

# Convergence of the Finite Element Method Applied to the Eigenvalue Problem $\Delta u + \lambda u = 0$

By

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## § 1. Introduction

This paper is concerned with the finite element approximation schemes for the eigenvalue problem:

$$(1) \quad \Delta u + \lambda u = 0 \quad \text{in } \mathcal{Q}$$

with the boundary condition

$$u = 0 \quad \text{on } S \text{ (Dirichlet type)}$$

or

$$\partial u / \partial n = 0 \quad \text{on } S \text{ (Neumann type)}$$

where  $\Delta$  is the Laplacian,  $\mathcal{Q}$  is a bounded domain in  $\mathbf{R}^2$ ,  $S$  is the piecewise smooth boundary of  $\mathcal{Q}$ , and  $n$  is the exterior normal.

We put the equation (1) into the weak form:

$$(2) \quad a(u, v) = \lambda(u, v) \quad \text{for any } v \in V$$

where

$$a(u, v) = \int_{\mathcal{Q}} (u_x v_x + u_y v_y) dx dy,$$

$$(u, v) = \int_{\mathcal{Q}} uv dx dy,$$

and where  $V = H_0^1(\mathcal{Q})$  or  $V = H^1(\mathcal{Q})$  according as the boundary condition is Dirichlet type or Neumann type. Here  $H^1(\mathcal{Q})$  is the Sobolev space of order 1 and  $H_0^1(\mathcal{Q})$  is a subset of  $H^1(\mathcal{Q})$  composed of functions vanish-

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ing on  $S$ . We apply to (2) the consistent mass (CM) and the lumped mass (LM) schemes, with the linear interpolation function. Then the corresponding equations may be written in the following forms:

$$(3) \quad K\widehat{U} = \widehat{\lambda}M_1\widehat{U} \quad \text{for the CM scheme,}$$

$$(4) \quad K\bar{U} = \bar{\lambda}M_2\bar{U} \quad \text{for the LM scheme}$$

where  $K$  is the stiffness matrix,  $M_1$  is the consistent mass matrix and  $M_2$  is the lumped mass matrix. Strang and Fix [3] have proved that the approximate eigenvalues and eigenfunctions in the CM scheme converge with a certain rate of convergence to the exact ones. However, for the LM scheme it seems that no such proof has been published explicitly.

In this paper, we propose the mixed mass (MM) scheme:

$$(5) \quad K\tilde{U} = \tilde{\lambda}M_3\tilde{U}$$

where

$$M_3 = \frac{1}{2}(M_1 + M_2)$$

and prove the convergence of the LM and the MM schemes with a certain rate of the convergence. Finally we give the numerical examples. The results by the three schemes show good agreements with the exact solutions. In particular, the MM scheme demonstrates more accurate numerical results than the other schemes.

## § 2. The Order of Convergence

We assume that every eigenvalue of (1) is distinct and simple. Further for simplicity we assume that the domain  $\mathcal{Q}$  is a convex polygon. Let us divide  $\mathcal{Q}$  into the triangles of finite numbers in the usual manner:

$$\bar{\mathcal{Q}} = \bigcup_{k=1}^m \bar{\mathcal{A}}_k,$$

$$\mathcal{A}_i \cap \mathcal{A}_j = \phi \quad (i \neq j)$$

where  $m$  is the number of the triangular elements. The finite number of nodes are denoted by  $P_1, P_2, \dots, P_n$ . We further assume that any

two adjacent triangles have an only common side. Now we define the finite dimensional spaces  $X^h \subset L^2(\Omega)$  and  $Y^h \subset V$  as follows:

$$X^h = \{\bar{\phi} : \bar{\phi} \text{ is a piecewise constant function in each triangular element in the sense stated below}\},$$

$$Y^h = \{\hat{\phi} : \hat{\phi} \text{ is a function which is linear in each triangular element and } \hat{\phi} \in V\}$$

where  $h$  is the length of the largest side of all triangular elements. ‘‘Piecewise constant’’ means, for instance,

$$\bar{\phi}(P) = \bar{\phi}(P_i), \quad P \in S_i (i=1, 2, 3)$$

where  $S_i$  is defined as in Figure 1. Then there exist the basis  $\{\bar{\phi}_1, \dots, \bar{\phi}_n\}$  and  $\{\hat{\phi}_1, \dots, \hat{\phi}_n\}$  for  $X^h$  and  $Y^h$ , respectively, such that

$$\bar{\phi}_i(P_j) = \hat{\phi}_i(P_j) = \delta_{ij}.$$

Every  $\bar{\phi} \in X^h$  or  $\hat{\phi} \in Y^h$  can be uniquely determined as

$$\begin{aligned} \bar{\phi} &= \sum_{i=1}^n \phi_i \bar{\phi}_i, \\ \hat{\phi} &= \sum_{i=1}^n \phi_i \hat{\phi}_i \end{aligned}$$

where  $\phi_1, \dots, \phi_n$  are nodal values. We say that  $\bar{\phi} \in X^h$  and  $\hat{\phi} \in Y^h$  are associated ( $\bar{\phi} \sim \hat{\phi}$ ), if they have a common nodal value at each nodal point.

