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Notes on Minimality and Ergodicity of Compact Abelian Group Extensions of Dynamics

By

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§ 0. Introduction and Definitions

W. Parry [5] introduced the notion of a G-extension of a topological dynamics, where G is a compact abelian group, and gave necessary and sufficient conditions for a G-extension of a minimal (respectively uniquely ergodic) topological dynamics to be minimal (uniquely ergodic). In the first part of this paper a proof of the Minimality Theorem of W. Parry without his "simple free" condition is given. In the purely measure-theoretic case W. Parry [6] introduced the notion of G-extension of type σ, where σ is an automorphism of G, and spectrally analysed it. In the second part of this paper a necessary and sufficient condition for a G-extension of an ergodic measure-preserving dynamics to be ergodic is shown. As particular cases of this result we have well-known necessary and sufficient conditions for a translation, a group-automorphism and an affine transformation on a compact group to be ergodic.

Throughout, G and $\widehat{G}$ will respectively denote a compact abelian metric group and its character-group. An element γ of $\widehat{G}$ is called n-periodic with respect to an automorphism σ of G if $\gamma\sigma\neq\gamma, \cdots, \gamma\sigma^{n-1}\neq\gamma$ and $\gamma\sigma^n=\gamma$ $(n\geqq 1)$. A topological dynamics (X, S) is a compact metric space X, together with a homeomorphism S. A topological dynamics (X_1, S_1) is conjugate to (X, S) if there is a homeomorphism τ of X onto X_1 such that the diagram

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$$\begin{array}{ccc} X & \xrightarrow{S} & X \\ \tau\downarrow & & \downarrow\tau \\ X_1 & \xrightarrow[S_1]{} & X_1 \end{array}$$

commutes. A set F is S-invariant if $SF=F$. An S-invariant closed set F is S-minimal if the only S-invariant closed subsets of F are F and $\emptyset$. (X, S) is minimal if X is S-minimal. Denote respectively by $C(X)$ and $C(X, K)$, the set of all continuous complex-valued functions defined on X and the set of all functions in $C(X)$ with absolute value 1.

A continuous G-action on X is a continuous map χ of $G\times X$ onto X such that $\chi(g, \chi(h, x))=\chi(gh, x)$ for x in X and g, h in G and $\chi(e, x)=x$ for x in X where e is the identity element of G. If the map χ is understood we shall write gx for $\chi(g, x)$. If (X, S) is a topological dynamics such that S commutes with the G-action (i.e. $Sgx=gSx$ for x in X and g in G) then S induces the homeomorphism S' on the G-orbit space X/G defined by $S'G(x)=G(Sx)$ where $G(x)=\{gx; g\in G\}$. If a topological dynamics (X_1, S_1) is conjugate to the topological dynamics $(X/G, S')$ we shall say that (X, S) is a G-extension of (X_1, S_1). (W. Parry [5]).

A measure-preserving dynamics (Ω, μ, φ) (in this paper) is a Lebesgue measure space (Ω, μ), $\mu(\Omega)=1$, together with a bimeasurable bijection φ such that $\mu(\varphi\Lambda)=\mu(\Lambda)$ for any measurable set Λ. For simplicity of notation, expressions involving sets or functions will be stated disregarding sets of measure zero. A measure-preserving dynamics $(\Omega_1, \mu_1, \varphi_1)$ is conjugate to (Ω, μ, φ) if there is a bimeasurable, measure-preserving bijection τ from (Ω, μ) onto (Ω_1, μ_1) such that the diagram

$$\begin{array}{ccc} \Omega & \xrightarrow{\varphi} & \Omega \\ \tau\downarrow & & \downarrow\tau \\ \Omega_1 & \xrightarrow[\varphi_1]{} & \Omega_1 \end{array}$$

commutes. (Ω, μ, φ) is ergodic if every measurable function f with $f(\varphi\omega)=f(\omega)$ is constant. Denote by $L^2(\Omega, \mu)$ the set of all square-integrable functions on Ω. A measurable G-action on (Ω, μ) is a measurable map

χ of $G\times\Omega$ onto Ω such that $\chi(g,\chi(h,\omega))=\chi(gh,\omega)$ for ω in Ω and g, h in G, $\chi(e,x)=x$ for x in Ω, and $\mu(\chi(g,A))=\mu(A)$ for any measurable set A and any g in G. If the map χ is understood we shall write $g\omega$ for $\chi(g,\omega)$. If (Ω,μ,φ) is a measure-preserving dynamics such that $\varphi g\omega=\sigma(g)\varphi\omega$ for ω in Ω and g in G for some automorphism σ of G, then φ induces the measure-preserving transformation φ' on the G orbit space Ω/G. If a measure-preserving dynamics $(\Omega_1,\mu_1,\varphi_1)$ is conjugate to the $(\Omega/G,\mu_{\Omega/G},\varphi')$ we shall say that (Ω,μ,φ) is a G-extension of type σ of $(\Omega_1,\mu_1,\varphi_1)$. (W. Parry [6]).

