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Traveling wave solutions to the Allen–Cahn equation

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Abstract. For the Allen–Cahn equation, it is well known that there is a monotone standing wave joining the balanced wells of the potential. In this paper we study the existence of traveling wave solutions for the Allen–Cahn equation on an infinite channel. Such traveling wave solutions possess a large number of oscillations and they are obtained with the aid of variational arguments.

1. Introduction

Let $(\xi, y) \in \Omega := \mathbb{R}^1 \times \Omega_y$, a cylinder with cross section Ω_y. Here Ω_y is a bounded open set in $\mathbb{R}^{N-1}$ with $C^{2,\alpha}$ boundary $\partial\Omega$, $\alpha \in (0,1)$. We are concerned with solutions for the Allen–Cahn equation

$$\begin{cases} u_t = u_{\xi\xi} + \Delta_y u + u(1-u^2), \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1.1}$$

To be specific, we investigate the existence of traveling wave solutions, i.e. solutions of the form $u(t,\xi,y) = v(\xi - ct, y)$ with speed $c \neq 0$.

The Allen–Cahn equation and related problems have attracted a lot of attention from different fields in mathematics (see e.g. [17, 30, 35] and the references therein). In particular, this nonlinear PDE serves as a model ([1, 4, 39]) in studying phase transition theory.

Traveling waves play an important role in understanding the dynamics of evolution systems ([3, 5, 6, 13, 14, 19, 20, 23, 31, 43]). There are many interesting results ([8, 21, 27–29, 36–38, 41, 42]) for the traveling wave solutions of

$$u_t = u_{\xi\xi} + \Delta_y u + g(u), \quad (\xi, y) \in \Omega,\ t > 0. \tag{1.2}$$

Particular examples include $g(u) = u(1-u)$ in the KPP equation and $g(u) = u(1-u) \times (u - \beta)$ with $\beta \in (0,1)$ in the Nagumo equation.

When a traveling wave solution is of the form $u(t,\xi,y) = v(\xi - ct)$, in which the wave profile does not depend on y, it is called a *planar traveling wave*. In the reference frame moving with speed c, a planar traveling wave is a solution of an ordinary differential equation. Such wave solutions have been successfully investigated by the shooting method

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([3, 19]). A traveling wave with zero speed is referred to as a standing wave. It is known that (1.1) possesses a planar standing wave joining 1 and -1, the two global minima of the potential $F(s) := \int_0^s g(t)\,dt = -\frac{s^2}{2} + \frac{s^4}{4}$.

For (1.1) we will prove existence of traveling wave solutions that possess a large number of oscillations. Due to the Dirichlet boundary condition $u|_{\partial\Omega} = 0$, such traveling wave solutions cannot be planar. Our proof is based on variational methods, and our minimax argument relies on the Ljusternik–Schnirelman theory.

In the one-dimensional case (or under Neumann boundary conditions) one can apply the same methods we use here to prove existence of an infinite number of planar traveling front solutions for (1.1) which satisfy $\lim_{t\to-\infty} u(\xi - ct) = 0$ and $\lim_{t\to\infty} u(\xi - ct) = 1$ or -1. In such cases, since the equation reduces to an ordinary differential equation, one can deduce the result also by phase plane analysis.

We next present – as motivation and illustration of our result – the phase plane analysis for one example of such planar traveling front solutions.

For a traveling wave of the form $u(t, x) = v(x - ct)$, the function v satisfies

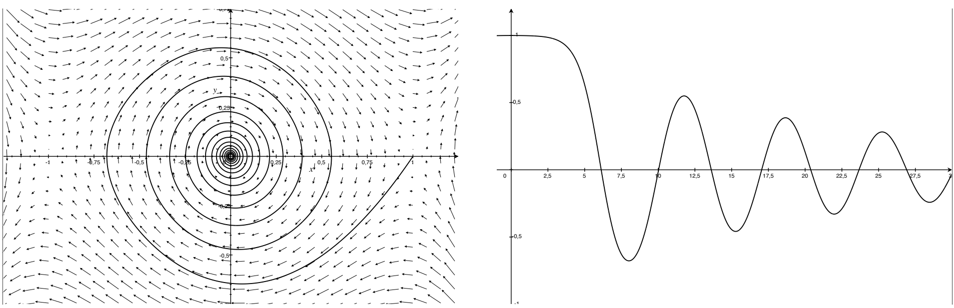
$$v'' = -cv' - v(1 - v^2).$$

Here $v = 0$, $p = v' = 0$ is an attracting focus if $c \in (0, 2)$ (repelling if $c \in (-2, 0)$). A quite detailed phase plane analysis of the equation for the case $|c| > 2$ can be found in [3, 19]. See Figure 1 for a description of the vector field of the corresponding system

$$\begin{cases} v' = p, \\ p' = -cp - v(1 - v^2), \end{cases}$$

when $c = 0.1$, the trajectory of the solution of the initial value problem $v(0) = 0.999$, $p(0) = 0$ and the graph of the solution $v(t)$. One can show that for all $0 < c < 1$, there are solutions which oscillate around zero (as $t \to +\infty$) infinitely many times.

To show a nonplanar traveling wave solution with oscillating behavior, we now introduce a variational argument. Following the ansatz proposed in [27], if $u(c(\xi - ct), y)$



(a) The vector field and the trajectory **(b)** The solution $v(t)$

Figure 1. The vector field $(p, -cp - v(1 - v^2))$, the trajectory of $(v(t), p(t))$ and the solution $v(t)$ when $c = 0.1$.

satisfies (1.1) then

$$\begin{cases} c^2(u_{xx}+u_x)+\Delta_y u+u(1-u^2)=0, \\ u|_{\partial\Omega}=0, \end{cases} \tag{1.3}$$

where $x=c(\xi-ct)$. Denote by L^p_w the Banach space of functions in $L^p_{\mathrm{loc}}(\Omega)$ equipped with the norm

$$\|u\|^p_{L^p_w}=\int_\Omega e^x|u|^p\,dx\,dy.$$

The appearance of weight function e^x is due to the first-order derivative term u_x in (1.3). Given $u\in H^1_{\mathrm{loc}}(\Omega)$ let $\|u\|^2_{\mathbf{E}}=\|u\|^2_{L^2_w}+\|u\|^2_{L^4_w}+\|u_x\|^2_{L^2_w}+\|\nabla_y u\|^2_{L^2_w}$. The set of functions with $\|u\|_{\mathbf{E}}<\infty$ is denoted by $\mathbf{E}$, while $\mathbf{E}_0$ is the completion of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|_{\mathbf{E}}$.

Define

$$\Phi_c[w]:=\frac{c^2}{2}\int_\Omega e^x w_x^2\,dx\,dy+\int_\Omega e^x\Big(\frac{1}{2}|\nabla_y w|^2+F(w)\Big)\,dx\,dy. \tag{1.4}$$

By the standard theory of calculus of variations, a critical point of Φ_c in $\mathbf{E}_0$ is a solution of (1.3) (see for example [33,40] and, for a more closely related setting, [23]). We remark that the choice of the ansatz $u(c(\xi-ct),y)$ instead of $u(\xi-ct,y)$ seems to be more convenient. It allows us to deal with Φ_c on function spaces with a fixed weight, for instance in studying the continuous dependence on c.