We will use the following notations and definitions:

$$(u, v) = \int_{\Omega} uv dx dy,$$

$$\|u\|^2 = (u, u),$$

$$a(u, v) = \int_{\Omega} (u_x v_x + u_y v_y) dx dy,$$

$$K : \text{stiffness matrix } K = \{a(\hat{\phi}_i, \hat{\phi}_j)\} \quad (1 \leq i, j \leq n),$$

$$M_1 : \text{CM matrix } M_1 = \{(\hat{\phi}_i, \hat{\phi}_j)\} \quad (1 \leq i, j \leq n),$$

$$M_2 : \text{LM matrix } M_2 = \{(\bar{\phi}_i, \bar{\phi}_j)\} \quad (1 \leq i, j \leq n),$$

$$M_3 : \text{MM matrix } M_3 = \{((\bar{\phi}_i, \bar{\phi}_j) + (\hat{\phi}_i, \hat{\phi}_j))/2\} \quad (1 \leq i, j \leq n),$$

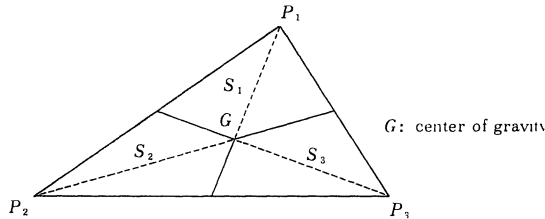


Figure 1. The region  $S_i$

$\lambda_i$  : the eigenvalues of (1) ( $\lambda_1 < \lambda_2 < \dots$ ),  
 $u_i$  : the eigenfunctions corresponding to  $\lambda_i$  ( $(u_i, u_j) = \delta_{ij}$ ),  
 $\hat{\lambda}_i$  : the eigenvalues of (3) ( $\hat{\lambda}_1 \leq \hat{\lambda}_2 \leq \dots \leq \hat{\lambda}_n$ ),  
 $\hat{U}_i$  : the eigenvectors corresponding to  $\hat{\lambda}_i$  ( $\hat{U}_i^t M_1 \hat{U}_j = \delta_{ij}$ ),  
 $\hat{v}_i$  : the eigenfunctions corresponding to  $\hat{\lambda}_i$

$$((\hat{v}_i, \hat{v}_j) = \delta_{ij}, (u_i, \hat{v}_i) \geq 0),$$

$\bar{\lambda}_i$  : the eigenvalues of (4) ( $\bar{\lambda}_1 \leq \bar{\lambda}_2 \leq \dots \leq \bar{\lambda}_n$ ),  
 $\bar{U}_i$  : the eigenvectors corresponding to  $\bar{\lambda}_i$  ( $\bar{U}_i^t M_2 \bar{U}_j = \delta_{ij}$ ),  
 $\{\bar{w}_i, \hat{w}_i\}$  : the eigenfunctions corresponding to  $\bar{\lambda}_i$

$$(\bar{w}_i \sim \hat{w}_i, (\bar{w}_i, \bar{w}_j) = \delta_{ij}, (u_i, \bar{w}_i) \geq 0),$$

$\tilde{\lambda}_i$  : the eigenvalues of (5) ( $\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \dots \leq \tilde{\lambda}_n$ ),  
 $\tilde{U}_i$  : the eigenvectors corresponding to  $\tilde{\lambda}_i$  ( $\tilde{U}_i^t M_3 \tilde{U}_j = \delta_{ij}$ ),  
 $\{\bar{z}_i, \hat{z}_i\}$  : the eigenfunctions corresponding to  $\tilde{\lambda}_i$

$$(\bar{z}_i \sim \hat{z}_i, ((\bar{z}_i, \bar{z}_j) + (\hat{z}_i, \hat{z}_j))/2 = \delta_{ij}, (u_i, \bar{z}_i) + (u_i, \hat{z}_i) \geq 0).$$

The solutions  $\{\hat{\lambda}, \hat{v}\}$  of the CM scheme,  $\{\bar{\lambda}, \bar{w}, \hat{w}\}$  of the LM scheme and  $\{\tilde{\lambda}, \bar{z}, \hat{z}\}$  of the MM scheme are defined as follows:

$$\begin{aligned}
 a(\hat{v}, \hat{u}) &= \hat{\lambda}(\hat{v}, \hat{u}), \quad \hat{v} \in Y^h \quad \text{for any } \hat{u} \in Y^h, \\
 a(\hat{w}, \hat{u}) &= \bar{\lambda}(\bar{w}, \hat{u}), \quad \bar{w} \in X^h, \quad \hat{w} \in Y^h, \quad \bar{w} \sim \hat{w} \\
 &\quad \text{for any } \bar{u} \in X^h, \quad \hat{u} \in Y^h, \quad \bar{u} \sim \hat{u}, \\
 a(\hat{z}, \hat{u}) &= \frac{\tilde{\lambda}}{2} \{(\bar{z}, \bar{u}) + (\hat{z}, \hat{u})\}, \quad \bar{z} \in X^h, \quad \hat{z} \in Y^h, \quad \bar{z} \sim \hat{z} \\
 &\quad \text{for any } \bar{u} \in X^h, \quad \hat{u} \in Y^h, \quad \bar{u} \sim \hat{u}.
 \end{aligned}$$

In order to prove our results, we prepare some lemmas.