§ 1. Minimality of a G-extension

Lemma 1. *Let (X,S) be a G-extension of a minimal topological dynamics. Then for any S-minimal set C, gC is S-minimal for any g in G and $X=\bigcup_{g\in G}gC$.*

Proof. It is easy to see that gC is S-minimal for any g in G. We denote by π the map from (X,S) to the minimal topological dynamics (X_1,S_1) defined by $\pi x=\tau^{-1}G(x)$. The set $\bigcup_{g\in G}gC$ is closed and S-invariant and the set $\pi(\bigcup_{g\in G}gC)$ is closed and S_1-invariant. From the minimality of (X_1,S_1) we have $\pi(\bigcup_{g\in G}gC)=X_1$. Therefore we have $\bigcup_{g\in G}gC=X$.

q.e.d.

Lemma 2. *Let Y be a compact topological space on which there is a continuous G-action such that $Y=\{gy;g\in G\}$ for some (any) point y in Y. And let Γ be the set of all γ in $\widehat{G}$ such that there exists an f_γ in $C(Y,K)$ with $f_\gamma(gy)=\gamma(g)f_\gamma(y)$ for y in Y and g in G. Then for h in G, $\gamma(h)=1$ for any γ in Γ if and only if $hy=y$ for any y in Y. In particular, $\Gamma=\{1\}$ if and only if Y is one point space.*

Proof. For f in $C(Y)$ and γ in $\widehat{G}$, put $f_\gamma(y)=\int\gamma(g)f(g^{-1}y)\,dg$ where dg is the Haar measure on G, then $f_\gamma(gy)=\gamma(g)f_\gamma(y)$ and $f_\gamma=0$ for γ not in Γ. Now $\gamma(h)=1$ for any γ in Γ, iff $f_\gamma(hy)=f_\gamma(y)$ for y in Y, any f in $C(Y)$ and any γ in $\widehat{G}$, iff $\int\gamma(g)\{f(g^{-1}hy)-f(g^{-1}y)\}\,dg=0$

for y in Y, any f in $C(Y)$ and any γ in $\widehat{G}$, iff $f(hy)=f(y)$ for y in Y, any f in $C(Y)$. All these hold iff $hy=y$ for y in Y. q.e.d.

Theorem 1. *Let (X, S) be a G-extension of a minimal topological dynamics. Then (X, S) is not minimal if and only if there exists a γ in $\widehat{G}$, $\gamma\neq 1$ and f in $C(X, K)$ such that $f(gx)=\gamma(g)f(x)$ and $f(Sx)=f(x)$ for any x in X and any g in G.*

Proof. Note that the quotient space X/C is Hausdorff (and compact) and apply Lemma 1 and Lemma 2. q.e.d.

Corollary 1. (H. Fürstenberg [2], W. Parry [5]). *Let (X, S) be a minimal topological dynamics and α be a continuous G-valued function defined on X. $\tilde{S}$ is a homeomorphism of the product space $X\times G$ defined by*

$$\tilde{S}(x, g)=(Sx, \alpha(x)g), \quad (x, g) \text{ in } X\times G.$$

Then the topological dynamics $(X\times G, \tilde{S})$ is not minimal if and only if there exists a γ in $\widehat{G}$, $\gamma\neq 1$ and an f in $C(X, K)$ such that

$$\gamma(\alpha(x))f(Sx)=f(x) \text{ for all } x \text{ in } X.$$

Proof. Consider the G-action $g(x, h)=(x, gh)$ on $X\times G$. $(X\times G, \tilde{S})$ is a G-extension of (X, S). Corollary 1 follows from Theorem 1. q.e.d.

