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Point Spectra of Non-Singular Flows

By

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Abstract

An example is given of a family of non-singular transformations whose point spectra are uncountable.

§ 1. Definition of the Point Spectrum

Let $(X, \mathcal{F}, P)$ be a Lebesgue measure space and let $\{T_t; t\in \boldsymbol{R}\}$ be a one-parameter group of null-measure-preserving transformations of X onto itself, where the map $X\times \boldsymbol{R}\to X((x, t)\to T_tx)$ is measurable. The non-singular flow $\{T_t; t\in \boldsymbol{R}\}$ is said to be ergodic if every measurable function f with $f(T_tx)=f(x)$ a.e.x for each $t\in \boldsymbol{R}$ is constant a.e.. A real number θ is called an element of the point spectrum of $\{T_t; t\in \boldsymbol{R}\}$ if there exists a non-zero measurable function φ with $\varphi(T_tx)=e^{i\theta t}\varphi(x)$ a.e.x for each $t\in \boldsymbol{R}$. If $\{T_t; t\in \boldsymbol{R}\}$ is ergodic we may consider $|\varphi(x)|=1$. We denote by $Sp(\{T_t\})$ the point spectrum of $\{T_t; t\in \boldsymbol{R}\}$. It is well known that if $\{T_t; t\in \boldsymbol{R}\}$ has a finite invariant measure equivalent to P $Sp(\{T_t\})$ is $\{0\}$ or a countable subset of the set of all real numbers.

§ 2. AC-Flow

For every integer $n\geqq 1$ let Ω_n be the l_n-point set $\{0, 1, \cdots, l_n-1\}$ for some integer $l_n\geqq 2$, $\mathfrak{B}_n$ be the algebra consisting all subsets of Ω_n and G_n be the group of all permutations of Ω_n. Let Ω be the product $\prod_{n=1}^{\infty}\Omega_n$ of the Ω_n, $n\geqq 1$. For a point ω of Ω, ω_n means the n-th coordinate of ω. We may consider $\mathfrak{B}_n$ as an algebra on Ω, denote by $\mathfrak{B}$ the algebra

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generated by $\bigcup_{j=1}^{\infty} \mathfrak{B}_j$, and by $\mathfrak{B}_n^*$ the σ-algebra generated by $\bigcup_{j=1}^{n} \mathfrak{B}_j$. We may consider G_n as a transformation group on Ω, denote by G the group generated by $\bigcup_{j=1}^{\infty} G_j$ and by G_n^* the group generated by $\bigcup_{j=1}^{n} G_j$.

We define a transformation T of Ω onto itself as follows: if $\omega_j = l_j - 1$ for $j = 1, 2, \cdots, n-1$ and $\omega_n \leqq l_n - 2$ then $(T\omega)_j = 0$ for $j = 1, 2, \cdots, n-1$, $(T\omega)_n = \omega_n + 1$ and $(T\omega)_j = \omega_j$ for $j = n+1, n+2, \cdots$. T is called the adding machine with measure μ if $\mu(A) = 0$ implies $\mu(TA) = 0$.

For given numbers $X_n(k) \in \boldsymbol{R}$, $k = 0, 1, \cdots, l_n - 1$, $n = 1, 2, \cdots$ such that $X_n(0) = 0$ and $X_n(k+1) > X_n(k) + \sum_{j=1}^{n-1} X_j(l_j - 1)$, $k = 0, 1, \cdots, l_n - 2$, $n = 1, 2, \cdots$, we define, if $\omega_j = l_j - 1$ for $j = 1, 2, \cdots, n-1$ and $\omega_n = k \leqq l_n - 2$, $f(\omega) = X_n(k+1) - X_n(k) - \sum_{j=1}^{n-1} X_j(l_j - 1)$. We may consider $X_n(\cdot)$ as a $\mathfrak{B}_n$-measurable function defined on Ω and we have $f(\omega) = \sum_{n=1}^{\infty} (X_n(T\omega) - X_n(\omega)) > 0$. Consider the following flow built under a function f. Let X be the set of all points (ω, s) of the product space $\Omega \times \boldsymbol{R}$ with $0 \leqq s < f(\omega)$, $\mathcal{F}$ be the restriction of the product σ-algebra $\mathfrak{B} \times \mathfrak{B}_{\boldsymbol{R}}$ to X and P be the restriction of the product measure $\mu \times e^{-s} ds$ to X, where ds is the uniform measure defined on the Borel field $\mathfrak{B}_{\boldsymbol{R}}$ of $\boldsymbol{R}$. A flow $\{T_t, t \in \boldsymbol{R}\}$ is defined on $(X, \mathcal{F}, P)$ as follows.