Consider a cross section Ω_y and the boundary value problem

$$\begin{cases} \Delta_y u+u(1-u^2)=0, \\ u|_{\partial\Omega_y}=0. \end{cases} \tag{1.5}$$

The existence of multiple solutions to (1.5) has been established by variational methods. These solutions are the critical points of the functional $J\colon H^1_0(\Omega_y)\to\mathbb{R}$ defined by

$$J[v]:=\int_{\Omega_y}\Big(\frac{1}{2}|\nabla_y v|^2+F(v)\Big)\,dy. \tag{1.6}$$

Denote by $\lambda_1<\lambda_2\le\lambda_3\le\cdots$ the eigenvalues of

$$\begin{cases} \Delta_y\psi+\lambda\psi=0, \\ \psi|_{\partial\Omega_y}=0. \end{cases} \tag{1.7}$$

Clearly $u\equiv 0$ is a trivial solution of (1.5) and $J[0]=0$. If $\lambda_1<1$, it is known ([9,25,32]) that there is a unique positive solution u_+ for (1.5) and $J[u_+]=\inf_{v\in H^1_0(\Omega_y)}J[v]<0$ (see Proposition 2.2). Moreover, both variational arguments and bifurcation results ([9,24–26]) show that if $\lambda_k<1$ then (1.5) has at least $2k$ nontrivial solutions with negative critical values.

For (1.1), it is known that the one with lower energy acts as an invader in the traveling wave while the other is being displaced through the flow (see also (4.12)). With the ansatz we are using, the results in the next theorem depend on c^2 only; thus we may take $c>0$.

Theorem 1.1. *Let Ω_y be a $C^{2,\alpha}$ bounded domain and λ_j be the eigenvalues of* (1.7). *Assume that $\lambda_1 < 1$.*

(i) *If $1 \le \lambda_2$, then for every $c \in (0, 2\sqrt{(1-\lambda_1)})$, there is a function $u(x, y)$ such that $w(t, \xi, y) = u(c(\xi - ct), y)$ is a traveling wave solution of* (1.1) *with wave speed c. Moreover, $u(x, y) \to 0$ as $x \to +\infty$ and $u(x, y) \to u_+(y)$ as $x \to -\infty$.*

(ii) *If J has a finite number of critical points in $H_0^1(\Omega_y)$, then for every $c \in (0, 2\sqrt{(1-\lambda_1)})$, there is a function $u(x, y)$ such that $w(t, \xi, y) = u(c(\xi - ct), y)$ is a traveling wave solution of* (1.1) *with wave speed c. Moreover, $u(x, y) \to u^*(y)$ as $x \to +\infty$ and $u(x, y) \to u_*(y)$ as $x \to -\infty$, where u_*, u^* are two critical points of J such that $J[u_*] < J[u^*]$.*

(iii) *Also $w(t, \xi, y) = -u(c(\xi - ct), y)$ is a traveling wave solution of* (1.1), *connecting 0 to $-u_+(y)$ in case* (i), *connecting $-u_*$ to $-u^*$ in case* (ii).

Remark 1.2. (a) The functional J has a finite number of critical points in $H_0^1(\Omega_y)$ if Ω_y is an interval. Indeed by [10, 11] there are $2k + 1$ critical points for J if $\lambda_k < 1 \le \lambda_{k+1}$.

(b) We believe that our existence result can be extended with similar techniques – at least when the nonlinearity is an odd function – to more general nonlinearities and boundary conditions .

For the scalar reaction–diffusion equation (1.2), the ordered method has been developed to show the existence of traveling waves on the cylindrical domain ([8, 42]). As a consequence of the maximum principle ([22]), such traveling front solutions possess certain monotonicity properties. For instance, let $v_+(y)$ and $v_-(y)$ be the stable solutions of

$$\begin{cases} v_t = \Delta_y v + g(v), \\ v|_{\partial\Omega_y} = 0. \end{cases} \tag{1.8}$$

If

$$v_+(y) > v_-(y) \text{ for all } y \in \Omega_y,$$

Vega ([42]) proved for (1.2) the existence of a traveling front solution $w(\xi - ct, y)$ which satisfies

$$v_+(y) > w(x, y) > v_-(y) \quad \text{for all } (x, y) \in \mathbb{R} \times \Omega_y. \tag{1.9}$$

Moreover, the method of moving planes and the sliding method ([7]) show that such a wave is strictly monotone in the x-direction. Based on an extension of the comparison technique, the ordered method has been generalized to studying traveling front solutions in monotone systems ([43]). In an earlier work ([21]), Gardner considered a discretization of (1.2) and applied the Conley index to establish an existence result similar to [42]. The monotonicity properties for traveling waves in combustion models have been studied in [6].

More recently, variational methods have been employed to investigate the traveling wave solutions for reaction–diffusion equations with Ginzburg–Landau or bistable-type nonlinearities. In [27–29], the authors proved existence of traveling waves via constrained minimization in a weighted Sobolev space like $\mathbf{E}_0$. This constrained minimization requires the traveling front solution to stay in the weighted Sobolev space and leads the solution to have certain monotonicity properties. Let us mention also [23], where existence has been proved via a renormalization of the associated functional.

The remainder of this paper is organized as follows. Section 2 begins with some known results as a preliminary. For the traveling wave solution with a given speed c, a variational formulation is introduced in Section 3 to establish a sequence of approximated solutions through a minimax scheme based on the Krasnosel'skii genus. Since the genus is increasing to infinity along this sequence of approximated solutions, it is expected ([24–26]) that the limit traveling wave solution possesses a large number of oscillations.

In using the variational approach to study traveling wave solutions, a commonly used weighted Sobolev space ([27–29]) is the Hilbert space $\mathbf{H}$ or $\mathbf{H}_0$ equipped with the norm $\|\cdot\|_{\mathbf{H}}$, where $\|u\|_{\mathbf{H}}^2 = \|u\|_{L_w^2}^2 + \|u_x\|_{L_w^2}^2 + \|\nabla_y u\|_{L_w^2}^2$. Choosing the space $\mathbf{E}_0$ enables us to work out the boundedness of the solutions and the compactness of Palais–Smale sequences in dealing with the minimax argument. Then in Section 4, utilizing a suitable limit procedure, we establish the traveling wave solutions. Moreover, as stated in Theorem 1.1, there are an infinite number of traveling wave solutions which are distinguished by their speed. To the best of our knowledge, using the minimax method to establish traveling wave solutions seems to be new and such a class of traveling front solutions have not been studied before.

We point out that in the existence results stated in Theorem 1.1, all the traveling front solutions do not belong to $\mathbf{E}_0$ or $\mathbf{H}_0$. Because of this fact, we need a delicate procedure in passing to the limit from the approximated solutions; however, this procedure does not keep track of the number of oscillations. It is not clear to us whether a direct argument is available for the proof of our result. A normalization technique has frequently been used in proving existence of heteroclinic orbits ([15, 16, 23, 34, 35]). In [23], with a different type of nonlinearity, the authors employed unconstrained minimization, after they worked out a normalization process with the aid of a supersolution and a subsolution. To adapt this approach to the minimax setting seems to be very difficult.

2. Preliminary

We state some useful inequalities whose proofs can be found in [29, 31].

