + \|L_3^\#(f_1, f_{2,n}-f_2, f_{3,n})(t)\|_{L^1_{x,v}} \\
&\qquad + \|L_3^\#(f_1, f_2, f_{3,n}-f_3)\|_{L^1_{x,v}} \\
&\quad \leq C\|L_3^\#(f_{1,n}-f_1, M_{\alpha,\beta}^{-\#}, M_{\alpha,\beta}^{-\#})(t)\|_{L^1_{x,v}} + C\|L_3^\#(M_{\alpha,\beta}^{-\#}, f_{2,n}-f_2, M_{\alpha,\beta}^{-\#})(t)\|_{L^1_{x,v}} \\
&\qquad + C\|L_3^\#(M_{\alpha,\beta}^{-\#}, M_{\alpha,\beta}^{-\#}, f_{3,n}-f_3)\|_{L^1_{x,v}} \\
&\quad \leq C_\beta(\|(f_{1,n}^\#(t)-f_1^\#(t))p_{\gamma_2,\gamma_3}\|_{L^1_{x,v}} + C_\beta\|(f_{2,n}^\#(t)-f_2^\#(t))p_{\gamma_2,\gamma_3}\|_{L^1_{x,v}} \\
&\qquad + C_\beta\|(f_{3,n}^\#(t)-f_3^\#(t))p_{\gamma_2,\gamma_3}\|_{L^1_{x,v}}),
\end{aligned} \quad (3.37)$$

where to obtain the last inequality we use (3.25) from Proposition 3.4 and $\|M_{\alpha,\beta}\|_{\mathcal{M}_{\alpha,\beta}}=1$. By (3.32) and the dominated convergence theorem, each of the terms in (3.37) goes to zero as $n\to\infty$ and (3.35) is proved. Combining (3.34)–(3.35), we obtain (3.31).

The gain operator convergence follows by a similar argument. ∎

3.4. A global estimate on the time average of the transported gain and loss operators

Here, we prove Proposition 3.7, which provides upper global bounds for the time average of the transported operators. These estimates will be essential to prove that the necessary beginning condition (4.49) for the convergence of the iteration holds globally in time for small enough initial data (see Section 5). For the binary case and soft potentials, these bounds were established in [2]. However, the presence of the ternary collisional operator requires new treatment which strongly relies on the properties of ternary interactions.

Before stating Proposition 3.7, we provide the following auxiliary estimate for the time integral of a traveling Maxwellian which will be used in the proof of the result for $n=d$ in the binary case and $n=2d$ in the ternary case.

Lemma 3.6. *Let $n\in\mathbb{N}$, $x_0,u_0\in\mathbb{R}^n$, with $u_0\neq 0$ and $\alpha>0$. Then the following estimate holds:*

$$\int_0^\infty e^{-\alpha|x_0-\tau u_0|^2}\,d\tau \leq \frac{\sqrt{\pi}}{2}\alpha^{-1/2}|u_0|^{-1}.$$

Proof. By the triangle inequality we have

$$\big|\tau|u_0|-|x_0|\big| \leq |x_0-\tau u_0| \Rightarrow e^{-\alpha|x_0-\tau u_0|^2} \leq e^{-\alpha(\tau|u_0|-|x_0|)^2} \quad \forall \tau\geq 0.$$

Therefore, integrating in τ, we obtain

$$\int_0^\infty e^{-\alpha|x_0-\tau u_0|^2}\,d\tau \le \int_0^\infty e^{-\alpha(\tau|u_0|-|x_0|)^2}\,d\tau \le \alpha^{-1/2}|u_0|^{-1}\int_0^\infty e^{-y^2}\,dy \le \frac{\sqrt{\pi}}{2}\alpha^{-1/2}|u_0|^{-1},$$

and the estimate is proved. ∎

We now state and prove Proposition 3.7. Given $f \in L^\infty([0,T),\mathcal{M}_{\alpha,\beta})$, recall from (2.63) the norm

$$|||f|||_\infty = \sup_{t\in[0,T)} \|f(t)\|_{\mathcal{M}_{\alpha,\beta}}.$$

Proposition 3.7. *Let* $0<T\le\infty$ *and* $\alpha,\beta>0$. *Then, for all* $f,g,h\in\mathcal{F}_T$ *with* $f^\#,g^\#,h^\#\in L^\infty([0,T),\mathcal{M}_{\alpha,\beta})$, *the following bounds hold for any* $t\in[0,T)$*:*

- *For the binary operators,*

$$\int_0^t |L_2^\#(f,g)(\tau)|\,d\tau,\quad \int_0^t |G_2^\#(f,g)(\tau)|\,d\tau \le K_\beta\alpha^{-1/2}M_{\alpha,\beta}|||f^\#|||_\infty|||g^\#|||_\infty; \quad (3.38)$$

- *for the ternary operators,*

$$\int_0^t |L_3^\#(f,g,h)(\tau)|\,d\tau,\quad \int_0^t |G_3^\#(f,g,h)(\tau)|\,d\tau \le K_\beta\alpha^{-1/2}M_{\alpha,\beta} \times |||f^\#|||_\infty|||g^\#|||_\infty|||h^\#|||_\infty; \quad (3.39)$$

- *for the mixed operators,*

$$\int_0^t |L^\#(f,g,h)|(\tau)\,d\tau,\quad \int_0^t |G^\#(f,g,h)(\tau)|\,d\tau \le K_\beta\alpha^{-1/2}M_{\alpha,\beta} \times |||f^\#|||_\infty|||g^\#|||_\infty(1+|||h^\#|||_\infty); \quad (3.40)$$

where

$$K_\beta = C_d\Big[\|b_2\|_{L^1(\mathbb{S}^{d-1})}\Big(\beta^{-d/2}+\frac{1}{d+\gamma_2-1}\Big) + \|b_3\|_{L^1(\mathbb{S}^{2d-1})}\Big(\beta^{-d}+\frac{1}{2d+\gamma_3-1}\Big)\Big]. \quad (3.41)$$

Proof. We prove each of the estimates separately.

- Proof of (3.38): As mentioned above, these bounds were established for the soft potential case in [2]. Here we also treat the hard potential case. Since $L_2^\#$, $G_2^\#$ are bilinear, we may assume without loss of generality that

$$|||f^\#|||_\infty = |||g^\#|||_\infty = 1. \quad (3.42)$$

Let us first prove it for the loss term. For any $t \in [0, T)$ and a.e. $(x, v) \in \mathbb{R}^{2d}$, relation (3.42), followed by an application of Lemma 3.6 for $n = d$, $x_0 = x$, $u_0 = u$, the fact that $-d + 1 < \gamma_2 \leq 1$ and an application of Lemma 3.2 (i) for $q_2 = \gamma_2 - 1$ imply

$$\begin{aligned}
&\int_0^t |L_2^\#(f, g)(\tau, x, v)|\, d\tau \\
&\quad \leq \|b_2\|_{L^1(\mathbb{S}^{d-1})} \int_0^t \int_{\mathbb{R}^d} |u|^{\gamma_2} |f(\tau, x + \tau v, v)|\, |g(\tau, x + \tau v, v_1)|\, dv_1\, d\tau \\
&\quad = \|b_2\|_{L^1(\mathbb{S}^{d-1})} \int_0^t \int_{\mathbb{R}^d} |u|^{\gamma_2} |f^\#(\tau, x, v)|\, |g^\#(\tau, x + \tau(v - v_1), v_1)|\, d\omega\, dv_1\, d\tau \\
&\quad \leq \|b_2\|_{L^1(\mathbb{S}^{d-1})} M_{\alpha,\beta}(x, v) \int_0^t \int_{\mathbb{R}^d} |u|^{\gamma_2} e^{-\alpha|x+\tau(v-v_1)|^2 - \beta|v_1|^2}\, dv_1\, d\tau \\
&\quad \leq \|b_2\|_{L^1(\mathbb{S}^{d-1})} M_{\alpha,\beta}(x, v) \int_{\mathbb{R}^d} |u|^{\gamma_2} e^{-\beta|v_1|^2} \int_0^\infty e^{-\alpha|x-\tau u|^2}\, d\tau\, dv_1 \\
&\quad \leq \|b_2\|_{L^1(\mathbb{S}^{d-1})} \frac{\sqrt{\pi}}{2} \alpha^{-1/2} M_{\alpha,\beta}(x, v) \int_{\mathbb{R}^d} |u|^{\gamma_2 - 1} e^{-\beta|v_1|^2}\, dv_1 \\
&\quad \leq \|b_2\|_{L^1(\mathbb{S}^{d-1})} \frac{\sqrt{\pi}}{2} \widetilde{K}^2_{\beta,\gamma_2-1} \alpha^{-1/2} M_{\alpha,\beta}(x, v) \\
&\quad \leq C_d \|b_2\|_{L^1(\mathbb{S}^{d-1})} \alpha^{-1/2} \Big(\beta^{-d/2} + \frac{1}{d + \gamma_2 - 1}\Big) M_{\alpha,\beta}(x, v),
\end{aligned} \tag{3.43}$$

where C_d is an appropriate constant depending on the dimension d. To obtain (3.43) we used (3.3) and the fact that $q_2 = \gamma_2 - 1 \leq 0$. Estimate (3.38) for the loss term follows.