**Lemma 1.** ([1]) *For any  $\bar{\phi} \in X^h$  and  $\hat{\phi} \in Y^h$  ( $\bar{\phi} \sim \hat{\phi}$ ), there exists a constant  $c$ , which is independent of  $h$ , such that*

$$\|\bar{\phi} - \hat{\phi}\|^2 \leq ch^2 a(\hat{\phi}, \hat{\phi}).$$

**Lemma 2.** ([1]) *For any  $\bar{\phi} \in X^h$  and  $\hat{\phi} \in Y^h$  ( $\bar{\phi} \sim \hat{\phi}$ ), it holds*

$$\|\bar{\phi}\| \geq \|\hat{\phi}\|.$$

**Lemma 3.** *For any  $\bar{\phi} \in X^h$  and  $\hat{\phi} \in Y^h$  ( $\bar{\phi} \sim \hat{\phi}$ ), there exists a constant  $C$ , which is independent of  $h$ , such that*

$$\|\bar{\phi}\|^2 - \|\hat{\phi}\|^2 \leq Cha(\hat{\phi}, \hat{\phi}).$$

*Proof.* By the Schwarz inequality, the Poincaré inequality and Lemma 1, we have

$$\begin{aligned} \|\bar{\phi}\|^2 - \|\hat{\phi}\|^2 &= 2(\hat{\phi}, \bar{\phi} - \hat{\phi}) + (\bar{\phi} - \hat{\phi}, \bar{\phi} - \hat{\phi}) \leq 2\|\hat{\phi}\| \cdot \|\bar{\phi} - \hat{\phi}\| + \|\bar{\phi} - \hat{\phi}\|^2 \\ &\leq C_0 \sqrt{a(\hat{\phi}, \hat{\phi})} \cdot \sqrt{c} h \sqrt{a(\hat{\phi}, \hat{\phi})} + ch^2 a(\hat{\phi}, \hat{\phi}) \\ &\leq Cha(\hat{\phi}, \hat{\phi}) \end{aligned}$$

where  $C_0$  and  $C = \max\{C_0\sqrt{c}, c \cdot \text{diam}(\mathcal{Q})\}$  are constants, which are independent of  $h$ . This completes the proof.<sup>(\*)</sup>

**Lemma 4.** (min-max principle) *Let  $S_i$  denote any  $i$ -dimensional subspace of  $V$ . Then, eigenvalues are characterized by the equation*

$$\lambda_i = \min_{S_i} \max_{\substack{u \in S_i \\ u \neq 0}} \frac{a(u, u)}{\|u\|^2}.$$

For the CM scheme, Strang and Fix[3] proved the following two theorems.

**Theorem 1.** *For small  $h$  and the fixed  $i$ , there exists a constant  $\hat{c}_i$ , which is independent of  $h$ , such that*

$$\lambda_i \leq \hat{\lambda}_i \leq \lambda_i + \hat{c}_i h^2.$$

**Theorem 2.** *For small  $h$  and the fixed  $i$ , there exists a constant  $\hat{c}_i$ , which is independent of  $h$ , such that*

$$\|u_i - \hat{v}_i\| \leq \hat{c}_i h^2.$$

Now we can prove the following results for the LM and the MM schemes. First we give error bounds for the eigenvalues.

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<sup>(\*)</sup> As a matter of fact, if the boundary condition is of the Neumann type, we have to replace  $a(u, v)$  by  $a(u, v) + \gamma(u, v)$  with a positive  $\gamma$  so that  $a(u, u) \geq \gamma \|u\|^2$ . However, if  $\mathcal{Q}$  has a certain symmetry and if we consider higher modes as in the numerical examples, we can apply the Poincaré inequality directly.

