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Analytic maps of parabolic and elliptic type with trivial centralisers

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Abstract. We prove that for a dense set of irrational numbers α, the analytic centraliser of the map $e^{2\pi i\alpha}z + z^2$ near 0 is trivial. We also prove that some analytic circle diffeomorphisms in the Arnol'd family, with irrational rotation numbers, have trivial centralisers. These provide the first examples of such maps with trivial centralisers.

1. Introduction

For $\alpha \in \mathbb{R}$, let $\mathcal{H}_\alpha^\omega$ denote the set of germs of holomorphic diffeomorphisms of $(\mathbb{C}, 0)$ of the form

$$h(z) = e^{2\pi i\alpha}z + O(z^2),$$

defined near 0. We also consider the class $\mathcal{C}_\alpha^\omega$ of orientation-preserving analytic diffeomorphisms of the circle $\mathbb{R}/\mathbb{Z}$ with rotation number α. Let $\mathcal{H}^\omega = \bigcup_{\alpha\in\mathbb{R}} \mathcal{H}_\alpha^\omega$ and $\mathcal{C}^\omega = \bigcup_{\alpha\in\mathbb{R}} \mathcal{C}_\alpha^\omega$.

The analytic *centraliser* of an element $h \in \mathcal{H}_\alpha^\omega$, denoted by $\mathrm{Cent}(h)$, is the set of elements of $\mathcal{H}^\omega$ which commute with h near 0. From a dynamical point of view, any element of $\mathrm{Cent}(h)$ is a conformal symmetry of the dynamics of h, that is, the conformal change of coordinates g which conjugate h to itself, $g^{-1} \circ h \circ g = h$. Evidently, $\mathrm{Cent}(h)$ forms a group, where the action is the composition of the elements. For every $k \in \mathbb{Z}$, a suitable restriction of the k-fold composition $h^{\circ k}$ is defined near 0 and belongs to $\mathrm{Cent}(h)$. If the only elements of $\mathrm{Cent}(h)$ are of the form $h^{\circ k}$ for some $k \in \mathbb{Z}$, it is said that h has a *trivial centraliser*. In the same fashion, for $h \in \mathcal{C}^\omega$, the collection $\mathrm{Cent}(h)$ of elements of $\mathcal{C}^\omega$ which commute with h enjoys the same features.

Theorem 1.1. *There is a dense set of* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ *such that* $\mathrm{Cent}(e^{2\pi i\alpha}z + z^2)$ *is trivial.*

The above theorem is proved using a successive perturbation argument and the following statement for parabolic maps which we prove in this paper.

Theorem 1.2. *For every* $p/q \in \mathbb{Q}$, $\mathrm{Cent}(e^{2\pi i p/q}z + z^2)$ *is trivial.*

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The main idea we employ to prove the above theorems also allows us to deal with analytic circle diffeomorphisms in the Arnol'd family,

$$S_{a,b}(x) = x + a + b\sin(2\pi x),$$

for $a \in \mathbb{R}$ and $b \in (0, 1/(2\pi))$.

Theorem 1.3. *For every $b \in (0, 1/(2\pi))$ there is $a \in \mathbb{R}$ such that* $\mathrm{Cent}(S_{a,b})$ *is trivial and the rotation number of $S_{a,b}$ belongs to $\mathbb{R} \setminus \mathbb{Q}$.*

Indeed, we prove that for each fixed $b \in (0, 1/(2\pi))$, the set of rotation numbers of the maps $S_{a,b}$ which have an irrational rotation number and a trivial centraliser is dense in $\mathbb{R}$. The above theorem is obtained from a successive perturbation argument and the analogue of Theorem 1.2 for maps $S_{a,b}$ with a parabolic cycle.

The main tool used to deal with parabolic maps is Écalle cylinders and horn maps, first studied and applied by Douady–Hubbard ([9]) and Lavaurs ([22]). Écalle ([10]) and Voronin ([31]) have shown that generic germs of analytic maps with a parabolic fixed point have a trivial local analytic centraliser at the fixed point. However, this argument does not apply to a specific map with a parabolic fixed point, and in particular, does not imply Theorem 1.2. Theorem 1.2 is proved in Section 2.

To our knowledge, Theorems 1.1 and 1.3 provide the first examples in $\mathcal{H}^\omega$ and $\mathcal{C}^\omega$ with irrational rotation numbers and trivial analytic centralisers. Below we briefly explain how these results fit in the frame of the dynamics of such analytic diffeomorphisms.

When an element $h \in \mathcal{H}^\omega_\alpha$, for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, is locally conformally conjugate to its linear part near 0, $\mathrm{Cent}(h)$ is a large set. That is, if $\phi^{-1} \circ h \circ \phi(w) = e^{2\pi i \alpha} w$ near 0, for some $\phi \in \mathcal{H}^\omega$, then for any $\mu \in \mathbb{C} \setminus \{0\}$, h commutes with the map $z \mapsto \phi(\mu\phi^{-1}(z))$. Indeed, here, $\mathrm{Cent}(h)$ is isomorphic to $\mathbb{C} \setminus \{0\}$. In some methods, the problem of understanding $\mathrm{Cent}(h)$ precedes the problem of local conjugation of h to its linear part. That is because, the space of solutions for the conjugation problem is the right-cosets of $\mathrm{Cent}(h)$; if ϕ is a solution of the conjugation problem, and $g \in \mathrm{Cent}(h)$, $g \circ \phi$ is also a solution of the conjugation problem. In this spirit, the size of $\mathrm{Cent}(h)$ may be thought of as a measure of linearisability of h near 0. The same argument applies to analytic circle diffeomorphisms.

For $h \in \mathcal{H}^\omega$, $\mathrm{Cent}(h)$ projects onto a subgroup of $\mathbb{R}/\mathbb{Z}$ through $g \mapsto \log g'(0)/(2\pi i)$. Similarly, for $h \in \mathcal{C}^\omega$, one maps $g \in \mathrm{Cent}(h)$ to its rotation number. Let $\mathcal{G}(h) \subset \mathbb{R}/\mathbb{Z}$ denote the image of this projection.

By remarkable results of Herman and Siegel ([17, 29]) there is a full-measure set $\mathcal{C} \subset \mathbb{R} \setminus \mathbb{Q}$ such that for every $\alpha \in \mathcal{C}$, any $h \in \mathcal{H}^\omega_\alpha \cup \mathcal{C}^\omega_\alpha$ is analytically linearisable. But, for a generic choice of α, there are $h \in \mathcal{H}^\omega_\alpha$ and $h \in \mathcal{C}^\omega_\alpha$ which are not linearisable ([1, 5]). We note that if f and g commute, and one of them is linearisable at 0, then the other one must also be linearisable through the same map. This implies that if $h \in \mathcal{H}^\omega_\alpha \cup \mathcal{C}^\omega_\alpha$ is not linearisable, then $\mathcal{G}(h) \subseteq (\mathbb{R} \setminus \mathcal{C})/\mathbb{Z}$. However, by a profound result of Moser ([24]), $\mathcal{G}(h)$ may not be an arbitrary subgroup of that set. That is because there is an arithmetic restriction on the rotation numbers of commuting non-linearisable maps. The optimal arithmetic condition for the linearisation of commuting maps in $\mathcal{H}_\alpha^\omega$, and in $\mathcal{C}_\alpha^\omega$, remains open. This complication is due to the rich structure of the local dynamics of such maps near 0; see [4, 27] and the references therein. However, a complete solution for smooth circle diffeomorphisms is presented in [14].

In [17, 32, 33], Herman and Yoccoz carried out a groundbreaking study of the centraliser and conjugation problem for circle diffeomorphisms and germs of holomorphic diffeomorphisms of $(\mathbb{C}, 0)$. In particular, Herman proved the existence of C^∞ circle diffeomorphisms with irrational rotation number having uncountably many C^∞ symmetries, and Yoccoz proved the existence of C^∞ circle diffeomorphisms with irrational rotation numbers and trivial centralisers. Pérez Marco in [27] elaborated a construction of Yoccoz to build elements $h \in \mathcal{H}^\omega$ and $h \in \mathcal{C}^\omega$, with irrational rotation number, such that $\mathcal{G}(h)$ is uncountable. His construction provides remarkable examples where $\mathcal{G}(h)$ contains infinitely many elements of finite order. In this paper we close the problem of the existence of maps in $\mathcal{H}^\omega$ and $\mathcal{C}^\omega$ with irrational rotation number and trivial centraliser. In light of the above discussions, our result shows that quadratic polynomials and the Arnol'd family provide the least linearisable elements in $\mathcal{H}^\omega$ and $\mathcal{C}^\omega$, respectively. This is consistent with the spirit of Yoccoz's argument in [32], that is, if some $e^{2\pi i\alpha}z + z^2$ is linearisable, then any element of $\mathcal{H}_\alpha^\omega$ is linearisable.