To prove (3.38) for the gain term we will use the identity

$$|x + \tau(v - v')|^2 + |x + \tau(v - v_1')|^2 = |x|^2 + |x + \tau(v - v_1)|^2, \tag{3.44}$$

which follows from the binary conservation of momentum and energy,

$$\begin{aligned} v' + v_1' &= v + v_1, \\ |v'|^2 + |v_1'|^2 &= |v|^2 + |v_1|^2. \end{aligned} \tag{3.45}$$

For any $t \in [0, T)$ and a.e. $(x, v) \in \mathbb{R}^{2d}$, (3.42) and (3.44)–(3.45) imply

$$\begin{aligned}
&\int_0^t |G_2^\#(f, g)(\tau, x, v)|\, d\tau \\
&\leq \int_0^t \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} |u|^{\gamma_2} b_2(\hat{u}\cdot\omega) |f(\tau, x + \tau v, v')|\, |g(\tau, x + \tau v, v_1')|\, d\omega\, dv_1\, d\tau \\
&= \int_0^t \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} |u|^{\gamma_2} b_2(\hat{u}\cdot\omega) |f^\#(\tau, x + \tau(v - v'), v')| \\
&\qquad\qquad \times |g^\#(\tau, x + \tau(v - v_1'), v_1')|\, d\omega\, dv_1\, d\tau \\
&\leq \int_0^t \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} |u|^{\gamma_2} b_2(\hat{u}\cdot\omega) e^{-\alpha(|x+\tau(v-v')|^2 + |x+\tau(v-v_1')|^2)} e^{-\beta(|v'|^2+|v_1'|^2)}\, d\omega\, dv_1\, d\tau \\
&= \|b_2\|_{L^1(\mathbb{S}^{d-1})} M_{\alpha,\beta}(x, v) \int_0^t \int_{\mathbb{R}^d} |u|^{\gamma_2} e^{-\alpha|x+\tau(v-v_1)|^2 - \beta|v_1|^2}\, dv_1\, d\tau.
\end{aligned} \tag{3.46}$$

Combining (3.46) with an identical argument to the one used for the loss term, we obtain

$$\int_0^t |G_2^\#(f,g)(\tau,x,v)|\,d\tau \le C_d \|b_2\|_{L^1(\mathbb{S}^{d-1})}\alpha^{-1/2}\Big(\beta^{-d/2} + \frac{1}{d+\gamma_2-1}\Big) M_{\alpha,\beta}(x,v), \tag{3.47}$$

and estimate (3.38) for the gain term follows.

- Proof of (3.39): Since $L_3^\#$, $G_3^\#$ are trilinear, we may assume without loss of generality that

$$|||f^\#|||_\infty = |||g^\#|||_\infty = |||h^\#|||_\infty = 1. \tag{3.48}$$

Let us first prove (3.39) for the loss term. For any $t \in [0,T)$ and a.e. $(x,v) \in \mathbb{R}^{2d}$, (3.48) implies

$$\begin{aligned}
&\int_0^t |L_3^\#(f,g,h)(\tau,x,v)|\,d\tau \\
&\quad \le \|b_3\|_{L^1(\mathbb{S}^{2d-1})} \int_0^t \int_{\mathbb{R}^{2d}} |\tilde{\boldsymbol{u}}|^{\gamma_3} |f(\tau,x+\tau v,v)|\,|g(\tau,x+\tau v,v_1)| \\
&\qquad\qquad \times |h(\tau,x+\tau v,v_2)|\,dv_1\,dv_2\,d\tau \\
&\quad = \|b_3\|_{L^1(\mathbb{S}^{2d-1})} \int_0^t \int_{\mathbb{R}^{2d}} |\tilde{\boldsymbol{u}}|^{\gamma_3} |f^\#(\tau,x,v)|\,|g^\#(\tau,x+\tau(v-v_1),v_1)| \\
&\qquad\qquad \times |h^\#(\tau,x+\tau(v-v_2),v_2)|\,dv_1\,dv_2\,d\tau \\
&\quad \le \|b_3\|_{L^1(\mathbb{S}^{2d-1})} M_{\alpha,\beta}(x,v) \int_0^t \int_{\mathbb{R}^{2d}} |\tilde{\boldsymbol{u}}|^{\gamma_3} e^{-\alpha(|x+\tau(v-v_1)|^2+|x+\tau(v-v_2)|^2)} \\
&\qquad\qquad \times e^{-\beta(|v_1|^2+|v_2|^2)}\,dv_1\,dv_2\,d\tau \\
&\quad \le \|b_3\|_{L^1(\mathbb{S}^{2d-1})} M_{\alpha,\beta}(x,v) \int_{\mathbb{R}^{2d}} |\tilde{\boldsymbol{u}}|^{\gamma_3} e^{-\beta(|v_1|^2+|v_2|^2)} \\
&\qquad\qquad \times \int_0^\infty e^{-\alpha|\boldsymbol{x}-\tau\boldsymbol{u}|^2}\,d\tau\,dv_1\,dv_2,
\end{aligned} \tag{3.49}$$

where in (3.49) we use the notation

$$\boldsymbol{x} := \begin{pmatrix} x \\ x \end{pmatrix} \in \mathbb{R}^{2d}, \quad \boldsymbol{u} = \begin{pmatrix} v_1 - v \\ v_2 - v \end{pmatrix} \in \mathbb{R}^{2d}.$$

Notice that by the triangle inequality and Young's inequality we have

$$\begin{aligned}
|\tilde{\boldsymbol{u}}|^2 &= |v-v_1|^2 + |v-v_2|^2 + |v_1-v_2|^2 \\
&\le |v-v_1|^2 + |v-v_2|^2 + (|v-v_1| + |v-v_2|)^2 \\
&\le 3(|v-v_1|^2 + |v-v_2|^2) \\
&= 3|\boldsymbol{u}|^2.
\end{aligned} \tag{3.50}$$

Therefore, an application of Lemma 3.6 for $n = 2d$, $x_0 = \boldsymbol{x}$, $u_0 = \boldsymbol{u}$, followed by (3.50), the fact that $-2d+1 < \gamma_3 \le 1$, and an application of Lemma 3.2 (ii) for $q_3 = \gamma_3 - 1$

yield

$$
\begin{aligned}
&\int_0^t |L_3^\#(f,g,h)(\tau,x,v)|\,d\tau \\
&\quad \leq \|b_3\|_{L^1(\mathbb{S}^{2d-1})} M_{\alpha,\beta}(x,v) \int_{\mathbb{R}^{2d}} |\tilde{\boldsymbol{u}}|^{\gamma_3} e^{-\beta(|v_1|^2+|v_2|^2)} \int_0^\infty e^{-\alpha|\boldsymbol{x}-\tau\boldsymbol{u}|^2}\,d\tau\,dv_1\,dv_2 \\
&\quad \leq \|b_3\|_{L^1(\mathbb{S}^{2d-1})} \frac{\sqrt{\pi}}{2}\alpha^{-1/2} M_{\alpha,\beta}(x,v) \int_{\mathbb{R}^{2d}} |\tilde{\boldsymbol{u}}|^{\gamma_3} |\boldsymbol{u}|^{-1} e^{-\beta(|v_1|^2+|v_2|^2)}\,dv_1\,dv_2 \\
&\quad \leq \|b_3\|_{L^1(\mathbb{S}^{2d-1})} \frac{\sqrt{3\pi}}{6}\alpha^{-1/2} M_{\alpha,\beta}(x,v) \int_{\mathbb{R}^{2d}} |\tilde{\boldsymbol{u}}|^{\gamma_3-1} e^{-\beta(|v_1|^2+|v_2|^2)}\,dv_1\,dv_2 \\
&\quad \leq \|b_3\|_{L^1(\mathbb{S}^{2d-1})} \frac{\sqrt{3\pi}}{6}\tilde{K}^3_{\beta,\gamma_3-1}\alpha^{-1/2} M_{\alpha,\beta}(x,v) \\
&\quad \leq C_d \|b_3\|_{L^1(\mathbb{S}^{2d-1})}\alpha^{-1/2}\Big(\beta^{-d} + \frac{1}{2d+\gamma_3-1}\Big) M_{\alpha,\beta}(x,v),
\end{aligned} \tag{3.51}
$$

where C_d is an appropriate constant depending on the dimension d. To obtain (3.51), we used (3.5) and the fact that $q_3 = \gamma_3 - 1 \leq 0$. Estimate (3.39) for the loss term follows.

