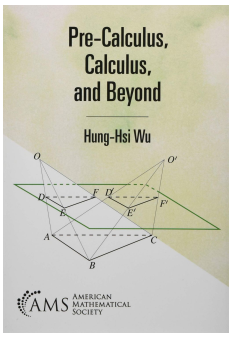
Book reviews

Pre-Calculus, Calculus, and Beyond by Hung-Hsi Wu

Reviewed by António de Bivar Weinholtz



This is the sixth and final book of a series covering the K-12 curriculum, as an instrument for the mathematical education of school teachers. It is the third and final volume of the series dedicated to high-school teachers. Unlike the two previous such volumes, which included topics that had already been treated in the series (to ensure that high-school teachers could have at their disposal a set of self-contained instruments for their mathematical education, expressly written for them, thus not neglecting the pre-requisites to what they have to teach), this final book is composed of entirely new topics.

The first chapter is dedicated to trigonometry and the definition of trigonometric functions with domain $\mathbb{R}$. It starts with the basic definitions, the general notion of extension of a function, then applied to extending trigonometric functions, with the use of the unit circle, to the interval $[-360, 360]$ and finally to $\mathbb{R}$. The laws of sines and cosines, as well as other basic trigonometric identities, such as the addition formulas, are proven in this general setting. It proceeds to the definitions of radian and the new trigonometric functions obtained by switching from degrees to radians, and to the definition of polar coordinates. Finally, trigonometric functions are put to use in the geometric interpretation of complex numbers and the derivation of the De Moivre and Euler formulas, with the exponential notation; applications are given to the study of n-th roots of unity, to a formulation of basic isometries in terms of complex numbers, and to the study of graphs of quadratic functions, with the use of rotations to eliminate the mixed term in general quadratic equations in two variables. The chapter concludes with the introduction of inverse trigonometric functions and a final section where the author analyzes the importance of these functions in the study of general periodic functions, which play a fundamental role in the physics of many phenomena, through Fourier series. More advanced treatments of trigonometric and general exponential functions are given a brief overview, which provides adequate complementary useful knowledge to the readers.

The following chapter proceeds with a rigorous treatment of real numbers. Thus it becomes finally possible to justify what had previously been called "FASM" (the "fundamental assumption of school mathematics") and enabled students to use real numbers, without betraying the basic principles of mathematical studies, from the moment it becomes mandatory for the development of their mathematical instruction, but before it is possible to include in the curriculum a rigorous treatment of the real line, due to the inner complexity of the subject. After an algebraic reformulation of the theory of rational numbers, the introduction of an extra axiom finally leads to the fundamental distinction of the sets of real numbers and rational numbers, that can then be identified with a dense subset of the real line. The concept of limit of a sequence of real numbers is defined and its basic properties are then presented, proved and applied to the rigorous treatment of some of the concepts and properties that had been previously accepted with the use of FASM, namely the existence and basic properties of positive n-th roots of positive real numbers and the fundamental theorem of similarity, followed by a whole chapter dedicated to a full study of the decimal expansion of a number, including repeating and non-repeating decimals, and using the concept and basic properties of infinite series.

A new chapter follows where the delicate concepts of length and area are treated as rigorously as possible at this stage, based on a list of fundamental principles for geometric measurements that are accepted as a guide to the foundation of those concepts, but the inherent difficulties of these topics are explained. In this framework, the author introduces the concept of rectifiable curve and identifies the problems one faces when trying to obtain a rigorous argument that leads to the formula for the circumference of a circle, postponing the final solution to the end of the volume, where a more advanced treatment is given of trigonometric functions, that is put to this use.

Some basic formulas for the area of elementary figures are revisited in this more general setting and obtained using the assumptions of this chapter; a famous proof of the Pythagorean theorem using the concept of area is finally given a proper formulation, whereas it is very often presented to students without the due care to observe that it depends on rather subtle and nontrivial concepts and properties of area and without some apparent geometrical properties being adequately proven. As the author explains in one of his illuminating pedagogical comments, this is another example of how misleading some rather common incoherences in the teaching of school mathematics can be.

After length and area, it is time for the introduction of some comments on three-dimensional geometry and the concept of volume. By the formulation of some elementary principles that, at this stage, have to be accepted without further foundation, the author proceeds to the proof of some basic facts on perpendicularity and parallelism of lines and planes in three-space and to the analysis of Cavalieri's principle, which leads to the formula for the volume of a sphere.

The two final chapters are dedicated to an introduction of derivatives and integrals of real-valued functions of one variable and their basic properties, and applications to trigonometric functions and to new formulations of the logarithmic and exponential functions; they start with the notions of limit of a function in a point and of continuity.

