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On the Asymptotic Behaviors of Transition Probability Densities of One-Dimensional Diffusion Processes

By

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§1. Introduction

Let $\mathbf{X}^{\cdot}=[X^{\cdot}(t), P_x^{\cdot}, x\in Q]$ be a one-dimensional diffusion processes on an interval Q ($\subseteqq R^1$) and $p(t, x, y)$ be its transition probability density with respect to the speed measure $m(dx)$. We are interested in the problem of describing the asymptotic behavior of $p(t, a, a)$ as $t\to 0[\infty]$ in terms of the speed measure m. Concerning this problem, there have been several works when $a\in Q$ is the regular left end point except a trap: By completing I. S. Kac's result [3], Kasahara [4] showed that $p(t, a, a)$ varies regularly in t if and only if $m[a, x)$ does so in x. I. S. Kac [2] discussed the condition of the convergence of integrals related to the spectral function corresponding to the measure $m(dx)$ and this result gives conditions for the convergence of the integrals $\int_{0+}\varphi(t)p(t, a, a)dt$ and $\int^{\infty}\varphi(t)p(t, a, a)dt$ in terms of the measure $m(dx)$ for some class of positive nonincreasing functions $\varphi(t)$.

In this paper we remove the restriction that a is the left end point and obtain the following results for the general case of an interior point or a regular end point. Our main results are following: First, we obtain some inequalities estimating the growth order of the function $p(t, a, a)$ when $t\to 0$ or $+\infty$ using some nonincreasing functions $F_a(t)$ defined in terms of the speed measure m. The proof of these inequalities is based on some inequalities similar to that of I. S. Kac [3] concerning

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$G(\alpha, a, a)=\int_0^\infty e^{-\alpha t}p(t, a, a)dt$ which we will prove by a probabilistic method. Second, using these inequalities we obtain necessary and sufficient conditions, in terms of the speed measure m, for the convergence of the integrals $\int_{0+}\varphi(t)p(t, a, a)dt$ and $\int^\infty \varphi(t)p(t, a, a)dt$ for *any* positive and nonincreasing function $\varphi(t)$.

Our results will be applied to obtain, for a two-dimensional diffusion process which is given as a direct product of one-dimensional diffusion processes, necessary and sufficient conditions for possibility of hitting a given point and for recurrence or transience of the diffusion.

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§2. Main Theorems

Let $l_1[l_2]$ be the left [right] end point Q. Let $m(dx)$ and $k(dx)$ be nonnegative Borel measures on (l_1, l_2), which are finite on each closed subinterval of (l_1, l_2), where $m(U)>0$ for any open set $U\ (\neq\phi)$. Denote the generator of $\mathbf{X}^\cdot$ by $\mathfrak{G}^\cdot$:

$$(\mathfrak{G}^\cdot u)(x)=\{u^+(dx)-u(x)k(dx)\}/m(dx),^{(1)} \qquad x\in Q.$$

If $l_i\in Q$, then the boundary condition is given by

$$(l_i-1) \qquad (-1)^i u^i(l_i)+k_i u(l_i)+m_i(\mathfrak{G}^\cdot u)(l_i)=0,$$

or

$$(l_i-2) \qquad (\mathfrak{G}^\cdot u)(l_i)=-\kappa_i u(l_i),$$

where $0\leqq k_i,\ m_i,\ \kappa_i<\infty$ and $u^1(l_1)\equiv u^+(l_1+)$, $u^2(l_2)\equiv u^-(l_2-)$.[(2)] If $l_i\in Q$, we extend $m(dx)$ and $k(dx)$ on Q so that

$$m(\{l_i\})=m_i,\ k(\{l_i\})=k_i \qquad \textit{if}\ \ (l_i-1),$$

$$m(\{l_i\})=k(\{l_i\})=0, \qquad \textit{if}\ \ (l_i-2).$$

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(1) (2) $u^+(x)=\lim_{y\downarrow x}\frac{u(y)-u(x)}{s(y)-s(x)}$, $u^-(x)=\lim_{y\uparrow x}\frac{u(x)-u(y)}{s(x)-s(y)}$,
where $s(x)$ is the scale, *i.e.* a continuous increasing function on (l_1, l_2).

Let Q^r be the set of all points $a \in Q$ which satisfy either of the following conditions:

(i) $l_1 < a < l_2$,

(ii) $a = l_i$ with $(l_i - 1)$ and $s(l_i)$ is bounded, where $s(l_1) \equiv s(l_1 +)$ and $s(l_2) \equiv s(l_2 -)$.