To prove (3.39) for the gain term we will use the identity

$$
\begin{aligned}
&|x+\tau(v-v^*)|^2 + |x+\tau(v-v_1^*)|^2 + |x+\tau(v-v_2^*)|^2 \\
&\qquad = |x|^2 + |x+\tau(v-v_1)|^2 + |x+\tau(v-v_2)|^2,
\end{aligned} \tag{3.52}
$$

following from the ternary conservation of momentum and energy,

$$
\begin{aligned}
v^* + v_1^* + v_2^* &= v + v_1 + v_2, \\
|v^*|^2 + |v_1^*|^2 + |v_2^*|^2 &= |v|^2 + |v_1|^2 + |v_2|^2.
\end{aligned} \tag{3.53}
$$

For any $t \in [0,T)$ and a.e. $(x,v) \in \mathbb{R}^{2d}$, by (3.48) and (3.52)–(3.53) we obtain

$$
\begin{aligned}
&\int_0^t |G_3^\#(f,g,h)(\tau)|\,d\tau \\
&\quad \leq \int_0^t \int_{\mathbb{S}^{2d-1}\times\mathbb{R}^{2d}} |\tilde{\boldsymbol{u}}|^{\gamma_3} b_3(\bar{\boldsymbol{u}}\cdot\boldsymbol{\omega},\omega_1\cdot\omega_2)|f(\tau,x+\tau v,v^*)|\,|g(\tau,x+\tau v,v_1^*)| \\
&\qquad\qquad \times |h(\tau,x+\tau v,v_2^*)|\,d\omega_1\,d\omega_2\,dv_1\,dv_2\,d\tau \\
&\quad = \int_0^t \int_{\mathbb{S}^{2d-1}\times\mathbb{R}^{2d}} |\tilde{\boldsymbol{u}}|^{\gamma_3} b_3(\bar{\boldsymbol{u}}\cdot\boldsymbol{\omega},\omega_1\cdot\omega_2)|f^\#(\tau,x+\tau(v-v^*),v^*)| \\
&\qquad\qquad \times |g^\#(\tau,x+\tau(v-v_1^*),v_1^*)|\,|h^\#(\tau,x+\tau(v-v_2^*),v_2^*)|\,d\omega_1\,d\omega_2\,dv_1\,dv_2\,d\tau \\
&\quad \leq \int_0^t \int_{\mathbb{S}^{2d-1}\times\mathbb{R}^{2d}} |\tilde{\boldsymbol{u}}|^{\gamma_3} b_3(\bar{\boldsymbol{u}}\cdot\boldsymbol{\omega},\omega_1\cdot\omega_2) e^{-\alpha(|x+\tau(v-v^*)|^2+|x+\tau(v-v_1^*)|^2+|x+\tau(v-v_2^*)|^2)} \\
&\qquad\qquad \times e^{-\beta(|v^*|^2+|v_1^*|^2+|v_2^*|^2)}\,d\omega_1\,d\omega_2\,dv_1\,dv_2\,d\tau \\
&\quad = \|b_3\|_{L^1(\mathbb{S}^{2d-1})} M_{\alpha,\beta}(x,v) \int_0^t \int_{\mathbb{R}^{2d}} |\tilde{\boldsymbol{u}}|^{\gamma_3} e^{-\alpha(|x+\tau(v-v_1)|^2+|x+\tau(v-v_2)|^2)} \\
&\qquad\qquad \times e^{-\beta(|v_1|^2+|v_2|^2)}\,dv_1\,dv_2\,d\tau
\end{aligned}
$$

$$\leq \|b_3\|_{L^1(\mathbb{S}^{2d-1})} M_{\alpha,\beta}(x,v) \int_{\mathbb{R}^{2d}} |\tilde{\boldsymbol{u}}|^{\gamma_3} e^{-\beta(|v_1|^2+|v_2|^2)} \times \int_0^\infty e^{-\alpha|\boldsymbol{x}-\tau\boldsymbol{u}|^2}\, d\tau\, dv_1\, dv_2. \tag{3.54}$$

Combining (3.54) with an identical argument to the one used for the loss case, we obtain

$$\int_0^t |G_3^{\#}(f,g,h)(\tau)|\, d\tau \leq C_d \|b_3\|_{L^1(\mathbb{S}^{2d-1})} \alpha^{-1/2} \Big(\beta^{-d} + \frac{1}{2d+\gamma_3-1}\Big) M_{\alpha,\beta}(x,v), \tag{3.55}$$

and estimate (3.39) for the gain term follows.

- Proof of (3.40): It follows directly from (3.38)–(3.39). ■

4. The Kaniel–Shinbrot iteration scheme and the associated linear problem

In this section we present the Kaniel–Shinbrot iteration scheme, which will then be used as the heart of the construction of a global solution in Section 5. This scheme is motivated by [13, 14]. However, the presence of the ternary collisional operator, in addition to the binary collisional operator, required a modification of the original construction.

In particular, we outline the construction of the Kaniel–Shinbrot iteration that we will use in this paper. Formally speaking, given an initial data f_0, we construct an increasing sequence $(l_n)_{n\in\mathbb{N}}$ and a decreasing sequence $(u_n)_{n\in\mathbb{N}}$, with $l_n \leq u_n$, through the iteration

$$\begin{aligned} \frac{dl_n}{dt} + v\cdot\nabla_x l_n &= G(l_{n-1}, l_{n-1}, l_{n-1}) - L(l_n, u_{n-1}, u_{n-1}),\\ l_n(0) &= f_0, \end{aligned} \tag{4.1}$$

$$\begin{aligned} \frac{du_n}{dt} + v\cdot\nabla_x u_n &= G(u_{n-1}, u_{n-1}, u_{n-1}) - L(u_n, l_{n-1}, l_{n-1}),\\ u_n(0) &= f_0. \end{aligned} \tag{4.2}$$

We will see that the sequences l_n, u_n converge to the same limit, namely a function f, which will be the solution of the binary–ternary Boltzmann equation (1.1).

To make things rigorous, we first study an associated linear problem, and then inductively apply these results, together with the estimates derived in Section 3, to establish that the Kaniel–Shinbrot iteration converges to a solution of (1.1), provided that an appropriate beginning condition is satisfied. This solution will be unique in the class of functions uniformly bounded by a Maxwellian.

4.1. The associated linear problem

Here, we prove well-posedness for a linear problem associated to the iteration scheme (4.1)–(4.2). More precisely, given some functions of time g, h, we show well-posedness up to time $0 < T \le \infty$ of the linear problem

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = h - L(f, g, g), & (t, x, v) \in (0, T) \times \mathbb{R}^d \times \mathbb{R}^d, \\ f(0) = f_0, & (x, v) \in \mathbb{R}^d \times \mathbb{R}^d. \end{cases} \tag{4.3}$$

Definition 4.1. Let $0 < T \le \infty$, $\alpha, \beta > 0$, $f_0 \in L^{1,+}_{x,v}$, $g^\# \in L^\infty([0,T), \mathcal{M}^+_{\alpha,\beta})$ and $h^\# \in L^1_{\text{loc}}([0,T), L^{1,+}_{x,v})$. We say that a function $f \in \mathcal{F}^+_T$ with

(i) $f^\# \in C^0([0,T), L^{1,+}_{x,v})$,

(ii) $L^\#(f, g, g) \in L^1_{\text{loc}}([0,T), L^{1,+}_{x,v})$,

(iii) $f^\#$ is weakly differentiable and satisfies

$$\begin{cases} \dfrac{df^\#}{dt} + L^\#(f, g, g) = h^\#, \\ f^\#(0) = f_0, \end{cases} \tag{4.4}$$

is a mild solution of (4.3) in $[0, T)$ with initial data $f_0 \in L^{1,+}_{x,v}$.

Remark 4.2. The differential equation of (4.4) is interpreted as an equality of distributions since all terms involved belong to $L^1_{\text{loc}}([0,T), L^{1,+}_{x,v})$.

Remark 4.3. Remarks 2.5 and 2.6 imply that a mild solution f to (4.3) belongs to $C^0([0,T), L^{1,+}_{x,v})$.

For technical reasons, we first prove well-posedness of (4.3) under the additional assumptions

$$f_0 \in \mathcal{M}^+_{\alpha,\beta}, \quad 0 \le h^\#(t) \le Ce^{-t^2} M_{\alpha,\beta}, \quad \forall t \in [0, T), \tag{4.5}$$

for some constant $C > 0$. Clearly, if (4.5) holds, then $f_0 \in L^{1,+}_{x,v}$ and $h^\# \in L^1_{\text{loc}}([0,T), L^{1,+}_{x,v})$, thus (4.5) is a stronger assumption than those appearing in Definition 4.1. This additional assumption will be removed later using an approximation argument.