$$
T_t(\omega, s) = \begin{cases} \left((T^{-n}\omega, s + t + \sum_{j=1}^{n} f(T^{-j}\omega) \right) \\ \qquad \text{if} - \sum_{j=1}^{n} f(T^{-j}\omega) \leqq s + t < - \sum_{j=1}^{n-1} f(T^{-j}\omega) \\ \qquad (n = 1, 2, \cdots) \\ \left((T^{n}\omega, s + t - \sum_{j=1}^{n-1} f(T^{j}\omega) \right) \\ \qquad \text{if} \sum_{j=0}^{n-1} f(T^{j}\omega) \leqq s + t < \sum_{j=0}^{n} f(T^{j}\omega) \\ \qquad (n = 0, 1, \cdots) \end{cases}
$$

We call this $\{T_t, t \in \boldsymbol{R}\}$ the AC-flow determined by T, μ and $\{X_n(\cdot)\}$. (AC for Adding machine and Ceiling function, or perhaps for Alain Connes.)

§ 3. Point Spectra of AC-Flows

Theorem. *Let $\{T_t; t \in \boldsymbol{R}\}$ be the AC-flow determined by T, μ and*

$\{X_n(\cdot)\}$.

(*a*) *If there exists a sequence* $\{C_n\}$ *of real numbers such that* $\exp\{i\theta \sum_{j=1}^{n} (X_j(\omega)-C_j)\}$ *converges a.e. as* $n\to\infty$, *then* $\theta\in \mathrm{S}p(\{T_t\})$.

(*b*) *If* μ *is an infinite product of measures* μ_n *on* Ω_n, $n=1, 2, \cdots$ *with* $\mu_n(\{k\})>0$, *then* T *is ergodic and the converse of* (*a*) *is true.*

Proof. It is easy to see that $\theta\in \mathrm{S}p(\{T_t\})$ iff there exists a non-zero measurable function $\varphi(\omega)$ with

$$\text{☆} \qquad \varphi(T\omega)=e^{i\theta f(\omega)}\varphi(\omega).$$

If $\varphi(\omega)=\exp\{i\theta \sum_{j=1}^{\infty} (X_j(\omega)-C_j)\}$ exists, it satisfies ☆. Let $\varphi(\omega)$ be a non-zero function satisfying ☆, we have

$$\exp\{i\theta \sum_{j=1}^{\infty} (X_j(T^k\omega)-X_j(\omega))\}=\frac{\varphi(T^k\omega)}{\varphi(\omega)}, \quad k=0, 1, 2, \cdots.$$

Since for any $g\in G_n^*$ and for a.e $\omega\in\Omega$, there exists an integer $k(g, \omega)$ such that $g\omega=T^{k(g,\omega)}\omega$,

$$\exp\{i\theta \sum_{j=1}^{n} (X_j(g\omega)-X_j(\omega))\}=\frac{\varphi(g\omega)}{\varphi(\omega)}, \text{ a.e.}\omega \text{ for each } g\in G_n{}^*.$$

Since $\varphi_n^*(\omega)=\varphi(\omega)\exp\{-i\theta \sum_{j=1}^{n} X_j(\omega)\}$ is a G_n^*-invariant function and since $\mathfrak{B}_n$'s are mutually independent σ-algebras on the assumption of (b), we have

$$\lim_{n\to\infty} \exp\{i\theta \sum_{j=1}^{n} X_j(\omega)\}E(\varphi_n^*(\omega)) = \lim_{n\to\infty} E((\varphi(\omega)/\mathfrak{B}_n^*)=\varphi(\omega).$$

Hence there exists a sequence $\{C_n\}$ of real numbers such that $\exp\{i\theta \sum_{j=1}^{n} (X_j(\omega)-C_j)\}$ converges a.e. as $n\to\infty$. (cf. [2])