**Theorem 3.** *For small  $h$  and the fixed  $i$ , there exists a constant  $c_i$ , which is independent of  $h$ , such that*

$$\lambda_i - c_i h \leq \bar{\lambda}_i \leq \tilde{\lambda}_i \leq \hat{\lambda}_i \leq \lambda_i + \hat{c}_i h^2.$$

*Proof.* By Lemma 2, we have

$$\frac{a(\hat{\phi}, \bar{\phi})}{\|\bar{\phi}\|^2} \leq \frac{a(\hat{\phi}, \hat{\phi})}{\frac{1}{2}(\|\bar{\phi}\|^2 + \|\hat{\phi}\|^2)} \leq \frac{a(\hat{\phi}, \hat{\phi})}{\|\hat{\phi}\|^2} \quad \text{for } \bar{\phi} \sim \hat{\phi} (\neq 0).$$

Therefore we obtain

$$\frac{U^*KU}{U^*M_2U} \leq \frac{U^*KU}{U^*M_3U} \leq \frac{U^*KU}{U^*M_1U} \quad \text{for any vector } U (\neq 0).$$

The eigenvalues  $\bar{\lambda}_i$ ,  $\tilde{\lambda}_i$  and  $\hat{\lambda}_i$  are also characterized by Lemma 4 in the following manner:

$$\begin{aligned} \bar{\lambda}_i &= \min_{v_i} \max_{\substack{u \in \bar{V}_i \\ u \neq 0}} \frac{U^*KU}{U^*M_2U}, \\ \tilde{\lambda}_i &= \min_{v_i} \max_{\substack{u \in \tilde{V}_i \\ u \neq 0}} \frac{U^*KU}{U^*M_3U}, \\ \hat{\lambda}_i &= \min_{v_i} \max_{\substack{u \in \hat{V}_i \\ u \neq 0}} \frac{U^*KU}{U^*M_1U} \end{aligned}$$

where  $V_i$  is any  $i$ -dimensional vector space of  $\mathbf{R}^n$ . Further we define the spaces  $\bar{V}_i$ ,  $\tilde{V}_i$  and  $\hat{V}_i$  as follows:

$$\begin{aligned} \bar{V}_i &= \text{Span}[\bar{U}_1, \dots, \bar{U}_i], \\ \tilde{V}_i &= \text{Span}[\tilde{U}_1, \dots, \tilde{U}_i], \\ \hat{V}_i &= \text{Span}[\hat{U}_1, \dots, \hat{U}_i]. \end{aligned}$$

Then, we have

$$\bar{\lambda}_i \leq \max_{u \in \bar{V}_i} \frac{U^*KU}{U^*M_2U} \leq \max_{u \in \tilde{V}_i} \frac{U^*KU}{U^*M_3U} = \tilde{\lambda}_i \leq \max_{u \in \hat{V}_i} \frac{U^*KU}{U^*M_3U} \leq \max_{u \in \hat{V}_i} \frac{U^*KU}{U^*M_1U} = \hat{\lambda}_i.$$

By Lemma 3, we have

$$\frac{a(\hat{\phi}, \bar{\phi})}{\|\bar{\phi}\|^2} \geq \frac{a(\hat{\phi}, \hat{\phi})}{\|\hat{\phi}\|^2 + Cha(\hat{\phi}, \hat{\phi})} \quad \text{for any } \bar{\phi} \sim \hat{\phi} (\neq 0).$$

Hence we have

$$\begin{aligned} \bar{\lambda}_i &= \max_{v \in \bar{V}_i} \frac{U^i K U}{U^i M_2 U} \geq \max_{v \in \bar{V}_i} \frac{U^i K U}{U^i M_1 U + Ch \cdot U^i K U} = \max_{v \in \bar{V}_i} \frac{1}{\frac{1}{\frac{U^i K U}{U^i M_1 U}} + Ch} \\ &\geq \frac{1}{\frac{1}{\hat{\lambda}_i} + Ch} \geq \frac{1}{\lambda_i + Ch}. \end{aligned}$$

Therefore, by Theorem 1, we obtain

$$\lambda_i + \hat{c}_i h^2 \geq \hat{\lambda}_i \geq \tilde{\lambda}_i \geq \bar{\lambda}_i \geq \lambda_i (1 + \lambda_i Ch)^{-1} \geq \lambda_i - c_i h.$$

This completes the proof.

The following two theorems give error bounds to the approximate eigenfunctions.

**Theorem 4.** *Let  $\{\bar{\lambda}_i, \bar{w}_i, \hat{w}_i\}$  be the LM solutions. For small  $h$  and the fixed  $i$ , there exists a constant  $\bar{c}_i$ , which is independent of  $h$ , such that*

$$\|u_i - \bar{w}_i\| \leq \bar{c}_i h,$$

$$\|u_i - \hat{w}_i\| \leq \bar{c}_i h.$$

*Proof.* Define two mappings  $P$  and  $L$  as follows:

$$P: V \rightarrow Y^h, \quad a(Pu, \hat{v}) = a(u, \hat{v}) \quad \text{for any } \hat{v} \in Y^h,$$

$$L: V \rightarrow X^h, \quad (Lu, \bar{v}) = (u, \hat{v}) \quad \text{for any } \bar{v} \in X^h, \hat{v} \in Y^h, \bar{v} \sim \hat{v}.$$