Lemma 2.1. *If $w(x,y)$ is such that $\|w\|_{L_w^2}^2 + \|w_x\|_{L_w^2}^2 + \|\nabla_y w\|_{L_w^2}^2 < +\infty$, then*

$$\int_r^{+\infty} \int_{\Omega_y} e^x w^2 \, dy \, dx \leq 4 \int_r^{+\infty} \int_{\Omega_y} e^x w_x^2 \, dy \, dx, \tag{2.1}$$

$$\int_{\Omega_y} w^2(r,y) \, dy \leq e^{-r} \int_r^{+\infty} \int_{\Omega_y} e^x w_x^2 \, dy \, dx, \tag{2.2}$$

for any $r \in \mathbb{R}$; in particular,

$$\int_\Omega e^x w^2 \, dx \, dy \le 4 \int_\Omega e^x w_x^2 \, dx \, dy. \tag{2.3}$$

For the nontrivial solutions of (1.5), some existence and uniqueness results can be found in [10, 11, 24, 26, 32]. Some of these results are stated in the next proposition.

Proposition 2.2. *Let Ω_y be a bounded open set in $\mathbb{R}^n$ and λ_i the eigenvalues of* (1.7).

(a) *If $\lambda_1 < 1$ then there is a unique positive solution u_+ for* (1.5) *and $J[u_+] = \inf_{v \in H_0^1(\Omega_y)} J[v] < 0$.*

(b) *If $\lambda_1 < 1 \le \lambda_2$ then u_+ and $-u_+$ are the only nontrivial solutions of* (1.5).

3. Variational framework

In this section, a variational framework will be used to construct an approximation to a traveling wave solution of (1.1). We will always assume $\lambda_1 < 1$ and $0 < c < 2\sqrt{1 - \lambda_1}$ without further comment.

Some of the results in this section, in particular Proposition 3.2 and Lemmas 3.11 and 3.12, can be found in [29]; see also [12] for the FitzHugh–Nagumo system. We include the proofs for completeness.

Let $\Omega_* := (-\infty, 0) \times \Omega_y$ and consider the following boundary value problem:

$$\begin{cases} c^2(u_{xx} + u_x) + \Delta_y u + u(1 - u^2) = 0 & \text{in } \Omega_*, \\ u|_{\partial\Omega_*} = 0. \end{cases} \tag{3.1}$$

Let $\mathbf{E}_*$ be the closure of $C_0^\infty(\Omega_*)$ in $\mathbf{E}$ and, for all $u \in \mathbf{E}_*$,

$$I_c[u] := \int_{-\infty}^0 \int_{\Omega_y} \Big(\frac{c^2}{2} u_x^2 + \frac{1}{2} |\nabla_y u|^2 - \frac{1}{2} u^2 + \frac{1}{4} u^4 \Big) e^x \, dx \, dy. \tag{3.2}$$

Proposition 3.1. *The functional $I_c \in C^1(\mathbf{E}_*; \mathbb{R})$ and is bounded from below.*

The proof is standard (e.g. [33]). We omit it.

Proposition 3.2. *Suppose that $u \in \mathbf{E}_*$ is a critical point of I_c, with $\|u\|_\infty < \infty$. Then $u \in C^{2,\alpha}(\Omega_*) \cap C^{1,\alpha}(\bar{\Omega}_*)$ and satisfies* (3.1)*. Moreover, $\|u\|_{C^{1,\alpha}((-\infty,0] \times \bar{\Omega}_y)}$ is bounded; in particular, u_x and $\nabla_y u$ are uniformly continuous in Ω_*.*

Proof. A critical point u satisfies

$$0 = I_c'[u]\phi = \int_{-\infty}^0 \int_{\Omega_y} (c^2 u_x \phi_x + \nabla_y u \cdot \nabla_y \phi - u\phi + u^3 \phi) e^x \, dx \, dy$$

for $\phi \in \mathbf{E}_*^\dagger$ (the dual of $\mathbf{E}_*$), in particular for all $\phi \in C_0^\infty(\Omega_*)$. Since u is bounded by assumption and e^x is bounded on bounded subsets of Ω_*, it immediately follows that

$u \in H^2_{\mathrm{loc}}(\Omega_*)$ (see for example [18, §6.3.1]). Then standard regularity theory ([22]) shows that $u \in C^{2,\alpha}(\Omega_*) \cap C^{1,\alpha}(\overline{\Omega}_*)$, and thus it is a classical solution of (3.1). ■

Lemma 3.3. *If $u \in \mathbf{E}_*$ and $I_c[u] \le 0$ then*

$$\int_{\Omega_*} e^x u^4\,dx\,dy \le 4\int_{\Omega_*} e^x\,dx\,dy = 4|\Omega_y|, \tag{3.3}$$

$$\int_{\Omega_*} e^x u^2\,dx\,dy \le 2\int_{\Omega_*} e^x\,dx\,dy = 2|\Omega_y|, \tag{3.4}$$

$$\int_{\Omega_*} e^x u_x^2\,dx\,dy \le \frac{2}{c^2}\int_{\Omega_*} e^x\,dx\,dy = \frac{2}{c^2}|\Omega_y| \tag{3.5}$$

and

$$\int_{\Omega_*} e^x |\nabla_y u|^2\,dx\,dy \le 2\int_{\Omega_*} e^x\,dx\,dy. \tag{3.6}$$

In particular,

$$\|u\|_{L^4_w} \le \sqrt{2}|\Omega_y|^{1/4} \quad \textit{for all } u \in \mathbf{E}_* \textit{ such that } I_c[u] \le 0. \tag{3.7}$$

Proof. By the Hölder inequality,

$$\int_{\Omega_*} e^x u^2\,dx\,dy \le \left(\int_{\Omega_*} e^x\,dx\,dy\right)^{\frac{1}{2}} \left(\int_{\Omega_*} e^x u^4\,dx\,dy\right)^{\frac{1}{2}}. \tag{3.8}$$

Clearly $I_c[u] \le 0$ implies that

$$\int_{\Omega_*} \left(\frac{c^2}{2}u_x^2\,dx\,dy + \frac{1}{2}|\nabla_y u|^2 + \frac{1}{4}u^4\right)e^x\,dx\,dy \le \int_{\Omega_*} \frac{1}{2}e^x u^2\,dx\,dy. \tag{3.9}$$

This together with (3.8) yields (3.3). Substituting (3.3) into (3.8) gives (3.4). Then (3.5) and (3.6) easily follow from (3.9). ■

Lemma 3.4. *Assume that $u_n \in \mathbf{E}_*$ is a sequence such that $I_c[u_n] \le 0$ and $\|u_n\|_{L^6_w(\Omega_*)} \le C$.*

Then there exists a subsequence u_{n_k} which converges weakly in $\mathbf{E}_*$ and strongly in $L^p_w(\Omega_*)$ for all $p \in [2,4]$ to a function $\bar{u} \in \mathbf{E}_*$.

Proof. It immediately follows from Lemma 3.3 that u_n is bounded in $\mathbf{E}_*$. From the boundedness of u_n, there exists a subsequence u_{n_k} which converges weakly to some $\bar{u} \in \mathbf{E}_*$ and strongly in $L^2([-L,0] \times \Omega_y)$ for all $L > 0$.

We next show that $u_{n_k} \to \bar{u}$ in $L^p_w(\Omega_*)$ if $p \in [2,4]$. Let us first remark that Lemma 3.3 implies that

$$\int_{\Omega_*} e^x |u_{n_k}|^p\,dx\,dy \le C$$

for $2 \le p \le 4$, and the same inequality holds for $\bar{u}$, with a constant C not depending on k.