Let us define the following functions:

$$\overline{U}_a(x) = \int_0^x m \circ t[s(a), s(a) + y) dy,$$

$$U_a(x) = \int_0^x m \circ t(s(a) - y, s(a)] dy,$$

$$U_a(x) = \int_0^x m \circ t(s(a) - y, s(a) + y) dy,$$

$$\overline{V}_a(x) = x m \circ t[s(a), s(a) + x),$$

$$\underline{V}_a(x) = x m \circ t(s(a) - x, s(a)],$$

$$V_a(x) = x m \circ t(s(a) - x, s(a) + x),$$

where $t(x)$ is the inverse function of $s(x)$ and $m \circ t(a, b] = m(t(a), t(b)]$. Let $\overline{\Phi}_a(x)\,[\overline{\Psi}_a(x)]$, $\underline{\Phi}_a(x)\,[\underline{\Psi}_a(x)]$ and $\Phi_a(x)\,[\Psi_a(x)]$ be the inverse functions of $x \mapsto \overline{U}_a(x)\,[\overline{V}_a(x)]$, $x \mapsto \underline{U}_a(x)\,[\underline{V}_a(x)]$ and $x \mapsto U_a(x)\,[V_a(x)]$ respectively. Let us put $P_a(t) = \int_0^t p(s, a, a) ds$.

Through this paper we shall introduce the following notation: we write

$$a(t) \asymp b(t) \quad as \quad t \downarrow 0 \ [t \uparrow \infty]$$

if and only if

$$0 < \varliminf_{\substack{t \downarrow 0 \\ [t \uparrow \infty]}} \frac{a(t)}{b(t)} \leq \varlimsup_{\substack{t \downarrow 0 \\ [t \uparrow \infty]}} \frac{a(t)}{b(t)} < \infty.$$

First we study the asymptotic behavior of $P_a(t)$ as $t \downarrow 0$.

Theorem 2.1. *For every* $a \in Q^r$,

$$P_a(t) \asymp F_a(t) \qquad as \quad t \downarrow 0,$$

where

$$F_a = \begin{cases} \bar{\Phi}_a \text{ or } \bar{\Psi}_a, & \text{if } a = l_1, \\ \underline{\Phi}_a \text{ or } \underline{\Psi}_a, & \text{if } a = l_2, \\ \Phi_a \text{ or } \Psi_a, & \text{if } l_1 < a < l_2. \end{cases}$$

Remark. In particular we see $\bar{\Phi}_a(t) \asymp \bar{\Psi}_a(t)$, $\underline{\Phi}_a(t) \asymp \underline{\Psi}_a(t)$ and $\Phi_a(t) \asymp \Psi_a(t)$ as $t \downarrow 0$ which can be verified directly.

Next we study the case $t \uparrow \infty$. We are interested only in the case $\lim_{t \uparrow \infty} P_a(t) = \infty$ ($a \in Q^r$), which holds if and only if the following conditions are satisfied:

(P–1) $k(Q) = 0$,

(P–2) *if* $s(l_1) > -\infty$, *then* $m(l_1, l_0] < \infty$ *and* $l_1 \in Q$ *with* $(l_1 - 1)$,[3]

(P–3) *if* $s(l_2) < \infty$, *then* $m[l_0, l_2) < \infty$ *and* $l_2 \in Q$ *with* $(l_2 - 1)$,[4]

where l_0 is any fixed point in (l_1, l_2). If, besides (P–1), (P–2) and (P–3), $m(Q) < \infty$ is satisfied, then we have $\lim_{t \uparrow \infty} P_a(t)/t = 1/m(Q)$. Therefore we consider the remaining case $m(Q) = \infty$. We may suppose $m[l_0, l_2) = \infty$ without loss of generality.

Theorem 2.2. *Assume* (P–1), (P–2), $s(l_2) = \infty$ *and* $m[l_0, l_2) = \infty$. *Then for every* $a \in Q^r$,

$$P_a(t) \asymp H_{a_0}(t) \qquad \text{as} \quad t \uparrow \infty,$$

where a_0 *is any fixed point in* (l_1, l_2) *and*

$$H_a = \begin{cases} \bar{\Phi}_a \text{ or } \bar{\Psi}_a, & \text{if } m(l_1, l_0] < \infty, \\ \Phi_a \text{ or } \Psi_a, & \text{if } m(l_1, l_0] = \infty. \end{cases}$$

These asymptotic behaviors of $P_a(t)$ are equivalent to those of the Green function. Let $G(\alpha, x, y)$ be the Green function of $\boldsymbol{X}^{\cdot}$, *i.e.* $G(\alpha,$

(3), (4) If (P–1) and (P–2) [(P–3)] are satisfied, then $(l_i - 1)$ means $(-1)^i u^i(l_i) + m^i(\mathfrak{G}^{\cdot} u)(l_i) = 0$.

$x, y)=\int_0^\infty e^{-\alpha t}p(t, x, y)dt$. Then Theorem 2.1 [2.2] is equivalent to the following Theorem 2.1′ [2.2′], which can be proved by using [5; Lemma 2.1] noting $G(\alpha, a, a)=\int_0^\infty e^{-\alpha t}P_a(dt)$ and $P_a(t)$ is positive, increasing and concave.