It is worth noting that the commutation problem for (the globally defined) rational functions of the Riemann sphere was studied by Fatou and Julia in the 1920s ([13, 18]) using iteration methods. A complete classification of such pairs was successfully obtained by Ritt ([28]), using topological and analytic methods, and was re-proved by Erëmenko ([11]) using modern iteration techniques. If iterates of g and h are not identical, modulo conjugation, they are either power maps, Chebyshev polynomials, or Lattès maps. The global commutation problem for entire functions of the complex plane still remains open, although substantial progress has been made so far; see for instance [2, 3, 15, 21, 25]. The global commutation problem on higher-dimensional complex spaces has been widely studied using iteration methods in recent years; see [6, 7, 19] and the references therein. For an extensive discussion on the centraliser and conjugation problems in low dimensions one may refer to [20, 30] and the more recent survey article [26].

2. Parabolic case

For $\alpha \in \mathbb{R}$, let

$$Q_\alpha(z) = e^{2\pi i\alpha}z + z^2.$$

Fix an arbitrary rational number $p/q \in \mathbb{Q}$ with $(p, q) = 1$. Also fix an arbitrary g in $\mathrm{Cent}(Q_{p/q})$.

The map $F = Q_{p/q}^{\circ q}$ has a parabolic fixed point at 0 with multiplier $+1$, and there are q attracting directions. It follows that the parabolic fixed point of F at 0 has multiplicity

$q+1$. That is, the Taylor series expansion of F near 0 is of the form

$$F(z) = Q_{p/q}^{\circ q}(z) = z + \sum_{k=q+1}^{2^q} a_k z^k, \tag{1}$$

with $a_{q+1} \neq 0$.

Lemma 2.1. *We have $g'(0)^q = 1$.*

Proof. Let $g(z) = \sum_{k=1}^{\infty} b_k z^k$ denote the Taylor series expansion of g about 0. Note that $F \circ g = g \circ F$ near 0. We may identify the coefficients of z^{q+1} in the power series expansions of $F \circ g$ and $g \circ F$, which gives us $b_{q+1} + b_1^{q+1} a_{q+1} = b_{q+1} + b_1 a_{q+1}$. Using $a_{q+1} \neq 0$, we conclude that $b_1^{q+1} = b_1$, and using $b_1 \neq 0$, since g is a local diffeomorphism, we must have $b_1^q = 1$. ■

By Lemma 2.1, there is an integer j with $0 \leq j \leq q-1$ such that $(Q_{p/q}^{\circ j} \circ g)'(0) = 1$. Consider the holomorphic map

$$G(z) = Q_{p/q}^{\circ j} \circ g, \tag{2}$$

which is defined near 0 and commutes with F.

Lemma 2.2. *The multiplicity of the fixed point of G at 0 is $q+1$. That is, $G(z) = z + \sum_{i=q+1}^{\infty} b_i z^i$, with $b_{q+1} \neq 0$.*

Proof. Assume that $G(z) = z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \cdots$ is a convergent Taylor series with $b_{n+1} \neq 0$. Observe that

$$\begin{aligned}
F \circ G(z) &= (z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \cdots) \\
&\quad + a_{q+1}(z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \cdots)^{q+1} \\
&\quad \ldots \\
&\quad + a_{q+j}(z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \cdots)^{q+j} \\
&\quad \ldots \\
&= (z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \cdots) \\
&\quad + a_{q+1}(z^{q+1} + b_{n+1}(q+1)z^{q+n+1} + \cdots) \\
&\quad \ldots \\
&\quad + a_{q+j}(z^{q+j} + b_{n+1}(q+j)z^{q+n+j} + \cdots) \\
&\quad \ldots .
\end{aligned}$$

The coefficient of z^{q+n+1} in the above expansion is

$$b_{q+n+1} + a_{q+1}b_{n+1}(q+1) + a_{q+n+1}.$$

Similarly, the coefficient of z^{n+q+1} in the expansion of $G \circ F$ is

$$a_{q+n+1} + b_{n+1}a_{q+1}(n+1) + b_{q+n+1}.$$

Since $F \circ G = G \circ F$ near 0, the above values must be the same. Using $a_{q+1} \neq 0$ and $b_{n+1} \neq 0$, we conclude that $q = n$. ∎

We shall use the theory of Leau–Fatou flowers, Fatou coordinates, and horn maps to exploit the local dynamics of F near 0. One may refer to [23] and [8] for the basic definitions and constructions we present below, although conventions may be different.

For $s > 0$, define the open sets

$$\Omega^s_{\mathrm{att}} = \big\{\zeta \in \mathbb{C} \mid \operatorname{Re}\zeta > s - |\operatorname{Im}\zeta|\big\}, \quad \Omega^s_{\mathrm{rep}} = \big\{\zeta \in \mathbb{C} \mid \operatorname{Re}\zeta < -s + |\operatorname{Im}\zeta|\big\}.$$

Also, consider the map $I \colon \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$,

$$I(z) = \frac{-1}{q a_{q+1} z^q}.$$

For $s > 0$ there are holomorphic and injective branches of I^{-1} defined on Ω^s_{att} and Ω^s_{rep}.

Consider two complex numbers v_{att} and v_{rep} such that

$$q a_{q+1} v^q_{\mathrm{att}} = -1, \quad v_{\mathrm{rep}} = e^{-\pi i/q} v_{\mathrm{att}}. \tag{3}$$

Evidently, $I(v_{\mathrm{att}}) = +1$ and $I(v_{\mathrm{rep}}) = -1$. For $s > 0$, there is an injective and holomorphic branch of I^{-1} defined on Ω^s_{att} such that $I^{-1}(\Omega^s_{\mathrm{att}})$ contains $\varepsilon v_{\mathrm{att}}$, for sufficiently small $\varepsilon > 0$. Similarly, there is an injective branch of I^{-1} defined on Ω^s_{rep} such that $I^{-1}(\Omega^s_{\mathrm{rep}})$ contains $\varepsilon v_{\mathrm{rep}}$, for sufficiently small $\varepsilon > 0$. From now on, we shall fix these choices of inverse branches for I^{-1} on Ω^s_{att} and Ω^s_{rep}. The choice of the inverse branch near infinity is independent of $s > 0$.

Let

$$\begin{aligned}
W_{\mathrm{att}} &= \big\{z \in \mathbb{C} \setminus \{0\} \mid |\arg(z/v_{\mathrm{att}})| \le \pi/q\big\},\\
W_{\mathrm{rep}} &= \big\{z \in \mathbb{C} \setminus \{0\} \mid |\arg(z/v_{\mathrm{rep}})| \le \pi/q\big\},\\
W'_{\mathrm{att}} &= \big\{z \in \mathbb{C} \setminus \{0\} \mid |\arg(z/v_{\mathrm{att}})| \le \pi/q - \pi/(4q)\big\},\\
W'_{\mathrm{rep}} &= \big\{z \in \mathbb{C} \setminus \{0\} \mid |\arg(z/v_{\mathrm{rep}})| \le \pi/q - \pi/(4q)\big\},
\end{aligned}$$

where arg denotes a branch of argument with values in $[-\pi, +\pi]$.