§ 4. Example 1

Put $\Omega_n=\{0, 1\}$, $\mu_n(0)=\frac{1}{1+\lambda}$, $\mu_n(1)=\frac{\lambda}{1+\lambda}$ and $X_n(0)=0$, $X_1(1)=2^n a$, $n=1, 2, \cdots$ for some $0<\lambda\leq 1$ and some a in $\boldsymbol{R}$. Let $\{T_t; t\in\boldsymbol{R}\}$ be the AC-flow determined by the product measure $\mu=\prod_{n=1}^{\infty} \mu_n$ and the

$\{X_n(\cdot)\}$. Then $Sp(\{T_t\}) = \frac{2\pi}{a} \times \{$2-adic rational numbers$\}$. In fact since for any 2-adic rational number θ $\exp\left\{i\frac{2\pi}{a}\theta X_n(\omega)\right\} = 1$ for sufficiently large n, $\exp\left\{i\frac{2\pi}{a}\theta\sum_{n=1}^{\infty}X_n(\omega)\right\}$ converges a.e.. Try next to assume that $\exp\left\{i\frac{2\pi}{a}\theta\sum_{n=1}^{\infty}(X_n(\omega)-C_n)\right\}$ converges a.e. for some not 2-adic rational number θ and some sequence $\{C_n\}$ of real numbers. Put $E = \left\{\omega\in\Omega;\ \exp\left(i\frac{2\pi}{a}\theta(X_n(\omega)-C_n)\right)\text{ converges to 1 as } n\to\infty\right\}$; we have $\mu(E)=1$. Looking at fractional parts of $\theta 2^n$, one sees there exists an infinite subset N_1 of positive integers such that $|\exp\{i2\pi\theta 2^n - 1\}| > \sqrt{2}$, $n\in N_1$. Since, from the Borel-Cantelli lemma, there exist an ω in E and an infinite subset N_2 of N_1 such that $\omega_n = 1$ for $n\in N_2$, $\lim_{\substack{n\in N_2\\ n\to\infty}}\exp(i2\pi\theta 2^n)\exp\left(-i\frac{2\pi}{a}\theta C_n\right) = 1$. Again from the Borel-Cantelli lemma there exist an ω in E and an infinite subset N_3 of N_2 such that $\omega_n = 0$ for $n\in N_3$, so $\lim_{\substack{n\in N_3\\ n\to\infty}}\exp\left(-i\frac{2\pi}{a}\theta C_n\right) = 1$. This is a contradiction and thus $\frac{2\pi}{a}\theta$ is not in $Sp(\{T_t\})$. (cf. [2])

§ 5. Example 2 (with uncountable spectra)

Put $M_1 = 2$, $M_n = 2^n M_{n-1}$ $(n = 2, 3, \cdots)$. For any real number θ there exists a unique sequence $\{l_1', l_2', \cdots, l_n', \cdots\}$ of integers such that $-2^{n-1} \leqq l_n' \leqq 2^{n-1}$, $n = 2, 3, \cdots$ and $-\frac{1}{2M_n} < \theta - \left(\frac{l_1'}{M_1} + \frac{l_2'}{M_2} + \cdots + \frac{l_n'}{M_n}\right) \leqq \frac{1}{2M_n}$, $n = 1, 2, \cdots$. For any sequence $\{l_1, l_2, \cdots, l_n, \cdots\}$ of integers with $|l_n| \leqq 2^{n-2}$ for sufficiently large n, the unique sequence $\{l_1', l_2', \cdots, l_n', \cdots\}$ of integers determined by a real number $\frac{l_1}{M_1} + \frac{l_2}{M_2} + \cdots + \frac{l_n}{M_n} + \cdots$ satisfies $l_n' = l_n$ for sufficiently large n, because $\left|\frac{l_n}{M_n} + \frac{l_{n+1}}{M_{n+1}} + \cdots\right| < \frac{1}{2M_{n-1}}$. Hence the set of all real numbers whose uniquely determined sequence $\{l_1', l_2', \cdots, l_n', \cdots\}$ satisfies $\sum_{n=1}^{\infty}\left(\frac{l_n'}{2^n}\right)^2 < +\infty$ is the set of all real numbers $\frac{l_1}{M_1} + \frac{l_2}{M_2} + \cdots + \frac{l_n}{M_n} + \cdots$ determined by a sequence $\{l_1, l_2, \cdots, l_n \cdots\}$ with $\sum_{n=1}^{\infty}\left(\frac{l_n}{2^n}\right)^2 < +\infty$. We denote by $\mathcal{M}$ the latter set. It is easy to see that $\mathcal{M}$ is an uncountable subgroup of the group of real numbers and that the complement of $\mathcal{M}$ is also uncountable.

