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Gradient flow for β-symplectic critical surfaces

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Abstract. In this paper we consider a gradient flow for the L_β-functional introduced by the authors in Han–Li–Sun (2018). We first prove that the "symplectic" property is preserved along the gradient flow. Then we prove a monotonicity formula and an ε-regularity theorem for the flow. As consequences, we show that the λ-tangent cone of the flow consists of finite flat planes. Another application is that we can show that the flow exists globally and converges to a holomorphic curve if the initial surface is sufficiently close to a holomorphic curve and the ambient Käher–Einstein surface has positive scalar curvature.

1. Introduction

Suppose that M is a Kähler surface. Let ω be the Kähler form on M and J be a complex structure compatible with ω. The Riemannian metric $\langle\, ,\rangle$ on M is defined by

$$\langle U, V\rangle = \omega(U, JV).$$

For a compact oriented real surface Σ which is smoothly immersed in M, one defines, following [5], the Kähler angle α of Σ in M by

$$\omega|_\Sigma = \cos\alpha \, d\mu_\Sigma,$$

where $d\mu_\Sigma$ is the area element of Σ of the induced metric form $\langle\, ,\rangle$. We say that Σ is a *holomorphic curve* if $\cos\alpha \equiv 1$, Σ is a *Lagrangian surface* if $\cos\alpha \equiv 0$ and Σ is a *symplectic surface* if $\cos\alpha > 0$.

The existence of holomorphic curves in a Kähler surface is a fundamental problem in differential geometry. Since holomorphic curves are always area minimizing in their homological class due to the Wirtinger inequality, we see that holomorphic curves are all stable symplectic minimal surfaces. Wolfson [23] showed that a symplectic minimal surface in a Kähler–Einstein surface with nonnegative scalar curvature must be holomorphic. Thus, we can look for holomorphic curves by finding the symplectic minimal surfaces in this case.

Furthermore, Chen–Li [3] and Wang [21] showed that the "symplectic" property is preserved along mean curvature flow. Therefore, an idea for approaching the existence of

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holomorphic curves is to look for symplectic minimal surfaces using the mean curvature flow starting from a symplectic surface, which we call "symplectic mean curvature flow". There are some interesting results on the study of symplectic mean curvature flow. For instance, Chen–Li [3] and Wang [21] showed that there are no Type I singularities for such a flow at finite time. There are also some interesting results on the study of Type II singularities for the symplectic mean curvature flow ([9, 13] etc.). However, since the flow is of codimension two and the normal bundle is much more complex, it is hard to clear out all singularities. On the other hand, Arezzo [2] constructed examples which show that a strictly stable minimal surface in a Kähler–Einstein surface with negative scalar curvature may not be holomorphic.

For this reason, we introduce a new idea to approach the existence of holomorphic curves using a variational method combined with the continuity method. More precisely, we consider a sequence of functionals [11]

$$L_\beta(\Sigma) = \int_\Sigma \frac{1}{\cos^\beta \alpha}\, d\mu,$$

where $\beta \geq 0$. The functional L_1 was introduced by Han–Li [8]. The critical point of the functional L_β in the class of symplectic surfaces in a Kähler surface is called a β-*symplectic critical surface*. We have proved that (cf. [11]) the Euler–Lagrange equation of the functional L_β is

$$\cos^3 \alpha \mathbf{H} - \beta(J(J\nabla \cos\alpha)^\top)^\perp = 0,$$

where $\mathbf{H}$ is the mean curvature vector of Σ in M, $(\ )^\top$ means tangential components of $(\ )$ and $(\ )^\perp$ means the normal components of $(\ )$. It is clear that holomorphic curves are β-symplectic critical surfaces for each β. When $\beta = 0$, the functional is exactly the area functional, and a 0-symplectic critical surface is exactly a symplectic minimal surface. We aim to deform, from a 0-symplectic critical surface (i.e., a minimal surface) to a holomorphic curve when β tends to infinity. We showed the openness (cf. [11]) and partial results on the compactness (cf. [10]) of the program.

An important step in the program concerns the existence of a β-symplectic critical surface for each fixed β. A natural idea is to consider the negative gradient flow of the functional L_β. For this purpose, let us recall the first variation formula for L_β:

Theorem 1.1 ([11]). *Let M be a Kähler surface. The first variation formula of the functional L_β is, for any smooth normal vector field $\mathbf{X}$ on Σ,*

$$\delta_X L_\beta = -(\beta+1)\int_\Sigma \frac{\mathbf{X}\cdot\mathbf{H}}{\cos^\beta\alpha}\, d\mu + \beta(\beta+1)\int_\Sigma \frac{\mathbf{X}\cdot(J(J\nabla\cos\alpha)^\top)^\perp}{\cos^{\beta+3}\alpha}\, d\mu,$$

where $\mathbf{H}$ is the mean curvature vector of Σ in M, and $(\ )^\top$ means tangential components of $(\ )$, $(\ )^\perp$ means the normal components of $(\ )$.

In this paper we will consider the negative gradient flow of L_β, i.e.,

$$\begin{cases} \dfrac{dF}{dt} = \dfrac{\cos^3\alpha\mathbf{H} - \beta(J(J\nabla\cos\alpha)^\top)^\perp}{\cos\alpha(\cos^2\alpha + \beta\sin^2\alpha)}, \\ F(\cdot, 0) = F_0, \end{cases} \tag{1.1}$$

where $F_0\colon \Sigma \to M$ is a smooth immersion. Denote $\Sigma_t := F(\Sigma, t)$. We set

$$\mathbf{f} = \frac{\cos^3\alpha\mathbf{H} - \beta(J(J\nabla\cos\alpha)^\top)^\perp}{\cos\alpha(\cos^2\alpha + \beta\sin^2\alpha)}.$$

It is clear that $\mathbf{f} \equiv 0$ if and only if Σ is a β-symplectic critical surface. (We refer to the proof of Theorem 2.3 for the derivation of the exact expression for $\mathbf{f}$.)

When $\beta = 0$, flow (1.1) reduces to the symplectic mean curvature flow, and when $\beta = 1$, flow (1.1) reduces to the flow considered by [12].

By the first variation formula, we see that along flow (1.1),

$$\begin{aligned} \frac{dL_\beta}{dt} &= -(\beta+1)\int_\Sigma \frac{|\cos^3\alpha\mathbf{H} - \beta(J(J\nabla\cos\alpha)^\top)^\perp|^2}{\cos^{\beta+4}\alpha(\cos^2\alpha + \beta\sin^2\alpha)}\,d\mu \\ &= -(\beta+1)\int_\Sigma \frac{\cos^2\alpha + \beta\sin^2\alpha}{\cos^{\beta+2}\alpha}|\mathbf{f}|^2\,d\mu. \end{aligned}$$

Thus, flow (1.1) is a gradient flow of the functional L_β. From [11, Proposition 3.1], we know that flow (1.1) is a parabolic system and the short-time existence can be shown by a standard argument. The first step to study flow (1.1) is to show that the symplectic property is preserved along this flow. This is exactly the first result in this paper.

Theorem 1.2. *The symplectic property is preserved along flow* (1.1).

Note that, compared with the case of symplectic mean curvature flow [3, 21], we do not need to assume that M is Kähler–Einstein when $\beta > 0$. Similar to mean curvature flow [14], we can show that the flow can be extended if the second fundamental form of Σ in M is uniformly bounded.

Huisken's monotonicity formula plays an important role in the study of mean curvature flow [15]. We can also prove the monotonicity formula for flow (1.1). For symplectic mean curvature flow, Chen–Li [3] and Wang [21] independently derived a new monotonicity formula and used it to rule out the possibility of Type I singularity. In this paper we will also prove monotonicity for flow (1.1) (see Section 3).

Theorem 1.3. *Let M^4 be a compact Kähler surface and Ψ be given by* (3.5)*. Then there are positive constants a, c_1 and c_2 depending only on M^4, F_0, r, t_0 and β, such that along* *flow* (1.1) *with* $\beta > 0$*, we have*

$$\begin{aligned}
&\frac{d}{dt}(e^{c_1\sqrt{t_0-t}}\Psi(X_0,t_0,t)) \\
&\quad \le -e^{c_1\sqrt{t_0-t}}\int_\Sigma \frac{1}{\cos^p\alpha}\Big\{\frac{1}{4}\Big|\frac{(F-X_0)^\perp}{t_0-t}+\mathbf{f}+\mathbf{H}\Big|^2 \\
&\qquad\qquad + \Big|\mathbf{H}+\frac{4p(\cos^4\alpha-\beta\sin^4\alpha)-\beta^2\sin^4\alpha}{3p\cos^2\alpha(\cos^2\alpha+\beta\sin^2\alpha)-\beta^2\sin^4\alpha}\mathbf{V}\Big|^2 \\
&\qquad\qquad + \frac{7}{8}|\mathbf{H}|^2+\frac{7}{8}\mathbf{V}|^2\Big\}\rho\, d\mu_t \\
&\quad + c_2 e^{c_1\sqrt{t_0-t}}.
\end{aligned}$$

Another important tool in the study of the singularities of mean curvature flow is the ε-regularity theorem due to White [22]. Using similar ideas, we can also prove the ε-regularity theorem for flow (1.1). We denote the parabolic ball by $P(X_0,t_0,r) := B_r(X_0)\times(t_0-r^2,t_0]$.

Theorem 1.4. *Let* Ψ *be given by* (3.5)*, Then there exist constants* $\varepsilon > 0$ *and* C_0*, depending on only on* β *and* M^4*, such that if* Σ_t *is a smooth solution to flow* (1.1) *in* $P(0,0,8\rho)$ *and*

$$\sup_{(X,t)\in P(X_0,t_0,r)} \Psi(X,t,t-r^2) < 1+\varepsilon,$$

for some $r \in (0,\rho)$*, then we have*

$$\sup_{P(X_0,t_0,\frac{r}{2})} |\mathbf{A}| \le \frac{C_0}{r}.$$

When the singularity occurs, we can rescale the flow near the singular point and obtain some limiting model in some sense, which we call a λ-tangent cone. Understanding the behaviors of the tangent cones is crucial to study the flow. For mean curvature flow, from Huisken's monotonicity formula [15], we know that the tangent flows are self-shrinkers. There are many important works on the classification of self-shrinkers for mean curvature flow (cf. [6, 16]). For Lagrangian mean curvature flow, we can obtain more information due to the extra geometric condition (cf. [4, 19]).

As a consequence of our monotonicity formula, we can show that the λ-tangent cone of flow (1.1) consists of finitely many unions of flat planes if it is nonempty.

We finally remark that in [12], we also derive a monotonicity formula for flow (1.1) with $\beta = 1$. The weight we used in [12] is $f(x) = e^{x^2}$. In the present paper we use the weight $f(x) = x^p$. The advantage is that we can prove the stability of flow (1.1) for surfaces near a holomorphic curve in a Kähler–Einstein surface with positive scalar curvature. The corresponding result for symplectic mean curvature flow is obtained by Han–Li [7].

Theorem 1.5. *For any $\beta \geq 0$, there is a constant $\varepsilon_0 > 0$, depending on β, Σ_0 and M, such that if Σ_0 is a closed symplectic surface in a compact Kähler–Einstein surface with positive scalar curvature $K_0 > 0$ and $\cos\alpha > 1 - \varepsilon_0$, then flow (1.1) starting from Σ_0 exists globally and converges to a holomorphic curve at infinity.*

Similar to symplectic mean curvature flow ($\beta = 0$), it is natural to ask whether we can divide the finite time singularities of flow (1.1) into two types and exclude Type I singularity [3, 15, 21]. To define the types of singularities, we first need derive the evolution equation for the second fundamental form. Then combining with the monotonicity formula derived in this paper, it is possible to exclude the Type I singularities. We will examine this problem in the future.

The subsequent sections are organized as follows: in Section 2, we derive the evolution equation for the Kähler angle and prove the symplectic property is preserved along flow (1.1); in Sections 3 and 4, we prove the weighted monotonicity formula and ε-regularity theorem for flow (1.1) respectively; in Section 5 we study the λ-tangent cone of the flow; in Section 6 we prove the stability of the flow for surfaces near a holomorphic curve in a Kähler–Einstein surface with positive scalar curvature.