Let U be a Jordan neighbourhood of 0 such that G is defined on U and both G and F are injective on U. Since $F'(0) = 1$ and $G'(0) = 1$, there is $\delta > 0$ such that $B(0, \delta) \subset U$ and

$$\begin{aligned}
F(W'_{\mathrm{att}} \cap B(0,\delta)) &\subset W_{\mathrm{att}}, \qquad & F(W'_{\mathrm{rep}} \cap B(0,\delta)) &\subset W_{\mathrm{rep}},\\
G(W'_{\mathrm{att}} \cap B(0,\delta)) &\subset W_{\mathrm{att}}, \qquad & G(W'_{\mathrm{rep}} \cap B(0,\delta)) &\subset W_{\mathrm{rep}}.
\end{aligned} \tag{4}$$

We may choose $r > 0$ such that

$$I^{-1}(\Omega^r_{\mathrm{att}}) \subset W'_{\mathrm{att}} \cap B(0,\delta), \quad I^{-1}(\Omega^r_{\mathrm{rep}}) \subset W'_{\mathrm{rep}} \cap B(0,\delta). \tag{5}$$

Now we may lift $F \colon W'_{\mathrm{att}} \cap B(0,\delta) \to W_{\mathrm{att}}$ and $F \colon W'_{\mathrm{rep}} \cap B(0,\delta) \to W_{\mathrm{rep}}$ via the change of coordinate $I(z) = \zeta$ to define injective holomorphic maps

$$\widetilde{F}_{\mathrm{att}} \colon \Omega^r_{\mathrm{att}} \to \mathbb{C} \quad \text{and} \quad \widetilde{F}_{\mathrm{rep}} \colon \Omega^r_{\mathrm{rep}} \to \mathbb{C}.$$

Straightforward calculations show that these maps are of the form

$$\widetilde{F}_{\text{att}}(\zeta) = \zeta + 1 + O(1/|\zeta|^{1/q}), \quad \widetilde{F}_{\text{rep}}(\zeta) = \zeta + 1 + O(1/|\zeta|^{1/q}),$$

as $|\zeta| \to +\infty$. There is $s > 0$ such that,

$$\begin{aligned} |\widetilde{F}_{\text{att}}(\zeta) - (\zeta + 1)| &\leq 1/4 \quad \forall \zeta \in \Omega^s_{\text{att}}, \\ |\widetilde{F}_{\text{rep}}(\zeta) - (\zeta + 1)| &\leq 1/4 \quad \forall \zeta \in \Omega^s_{\text{rep}}. \end{aligned}$$

There are injective holomorphic maps

$$\Phi_{\text{att}} \colon \Omega^s_{\text{att}} \to \mathbb{C}, \quad \Phi_{\text{rep}} \colon \Omega^s_{\text{rep}} \to \mathbb{C},$$

such that

$$\begin{aligned} \Phi_{\text{att}} \circ \widetilde{F}_{\text{att}} &= \Phi_{\text{att}} + 1 \quad \text{on } \Omega^s_{\text{att}}, \\ \Phi_{\text{rep}} \circ \widetilde{F}_{\text{rep}} &= \Phi_{\text{rep}} + 1 \quad \text{on } \widetilde{F}^{-1}_{\text{rep}}(\Omega^s_{\text{rep}}). \end{aligned}$$

It is known that

$$|\Phi_{\text{att}}(\zeta)/\zeta - 1| \to 0 \quad \text{as } \operatorname{Re} \zeta \to +\infty, \tag{6}$$

$$|\Phi_{\text{rep}}(\zeta)/\zeta - 1| \to 0 \quad \text{as } \operatorname{Re} \zeta \to -\infty. \tag{7}$$

Let us define

$$\mathcal{P}^s_{\text{att}} = I^{-1}(\Omega^s_{\text{att}}), \quad \mathcal{P}^s_{\text{rep}} = I^{-1}(\Omega^s_{\text{rep}}).$$

Then the injective holomorphic maps

$$\phi_{\text{att}} = \Phi_{\text{att}} \circ I \colon \mathcal{P}^s_{\text{att}} \to \mathbb{C}, \quad \phi_{\text{rep}} = \Phi_{\text{rep}} \circ I \colon \mathcal{P}^s_{\text{rep}} \to \mathbb{C}$$

satisfy

$$\begin{aligned} \phi_{\text{att}} \circ F &= \phi_{\text{att}} + 1 \qquad \text{on } \mathcal{P}^s_{\text{att}}, \\ \phi_{\text{rep}} \circ F &= \phi_{\text{rep}} + 1 \qquad \text{on } F^{-1}(\mathcal{P}^s_{\text{rep}}). \end{aligned} \tag{8}$$

The map ϕ_{att} is an *attracting Fatou coordinate* for F, and ϕ_{rep} is a *repelling Fatou coordinate* for F.

Let

$$\mu = b_{q+1}/a_{q+1}.$$

Also, for $c \in \mathbb{C}$, let $T_c \colon \mathbb{C} \to \mathbb{C}$ denote the translation by c; $T_c(z) = z + c$.

Lemma 2.3. *There is $t \geq 0$ such that*

(i) $G(z) = \phi^{-1}_{\text{att}} \circ T_\mu \circ \phi_{\text{att}}(z)$ *for all* $z \in \mathcal{P}^t_{\text{att}}$,

(ii) $G(z) = \phi^{-1}_{\text{rep}} \circ T_\mu \circ \phi_{\text{rep}}(z)$ *for all* $z \in \mathcal{P}^t_{\text{rep}}$.

Proof. Part (i): By equations (4) and (5), we may lift $G\colon W'_{\mathrm{att}} \cap B(0,\delta) \to W_{\mathrm{att}}$ via the change of coordinate $I(z) = \zeta$ to define an injective holomorphic map $\widetilde{G}_{\mathrm{att}}\colon \Omega^r_{\mathrm{att}} \to \mathbb{C}$. We note that $\widetilde{G}_{\mathrm{att}}$ is of the form

$$\widetilde{G}_{\mathrm{att}}(\zeta) = \zeta + \frac{b_{q+1}}{a_{q+1}} + O\Big(\frac{1}{|\zeta|^{1/q}}\Big) \quad \text{as } |\zeta| \to +\infty.$$

In particular, if $|\zeta|$ is large enough, $|\widetilde{G}_{\mathrm{att}}(\zeta) - (\zeta + \mu)| \leq 1$. This implies that there is $t > s$ such that

$$\widetilde{G}_{\mathrm{att}}(\Omega^t_{\mathrm{att}}) \subset \Omega^s_{\mathrm{att}}.$$

Let

$$V = \Phi_{\mathrm{att}}(\Omega^t_{\mathrm{att}}).$$

Note that since $\widetilde{F}_{\mathrm{att}}(\Omega^t_{\mathrm{att}}) \subset \Omega^t_{\mathrm{att}}$, $V + 1 \subset V$. By equation (6), if $\operatorname{Re}\zeta$ is large enough, $|\Phi_{\mathrm{att}}(\zeta) - \zeta| \leq |\zeta|/3$. This implies that V contains some right half-plane, and hence

$$V/\mathbb{Z} = \mathbb{C}/\mathbb{Z}.$$

Consider the injective holomorphic map

$$\widehat{G}_{\mathrm{att}} = \Phi_{\mathrm{att}} \circ \widetilde{G}_{\mathrm{att}} \circ \Phi^{-1}_{\mathrm{att}}\colon V \to \mathbb{C}.$$

Since F commutes with G near 0, $\widetilde{F}_{\mathrm{att}}$ commutes with $\widetilde{G}_{\mathrm{att}}$ on the common domain of definition Ω^t_{att}. Therefore, for $w \in V$, we have

$$\begin{aligned}
\widehat{G}_{\mathrm{att}} \circ T_1(w) &= \Phi_{\mathrm{att}} \circ \widetilde{G}_{\mathrm{att}} \circ \Phi^{-1}_{\mathrm{att}} \circ T_1(w) \\
&= \Phi_{\mathrm{att}} \circ \widetilde{G}_{\mathrm{att}} \circ \widetilde{F}_{\mathrm{att}} \circ \Phi^{-1}_{\mathrm{att}}(w) \\
&= \Phi_{\mathrm{att}} \circ \widetilde{F}_{\mathrm{att}} \circ \widetilde{G}_{\mathrm{att}} \circ \Phi^{-1}_{\mathrm{att}}(w) \\
&= T_1 \circ \Phi_{\mathrm{att}} \circ \widetilde{G}_{\mathrm{att}} \circ \Phi^{-1}_{\mathrm{att}}(w) = T_1 \circ \widehat{G}_{\mathrm{att}}(w).
\end{aligned}$$

Since $V/\mathbb{Z} = \mathbb{C}/\mathbb{Z}$, the above relation implies that $\widehat{G}_{\mathrm{att}}$ induces a well-defined injective holomorphic map from $\mathbb{C}/\mathbb{Z}$ to $\mathbb{C}/\mathbb{Z}$. Thus, $\widehat{G}_{\mathrm{att}}$ is a translation on $V/\mathbb{Z}$, and hence, $\widehat{G}_{\mathrm{att}}$ is a translation on V, say T_τ. However, since $\Phi'_{\mathrm{att}}(\zeta) \to +1$, as $\operatorname{Re}\zeta \to +\infty$, and $\widetilde{G}_{\mathrm{att}}(\zeta)$ is asymptotically a translation by μ near $+\infty$, we must have $\tau = \mu$. That is, $\widehat{G}_{\mathrm{att}} = T_\mu$.

