Book reviews

An Introduction to Singular Stochastic PDEs: Allen–Cahn Equations, Metastability, and Regularity Structures
by Nils Berglund

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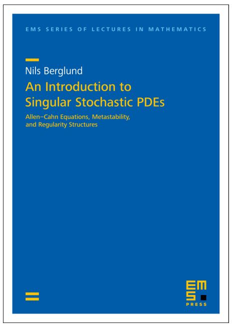

The past decade has seen fast paced progress in our understanding of stochastic partial differential equations (SPDEs), especially of the so-called *singular SPDEs*, and this nice little book provides a gentle introduction to the subject. The author wisely eschews the construction of a general theory and instead chooses to focus on the example of the stochastic Allen–Cahn equation, which allows to showcase increasing levels of complexity by varying the dimension of the underlying space.

The deterministic Allen–Cahn equation is the model for phase separation given by

$$\partial_t u = \Delta u + u - u^3, \tag{AC}$$

where u is a real-valued function of time and of d-dimensional space. It clearly admits $u = \pm 1$ as stable stationary states (assuming the spatial variable takes values in a domain without boundaries, like $\mathbf{R}^d$ or the torus $\mathbf{T}^d$, or that the equation is endowed with Neumann boundary conditions) and $u = 0$ as an unstable state.[1] The main subject of study of the book under review is then the behaviour of (AC) under the addition of random noise. More precisely, writing ξ for *space-time white noise*, namely a centred Gaussian random distribution with covariance formally given by $\mathbf{E}\,\xi(s,x)\xi(t,y) = \delta(t-s)\delta(x-y)$, where δ denotes the Dirac distribution, one considers the model

$$\partial_t u = \Delta u + u - u^3 + \sqrt{2T}\xi. \tag{SAC}$$

[1] Depending on the size of the domain, the dynamics can admit further non-trivial saddle points, but the author mostly assumes that the domain is small enough so that this doesn't happen.

Here, the parameter $T \geq 0$ is interpreted as the "temperature" of the system, which is justified in view of formula (BG) below.

The author then studies two types of questions. First, there are "local" questions around the existence and uniqueness of solutions. In the present case however, there is actually an even more basic question that arises, namely, what does (SAC) actually mean? The operation $u \mapsto u^3$ plainly makes sense if u is a (random) function but, since ξ is only a distribution, it is a priori not clear whether (SAC) admits function-valued solutions. In fact, it turns out that this is the case if and only if $d < 2$, so that, in higher dimensions, there is a non-trivial question as to how to even interpret (SAC). The second type of questions studied in this book are "global" questions regarding our solutions. This includes of course the question of global well-posedness, but also the question of the description of the invariant measure for the Markov process generated by (SAC).

Another global question that is being systematically addressed is that of the metastability of the ± 1 steady states. For this, one considers (SAC) at low temperature, namely with T very small. In this case, if one starts with the initial condition $u_0 = 1$, say, then one would expect the solution to remain within a small neighbourhood of 1 for a very long duration. A natural question then is how long it typically takes for the noise to kick the solution over to a neighbourhood of the other stable steady state -1. This question is being tackled using potential-theoretic methods and the book also serves as a nice introduction to this subject.

Regarding the structure of the book, it proceeds by increasing dimension of the underlying physical space, which neatly corresponds to an increase in sophistication of the methods required. Chapter 2 actually starts with "dimension 0", namely the case where the "space" is a finite set Λ of points and the linear operator Δ is a finite-difference operator. In this case, the local questions mentioned above are trivial and one focuses on the global questions. One of the main features of (AC) is that it is a gradient flow for the energy functional

$$V(u) = \int \Big(\frac{|\nabla u|^2}{2} + \frac{u^4}{4} - \frac{u^2}{2} \Big)\, dx, \tag{V}$$

which yields sufficient control on (SAC) to get global solutions. (In the discrete, "zero-dimensional" case, the integral is with respect to the counting measure on Λ and the gradient is a finite difference.) One furthermore shows that the Boltzmann–Gibbs measure

$$\mu_T(du) = Z^{-1} \exp\bigl(-V(u)\,/\,T\bigr)\,du, \tag{BG}$$

where Z is a normalisation constant and du denotes the Lebesgue measure on $\mathbf{R}^\Lambda$, is invariant for the dynamics. The *plat de résistance* of this chapter is a sketch of the proof of the Eyring–Kramers law: provided that 0 is the only saddle point for V, the expected time to go from $+1$ to -1 is asymptotically as $T \to 0$ of order

$$\frac{2\pi}{|\mu_0|}\sqrt{\left|\frac{\det \operatorname{Hess} V(0)}{\det \operatorname{Hess} V(1)}\right|}\, \exp\bigl(\bigl(V(0)-V(1)\bigr)\,/\,T\bigr)\bigl(1+\mathcal{O}(T)\bigr), \tag{EK}$$

with μ_0 the lowest eigenvalue of the Hessian $\operatorname{Hess} V(0)$.