2. Preserving the symplectic property

In this section we will prove our first result. To start, we set $\Sigma_t = F(\Sigma, t)$ with $\Sigma_0 = \Sigma$. In the following we will choose a frame $\{e_1, e_2, v_3, v_4\}$ at a fixed point $p \in \Sigma$ so that $\{e_1, e_2\}$ spans $T\Sigma$, $\{v_3, v_4\}$ spans $N\Sigma$ and the complex structure takes the form

$$J = \begin{pmatrix} 0 & \cos\alpha & \sin\alpha & 0 \\ -\cos\alpha & 0 & 0 & -\sin\alpha \\ -\sin\alpha & 0 & 0 & \cos\alpha \\ 0 & \sin\alpha & -\cos\alpha & 0 \end{pmatrix}. \tag{2.1}$$

From now on we will agree on the following index ranges:

$$1 \leq i, j, k, l \leq 2, \quad 3 \leq \alpha, \beta, \gamma \leq 4, \quad 1 \leq A, B, C \leq 4.$$

We can write

$$\mathbf{A} = \mathbf{A}^\alpha v_\alpha, \quad \mathbf{H} = H^\alpha v_\alpha,$$

where $\mathbf{A}^\alpha = (h^\alpha_{ij})$ is the second fundamental form of Σ in M and $H^\alpha = g^{ij} h^\alpha_{ij} = h^\alpha_{ii}$. By the Weingarten equation, we have

$$h^\alpha_{ij} = -\langle \overline{\nabla}_{e_i} v_\alpha, e_j \rangle = \langle v_\alpha, \overline{\nabla}_{e_i} e_j \rangle = h^\alpha_{ji},$$

where $\overline{\nabla}$ is the Levi-Civita connection on M.

It can be checked that (cf. [8, equation (2.5)])

$$\nabla_{e_1}\alpha = -(h_{11}^4 + h_{12}^3), \quad \nabla_{e_2}\alpha = -(h_{12}^4 + h_{22}^3).$$

If we set $\mathbf{V} = \partial_2\alpha v_3 + \partial_1\alpha v_4$, it is easy to see that flow (1.1) can be rewritten as

$$\frac{dF}{dt} = \frac{\cos^2\alpha\mathbf{H} - \beta\sin^2\alpha\mathbf{V}}{\cos^2\alpha + \beta\sin^2\alpha} \equiv \mathbf{f}.$$

The evolution of the area form along flow (1.1) is given in the next lemma:

Lemma 2.1. *Along the flow (1.1), the induced area form satisfies*

$$\frac{d}{dt}d\mu_t = \frac{1}{2}\operatorname{tr}_g\frac{\partial g}{\partial t}\,d\mu_t = \frac{-\cos^2\alpha|\mathbf{H}|^2 + \beta\sin^2\alpha\mathbf{V}\cdot\mathbf{H}}{\cos^2\alpha + \beta\sin^2\alpha}\,d\mu_t.$$

Proof. We compute it in local normal coordinates:

$$\begin{aligned}\frac{\partial}{\partial t}g_{ij} &= \frac{\partial}{\partial t}\Big\langle\frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j}\Big\rangle = \Big\langle\frac{\partial \mathbf{f}}{\partial x^i}, \frac{\partial F}{\partial x^j}\Big\rangle + \Big\langle\frac{\partial F}{\partial x^i}, \frac{\partial \mathbf{f}}{\partial x^j}\Big\rangle\\ &= -2\Big\langle\mathbf{f}, \frac{\partial F}{\partial x^i\partial x^j}\Big\rangle = -2\langle\mathbf{f}, h_{ij}^\alpha v_\alpha\rangle,\end{aligned}$$

thus

$$\frac{d}{dt}d\mu_t = \frac{1}{2}\operatorname{tr}_g\frac{\partial g}{\partial t}\,d\mu_t = -\mathbf{f}\cdot\mathbf{H}\,d\mu_t = \frac{-\cos^2\alpha|\mathbf{H}|^2 + \beta\sin^2\alpha\mathbf{V}\cdot\mathbf{H}}{\cos^2\alpha + \beta\sin^2\alpha}\,d\mu_t. \qquad \blacksquare$$

We also recall the following elliptic equation of the Kähler angle:

Proposition 2.2 ([8]). *If Σ is a closed symplectic surface which is smoothly immersed in M with the Kähler angle α, then α satisfies the equation*

$$\begin{aligned}\Delta\cos\alpha &= \cos\alpha(-|h_{1k}^3 - h_{2k}^4|^2 - |h_{1k}^4 + h_{2k}^3|^2)\\ &\quad + \sin\alpha(H_{,1}^4 + H_{,2}^3) - \sin^2\alpha\operatorname{Ric}(Je_1, e_2),\end{aligned}$$

where Ric *is the Ricci curvature tensor of M and $H_{,i}^\alpha = \langle\overline{\nabla}_{e_i}^N\mathbf{H}, v_\alpha\rangle$.*

Now we can derive the evolution equation of $\cos\alpha$ along flow (1.1).

Theorem 2.3. *Let M be a Kähler surface. Assume that α is the Kähler angle of Σ_t which evolves by flow (1.1). Then $\cos\alpha$ satisfies the equation*

$$\begin{aligned}\frac{\partial}{\partial t}\cos\alpha &= \Delta\cos\alpha + \frac{\cos^2\alpha\sin^2\alpha}{\cos^2\alpha + \beta\sin^2\alpha}\operatorname{Ric}(Je_1, e_2) + \frac{\cos^3\alpha}{\cos^2\alpha + \beta\sin^2\alpha}|\mathbf{H}|^2\\ &\quad + \frac{\cos\alpha[(\beta^2 - 2\beta)\sin^4\alpha + \beta\sin^2\alpha\cos^2\alpha + 2\cos^4\alpha]}{(\cos^2\alpha + \beta\sin^2\alpha)^2}|\mathbf{V}|^2\\ &\quad + \frac{2\cos\alpha(\cos^4\alpha - \beta\sin^4\alpha)}{(\cos^2\alpha + \beta\sin^2\alpha)^2}\mathbf{H}\cdot\mathbf{V},\end{aligned} \tag{2.2}$$

where $\{e_1, e_2, v_3, v_4\}$ is an orthonormal basis of T_pM such that J takes the form (2.1).

Proof. We will compute it pointwise in the normal coordinate system. Suppose the flow is given by $\frac{\partial F}{\partial t} = \mathbf{f}$. Using the fact that $\overline{\nabla}\omega = 0$ and Lemma 2.1, we have

$$\begin{aligned}\frac{\partial}{\partial t}\cos\alpha &= \frac{\partial}{\partial t}\Big(\frac{\omega(e_1, e_2)}{\sqrt{\det(g_t)}}\Big) = \omega(\overline{\nabla}_{e_1}\mathbf{f}, e_2) - \omega(\overline{\nabla}_{e_2}\mathbf{f}, e_1) - \frac{1}{2}\cos\alpha\,\mathrm{tr}_g\,\frac{\partial g}{\partial t}\\ &= \omega(\overline{\nabla}_{e_1}\mathbf{f}, e_2) - \omega(\overline{\nabla}_{e_2}\mathbf{f}, e_1) + \cos\alpha\,\mathbf{f}\cdot\mathbf{H}.\end{aligned}$$

By breaking $\overline{\nabla}_{e_1}\mathbf{f}$ and $\overline{\nabla}_{e_2}\mathbf{f}$ into normal and tangential parts, we get

$$\begin{aligned}\omega(\overline{\nabla}_{e_1}\mathbf{f}, e_2) - \omega(\overline{\nabla}_{e_2}\mathbf{f}, e_1) &= \omega(\overline{\nabla}^N_{e_1}\mathbf{f}, e_2) - \omega(\overline{\nabla}^N_{e_2}\mathbf{f}, e_1) + \omega(\overline{\nabla}^T_{e_1}\mathbf{f}, e_2) - \omega(\overline{\nabla}^T_{e_2}\mathbf{f}, e_1)\\ &= \omega(\overline{\nabla}^N_{e_1}\mathbf{f}, e_2) - \omega(\overline{\nabla}^N_{e_2}\mathbf{f}, e_1) - \langle\overline{\nabla}^T_{e_1}\mathbf{f}, Je_2\rangle + \langle\overline{\nabla}^T_{e_2}\mathbf{f}, Je_1\rangle\\ &= \omega(\overline{\nabla}^N_{e_1}\mathbf{f}, e_2) - \omega(\overline{\nabla}^N_{e_2}\mathbf{f}, e_1) + \cos\alpha(\langle\overline{\nabla}^T_{e_1}\mathbf{f}, e_1\rangle + \langle\overline{\nabla}^T_{e_2}\mathbf{f}, e_2\rangle)\\ &= \omega(\overline{\nabla}^N_{e_1}\mathbf{f}, e_2) - \omega(\overline{\nabla}^N_{e_2}\mathbf{f}, e_1) - \cos\alpha(\langle\mathbf{f}, \overline{\nabla}_{e_1}e_1 + \overline{\nabla}_{e_2}e_2\rangle)\\ &= \omega(\overline{\nabla}^N_{e_1}\mathbf{f}, e_2) - \omega(\overline{\nabla}^N_{e_2}\mathbf{f}, e_1) - \cos\alpha\,\mathbf{f}\cdot\mathbf{H},\end{aligned}$$

where we have used the fact that $\langle f, e_i\rangle = 0$. Therefore, we have

$$\frac{\partial}{\partial t}\cos\alpha = \omega(\overline{\nabla}^N_{e_1}\mathbf{f}, e_2) - \omega(\overline{\nabla}^N_{e_2}\mathbf{f}, e_1).$$

For the present purpose, we assume that

$$\mathbf{f} = u\Big(\frac{1}{\cos\alpha}\Big)(\cos^3\alpha\mathbf{H} - \beta(J(J\nabla\cos\alpha)^\top)^\perp) = u\Big(\frac{1}{\cos\alpha}\Big)\cos\alpha(\cos^2\alpha\mathbf{H} - \beta\sin^2\alpha\mathbf{V}),$$

where u is a smooth positive function to be determined later. Then we have

$$\begin{aligned}\frac{\partial}{\partial t}\cos\alpha &= \omega(\overline{\nabla}^N_{e_1}\mathbf{f}, e_2) - \omega(\overline{\nabla}^N_{e_2}\mathbf{f}, e_1)\\ &= \omega(\overline{\nabla}^N_{e_1}(u\cos^3\alpha\mathbf{H} - \beta u\cos\alpha\sin^2\alpha\mathbf{V}), e_2)\\ &\quad - \omega(\overline{\nabla}^N_{e_2}(u\cos^3\alpha\mathbf{H} - \beta u\cos\alpha\sin^2\alpha\mathbf{V}), e_1)\\ &= \langle\overline{\nabla}^N_{e_1}(u\cos^3\alpha\mathbf{H} - \beta u\cos\alpha\sin^2\alpha\mathbf{V}), -Je_2\rangle\\ &\quad + \langle\overline{\nabla}^N_{e_2}(u\cos^3\alpha\mathbf{H} - \beta u\cos\alpha\sin^2\alpha\mathbf{V}), Je_1\rangle\\ &= \sin\alpha\big[\langle\overline{\nabla}^N_{e_1}(u\cos^3\alpha\mathbf{H} - \beta u\cos\alpha\sin^2\alpha\mathbf{V}), v_4\rangle\\ &\qquad + \langle\overline{\nabla}^N_{e_2}(u\cos^3\alpha\mathbf{H} - \beta u\cos\alpha\sin^2\alpha\mathbf{V}), v_3\rangle\big]\\ &= \sin\alpha\Big[u\cos^3\alpha(H^4_{,1} + H^3_{,2}) - \beta u\cos\alpha\sin^2\alpha(V^4_{,1} + V^3_{,2})\\ &\qquad + (u'\sin\alpha\cos\alpha - 3u\cos^2\alpha\sin\alpha)(H^4\partial_1\alpha + H^3\partial_2\alpha)\\ &\qquad - \beta\Big(u'\frac{\sin^3\alpha}{\cos\alpha} - u\sin^3\alpha + 2u\sin\alpha\cos^2\alpha\Big)(V^4\partial_1\alpha + V^3\partial_2\alpha)\Big]\end{aligned}$$

$$
\begin{aligned}
&= u\sin\alpha\cos^3\alpha(H^4_{,1}+H^3_{,2}) - \beta u\cos\alpha\sin^3\alpha\Delta\alpha \\
&\quad + \sin^2\alpha\cos\alpha(u' - 3u\cos\alpha)\mathbf{H}\cdot\mathbf{V} \\
&\quad - \beta\sin^2\alpha\Big(u'\frac{\sin^2\alpha}{\cos\alpha} - u\sin^2\alpha + 2u\cos^2\alpha\Big)|\mathbf{V}|^2.
\end{aligned}
$$