Lemma 4.4. *Let $0 < T \le \infty$ and $\alpha, \beta > 0$. Consider f_0, h satisfying (4.5) and $g^\# \in L^\infty([0,T), \mathcal{M}^+_{\alpha,\beta})$. Then there exists a mild solution f of (4.3) with $f^\# \in L^\infty([0,T), \mathcal{M}^+_{\alpha,\beta})$. Moreover, $\|f^\#(\cdot)\|_{L^1_{x,v}}$ is absolutely continuous and satisfies*

$$\|f^\#(t)\|_{L^1_{x,v}} + \int_0^t \|L^\#(f, g, g)(\tau)\|_{L^1_{x,v}}\, d\tau = \|f_0\|_{L^1_{x,v}} + \int_0^t \|h^\#(\tau)\|_{L^1_{x,v}}\, d\tau \quad \forall t \in [0, T). \tag{4.6}$$

Proof. Since $g^\# \in L^\infty([0,T), \mathcal{M}^+_{\alpha,\beta})$, Lemma 3.3 (i) implies

$$0 \le R^\#(g, g)(t) \le C_\beta |||g^\#|||_\infty (1 + |||g^\#|||_\infty) p_{\gamma_2,\gamma_3} \quad \forall t \in [0, T), \tag{4.7}$$

for some constant $C_\beta > 0$ depending on β. We define f by

$$f^\#(t) := f_0 \exp\Big(-\int_0^t R^\#(g,g)(\sigma)\,d\sigma\Big) + \int_0^t h^\#(\tau)\exp\Big(-\int_\tau^t R^\#(g,g)(\sigma)\,d\sigma\Big)\,d\tau, \quad t \in [0,T). \tag{4.8}$$

By (4.5), (4.7) and the fact $f_0 \in \mathcal{M}^+_{\alpha,\beta}$, $f^\#$ is well defined and satisfies the bound

$$0 \le f^\#(t) \le f_0 + \int_0^t h^\#(\tau)\,d\tau \le \Big(\|f_0\|_{\mathcal{M}_{\alpha,\beta}} + C\frac{\sqrt{\pi}}{2}\Big)M_{\alpha,\beta} \quad \forall t \in [0,T), \tag{4.9}$$

thus $f \ge 0$ and

$$f^\# \in L^\infty([0,T), \mathcal{M}^+_{\alpha,\beta}). \tag{4.10}$$

Let us now show that $f^\# \in C^0([0,T), L^{1,+}_{x,v})$. For any $t, s \in [0,T)$, expression (4.8) yields

$$\begin{aligned}|f^\#(t) - f^\#(s)| &= \bigg|\Big[f_0\exp\Big(-\int_0^s R^\#(g,g)(\sigma)\,d\sigma\Big) + \int_0^s h^\#(\tau)\exp\Big(-\int_\tau^s R^\#(g,g)(\sigma)\,d\sigma\Big)\,d\tau\Big] \\ &\quad\times\Big[\exp\Big(-\int_s^t R^\#(g,g)(\sigma)\,d\sigma\Big) - 1\Big] + \int_s^t h^\#(\tau)\exp\Big(-\int_\tau^t R^\#(g,g)(\sigma)\,d\sigma\Big)\,d\tau\bigg|,\end{aligned}$$

and therefore by (4.5), (4.7), we may find a positive constants $C_{f_0,g,h} > 0$ such that

$$|f^\#(t) - f^\#(s)| \le C_{f_0,g,h}M_{\alpha,\beta}(1 - e^{-C_{f_0,g,h}|t-s|p_{\gamma_2,\gamma_3}}) + C_{f_0,g,h}|t-s|M_{\alpha,\beta} \quad \forall t \in [0,T). \tag{4.11}$$

Using the elementary inequality $1 - e^{-x} \le x$, for all $x \ge 0$, we obtain

$$|f^\#(t) - f^\#(s)| \le 2C_{f_0,g,h}|t-s|p_{\gamma_2,\gamma_3}M_{\alpha,\beta} \quad \forall t \in [0,T). \tag{4.12}$$

Integrating (4.12) we obtain

$$\|f^\#(t) - f^\#(s)\|_{L^1_{x,v}} \le 2C_{f_0,g,h}|t-s| \quad \forall t,s \in [0,T), \tag{4.13}$$

since $p_{\gamma_2,\gamma_3}M_{\alpha,\beta} \in L^{1,+}_{x,v}$. We conclude that $f^\# \in C^0([0,T), L^{1,+}_{x,v})$, and therefore $f \in C^0([0,T), L^{1,+}_{x,v})$. In particular, bound (4.13) implies that f is actually Lipschitz continuous.

Since $f^\#, g^\# \in L^\infty([0,T), \mathcal{M}^+_{\alpha,\beta})$, Lemma 3.3 (ii) implies

$$L^\#(f,g,g) \in L^\infty([0,T), L^{1,+}_{x,v}) \subseteq L^1_{\text{loc}}([0,T), L^{1,+}_{x,v}). \tag{4.14}$$

Finally, by (4.5), (4.14), representation (4.8) and the dominated convergence theorem, we conclude that $f^\#$ is weakly differentiable and satisfies

$$\begin{cases}\dfrac{df^\#}{dt} + L^\#(f,g,g) = h^\#, \\ f^\#(0) = f_0,\end{cases} \tag{4.15}$$

and thus it is a mild solution of (4.3).

Integrating (4.15), the fundamental theorem of calculus and the fact that $f^\# \in C^0([0,T), L^{1,+}_{x,v})$, $L^\#(f,g,g)$ and $h^\# \in L^1_{\mathrm{loc}}([0,T), L^{1,+}_{x,v})$, imply

$$f^\#(t) + \int_0^t L^\#(f,g,g)(\tau)\,d\tau = f_0 + \int_0^t h^\#(\tau)\,d\tau \quad \forall t \in [0,T). \tag{4.16}$$

Using the non-negativity of all terms involved in (4.16), and Fubini's theorem, we obtain (4.6) and absolute continuity of $\|f(t)\|_{L^1_{x,v}}$ follows. The proof is complete. ∎

Since the gain operator does not satisfy (4.5), it will be convenient to relax assumption (4.5) to $f_0 \in L^{1,+}_{x,v}$, $h^\# \in L^1_{\mathrm{loc}}([0,T), L^{1,+}_{x,v})$. As in [14], the idea is to approximate f_0, $h^\#$ in the $L^1_{x,v}$-norm with a monotone sequence of solutions occurring from a repeated application of Lemma 4.4. We obtain the following well-posedness result.

Proposition 4.5. *Let $0 < T \le \infty$ and $\alpha, \beta > 0$. Consider $f_0 \in L^{1,+}_{x,v}$, $g^\# \in L^\infty([0,T), \mathcal{M}^+_{\alpha,\beta})$ and $h^\# \in L^1_{\mathrm{loc}}([0,T), L^{1,+}_{x,v})$. Then there exists a unique mild solution f of (4.3). In particular, $f^\#$ is given by*

$$f^\#(t) := f_0 \exp\left(-\int_0^t R^\#(g,g)(\sigma)\,d\sigma\right) + \int_0^t h^\#(\tau) \exp\left(-\int_\tau^t R^\#(g,g)(\sigma)\,d\sigma\right) d\tau, \quad t \in [0,T). \tag{4.17}$$

Proof.

- Existence: Given $n \in \mathbb{N}$, let us define

$$f_{0,n} := \begin{cases} f_0 & \text{if } f_0 \le n M_{\alpha,\beta}, \\ n M_{\alpha,\beta} & \text{if } f_0 > n M_{\alpha,\beta}, \end{cases} \tag{4.18}$$

and

$$h^\#_n(t) := \begin{cases} h^\#(t) & \text{if } h^\#(t) \le n e^{-t^2} M_{\alpha,\beta}, \\ n e^{-t^2} M_{\alpha,\beta} & \text{if } h^\#(t) > n e^{-t^2} M_{\alpha,\beta}. \end{cases} \tag{4.19}$$

It is clear that $f_{0,n}$, h_n satisfy condition (4.5) for all $n \in \mathbb{N}$ and that

$$0 \le f_{0,n} \nearrow f_0 \quad \text{as } n \to \infty, \tag{4.20}$$

$$\forall t \in [0,T): \quad 0 \le h^\#_n(t) \nearrow h^\#(t) \quad \text{as } n \to \infty. \tag{4.21}$$

Then the monotone convergence theorem yields that

$$\|f_{0,n}\|_{L^1_{x,v}} \nearrow \|f_0\|_{L^1_{x,v}} \quad \text{as } n \to \infty, \tag{4.22}$$

$$\forall t \in [0,T): \quad \|h^\#_n(t)\|_{L^1_{x,v}} \nearrow \|h^\#(t)\|_{L^1_{x,v}} \quad \text{as } n \to \infty. \tag{4.23}$$

Moreover, since $f_0 \in L^1_{x,v}$ and $h^\# \in L^1_{\mathrm{loc}}([0,T), L^{1,+}_{x,v})$, relations (4.20)–(4.21) and the dominated convergence theorem yield

$$f_{0,n} \xrightarrow{L^1_{x,v}} f_0 \quad \text{as } n \to \infty, \tag{4.24}$$

$$\text{for a.e. } t \in [0,T): \qquad h_n^\#(t) \xrightarrow{L^1_{x,v}} h^\#(t) \qquad \text{as } n \to \infty, \tag{4.25}$$

$$\forall t \in [0,T): \qquad \int_0^t h_n^\#(\tau)\,d\tau \xrightarrow{L^1_{x,v}} \int_0^t h^\#(\tau)\,d\tau \quad \text{as } n \to \infty. \tag{4.26}$$