Theorem 2.1′. *For every* $a \in Q^r$,

$$G(\alpha, a, a) \asymp F_a(1/\alpha) \qquad \text{as} \quad \alpha \uparrow \infty.$$

Theorem 2.2′. *Assume* (P–1), (P–2), $s(l_2)=\infty$ *and* $m[l_0, l_2)=\infty$. *Then for every* $a \in Q^r$,

$$G(\alpha, a, a) \asymp H_{a_0}(1/\alpha) \qquad \text{as} \quad \alpha \downarrow 0,$$

where a_0 *is any fixed point in* (l_1, l_2).

Since $\lim_{\alpha \downarrow 0} G(\alpha, a, b)/G(\alpha, c, d)=1$ $(a, b, c, d \in Q^r)$ under the assumption of Theorem 2.2 or Theorem 2.2′, Theorem 2.2 and Theorem 2.2′ may be summarized in the following table:

Table 2.1

$k(Q)=0$	$s(l_2)$	$<\infty, l_2 \in Q, (l_2-1)$	$=\infty$
$s(l_1)$	$m[l_0, l_2)$ / $m(l_1, l_0]$	$<\infty$	$=\infty$
$>-\infty$ $l_1 \in Q$ (l_1-1)	$<\infty$	$\lim_{\alpha \downarrow 0} \alpha G(\alpha, a, b)=1/m(Q)$ $\lim_{t \uparrow \infty} P_a(t)/t=1/m(Q)$	$G(\alpha, a, b) \asymp \bar{\Phi}_c(1/\alpha)$ *or* $\bar{\Psi}_c(1/\alpha)$ *as* $\alpha \downarrow 0$ $P_a(t) \asymp \bar{\Phi}_c(t)$ *or* $\bar{\Psi}_c(t)$ *as* $t \uparrow \infty$
$=\infty$	$=\infty$	$G(\alpha, a, b) \asymp \underline{\Phi}_c(1/\alpha)$ *or* $\underline{\Psi}_c(1/\alpha)$ *as* $\alpha \downarrow 0$ $P_a(t) \asymp \underline{\Phi}_c(t)$ *or* $\underline{\Psi}_c(t)$ *as* $t \uparrow \infty$	$G(\alpha, a, b) \asymp \Phi_c(1/\alpha)$ *or* $\Psi_c(1/\alpha)$ *as* $\alpha \downarrow 0$ $P_a(t) \asymp \Phi_c(t)$ *or* $\Psi_c(t)$ *as* $t \uparrow \infty$

where $a, b \in Q^r$ and $c\ (\neq l_i)$ is any fixed point in Q^r.

§3. Proofs of the Main Theorems

First, we obtain some inequalities on $G(\alpha, x, y)$ to prove Theorem 2.1′ and Theorem 2.2′. For this we introduce the *conservative* diffusion $\boldsymbol{X}=[X(t), P_x, x\in Q]$ with the same $m(dx)$ and $s(x)$ as $\boldsymbol{X}^{\cdot}$. In other words, its generator $\mathfrak{G}$ is given by $(\mathfrak{G}u)(x)=u^+(dx)/m(dx)$ and if $l_i\in Q$, the boundary condition is

$$(-1)^i u^i(l_i)+m_i(\mathfrak{G}u)(l_i)=0, \qquad \textit{in case} \quad (l_i-1),$$

$$(\mathfrak{G}u)(l_i)=0, \qquad \textit{in case} \quad (l_i-2).$$

Denoting the local time at x by $\mathfrak{t}(t, x)$ (*i.e.* $\int_A \mathfrak{t}(t, x)m(dx)=\int_0^t I_A(X(s))ds$, $A\in\mathfrak{B}(Q)$), we have

$$G(\alpha, a, b)=E_a\left[\int_0^\infty e^{\mathfrak{k}(t,\alpha)}\mathfrak{t}(dt, b)\right],^{(5)}$$

where $\mathfrak{k}(t, \alpha)=-\alpha t-\int_Q \mathfrak{t}(t, x)k(dx)-\mathfrak{f}_1(t)-\mathfrak{f}_2(t)$,

$$\mathfrak{f}_i(t)=\begin{cases}\kappa_i\times \textit{Lebesgue measure of } \{s;\ X(s)=l_i,\ s\leqq t\}, \\ \qquad\qquad\qquad\qquad \textit{if } l_i\in Q \textit{ and } (l_i-2), \\ 0, \qquad \textit{otherwise.}\end{cases}$$

Define σ_x by

$$\sigma_x=\min\{t;\ X(t)=x,\ t\geqq 0\} \qquad (x\in Q).$$

It follows from the strict Markov property that

$$(3.1) \qquad G(\alpha, a, a)=\frac{h_{\xi\eta}(a, a, \alpha)}{1-g_{\xi\eta}(a, \alpha)f(\xi, a, \alpha)-g_{\eta\xi}(a, \alpha)f(\eta, a, \alpha)},$$

$$\textit{for} \quad a, \xi, \eta\in Q^r \qquad (\xi<a<\eta),$$

and in particular if $m(l_1, l_0]<\infty$ and (P–2) is satisfied, then

(5) See [1; §5.6].