Given $\varepsilon > 0$, since

$$\begin{aligned}
&\int_{-\infty}^{-L}\int_{\Omega_y} e^x |u_{n_k} - \bar{u}|^2\, dy\, dx \\
&\quad \le \left(\int_{-\infty}^{-L}\int_{\Omega_y} e^x\, dy\, dx\right)^{1/2}\left(\int_{-\infty}^{-L}\int_{\Omega_y} e^x |u_{n_k} - \bar{u}|^4\, dy\, dx\right)^{1/2} \\
&\quad \le |\Omega_y|^{1/2} e^{-L/2}\left(\int_{-\infty}^{0}\int_{\Omega_y} e^x |u_{n_k} - \bar{u}|^4\, dy\, dx\right)^{1/2} \\
&\quad \le \widetilde{C} e^{-L/2},
\end{aligned}$$

we take $L > 0$ such that

$$\int_{-\infty}^{-L}\int_{\Omega_y} e^x |u_{n_k} - \bar{u}|^2\, dy\, dx \le \widetilde{C} e^{-L/2} < \frac{\varepsilon^2}{2}$$

and then $k_0 \in \mathbb{N}$ such that

$$\int_{-L}^{0}\int_{\Omega_y} e^x |u_{n_k} - \bar{u}|^2\, dy\, dx \le \frac{\varepsilon^2}{2} \quad \text{for all } k \ge k_0.$$

Then for all $k \ge k_0$,

$$\begin{aligned}
\|u_{n_k} - \bar{u}\|^2_{L^2_w(\Omega_*)} &= \int_{-\infty}^{0}\int_{\Omega_y} e^x |u_{n_k} - \bar{u}|^2\, dy\, dx \\
&= \int_{-\infty}^{-L}\int_{\Omega_y} e^x |u_{n_k} - \bar{u}|^2\, dy\, dx \\
&\quad + \int_{-L}^{0}\int_{\Omega_y} e^x |u_{n_k} - \bar{u}|^2\, dy\, dx < \varepsilon^2
\end{aligned}$$

and $\|u_{n_k} - \bar{u}\|^2_{L^2_w(\Omega_*)} \to 0$.

Observe that

$$\begin{aligned}
\|u_{n_k} - \bar{u}\|^4_{L^4_w(\Omega_*)} &= \int_{-\infty}^{0}\int_{\Omega_y} e^x |u_{n_k} - \bar{u}|^4\, dy\, dx \\
&\le \left(\int_{\Omega_*} e^x |u_{n_k} - \bar{u}|^2\, dy\, dx\right)^{1/2}\left(\int_{\Omega_*} e^x |u_{n_k} - \bar{u}|^6\, dy\, dx\right)^{1/2} \\
&\le C\|u_{n_k} - \bar{u}\|_{L^2_w(\Omega_*)}
\end{aligned}$$

by the boundedness of u_{n_k} in $L^6_w(\Omega_*)$. Now the lemma follows. ■

The following lemma states the compactness condition – essentially the Palais–Smale condition – which will be used via the minimax procedure to find the solutions on the half-cylinder Ω_*.

Lemma 3.5. *Assume that $u_n \in \mathbf{E}_*$ is a sequence such that $\|u_n\|_{L^6_w} \le C$, $I_c[u_n] \to b \le 0$ and $I_c'[u_n] \to 0$.*

Then there exists a subsequence u_{n_k} such that $u_{n_k} \to \bar{u} \in \mathbf{E}_*$, $I_c[\bar{u}] = b$ and $I_c'[\bar{u}] = 0$.

Proof. Lemma 3.4 implies that there is a subsequence u_{n_k} which converges weakly in $\mathbf{E}_*$ and strongly in $L^2_w(\Omega_*)$ to a function $\bar{u} \in \mathbf{E}_*$. Then

$$\begin{aligned}
&\int_{\Omega_*} [c^2|(u_{n_k})_x - \bar{u}_x|^2 + |\nabla_y u_{n_k} - \nabla_y \bar{u}|^2] e^x \, dx\, dy \\
&\quad = \int_{\Omega_*} [c^2 (u_{n_k})_x((u_{n_k})_x - \bar{u}_x) + \nabla_y u_{n_k}(\nabla_y u_{n_k} - \nabla_y \bar{u})] e^x \, dx\, dy \\
&\quad\quad - \int_{\Omega_*} [c^2 \bar{u}_x((u_{n_k})_x - \bar{u}_x) + \nabla_y \bar{u}(\nabla_y u_{n_k} - \nabla_y \bar{u})] e^x \, dx\, dy \\
&\quad = \langle I_c'[u_{n_k}], u_{n_k} - \bar{u}\rangle + \int_{\Omega_*} [u_{n_k}(u_{n_k} - \bar{u}) - u_{n_k}^3 (u_{n_k} - \bar{u})] e^x \, dx\, dy \\
&\quad\quad - \int_{\Omega_*} [c^2 \bar{u}_x((u_{n_k})_x - \bar{u}_x) + \nabla_y \bar{u}(\nabla_y u_{n_k} - \nabla_y \bar{u})] e^x \, dx\, dy,
\end{aligned}$$

which converges to zero since $u_{n_k} - \bar{u}$ is bounded, converges weakly in $\mathbf{E}_*$, strongly in $L^2_w(\Omega_*)$ to 0, while u_{n_k} is bounded in $L^6_w(\Omega_*)$. This immediately implies that $I_c'[\bar{u}] = 0$ and $I_c[\bar{u}] = b$. ∎

Lemma 3.6. *There exists $L_* > 0$ such that*

$$\|I_c'[u] - I_c'[v]\|_{\mathbf{E}_*^\dagger} \le L_* \|u - v\|_{\mathbf{E}_*}$$

if $u, v \in \mathbf{E}_*$, $I_c[u] \le 0$ and $I_c[v] \le 0$.

Proof. Suppose that $u, v \in \mathbf{E}_*$ and $I_c[u] \le 0$, $I_c[v] \le 0$. Set $h = u - v$; then for all $\phi \in \mathbf{E}_*$,

$$(I_c'[u] - I_c'[v])[\phi] = \int_{\Omega_*} (c^2 h_x \phi_x + \nabla_y h \cdot \nabla_y \phi - h\phi + (u^3 - v^3)\phi) e^x \, dx\, dy.$$

By (3.7) and the Hölder inequality,

$$\begin{aligned}
&\int_{\Omega_*} ((u^3 - v^3)\phi) e^x \, dx\, dy \\
&\quad = \int_{\Omega_*} ((u - v)(u^2 + uv + v^2)\phi) e^x \, dx\, dy \\
&\quad \le \|u - v\|_{L^4_w} (\|u\|^2_{L^4_w} + \|u\|_{L^4_w}\|v\|_{L^4_w} + \|v\|^2_{L^4_w}) \|\phi\|_{L^4_w} \\
&\quad \le C\|u - v\|_{L^4_w} \|\phi\|_{L^4_w},
\end{aligned}$$

with the constant C not depending on u or v. Thus the proof is complete. ∎

To introduce a minimax procedure, we recall the definition of the Krasnosel'skii genus. Details can be found in [2,33,40].

Let $A \subset \mathbf{E}_* \setminus \{0\}$, A is closed in $\mathbf{E}_*$ and $A = -A$. The genus of A is defined as

$$\gamma(A) = \inf\{n \in \mathbb{N} \mid \exists \phi \in C(A, \mathbb{R}^n \setminus \{0\}), \phi \text{ odd}\}.$$

If such a ϕ does not exist we define $\gamma(A) = +\infty$, while $\gamma(\emptyset) = 0$.