By the approximation theorem ([3]), there exists a constant  $C_1$ , which is independent of  $h$ , such that

$$\|u_i - Pu_i\| \leq C_1 h,$$

$$\|u_i - Lu_i\| \leq C_1 h.$$

Since  $a(Pu_i, \hat{w}_j) = a(u_i, \hat{w}_j)$  by the definition of the weak solution and since  $(Lu_i, \bar{w}_j) = (u_i, \hat{w}_j)$ , we have

$$\begin{aligned} (\bar{\lambda}_j - \lambda_i) (\bar{P}u_i, \bar{w}_j) &= \bar{\lambda}_j (\bar{P}u_i, \bar{w}_j) - \lambda_i (\bar{P}u_i, \bar{w}_j) = a(Pu_i, \hat{w}_j) - \lambda_i (\bar{P}u_i, \bar{w}_j) \\ &= a(u_i, \hat{w}_j) - \lambda_j (\bar{P}u_i, \bar{w}_j) = \lambda_i (u_i, \hat{w}_j) - \lambda_i (\bar{P}u_i, \bar{w}_j) \end{aligned}$$

$$= \lambda_i(Lu_i, \bar{w}_j) - \lambda_i(\bar{P}u_i, \bar{w}_j) = \lambda_i(Lu_i - \bar{P}u_i, \bar{w}_j)$$

where  $Pu_i \sim \bar{P}u_i \in X^h$ . Because the set  $[\bar{w}_1, \dots, \bar{w}_n]$  forms an orthonormal basis for  $X^h$ , we have

$$\bar{P}u_i = \sum_{j=1}^n (\bar{P}u_i, \bar{w}_j) \bar{w}_j.$$

Since  $\lambda_i$  is distinct from the other eigenvalues, for sufficiently small  $h$ , there exists a constant  $C_2$ , which is independent of  $h$ , such that

$$|\lambda_i / (\bar{\lambda}_j - \lambda_i)| \leq C_2 \quad \text{for all } j (\neq i).$$

Hence, putting  $\beta = (\bar{P}u_i, \bar{w}_i)$ , there exists a constant  $C_3$ , which is independent of  $h$ , such that

$$\begin{aligned} \|\bar{P}u_i - \beta \bar{w}_i\|^2 &= \sum_{j \neq i} (\bar{P}u_i, \bar{w}_j)^2 = \sum_{j \neq i} \left( \frac{\lambda_i}{\bar{\lambda}_j - \lambda_i} \right)^2 (Lu_i - \bar{P}u_i, \bar{w}_j)^2 \\ &\leq C_2^2 \sum_{j \neq i} (Lu_i - \bar{P}u_i, \bar{w}_j)^2 \leq C_2^2 \|Lu_i - \bar{P}u_i\|^2 \\ &\leq C_2^2 \{ \|Lu_i - u_i\| + \|u_i - Pu_i\| + \|Pu_i - \bar{P}u_i\| \}^2 \leq C_3^2 h^2. \end{aligned}$$

Therefore, there exists a constant  $C_4$ , which is independent of  $h$ , such that

$$\|u_i - \beta \bar{w}_i\| \leq \|u_i - \bar{P}u_i\| + \|\bar{P}u_i - \beta \bar{w}_i\| \leq C_4 h.$$

By the triangle inequality, we have

$$\|u_i\| - \|u_i - \beta \bar{w}_i\| \leq \|\beta \bar{w}_i\| \leq \|u_i\| + \|u_i - \beta \bar{w}_i\|.$$

Recalling that  $u_i$  and  $\bar{w}_i$  are normalized, and choosing the sign so that  $\beta \geq 0$ , we have

$$|\beta - 1| \leq \|u_i - \beta \bar{w}_i\|.$$

Therefore, there exist constants  $C_5$  and  $C_6$ , which are independent of  $h$ , such that

$$\begin{aligned} \|u_i - \bar{w}_i\| &\leq \|u_i - \beta \bar{w}_i\| + \|\beta \bar{w}_i - \bar{w}_i\| = \|u_i - \beta \bar{w}_i\| + |\beta - 1| \leq 2 \|u_i - \beta \bar{w}_i\| \\ &\leq 2C_4 h = C_5 h, \end{aligned}$$

$$\|u_i - \hat{w}_i\| \leq \|u_i - \bar{w}_i\| + \|\bar{w}_i - \hat{w}_i\| \leq C_6 h.$$

The proof is complete by taking  $\bar{c}_i = \max\{C_5, C_6\}$ .