(3.2) $$G(\alpha, a, a)=\frac{h_{\xi\xi}(a, a, \alpha)}{1-f(a, \xi, \alpha)f(\xi, a, \alpha)}, \qquad for \quad a, \xi \in Q^r \quad (a<\xi),$$

where

$$f(x, y, \alpha)=E_x[e^{\mathfrak{k}(\sigma_y, \alpha)}],$$

$$g_{\xi\eta}(x, \alpha)=E_x[e^{\mathfrak{k}(\sigma_\xi, \alpha)}; \sigma_\xi<\sigma_\eta],$$

$$h_{\xi\eta}(x, y, \alpha)=E_x\left[\int_0^{\sigma_\xi \wedge \sigma_\eta} e^{\mathfrak{k}(t, \alpha)}\mathfrak{t}(dt, y)\right].$$

Assume that points x, y, ξ, η in f, g, h are in Q^r. For simplicity we write $i_\alpha(dx)=\alpha m(dx)+k(dx)$ and

$$\bar{I}_a(x, \alpha)=\int_a^x i_\alpha[a, y)s(dy),$$

$$\underset{\sim}{I}_a(x, \alpha)=\int_x^a i_\alpha(y, a)s(dy),$$

$$\bar{J}_a(x, \alpha)=(s(x)-s(a))i_\alpha[a, x),$$

$$\underset{\sim}{J}_a(x, \alpha)=(s(a)-s(x))i_\alpha(x, a).$$

We show the following inequalities, which are analogous to I. S. Kac's one [3; Lemma 2.7].

Lemma 3.1. *Let a be any point of Q^r $(a<l_2)$. Let $\bar{K}_a[\underset{\sim}{K}_a]$ be either of functions $\bar{I}_a$ or $\bar{J}_a[\underset{\sim}{I}_a$ or $\underset{\sim}{J}_a]$. Then for any $\xi, \eta \in Q^r$ such that $\xi<a<\eta$,*

(3.3) $$\varepsilon\delta/\{\varepsilon+\delta+\varepsilon\bar{K}_a(\eta, \alpha)+\delta\underset{\sim}{K}_a(\xi, \alpha)\}$$
$$\leqq G(\alpha, a, a)$$
$$\leqq \varepsilon\delta\{\varepsilon+\delta+\varepsilon\bar{K}_a(\eta, \alpha)+\delta\underset{\sim}{K}_a(\xi, \alpha)\}/\{\varepsilon^2\bar{K}_a(\eta, \alpha)+\delta^2\underset{\sim}{K}_a(\xi, \alpha)\},$$

where $\varepsilon=s(a)-s(\xi)$ and $\delta=s(\eta)-s(a)$. In particular, if $m(l_1, l_0]<\infty$ and (P–2) is satisfied, then for every $\xi \in Q^r$ such that $a<\xi$,

(3.4) $$\delta/\{1+\delta W(a, \alpha)+\bar{K}_a(\xi, \alpha)\}\leqq G(\alpha, a, a)\leqq\delta\{1+1/\bar{K}_a(\xi, \alpha)\},$$

where $\delta = s(\xi) - s(a)$ *and*

$$W(a, \alpha) = \begin{cases} \int_{[l_1, a)\cap Q} f(x, a, \alpha) i_\alpha(dx), & \text{if } a > l_1, \\ 0, & \text{if } a = l_1. \end{cases}$$

Proof. First we show (3.3). Since $h_{\xi\eta}(x, y, \alpha)$ satisfies

$$h_{\xi\eta}(x, y, \alpha) = e_{\xi\eta}(x, y) - \int_\xi^\eta e_{\xi\eta}(x, z) h_{\xi\eta}(z, y, \alpha) i_\alpha(dz),^{(6)}$$

for $x, y \in (\xi, \eta)$, where $e_{\xi\eta}(x, y) = e_{\xi\eta}(y, x) = \{s(x) - s(\xi)\}\{s(\eta) - s(y)\}/\{s(\eta) - s(\xi)\}$ $(\xi \leqq x \leqq y \leqq \eta)$, by using $h_{\xi\eta}(x, y, \alpha) \leqq h_{\xi\eta}(y, y, \alpha)$ we have

$$\begin{aligned} h_{\xi\eta}(a, a, \alpha) &\geqq \varepsilon\delta / \{\varepsilon + \delta + \varepsilon \bar{I}_a(\eta, \alpha) + \delta \underset{\sim}{I}_a(\xi, \alpha)\} \\ &\geqq \varepsilon\delta / \{\varepsilon + \delta + \varepsilon \bar{J}_a(\eta, \alpha) + \delta \underset{\sim}{J}_a(\xi, \alpha)\}. \end{aligned}$$

Because of $G(\alpha, a, a) \geqq h_{\xi\eta}(a, a, \alpha)$ and (3.1), we obtain the lower estimate of $G(\alpha, a, a)$.