We now introduce the minimax classes

$$\Gamma_k = \{A \subset \mathbf{E}_* \setminus \{0\} \mid A \text{ closed}, A = -A \text{ and } \gamma(A) \geq k\}$$

and

$$\widehat{\Gamma}_k = \{A \in \Gamma_k \mid \|u(x, y)\|_{L^\infty(\Omega_*)} \leq 1 \text{ for all } u \in A\}.$$

The minimax levels corresponding to Γ_k and $\widehat{\Gamma}_k$ are defined as

$$c_k = \inf_{A \in \Gamma_k} \sup_{u \in A} I_c[u], \quad \hat{c}_k = \inf_{A \in \widehat{\Gamma}_k} \sup_{u \in A} I_c[u].$$

Proposition 3.7. *For all $k \in \mathbb{N}$,*

$$c_k = \hat{c}_k < 0. \tag{3.10}$$

Proof. Take $L > 0$ (to be fixed later) and consider the linear problem

$$\begin{cases} -c^2(u_{xx} + u_x) - \Delta_y u - u = \lambda_1 u & \text{in } \Omega_L = [-L, 0] \times \Omega_y, \\ u = 0 & \text{on } \partial\Omega_L, \end{cases}$$

where λ_1 is the first eigenvalue of (1.7) and φ_+ the corresponding eigenfunction. Set

$$\phi_k(x, y) = e^{-x/2} \sin\Big(\frac{k\pi}{L} x\Big) \varphi_+(y).$$

Notice that $\phi_k(x, y) = 0$ for $(x, y) \in \partial\Omega_L$ and it is a solution of

$$-c^2(\phi_{xx} + \phi_x) - \Delta_y \phi - \phi = \Big[c^2\Big(\frac{1}{4} + \frac{k^2\pi^2}{L^2}\Big) + (\lambda_1 - 1)\Big]\phi. \tag{3.11}$$

Multiplying (3.11) by ϕe^x and integrating it, we obtain

$$\begin{aligned} &\int_{-L}^0 \int_{\Omega_y} [-c^2(\phi_{xx} + \phi_x) - \Delta_y \phi - \phi] \phi e^x \, dy \, dx \\ &\qquad = \Big[c^2\Big(\frac{1}{4} + \frac{k^2\pi^2}{L^2}\Big) + (\lambda_1 - 1)\Big] \int_{-L}^0 \int_{\Omega_y} \phi^2 e^x \, dy \, dx. \end{aligned}$$

Set

$$Q_L[\phi] = \frac{1}{2} \int_{-L}^0 \int_{\Omega_y} [c^2 \phi_x^2 + |\nabla_y \phi|^2 - \phi^2] e^x \, dy \, dx.$$

Since

$$2Q_L[\phi] = \int_{-L}^{0}\int_{\Omega_y} [-c^2((\phi_x\phi e^x)_x - \phi_x^2 e^x) + |\nabla_y\phi|^2 e^x - \phi^2 e^x]\, dy\, dx$$
$$= \int_{-L}^{0}\int_{\Omega_y} [-c^2(\phi_{xx} + \phi_x) - \Delta_y\phi - \phi]\phi e^x\, dy\, dx,$$

it follows that $Q_L[\phi_k] < 0$ if

$$c^2\Big(\frac{1}{4} + \frac{k^2\pi^2}{L^2}\Big) + (\lambda_1 - 1) < 0.$$

In fact, for every given $k \in \mathbb{N}$, if L is large enough then $Q_L[\phi_k] < 0$, because $\lambda_1 < 1$ and $c < 2\sqrt{(1-\lambda_1)}$. Clearly,

$$Q_L[\alpha_i\phi_i + \alpha_j\phi_j] = \alpha_i^2 Q_L[\phi_i] + \alpha_j^2 Q_L[\phi_j].$$

We now extend the functions $\phi_i(x, y)$ to be defined on Ω_* by setting $\phi_i(x, y) = 0$ for all $x < -L$.

Denoting by S^k the unit sphere in $\mathbb{R}^k$, we consider an odd map

$$\xi\colon S^k \to \mathbf{E}_*, \quad \xi(\alpha_1, \alpha_2, \ldots, \alpha_k) = \varepsilon\sum_{i=1}^{k}\alpha_i\phi_i.$$

By direct calculation,

$$I_c[\varepsilon\xi(\alpha_1, \ldots, \alpha_k)] = \varepsilon^2\sum_{i=1}^{k}\alpha_i^2 Q_L[\phi_i] + \varepsilon^4\int_{-L}^{0}\int_{\Omega_y}\frac{|\sum_{i=1}^{k}\alpha_i\phi_i|^4}{4}e^x\, dx\, dy,$$

which is negative for all $(\alpha_1, \ldots, \alpha_k) \in S^k$, provided that we pick L to be large enough and ε small enough. Then for such an L and ε, set $A = \xi(S^k) \subset \mathbf{E}_* \setminus \{0\}$. It is clear that $A = -A$. Since any odd map $h\colon A \to \mathbb{R}^m \setminus \{0\}$ gives rise to an odd map $h \circ \xi\colon S^k \to \mathbb{R}^m \setminus \{0\}$, and $\gamma(S^k) = k$, we conclude that

$$\gamma(A) \geq k \quad \text{and} \quad A \in \Gamma_k.$$

Consequently,

$$c_k = \inf_{A\in\Gamma_k}\sup_{u\in A} I_c[u] < 0$$

for all $k \in \mathbb{N}$.

Recall that

$$\hat{c}_k = \inf_{A\in\widehat{\Gamma}_k}\sup_{u\in A} I_c[u],$$

which implies $c_k \leq \hat{c}_k$. Setting a truncated function $\hat{u}$ from u to

$$\hat{u}(x, y) = \begin{cases} u(x, y) & \text{if } |u(x, y)| \leq 1, \\ \dfrac{u(x, y)}{|u(x, y)|} & \text{if } |u(x, y)| > 1, \end{cases}$$

yields $I_c[\hat{u}] \le I_c[u]$ for all $u \in \mathbf{E}_*$. Given an $A \in \Gamma_k$, define

$$\widehat{A} = \{\hat{u} \mid u \in A\}.$$

Since the map $u \mapsto \hat{u}$ is continuous in $\mathbf{E}_*$, we conclude that $\widehat{A} \in \widehat{\Gamma}_k$ and $c_k = \hat{c}_k$. The proof of (3.10) is complete. ∎

Proposition 3.8. *For $k \in \mathbb{N}$, let $\widehat{A}_n \in \widehat{\Gamma}_k$ be such that*

$$c_k \le \sup_{u \in \widehat{A}_n} I_c[u] \le c_k + \frac{1}{n} < 0.$$

Then there is $u_n \in \widehat{A}_n$ such that

$$c_k - \frac{2}{n} \le I_c[u_n] \le c_k + \frac{1}{n}, \quad \|I_c'[u_n]\|_{\mathbf{E}_*^\dagger} \le 8\sqrt{\frac{L_*}{n}},$$

where L_* is given by Lemma 3.6.