Using Proposition 2.2 and the definition of $\mathbf{V}$, we see that

$$
\begin{aligned}
\frac{\partial}{\partial t}\cos\alpha &= u\cos^3\alpha\big[\Delta\cos\alpha + \cos\alpha(|h^3_{1k} - h^4_{2k}|^2 + |h^4_{1k} + h^3_{2k}|^2) \\
&\qquad\qquad + \sin^2\alpha\,\mathrm{Ric}(Je_1, e_2)\big] \\
&\quad + \beta u\cos\alpha\sin^2\alpha(\Delta\cos\alpha + \cos\alpha|\nabla\alpha|^2) + \sin^2\alpha\cos\alpha(u' - 3u\cos\alpha)\mathbf{H}\cdot\mathbf{V} \\
&\quad - \beta\sin^2\alpha\Big(u'\frac{\sin^2\alpha}{\cos\alpha} - u\sin^2\alpha + 2u\cos^2\alpha\Big)|\mathbf{V}|^2 \\
&= u\cos\alpha(\cos^2\alpha + \beta\sin^2\alpha)\Delta\cos\alpha + u\cos^4\alpha(|h^3_{1k} - h^4_{2k}|^2 + |h^4_{1k} + h^3_{2k}|^2) \\
&\quad + u\cos^3\alpha\sin^2\alpha\,\mathrm{Ric}(Je_1, e_2) + \sin^2\alpha\cos\alpha(u' - 3u\cos\alpha)\mathbf{H}\cdot\mathbf{V} \\
&\quad - \beta\sin^2\alpha\Big(u'\frac{\sin^2\alpha}{\cos\alpha} - u\sin^2\alpha + u\cos^2\alpha\Big)|\mathbf{V}|^2.
\end{aligned}
$$

It can be checked that

$$
|h^3_{1k} - h^4_{2k}|^2 + |h^4_{1k} + h^3_{2k}|^2 = |\mathbf{H}|^2 + 2|\mathbf{V}|^2 + 2\langle\mathbf{H}, \mathbf{V}\rangle.
$$

Therefore, we obtain

$$
\begin{aligned}
\frac{\partial}{\partial t}\cos\alpha &= u\cos\alpha(\cos^2\alpha + \beta\sin^2\alpha)\Delta\cos\alpha + u\cos^3\alpha\sin^2\alpha\,\mathrm{Ric}(Je_1, e_2) \\
&\quad + u\cos^4\alpha|\mathbf{H}|^2 + [2u\cos^4\alpha + \sin^2\alpha\cos\alpha(u' - 3u\cos\alpha)]\mathbf{H}\cdot\mathbf{V} \\
&\quad + \Big[2u\cos^4\alpha - \beta\sin^2\alpha\Big(u'\frac{\sin^2\alpha}{\cos\alpha} - u\sin^2\alpha + u\cos^2\alpha\Big)\Big]|\mathbf{V}|^2.
\end{aligned}
$$

By setting $x = \frac{1}{\cos\alpha}$, we have

$$
\cos\alpha(\cos^2\alpha + \beta\sin^2\alpha) = \frac{1}{x}\Big(\beta + \frac{1-\beta}{x^2}\Big) = \frac{\beta x^2 + 1 - \beta}{x^3}.
$$

Hence, if we choose $u(x) = \frac{x^3}{\beta x^2 + 1 - \beta}$, then we have

$$
u'(x) = \frac{\beta x^4 + 3(1-\beta)x^2}{(\beta x^2 + 1 - \beta)^2},
$$

so that

$$
u\Big(\frac{1}{\cos\alpha}\Big) = \frac{1}{\cos\alpha(\cos^2\alpha + \beta\sin^2\alpha)}
$$

and

$$
u'\Big(\frac{1}{\cos\alpha}\Big) = \frac{\beta + 3(1-\beta)\cos^2\alpha}{(\cos^2\alpha + \beta\sin^2\alpha)^2}.
$$

For this choice of u, we have

$$\mathbf{f} = \frac{\cos^2\alpha \mathbf{H} - \beta \sin^2\alpha \mathbf{V}}{\cos^2\alpha + \beta \sin^2\alpha}.$$

By direct computation, we finally obtain

$$\begin{aligned}\frac{\partial}{\partial t}\cos\alpha &= \Delta\cos\alpha + \frac{\cos^2\alpha\sin^2\alpha}{\cos^2\alpha+\beta\sin^2\alpha}\mathrm{Ric}(Je_1, e_2) + \frac{\cos^3\alpha}{\cos^2\alpha+\beta\sin^2\alpha}|\mathbf{H}|^2 \\ &\quad + \frac{\cos\alpha[(\beta^2-2\beta)\sin^4\alpha + \beta\sin^2\alpha\cos^2\alpha + 2\cos^4\alpha]}{(\cos^2\alpha+\beta\sin^2\alpha)^2}|\mathbf{V}|^2 \\ &\quad + \frac{2\cos\alpha(\cos^4\alpha - \beta\sin^4\alpha)}{(\cos^2\alpha+\beta\sin^2\alpha)^2}\mathbf{H}\cdot\mathbf{V}.\end{aligned}$$

This proves the theorem. ∎

The above theorem implies that the symplectic property is preserved along flow (1.1). Note that we do not need to assume that M is a Kähler–Einstein surface if $\beta > 0$.

Corollary 2.4. *Let M be a compact Kähler surface and Σ_0 be a closed symplectic surface in M. Then along flow (1.1) with $\beta > 0$, if Σ_0 is symplectic, then at each time t, Σ_t is symplectic. In particular, suppose that $|\mathrm{Ric}_M| \le K_1$. Then we have*

$$\min_{\Sigma_t}\cos\alpha \ge e^{-\frac{K_1 t}{\beta}}\min_{\Sigma_0}\cos\alpha, \tag{2.3}$$

as long as the smooth solution exists.

Proof. We can rewrite (2.2) as

$$\begin{aligned}\frac{\partial}{\partial t}\cos\alpha &= \Delta\cos\alpha + \frac{\cos^2\alpha\sin^2\alpha}{\cos^2\alpha+\beta\sin^2\alpha}\mathrm{Ric}(Je_1, e_2) + \frac{\cos^3\alpha}{\cos^2\alpha+\beta\sin^2\alpha}|\mathbf{H}|^2 \\ &\quad + \frac{\cos\alpha[(\beta^2-2\beta)\sin^4\alpha + \beta\sin^2\alpha\cos^2\alpha + 2\cos^4\alpha]}{(\cos^2\alpha+\beta\sin^2\alpha)^2}|\mathbf{V}|^2 \\ &\quad + \frac{2\cos\alpha(\cos^4\alpha - \beta\sin^4\alpha)}{(\cos^2\alpha+\beta\sin^2\alpha)^2}\mathbf{H}\cdot\mathbf{V} \\ &\ge \Delta\cos\alpha - K_1\frac{\cos^2\alpha\sin^2\alpha}{\cos^2\alpha+\beta\sin^2\alpha} + \frac{\cos^3\alpha}{\cos^2\alpha+\beta\sin^2\alpha}|\mathbf{H}|^2 \\ &\quad + \frac{\cos\alpha[(\beta^2-2\beta)\sin^4\alpha + \beta\sin^2\alpha\cos^2\alpha + 2\cos^4\alpha]}{(\cos^2\alpha+\beta\sin^2\alpha)^2}|\mathbf{V}|^2 \\ &\quad + \frac{2\cos\alpha(\cos^4\alpha - \beta\sin^4\alpha)}{(\cos^2\alpha+\beta\sin^2\alpha)^2}\mathbf{H}\cdot\mathbf{V}\end{aligned}$$

$$\geq \Delta \cos\alpha - \frac{K_1}{\beta}\cos\alpha + \frac{\cos^3\alpha}{\cos^2\alpha + \beta\sin^2\alpha}|\mathbf{H}|^2 + \frac{\cos\alpha[(\beta^2 - 2\beta)\sin^4\alpha + \beta\sin^2\alpha\cos^2\alpha + 2\cos^4\alpha]}{(\cos^2\alpha + \beta\sin^2\alpha)^2}|\mathbf{V}|^2 + \frac{2\cos\alpha(\cos^4\alpha - \beta\sin^4\alpha)}{(\cos^2\alpha + \beta\sin^2\alpha)^2}\mathbf{H}\cdot\mathbf{V}.$$

Notice that $\mathbf{V} = \partial_2\alpha v_3 + \partial_1\alpha v_4$. Then (2.3) follows from the maximum principle. ■

When the ambient manifold is a Kähler–Einstein surface, we have $\mathrm{Ric} = \frac{K_0}{4}\bar{g}$, where $\bar{g}$ and K_0 are the Kähler metric and the scalar curvature of M, respectively.

Corollary 2.5. *Let M be a Kähler–Einstein surface with scalar curvature K_0. Assume that α is the Kähler angle of Σ_t which evolves by flow (1.1) for $\beta \geq 0$. Then $\cos\alpha$ satisfies the equation*

$$\frac{\partial}{\partial t}\cos\alpha = \Delta\cos\alpha + \frac{K_0\cos^3\alpha\sin^2\alpha}{4(\cos^2\alpha + \beta\sin^2\alpha)} + \frac{\cos^3\alpha}{\cos^2\alpha + \beta\sin^2\alpha}|\mathbf{H}|^2 + \frac{\cos\alpha[(\beta^2 - 2\beta)\sin^4\alpha + \beta\sin^2\alpha\cos^2\alpha + 2\cos^4\alpha]}{(\cos^2\alpha + \beta\sin^2\alpha)^2}|\mathbf{V}|^2 + \frac{2\cos\alpha(\cos^4\alpha - \beta\sin^4\alpha)}{(\cos^2\alpha + \beta\sin^2\alpha)^2}\mathbf{H}\cdot\mathbf{V}. \tag{2.4}$$

Corollary 2.6. *Let M be a compact Kähler–Einstein surface with scalar curvature K_0 and Σ be a closed symplectic surface in M. Then along flow (1.1) with $\beta \geq 0$, if Σ_0 is symplectic, then at each time t, Σ_t is symplectic. If we set $K_1 = \max\{-K_0, 0\}$, then we have*

$$\min_{\Sigma_t}\cos\alpha \geq e^{-\frac{K_1}{4}t}\min_{\Sigma_0}\cos\alpha, \tag{2.5}$$

as long as the smooth solution exists. In particular, if M is a Kähler–Einstein surface with scalar curvature $K_0 \geq 0$, then we have

$$\min_{\Sigma_t}\cos\alpha \geq \min_{\Sigma_0}\cos\alpha, \tag{2.6}$$

as long as the smooth solution exists.