For $z \in \mathcal{P}^t_{\mathrm{att}}$, we have

$$\begin{aligned}
\phi^{-1}_{\mathrm{att}} \circ T_\mu \circ \phi_{\mathrm{att}} &= I^{-1} \circ \Phi^{-1}_{\mathrm{att}} \circ T_\mu \circ \Phi_{\mathrm{att}} \circ I \\
&= I^{-1} \circ \Phi^{-1}_{\mathrm{att}} \circ \widehat{G}_{\mathrm{att}} \circ \Phi_{\mathrm{att}} \circ I = I^{-1} \circ \widetilde{G}_{\mathrm{att}} \circ I = G.
\end{aligned}$$

Part (ii): As in the previous part, we may lift $G\colon W'_{\mathrm{rep}} \cap B(0,\delta) \to W_{\mathrm{rep}}$ to obtain an injective holomorphic map $\widetilde{G}_{\mathrm{rep}}\colon \Omega^r_{\mathrm{rep}} \to \mathbb{C}$ of the form $\widetilde{G}_{\mathrm{rep}} = \zeta + \mu + o(1)$ as $|\zeta| \to +\infty$. Then one may repeat the argument in part (i) with $\widetilde{F}_{\mathrm{rep}}$ and Φ_{rep}. ∎

Let B denote the set of $z \in \mathbb{C}$ such that $F^{\circ n}$ uniformly converges to 0 on a neighbourhood of z as $n \to +\infty$. Evidently, $\mathcal{P}^s_{\text{att}}$ is a connected open set and is contained in B. Let B_1 denote the connected component of B which contains $\mathcal{P}^s_{\text{att}}$. That is, B_1 is the immediate basin of attraction of 0 in the direction of v_{att}. The set B_1 is a Jordan domain. For every $z \in B_1$, there is $k \in \mathbb{N}$ with $F^{\circ k}(z) \in \mathcal{P}^s_{\text{att}}$. By the maximum principle, B_1 is a simply connected subset of $\mathbb{C}$. We may employ the functional relation in equation (8), to extend $\phi_{\text{att}}\colon \mathcal{P}^s_{\text{att}} \to \mathbb{C}$ to a holomorphic map

$$\phi_{\text{att}}\colon B_1 \to \mathbb{C},$$

such that $\phi_{\text{att}} \circ F = \phi_{\text{att}} + 1$ over all of B_1.

By equation (7), if $\operatorname{Re} \zeta$ is small enough, we have $|\Phi_{\text{rep}}(\zeta) - \zeta| \leq |\zeta|/3$. This implies that $\Phi_{\text{rep}}(\Omega^s_{\text{rep}})$ contains a left half-plane. Let us choose $r > 0$ such that

$$\Pi = \big\{w \in \mathbb{C} \mid -r - |\mu| - 1 < \operatorname{Re} w < -r\big\} \subset \Phi_{\text{rep}}(\Omega^s_{\text{rep}}).$$

It follows that for all $w \in \Pi$ with $\operatorname{Im} w$ sufficiently large, $\Phi^{-1}_{\text{rep}}(w) \in \Omega^s_{\text{att}}$. Therefore, because of our choice of consecutive attracting and repelling directions in equation (3), for all such $w \in \Pi$, $\phi^{-1}_{\text{rep}}(w) \in B_1$. However, for some $w \in \Pi$, $\phi^{-1}_{\text{rep}}(w) \notin B_1$, which is because $\phi^{-1}_{\text{rep}}(\Pi)$ crosses a repelling direction at the 0 fixed point.

Let Π' denote the connected component of the set $\{w \in \Pi \mid \phi^{-1}_{\text{rep}}(w) \in B_1\}$ which contains the top end of Π. We may consider the map

$$h = \phi_{\text{att}} \circ \phi^{-1}_{\text{rep}}\colon \Pi' \to \mathbb{C}.$$

This is a *horn map* of F. By the functional equations for ϕ_{att} and ϕ_{rep}, we must have $h(\zeta + 1) = h(\zeta) + 1$, whenever both sides of the equation are defined. This relation can be used to extend h onto the set $\Pi' + \mathbb{Z}$, which is the natural maximal domain of definition of this map (it cannot be extended across any point on the boundary). The map h projects down to a holomorphic map

$$H\colon \operatorname{Dom} H \to \mathbb{C},$$

on a punctured neighbourhood of 0 so that $H(e^{2\pi i\zeta}) = e^{2\pi i h(\zeta)}$. As both maps Φ_{att} and Φ_{rep} map infinity to infinity, we have $\operatorname{Im} h(\zeta) \to +\infty$ as $\operatorname{Im} \zeta \to +\infty$. This implies that H has a removable singularity at 0. That is, $\operatorname{Dom} H$ contains a neighbourhood of 0. In the same fashion, H has a natural maximal domain of definition which is a Jordan neighbourhood of 0. It is obtained from projecting the maximal domain of definition of h.[1]

Lemma 2.4. *The map H has infinitely many critical points, all mapped to the same value.*

Proof. Let c_1 denote the unique critical point of F within B_1. The map ϕ_{att} has a simple critical point at c_1. It follows from equation (8) that any $z \in B_1$ which is mapped to c_1

[1]The map H is only unique modulo pre-composition and post-composition by linear maps of the form $w \mapsto \lambda w$. This is due to the freedom in the choice of ϕ_{att} and ϕ_{rep} up to post-compositions with translations. However, we are not concerned with those choices here.

under some iterate of F is a critical point of $\phi_{\rm att}$. The set of accumulation points of such points is equal to the boundary of B_1. In particular, there are infinitely many such points near any point in ∂B_1. Since $\phi_{\rm rep}^{-1}$ is conformal on Π', it follows that h has infinitely many critical points near any point in $\partial(\Pi' + \mathbb{Z})$.

On the other hand, by equation (8), those critical points of $\phi_{\rm att}$ are mapped to $\phi_{\rm att}(c_1)$, $\phi_{\rm att}(c_1) - 1$, $\phi_{\rm att}(c_1) - 2, \ldots$. Again, since $\phi_{\rm rep}^{-1}$ is conformal on Π', we conclude that the only critical values of h are at $\phi_{\rm att}(c_1), \phi_{\rm att}(c_1) - 1, \phi_{\rm att}(c_1) - 2, \ldots$. Evidently, all those points project to the same value in the range of H. ∎

Lemma 2.5. *The map H commutes with $\xi \mapsto e^{2\pi i\mu}\xi$ near* 0.

Proof. By Lemma 2.3, $G = \phi_{\rm att}^{-1} \circ T_\mu \circ \phi_{\rm att}$ on $\mathcal{P}_{\rm att}^t$ and $G = \phi_{\rm rep}^{-1} \circ T_\mu \circ \phi_{\rm rep}$ on $\mathcal{P}_{\rm rep}^t$. Thus,

$$\phi_{\rm att}^{-1} \circ T_\mu \circ \phi_{\rm att} = \phi_{\rm rep}^{-1} \circ T_\mu \circ \phi_{\rm rep}$$

at any point in $\mathcal{P}_{\rm att}^t \cap \mathcal{P}_{\rm rep}^t$ where both sides of the equation are defined. Equivalently,

$$T_\mu \circ \phi_{\rm att} \circ \phi_{\rm rep}^{-1} = \phi_{\rm att} \circ \phi_{\rm rep}^{-1} \circ T_\mu,$$

whenever both sides of the equation are defined. We note that $T_\mu^{-1}(\Pi') \cap \Pi'$ is a non-empty open set, where both sides of the above equation are defined. This implies that the horn map h commutes with T_μ. Hence, H commutes with the map $\xi \mapsto e^{2\pi i\mu}\xi$. ∎