Proof. The proof is based on the deformation theory. Suppose the assertion is false; then

$$\|I_c'[v]\|_{\mathbf{E}_*^\dagger} > 8\sqrt{\frac{L_*}{n}}$$

for all $v \in \widehat{A}_n$ such that $c_k - \frac{2}{n} \le I_c[v] \le c_k + \frac{1}{n}$. Let $\delta = \frac{2}{\sqrt{nL_*}}$. If $I_c[u] \in [c_k - \frac{2}{n}, c_k + \frac{2}{n}]$ and $\|u - v\|_{\mathbf{E}_*} < 2\delta$, invoking Lemma 3.6 yields

$$\begin{aligned}\|I_c'[u]\|_{\mathbf{E}_*^\dagger} &\ge \|I_c'[v]\|_{\mathbf{E}_*^\dagger} - \|I_c'[v] - I_c'[u]\|_{\mathbf{E}_*^\dagger} \\ &\ge 8\sqrt{\frac{L_*}{n}} - 2L_*\delta = 4\sqrt{\frac{L_*}{n}}.\end{aligned}$$

We can then apply [44, Lemmas 2.3, 3.1] with $S = \widehat{A}_k$ and $\varepsilon = \frac{1}{n}$, since

$$\frac{8\varepsilon}{\delta} = \frac{8}{n}\frac{\sqrt{nL_*}}{2} = 4\sqrt{\frac{L_*}{n}}.$$

A consequence of the above lemmas provides a deformation $\eta\colon [0, 1] \times \mathbf{E}_* \to \mathbf{E}_*$, odd in the second variable, and satisfying

$$I_c[u] < c_k - \frac{1}{n} \quad \text{for all } u \in \eta(1, \widehat{A}_n).$$

Now we have reached a contradiction, since η is odd in the second variable shows $\eta(1, \widehat{A}_n) \in \Gamma_k$. ∎

The following result follows from an application of the Ljusternik–Schnirelman theory. We refer to [26, 33, 44] for related applications to differential equations.

Proposition 3.9. *Let $\lambda_1 < 1$ and $c < 2\sqrt{(1-\lambda_1)}$. Then there exist a sequence of critical points $\{\hat{u}_k\}$ of I_c such that $I_c[\hat{u}_k] \le I_c[\hat{u}_{k+1}] < 0$, $|\hat{u}(x,y)| \le 1$ for all $(x,y) \in \Omega_*$,*

$$\lim_{k\to+\infty} I_c[\hat{u}_k] = 0 \tag{3.12}$$

and

$$\int_{-\infty}^{0}\int_{\Omega_y} e^x (\hat{u}_k)_x^2\, dx\, dy > 0. \tag{3.13}$$

Proof. It has been shown that $\hat{c}_k = c_k$ is a critical level. Following from Proposition 3.8, we can find a Palais–Smale sequence v_n at level c_k such that $|v_n(x,y)| \le 1$ for all $(x,y) \in \Omega_*$. By Lemma 3.5 we deduce that v_n converge to a critical point $\hat{u}_k$ at level c_k such that $|\hat{u}_k(x,y)| \le 1$ for all $(x,y) \in \Omega_*$. Thus we get a sequence of critical points $\{\hat{u}_k\}$ such that $I_c[\hat{u}_k] = c_k$.

If $\int_{-\infty}^{0}\int_{\Omega_y} e^x(\hat{u}_k)_x^2\, dx\, dy = 0$ for some $k \in \mathbb{N}$, then $\hat{u}_k \equiv 0$; however, this would violate $I_c[\hat{u}_k] < 0$, thus (3.13) must hold.

To show (3.12), we can follow a variant of a rather standard procedure (see e.g. [2, Theorem 10.10]). Let

$$\mathcal{B}_C = \{u \in \mathbf{E}_* \mid \|u\|_{L^6_w} \le C\} \quad \text{and} \quad I_c^d = \{u \in \mathbf{E}_* \mid I_c[u] < d\}.$$

Notice that if $u \in \mathbf{E}_*$ and $\|u\|_{L^\infty(\Omega_*)} \le 1$ then $\|u\|_{L^6_w} \le |\Omega_y|^{1/6}$. Since $|u(x,y)| \le 1$ implies $u \in \mathcal{B}_C$ for all $C \ge \bar{C} = |\Omega_y|^{1/6}$, we can, for each $k \in \mathbb{N}$, find a set $A \subset \mathcal{B}_C \cap I_c^0$ with genus k.

Suppose that

$$\lim_{k\to+\infty} I_c[\hat{u}_k] = \lim_{k\to+\infty} c_k = \chi < 0.$$

Then $\gamma(I_c^{\chi+\varepsilon} \cap \mathcal{B}_{\bar{C}}) = +\infty$ for all $\varepsilon > 0$ such that $\chi + \varepsilon < 0$. Since the set

$$\hat{Z}_\chi = \{u \in \mathbf{E}_* \mid I_c[u] = \chi \text{ and } I_c'[u] = 0 \text{ and } \|u\|_{L^6_w(\Omega_*)} \le \bar{C}\}$$

is compact in $\mathbf{E}_*$, using a property of genus, we can find a neighborhood U of $\hat{Z}_\chi$ which has finite genus, say $\gamma(U) = k_0 < +\infty$. Let $\mathcal{A} = I_c^{\chi+\varepsilon} \cap \mathcal{B}_{\bar{C}}$. As in proving Proposition 3.8, since the Palais–Smale condition holds in $\mathcal{A}$, when ε is small enough we can find a deformation η such that

$$\eta(1, \mathcal{A} \setminus U) \in I_c^{\chi-\varepsilon}.$$

This implies that

$$\gamma(\mathcal{A} \setminus U) \le \gamma(\eta(1, \mathcal{A} \setminus U)) = k_1 < +\infty.$$

Then $\mathcal{A} = (\mathcal{A} \setminus U) \cup (\mathcal{A} \cap U)$ gives

$$\gamma(\mathcal{A}) \le \gamma(\mathcal{A} \setminus U) + \gamma(\mathcal{A} \cap U) \le k_1 + k_0 < +\infty,$$

which leads to a contradiction. ∎

Lemma 3.10. *If $\{\hat{u}_k\}$ is the sequence of critical points obtained by Proposition 3.9, then*

$$\lim_{k\to+\infty}\int_{\Omega_*} e^x \hat{u}_k^4\,dx\,dy = 0, \tag{3.14}$$

$$\lim_{k\to+\infty}\int_{\Omega_*} e^x \hat{u}_k^2\,dx\,dy = 0, \tag{3.15}$$

$$\lim_{k\to+\infty}\int_{\Omega_*} e^x (\hat{u}_k)_x^2\,dx\,dy = 0, \tag{3.16}$$

$$\lim_{k\to+\infty}\int_{\Omega_*} e^x |\nabla_y \hat{u}_k|^2\,dx\,dy = 0. \tag{3.17}$$

Proof. Since

$$\int_{\Omega_*} (c^2(\hat{u}_k)_x^2\,dx\,dy + |\nabla_y \hat{u}_k|^2 - F'(\hat{u}_k)u_k)e^x\,dx\,dy = \langle I_c'[\hat{u}_k], \hat{u}_k\rangle = 0,$$

it follows that

$$\begin{aligned}
0 &= \lim_{k\to+\infty} I_c[\hat{u}_k] \\
&= \lim_{k\to+\infty}\int_{\Omega_*}\Big(\frac{c^2}{2}(\hat{u}_k)_x^2\,dx\,dy + \frac{1}{2}|\nabla_y \hat{u}_k|^2 + F(\hat{u}_k)\Big)e^x\,dx\,dy \\
&= \lim_{k\to+\infty} -\frac{1}{4}\int_{\Omega_*} e^x \hat{u}_k^4\,dx\,dy.
\end{aligned} \tag{3.18}$$

This together with (3.8) yields

$$\lim_{k\to+\infty}\int_{\Omega_*} e^x \hat{u}_k^2\,dx\,dy = 0.$$

Combining with (3.18) completes the proof. ∎

Lemma 3.11. *If $u \in \mathbf{E}_*$ is a bounded critical point of I_c then $u_x \in L^2(\Omega_*)$ and*

$$\lim_{x\to-\infty} u_x(x, y) = 0 \tag{3.19}$$

uniformly in y.