Let $f_n \in \mathcal{F}_T^+$ be the mild solution to the problem

$$\begin{cases} \dfrac{df_n}{dt} + v \cdot \nabla_x f_n = h_n - L(f_n, g, g), \\ f_n(0) = f_{0,n}, \end{cases} \tag{4.27}$$

constructed in Lemma 4.4. Let us note that Lemma 4.4 is applicable for all $n \in \mathbb{N}$ since $f_{0,n}, h_n$ satisfy (4.5). Hence, $f_n^\#$ satisfies

$$\begin{cases} \dfrac{df_n^\#}{dt} + L^\#(f_n, g, g) = h_n^\#, \\ f_n^\#(0) = f_{0,n}, \end{cases} \tag{4.28}$$

and is given by the formula

$$\begin{aligned} f_n^\#(t) := {} & f_{0,n} \exp\Big(-\int_0^t R^\#(g,g)(\sigma)\,d\sigma\Big) \\ & + \int_0^t h_n^\#(\tau) \exp\Big(-\int_\tau^t R^\#(g,g)(\sigma)\,d\sigma\Big)\,d\tau, \quad t \in [0,T). \end{aligned} \tag{4.29}$$

Also by (4.6), given $t \in [0,T)$, we have the bound

$$\begin{aligned} \sup_{n\in\mathbb{N}} \|f_n^\#(t)\|_{L^1_{x,v}} & \le \sup_{n\in\mathbb{N}} \Big(\|f_{0,n}\|_{L^1_{x,v}} + \int_0^t \|h_n^\#(\tau)\|_{L^1_{x,v}}\,d\tau \Big) \\ & \le \|f_0\|_{L^1_{x,v}} + \int_0^t \|h^\#(\tau)\|_{L^1_{x,v}}\,d\tau < \infty, \end{aligned} \tag{4.30}$$

where to obtain the last bound we use (4.22)–(4.23), the fact that $R^\#(g,g) \ge 0$ (by monotonicity of $R^\#$ and $g \ge 0$), $f_0 \in L^1_{x,v}$ and $h^\# \in L^1_{\mathrm{loc}}([0,T), L^{1,+}_{x,v})$.

Since the sequences $(f_{0,n})_n$, $(h_n^\#(t))_n$ are increasing and non-negative for all $t \in [0,T)$, formula (4.29) implies that the sequence $(f_n^\#(t))_n$ is increasing for all $t \in [0,T)$. Let us define

$$f^\#(t) := \lim_{n\to\infty} f_n^\#(t).$$

Clearly $f \ge 0$. By the monotone convergence theorem and bound (4.30) we obtain that $f^\#(t) \in L^{1,+}_{x,v}$ for all $t \in [0,T)$. Then the dominated convergence theorem implies

$$\forall t \in [0,T): \quad f_n^\#(t) \xrightarrow{L^1_{x,v}} f^\#(t) \quad \text{as } n \to \infty. \tag{4.31}$$

Moreover, we have

$$\begin{aligned} \forall t \in [0,T): \quad L^\#(f_n, g, g)(t) & = f_n^\#(t) R^\#(g,g)(t) \\ & \nearrow f^\#(t) R^\#(g,g)(t) = L^\#(f,g,g)(t) \quad \text{as } n \to \infty, \end{aligned} \tag{4.32}$$

since $R^{\#}(g,g)(t) \geq 0$ by monotonicity of $R^{\#}$ and the fact that $g \geq 0$. By the monotone convergence theorem we obtain

$$\forall t \in [0,T): \quad \int_0^t \|L^{\#}(f_n,g,g)(\tau)\|_{L^1_{x,v}}\,d\tau \nearrow \int_0^t \|L^{\#}(f,g,g)(\tau)\|_{L^1_{x,v}}\,d\tau \quad \text{as } n\to\infty. \tag{4.33}$$

Therefore, for any $t \in [0,T)$, equation (4.6) implies

$$\int_0^t \|L^{\#}(f,g,g)(\tau)\|_{L^1_{x,v}}\,d\tau = \sup_{n\in\mathbb{N}} \int_0^t \|L^{\#}(f_n,g,g)(\tau)\|_{L^1_{x,v}}\,d\tau \tag{4.34}$$

$$\leq \sup_{n\in\mathbb{N}}\left(\|f_{0,n}\|_{L^1_{x,v}} + \int_0^t \|h_n^{\#}(\tau)\|_{L^1_{x,v}}\,d\tau\right)$$

$$\leq \|f_0\|_{L^1_{x,v}} + \int_0^t \|h^{\#}(\tau)\|_{L^1_{x,v}}\,d\tau < \infty, \tag{4.35}$$

since $f_0 \in L^1_{x,v}$ and $h^{\#} \in L^1_{\mathrm{loc}}([0,T), L^{1,+}_{x,v})$, and thus

$$L^{\#}(f,g,g)(t) \in L^1_{x,v} \quad \text{for a.e. } t \in [0,T), \tag{4.36}$$

$$L^{\#}(f,g,g) \in L^1_{\mathrm{loc}}([0,T), L^{1,+}_{x,v}). \tag{4.37}$$

By (4.32), (4.36) and the dominated convergence theorem, for a.e. $t \in [0,T)$ we have

$$L^{\#}(f_n,g,g)(t) \xrightarrow{L^1_{x,v}} L^{\#}(f,g,g)(t) \quad \text{as } n\to\infty, \tag{4.38}$$

and by (4.37) and another application of the dominated convergence theorem we obtain

$$\int_0^t L^{\#}(f_n,g,g)(\tau)\,d\tau \xrightarrow{L^1_{x,v}} \int_0^t L^{\#}(f,g,g)(\tau)\,d\tau \quad \forall t \in [0,T). \tag{4.39}$$

Since $f_n^{\#}$ satisfies (4.28), the fundamental theorem of calculus and the fact that $f_n^{\#} \in C^0([0,T), L^{1,+}_{x,v})$, $L^{\#}(f_n,g,g)$ and $h_n^{\#} \in L^1_{\mathrm{loc}}([0,T), L^{1,+}_{x,v})$ imply

$$f_n^{\#}(t) + \int_0^t L^{\#}(f_n,g,g)(\tau)\,d\tau = f_{0,n} + \int_0^t h_n^{\#}(\tau)\,d\tau \quad \forall t \in [0,T), \quad \forall n \in \mathbb{N}. \tag{4.40}$$

Using (4.31), (4.39), (4.24) and (4.26), we let $n \to \infty$ in (4.40) to obtain

$$f^{\#}(t) + \int_0^t L^{\#}(f,g,g)(\tau)\,d\tau = f_0 + \int_0^t h^{\#}(\tau)\,d\tau \quad \forall t \in [0,T), \quad \forall n \in \mathbb{N}, \tag{4.41}$$

and thus $f^{\#} \in C^0([0,T), L^{1,+}_{x,v})$, $f^{\#}$ is weakly differentiable and satisfies (4.4). We conclude that f is a mild solution of (4.3). Moreover, since $g \geq 0$ we may take the limit as $n \to \infty$ on both sides of (4.29) to obtain (4.17).

• Uniqueness: Since the problem is linear it suffices to show that if f is a solution of (4.3) with $f_0 = 0$ and $h = 0$, then $f = 0$.

Assume f is a mild solution of (4.3) with $f_0 = 0$ and $h = 0$, i.e. $f \geq 0$, $f^\# \in C^0([0,T), L^{1,+}_{x,v})$, $L^\#(f,g,g) \in L^1_{\text{loc}}([0,T), L^{1,+}_{x,v})$ and $f^\#$ is weakly differentiable and satisfies

$$\begin{cases} \dfrac{df^\#}{dt} + L^\#(f,g,g) = 0, \\ f^\#(0) = 0. \end{cases} \tag{4.42}$$

Then (4.42), the fundamental theorem of calculus and the facts $f^\# \in C^0([0,T), L^{1,+}_{x,v})$, $L^\#(f,g,g) \in L^1_{\text{loc}}([0,T), L^{1,+}_{x,v})$ imply

$$f^\#(t) = -\int_0^t L^\#(f,g,g)(\tau)\,d\tau = -\int_0^t f^\#(\tau)R^\#(g,g)(\tau)\,d\tau \quad \forall t \in [0,T). \tag{4.43}$$

We claim the following:

Claim. *For any compact set $K \subseteq \mathbb{R}^d \times \mathbb{R}^d$, we have $\|f^\#(t)\|_{L^1_{x,v}(K)} = 0$ for all $t \in [0,T)$.*

Proof of the claim. Fix any compact set $K \subseteq \mathbb{R}^d \times \mathbb{R}^d$. By (4.43), Fubini's theorem, Lemma 3.3 (i) and the fact that p_{γ_2,γ_3} is continuous, we obtain

$$\begin{aligned} \|f^\#(t)\|_{L^1_{x,v}(K)} &\leq \int_0^t \|f^\#(\tau)R^\#(g,g)(\tau)\|_{L^1_{x,v}(K)}\,d\tau \\ &\leq C_\beta |||g^\#|||_\infty (1 + |||g^\#|||_\infty) \int_0^t \|p_{\gamma_2,\gamma_3} f^\#(\tau)\|_{L^1_{x,v}(K)}\,d\tau \\ &\leq C_{K,\beta} |||g^\#|||_\infty (1 + |||g^\#|||_\infty) \int_0^t \|f^\#(\tau)\|_{L^1_{x,v}(K)}\,d\tau. \end{aligned} \tag{4.44}$$

Since $f^\# \in C^0([0,T), L^{1,+}_{x,v})$, the map $t \in [0,T) \to \|f^\#(t)\|_{L^1_{x,v}(K)} \in [0,\infty)$ is continuous, and thus (4.44) and Gronwall's inequality imply that

$$\|f^\#(t)\|_{L^1_{x,v}(K)} = 0 \quad \forall t \in [0,T).$$

The claim is proved. ■

Consider now a sequence of compact sets $(K_m)_m \nearrow \mathbb{R}^d \times \mathbb{R}^d$. By the claim above and the monotone convergence theorem, we have

$$\|f^\#(t)\|_{L^1_{x,v}} = \lim_{m\to\infty} \|f^\#(t)\|_{L^1_{x,v}(K_m)} = 0 \quad \forall t \in [0,T).$$

Since $f^\# \geq 0$, we obtain $f^\# = 0$ and hence $f = 0$. Uniqueness is proved. ■

The following comparison corollary comes immediately by the monotonicity of $R^\#$ and representation (4.17).