Since $f(\cdot, \xi, \alpha)$ is a solution of $\mathfrak{G}u = \alpha u$ and satisfies

$$(-1)^i f^i(l_i, \xi, \alpha) + (k_i + \alpha m_i) f_i(l_i, \xi, \alpha) = 0, \qquad \text{if } l_i \in Q \text{ and } (l_i - 1),$$

we have the estimate

$$(3.5) \qquad f(a, \xi, \alpha) \leqq \begin{cases} (1 + \bar{I}_a(\xi, \alpha))^{-1}, & \text{if } a < \xi, \\ (1 + \underset{\sim}{I}_a(\xi, \alpha))^{-1}, & \text{if } a > \xi. \end{cases}$$

On the other hand, it is easy to see that $g_{\xi\eta}(x, \alpha)$ is nonincreasing whereas $g_{\eta\xi}(x, \alpha)$ is nondecreasing and that they satisfy

$$g_{\xi\eta}(x, \alpha) = \frac{s(\eta) - s(x)}{s(\eta) - s(\xi)} - \int_\xi^\eta e_{\xi\eta}(x, y) g_{\xi\eta}(y, \alpha) i_\alpha(dy),$$

$$g_{\eta\xi}(x, \alpha) = \frac{s(x) - s(\xi)}{s(\eta) - s(\xi)} - \int_\xi^\eta e_{\xi\eta}(x, y) g_{\eta\xi}(y, \alpha) i_\alpha(dy),^{(7)}$$

for $x \in [\xi, \eta]$, and hence

(6) The proof is similar to [1; §5.6 (16)].

(7) See [1; §5.6 (12)].

(3.6) $$g_{\xi\eta}(a, \alpha) \leqq \left\{\frac{\varepsilon+\delta}{\delta} + \underline{I}_a(\xi, \alpha)\right\}^{-1},$$

$$g_{\eta\xi}(a, \alpha) \leqq \left\{\frac{\varepsilon+\delta}{\varepsilon} + \bar{I}_a(\eta, \alpha)\right\}^{-1}.$$

We combine (3.5) and (3.6) to obtain

$$f(\xi, a, \alpha) g_{\xi\eta}(a, \alpha) \leqq \left\{\frac{\varepsilon+\delta}{\delta} + \underline{J}_a(\xi, \alpha)\right\}^{-1}$$

$$\leqq \left\{\frac{\varepsilon+\delta}{\delta} + \underline{I}_a(\xi, \alpha)\right\}^{-1},$$

$$f(\eta, a, \alpha) g_{\eta\xi}(a, \alpha) \leqq \left\{\frac{\varepsilon+\delta}{\varepsilon} + \bar{J}_a(\eta, \alpha)\right\}^{-1}$$

$$\leqq \left\{\frac{\varepsilon+\delta}{\varepsilon} + \bar{I}_a(\eta, \alpha)\right\}^{-1}.$$

Since $h_{\xi\eta}(a, a, \alpha) \leqq e_{\xi\eta}(a, a) = \varepsilon\delta/(\varepsilon+\delta)$, by (3.1) we have the upper estimate of $G(\alpha, a, a)$ in (3.3).

Next we show (3.4). Since $h_{\xi\xi}(x, y, \alpha)$ satisfies

$$h_{\xi\xi}(x, y, \alpha) = e_\xi(x, y) - \int_{[l_1, \xi)} e_\xi(x, z) h_{\xi\xi}(z, y, \alpha) i_\alpha(dz),^{(8)}$$

for $x, y < \xi$, where $e_\xi(x, y) = e_\xi(y, x) = s(\xi) - s(y)$ $(x \leqq y \leqq \xi)$, by using $h_{\xi\xi}(x, y, \alpha) \leqq h_{\xi\xi}(y, y, \alpha)$ we have

$$h_{\xi\xi}(a, a, \alpha) \geqq \delta/\{1 + \delta W(a, \alpha) + \bar{I}_a(\xi, \alpha)\}$$

$$\geqq \delta/\{1 + \delta W(a, \alpha) + \bar{J}_a(\xi, \alpha)\}.$$

Therefore we obtain the lower estimate in (3.4) because $G(\alpha, a, a) \geqq h_{\xi\xi}(a, a, \alpha)$ and (3.2).

The upper estimate follows from (3.2), $h_{\xi\xi}(a, a, \alpha) \geqq e_\xi(a, a) = \delta$ and

$$f(a, \xi, \alpha) f(\xi, a, \alpha) \leqq \{1 + \bar{J}_a(\xi, a)\}^{-1}$$

$$\leqq \{1 + \bar{I}_a(\xi, a)\}^{-1}, \qquad \textit{if} \quad a < \xi,$$

(8) The proof is similar to [1; §5.6 (16)].

where we used (3.5). Thus we obtain the assertion of the lemma. Q.E.D.