Proof. First, applying the maximum principle to (2.4), we see that $\cos\alpha > 0$ is preserved along flow (1.1). Then we have from (2.4) that

$$\frac{\partial}{\partial t}\cos\alpha \geq \Delta\cos\alpha - \frac{K_1}{4}\cos\alpha + \frac{\cos^3\alpha}{\cos^2\alpha + \beta\sin^2\alpha}|\mathbf{H}|^2 + \frac{\cos\alpha[(\beta^2 - 2\beta)\sin^4\alpha + \beta\sin^2\alpha\cos^2\alpha + 2\cos^4\alpha]}{(\cos^2\alpha + \beta\sin^2\alpha)^2}|\mathbf{V}|^2 + \frac{2\cos\alpha(\cos^4\alpha - \beta\sin^4\alpha)}{(\cos^2\alpha + \beta\sin^2\alpha)^2}\mathbf{H}\cdot\mathbf{V}.$$

Notice that $\mathbf{V} = \partial_2\alpha v_3 + \partial_1\alpha v_4$. Then (2.5) follows from the maximum principle and (2.6) follows directly from (2.5). ∎

3. Monotonicity formula

In this section we will prove the monotonicity formula for flow (1.1). Namely, we will consider the flow

$$\begin{cases} \dfrac{dF}{dt} = \dfrac{\cos^2\alpha\mathbf{H} - \beta\sin^2\alpha\mathbf{V}}{\cos^2\alpha + \beta\sin^2\alpha} \equiv \mathbf{f}, \\ F(\cdot, 0) = F_0, \end{cases} \tag{3.1}$$

where $F_0 \colon \Sigma \to M$ is a smooth immersion. Denote $\Sigma_t := F(\Sigma, t)$. By Corollaries 2.4 and 2.6, we may assume that

$$\cos\alpha \geq \delta > 0$$

on $[0, t_0)$ for $t_0 < \infty$. We will always assume M to be Kähler–Einstein if $\beta = 0$ in this section.

First, we can argue in the same way as we did in the proof of the long-time existence theorem for $\beta = 1$ [12, Theorem 3.1] to show that the singularity of flow (3.1) is characterized by the maximal norm of the second fundamental form of Σ in M.

Theorem 3.1. *Let M be a Kähler surface and Σ be a closed surface. Let $F \colon \Sigma \times [0, T) \to M$ be a smooth solution to flow (3.1). Set $\Sigma_t = F(\Sigma, t)$. If*

$$\max_{\Sigma_t} |A|^2 \leq \Lambda$$

for all $t \in [0, T)$, then the flow can be extended smoothly to an interval $[0, T + \varepsilon)$ for some $\varepsilon > 0$.

We first consider the monotonicity formula for the flow in $\mathbb{R}^4$. Recall that the backward heat kernel is defined by

$$\rho(X, t) := \frac{1}{4\pi(t_0 - t)} e^{-\frac{|X - X_0|^2}{4(t_0 - t)}}. \tag{3.2}$$

We will prove the following proposition:

Proposition 3.2. *Let $M = \mathbb{R}^4$. Then there exists a constant $p_0 > 0$ depending on β and δ, such that along flow (3.1) for $\beta \geq 0$ and $p \geq p_0$, we have*

$$\begin{aligned} &\frac{d}{dt}\int_\Sigma \frac{1}{\cos^p\alpha}\rho\, d\mu_t \\ &\quad \leq -\int_\Sigma \frac{1}{\cos^p\alpha}\biggl\{\frac{1}{4}\biggl|\frac{(F - X_0)^\perp}{t_0 - t} + \mathbf{f} + \mathbf{H}\biggr|^2 + |\mathbf{H}|^2 + |\mathbf{V}|^2 \\ &\qquad\qquad + \biggl|\mathbf{H} + \frac{4p(\cos^4\alpha - \beta\sin^4\alpha) - \beta^2\sin^4\alpha}{3p\cos^2\alpha(\cos^2\alpha + \beta\sin^2\alpha) - \beta^2\sin^4\alpha}\mathbf{V}\biggr|^2\biggr\}\rho\, d\mu_t. \end{aligned}$$

Proof. By (2.2), we know that $\cos\alpha$ satisfies the equation

$$
\begin{aligned}
\frac{\partial}{\partial t}\cos\alpha = \Delta\cos\alpha + \frac{\cos^3\alpha}{\cos^2\alpha+\beta\sin^2\alpha}|\mathbf{H}|^2 \\
+ \frac{\cos\alpha[(\beta^2-2\beta)\sin^4\alpha+\beta\sin^2\alpha\cos^2\alpha+2\cos^4\alpha]}{(\cos^2\alpha+\beta\sin^2\alpha)^2}|\mathbf{V}|^2 \\
+ \frac{2\cos\alpha(\cos^4\alpha-\beta\sin^4\alpha)}{(\cos^2\alpha+\beta\sin^2\alpha)^2}\mathbf{H}\cdot\mathbf{V}.
\end{aligned}
$$

In particular, we have

$$
\begin{aligned}
&\Big(\frac{\partial}{\partial t}-\Delta\Big)\Big(\frac{1}{\cos\alpha}\Big)\\
&= -\frac{1}{\cos^2\alpha}\Big(\frac{\partial}{\partial t}-\Delta\Big)\cos\alpha - 2\frac{|\nabla\cos\alpha|^2}{\cos^3\alpha}\\
&= -\frac{\cos\alpha}{\cos^2\alpha+\beta\sin^2\alpha}|\mathbf{H}|^2\\
&\quad - \frac{(\beta^2-2\beta)\sin^4\alpha+\beta\sin^2\alpha\cos^2\alpha+2\cos^4\alpha}{\cos\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}|\mathbf{V}|^2\\
&\quad - \frac{2(\cos^4\alpha-\beta\sin^4\alpha)}{\cos\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}\mathbf{H}\cdot\mathbf{V} - 2\frac{\sin^2\alpha}{\cos^3\alpha}|\mathbf{V}|^2\\
&= -\frac{\cos\alpha}{\cos^2\alpha+\beta\sin^2\alpha}|\mathbf{H}|^2 - \frac{2(\cos^4\alpha-\beta\sin^4\alpha)}{\cos\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}\mathbf{H}\cdot\mathbf{V}\\
&\quad - \frac{2\beta^2\sin^6\alpha+(\beta^2+2\beta)\sin^4\alpha\cos^2\alpha+(\beta+2)\sin^2\alpha\cos^4\alpha+2\cos^6\alpha}{\cos^3\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}|\mathbf{V}|^2.
\end{aligned}
$$

Let f be a positive function to be determined later. Then we have

$$
\begin{aligned}
&\Big(\frac{\partial}{\partial t}-\Delta\Big)f\Big(\frac{1}{\cos\alpha}\Big)\\
&= f'\Big(\frac{\partial}{\partial t}-\Delta\Big)\frac{1}{\cos\alpha} - f''\Big|\nabla\frac{1}{\cos\alpha}\Big|^2\\
&= f'\Big[-\frac{\cos\alpha}{\cos^2\alpha+\beta\sin^2\alpha}|\mathbf{H}|^2 - \frac{2(\cos^4\alpha-\beta\sin^4\alpha)}{\cos\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}\mathbf{H}\cdot\mathbf{V}\\
&\quad - \frac{2\beta^2\sin^6\alpha+(\beta^2+2\beta)\sin^4\alpha\cos^2\alpha+(\beta+2)\sin^2\alpha\cos^4\alpha+2\cos^6\alpha}{\cos^3\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}|\mathbf{V}|^2\Big]\\
&\quad - f''\frac{\sin^2\alpha}{\cos^4\alpha}|\mathbf{V}|^2.
\end{aligned}
$$

By (3.2), we see that along flow (3.1),

$$\frac{\partial}{\partial t}\rho(X,t) = \Big(\frac{1}{t_0-t} - \frac{\langle F-X_0,\mathbf{f}\rangle}{2(t_0-t)} - \frac{|X-X_0|^2}{4(t_0-t)^2}\Big)\rho.$$

We also have

$$\nabla\rho(X,t) = -\rho\frac{\langle X-X_0,\nabla X\rangle}{2(t_0-t)}$$

and

$$\Delta\rho(X,t) = \Big(\frac{|(X-X_0)^T|^2}{4(t_0-t)^2} - \frac{\langle F-X_0,\mathbf{H}\rangle}{2(t_0-t)} - \frac{1}{t_0-t}\Big)\rho.$$

Hence we have

$$\Big(\frac{\partial}{\partial t}+\Delta\Big)\rho(X,t) = -\Big(\frac{\langle F-X_0,\mathbf{f}+\mathbf{H}\rangle}{2(t_0-t)} + \frac{|(X-X_0)^\perp|^2}{4(t_0-t)^2}\Big)\rho. \tag{3.3}$$

Then we compute that

$$\begin{aligned}
&\frac{d}{dt}\int_\Sigma f\Big(\frac{1}{\cos\alpha}\Big)\rho\,d\mu_t\\
&= \int_\Sigma\Big(\frac{\partial}{\partial t}f\Big(\frac{1}{\cos\alpha}\Big)\Big)\rho\,d\mu_t + \int_\Sigma f\Big(\frac{\partial}{\partial t}\rho\Big)\,d\mu_t - \int_\Sigma f\rho\langle\mathbf{f},\mathbf{H}\rangle\,d\mu_t\\
&= \int_\Sigma\Big(\frac{\partial}{\partial t}-\Delta\Big)f\Big(\frac{1}{\cos\alpha}\Big)\rho\,d\mu_t + \int_\Sigma f\Big(\frac{\partial}{\partial t}+\Delta\Big)\rho\,d\mu_t - \int_\Sigma f\rho\langle\mathbf{f},\mathbf{H}\rangle\,d\mu_t\\
&= -\int_\Sigma\Big\{f'\frac{\cos\alpha}{\cos^2\alpha+\beta\sin^2\alpha}|\mathbf{H}|^2 + f'\frac{2(\cos^4\alpha-\beta\sin^4\alpha)}{\cos\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}\mathbf{H}\cdot\mathbf{V}\\
&\qquad + f'\frac{2\beta^2\sin^6\alpha+(\beta^2+2\beta)\sin^4\alpha\cos^2\alpha+(\beta+2)\sin^2\alpha\cos^4\alpha+2\cos^6\alpha}{\cos^3\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}|\mathbf{V}|^2\\
&\qquad + f''\frac{\sin^2\alpha}{\cos^4\alpha}|\mathbf{V}|^2\Big\}\rho\,d\mu_t\\
&\quad - \int_\Sigma f\Big(\frac{\langle F-X_0,\mathbf{f}+\mathbf{H}\rangle}{2(t_0-t)} + \frac{|(X-X_0)^\perp|^2}{4(t_0-t)^2} + \langle\mathbf{f},\mathbf{H}\rangle\Big)\rho\,d\mu_t\\
&= -\int_\Sigma\Big\{f'\frac{\cos\alpha}{\cos^2\alpha+\beta\sin^2\alpha}|\mathbf{H}|^2 + f'\frac{2(\cos^4\alpha-\beta\sin^4\alpha)}{\cos\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}\mathbf{H}\cdot\mathbf{V}\\
&\qquad + f'\frac{2\beta^2\sin^6\alpha+(\beta^2+2\beta)\sin^4\alpha\cos^2\alpha+(\beta+2)\sin^2\alpha\cos^4\alpha+2\cos^6\alpha}{\cos^3\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}|\mathbf{V}|^2\\
&\qquad + f''\frac{\sin^2\alpha}{\cos^4\alpha}|\mathbf{V}|^2 + \frac{1}{4}\Big|\frac{(F-X_0)^\perp}{t_0-t}+\mathbf{f}+\mathbf{H}\Big|^2 f - \frac{|\mathbf{f}+\mathbf{H}|^2}{4}f + \langle\mathbf{f},\mathbf{H}\rangle f\Big\}\rho\,d\mu_t.
\end{aligned}$$