Corollary 4.6. *Let* $0 < T \leq \infty$ *and* $\alpha, \beta > 0$. *Consider* $f_{0,1}, f_{0,2} \in L^{1,+}_{x,v}$, $g_1, g_2 \in L^\infty([0,T), \mathcal{M}^+_{\alpha,\beta})$ *and* $h_1, h_2 \in L^1_{\text{loc}}([0,T), L^{1,+}_{x,v})$, *with*

$$f_{0,1} \leq f_{0,2}, \quad g_1^\# \geq g_2^\#, \quad h_1^\# \leq h_2^\#.$$

Let f_i, $i \in \{1,2\}$ *be the corresponding unique solution of* (4.3) *with* $f_0 := f_{0,i}$, $g := g_i$ *and* $h := h_i$. *Then* $f_1 \leq f_2$.

Proof. We have $g_1^\# \geq g_2^\# \Rightarrow g_1 \geq g_2$. By monotonicity of $R^\#$ we obtain $R^\#(g_1, g_1) \geq R^\#(g_2, g_2)$. The claim then comes immediately by representation (4.17). ∎

4.2. The Kaniel–Shinbrot iteration

Now we will use well-posedness of the associated linear problem and the estimates developed in Section 3 to prove convergence of the Kaniel–Shinbrot iteration to the unique solution of (1.1) in the class of functions bounded by a Maxwellian, if an appropriate beginning condition is satisfied.

Let $0 < T \leq \infty$ and $\alpha, \beta > 0$. Consider a function $f_0 \in \mathcal{M}^+_{\alpha,\beta}$ and a pair of functions $(l_0^\#, u_0^\#) \in \mathcal{M}^+_{\alpha,\beta} \times \mathcal{M}^+_{\alpha,\beta}$. By Lemma 3.3 (ii) we have that $G^\#(l_0, l_0, l_0), G^\#(u_0, u_0, u_0) \in L^\infty([0,T), L^{1,+}_{x,v})$. Applying Proposition 4.5 with h either $G(l_0, l_0, l_0)$ or $G(u_0, u_0, u_0)$, we find unique functions l_1, u_1 such that l_1 is the mild solution of

$$\begin{aligned} \frac{dl_1}{dt} + v \cdot \nabla_x l_1 &= G(l_0, l_0, l_0) - L(l_1, u_0, u_0), \\ l_1(0) &= f_0, \end{aligned} \tag{4.45}$$

and u_1 is the mild solution of

$$\begin{aligned} \frac{du_1}{dt} + v \cdot \nabla_x u_1 &= G(u_0, u_0, u_0) - L(u_1, l_0, l_0), \\ u_1(0) &= f_0. \end{aligned} \tag{4.46}$$

We obtain the following result:

Theorem 4.7. *Let* $0 < T \leq \infty$, $\alpha, \beta > 0$ *and*

$$\begin{aligned} K_\beta = C_d \Big[&\|b_2\|_{L^1(\mathbb{S}^{d-1})} \Big(\beta^{-d/2} + \frac{1}{d + \gamma_2 - 1} \Big) \\ &+ \|b_3\|_{L^1(\mathbb{S}^{2d-1})} \Big(\beta^{-d} + \frac{1}{2d + \gamma_3 - 1} \Big) \Big], \end{aligned}$$

be the constant given in (3.41). *Consider* $f_0 \in \mathcal{M}^+_{\alpha,\beta}$ *and* $(l_0^\#, u_0^\#) \in \mathcal{M}^+_{\alpha,\beta} \times \mathcal{M}^+_{\alpha,\beta}$ *with*

$$\|u_0^\#\|_{\mathcal{M}_{\alpha,\beta}} < \lambda_{\alpha,\beta}, \tag{4.47}$$

where

$$\lambda_{\alpha,\beta} = \min\Big\{ \frac{\alpha^{1/2}}{24 K_\beta}, \frac{\alpha^{1/4}}{2\sqrt{6K_\beta}} \Big\}. \tag{4.48}$$

Let l_1, u_1 be the mild solutions to (4.45) and (4.46) respectively, and assume that the following beginning condition holds:

$$0 \le l_0^\# \le l_1^\#(t) \le u_1^\#(t) \le u_0^\# \quad \forall t \in [0,T). \tag{4.49}$$

Then we conclude the following:

(i) *There are unique sequences $(l_n)_n$, $(u_n)_n$ such that, for any $n \in \mathbb{N}$, l_n, u_n are the mild solution to (4.1), (4.2) respectively. Moreover, for any $n \in \mathbb{N}$ we have*

$$0 \le l_0^\# \le l_1^\#(t) \le \cdots \le l_n^\#(t) \le u_n^\#(t) \le \cdots \le u_1^\#(t) \le u_0^\# \quad \forall t \in [0,T). \tag{4.50}$$

(ii) *For all $t \in [0,T)$, the sequences $(l_n^\#(t))_n$, $(u_n^\#(t))_n$ converge in $\mathcal{M}_{\alpha,\beta}$. Let us define*

$$l^\#(t) := \lim_{n\to\infty} l_n^\#(t), \quad u^\#(t) := \lim_{n\to\infty} u_n^\#(t), \quad t \in [0,T).$$

Then we conclude that

$$l^\#, u^\# \in C^0([0,T), L^{1,+}_{x,v}) \cap L^\infty([0,T), \mathcal{M}^+_{\alpha,\beta}),$$
$$L^\#(l,u,u),\ \ L^\#(u,l,l),\ \ G^\#(l,l,l),\ \ G^\#(u,u,u) \in L^\infty([0,T), L^{1,+}_{x,v})$$

and the following integral equations are satisfied:

$$l^\#(t) + \int_0^t L^\#(l,u,u)(\tau)\,d\tau = f_0 + \int_0^t G^\#(l,l,l)(\tau)\,d\tau \quad \forall t \in [0,T), \tag{4.51}$$
$$u^\#(t) + \int_0^t L^\#(u,l,l)(\tau)\,d\tau = f_0 + \int_0^t G^\#(u,u,u)(\tau)\,d\tau \quad \forall t \in [0,T). \tag{4.52}$$

(iii) *The limits l, u coincide, i.e. $u = l$.*

(iv) *Let us define $f := l = u$. Then f is the unique mild solution of the binary–ternary Boltzmann equation (1.1) in $[0,T)$, with initial data $f_0 \in \mathcal{M}^+_{\alpha,\beta}$ satisfying*

$$|||f^\#|||_\infty \le \|u_0^\#\|_{\mathcal{M}_{\alpha,\beta}}. \tag{4.53}$$

Remark 4.8. The uniqueness claimed above holds in the class of solutions satisfying (4.53).

Proof of Theorem 4.7.

(i) We will construct sequences $(l_n)_n$, $(u_n)_n$ satisfying (4.1)–(4.50) inductively.