Lemma 2.6. *The constant μ belongs to $\mathbb{Z}$.*

Proof. First note that Dom H is invariant under multiplication by $e^{2\pi i\mu}$. Let c denote a critical point of H. Differentiating $H(e^{2\pi i\mu}\xi) = e^{2\pi i\mu}H(\xi)$ at c, we note that $e^{2\pi i\mu}c$ is a critical point of H. However, $H(e^{2\pi i\mu}c) = e^{2\pi i\mu}H(c)$ is a critical value of H. By Lemma 2.4, we must have $H(c) = e^{2\pi i\mu}H(c)$, and using $H(c) \neq 0$, we conclude that $\mu \in \mathbb{Z}$. ∎

Proof of Theorem 1.2. By Lemma 2.3, $G = \phi_{\rm att}^{-1} \circ T_\mu \circ \phi_{\rm att}$ on $\mathcal{P}_{\rm att}^t$, and by Lemma 2.6, μ is an integer. Thus, on $\mathcal{P}_{\rm att}^t$,

$$G = \phi_{\rm att}^{-1} \circ T_1^{\circ\mu} \circ \phi_{\rm att} = (\phi_{\rm att}^{-1} \circ T_1 \circ \phi_{\rm att}) \circ (\phi_{\rm att}^{-1} \circ T_1 \circ \phi_{\rm att}) \circ \cdots \circ (\phi_{\rm att}^{-1} \circ T_1 \circ \phi_{\rm att}) = F^{\circ\mu}.$$

As $\mathcal{P}_{\rm att}^t$ is a non-empty open set, we must have $G = F^{\circ\mu}$ on a neighbourhood of 0.

Looking back at definitions (1) and (2), we conclude that $(Q_{p/q}^{\circ q})^{\circ\mu} = Q_{p/q}^{\circ j} \circ g$, on a neighbourhood of 0, for some $0 \le j \le q-1$. Thus, $g = Q_{p/q}^{\circ(q\mu - j)}$ near 0. ∎

3. Elliptic case

Let $g(z) = \sum_{k=1}^{\infty} g_k z^k \in \mathrm{Cent}(Q_\alpha)$. It is easy to see that $|g_1| = 1$. Let us say that g is *r-good*, if $|g_k| \le r^{1-k}$ for all $k \ge 1$. Note that if g is r-good, then it is defined and holomorphic on the disk $|z| < r$. Moreover, the set of r-good maps forms a closed set with respect to the topology of uniform convergence on compact subsets of $|z| < r$.

Lemma 3.1. *For every $p/q \in \mathbb{Q}$ and every $r > 0$, $Q_{p/q}^{\circ k}$ is r-good for only finitely many values of $k \in \mathbb{Z}$.*

Proof. As $Q_{p/q}$ has a parabolic fixed point at 0, the family of iterates $\{Q_{p/q}^{\circ k}\}_{k \geq 0}$ and $\{Q_{p/q}^{\circ -k}\}_{k \geq 0}$ have no uniformly convergent subsequence on any neighbourhood of 0. ∎

We let

$$K(p/q, r) = \{k \in \mathbb{Z};\, Q_{p/q}^{\circ k} \text{ is } r\text{-good}\}.$$

By the above lemma, $K(p/q, r)$ is a finite set.

Lemma 3.2. *For every $p/q \in \mathbb{Q}$ and every $r > 0$, there exists $\delta(p/q, r) > 0$ such that for every $p'/q' \in \mathbb{Q}$ with $|p'/q' - p/q| \leq \delta(p/q, r)$ we have $K(p'/q', r) \subseteq K(p/q, r)$.*

Proof. Because the set of r-good holomorphic maps is closed, there is $N(r)$ such that any r-good map has less than $N(r)$ critical points in the disk $|z| < r/2$.

As L tends to $+\infty$, the set of the critical points of $Q_{p/q}^{\circ L}$ increases, and accumulates on 0. Let $L \in \mathbb{N}$ be such that $Q_{p/q}^{\circ L}$ has at least $N(r)$ critical points in the open disk $|z| < r/2$. If p'/q' is close enough to p/q, then $Q_{p'/q'}^{\circ L}$ has at least $N(r)$ critical points in the open disk $|z| < r/2$. For $l \geq L$, $Q_{p'/q'}^{\circ l}$ has at least all those critical points, so it is not r-good.

Let $M \in \mathbb{N}$ be such that $Q_{p/q}^{\circ -M}$, and hence $Q_{p/q}^{\circ -m}$ for any $m \geq M$, does not extend to the open disk $|z| < r$. Then the same is true for p'/q' close to p/q.

Finally, if $k \notin K(p/q, r)$ and $-M \leq k \leq L$, $Q_{p'/q'}^{\circ k}$ is not r-good if p'/q' is too close to p/q, because otherwise one could take limits to conclude that $Q_{p/q}^{\circ k}$ is r-good. ∎

Lemma 3.3. *For every $p/q \in \mathbb{Q}$, every $r > 0$, and every $\varepsilon > 0$, there exists $\kappa(p/q, r, \varepsilon) > 0$ which satisfies the following. For every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ with $|\alpha - p/q| \leq \kappa(p/q, r, \varepsilon)$, and every $g(z) = e^{2\pi i\beta} z + O(z^2)$ which commutes with Q_α and is r-good, there exists $k \in K(p/q, r)$ such that $|\beta - kp/q| < \varepsilon \mod \mathbb{Z}$.*[2]

Proof. If the result does not hold, we may take a sequence $\alpha_n \to p/q$ and r-good maps $g_n(z) = e^{2\pi i\beta_n} z + O(z^2)$ which commute with Q_{α_n}. By the closedness of the set of r-good maps, we may choose a convergent subsequence of the g_n converging to a limit g which is r-good and commutes with $Q_{p/q}$. Then g will not be of the form $Q_{p/q}^{\circ k}$ for some $k \in K(p/q, r)$. This contradicts Theorem 1.2 and Lemma 3.1. ∎

Lemma 3.4. *For every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, if a holomorphic germ of the form $g(z) = e^{2\pi ik\alpha} z + O(z^2)$, for some $k \in \mathbb{Z}$, commutes with Q_α, then $g = Q_\alpha^{\circ k}$ near 0.*

Proof. By considering $Q_\alpha^{\circ -k} \circ g$ instead, we may assume that $k = 0$. Then, by an inductive argument, one may show that the coefficients of the Taylor series expansion of g, except the first term, must be 0. That is, $g(z) = z$. ∎

[2] By the inequality $|x - y| < r \mod \mathbb{Z}$ we mean the length of the shortest arc on $\mathbb{R}/\mathbb{Z}$ from $x/\mathbb{Z}$ to $y/\mathbb{Z}$ is less than r.

Proof of Theorem 1.1. Start with any rational number p_1/q_1. We inductively define a strictly increasing sequence of rational numbers p_n/q_n, for $n \geq 1$, so that for all $1 \leq l \leq j < n$ we have

$$|p_n/q_n - p_j/q_j| < \delta(p_j/q_j, 1/j), \tag{9}$$

$$|p_n/q_n - p_j/q_j| < \kappa(p_j/q_j, 1/l, 1/j), \tag{10}$$

$$|p_n/q_n - p_j/q_j| < 1/q_j^2. \tag{11}$$

Let $\alpha = \lim_{n\to\infty} p_n/q_n$. Since the sequence p_n/q_n is strictly increasing, it follows from equation (11) that $q_n \to \infty$ as $n \to \infty$, and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Taking the limit as $n \to \infty$ in equation (10), we note that $|\alpha - p_j/q_j| \leq \kappa(p_j/q_j, 1/l, 1/j)$ for every $1 \leq l \leq j$.

Assume that $g(z) = e^{2\pi i\beta} z + O(z^2)$ is a germ of a holomorphic map which commutes with Q_α. Then there is $l \geq 1$ such that g is $(1/l)$-good.

By equation (9) and Lemma 3.2, we obtain $K(p_j/q_j, 1/l) \subseteq K(p_l/q_l, 1/l)$ for $1 \leq l \leq j$.