We are ready to prove Theorem 2.1′ and Theorem 2.2′.

Proof of Theorem 2.1′. Suppose $l_1 < a < l_2$. There is a $\delta > 0$ such that $t(s(a)-\delta)$, $t(s(a)+\delta) \in Q^r$. Taking $\xi = t(s(a)-\delta)$ and $\eta = t(s(a)+\delta)$ in (3.3), we have

$$\begin{aligned}
&\delta/\{2+\bar{K}_a(t(s(a)+\delta), \alpha)+\underset{\sim}{K}_a(t(s(a)-\delta), \alpha)\} \\
&\quad \leqq G(\alpha, a, a) \\
&\quad \leqq \delta(1+2/\alpha V_a(\delta)) \\
&\quad \leqq \delta(1+2/\alpha U_a(\delta)),
\end{aligned}$$

where U_a and V_a are the functions defined in Section 2, and hence setting $\delta = F_a(1/\alpha)$, we have

$$\begin{aligned}
(3.7)\qquad &F_a(1/\alpha)/\{3+F_a(1/\alpha)k\circ t(s(a)-F_a(1/\alpha), s(a)+F_a(1/\alpha))\} \\
&\quad \leqq G(\alpha, a, a) \\
&\quad \leqq 3F_a(1/\alpha),
\end{aligned}$$

for every $\alpha > 0$ such that $t(s(a) \pm F_a(1/\alpha)) \in Q^r$. Since $\lim_{x \downarrow 0} F_a(x) = 0$, we have

$$1/3 \leqq \varliminf_{\alpha \uparrow \infty} G(\alpha, a, a)/F_a(1/\alpha) \leqq \varlimsup_{\alpha \uparrow \infty} G(\alpha, a, a)/F_a(1/\alpha) \leqq 3.$$

By the same way in case $a = l_i$ we obtain

$$1/2 \leqq \varliminf_{\alpha \uparrow \infty} G(\alpha, a, a)/F_a(1/\alpha) \leqq \varlimsup_{\alpha \uparrow \infty} G(\alpha, a, a)/F_a(1/\alpha) \leqq 2.$$

Thus the theorem is proved. Q.E.D.

Proof of Theorem 2.2′. In case $m(l_1, l_0] = \infty$ the result is obvious from (3.7) and $\lim_{\alpha \downarrow 0} G(\alpha, a, a)/G(\alpha, b, b) = 1$ $(a, b \in Q^r)$. In case $m(l_1, l_0] < \infty$ by the same method as in the proof of Theorem 2.1′ it follows from (3.4) that

$$H_a(1/\alpha)/\{2+\alpha H_a(1/\alpha)m([l_1, a)\cap Q)\}\leqq G(\alpha, a, a)\leqq 2H_a(1/\alpha).$$

Put $M_a=\overline{U}_a$ or $\overline{V}_a$. Since $M_a(x)\geqq xm(a, A)/2$ for all $x>B\equiv 2(s(A)-s(a))$, where $A\ (>a)$ is any fixed number, we have $\alpha H_a(1/\alpha)\leqq 2/m(a, A)$ for all $\alpha\leqq 1/M_a(B)$, so that

$$\varliminf_{\alpha\downarrow 0} G(\alpha, a, a)/H_a(1/\alpha)\geqq 1/\{2+2m([l_1, a)\cap Q)/m(a, A)\}.$$

Since $m[l_0, l_2)=\infty$ and A is arbitrary, letting $A\uparrow l_2$, we obtain

$$1/2\leqq \varliminf_{\alpha\downarrow 0} G(\alpha, a, a)/H_a(1/\alpha)\leqq \varlimsup_{\alpha\downarrow 0} G(\alpha, a, a)/H_a(1/\alpha)\leqq 2,$$

which completes the proof. Q.E.D.

§4. Integral Characteristics

Let φ be a positive nonincreasing function on $(0, \delta)[(\delta, \infty)]$ for some $\delta\in(0, \infty)$. By integration by parts and Theorem 2.1 [2.2] we have Theorem 4.1 [4.2] immediately.