§ 2. Ergodicity of a G-extension

Lemma 3. (W. Parry [6]). *Let (Ω, μ, φ) be a measure-preserving dynamics such that $\varphi(g\omega)=\sigma(g)\varphi(\omega)$ for ω in Ω and g in G for some automorphism σ of G.*

Let V_γ $(\gamma\in\widehat{G})$ be the set of all f_γ in $L^2(\Omega, \mu)$ such that $f_\gamma(g\omega)=\gamma(g)f_\gamma(\omega)$ with ω in Ω and g in G. Then

(1) *$L^2(\Omega, \mu)=\sum_{\gamma\in\widehat{G}}\oplus V_\gamma$ (orthogonal sum) and*

(2) *if f_γ is in V_γ then $f_\gamma\varphi$ is in $V_{\gamma\sigma}$.*

Proof. (1) For an f_γ in V_γ and an $f_{\gamma'}$ in $V_{\gamma'}$ we have

$$\int f_\gamma(\omega)\overline{f_{\gamma'}(\omega)}\,d\mu(\omega) = \int f_\gamma(g\omega)\overline{f_{\gamma'}(g\omega)}\,d\mu(\omega)$$

$$= \gamma(g)\overline{\gamma(g')}\int f_\gamma(\omega)\overline{f_{\gamma'}(\omega)}\,d\mu(\omega).$$

If $\gamma\neq\gamma'$, $\gamma(g)\overline{\gamma'(g)}\neq 1$ for some g in G, and so f_γ is orthogonal to $f_{\gamma'}$. Suppose that f in $L^2(\Omega,\mu)$ is orthogonal to any function in $\bigcup_{\gamma\in\widehat{G}} V_\gamma$. Put $f_\gamma(\omega) = \int \gamma(g) f(g^{-1}\omega)\,dg$ for γ in $\widehat{G}$, then f_γ is in V_γ. We have

$$\int f_\gamma(\omega)\overline{f_\gamma(\omega)}\,d\mu(\omega) = \int f_\gamma(\omega)\int\overline{\gamma(g)f(g^{-1}\omega)}\,dg\,d\mu(\omega)$$

$$= \int\int f_\gamma(g^{-1}\omega)\overline{f(g^{-1}\omega)}\,d\mu(\omega)\,dg$$

$$= \int f_\gamma(\omega)\overline{f(\omega)}f\mu(\omega) = 0\,.$$

Hence $f_\gamma(\omega)=0$ for ω in Ω and γ in $\widehat{G}$, and thus $f(\omega)=0$ for ω in Ω. Assersion (2) follows from the equation

$$f_\gamma(\varphi g\omega) = f_\gamma(\sigma(g)\varphi\omega) = \gamma(\sigma(g))f_\gamma(\varphi\omega). \qquad \text{q.e.d.}$$

Theorem 2. *Let* (Ω,μ,φ) *be a G-extension of type* σ *of an ergodic measure-preserving dynamics. Then* φ *is not ergodic if and only if there exists a positive integer* n *and a* γ *in* $\widehat{G}$, *n-periodic with respect to* σ *and not equal to* 1, *and an* f_γ *in* $L^2(\Omega,\mu)$, $f_\gamma\neq 0$, *such that* $f_\gamma(\varphi^n\omega)=f_\gamma(\omega)$ *and* $f_\gamma(g\omega)=\gamma(g)f_\gamma(\omega)$, *for* ω *in* Ω *and* g *in* G.

Proof. Let f_γ be a function which satisfies the conditions of Theorem 2. Put $f(\omega)=f_\gamma(\omega)+f_\gamma(\varphi\omega)+\cdots+f_\gamma(\varphi^{n-1}\omega)$. Then f is in $V_\gamma\oplus V_{\gamma\sigma}\oplus\cdots\oplus V_{\gamma\sigma^{n-1}}$ and $f(\varphi\omega)=f(\omega)$ *for* ω *in* Ω. That is, f is not constant and φ-invariant. Hence φ is not ergodic. Conversely, let f be a not constant function drawn from $L^2(\Omega,\mu)$ such that $f\varphi=f$, *and let* $f=\sum_{\gamma\in\widehat{G}}\oplus f_\gamma$ with f_γ in V_γ be the direct sum decomposition of f. Then $f\varphi=\sum_{\gamma\in\widehat{G}}\oplus f_\gamma\varphi$ where $f_\gamma\varphi$ is in $V_{\gamma\sigma}$. From $f\varphi=f$ we have $f_\gamma\varphi=f_{\gamma\sigma}$ and $\|f_\gamma\|_{L^2}=\|f_{\gamma\sigma}\|_{L^2}$ for γ in $\widehat{G}$. From the orthogonality of f_γ's we have $f_\gamma=0$ if γ is not periodic w.r.t. σ. Since any φ-invariant, G-invariant function is constant from the ergodic assumption, there exists a positive integer n and an n-periodic γ in $\widehat{G}$, $\gamma\neq 1$ such that $f_\gamma\neq 0$. We have $f_\gamma(\varphi^n\omega)=f_{\gamma\sigma^n}(\omega)=f_\gamma(\omega)$ for ω in Ω. q.e.d.