- $n = 1$: l_1, u_1 satisfy (4.1) for $k = 1$ by assumption. Moreover, for $k = 1$, (4.50) reduces to assumption (4.49).
- Assume we have constructed $l_1, \dots, l_{n-1}, u_1, \dots, u_{n-1}$ satisfying (4.1) and

$$l_0^\# \le l_1^\#(t) \le \cdots \le l_{n-1}^\#(t) \le u_{n-1}^\#(t) \le \cdots \le u_1^\#(t) \le u_0^\# \quad \forall t \in [0,T). \tag{4.54}$$

Let l_n, u_n be the mild solutions of (4.1), (4.2), respectively, for $k = n$, given by Proposition 4.5. Having in mind (4.54), in order to prove (4.50) it suffices to show

$$l^{\#}_{n-1}(t) \leq l^{\#}_n(t) \leq u^{\#}_n(t) \leq u^{\#}_{n-1}(t) \quad \forall t \in [0, T). \tag{4.55}$$

Fix any $t \in [0, T)$. Then (4.54) and Proposition 3.1, which gives monotonicity of $G^{\#}$, yield that for any $t \in [0, T)$ we have

$$\begin{aligned} G^{\#}(l_{n-2}, l_{n-2}, l_{n-2})(t) &\leq G^{\#}(l_{n-1}, l_{n-1}, l_{n-1})(t) \\ &\leq G^{\#}(u_{n-1}, u_{n-1}, u_{n-1})(t) \\ &\leq G^{\#}(u_{n-2}, u_{n-2}, u_{n-2})(t). \end{aligned} \tag{4.56}$$

Using (4.54), (4.56) and Corollary 4.6 with

$$g^{\#}_1 = u^{\#}_{n-2}, \quad g^{\#}_2 = u^{\#}_{n-1}, \quad h^{\#}_1 = G^{\#}(l_{n-2}, l_{n-2}, l_{n-2}), \quad h^{\#}_2 = G^{\#}(l_{n-1}, l_{n-1}, l_{n-1}),$$

we obtain $l^{\#}_{n-1} \leq l^{\#}_n$. Similarly, using Corollary 4.6 for $g^{\#}_1 = u^{\#}_{n-1}$, $g^{\#}_2 = l^{\#}_{n-1}$, $h^{\#}_1 = G^{\#}(l_{n-1}, l_{n-1}, l_{n-1})$, $h^{\#}_2 = G^{\#}(u_{n-1}, u_{n-1}, u_{n-1})$, we obtain $l^{\#}_n \leq u^{\#}_n$, and using it for $g^{\#}_1 = l^{\#}_{n-1}$, $g^{\#}_2 = l^{\#}_{n-2}$, $h^{\#}_1 = G^{\#}(u_{n-1}, u_{n-1}, u_{n-1})$, $h^{\#}_2 = G^{\#}(u_{n-2}, u_{n-2}, u_{n-2})$, we obtain $u^{\#}_n \leq u^{\#}_{n-1}$. Condition (4.55) is proved and the claim follows.

(ii) To prove convergence, notice that (4.50) implies that, for any $t \in [0, T)$, the sequence $(l^{\#}_n(t))_n$ is increasing and upper bounded and the sequence $(u^{\#}_n(t))_n$ is decreasing and lower bounded, thus they are convergent. Let us define

$$l^{\#}(t) := \lim_{n\to\infty} l^{\#}_n(t), \quad u^{\#}(t) := \lim_{n\to\infty} u^{\#}_n(t), \quad t \in [0, T).$$

Since $u^{\#}_0 \in \mathcal{M}^+_{\alpha,\beta}$, estimate (4.50) actually implies that the convergence takes place in $\mathcal{M}_{\alpha,\beta}$ and that $l^{\#}, u^{\#} \in L^\infty([0, T), \mathcal{M}^+_{\alpha,\beta})$. Thus relations (3.20) from Lemma 3.3 imply that

$$L^{\#}(l, u, u), \;\; L^{\#}(u, l, l), \;\; G^{\#}(l, l, l), \;\; G^{\#}(u, u, u) \in L^\infty([0, T), L^{1,+}_{x,v}). \tag{4.57}$$

Moreover, since for any $t \in [0, T)$ we have

$$(l^{\#}_n, u^{\#}_{n-1}, u^{\#}_{n-1})(t) \xrightarrow{\mathcal{M}_{\alpha,\beta}} (l^{\#}, u^{\#}, u^{\#})(t), \quad (u^{\#}_n, l^{\#}_{n-1}, l^{\#}_{n-1})(t) \xrightarrow{\mathcal{M}_{\alpha,\beta}} (u^{\#}, l^{\#}, l^{\#})(t),$$

as $n \to \infty$, Corollary 3.5 implies that for any $t \in [0, T)$ we have

$$\begin{aligned} L^{\#}(l_n, u_{n-1}, u_{n-1})(t) &\xrightarrow{L^1_{x,v}} L^{\#}(l, u, u), \\ L^{\#}(u_n, l_{n-1}, l_{n-1})(t) &\xrightarrow{L^1_{x,v}} L^{\#}(u, l, l). \end{aligned} \tag{4.58}$$

Similarly, for any $t \in [0, T)$, we obtain

$$\begin{aligned} G^{\#}(l_{n-1}, l_{n-1}, l_{n-1})(t) &\xrightarrow{L^1_{x,v}} G^{\#}(l, l, l), \\ G^{\#}(u_{n-1}, u_{n-1}, u_{n-1})(t) &\xrightarrow{L^1_{x,v}} G^{\#}(u, u, u). \end{aligned} \tag{4.59}$$

Moreover, by relation (4.50), monotonicity of $L^{\#}$, $G^{\#}$ and the fact that $u_0^{\#} \in \mathcal{M}_{\alpha,\beta}^{+}$, Lemma 3.3 implies

$$\begin{aligned} L^{\#}(l_n, u_{n-1}, u_{n-1}),\quad G^{\#}(l_{n-1}, l_{n-1}, l_{n-1}) &\in L^{\infty}([0,T), L^1_{x,v}) \quad \forall n \in \mathbb{N}, \\ L^{\#}(u_n, l_{n-1}, l_{n-1}),\quad G^{\#}(u_{n-1}, u_{n-1}, u_{n-1}) &\in L^{\infty}([0,T), L^1_{x,v}) \quad \forall n \in \mathbb{N}. \end{aligned} \tag{4.60}$$

Recalling Definition 2.7, the initial value problems (4.1), (4.2) and the fundamental theorem of calculus imply that for all $n \in \mathbb{N}$ we have

$$\begin{aligned} l_n^{\#}(t) &+ \int_0^t L^{\#}(l_n, u_{n-1}, u_{n-1})(\tau)\, d\tau \\ &= f_0 + \int_0^t G^{\#}(l_{n-1}, l_{n-1}, l_{n-1})(\tau)\, d\tau \quad \forall t \in [0,T), \end{aligned} \tag{4.61}$$

$$\begin{aligned} u_n^{\#}(t) &+ \int_0^t L^{\#}(u_n, l_{n-1}, l_{n-1})(\tau)\, d\tau \\ &= f_0 + \int_0^t G^{\#}(u_{n-1}, u_{n-1}, u_{n-1})(\tau)\, d\tau \quad \forall t \in [0,T). \end{aligned} \tag{4.62}$$

Letting $n \to \infty$ and using the dominated convergence theorem, we obtain (4.51)–(4.52). The fact that $l^{\#}, u^{\#} \in C^0([0,T), L^1_{x,v})$ easily follows from (4.51)–(4.52).

(iii) Since $l_n^{\#} \leq u_n^{\#}$ by (4.50), letting $n \to \infty$ we obtain

$$0 \leq l_0^{\#} \leq l^{\#}(t) \leq u^{\#}(t) \leq u_0^{\#} \quad \forall t \in [0,T). \tag{4.63}$$

Subtracting (4.51) from (4.52) and using (4.63) and the triangle inequality, we obtain

$$\begin{aligned} |u^{\#}(t) - l^{\#}(t)| \leq \int_0^t & |G^{\#}(u,u,u)(\tau) - G^{\#}(l,l,l)(\tau)| \\ &+ |L^{\#}(l,u,u)(\tau) - L^{\#}(u,l,l)(\tau)|\, d\tau. \end{aligned} \tag{4.64}$$

Let us estimate the right-hand side of (4.64). Recalling (2.38), the triangle inequality yields

$$\begin{aligned} &\int_0^t |G^{\#}(u,u,u)(\tau) - G^{\#}(l,l,l)(\tau)|\, d\tau \\ &\quad \leq \int_0^t |G_2^{\#}(u,u)(\tau) - G_2^{\#}(l,l)(\tau)| + |G_3^{\#}(u,u,u)(\tau) - G_3^{\#}(l,l,l)(\tau)|\, d\tau. \end{aligned} \tag{4.65}$$

Bilinearity of $G_2^{\#}$, the triangle inequality, bound (3.38) from Proposition 3.7 and the right-hand side inequality of (4.63) yield

$$\begin{aligned} \int_0^t |G_2^{\#}(u,u)(\tau) - G_2^{\#}(l,l)(\tau)|\, d\tau &\leq \int_0^t |G_2^{\#}(u-l,u)(\tau)| + |G_2^{\#}(l,u-l)(\tau)|\, d\tau \\ &\leq K_\beta \alpha^{-1/2} M_{\alpha,\beta} \|u^{\#} - l^{\#}\|_{L^\infty([0,T)],\mathcal{M}_{\alpha,\beta})} (|||u^{\#}|||_\infty + |||l^{\#}|||_\infty) \\ &\leq 2K_\beta \alpha^{-1/2} M_{\alpha,\beta} \|u_0^{\#}\|_{\mathcal{M}_{\alpha,\beta}} |||u^{\#} - l^{\#}|||_\infty. \end{aligned} \tag{4.66}$$