Recalling that

$$\mathbf{f} = \frac{\cos^2\alpha\mathbf{H}-\beta\sin^2\alpha\mathbf{V}}{\cos^2\alpha+\beta\sin^2\alpha},$$

we have

$$
\begin{aligned}
& f'\frac{\cos\alpha}{\cos^2\alpha+\beta\sin^2\alpha}|\mathbf{H}|^2+f'\frac{2(\cos^4\alpha-\beta\sin^4\alpha)}{\cos\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}\mathbf{H}\cdot\mathbf{V}\\
&\quad+f'\frac{2\beta^2\sin^6\alpha+(\beta^2+2\beta)\sin^4\alpha\cos^2\alpha+(\beta+2)\sin^2\alpha\cos^4\alpha+2\cos^6\alpha}{\cos^3\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}|\mathbf{V}|^2\\
&\quad+f''\frac{\sin^2\alpha}{\cos^4\alpha}|\mathbf{V}|^2-\frac{|\mathbf{f}+\mathbf{H}|^2}{4}f+\langle\mathbf{f},\mathbf{H}\rangle f\\
&=f'\frac{\cos\alpha}{\cos^2\alpha+\beta\sin^2\alpha}|\mathbf{H}|^2+f'\frac{2(\cos^4\alpha-\beta\sin^4\alpha)}{\cos\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}\mathbf{H}\cdot\mathbf{V}+f''\frac{\sin^2\alpha}{\cos^4\alpha}|\mathbf{V}|^2\\
&\quad+f'\frac{2\beta^2\sin^6\alpha+(\beta^2+2\beta)\sin^4\alpha\cos^2\alpha+(\beta+2)\sin^2\alpha\cos^4\alpha+2\cos^6\alpha}{\cos^3\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}|\mathbf{V}|^2\\
&\quad-\frac{|(2\cos^2\alpha+\beta\sin^2\alpha)\mathbf{H}-\beta\sin^2\alpha\mathbf{V}|^2}{4(\cos^2\alpha+\beta\sin^2\alpha)^2}f+\frac{\langle\cos^2\alpha\mathbf{H}-\beta\sin^2\alpha\mathbf{V},\mathbf{H}\rangle}{\cos^2\alpha+\beta\sin^2\alpha}f\\
&=\Big(\frac{f'\cos\alpha}{\cos^2\alpha+\beta\sin^2\alpha}-\frac{\beta^2 f\sin^4\alpha}{4(\cos^2\alpha+\beta\sin^2\alpha)^2}\Big)|\mathbf{H}|^2\\
&\quad+\Big(\frac{2f'(\cos^4\alpha-\beta\sin^4\alpha)}{\cos\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}-\frac{\beta^2 f\sin^4\alpha}{2(\cos^2\alpha+\beta\sin^2\alpha)^2}\Big)\mathbf{H}\cdot\mathbf{V}\\
&\quad+\Big(f'\frac{2\beta^2\sin^6\alpha+(\beta^2+2\beta)\sin^4\alpha\cos^2\alpha+(\beta+2)\sin^2\alpha\cos^4\alpha+2\cos^6\alpha}{\cos^3\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}\\
&\qquad+f''\frac{\sin^2\alpha}{\cos^4\alpha}-\frac{\beta^2 f\sin^4\alpha}{4(\cos^2\alpha+\beta\sin^2\alpha)^2}\Big)|\mathbf{V}|^2.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_\Sigma f\Big(\frac{1}{\cos\alpha}\Big)\rho\,d\mu_t\\
&=-\int_\Sigma\Big\{\frac14\Big|\frac{(F-X_0)^\perp}{t_0-t}+\mathbf{f}+\mathbf{H}\Big|^2 f\\
&\qquad+\Big(\frac{f'\cos\alpha}{\cos^2\alpha+\beta\sin^2\alpha}-\frac{\beta^2 f\sin^4\alpha}{4(\cos^2\alpha+\beta\sin^2\alpha)^2}\Big)|\mathbf{H}|^2\\
&\qquad+\Big(\frac{2f'(\cos^4\alpha-\beta\sin^4\alpha)}{\cos\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}-\frac{\beta^2 f\sin^4\alpha}{2(\cos^2\alpha+\beta\sin^2\alpha)^2}\Big)\mathbf{H}\cdot\mathbf{V}\\
&\qquad+\Big(f'\frac{2\beta^2\sin^6\alpha+(\beta^2+2\beta)\sin^4\alpha\cos^2\alpha+(\beta+2)\sin^2\alpha\cos^4\alpha+2\cos^6\alpha}{\cos^3\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}\\
&\qquad\quad+f''\frac{\sin^2\alpha}{\cos^4\alpha}-\frac{\beta^2 f\sin^4\alpha}{4(\cos^2\alpha+\beta\sin^2\alpha)^2}\Big)|\mathbf{V}|^2\Big\}\rho\,d\mu_t. \qquad (3.4)
\end{aligned}
$$

Now we take $f(x)=x^p$ with $p\ge 1$ to be determined. Then

$$
f'=\frac{p}{x}f,\quad f''=\frac{p(p-1)}{x^2}f.
$$

Hence, we have

$$
\begin{aligned}
&\Big(\frac{f'\cos\alpha}{\cos^2\alpha+\beta\sin^2\alpha}-\frac{\beta^2 f\sin^4\alpha}{4(\cos^2\alpha+\beta\sin^2\alpha)^2}\Big)|\mathbf{H}|^2\\
&\quad+\Big(\frac{2f'(\cos^4\alpha-\beta\sin^4\alpha)}{\cos\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}-\frac{\beta^2 f\sin^4\alpha}{2(\cos^2\alpha+\beta\sin^2\alpha)^2}\Big)\mathbf{H}\cdot\mathbf{V}\\
&\quad+\Big(f'\frac{2\beta^2\sin^6\alpha+(\beta^2+2\beta)\sin^4\alpha\cos^2\alpha+(\beta+2)\sin^2\alpha\cos^4\alpha+2\cos^6\alpha}{\cos^3\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}\\
&\qquad+f''\frac{\sin^2\alpha}{\cos^4\alpha}-\frac{\beta^2 f\sin^4\alpha}{4(\cos^2\alpha+\beta\sin^2\alpha)^2}\Big)|\mathbf{V}|^2\\
&=\Big(\frac{p\cos^2\alpha}{\cos^2\alpha+\beta\sin^2\alpha}-\frac{\beta^2\sin^4\alpha}{4(\cos^2\alpha+\beta\sin^2\alpha)^2}\Big)f|\mathbf{H}|^2\\
&\quad+\Big(\frac{2p(\cos^4\alpha-\beta\sin^4\alpha)}{(\cos^2\alpha+\beta\sin^2\alpha)^2}-\frac{\beta^2\sin^4\alpha}{2(\cos^2\alpha+\beta\sin^2\alpha)^2}\Big)f\,\mathbf{H}\cdot\mathbf{V}\\
&\quad+\Big(\frac{p[2\beta^2\sin^6\alpha+(\beta^2+2\beta)\sin^4\alpha\cos^2\alpha+(\beta+2)\sin^2\alpha\cos^4\alpha+2\cos^6\alpha]}{\cos^2\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}\\
&\qquad+\frac{p(p-1)\sin^2\alpha}{\cos^2\alpha}-\frac{\beta^2\sin^4\alpha}{4(\cos^2\alpha+\beta\sin^2\alpha)^2}\Big)f|\mathbf{V}|^2\\
&=\Big(\frac{3p\cos^2\alpha}{4(\cos^2\alpha+\beta\sin^2\alpha)}-\frac{\beta^2\sin^4\alpha}{4(\cos^2\alpha+\beta\sin^2\alpha)^2}\Big)f|\mathbf{H}+D_1\mathbf{V}|^2\\
&\quad+\frac{p\cos^2\alpha}{4(\cos^2\alpha+\beta\sin^2\alpha)}f|\mathbf{H}|^2+D_2 f|\mathbf{V}|^2,
\end{aligned}
$$

where

$$
D_1=\frac{4p(\cos^4\alpha-\beta\sin^4\alpha)-\beta^2\sin^4\alpha}{3p\cos^2\alpha(\cos^2\alpha+\beta\sin^2\alpha)-\beta^2\sin^4\alpha}
$$

and

$$
\begin{aligned}
D_2=&\frac{p[2\beta^2\sin^6\alpha+(\beta^2+2\beta)\sin^4\alpha\cos^2\alpha+(\beta+2)\sin^2\alpha\cos^4\alpha+2\cos^6\alpha]}{\cos^2\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}\\
&+\frac{p(p-1)\sin^2\alpha}{\cos^2\alpha}-\frac{\beta^2\sin^4\alpha}{4(\cos^2\alpha+\beta\sin^2\alpha)^2}\\
&-\frac{[4p(\cos^4\alpha-\beta\sin^4\alpha)-\beta^2\sin^4\alpha]^2}{4(\cos^2\alpha+\beta\sin^2\alpha)^2[3p\cos^2\alpha(\cos^2\alpha+\beta\sin^2\alpha)-\beta^2\sin^4\alpha]}.
\end{aligned}
$$

Since $\cos\alpha\geq\delta>0$, we see that there exists a positive constant p_0 depending on δ and β, such that for all $p\geq p_0$, we have

$$
\frac{3p\cos^2\alpha}{4(\cos^2\alpha+\beta\sin^2\alpha)}-\frac{\beta^2\sin^4\alpha}{4(\cos^2\alpha+\beta\sin^2\alpha)^2}\geq 3
$$

and $D_2 \geq 1$. For $f(x) = x^p$ with $p \geq p_0$, (3.4) gives us that

$$\begin{aligned}
&\frac{d}{dt}\int_\Sigma \frac{1}{\cos^p\alpha}\rho\, d\mu_t \\
&\quad \leq -\int_\Sigma \frac{1}{\cos^p\alpha}\Big\{\frac{1}{4}\Big|\frac{(F-X_0)^\perp}{t_0-t}+\mathbf{f}+\mathbf{H}\Big|^2+|\mathbf{H}|^2+|\mathbf{V}|^2 \\
&\qquad\qquad + \Big|\mathbf{H}+\frac{4p(\cos^4\alpha-\beta\sin^4\alpha)-\beta^2\sin^4\alpha}{3p\cos^2\alpha(\cos^2\alpha+\beta\sin^2\alpha)-\beta^2\sin^4\alpha}\mathbf{V}\Big|^2\Big\}\rho\, d\mu_t.
\end{aligned}$$

This proves the proposition. ∎

Next we will consider the monotonicity formula for flow (3.1) in a Kähler surface M. Let i_M be the injectivity radius of M. We choose a cut-off function $\phi \in C_0^\infty(B_{2r}(X_0))$ with $\phi \equiv 1$ in $B_r(X_0)$, where $X_0 \in M$, $0 < 2r < i_M$. Choose normal coordinates in $B_{2r}(X_0)$ and express F using the coordinates (F^1, F^2, F^3, F^4) as a surface in $\mathbb{R}^4$. We define

$$\Psi(X_0,t_0,t) := \int_{\Sigma_t}\frac{1}{\cos^p\alpha}\phi(F)\rho\, d\mu_t. \tag{3.5}$$

Then we have the following theorem:

Theorem 3.3. *Let M^4 be a compact Kähler surface. Then there are positive constants p_0, c_1 and c_2 depending only on M^4, F_0, r, t_0 and β, such that along flow (3.1) with $\beta \geq 0$ and $p \geq p_0$, we have*

$$\begin{aligned}
&\frac{d}{dt}(e^{c_1\sqrt{t_0-t}}\Psi(X_0,t_0,t)) \\
&\quad \leq -e^{c_1\sqrt{t_0-t}}\int_\Sigma \frac{1}{\cos^p\alpha}\Big\{\frac{1}{4}\Big|\frac{(F-X_0)^\perp}{t_0-t}+\mathbf{f}+\mathbf{H}\Big|^2 \\
&\qquad\qquad + \Big|\mathbf{H}+\frac{4p(\cos^4\alpha-\beta\sin^4\alpha)-\beta^2\sin^4\alpha}{3p\cos^2\alpha(\cos^2\alpha+\beta\sin^2\alpha)-\beta^2\sin^4\alpha}\mathbf{V}\Big|^2 \\
&\qquad\qquad + \frac{7}{8}|\mathbf{H}|^2+\frac{7}{8}\mathbf{V}|^2\Big\}\rho\, d\mu_t \\
&\qquad + c_2 e^{c_1\sqrt{t_0-t}}.
\end{aligned}\tag{3.6}$$