**Theorem 5.** *Let  $\{\tilde{\lambda}_i, \bar{z}_i, \hat{z}_i\}$  be the MM solutions. For small  $h$  and the fixed  $i$ , there exists a constant  $\tilde{c}_i$ , which is independent of  $h$ , such that*

$$\|u_i - \bar{z}_i\|^2 + \|u_i - \hat{z}_i\|^2 \leq \tilde{c}_i h^2.$$

*Proof.* We introduce two spaces  $H = L^2(\Omega) \times V$  and

$$\tilde{H} = \{ \{\bar{u}, \hat{u}\} : \bar{u} \in X^h, \hat{u} \in Y^h, \bar{u} \sim \hat{u} \}.$$

Addition and scalar multiplication are defined in the obvious manner and the inner product and the norm are defined by

$$[\{u, v\}, \{w, z\}] = \frac{1}{2} \{ (u, w) + (v, z) \},$$

$$\| \{u, v\} \|^2 = [ \{u, v\}, \{u, v\} ].$$

Define two mappings  $P$  and  $Q$  as follows:

$$P: V \rightarrow Y^h, \quad a(Pu, \hat{v}) = a(u, \hat{v}) \quad \text{for any } \hat{v} \in Y^h,$$

$$Q: H \rightarrow \tilde{H}, \quad [Q\{u, v\}, \{\bar{w}, \hat{w}\}] = [ \{u, v\}, \{\bar{w}, \hat{w}\} ]$$

$$\text{for any } \{\bar{w}, \hat{w}\} \in \tilde{H}.$$

By the approximation theorem ([3]), there exists a constant  $\tilde{C}_1$ , which is independent of  $h$ , such that

$$\|u_i - Pu_i\| \leq \tilde{C}_1 h,$$

$$\| \{u_i, u_i\} - Q\{u_i, u_i\} \| \leq \tilde{C}_1 h.$$

Since  $a(Pu_i, \hat{z}_j) = a(u_i, \hat{z}_j)$  by the definition of the weak solution, and since  $[Q\{u_i, u_i\}, \{\bar{z}_j, \hat{z}_j\}] = [ \{u_i, u_i\}, \{\bar{z}_j, \hat{z}_j\} ]$ , we have

$$\begin{aligned} (\tilde{\lambda}_j - \lambda_i) [ \{\bar{P}u_i, Pu_i\}, \{\bar{z}_j, \hat{z}_j\} ] &= \tilde{\lambda}_j [ \{\bar{P}u_i, Pu_i\}, \{\bar{z}_j, \hat{z}_j\} ] \\ &\quad - \lambda_i [ \{\bar{P}u_i, Pu_i\}, \{\bar{z}_j, \hat{z}_j\} ] \\ &= a(Pu_i, \hat{z}_j) - \lambda_i [ \{\bar{P}u_i, Pu_i\}, \{\bar{z}_j, \hat{z}_j\} ] \\ &= a(u_i, \hat{z}_j) - \lambda_i [ \{\bar{P}u_i, Pu_i\}, \{\bar{z}_j, \hat{z}_j\} ] \\ &= \lambda_i (u_i, \hat{z}_j) - \lambda_i [ \{\bar{P}u_i, Pu_i\}, \{\bar{z}_j, \hat{z}_j\} ] \\ &= \lambda_i [ \{u_i, u_i\}, \{\bar{z}_j, \hat{z}_j\} ] - \lambda_i [ \{\bar{P}u_i, Pu_i\}, \{\bar{z}_j, \hat{z}_j\} ] \\ &= \lambda_i [ Q\{u_i, u_i\}, \{\bar{z}_j, \hat{z}_j\} ] - \lambda_i [ \{\bar{P}u_i, Pu_i\}, \{\bar{z}_j, \hat{z}_j\} ] \end{aligned}$$

$$= \lambda_i [ \{Q\{u_i, u_i\} - \{\overline{Pu}_i, Pu_i\}\}, \{\bar{z}_j, \hat{z}_j\} ]$$

where  $Pu_i \sim \overline{Pu}_i \in X^n$ . Because the set  $[\{\bar{z}_1, \hat{z}_1\}, \dots, \{\bar{z}_n, \hat{z}_n\}]$  forms an orthonormal basis for  $\tilde{H}$ , we have

$$\{\overline{Pu}_i, Pu_i\} = \sum_{j=1}^n [ \{\overline{Pu}_i, Pu_i\}, \{\bar{z}_j, \hat{z}_j\} ] \{\bar{z}_j, \hat{z}_j\}.$$

On the other hand, there exists a separation constant  $\tilde{C}_2$ , which is independent of  $h$ , such that

$$|\lambda_i / (\tilde{\lambda}_j - \lambda_i)| \leq \tilde{C}_2 \quad \text{for all } j (\neq i),$$

since the eigenvalue  $\lambda_i$  is distinct from the other eigenvalues. Hence, putting  $\beta = [ \{\overline{Pu}_i, Pu_i\}, \{\bar{z}_i, \hat{z}_i\} ]$ , there exists a constant  $\tilde{C}_3$ , which is independent of  $h$ , such that