Proof. Multiplying (1.3) by u_x and integrating by parts, we get

$$\begin{aligned}
c^2\int_{x_n}^0\int_{\Omega_y} u_x^2\,dy\,dx = &-\int_{\Omega_y}\Big(\frac{c^2}{2}u_x^2 - \frac{1}{2}|\nabla_y u|^2 - F(u)\Big)\,dy\,\Big|_{x=x_n}^{x=0} \\
&- \int_{\partial\Omega_y}\int_{x_n}^0 u_x\frac{\partial u}{\partial \nu_y}\,dx\,d\sigma_y,
\end{aligned} \tag{3.20}$$

where ν_y is a normal vector to $\partial\Omega_y$ on which $d\sigma_y$ is a surface element. The last term of (3.20) vanishes since $u_x \equiv 0$ on $\partial\Omega_y$ due to the boundary conditions and hence

$$c^2 \int_{x_n}^0 \int_{\Omega_y} u_x^2 \, dy \, dx = -\int_{\Omega_y} \Big(\frac{1}{2}|\nabla_y u(x_n, y)|^2 + F(u(x_n, y))\Big) \, dy + \frac{c^2}{2} \int_{\Omega_y} (u_x^2(x_n, y) - u_x^2(0, y)) \, dy. \tag{3.21}$$

Using the facts that u and ∇u are uniformly bounded, we arrive at

$$\int_{x_n}^0 \int_{\Omega_y} u_x^2 \, dx \, dy \le C \tag{3.22}$$

with C being a constant independent of n. Passing to the limit as $n \to \infty$ yields $u_x \in L^2(\Omega_*)$. Then (3.19) follows, since u_x is uniformly continuous in Ω_*. ∎

Lemma 3.12. *Suppose that J has only isolated critical points in $H_0^1(\Omega_y)$ and u is a nonconstant critical point of I_c obtained by Proposition 3.9. Then*

$$\lim_{x \to -\infty} u(x, y) = v(y) \quad \textit{uniformly in } y \tag{3.23}$$

and v is a critical point of J with $J[v] < 0$. Furthermore, if $\lambda_1 < 1 \le \lambda_2$ then $v = u_+$ or $-u_+$.

Proof. We first show that for any sequence $x_n \to -\infty$ there exist a subsequence x_{n_k} and a critical point $v(y)$ of J such that

$$u(x_{n_k}, y) \to v(y) \quad \text{in } C^1(\bar{\Omega}_0),$$

where $\Omega_0 = (-1, 0) \times \Omega_y$.

Take any sequence $x_n \to -\infty$. By Proposition 3.2, for all $n \in \mathbf{N}$, $\|u(x + x_n, y)\|_{C^{1,\alpha}(\bar{\Omega}_0)}$ are uniformly bounded. Hence along a subsequence x_{n_k},

$$u(x + x_{n_k}, y) \to v(x, y) \quad \text{in } C^1(\bar{\Omega}_0). \tag{3.24}$$

It follows from (3.19) that $v_x(x, y) \equiv 0$; thus $v \in C^1(\bar{\Omega}_y)$, a function which depends on y only.

Let $\phi \in H_0^1(\Omega_y)$. Multiplying (1.3) by ϕ and integrating over Ω_0, we get

$$c^2 \int_{\Omega_y} (u_x(x + x_{n_k}, y) + u(x + x_{n_k}, y)) \cdot \phi(y) \, dy \, \Big|_{x=0}^{x=1} - \int_{\Omega_0} [\nabla_y u(x + x_{n_k}, y) \cdot \nabla_y \phi(y) - f(u(x + x_{n_k}, y))\phi(y)] \, dy \, dx = 0.$$

Passing to the limit as $k \to \infty$, we use (3.19) and (3.24) to obtain

$$\int_{\Omega_0} [\nabla_y v(y) \cdot \nabla_y \phi(y) - f(v(y))\phi(y)] \, dx \, dy = 0.$$

Then

$$\int_{\Omega_y} [\nabla_y v \cdot \nabla_y \phi - f(v)\phi]\, dy = 0,$$

which shows v is a critical point of J and our claim holds. Invoking (3.19) and letting $k \to \infty$ in (3.20), we also get

$$\begin{aligned} J[v] &= \int_{\Omega_y} \left[\frac{1}{2}|\nabla_y v|^2 + F(v)\right] dy \\ &= -c^2 \int_{\Omega_*} u_x^2\, dy\, dx - \frac{c^2}{2} \int_{\Omega_y} u_x^2(0, y)\, dy < 0. \end{aligned} \tag{3.25}$$

From the above equality we deduce that, while v in principle depends on the sequence $\{x_n\}$ and its subsequence n_k, the critical value $J[v]$ does not.

To show (3.23), we claim

$$u(x + x_n, y) \to v(y) \quad \text{in } C^1(\bar{\Omega}_y) \text{ along any sequence } x_n \to -\infty.$$

For otherwise, there exists a decreasing sequence $x_n \to -\infty$ such that

$$u(x + x_n, y) \to \tilde{v}(y) \;\text{ if } n \text{ is odd}, \quad u(x + x_n, y) \to v(y) \;\text{ if } n \text{ is even} \tag{3.26}$$

and

$$\kappa := \|\tilde{v} - v\|_{C(\bar{\Omega}_y)} > 0.$$

It follows from (3.25) that

$$E(v) = E(\tilde{v}).$$

(i) Suppose that there exists a decreasing sequence $x_n \to -\infty$ such that (3.26) holds and $|x_{n+1} - x_n| \le M$ for all $n \in \mathbb{N}$. If n is large then there exist $y \in \Omega_y$ and $\xi_n \in (x_{n+1}, x_n)$ such that

$$|u_x(\xi_n, y)| = \frac{|u(x_n, y) - u(x_{n+1}, y)|}{|x_n - x_{n+1}|} \ge \frac{\kappa}{3M}.$$

This contradicts (3.19).

(ii) It remains to treat the case of $|x_{n+1} - x_n| \to \infty$, $\|u(x + x_n, y) - \tilde{v}(y)\|_{C^1(\bar{\Omega}_y)} \to 0$ and $\|u(x + x_{n+1}, y) - v(y)\|_{C^1(\bar{\Omega}_y)} \to 0$ as $n \to \infty$. From Lemma 3.11, we know $u_x \in L^2(\Omega_*)$. Hence there exists a sequence $\{\eta_n\}$ with $\lim_{n\to\infty} \eta_n = 0$ such that

$$\|u(x + x_n, y) - \tilde{v}(y)\|_{C(\bar{\Omega}_0)} < \eta_n,$$

$$\|u(x + x_{n+1}, y) - v(y)\|_{C(\bar{\Omega}_0)} < \eta_n$$

and

$$\int_{x_n}^{x_{n+1}} \int_{\Omega_y} |u_x(x, y)|^2\, dy\, dx < \eta_n.$$

Since J has only isolated critical points in $H_0^1(\Omega_y)$, there exist $\kappa_1, \kappa_2 \in (0, \kappa)$ such that w is not a critical point of J if $\kappa_1 \le \|w - v\|_{C(\bar{\Omega}_y)} \le \kappa_2$. Since $\|u(x + \xi, y)\|_{C(\bar{\Omega}_0)}$ is continuous with respect to ξ, there exists $\bar{\xi}_n \in (x_{n+1}, x_n)$ such that $\|u(x+\bar{\xi}_n, y) - v\|_{C(\bar{\Omega}_0)} = \frac{\kappa_1+\kappa_2}{2}$. Arguing like in (i), we see that $|\bar{\xi}_n - x_n| \to \infty$ and $|\bar{\xi}_n - x_{n+1}| \to \infty$ as $n \to \infty$. Set $V_n(x, y) = u(x+\bar{\xi}_n, y)$. We then know that, along a subsequence, still denoted by $\{V_n\}$, we have that $V_n(x, y) \to V(y)$ in $C^1(\bar{\Omega}_0)$ with $V(y)$ a critical point of J. This is not possible since

$$\|V(y) - v(y)\|_{C(\bar{\Omega}_y)} = \lim_{n\to-\infty} \|u(x+\bar{\xi}_n, y) - v(y)\|_{C(\bar{\Omega}_0)} = \frac{\kappa_1+\kappa_2}{2}.$$

The last assertion follows from Proposition 2.2. Now the proof is complete. ■

In the one-dimensional case, Heinz ([25,26]) obtained the variational characterizations which link the critical levels $\{c_k\}$ to the nodal properties of $\{\hat{u}_k\}$.