Proof. Note that

$$\Delta F = \mathbf{H} + g^{ij}\Gamma_{ij}^\alpha v_\alpha,$$

where $\{v_\alpha\}_{\alpha=3,4}$ is a basis of $N\Sigma_t$, g_{ij} is the induced metric on Σ, (g^{ij}) is the inverse of (g_{ij}) and Γ_{ij}^α is the Christoffel symbol on M. Then (3.3) reads

$$\Big(\frac{\partial}{\partial t}+\Delta\Big)\rho(X,t) = -\Big(\frac{\langle F-X_0, \mathbf{f}+\mathbf{H}+g^{ij}\Gamma_{ij}^\alpha v_\alpha\rangle}{2(t_0-t)}+\frac{|(X-X_0)^\perp|^2}{4(t_0-t)^2}\Big)\rho.$$

Using Theorem 2.3, we have

$$\begin{aligned}
&\Big(\frac{\partial}{\partial t}-\Delta\Big)f\Big(\frac{1}{\cos\alpha}\Big)=f'\Big(\frac{\partial}{\partial t}-\Delta\Big)\frac{1}{\cos\alpha}-f''\Big|\nabla\frac{1}{\cos\alpha}\Big|^2\\
&=f'\Big[-\frac{\cos\alpha}{\cos^2\alpha+\beta\sin^2\alpha}|\mathbf{H}|^2-\frac{2(\cos^4\alpha-\sin^4\alpha)}{\cos\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}\mathbf{H}\cdot\mathbf{V}\\
&\qquad-\frac{2\beta^2\sin^6\alpha+(\beta^2+2\beta)\sin^4\alpha\cos^2\alpha+(\beta+2)\sin^2\alpha\cos^4\alpha+2\cos^6\alpha}{\cos^3\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}|\mathbf{V}|^2\Big]\\
&\quad-f''\frac{\sin^2\alpha}{\cos^4\alpha}|\mathbf{V}|^2-f'\frac{\sin^2\alpha}{\cos^2\alpha+\beta\sin^2\alpha}\mathrm{Ric}(Je_1,e_2),
\end{aligned}\tag{3.7}$$

where $f(x)=x^p$. We also have

$$\frac{\partial}{\partial t}\phi(F)=\langle D\phi,\mathbf{f}\rangle.$$

Suppose $|\mathrm{Ric}_M|\le K_1$. Then from the proof of Proposition 3.2, we have for $p\ge p_0$,

$$\begin{aligned}
&\frac{d}{dt}\Psi(X_0,t_0,t)\\
&\qquad=\frac{d}{dt}\int_{\Sigma_t}\frac{1}{\cos^p\alpha}\phi(F)\rho\,d\mu_t\\
&\qquad=\int_\Sigma\phi\rho\Big(\frac{\partial}{\partial t}-\Delta\Big)f\,d\mu_t+\int_\Sigma\phi f\Big(\frac{\partial}{\partial t}+\Delta\Big)\rho\,d\mu_t\\
&\qquad\quad+\int_{\Sigma_t}\Big(\frac{\partial}{\partial t}\phi(F)\Big)f\rho\,d\mu_t-\int_\Sigma f\phi\rho\langle\mathbf{f},\mathbf{H}\rangle\,d\mu_t\\
&\qquad\quad+\int_{\Sigma_t}\phi\rho\Delta f\Big(\frac{1}{\cos\alpha}\Big)\,d\mu_t-\int_{\Sigma_t}\phi f\Delta\rho\,d\mu_t\\
&\qquad\le-\int_\Sigma\frac{1}{\cos^p\alpha}\Big\{\frac{1}{4}\Big|\frac{(F-X_0)^\perp}{t_0-t}+\mathbf{f}+\mathbf{H}\Big|^2+|\mathbf{H}|^2+|\mathbf{V}|^2\\
&\qquad\qquad+\Big|\mathbf{H}+\frac{4p(\cos^4\alpha-\beta\sin^4\alpha)-\beta^2\sin^4\alpha}{3p\cos^2\alpha(\cos^2\alpha+\beta\sin^2\alpha)-\beta^2\sin^4\alpha}\mathbf{V}\Big|^2\Big\}\rho\,d\mu_t\\
&\qquad\quad+\int_{\Sigma_t}\langle D\phi,\mathbf{f}\rangle f\rho\,d\mu_t+\int_{\Sigma_t}f\rho\Delta\phi\,d\mu_t+2\int_{\Sigma_t}f\langle\nabla\rho,\nabla\phi\rangle\,d\mu_t\\
&\qquad\quad-\int_{\Sigma_t}\phi f\rho\frac{\langle F-X_0,g^{ij}\Gamma_{ij}^\alpha v_\alpha\rangle}{2(t_0-t)}\,d\mu_t\\
&\qquad\quad+K_1\int_{\Sigma_t}f'\phi\rho\frac{\sin^2\alpha}{\cos^2\alpha+\beta\sin^2\alpha}\,d\mu_t.
\end{aligned}$$

As in the proof of [4, Proposition 2.1] (see [4, equation (13)]), we see that

$$\Big|\frac{\langle F-X_0,g^{ij}\Gamma_{ij}^\alpha v_\alpha\rangle}{2(t_0-t)}\Big|\rho\le c_1\frac{\rho(F,t)}{\sqrt{t_0-t}}+C.$$

Furthermore, since $\phi \in C_0^\infty(B_{2r}(X_0), \mathbb{R}^+)$, we have

$$\frac{|D\phi|^2}{\phi} \le 2 \max_{\phi>0} |D^2\phi|^2.$$

We also have

$$|\mathbf{f}|^2 = \Big|\frac{\cos^2\alpha \mathbf{H} - \beta \sin^2\alpha \mathbf{V}}{\cos^2\alpha + \beta\sin^2\alpha}\Big|^2 \le |\mathbf{H}|^2 + |\mathbf{V}|^2.$$

Hence we have

$$\begin{aligned} |\langle D\phi, \mathbf{f}\rangle| f\rho &\le \frac{1}{8}|\mathbf{f}|^2\phi f\rho + 2\frac{|D\phi|^2}{\phi} f\rho \\ &\le \frac{1}{8}(|\mathbf{H}|^2 + |\mathbf{V}|^2)\phi f\rho + 2\frac{|D\phi|^2}{\phi} f\rho. \end{aligned}$$

Note that $\nabla\phi = 0$ in $B_r(X_0)$ so that $|\rho\Delta\phi|$ and $\langle\nabla\rho, \nabla\phi\rangle$ are bounded in $B_{2r}(X_0)$. We also note by the choice of f that

$$0 \le f' \sin^2\alpha = p\sin^2\alpha\cos\alpha f \le pf.$$

Hence we have

$$\begin{aligned} &\frac{d}{dt}\Psi(X_0, t_0, t) \\ &\quad \le -\int_\Sigma \frac{1}{\cos^p\alpha}\Big\{\frac{1}{4}\Big|\frac{(F-X_0)^\perp}{t_0-t} + \mathbf{f} + \mathbf{H}\Big|^2 + \frac{7}{8}|\mathbf{H}|^2 + \frac{7}{8}|\mathbf{V}|^2 \\ &\qquad + \Big|\mathbf{H} + \frac{4p(\cos^4\alpha - \beta\sin^4\alpha) - \beta^2\sin^4\alpha}{3p\cos^2\alpha(\cos^2\alpha+\beta\sin^2\alpha) - \beta^2\sin^4\alpha}\mathbf{V}\Big|^2\Big\}\phi\rho\, d\mu_t \\ &\quad + \frac{c_1}{\sqrt{t_0-t}}\Psi + C\int_{\Sigma_t}\frac{1}{\cos^p\alpha}\, d\mu_t. \end{aligned} \tag{3.8}$$

Since (1.1) is the negative gradient flow of the functional $L_\beta = \int_\Sigma \frac{1}{\cos^\beta\alpha}\, d\mu_t$, we know that

$$\int_{\Sigma_t}\frac{1}{\cos^\beta\alpha}\, d\mu_t \le \int_\Sigma \frac{1}{\cos^\beta\alpha}\, d\mu_0,$$

for each $t \in [0, t_0)$. In particular,

$$\text{Area}(\Sigma_t) \le \int_{\Sigma_t}\frac{1}{\cos^\beta\alpha}\, d\mu_t \le \int_\Sigma \frac{1}{\cos^\beta\alpha}\, d\mu_0 = L_\beta(\Sigma_0). \tag{3.9}$$

By (2.3), we know that $\cos\alpha \ge \delta > 0$ on $[0, t_0)$ for some constant $\delta > 0$ depending on Σ_0 and t_0 whenever the flow has a smooth solution on $[0, t_0)$. Hence we have that $f \le \frac{1}{\delta^p}$ for $t \in [0, t_0)$, so that

$$\int_{\Sigma_t} f\, d\mu_t \le \frac{1}{\delta^p} L_\beta(\Sigma_0). \tag{3.10}$$

Therefore, we have from (3.8) that

$$
\begin{aligned}
&\frac{d}{dt}\Psi(X_0,t_0,t)\\
&\quad\le -\int_\Sigma \frac{1}{\cos^p\alpha}\Big\{\frac{1}{4}\Big|\frac{(F-X_0)^\perp}{t_0-t}+\mathbf{f}+\mathbf{H}\Big|^2+\frac{7}{8}|\mathbf{H}|^2+\frac{7}{8}|\mathbf{V}|^2\\
&\qquad\qquad + \Big|\mathbf{H}+\frac{4p(\cos^4\alpha-\beta\sin^4\alpha)-\beta^2\sin^4\alpha}{3p\cos^2\alpha(\cos^2\alpha+\beta\sin^2\alpha)-\beta^2\sin^4\alpha}\mathbf{V}\Big|^2\Big\}\phi\rho\, d\mu_t\\
&\qquad + \frac{c_1}{\sqrt{t_0-t}}\Psi + c_2.
\end{aligned}
$$

This implies the desired estimate. ∎

4. ε-regularity theorem

The first consequence of the monotonicity formula is the ε-regularity theorem, which plays the important role in the singularity analysis for flow (1.1). We denote the parabolic ball by $P(X_0,t_0,r) := B_r(X_0)\times(t_0-r^2,t_0]$. Using the notation of Section 3, we have the following theorem:

Theorem 4.1. *Let Ψ be given by (3.5). Then there exist constants $\varepsilon>0$ and C_0, depending on only on β and M^4, such that if Σ_t is a smooth solution to flow (1.1) in $P(0,0,8\rho)$,*

$$\sup_{(X,t)\in P(X_0,t_0,r)} \Psi(X,t,t-r^2) < 1+\varepsilon,$$

for some $r\in(0,\rho)$, then we have

$$\sup_{P(X_0,t_0,\frac{r}{2})} |\mathbf{A}| \le \frac{C_0}{r}.$$

Proof. We prove it by contradiction. Assume that the conclusion is false. Then there is a sequence of smooth solutions of flow (1.1), say, (Σ_t^j), in $P(0,0,8\rho_j)$ for some $\rho_j>1$ with

$$\sup_{(X,t)\in P(0,0,1)} \Psi_{\Sigma^j}(X,t,t-1) < 1+\frac{1}{j}, \tag{4.1}$$

but there are points $(X_j,t_j)\in P(0,0,\frac{1}{2})$, with $|\mathbf{A}|(X_j,t_j)>j$.