$$\begin{aligned} \|\| \{\overline{Pu}_i, Pu_i\} - \beta \{\bar{z}_i, \hat{z}_i\} \|\|^2 &= \sum_{j \neq i} [ \{\overline{Pu}_i, Pu_i\}, \{\bar{z}_j, \hat{z}_j\} ]^2 \\ &= \sum_{j \neq i} \left( \frac{\lambda_i}{\tilde{\lambda}_j - \lambda_i} \right)^2 [ \{Q\{u_i, u_i\} - \{\overline{Pu}_i, Pu_i\}\}, \{\bar{z}_j, \hat{z}_j\} ]^2 \\ &\leq \tilde{C}_2^2 \|\| \{Q\{u_i, u_i\} - \{\overline{Pu}_i, Pu_i\}\} \|\|^2 \\ &\leq \tilde{C}_2^2 \{ \|\| Q\{u_i, u_i\} - \{u_i, u_i\} \|\| + \|\| \{u_i, u_i\} - \{\overline{Pu}_i, Pu_i\} \|\| \}^2 \\ &\leq \tilde{C}_3^2 h^2. \end{aligned}$$

Therefore, there exists a constant  $\tilde{C}_4$ , which is independent of  $h$ , such that

$$\begin{aligned} \|\| \{u_i, u_i\} - \beta \{\bar{z}_i, \hat{z}_i\} \|\| &\leq \|\| \{u_i, u_i\} - \{\overline{Pu}_i, Pu_i\} \|\| + \|\| \{\overline{Pu}_i, Pu_i\} - \beta \{\bar{z}_i, \hat{z}_i\} \|\| \\ &\leq \tilde{C}_4 h. \end{aligned}$$

By the triangle inequality, we have

$$\begin{aligned} \|\| \{u_i, u_i\} \|\| - \|\| \{u_i, u_i\} - \beta \{\bar{z}_i, \hat{z}_i\} \|\| &\leq \|\| \beta \{\bar{z}_i, \hat{z}_i\} \|\| \\ &\leq \|\| \{u_i, u_i\} \|\| + \|\| \{u_i, u_i\} - \beta \{\bar{z}_i, \hat{z}_i\} \|\|. \end{aligned}$$

Choosing the sign so that  $\beta \geq 0$ , we obtain

$$|\beta - 1| \leq \|\| \{u_i, u_i\} - \beta \{\bar{z}_i, \hat{z}_i\} \|\|,$$

since  $\|\| \{u_i, u_i\} \|\| = \|\| u_i \|\| = 1$ . Therefore, we have

$$\begin{aligned} \|\| \{u_i, u_i\} - \{\bar{z}_i, \hat{z}_i\} \|\| &\leq \|\| \{u_i, u_i\} - \beta \{\bar{z}_i, \hat{z}_i\} \|\| + \|\| \beta \{\bar{z}_i, \hat{z}_i\} - \{\bar{z}_i, \hat{z}_i\} \|\| \\ &\leq 2 \|\| \{u_i, u_i\} - \beta \{\bar{z}_i, \hat{z}_i\} \|\| \leq 2\tilde{C}_4 h. \end{aligned}$$

By taking  $\tilde{c}_i = 8\tilde{C}_4^2$ , we can obtain

$$\|u_i - \bar{z}_i\|^2 + \|u_i - \hat{z}_i\|^2 \leq \tilde{c}_i h^2.$$

This completes the proof.

### § 3. Numerical Examples

For the purpose of checking the numerical accuracy of the finite element schemes, we have dealt with two examples of the rectangular domain  $(L_x \times L_y)$ . The first example is the following equation:

$$\begin{aligned} \Delta u + \lambda u &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } S \text{ (Dirichlet type)}. \end{aligned}$$

The exact eigenvalues and the corresponding eigenfunctions are given by

$$\begin{aligned} \lambda &= \pi^2 \left\{ \left( \frac{m}{L_x} \right)^2 + \left( \frac{n}{L_y} \right)^2 \right\}, \\ u &= \sin \frac{m\pi x}{L_x} \cdot \sin \frac{n\pi y}{L_y}, \quad (m, n = 1, 2, 3, \dots). \end{aligned}$$

As the second example, we have dealt with the room acoustic wave equation for steady state confined by a rigid wall. The equation is the following form:

$$\begin{aligned} \Delta P + \frac{\omega^2}{c^2} P &= 0 \quad \text{in } \Omega \\ \partial P / \partial n &= 0 \quad \text{on } S \text{ (Neumann type)} \end{aligned}$$

where  $P$  is the sound pressure,  $c$  is the sound speed and  $\omega$  is the angular frequency. The acoustic analysis is to determine the normal frequency and the corresponding eigenfunction  $P$ . The exact normal frequency values  $f$  and the corresponding eigenfunctions  $P$  are given by

$$\begin{aligned} f = 2\pi / \omega &= \frac{2}{c} \sqrt{\left( \frac{m}{L_x} \right)^2 + \left( \frac{n}{L_y} \right)^2}, \\ P &= \cos \frac{m\pi x}{L_x} \cdot \cos \frac{n\pi y}{L_y}, \quad (m, n = 0, 1, 2, \dots). \end{aligned}$$