The trilinearity of $G_3^\#$, the triangle inequality, bound (3.39) from Proposition 3.7 and the right-hand side of (4.63) yield

$$\begin{aligned}\int_0^t |G_3^\#(u,u,u)(\tau) - G_3^\#(l,l,l)(\tau)|\,d\tau \\ &\leq \int_0^t |G_3^\#(u-l,u,u)(\tau)| + |G_3^\#(l,u-l,u)(\tau)| + |G_3^\#(l,l,u-l)(\tau)|\,d\tau \\ &\leq K_\beta \alpha^{-1/2} M_{\alpha,\beta} |||u^\# - l^\#|||_\infty (|||u^\#|||_\infty^2 + |||u^\#|||_\infty |||l^\#|||_\infty + |||l^\#|||_\infty^2) \\ &\leq 3K_\beta \alpha^{-1/2} M_{\alpha,\beta} \|u_0^\#\|_{\mathcal{M}_{\alpha,\beta}}^2 |||u^\# - l^\#|||_\infty. \end{aligned} \tag{4.67}$$

Then estimates (4.66)–(4.67) yield

$$\begin{aligned}\int_0^t |G^\#(u,u,u)(\tau) - G^\#(l,l,l)(\tau)|\,d\tau \\ &\leq 6K_\beta \alpha^{-1/2} M_{\alpha,\beta} (\|u_0^\#\|_{\mathcal{M}_{\alpha,\beta}} + \|u_0^\#\|_{\mathcal{M}_{\alpha,\beta}}^2) |||u^\# - l^\#|||_\infty. \end{aligned} \tag{4.68}$$

By a similar argument, using (3.38), (3.39) instead, we also have

$$\begin{aligned}\int_0^t |L^\#(l,u,u)(\tau) - L^\#(u,l,l)(\tau)|\,d\tau \\ &\leq 6K_\beta \alpha^{-1/2} M_{\alpha,\beta} (\|u_0^\#\|_{\mathcal{M}_{\alpha,\beta}} + \|u_0^\#\|_{\mathcal{M}_{\alpha,\beta}}^2) |||u^\# - l^\#|||_\infty. \end{aligned} \tag{4.69}$$

Combining (4.64), (4.68)–(4.69), we obtain

$$|u^\#(t) - l^\#(t)| \leq 12K_\beta \alpha^{-1/2} M_{\alpha,\beta} (\|u_0^\#\|_{\mathcal{M}_{\alpha,\beta}} + \|u_0^\#\|_{\mathcal{M}_{\alpha,\beta}}^2) |||u^\# - l^\#|||_\infty \quad \forall t \in [0,T),$$

which is equivalent to

$$|||u^\# - l^\#|||_\infty \leq 12K_\beta \alpha^{-1/2} (\|u_0^\#\|_{\mathcal{M}_{\alpha,\beta}} + \|u_0^\#\|_{\mathcal{M}_{\alpha,\beta}}^2) |||u^\# - l^\#|||_\infty. \tag{4.70}$$

Notice though that (4.47)–(4.48) yield

$$12K_\beta \alpha^{-1/2} (\|u_0^\#\|_{\mathcal{M}_{\alpha,\beta}} + \|u_0^\#\|_{\mathcal{M}_{\alpha,\beta}}^2) < 1;$$

hence (4.70) yields $u = l$.

(iv) To prove existence, let us define f by

$$f^\# := l^\# = u^\# \in C^0([0,T), L^{1,+}_{x,v}) \cap L^\infty([0,T), \mathcal{M}^+_{\alpha,\beta}).$$

Then either (4.51) or (4.52) implies

$$f^\#(t) + \int_0^t L^\#(f,f,f)(\tau)\,d\tau = f_0 + \int_0^t G^\#(f,f,f)(\tau)\,d\tau \quad \forall t \in [0,T),$$

and therefore

$$\begin{cases} \dfrac{df^\#}{dt} + L^\#(f,f,f) = G^\#(f,f,f), \\ f^\#(0) = f_0. \end{cases}$$

Recalling Definition 2.7, we conclude that f is a mild solution to the binary–ternary Boltzmann equation (1.1) with initial data f_0. Bound (4.53) directly follows from (4.63).

Uniqueness of solutions satisfying (4.53) follows similarly to the proof of (iii) using a bilinearity–trilinearity argument and Proposition 3.7. Clearly, condition (4.53) is needed to have a contraction. ∎

5. Global well-posedness near vacuum

In this final section we prove the main result of this paper, stated in Theorem 2.10, which gives global well-posedness of (1.1) near vacuum, in the interval $[0, T)$, where $0 < T \leq \infty$. To prove this result we will rely on the time average bound of the gain term from Proposition 3.7.

Proof of Theorem 2.10. Consider $f_0 \in \mathcal{M}^+_{\alpha,\beta}$ satisfying (2.76) and let us define $l_0^\# = 0$, $u_0^\# = C_{\text{out}} M_{\alpha,\beta}$, where

$$C_{\text{out}} = \frac{1 - \sqrt{1 - 48 K_\beta \alpha^{-1/2}\big(1 + \frac{\alpha^{1/4}}{2\sqrt{6K_\beta}}\big)\|f_0\|_{\mathcal{M}_{\alpha,\beta}}}}{24 K_\beta \alpha^{-1/2}\big(1 + \frac{\alpha^{1/4}}{2\sqrt{6K_\beta}}\big)} \tag{5.1}$$

and K_β is given by (2.77). The reasoning behind defining C_{out} will become clear in (5.8). Notice that due to (2.76), $u_0^\#$ is well defined. In order to conclude the proof, we will use Theorem 4.7. Recalling from (4.48) that

$$\lambda_{\alpha,\beta} = \min\Big\{\frac{\alpha^{1/2}}{24K_\beta}, \frac{\alpha^{1/4}}{2\sqrt{6K_\beta}}\Big\},$$

(5.1) and (2.76) yield

$$\|u_0^\#\|_{\mathcal{M}_{\alpha,\beta}} = C_{\text{out}} < \lambda_{\alpha,\beta}, \tag{5.2}$$

and thus the conditions of Theorem 4.7 are satisfied. By Theorem 4.7, it suffices to prove that the beginning condition (4.49) for the approximating sequences generated by $f_0 \in \mathcal{M}^+_{\alpha,\beta}$ and the pair of functions $(l_0^\#, u_0^\#) \in \mathcal{M}^+_{\alpha,\beta} \times \mathcal{M}^+_{\alpha,\beta}$ is satisfied. Indeed, by the iteration scheme (4.45), we have

$$\begin{aligned}
\frac{dl_1^\#}{dt} + l_1^\# R^\#(u_0, u_0) &= 0,\\
\frac{du_1^\#}{dt} &= G^\#(u_0, u_0, u_0),\\
u_1^\#(0) = l_1^\#(0) &= f_0,
\end{aligned}$$

and therefore, we obtain

$$l_1^\#(t) = f_0 \exp\Big(-\int_0^t R^\#(u_0, u_0)(\tau)\, d\tau\Big), \quad t \in [0, T), \tag{5.3}$$

$$u_1^\#(t) = f_0 + \int_0^t G^\#(u_0, u_0, u_0)(\tau)\, d\tau, \qquad t \in [0, T). \tag{5.4}$$

Since $u_0 \geq 0$, formulas (5.3)–(5.4) together with Proposition 3.1 imply

$$0 = l_0^\# \leq l_1^\#(t) \leq u_1^\#(t) \quad \forall t \in [0, T). \tag{5.5}$$

It remains to prove that

$$u_1^\#(t) \leq u_0^\# \quad \forall t \in [0, T). \tag{5.6}$$

By representation (5.4) and (3.40) from Proposition 3.7, we obtain

$$\begin{aligned} u_1^\#(t) &\leq \|f_0\|_{\mathcal{M}_{\alpha,\beta}} M_{\alpha,\beta} + K_\beta \alpha^{-1/2} M_{\alpha,\beta} \|u_0^\#\|^2_{\mathcal{M}_{\alpha,\beta}} (1 + \|u_0^\#\|_{\mathcal{M}_{\alpha,\beta}}) \\ &\leq M_{\alpha,\beta}\Big[\|f_0\|_{\mathcal{M}_{\alpha,\beta}} + K_\beta \alpha^{-1/2}\Big(1 + \frac{\alpha^{1/4}}{2\sqrt{6K_\beta}}\Big) C_{\text{out}}^2\Big], \end{aligned} \tag{5.7}$$

where to obtain (5.7) we use the fact that $u_0^\# = C_{\text{out}} M_{\alpha,\beta}$ and (5.2). Recalling (5.1), we notice that C_{out} satisfies

$$\|f_0\|_{\mathcal{M}_{\alpha,\beta}} + 12K_\beta \alpha^{-1/2}\Big(1 + \frac{\alpha^{1/4}}{2\sqrt{6K_\beta}}\Big) C_{\text{out}}^2 = C_{\text{out}}, \tag{5.8}$$

and thus (5.7) implies

$$u_1^\#(t) \leq C_{\text{out}} M_{\alpha,\beta} = u_0^\# \quad \forall t \in [0, T).$$

Estimate (5.6) is proved and the claim of Theorem 2.10 follows. ■

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