Using the so-called point selection technique, we can find space-time points $(Y_j,s_j)\in P(0,0,\frac{3}{4})$ with $Q_j=|\mathbf{A}|(Y_j,s_j)>j$ such that

$$\sup_{P(Y_j,t_j,\frac{j}{10Q_j})} |\mathbf{A}| \le 2Q_j. \tag{4.2}$$

We describe the process of the point selection technique. For fixed j, if $(Y_j^0,s_j^0)=(X_j,t_j)$ satisfies (4.2) with $Q_j^0=|\mathbf{A}|(Y_j^0,s_j^0)$, then we are done. Otherwise, there is a point $(Y_j^1, s_j^1) \in P(Y_j^0, s_j^0, \frac{j}{10Q_j^0})$ with $Q_j^1 = |\mathbf{A}|(Y_j^1, s_j^1) > 2Q_j^0$. If (Y_j^1, s_j^1) satisfies (4.2), then we are done. Otherwise, there is point $(Y_j^2, s_j^2) \in P(Y_j^1, s_j^1, \frac{j}{10Q_j^1})$ with $Q_j^2 = |\mathbf{A}|(Y_j^2, s_j^2) > 2Q_j^1$, etc. Note that the parabolic distances between (X_j, t_j) and (Y_j^i, s_j^i) are bounded from above by

$$\frac{j}{10Q_j^0}\Big(1 + \frac{1}{2} + \frac{1}{4} + \cdots\Big) = \frac{j}{5Q_j^0} < \frac{1}{5}.$$

Since $(X_j, t_j) \in P(0, 0, \frac{1}{2})$, we see that $(Y_j^i, s_j^i) \in P(0, 0, \frac{7}{10})$. Since the flow is smooth, the iteration terminates after a finite number of steps, and the last point of the iteration lies in $P(0, 0, \frac{3}{4})$ and satisfies (4.2).

Now we can consider the rescaled flow (in normal coordinates)

$$\widetilde{\Sigma}_s^j := Q_j(\Sigma^j_{s_j + Q_j^{-2}s} - Y_j).$$

Then the rescaled flow satisfies $|\mathbf{A}|(0,0) = 1$ and $\sup_{P(0,0,\frac{j}{10})} |\mathbf{A}| \leq 2$. By the standard parabolic estimates, we can then obtain a nonflat smooth global limit in $\mathbb{C}^2$.

On the other hand, by the scale-invariant property of Ψ and (4.1), we see that the limit of Ψ is equal to 1 identically. However, the monotonicity formula (Theorem 3.3) implies that the limit must be a flat plane. This gives the desired contradiction. ∎

By continuity and the monotonicity formula, we see that if $\Psi < 1 + \frac{\varepsilon}{2}$ holds at some point and some scale, then $\Psi < 1 + \varepsilon$ holds at all nearby points and somewhat smaller scales. Thus we have the following corollary:

Corollary 4.2. *Under the assumption of the above theorem, there exists a constant $\varepsilon > 0$, depending on only on p, δ, M^4, such that if*

$$\Psi(X_0, t_0, t_0 - r^2) < 1 + \varepsilon,$$

then

$$\sup_{P(X_0, t_0, \frac{r}{2})} |\mathbf{A}| \leq \frac{C_0}{r}.$$

In particular, (X_0, t_0) is a regular point of flow (1.1).

5. Flatness of the λ-tangent cones

In this section we will use the monotonicity formula obtained in the Section 3 to show that the λ-tangent cones of flow (3.1) are the unions of flat planes.

Suppose that (X_0, T) is a singular point of flow (3.1), where T is the first singular time. We now describe the rescaling process around (X_0, T). As in the previous section, we choose normal coordinates centered at X_0 with radius r ($0 < r < \frac{i_M}{2}$), using the exponential map. We express F in its coordinates functions. For any $t < 0$, we set

$$F_\lambda(x,t) = \lambda(F(x, T + \lambda^{-2}t) - X_0),$$

where λ are positive constants which go to infinity. The scaled surfaces are denoted by $\Sigma_t^\lambda = F_\lambda(\Sigma, t)$ on which $d\mu_t^\lambda$ are the area elements obtained from $d\mu_t$.

If g^λ is the metric on Σ_t^λ, it is clear that

$$g_{ij}^\lambda = \lambda^2 g_{ij}, \quad (g^\lambda)^{ij} = \lambda^{-2} g^{ij}.$$

It is easy to check that

$$\begin{aligned} \frac{\partial F_\lambda}{\partial t} &= \lambda^{-1}\frac{\partial F}{\partial t}, \\ \mathbf{H}_\lambda &= \lambda^{-1}\mathbf{H}, \\ \mathbf{V}_\lambda &= \lambda^{-1}\mathbf{V}, \\ |\mathbf{A}_\lambda|^2 &= \lambda^{-2}|\mathbf{A}|^2. \end{aligned}$$

It follows that the scaled surface also evolves by the flow

$$\frac{\partial F_\lambda}{\partial t} = \frac{\cos^2\alpha_\lambda \mathbf{H}_\lambda - \beta\sin^2\alpha_\lambda \mathbf{V}_\lambda}{\cos^2\alpha_\lambda + \beta\sin^2\alpha_\lambda} \equiv \mathbf{f}_\lambda.$$

The weighted monotonicity formula leads to the following integral estimates.

Proposition 5.1. *Let M be a Kähler surface. If the initial compact surface is symplectic, then for any $R > 0$ and any $-\infty < s_1 < s_2 < 0$, we have*

$$\int_{s_1}^{s_2}\int_{\Sigma_t^\lambda\cap B_R(0)} |\mathbf{H}_\lambda|^2\, d\mu_t^\lambda\, dt \to 0 \quad \text{as } \lambda\to\infty, \tag{5.1}$$

$$\int_{s_1}^{s_2}\int_{\Sigma_t^\lambda\cap B_R(0)} |\mathbf{V}_\lambda|^2\, d\mu_t^\lambda\, dt \to 0 \quad \text{as } \lambda\to\infty, \tag{5.2}$$

$$\int_{s_1}^{s_2}\int_{\Sigma_t^\lambda\cap B_R(0)} |\mathbf{f}_\lambda|^2\, d\mu_t^\lambda\, dt \to 0 \quad \text{as } \lambda\to\infty \tag{5.3}$$

and

$$\int_{s_1}^{s_2}\int_{\Sigma_t^\lambda\cap B_R(0)} |F_\lambda^\perp|^2\, d\mu_t^\lambda\, dt \to 0 \quad \text{as } \lambda\to\infty. \tag{5.4}$$

Proof. For any $R > 0$, we choose a cut-off function $\phi_R \in C_0^\infty(B_{2R}(0))$ with $\phi_R \equiv 1$ in $B_R(0)$, where $B_\rho(0)$ is the metric ball centered at 0 with radius ρ in $\mathbb{R}^4$. For any fixed $t < 0$, flow (3.1) has a smooth solution near $T + \lambda^{-2}t < T$ for sufficiently large λ, since $T > 0$ is the first blow-up time of the flow. Set

$$f_\lambda = \frac{1}{\cos^p\alpha_\lambda}.$$

It is clear that

$$\int_{\Sigma_t^\lambda} f_\lambda \frac{1}{0-t}\phi_R(F_\lambda)\exp\Big(-\frac{|F_\lambda|^2}{4(0-t)}\Big)\,d\mu_t^\lambda$$
$$= \int_{\Sigma_{T+\lambda^{-2}t}} f_\lambda \phi(F_\lambda)\frac{1}{T-(T+\lambda^{-2}t)}\exp\Big(-\frac{|F(x,T+\lambda^{-2}t)-X_0|^2}{4(T-(T+\lambda^{-2}t))}\Big)\,d\mu_t,$$

where ϕ is the function defined in the definition of Ψ. Note that $T+\lambda^{-2}t \to T$ for any fixed t as $\lambda \to \infty$. By (3.6),

$$\frac{\partial}{\partial t}(e^{c_1\sqrt{t_0-t}}\Psi) \le c_2 e^{c_1\sqrt{t_0-t}},$$

and it then follows that $\lim_{t\to t_0} e^{c_1\sqrt{t_0-t}}\Psi$ exists. This implies, by taking $t_0 = T$ and $t = T+\lambda^{-2}s$, that for any fixed s_1 and s_2 with $-\infty < s_1 < s_2 < 0$,

$$\begin{aligned} &e^{c_1\sqrt{T-(T+\lambda^{-2}s_2)}}\int_{\Sigma_{s_2}^\lambda} f_\lambda\phi_R\frac{1}{0-s_2}\exp\Big(-\frac{|F_\lambda|^2}{4(0-s_2)}\Big)\,d\mu_{s_2}^\lambda\\ &\quad - e^{c_1\sqrt{T-(T+\lambda^{-2}s_1)}}\int_{\Sigma_{s_1}^\lambda} f_\lambda\phi_R\frac{1}{0-s_1}\exp\Big(-\frac{|F_\lambda|^2}{4(0-s_1)}\Big)\,d\mu_{s_1}^\lambda\\ &\to 0 \quad \text{as } \lambda\to\infty. \end{aligned} \tag{5.5}$$

Integrating (3.6) from s_1 to s_2 yields

$$\begin{aligned} &-e^{c_1\sqrt{-\lambda^{-2}s_2}}\int_{\Sigma_{s_2}^\lambda} f_\lambda\phi_R\frac{1}{4\pi(0-s_2)}\exp\Big(-\frac{|F_\lambda|^2}{4(0-s_2)}\Big)\,d\mu_{s_2}^\lambda\\ &+e^{c_1\sqrt{-\lambda^{-2}s_1}}\int_{\Sigma_{s_1}^\lambda} f_\lambda\phi_R\frac{1}{4\pi(0-s_1)}\exp\Big(-\frac{|F_\lambda|^2}{4(0-s_1)}\Big)\,d\mu_{s_1}^\lambda\\ &\ge \int_{s_1}^{s_2} e^{c_1\sqrt{-\lambda^{-2}t}}\int_{\Sigma_t^\lambda} f_\lambda\phi_R\rho(F_\lambda,t)\Big|\frac{(F_\lambda)^\perp}{t_0-t}+\mathbf{f}_\lambda+\mathbf{H}_\lambda\Big|^2\,d\mu_t^\lambda\,dt\\ &+\int_{s_1}^{s_2} e^{c_1\sqrt{-\lambda^{-2}t}}\int_{\Sigma_t^\lambda} f_\lambda\phi_R\rho(F_k,t)\\ &\qquad\cdot\Big|\mathbf{H}_\lambda+\frac{4p(\cos^4\alpha-\beta\sin^4\alpha)-\beta^2\sin^4\alpha}{3p\cos^2\alpha(\cos^2\alpha+\beta\sin^2\alpha)-\beta^2\sin^4\alpha}\mathbf{V}_\lambda\Big|^2\,d\mu_t^\lambda\,dt\\ &+\int_{s_1}^{s_2} e^{c_1\sqrt{-\lambda^{-2}t}}\int_{\Sigma_t^\lambda}\Big(\frac{1}{8}|\mathbf{H}_\lambda|^2+\frac{1}{8}|\mathbf{V}_\lambda|^2\Big)f_\lambda\phi_R\rho(F_\lambda,t)\,d\mu_t^\lambda\,dt\\ &-c_2\lambda^{-2}(s_2-s_1)e^{c_1\lambda^{-1}\sqrt{-s_1}}. \end{aligned} \tag{5.6}$$

Putting (5.5) and (5.6) together, we have

$$\lim_{\lambda\to\infty}\int_{s_1}^{s_2}\int_{\Sigma_t^\lambda}\phi_R\rho(F_k,t)|\mathbf{H}_\lambda|^2\,d\mu_t^\lambda\,dt = 0$$

and

$$\lim_{\lambda\to\infty}\int_{s_1}^{s_2}\int_{\Sigma_t^\lambda}\phi_R\rho(F_k,t)|\mathbf{V}_\lambda|^2\,d\mu_t^\lambda\,dt=0,$$

which yield (5.1) and (5.2) respectively, and

$$\lim_{\lambda\to\infty}\int_{s_1}^{s_2}\int_{\Sigma_t^\lambda}\phi_R\rho(F_\lambda,t)\Big|\frac{(F_\lambda)^\perp}{t_0-t}+\mathbf{f}_\lambda+\mathbf{H}_\lambda\Big|^2\,d\mu_t^\lambda=0.$$

Recall that

$$\mathbf{f}_\lambda+\mathbf{H}_\lambda=\frac{(2\cos^2\alpha_\lambda+\beta\sin^2\alpha_\lambda)\mathbf{H}_\lambda-\beta\sin^2\alpha_\lambda\mathbf{V}_\lambda}{\cos^2\alpha_\lambda+\beta\sin^2\alpha_\lambda}.$$

Hence (5.1) and (5.2) imply (5.4), and (5.3) is a consequence of (5.1) and (5.2). ∎

Lemma 5.2. *For any $R>0$ and any $t<0$, for sufficiently large λ,*

$$\mu_t^\lambda(\Sigma_t^\lambda\cap B_R(0))\le CR^2, \tag{5.7}$$

where $B_R(0)$ is a metric ball in $\mathbb{R}^4$ and $C>0$ is independent of λ.