Chapter 3 proceeds to the continuum one-dimensional case. In this case, while (V) still has an obvious meaning, interpreting (BG) and (EK) is a bit more tricky. In the case of the Boltzmann–Gibbs measure, the problem is that there is no Lebesgue measure in infinite dimensions, while the problem with (EK) is that $\operatorname{Hess} V$ is of the form "Laplacian plus constant", so that it is an unbounded operator. Both of these difficulties can be resolved in relatively straightforward ways, in particular the ratio of determinants in (EK) is nothing but the Fredholm determinant $\det(1 - 3(2-\Delta)^{-1})$, but this gives the author a good opportunity to introduce some of the basic concepts in the study of stochastic PDEs, including a solution theory for (SAC), Schauder theory, the description of space-time white noise, etc.

This lays a good foundation on which to build the study the two-dimensional case in Chapter 4. It is in this case that, for the first time, the word "singular" appearing in the title of the book takes its meaning. Indeed, considering solutions to the *linear* stochastic heat equation

$$\partial_t v = \Delta v + \sqrt{2T}\xi,$$

one already finds that these are no longer function-valued in dimension two, but instead do at best take values in some Besov spaces with strictly negative regularity index. As a consequence, it is unclear a priori what "being a solution to (SAC)" actually means in this case. The author gives a short introduction to Wick calculus, which permits to give a meaning to "renormalised" powers $v^{\diamond p}$ of v by means of a suitable approximation procedure. For example, one has $v^{\diamond 2} = \lim_{\varepsilon\to 0}(v_\varepsilon^2 - C_\varepsilon)$, where v_ε is some smooth approximation to v and C_ε is a suitable chosen (and typically diverging as $\varepsilon \to 0$) sequence of constants. It is then natural to *define* solutions to (SAC) by setting $u = v + w$ and looking for w solving

$$\partial_t w = v + w - v^{\diamond 3} - 3v^{\diamond 2}w - 3vw^2 - w^3. \tag{$\star$}$$

It turns out that this not only provides a well-defined solution theory, but u can be approximated by solutions to a version of (SAC) with smoothened noise, provided that the nonlinearity $-u^3$ is replaced by $3C_\varepsilon u - u^3$. A very interesting consequence discussed in Section 4.6 is that the effect of renormalisation is to turn the Fredholm determinant appearing in the Eyring–Kramers formula, which is no longer well-defined since $(2-\Delta)^{-1}$ is no longer trace class, into the well-defined Carleman–Fredholm determinant $\det_2$.

Chapter 5 finally deals with the three-dimensional case. There, while it is still possible to define $v^{\diamond 2}$ and $v^{\diamond 3}$ as random distributions, the equation ($\star$) for the remainder term is itself ill-posed. Dealing with this problem was one of the original motivations for the development of the theory of regularity structures. Building on the concepts introduced in the previous parts, the main goal of this last chapter is to provide an introduction to the various aspects of this theory (reconstruction theorem, lift of various operations, renormalisation, etc.) in the context of the problem of building a robust solution theory for (SAC). Note that in this case, while a Freidlin–Wentzell type large deviations result is still available and is briefly discussed in Section 5.7, the interpretation and justification of the Eyring–Kramers formula is still an open problem, to the best of the reviewer's knowledge.

As the reader may have come to suspect by now, a complete mathematical treatise of all the aspects mentioned here would take much more space than the roughly 200 pages of this short book. Instead, the style chosen by the author is to provide details for some of the simpler proofs and only rough sketches of the main steps for many of the more advanced statements. This strikes a nice balance between self-contained proofs and references to more advanced material and makes the book a must read for anyone with a graduate-level background in probability and analysis who is interested in a quick introduction to the modern tools used in the analysis of singular SPDEs.

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