By Lemma 3.3, for every $j \geq l$, there exists $k \in \mathbb{Z}$ with $k \in K(p_j/q_j, 1/l) \subseteq K(p_l/q_l, 1/l)$ such that $|\beta - kp_j/q_j| < 1/j \mod \mathbb{Z}$. Taking limits of the latter inequality, as $j \to \infty$, we obtain $\beta = k\alpha$, for some k in the same range. Combining this with Lemma 3.4, we conclude that $g = Q_\alpha^{\circ k}$ near 0. ∎

4. Circle maps

We shall employ techniques from complex dynamics to study the analytic symmetries of the maps $S_{a,b}$. So we consider the complexified family of maps $S_{a,b}(z) = z + a + b\sin(2\pi z)$, for $z \in \mathbb{C}$, but for real values of a and b. Using the projection $z \mapsto e^{2\pi i z}$ from $\mathbb{C}$ to $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $S_{a,b}$ induces the holomorphic map

$$f_{a,b}(w) = e^{2\pi i a} w e^{\pi b(w - 1/w)}$$

from $\mathbb{C}^*$ to $\mathbb{C}^*$. Evidently, $f_{a,b}$ preserves the unit circle $\mathbb{T} = \{w \in \mathbb{C}; |w| = 1\}$. Since the map $S_{a,b}$ commutes with the complex conjugation map, the map $f_{a,b}$ commutes with the map $\tau(w) = 1/\overline{w}$. For $a \in \mathbb{R}$ and $b \in (0, 1/(2\pi))$, $f_{a,b}$ is a diffeomorphism of $\mathbb{T}$. We first aim to prove the analogue of Theorem 1.2 for the maps $f_{a,b}$, that is, Theorem 4.3 stated below.

Let us fix an arbitrary $f_{a,b}$, with $a \in \mathbb{R}$ and $b \in (0, 1/(2\pi))$, which has a parabolic cycle on $\mathbb{T}$, say $\{w_j\}_{j=1}^n$, of period $n \geq 1$. By relabelling if necessary, we may assume that $f_{a,b}(w_j) = w_{j+1}$, with the subscripts calculated modulo n.[3] Consider the map

$$F_{a,b} = f_{a,b}^{\circ n} \colon \mathbb{C}^* \to \mathbb{C}^*.$$

[3] We will use this convention in relation to indices and iterates, in similar situations.

Each w_j is a parabolic fixed point of $F_{a,b}$ with multiplier $+1$. For $1 \le j \le n$, let $U_j \subset \mathbb{C}^*$ denote the immediate basin of attraction of w_j for the iterates of $F_{a,b}$. That is, U_j is the union of the connected components of the basin of attraction of w_j which contain w_j on their boundary. A priori, each U_j may have several components. However, we shall show in a moment that there is a rather simple scenario here. The following lemma is a special case of a more general result by Geyer ([16, Thm 4.4]).

Lemma 4.1. *For every $1 \le j \le n$, U_j consists of a single connected component, which is invariant under τ, and contains precisely two distinct critical points of $F_{a,b}$. Moreover, $\bigcup_{j=1}^n \bar{U}_j = \mathbb{T}$.*

Proof. The critical points of $f_{a,b}$ are the solutions of the equation

$$f'_{a,b}(w) = e^{2\pi i a} e^{\pi b(w-1/w)}(1 + \pi b(w + 1/w)) = 0.$$

Evidently, if w is a solution of this equation, then $\overline{w}$, $1/w$, and $1/\overline{w}$ are also solutions of the above equation. However, because $f_{a,b}$ has no critical points on $\mathbb{T}$, for $b \in (0, 1/(2\pi))$, and the above equation has two distinct solutions, we must have $w = \overline{w}$. As $b > 0$, it follows that the distinct solutions of the above equation are of the form c_1 and $c_2 = \tau(c_1)$, for some $c_1 \in (-1, 0)$.

Since $F_{a,b}$ commutes with τ and $\tau(w_j) = w_j$, it follows that $\tau(U_j) = U_j$ for all $1 \le j \le n$. By a classical result of Fatou, see [23], the immediate basin of attraction of the parabolic cycle $\{w_j\}_{j=1}^n$, which is $\bigcup_{j=1}^n U_j$, contains at least one critical point of $f_{a,b}$. Thus, there is k in $\{1, 2, \dots, n\}$ such that c_1 or c_2 belongs to U_k. Then, since U_k is invariant under τ and $\{c_1, c_2\}$ is also invariant under τ, we conclude that both c_1 and c_2 belong to U_k. Moreover, because $f_{a,b}$ has only two critical points, c_1 and c_2 are the only critical points of $f_{a,b}$ in U_k, and there are no critical points of $f_{a,b}$ in the other domains U_j for $j \ne k$.

By the maximum principle, every connected component of each U_j is a simply connected region, and $f_{a,b}$ maps each connected component of U_j to a connected component of U_{j+1}. Since $f_{a,b}$ has only two critical points, and those belong to U_k, the map $f : U_j \to U_{j+1}$ is conformal, unless $j = k$. Because the critical points of $F_{a,b}$ are some pre-images of the critical points of $f_{a,b}$, it follows that each U_j contains exactly two distinct critical points of $F_{a,b}$.

Because $F'_{a,b}(w_j) = 1$ for all j, every connected component of each U_j is invariant under $F_{a,b}$. Thus, each component of U_j contains at least one critical point of $F_{a,b}$. As U_j contains precisely two critical points of $F_{a,b}$, we conclude that each U_j consists of at most two connected components.

Because $\mathbb{T}$ is invariant under $F_{a,b}$, the two tangent directions to $\mathbb{T}$ at w_k must be either attraction or repulsion directions for the Leau–Fatou flowers at w_k. There are three possibilities:

(i) both of those directions at w_k are repelling;

(ii) both of those directions at w_k are attracting;

(iii) one of those directions at w_k is attracting and the other one is repelling.

Case (i) cannot occur. If both of those directions at w_k are repelling, then both tangent directions to $\mathbb{T}$ at any other w_j must be repelling, due to the existence of a local conjugacy by a suitable iterate of $f_{a,b}$. Consider an arc of $\mathbb{T}$ cut off by w_k and w_{k+1} which does not contain any other w_l. This arc is invariant under $F_{a,b}$, and the orbits near each end of the arc are moved away from that end. This implies that there is another fixed point of $F_{a,b}$ in the interior of that arc, which is either attracting or parabolic. This is a contradiction since such a cycle of $f_{a,b}$ requires its own critical points distinct from the grand orbit of c_1 and c_2, which does not exist.

Case (ii) cannot occur as well. If both of those directions at w_k are attracting, then there are two distinct components of U_k which both meet $\mathbb{T}$. Each of those components contains at least one critical point of $F_{a,b}$ and is invariant under τ (τ acts as the identity map on $\mathbb{T}$). Since the critical points of $F_{a,b}$ are not on $\mathbb{T}$ and are invariant under τ, it follows that each of those components of U_k contains at least two critical points of $F_{a,b}$. Therefore, there must be at least four critical points of $F_{a,b}$ in U_k, which is a contradiction.

In case (iii), there is a connected component of U_k which meets $\mathbb{T}$. Since τ acts as the identity map on $\mathbb{T}$, and U_k is invariant under τ, that connected component of U_k must be invariant under τ. In particular, that component of U_k contains two critical points of $F_{a,b}$. Combining with the above paragraphs, we conclude that U_k has a single connected component. Since each U_j is conformally mapped to U_k by a suitable iterate of $f_{a,b}$, each U_j consists of a single component, containing precisely two critical points of $F_{a,b}$.

From case (iii) we can also note that $\bigcup_{j=1}^{n} \bar{U}_j = \mathbb{T}$. In this case, at each w_j one tangent direction to $\mathbb{T}$ is attracting, and one tangent direction to $\mathbb{T}$ is repelling. Fix an arbitrary w_j, and let ℓ_j be the arc of $\mathbb{T}$ cut off by w_j and w_{j+1} which does not contain any other w_l. This arc is invariant under $F_{a,b}$, and all orbits in this arc must tend to the same end point of ℓ_j. Otherwise, there must be a fixed point of $F_{a,b}$ in the interior of ℓ_j, which is either attracting or parabolic. As in case (i), this is a contradiction. ■

By relabelling the points w_i, and U_i accordingly, we may assume that U_1 contains the critical points c_1 and c_2 of $f_{a,b}$ (the integer k in the proof of Lemma 4.1 is 1).