4. Passing to the limit from approximate solutions

Let $\{\hat{u}_k\}$ be a sequence of solutions obtained in Section 3. First we consider the case that $\lambda_1 < 1 \le \lambda_2$. Then along a subsequence

$$\lim_{x\to-\infty} \hat{u}_k(x, y) = u_+(y) \tag{4.1}$$

or

$$\lim_{x\to-\infty} \hat{u}_k(x, y) = -u_+(y).$$

We may assume (4.1) holds, for otherwise taking $-\hat{u}_k$ will do.

By (3.16) and Proposition 3.9,

$$I_c[\hat{u}_k] \le I_c[\hat{u}_{k+1}] < 0, \tag{4.2}$$

$$\lim_{k\to+\infty} I_c[\hat{u}_k] = 0, \tag{4.3}$$

$$\lim_{k\to+\infty} \int_{\Omega_*} e^x (\hat{u}_k)_x^2 \, dx\, dy = 0, \tag{4.4}$$

while from (3.5) and (2.3) we deduce

$$0 < \int_{\Omega_*} e^x (\hat{u}_k)_x^2 \, dx\, dy \le \frac{2}{c^2} \int_{\Omega_*} e^x \, dx\, dy, \tag{4.5}$$

$$\int_{\Omega_*} e^x \hat{u}_k^2 \, dx\, dy \le 4 \int_{\Omega_*} e^x (\hat{u}_k)_x^2 \, dx\, dy. \tag{4.6}$$

Furthermore, using (3.9) yields

$$\int_{\Omega_*} e^x \hat{u}_k^4 \, dx\, dy \le 2 \int_{\Omega_*} e^x \hat{u}_k^2 \, dx\, dy, \tag{4.7}$$

$$\int_{\Omega_*} e^x |\nabla_y \hat{u}_k|^2 \, dx\, dy \le \int_{\Omega_*} e^x \hat{u}_k^2 \, dx\, dy. \tag{4.8}$$

Proof of Theorem 1.1. We prove (i) first. Let $\mu = \|v_+\|_{H_0^1(\Omega_y)}$ and x_k be the largest real number $\bar{x}$ satisfying

$$\int_{\bar{x}-1}^{\bar{x}} \int_{\Omega_y} (|\nabla_y \hat{u}_k(x,y) - \nabla_y v_+(y)|^2 + |\hat{u}_k(x,y) - v_+(y)|^2)\, dy\, dx = \frac{\mu}{8} \tag{4.9}$$

and

$$\int_{z-1}^{z} \int_{\Omega_y} (|\nabla_y \hat{u}_k(x,y) - \nabla_y v_+(y)|^2 + |\hat{u}_k(x,y) - v_+(y)|^2)\, dy\, dx < \frac{\mu}{8}$$

if $z < \bar{x}$. From (4.4), (4.6), (4.7) and (4.8), we deduce that for all $z < 0$,

$$\int_{z-1}^{z} \int_{\Omega_y} (|\nabla_y \hat{u}_k(x,y)|^2 + |\hat{u}_k(x,y)|^2)\, dy\, dx \to 0$$

as $k \to +\infty$. This implies $x_k \to -\infty$. Define

$$w_k(x,y) = \begin{cases} \hat{u}_k(x + x_k, y) & \text{if } x \le -x_k, \\ 0 & \text{if } x > -x_k. \end{cases} \tag{4.10}$$

It is clear that $w_k(x,y) \to v_+(y)$ as $x \to -\infty$ and $w_k(x,y) \to 0$ as $x \to +\infty$. Along a subsequence $w_k(x,y) \to U(x,y)$ in C^2_{loc}, and U is a bounded solution of (1.3). By (4.9),

$$\int_{-1}^{0} \int_{\Omega_y} (|\nabla_y U(x,y) - \nabla_y v_+(y)|^2 + |U(x,y) - v_+(y)|^2)\, dy\, dx = \frac{\mu}{8},$$

which ensures that $U(x,y)$ is a nontrivial solution of (1.3). We remark that for all $a, b \in \mathbb{R}$ and $a < b$,

$$\int_a^b \int_{\Omega_y} U_x^2(x,y)\, dx\, dy \le \lim_{k\to+\infty} \int_a^b \int_{\Omega_y} (w_k)_x^2(x,y)\, dx\, dy.$$

From the proof of (3.22), we know

$$\int_{-\infty}^{0} \int_{\Omega_y} (\hat{u}_k)_x^2\, dx\, dy$$

is bounded. Hence $U_x \in L^2(\Omega)$ and $U_x(x,y) \to 0$ as $x \to \pm\infty$. Arguing like Lemma 3.12, we deduce that

$$v_{-\infty}(y) = \lim_{x\to-\infty} U(x,y), \quad v_\infty(y) = \lim_{x\to+\infty} U(x,y),$$

where $v_{-\infty}$ and v_∞ are the solutions of (1.5).

As in the proof of (3.21), we have

$$\begin{aligned} c^2 \int_a^b \int_{\Omega_y} U_x^2\, dy\, dx = &- \int_{\Omega_y} \left(\frac{1}{2}|\nabla_y U(b,y)|^2 + F(U(b,y))\right) dy \\ &+ \int_{\Omega_y} \left(\frac{1}{2}|\nabla_y U(a,y)|^2 + F(U(a,y))\right) dy \\ &+ \frac{c^2}{2} \int_{\Omega_y} (U_x^2(a,y) - U_x^2(b,y))\, dy. \end{aligned} \tag{4.11}$$

Letting $a \to -\infty$ and $b \to +\infty$ gives

$$-J[v_{-\infty}] + J[v_\infty] = c^2 \int_{-\infty}^{\infty} \int_{\Omega_y} U_x^2 \, dy \, dx > 0. \tag{4.12}$$

This implies $v_{-\infty} = u_+$ and $v_\infty = 0$, which completes the proof of (i).

The proof of (iii) is trivial. It remains to show (ii). Since J has a finite number of critical points in $H_0^1(\Omega_y)$, there is a subsequence of $\{\hat{u}_k\}$, still denoted by $\{\hat{u}_k\}$, such that

$$\lim_{x \to -\infty} \hat{u}_k(x, y) = u_*(y)$$

and u_* is a solution of (1.5). With a slight modification, we obtain a bounded nontrivial solution $U(x, y)$ of (1.3) and (4.12) holds. Clearly $v_{-\infty} = u_*$. Then setting $v_\infty = u^*$ completes the proof. ∎

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