Theorem 4.1. *Fix any $a\in Q^r$. In order that the integral*

$$\int_0^\delta \varphi(t)p(t, a, a)dt$$

converges, it is necessary and sufficient that the integral

$$\int_0^l \varphi(u_a(x))dx$$

converges for some $l>0$, where

$$u_a=\begin{cases} \overline{U}_a \text{ or } \overline{V}_a, & \text{if } a=l_1, \\ \underline{U}_a \text{ or } \underline{V}_a, & \text{if } a=l_2, \\ U_a \text{ or } V_a, & \text{if } l_1<a<l_2. \end{cases}$$

Theorem 4.2. *Assume* (P–1), (P–2) *and* (P–3). *In order that the* *integral*

$$\int_{\delta}^{\infty} \varphi(t) p(t, a, a) dt$$

converges for some (and hence any) $a \in Q^r$, *it is necessary and sufficient that the integral*

$$\int_{l}^{\infty} \varphi(v_b(x)) dx$$

converges for some (and hence any) $b \in (l_1, l_2)$ *and some* $l > 0$, *where* $v_b(x)$ *is the function given in the following table*:

Table 4.1

$v_b(x)=$	$s(l_2)$	$<\infty,\ l\in Q,\ (l_2-1)$	$=\infty$
$s(l_1)$	$m[l_0, l_2)$ \ $m(l_1, l_0]$	$<\infty$	$=\infty$
$>-\infty$, $l_1 \in Q$, (l_1-1)	$<\infty$	x	$\overline{U}_b(x)$ *or* $\overline{V}_b(x)$
$=-\infty$	$=\infty$	$\underline{U}_b(x)$ *or* $\underline{V}_b(x)$	$U_b(x)$ *or* $V_b(x)$

§5. Applications to Two-Dimensional Direct Product Diffusion Processes

Let $X^i = [X^i(t), P^i_x, x \in Q^i]$ $(i=1, 2)$ be a conservative one-dimensional diffusion process with the generator $(\mathfrak{G}^i u)(x) = u^{i+}(dx)/m^i(dx)$, where $u^{i+}(x)$ denotes the right derivative by $s^i(x)$. Assume that X^i satisfies the conditions (P–1), (P–2) and (P–3) for each i. We define a two-dimensional diffusion process X on $Q = Q^1 \times Q^2$ by $X = [X(t) = (X^1(t), X^2(t)), P_x = P^1_{x^1} \times P^2_{x^2}, x = (x^1, x^2) \in Q]$. Let $G(\alpha, x, y)$ be its Green function. On the possibility of hitting a single point a $(\in Q^r \equiv Q^{1r} \times Q^{2r})$

for the sample path of $\boldsymbol{X}$, it is well known that $P_x(\sigma_a<\infty)>0$ $(x\in Q^r)$ if and only if $G(\alpha, a, a)<\infty$. Also it is well known that the process is recurrent if and only if $\lim_{\alpha\downarrow 0} G(\alpha, a, b)=\infty$ $(a, b\in Q^r)$. Combining these, we see that $P_x(\sigma_a<\infty)=1$ $(x\in Q^r)$ if and only if $G(\alpha, a, a)<\infty$ and $\lim_{\alpha\downarrow 0} G(\alpha, a, a)=\infty$. Thus denoting $\bar{\Phi}^i_a(x)$ etc. by the inverse function of $x\mapsto \bar{U}^i_a(x)\equiv\int_0^x m^i\circ t^i[s^i(a), s^i(a)+y)\,dy$ etc., we have the following results from Theorem 4.1. and Theorem 4.2.

Theorem 5.1. *Fix any point* $a=(a^1, a^2)\in Q^r$. *In order to* $P_x(\sigma_a<\infty)>0$ *for all* $x\in Q^r$, *it is necessary and sufficient that the integral*

$$\int_0^\delta \{\mu^1_{a^1}(x)\mu^2_{a^2}(x)\}^{-1}dx \tag{5.1}$$

converges for some $\delta>0$, *where*

$$\mu^i_{a^i}(x)=\begin{cases} m^i\circ t^i[s^i(a^i), s^i(a^i)+\bar{\Phi}^i_{a^i}(x)), & \text{if } a^i=l^i_1, \\ m^i\circ t^i(s^i(a^i)-\underline{\Phi}^i_{a^i}(x), s^i(a^i)], & \text{if } a^i=l^i_2, \\ m^i\circ t^i(s^i(a^i)-\Phi^i_{a^i}(x), s^i(a^i)+\Phi^i_{a^i}(x)), & \\ & \text{if } l^i_1<a^i<l^i_2. \end{cases}$$

Theorem 5.2. *In order that* $\boldsymbol{X}$ *is recurrent it is necessary and sufficient that the integral*

$$\int_l^\infty \{\nu^1_{a^1}(x)\nu^2_{a^2}(x)\}^{-1}dx \tag{5.2}$$

diverges for some $l>0$ *and some (and hence all)* $a=(a^1, a^2)\in Q^r$, *where* $\nu^i_{a^i}(x)$ *is the function given in Table* 5.1.

Theorem 5.3. *Fix any point* $a=(a^1, a^2)\in Q^r$. *In order to* $P_x(\sigma_a<\infty)=1$ *for all* $x\in Q^r$, *it is necessary and sufficient that the integral* (5.1) *converges for some* $\delta>0$ *and the integral* (5.2) *diverges for some* $l>0$.