Since the immediate parabolic basin of $F_{a,b}$ at w_1, U_1, has a single connected component, it follows that the multiplicity of the parabolic fixed point at w_1 is equal to $+2$. As in the previous section, there are attracting and repelling Fatou coordinates

$$\phi_{\text{att}}\colon \mathcal{P}_{\text{att}} \to \mathbb{C}, \quad \phi_{\text{rep}}\colon \mathcal{P}_{\text{rep}} \to \mathbb{C},$$

satisfying the functional equations

$$\phi_{\text{att}} \circ F_{a,b} = \phi_{\text{att}} + 1, \quad \phi_{\text{rep}} \circ F_{a,b} = \phi_{\text{rep}} + 1,$$

with $\phi_{\text{att}}(\mathcal{P}_{\text{att}}) = \Omega^s_{\text{att}}$ and $\phi_{\text{rep}}(\mathcal{P}_{\text{rep}}) = \Omega^s_{\text{rep}}$ for some $s > 0$, $F_{a,b}^{\circ j}$ converges to w_1 uniformly on compact subsets of $\mathcal{P}_{\text{att}}$ as $j \to +\infty$, and the local inverse maps $F_{a,b}^{\circ -j}$ converge to w_1 uniformly on compact subsets of $\mathcal{P}_{\text{rep}}$ as $j \to \infty$. The attracting coordinate may be extended to a holomorphic map $\phi_{\text{att}}\colon U_1 \to \mathbb{C}$ using the above functional equation.

The map

$$h = \phi_{\text{att}} \circ \phi_{\text{rep}}^{-1}$$

has a maximal domain of definition, which is $\phi_{\text{rep}}(U_1) + \mathbb{Z}$. As in Section 2, this induces a holomorphic map H defined on a neighbourhood of 0, with $H(0) = 0$.

Lemma 4.2. *The horn map H has infinitely many critical points, which are mapped to critical values v_1 and v_2 satisfying* $\arg v_1 = \arg v_2$.

Proof. The pre-images of c_1 and c_2 under iterates of $F_{a,b}\colon U_1 \to U_1$ are critical points of ϕ_{att}. The set of the accumulation points of those pre-images is equal to the boundary of U_1 (which is contained in the Julia set of $F_{a,b}$). By the functional equation for ϕ_{att}, ϕ_{att} maps those critical points into the set $\phi_{\text{att}}(c_1) + \mathbb{Z}$ or $\phi_{\text{att}}(c_2) + \mathbb{Z}$. On the other hand, ϕ_{rep}^{-1} is conformal on Ω^s_{rep}. This implies that the only critical values of h are contained in $(\phi_{\text{att}}(c_1) + \mathbb{Z}) \cup (\phi_{\text{att}}(c_2) + \mathbb{Z})$. Also, h has infinitely many critical points near any point on the boundary of its domain of definition. This implies that near any point on the boundary of definition of H there are infinitely many critical points of H.

Since $F_{a,b}$ is τ-symmetric, both ϕ_{att} and ϕ_{rep} are τ-symmetric. That is, by a suitable choice of normalisation for ϕ_{att} and ϕ_{rep}, we have $\phi_{\text{att}} \circ \tau = \overline{\phi_{\text{att}}}$ and $\phi_{\text{rep}} \circ \tau = \overline{\phi_{\text{rep}}}$. This is due to the uniqueness of Fatou coordinates up to translations by constants. Combining with the above paragraph, we conclude that $\overline{\phi_{\text{att}}(c_1)} = \phi_{\text{att}}(c_2)$. This implies that the critical values of h are complex conjugate, and hence, the critical values of H have the same argument. ∎

Theorem 4.3. *Assume that $f_{a,b}$ has a parabolic cycle on $\mathbb{T}$, for some $a \in \mathbb{R}$ and $b \in (0, 1/(2\pi))$. Then* $\mathrm{Cent}(f_{a,b})$ *is trivial.*

Proof. Fix an arbitrary $f_{a,b}$ with a parabolic cycle $\{w_j\}_{j=1}^n$ of period $n \geq 1$. Let us also fix an arbitrary $g \in \mathrm{Cent}(f_{a,b})$. The commutation implies that $g(w_1)$ is a periodic point of period n for $f_{a,b}$, which lies on $\mathbb{T}$. By Lemma 4.1, $f_{a,b}$ has a unique periodic cycle on $\mathbb{T}$, which is $\{w_j\}_{j=1}^n$. Therefore, there is an integer $k \geq 1$ such that $f_{a,b}^{\circ k} \circ g(w_1) = w_1$. Let us define the analytic map

$$G = f_{a,b}^{\circ k} \circ g\colon \mathbb{T} \to \mathbb{T}.$$

As $F_{a,b} = f_{a,b}^{\circ n}$ commutes with G, $F_{a,b}(w_1) = w_1$, $F'_{a,b}(w_1) = 1$, we may repeat Lemma 2.1 to conclude that $G'(w_1) = 1$. On the other hand, since the multiplicity of the fixed point of $F_{a,b}$ at w_1 is equal to $+2$, we may repeat Lemma 2.2 to conclude that the multiplicity of the fixed point of G at w_1 is also equal to $+2$. That is, G is of the form

$$G(w) = G(w_1) + (w - w_1) + b_2(w - w_1)^2 + \cdots$$

near 0, with $b_2 \neq 0$. As in the previous section, we must have $G = \phi_{\text{att}}^{-1} \circ T_\mu \circ \phi_{\text{att}}$ on $\mathcal{P}_{\text{att}}$ and $G = \phi_{\text{rep}}^{-1} \circ T_\mu \circ \phi_{\text{rep}}$ on $\mathcal{P}_{\text{rep}}$, where $\mu = 2b_2/F''_{a,b}(0)$. Repeating Lemma 2.5, we conclude that H must commute with the rotation $\xi \mapsto e^{2\pi i \mu}\xi$ near 0. Now, as in the proof of Lemma 2.6, we use Lemma 4.2 instead of Lemma 2.4, to say that if c is a critical point of H, then we must have $\arg H(c) = \arg(e^{2\pi i\mu} H(c))$. This implies that $\operatorname{Re} \mu \in \mathbb{Z}$. On the other hand, if $\operatorname{Im} \mu \neq 0$, since the domain of definition of H is invariant under $\xi \mapsto e^{2\pi i\mu}$, we conclude that H is defined over all of $\mathbb{C}$. But this is a contraction since H has infinitely many critical points in a bounded region of the plane. Therefore, $\mu \in \mathbb{Z}$, and hence $G = F_{a,b}^{\circ\mu}$. This completes the proof of Theorem 4.3 ■

Given $r > 1$, we say that an analytic map $g\colon \mathbb{T} \to \mathbb{T}$ is r-good if g is holomorphic on the annulus $1/r < |z| < r$ and maps that annulus into the annulus $1/2 < |z| < 2$. By the Schwarz–Pick lemma, for every $r > 1$ the class of r-good analytic maps of $\mathbb{T}$ forms a closed set of maps, in the topology of uniform convergence on compact subsets of the annulus $1/r < |z| < r$. Evidently, every analytic homeomorphism of T is r-good for some $r > 1$.

For $a \in \mathbb{R}$ and $b \in (0, 1/(2\pi))$, $f_{a,b}\colon \mathbb{T} \to \mathbb{T}$ is a homeomorphism. By the classic work of Poincaré on circle maps, $f_{a,b}$ has a well-defined rotation number $\rho(f_{a,b})$ which describes the asymptotic rate of rotation of orbits of $f_{a,b}$ around $\mathbb{T}$. For any $b \in (0, 1/(2\pi))$, the map $a \mapsto \rho(f_{a,b})$ is an increasing and continuous function of $a \in \mathbb{R}$. However, this map takes any rational value on a closed interval with non-empty interior.

By the classic work of Arnol'd ([1]), the parameter space $\mathbb{R} \times (0, 1/(2\pi))$ of the family $f_{a,b}$ is well understood in terms of these rotation numbers. We briefly explain the relevant bits below. For every $r \in \mathbb{Q}$, the set of $(a, b) \in \mathbb{R} \times (0, 1/(2\pi))$ such that $\rho(f_{a,b}) = r$ is known as an Arnol'd tongue. Let us denote the union of all those tongues by

$$L = \{(a, b) \in \mathbb{R} \times (0, 1/(2\pi)) \mid \rho(f_{a,b}) \in \mathbb{Q}\}.$$

Each component of L has non-empty interior, and is bounded by two disjoint arcs, each one connecting the horizontal line $b = 0$ to the horizontal line $b = 1/(2\pi)$. The boundary arcs of each component of L land at the same point on the horizontal line $b = 0$ (which is the tip of that tongue). Let us consider the union of all those pairs of boundary curves, that is,

$$P = \{(a, b) \in L \mid (a, b) \in \partial L\}.$$

In particular, each component of P is an arc connecting the horizontal line $b = 0$ to the horizontal line $b = 1/(2\pi)$. Indeed, each such arc is the graph of a function of $b \in (0, 1/(2\pi))$. We may naturally decompose the set P as

$$P = P^l \cup P^r,$$

where P^l consists of all arc components of P which lie on the left-hand side of the corresponding component of L. Similarly, P^r consists of all arc components of P which lie on the right-hand side of the corresponding component of L. Any component of P^l is accumulated from the left-hand side by components of P^l and P^r. Similarly, any component of P^r is accumulated from the right-hand side by components of P^l and P^r.