Corollary 2. *Let (Ω, μ, φ) be an ergodic measure-preserving dynamics, $\alpha(\omega)$ be a measurable G-valued function and σ be an automorphism of G. $\tilde{\varphi}$ is a measure-preserving transformation of the product $\Omega\times G$ defined by*

$$\tilde{\varphi}(\omega, g)=(\varphi\omega, \alpha(\omega)\sigma(g)), \quad (\omega, g) \text{ in } \Omega\times G.$$

Then $(\Omega\times G, \mu\times dg, \tilde{\varphi})$ is not ergodic if and only if there exists a positive integer n and a γ in $\widehat{G}$, n-periodic with respect to σ and not equal to 1, and an f in $L^2(\Omega, \mu)$, $f\neq 0$ such that $\gamma(\alpha(\varphi^{n-1}\omega)\sigma(\alpha(\varphi^{n-2}\omega))\cdots\sigma^{n-1}(\alpha(\omega)))f(\varphi^n\omega)=f(\omega)$ for ω in Ω.

Proof. Consider the G-action $g(\omega, h)=(\omega, gh)$ on $\Omega\times G$. $(\Omega\times G, \mu\times dg, \tilde{\varphi})$ is a G-extension of type σ of (Ω, μ, φ). Corollary 2 follows from Theorem 2. q.e.d.

Corollary 3. (1) *When σ of Corollary 2 is the identity, $\tilde{\varphi}$ is not ergodic if and only if there exists a γ in $\widehat{G}$, $\gamma\neq 1$, and a measurable function f such that $|f(\omega)|=1$, and $\gamma(\alpha(\omega))f(\varphi\omega)=f(\omega)$ for ω in Ω.* (H. Anzai [1]).

(2) *When $\alpha(\omega)=h$ for ω in Ω and G is connected, $\tilde{\varphi}$ of Corollary 2 is not ergodic if and only if* (i) *there exists an $n\geqq 2$ and an n-periodic γ in G, or* (ii) *there exists a* 1*-periodic γ in $\widehat{G}$, $\gamma\neq 1$, and a measurable function f, such that $|f(\omega)|=1$ and $\gamma(h)f(\varphi\omega)=f(\omega)$ for ω in Ω, that is, $\gamma(h)^{-1}$ is in the point spectrum of φ.*

Proof. (1) Clear from Corollary 2.

(2) If $n\geqq 2$ and γ be n-periodic, put $\gamma_1=\frac{\gamma\sigma}{\gamma}$. Then γ_1 is in $\widehat{G}$, $\gamma_1\neq 1$ and $\gamma_1\sigma^n=\gamma_1$. Let n_1 be the period of γ_1; we may represent n as n_1p where p is a positive integer. If $\frac{\gamma_1^p\sigma^k}{\gamma_1^p}=\left(\frac{\gamma_1\sigma^k}{\gamma_1}\right)^p=1$ for a positive integer k, we have $\frac{\gamma_1\sigma^k}{\gamma_1}=1$ from the connectedness of G. This means that γ_1^p

is also n_1-periodic. Since $\gamma_1^p(h\sigma h\cdots\sigma^{n_1-1}h)=\gamma_1(h\sigma h\cdots\sigma^{n-1}h)=1$, $\gamma_1^p(h\sigma h\cdots\sigma^{n_1-1}h)f(\varphi^{n_1-1}\omega)=f(\omega)$ for any constant function f. The rest of the proof is obvious. q.e.d.

Corollary 4. (1) *The affine transformation* $g\mapsto h\sigma(g)$ *on connected* G *is not ergodic if and only if there exists an* n*-periodic* γ *in* $\hat{G}$ *with* $n\geqq 2$ *or there exists a* 1*-periodic* γ *in* $\hat{G}$, $\gamma\neq 1$ *with* $\gamma(h)=1$ (F. Hahn [3]).
(2) *The group automorphism* $g\mapsto\sigma(g)$ *on* G *is not ergodic iff there exists an* n*-periodic* γ *in* $\hat{G}$, $\gamma\neq 1$ *for some* $n\geqq 1$. (P.R. Halmos [4]).
(3) *The translation* $g\mapsto hg$ *on* G *is not ergodic iff there exists a* γ *in* $\hat{G}$, $\gamma\neq 1$ *with* $\gamma(h)=1$.

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