As in the previous volume, this one also contains a very helpful Appendix with a list of assumptions, definitions, theorems and lemmas from the companion volumes. I strongly recommend reading first the review of the first volume (António de Bivar Weinholtz, Book review, "Understanding numbers in elementary school mathematics" by Hung-Hsi Wu, Eur. Math. Soc. Mag. **122** (2021), pp. 66–67). There, one can find the reasons why I deem this set of books a milestone in the struggle for a sound mathematical education of youths. I shall not repeat here all the historical and scientific arguments that sustain this claim, but I have to restate, regarding this final volume, that although it is written for high-school teachers, as an instrument for their mathematical education (both during pre-service years and for their professional development), and to provide a resource for authors of textbooks, the set of its potential readers should not be restricted to those for which it was primarily intended; it should include anyone with the basic ability to appreciate the beauty of the use of human reasoning in our quest to understand the world and the capacity and will to make the necessary efforts, which are required here as for any worthwhile enterprise. Of course, as the content and presentation of the three last volumes of the series is of a more advanced nature, a wider mathematical background is required. This volume being the last of the series, we are now able to fully appreciate the magnitude of the enterprise undertaken by Prof. Wu and how it is indisputable, as I wrote before, that with this set of books at hand there is no excuse left for school (including high-school) teachers, textbook authors and government officials to persist in the unfortunate practice of trying to serve to school students mathematics in a way that is in fact unlearnable…

Like the previous two volumes, this one is punctuated with pedagogical comments that give extremely useful advice regarding what content details should be used in classrooms and which are essentially meant to teachers; mathematical comments are also added to the main text, in order to extend the views of the reader whenever it helps to clarify the subject in question. To the readers interested in the full scope of the pedagogical comments of this volume I also recommend the lecture of my preceding review (António de Bivar Weinholtz, Book review, "Teaching school mathematics: Algebra" by Hung-Hsi Wu, Eur. Math. Soc. Mag. **125** (2022), pp. 50–52), where a detailed description is made of what the author considers to be the main characteristics of mathematics and how they have been neglected in schools for such a long period of time and replaced by what he calls "Textbook School Mathematics" (TSM); the same concern is present in all the topics treated in the present volume.

As it is almost inevitable in any printed book, there are some minor misprints that can be easily detected and corrected by the reader. I just point out some details in formal definitions that deserve some attention.

The definition of the *i*-th term of a sequence (p. 118) as the value assigned by the function (that the sequence is, by definition) to *i* is commonly found in these same terms in many mathematical texts, but it can lead to some awkward consequences; for instance, it is then not strictly true that "every sequence has infinitely many terms", as the appreciation of this statement, with the above given definition, depends on the number of *values* of the sequence rather than on the fact that its *domain* (the set of natural numbers) is infinite. With this definition, a *constant* sequence would have only one term…. A formal definition that would allow us to state that the number of terms of a sequence is always infinite, one for each whole number, could be to identify the *i*-th term of the sequence (s_n) with the ordered pair (i, s_i).

In the definition of convergence of regions (p. 230), apart from the stated condition on the boundaries, one needs some extra condition, as, for instance, the coincidence of the approximating regions with the limit region R outside a "vanishing neighborhood" of the boundary of R. This condition is very easily verified in all the cases where Theorem 4.3 (convergence theorem for area) is applied in this book, and also in the graphical examples that are used in the treatment of area; this treatment, of course, has to rely on some intuitive assumptions at this stage.

Also, the definition of the limit in a point x_0 of a real-valued function defined in a subset I of $\mathbb{R}$ (p. 286) adopted in this book is what we can call the "exclusive" limit, inasmuch as, to "test" the limit of the function, one only considers sequences in I with limit x_0 that never assume the value x_0, as opposed to what we can call the "inclusive" limit definition, where we can also consider such sequences that can assume the value x_0; but in the case of this "exclusive" limit, to ensure that the limit is unique, when it exists, one has to assume as well that x_0 is the limit of a sequence in I that never assumes the value x_0 (x_0 is then usually called an *accumulation point* of I). It is not enough to ensure that it is just the limit of a sequence in I (a *limit point*); if x_0 is what is usually called an *isolated point* of I, i.e., if it is a limit point but not an accumulation point of the domain, with this "exclusive" definition of limit, the function would have every number as limit in x_0 (because to contradict this fact one would have to find a sequence in I that has x_0 as its limit, never assuming the value x_0; but this contradicts the definition of an isolated point). So, either one only considers domains with no isolated points, or one has to define this kind of limit only in accumulation points and not in general limit points of the domain, as the uniqueness of the limit is an essential feature of this concept. Strictly speaking, when considering the algebra of limits of functions, like in Lemma 6.2 (p. 290), one also has to be careful to consider only accumulation points of the domain of the functions obtained by performing each algebraic operation in the pair of functions, as it is not mandatory that if a point is an accumulation point of the domain of each function in the pair it will also have this property with respect to the intersection of domains.

Finally, the definition of continuity (p. 289) is not affected by these subtleties, as it is not dependent on the definition of limit of functions (only in the intuitive motivation of this concept a link is established with limits). In fact, with the alternative ("inclusive") definition of the limit of a function, to be continuous in a point of the domain could simply be defined as having a limit in that point; however, if one aimed to use the adopted "exclusive" limit definition of continuity, one would have to treat separately the isolated points of the domain. Nevertheless, this leads to the conclusion that in the proof of Lemma 6.3, on the "algebra of continuity", one cannot fully rely on Lemma 6.2; once again we could be spared all these subtleties either if one excluded domains with isolated points, or if one considered the "inclusive" definition of limit (in this case, however, with some care also with domains in the algebra of limits).

All these details should not, of course, be brought to a high school classroom, although they can be of some use to teachers.

As in the previous volumes of this series, on each topic the author provides the reader with numerous illuminating activities, and an excellent choice of a wide range of exercises.

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