For $(a, b) \in L$, $f_{a,b}$ has an attracting or parabolic periodic cycle on $\mathbb{T}$. When $(a, b) \in P$ the unique periodic cycle of $f_{a,b}$ on $\mathbb{T}$ is parabolic. For a more detailed description of the dynamics of $f_{a,b}$ on $\mathbb{C}^*$ one may refer to [12] and the references therein. However, we do not need further information about the dynamics of these maps.

For a fixed b, the map $a \mapsto \rho(f_{a,b})$ is locally strictly increasing at irrational values, that is, if $\rho(f_{a,b}) \in \mathbb{R} \setminus \mathbb{Q}$ for some a, then for all $a' > a$, $\rho(f_{a',b}) > \rho(f_{a,b})$. It follows that when $r \in \mathbb{R} \setminus \mathbb{Q}$, the set of $(a,b) \in \mathbb{R} \times (0, 1/(2\pi))$ such that $\rho(f_{a,b}) = r$ is a simple curve connecting the horizontal line $b = 0$ to the horizontal line $b = 1/(2\pi)$. Also, each such arc is the graph of a function of $b \in (0, 1/(2\pi))$.

Lemma 4.4. *For every $(a,b) \in P$ and every $r > 1$, $f_{a,b}^{\circ k}$ is r-good for only finitely many values of k.*

Proof. For $(a,b) \in P$, $f_{a,b}$ has a parabolic cycle on $\mathbb{T}$, say $\{w_j\}_{j=1}^n$ for some $n \geq 1$. The family of iterates $\{f_{a,b}^{\circ k}\}_{k\geq 0}$ and $\{f_{a,b}^{\circ -k}\}_{k\geq 0}$ have no uniformly convergent subsequence on any neighbourhood of w_1. ∎

For $(a,b) \in P$ and $r > 1$, we define

$$K'(a,b,r) = \big\{k \in \mathbb{Z};\ f_{a,b}^{\circ k} \text{ is } r\text{-good}\big\}.$$

Lemma 4.5. *For every $(a,b) \in P$ and every $r > 1$, there exists $\delta'(a,b,r) > 0$ such that for every $(a',b) \in P$ with $|a' - a| \leq \delta'(a,b,r)$ we have*

$$K'(a',b,r) \subseteq K'(a,b,r).$$

Proof. This is the same as the proof of Lemma 3.2. ∎

Note that the set of (a',b) satisfying the conditions in the above lemma is not empty, regardless of the relation between $\delta'(a,b,r)$ and the width of the component of L containing (a,b). That is because, as we mentioned above, if $(a,b) \in P^l$, then (a,b) is accumulated from the left-hand side by elements $(a',b) \in P^l$ and also by elements $(a',b) \in P^r$. Similarly, if $(a,b) \in P^r$, then (a,b) is accumulated from the right-hand side by elements $(a',b) \in P^r$, and also by elements $(a',b) \in P^l$.

Lemma 4.6. *For every $(a,b) \in P$, every $r > 0$, and every $\varepsilon > 0$, there exists $\kappa'(a,b,r,\varepsilon) > 0$ which satisfies the following. For every $a' \in \mathbb{R}$ satisfying $|a' - a| \leq \kappa'(a,b,r,\varepsilon)$ and $\rho(f_{a',b}) \in \mathbb{R} \setminus \mathbb{Q}$, and every r-good map g which commutes with $f_{a',b}$, there exists $k \in K'(a,b,r)$ such that*

$$|\rho(g) - k\rho(f_{a,b})| < \varepsilon \bmod \mathbb{Z}.$$

Proof. The proof is identical to the one for Lemma 3.3. Here one uses the continuity of the map $x \mapsto \rho(f_{x,b})$, for $x \in \mathbb{R}$. ∎

Lemma 4.7. *Assume that $\rho(f_{a,b}) \in \mathbb{R} \setminus \mathbb{Q}$. If $g\colon \mathbb{T} \to \mathbb{T}$ is an analytic map which commutes with $f_{a,b}$ and $\rho(g) = k\rho(f)$ for some $k \in \mathbb{Z}$, then $g = f^{\circ k}$ on $\mathbb{T}$.*

Proof. By considering $f_{a,b}^{\circ -k} \circ g$ instead, we may assume that $\rho(g) = 0$. By Poincaré's theorem, g has a fixed point, and then by the commutation of $f_{a,b}$ and g, any iterate of that fixed point by $f_{a,b}$ must be a fixed point of g. Since the orbit of any point in $\mathbb{T}$ by $f_{a,b}$ is dense on $\mathbb{T}$, g has a dense set of fixed points. Thus, g is the identity map on $\mathbb{T}$. ∎

Proof of Theorem 1.3. The proof is similar to the one for Theorem 1.1, using Theorem 4.3 instead of Theorem 1.2. Fix an arbitrary $b \in (0, 1/(2\pi))$, and start with an arbitrary $a_1 \in \mathbb{R}$ such that $(a_1, b) \in P^r$. We inductively define a strictly increasing sequence of parameters $a_n \in \mathbb{R}$, for $n \geq 1$, such that for all $1 \leq l \leq j \leq n$ we have

$$(a_n, b) \in P^r, \tag{12}$$

$$|a_n - a_l| < \delta'(a_l, b, 1 + 1/l), \tag{13}$$

$$|a_n - a_j| < \kappa'(a_j, b, 1 + 1/l, 1/j), \tag{14}$$

$$|\rho(f_{a_n,b}) - \rho(f_{a_j,b})| < 1/q_j^2, \tag{15}$$

where $p_j/q_j = \rho(f_{a_j,b}) \in \mathbb{Q}$ and $(p_j, q_j) = 1$.

Because the sequence a_n is increasing and bounded, $a = \lim_{n\to\infty} a_n$ exists and belongs to $\mathbb{R}$. Also since the sequence a_n is strictly increasing, and belongs to P^r, the sequence p_n/q_n must be strictly increasing. It follows from equation (15) that $q_n \to \infty$ as $n \to \infty$, and $\rho(f_{a,b}) \in \mathbb{R} \setminus \mathbb{Q}$.

Taking the limit as $n \to \infty$ in equation (14), we note that

$$|a - a_j| \leq \kappa'(a_j, b, 1 + 1/l, 1/j)$$

for every $1 \leq l \leq j$.

Assume that g is an orientation-preserving analytic homeomorphism of $\mathbb{T}$ which commutes with $f_{a,b}$. There is $l \geq 1$ such that g is $(1 + 1/l)$-good.

By equation (13) and Lemma 4.5, we obtain $K'(a_j, b, 1 + 1/l) \subseteq K'(a_l, b, 1 + 1/l)$ for $1 \leq l \leq j$.

Applying Lemma 4.6 (with $(a, b) = (a_j, b)$, $r = 1 + 1/l$, $\varepsilon = 1/j$, and $a' = a$) we conclude that for every $j \geq l$, there exists an integer $k \in K'(a_j, b, 1 + 1/l) \subseteq K'(a_l, b, 1 + 1/l)$ such that $|\rho(g) - kp_j/q_j| < 1/j \mod \mathbb{Z}$. Taking limits of the latter inequality, as $j \to \infty$, we obtain $\rho(g) = k\rho(f_{a,b})$ for some k in the same range. Combining with Lemma 4.7, we conclude that $g = f_{a,b}^{\circ k}$ on $\mathbb{T}$.

In a similar fashion, for every $b \in (0, 1/(2\pi))$, one can use a decreasing sequence of parameters $a_n \in \mathbb{R}$ with $(a_n, b) \in P^l$, to build limiting parameters a such that $f_{a,b}$ has a trivial centraliser. ∎

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