Put $\Omega_n=\{0,1\}$, $X_n(0)=0$, $X_n(1)=M_n a$, $n=1,2,\cdots$ for some a in $\boldsymbol{R}$ and let $\{T_t; t\in\boldsymbol{R}\}$ be the AC-flow determined by the adding machine with an arbitrary measure μ and the $\{X_n(\cdot)\}$. For a real number $\theta=\frac{l_1}{M_1}+\frac{l_2}{M_2}+\cdots+\frac{l_n}{M_n}+\cdots$ denote by $Y_n(\omega_n)$ the fractional part of $\frac{\theta}{a}X_n(\omega)$, $n=1,2,\cdots$, we have $Y_n(0)=0$ and $Y_n(1)=M_n\left(\frac{l_{n+1}}{M_{n+1}}+\frac{l_{n+2}}{M_{n+2}}+\cdots\right)$, $n=1,2,\cdots$. All real numbers $\frac{2\pi}{a}\left(\frac{l_1}{M_1}+\frac{l_2}{M_2}+\cdots+\frac{l_n}{M_n}+\cdots\right)$ with $|l_n|\leqq 1$, $n=1,2,\cdots$ are in $\mathrm{S}p(\{T_t\})$, since $\sum_{n=1}^{\infty}|Y_n(\cdot)|\leqq\sum_{n=1}^{\infty}M_n\left(\frac{1}{M_{n+1}}+\frac{1}{M_{n+2}}+\cdots\right)\leqq\sum_{n=1}^{\infty}\frac{1}{2^n}<+\infty$.

If μ is an infinite product of measures μ_n on Ω_n, $n=1,2,\cdots$ such that $\mu_n(0)=\frac{1}{1+\lambda}$, $\mu_n(1)=\frac{\lambda}{1+\lambda}$, $n=1,2,\cdots,(0<\lambda\leqq 1)$, we have $\mathrm{S}p(\{T_t\})=\frac{2\pi}{a}\mathcal{M}$. In fact $\exp\{2\pi i\sum_{n=1}^{\infty}(Y_n(\omega)-C_n)\}$ converges a.e. for some seqence $\{C_n\}$ of real numbers iff $\sum_{n=1}^{\infty}V(Y_n(\omega))<+\infty$, iff $\sum_{n=1}^{\infty}M_n^2\left(\frac{l_{n+1}}{M_{n+1}}+\frac{l_{n+2}}{M_{n+2}}+\cdots\right)^2<+\infty$ and iff $\sum_{n=1}^{\infty}M_n^2\left(\frac{l_{n+1}}{M_{n+1}}\right)^2<+\infty$.

§ 6. Remarks Related to the Weak Equivalence Theory of Non-Singular Transformations

(1) J. Woods [5] constructed ITPFI factors with an uncountable ρ-set. A. Connes [1] constructed an adding machine with infinite product measure whose associated flow was the AC-flow determined by the adding machine and some $\{X_n(\cdot)\}$. Using his method we can construct an adding machine with infinite product measure whose associated flow is the AC-flow of Example 2. This has uncountable T-set $\frac{2\pi}{a}\mathcal{M}$. (cf. [3] and [4])

(2) We may take as parameter set the set $\boldsymbol{Z}$ of all integers instead of $\boldsymbol{R}$ as in this paper. In this case we would say AC-transformation instead of AC-flow. If an adding machine T has no σ-finite invariant measure equivalent to μ, every AC-transformations determined by T and some ceiling is weakly equivalent to T. The AC-transformations of

Example 1 for different prime numbers a have mutually different spectra. So they are weakly equivalent (of type III_λ, $0<\lambda<1$), but not equivalent.

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