Table 1, 2 and Figure 2, 3 show the results of the computations in comparison with the exact eigenvalues and the normal eigenfrequency values (in Hz) for a  $2.0 \times 1.1$  rectangular domain. They demonstrate

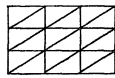
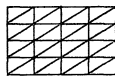
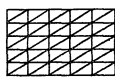
that the eigenvalues and normal frequency values converge with the mesh size and the CM scheme gives the upper bound and the LM scheme gives the lower bound for the exact values. And the MM scheme gives more accurate approximations between the CM scheme and the LM scheme.

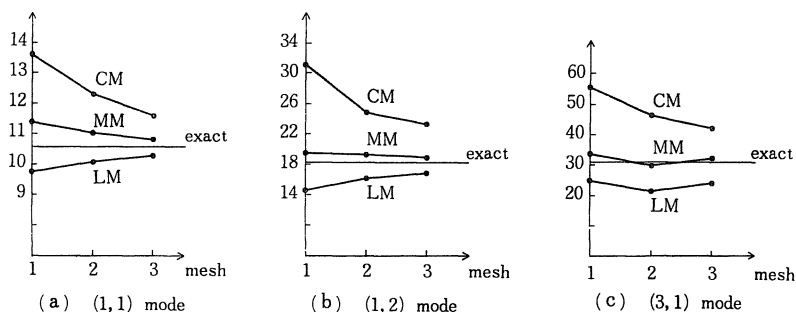
We have used FACOM 230-28 computer in Ehime University and FACOM 230-75 in Kyoto University.

**Remark.** As a natural extension of the MM scheme, we can propose the generalized mixed mass (GMM) scheme with a parameter  $\alpha$  ( $0 \leq \alpha \leq 1$ ):

$$K\tilde{U} = \tilde{\lambda} \{ \alpha M_1 + (1 - \alpha) M_2 \} \tilde{U}.$$

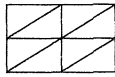
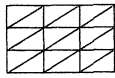
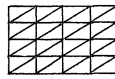
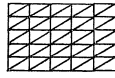
**Table 1.** The results for Dirichlet type.

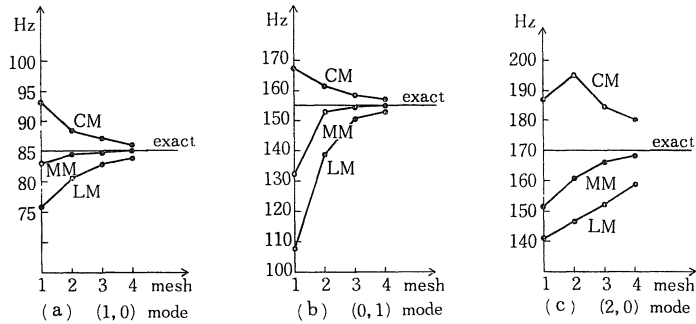
Mesh Configuration (Size: $2.0 \times 1.1$ )		Scheme	Eigenvalues		
			(1, 1) Mode	(1, 2) Mode	(3, 1) Mode
1		CM	13.66	30.62	54.64
		MM	11.34	19.44	33.75
		LM	9.69	14.19	24.56
2		CM	12.31	25.29	46.89
		MM	11.09	19.44	30.17
		LM	10.09	15.75	21.40
3		CM	11.69	22.61	42.57
		MM	10.94	19.12	31.40
		LM	10.28	16.53	24.25
Exact			10.62	18.03	30.36



**Figure 2.** Convergence of the eigenvalues (Dirichlet type)

**Table 2.** The results for Neumann type.

Mesh Configuration (Size: 2.0×1.1m)		Scheme	Normal Frequencies (Hz)		
			(1, 0) Mode	(0, 1) Mode	(2, 0) Mode
1		CM	93.1	167.6	187.5
		MM	83.3	131.6	151.5
		LM	75.8	107.0	140.3
2		CM	88.7	160.9	195.6
		MM	84.6	153.6	160.8
		LM	81.0	138.6	146.8
3		CM	87.1	158.2	186.0
		MM	84.9	154.2	166.9
		LM	82.8	150.4	151.2
4		CM	86.4	157.0	180.7
		MM	85.0	154.4	168.7
		LM	83.6	151.9	158.3
Exact			85.0	154.5	170.0



**Figure 3.** Convergence of the normal frequencies (Neumann type)

The GMM scheme includes as its special cases

- (i) CM scheme ( $\alpha = 1$ ),
- (ii) MM scheme ( $\alpha = 1/2$ ),
- (iii) LM scheme ( $\alpha = 0$ ).

We can prove the convergence of the GMM scheme using the same techniques used for the MM scheme. The proof will be published in a forthcoming paper.

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