As an immediate result of Theorem 5.1 we obtain the following:

Corollary 5.4. *Two-dimensional Lebesgue measure of* $\{a \in Q^r;$

Table 5.1

$v^i_{a^i}(x)=$	$s^i(l^i_2)$	$<\infty,\ l^i_2 \in Q^i,\ (l^i_2-1)$	$=\infty$
$s^i(l^i_1)$	$m^i[l^i_0, l^i_2)$ \ $m^i(l^i_1, l^i_0]$	$<\infty$	$=\infty$
$>-\infty$ $l^i_1 \in Q^i$ (l^i_1-1)	$<\infty$	*In this case* (5.2) *always diverges, therefore* **X** *is recurrent.*	$m^i \circ t^i[s^i(a^i), s^i(a^i)+\bar{\Phi}^i_{a^i}(x))$
$=-\infty$	$=\infty$	$m^i \circ t^i(s^i(a^i)-\underline{\Phi}^i_{a^i}(x), s^i(a^i)]$	$m^i \circ t^i(s^i(a^i)-\Phi^i_{a^i}(x), s^i(a^i)+\Phi^i_{a^i}(x))$

Table 5.2

$\int_{0+}\{U^2_{a^2}(x)\}^{-1/2}dx$ \ $\int^{\infty}\{U^2_{a^2}(x)\}^{-1/2}dx$	$=\infty$	$<\infty$
$=\infty$	*recurrent* $p(a)=0$	$p(a)=1$
$<\infty$	*transient* $p(a)=0$	$0<p(a)<1$

Table 5.3

γ	$-1<\gamma<0$	$\gamma=0$	$\gamma>0$
X	*recurrent*		*transient*
$a \in A^\circ$	$p(a)=1$	$p(a)=0$	
$a \notin A^\circ$	$p(a)=0$		

$P_x(\sigma_a<\infty)>0$ *for every* $x\in Q^r\}=0$

Example 5.5. Let $Q^i=R^1$, $s^i(x^i)=x^i$ and $m^1(dx^1)=constant\times dx^1$ (i.e. $\boldsymbol{X}^1$ may be considered as a one-dimensional Brownian motion). Then $U^2_{a^2}(x)=\int_0^x m^2(a^2-y,\ a^2+y)dy$ by the definition. Putting $p(a)=P.(\sigma_a<\infty)$, for any fixed point $a=(a^1,a^2)$ we obtain Table 5.2. In particular if $m^2(dx^2)=constant\times|x^2|^\gamma dx^2$ $(\gamma>-1)$, then we have Table 5.3, where $A^\circ=\{(x^1,0);\ x^1\in R^1\}$. In the case $\gamma=0$ these properties are well known, for $\boldsymbol{X}$ may be considered as a two-dimensional Brownian motion.

Example 5.6. Let $Q^i=R^1$, $s^i(x^i)=x^i$ and $m^1(dx^1)=constant\times|x^1|^\gamma dx^1$ $(\gamma>-1)$. Then for any fixed point $a=(a^1,a^2)$ we obtain the following table.

Table 5.4

$\int^{\infty}\{U^2_{a^2}(x)\}^{-(\gamma+1)/(\gamma+2)}dx$	$a\in B^\circ$		$a\notin B^\circ$	
	$\int_{0+}\{U^2_{a^2}(x)\}^{-(\gamma+1)/(\gamma+2)}dx$		$\int_{0+}\{U^2_{a^2}(x)\}^{-1/2}dx$	
	$=\infty$	$<\infty$	$=\infty$	$<\infty$
$=\infty$	*recurrent*			
	$p(a)=0$	$p(a)=1$	$p(a)=0$	$p(a)=1$
$<\infty$	*transient*			
	$p(a)=0$	$0<p(a)<1$	$p(a)=0$	$0<p(a)<1$

where $B^\circ=\{(0,x^2):\ x^2\in R^1\}$.

References

[1] Itô, K. and Mckean Jr., H. P., *Diffusion processes and their sample paths*, Springer, Berlin-Heidelberg-New York (1965).

[2] Kac, I. S., Integral characteristics of the growth of spectral functions for generalized second order boundary problems with conditions at a regular end, *Izv. Akad. Nauk. SSSR Ser. Mat.*, **38** (1971), 154–184 (in Russian), *Math. USSR-Izv.*, **5** (1971), 161–191 (Engl. transl.).

[3] ———, Generalization of an asymptotic formula of V. A. Marcenko for

spectral functions of a second order boundary value problem, *Izv. Akad. Nauk, SSSR Ser. Mat.*, **37** (1973), 422–436 (in Russian), *Mat. USSR-Izv.*, **7** (1973), 422–436 (Engl. transl.).

[4] Kasahara, Y., Spectral theory of generalized second order differential operators and its applications to Markov processes, *Japan J. Math.*, **1** (1975), 67–84.

[5] Tomisaki, M., Comparison theorems in generalized diffusion processes, *Mem. Fac. Sci. Kyushu Univ.*, **30** (1976), 247–256.