Proof. We will first prove inequality (5.7). We will use C below for uniform positive constants which are independent of R and λ. Straightforward computation shows

$$\begin{aligned}\mu_t^\lambda(\Sigma_t^\lambda\cap B_R(0))&=\lambda^2\int_{\Sigma_{T+\lambda^{-2}t}\cap B_{\lambda^{-1}R}(X_0)}d\mu_t\\&=R^2(\lambda^{-1}R)^{-2}\int_{\Sigma_{T+\lambda^{-2}t}\cap B_{\lambda^{-1}R}(X_0)}d\mu_t\\&\le CR^2\int_{\Sigma_{T+\lambda^{-2}t}\cap B_{\lambda^{-1}R}(X_0)}f\frac{1}{4\pi(\lambda^{-1}R)^2}e^{-\frac{|X-X_0|^2}{4(\lambda^{-1}R)^2}}\,d\mu_t\\&=CR^2\Psi(X_0,T+(\lambda^{-1}R)^2+\lambda^{-2}t,T+\lambda^{-2}t).\end{aligned}$$

By the monotonicity inequality (3.6), we have

$$\begin{aligned}\mu_t^\lambda(\Sigma_t^\lambda\cap B_R(0))&\le CR^2\big(\Psi(X_0,T+(\lambda^{-1}R)^2+\lambda^{-2}t,T/2)+C\big)\\&\le C\frac{R^2}{T}\Big(\int_{\Sigma_{\frac{T}{2}}}f\,d\mu_{\frac{T}{2}}+C\Big)\\&\le C\frac{R^2}{T}\Big(\frac{1}{\delta^p}L_\beta(\Sigma_0)+C\Big)\le CR^2,\end{aligned}$$

where we have used (3.10) ∎

Fix $t_0<0$. By (5.7), for any $R>0$, we see that the total measure of $(\Sigma_{t_0}^\lambda\cap B_R(0),\mu_{t_0}^\lambda)$ is bounded from above by CR^2. The compactness theorem of measures (cf. [20, Theorem 4.4]) implies that there is a subsequence $\lambda_i(R)\to\infty$ of λ such that

$(\Sigma_{t_0}^{\lambda_i(R)} \cap B_R(0), \mu_{t_0}^{\lambda_i(R)}) \to (\Sigma_{t_0}^{\infty} \cap B_R(0), \mu_{t_0}^{\infty})$ in the sense of measure. Using the diagonal subsequence argument, we conclude that, there is a subsequence $\lambda_k \to \infty$ such that $(\Sigma_{t_0}^{\lambda_k}, \mu_{t_0}^{\lambda_k}) \to (\Sigma_{t_0}^{\infty}, \mu_{t_0}^{\infty})$ in the sense of measures.

We now show that, for any $t < 0$, the subsequence λ_k which we have chosen above satisfies $(\Sigma_t^{\lambda_k}, \mu_t^{\lambda_k}) \to (\Sigma_{t_0}^{\infty}, \mu_{t_0}^{\infty})$ in the sense of measure. And consequently the limiting surface $(\Sigma_{t_0}^{\infty}, \mu_{t_0}^{\infty})$ is independent of t_0.

Lemma 5.3. *For any $t < 0$, the sequence $\lambda_k \to \infty$ we chose above satisfies $(\Sigma_t^{\lambda_k}, \mu_t^{\lambda_k}) \to (\Sigma^{\infty}, \mu^{\infty})$ in the sense of measure, where $(\Sigma^{\infty}, \mu^{\infty})$ is independent of t. The multiplicity of Σ^{∞} is finite.*

Proof. Note that the standard formula for flow (1.1),

$$\frac{d}{dt}\int_{\Sigma_t^\lambda} \phi \, d\mu_t^\lambda = -\int_{\Sigma_t^\lambda} (\phi \mathbf{H}_\lambda \cdot \mathbf{f}_\lambda + D\phi \cdot \mathbf{f}_\lambda) \, d\mu_t^\lambda \tag{5.8}$$

is valid for any test function $\phi \in C_0^\infty(M)$.

Then, for any given $t < 0$, integrating (5.8) yields

$$\begin{aligned}\int_{\Sigma_t^{\lambda_k}} \phi \, d\mu_t^{\lambda_k} - \int_{\Sigma_{t_0}^{\lambda_k}} \phi \, d\mu_{t_0}^{\lambda_k} &= \int_t^{t_0} \int_{\Sigma_t^{\lambda_k}} (\phi \mathbf{H}_{\lambda_k} \cdot \mathbf{f}_{\lambda_k} + D\phi \cdot \mathbf{f}_{\lambda_k}) \, d\mu_t^{\lambda_k} \, dt \\ &\to 0 \text{ as } k \to \infty, \quad \text{by (5.1) and (5.3).}\end{aligned}$$

So, for any fixed $t < 0$, $(\Sigma_t^{\lambda_k}, \mu_t^{\lambda_k}) \to (\Sigma_{t_0}^{\infty}, \mu_{t_0}^{\infty})$ in the sense of measures as $k \to \infty$. We denote $(\Sigma_{t_0}^{\infty}, \mu_{t_0}^{\infty})$ by $(\Sigma^{\infty}, \mu^{\infty})$, which is independent of t_0.

Inequality (5.7) yields a uniform upper bound on $R^{-2}\mu_t^{\lambda_k}(\Sigma_t^{\lambda_k} \cap B_R(0))$, which yields finiteness of the multiplicity of Σ^{∞}. ∎

Definition 5.4. Let (X_0, T) be a singular point of flow (3.1) of a closed symplectic surface Σ_0 in a compact Kähler surface M. We call $(\Sigma^{\infty}, d\mu^{\infty})$ obtained in Lemma 5.3 *a λ-tangent cone of flow* (3.1) *at* (X_0, T).

In the remaining part of this section we prove that the λ-tangent cones are flat. And for simplicity in notation, we write $\Sigma_t^{\lambda_k}$ as Σ_t^k.

A k-varifold is a Radon measure on $G^k(M)$, where $G^k(M)$ is the Grassmann bundle of all k-planes tangent to M. Allard's compactness theorem for rectifiable varifolds ([1, Theorem 6.4]; also see [17, Theorem 1.9] and [20, Theorem 42.7]) can be stated as follows.

Theorem 5.5 (Allard's compactness theorem). *Let (V_i, μ_i) be a sequence of rectifiable k-varifolds in M with*

$$\sup_{i \geq 1}(\mu_i(U) + |\delta V_i|(U)) < \infty \quad \textit{for each } U \subset\subset M.$$

Then there is a rectifiable varifold (V, μ) of locally bounded first variation and a subsequence, which we also denote by (V_i, μ_i), such that

(i) *convergence of measures:* $\mu_i \to \mu$ *as Radon measures on* M*;*

(ii) *convergence of tangent planes:* $V_i \to V$ *as Radon measures on* $G^k(M)$*;*

(iii) *convergence of first variations:* $\delta V_i \to \delta V$ *as* TM*-valued Radon measures;*

(iv) *lower semi-continuity of total first variations:* $|\delta V| \le \liminf_{i\to\infty} |\delta V_i|$ *as Radon measures.*

We first show that the λ-tangent cones are rectifiable and stationary. The proof is similar to those of [4, Proposition 3.1 and Theorem 4.1].

Proposition 5.6. *Let* M *be a compact Kähler surface. If the initial compact surface is symplectic, then the* λ*-tangent cone* Σ^∞ *is rectifiable and stationary.*

Proof. We set

$$E_R = \left\{t \in (-\infty, 0) \mid \liminf_{k\to\infty} \int_{\Sigma_t^k \cap B_R(0)} (|\mathbf{H}_k|^2 + |\mathbf{V}_k|^2)\, d\mu_t^k \neq 0\right\}$$

and

$$E = \bigcup_{R>0} E_R.$$

Denote the measures of E_R and E by $|E_R|$ and $|E|$ respectively. It is clear from (5.2), (5.3) and (5.1) that $|E_R| = 0$ for any $R > 0$. So $|E| = 0$.

Choose $t \notin E$. Let V_t^k be the varifold defined by Σ_t^k. It is explained in the previous section, that V_t^k is well defined in $B_R(0) \subset \mathbb{R}^4$ for any $R > 0$ when k is sufficiently large. By the definition of varifolds, we have

$$V_t^k(\psi) = \int_{\Sigma_t^k} \psi(x, T\Sigma_t^k)\, d\mu_t^k$$

for any $\psi \in C_0^0(G^2(\mathbb{R}^4), R)$, where $G^2(\mathbb{R}^4)$ is the Grassmannian bundle of all 2-planes tangent to Σ_t^∞ in $\mathbb{R}^4$. For each smooth surface Σ_t^k, the first variation δV_t^k of V_t^k (cf. [1], [20, Section 39.4] and [17, Section 1.7]) is that, for any smooth vector field X with support in $B_R(0)$,

$$\delta V_t^k(X) = -\int_{\Sigma_t^k \cap B_R(0)} X \cdot \mathbf{H}_k\, d\mu_t^k,$$

so by the area upper bound (5.7),

$$|\delta V_t^k(X)| \le CR\|X\|_{L^\infty(B_R(0))} \left(\int_{\Sigma_t^k \cap B_R(0)} |\mathbf{H}_k|^2\, d\mu_t^k\right)^{1/2}. \tag{5.9}$$

We therefore have that, for any $R > 0$,

$$\mu_t^k(B_R(0)) + \delta V_t^k(B_R(0)) \le C(R).$$

By Allard's compactness theorem, there exists a subsequence, which we also denote by (V_t^k, μ_t^k), such that $(V_t^k, \mu_t^k) \to (V_t^\infty, \mu_t^\infty)$ and the conclusions in Theorem 5.5 hold in $B_R(0)$. By the diagonal subsequence argument, there exists a subsequence which we also denote by (V_t^k, μ_t^k) such that $(V_t^k, \mu_t^k) \to (V_t^\infty, \mu_t^\infty)$ and satisfies (i)–(iv) in Theorem 5.5 in $\mathbb{R}^4$.

Because $t \notin E$, by (5.9), we see that $\delta V_t^k \to 0$ at t as $k \to \infty$ by applying Theorem 5.5. Using the same argument as in the proof of [4, Theorem 4.1], we can show that Σ^∞ is rectifiable. Furthermore, by (iii) in Theorem 5.5, we have

$$-\mu^\infty \lfloor \mathbf{H}_\infty = \delta V^\infty = \lim_{k\to\infty} \delta V_t^k = 0.$$

Therefore Σ^∞ is stationary. ∎

Theorem 5.7. *Let M be a compact Kähler surface. If the initial compact surface is symplectic and $T > 0$ is the first blow-up time of flow (3.1), then the λ-tangent cone Σ^∞ of flow (3.1) with $\beta \geq 0$ at (X_0, T) is a finite union of planes if it is not the empty set.*

Proof. Since Σ^∞ is not empty, without loss of any generality, we may assume $0 \in \Sigma^\infty$, where 0 is the origin of $\mathbb{R}^4$. There is a sequence of points $X_k \in \Sigma_t^k$ satisfying $X_k \to 0$ as $k \to \infty$. By Proposition 5.1, for any s_1 and s_2 with $-\infty < s_1 < s_2 < 0$ and any $R > 0$, we have

$$\int_{s_1}^{s_2} \int_{\Sigma_t^k \cap B_R(0)} |F_k^\perp|^2 \, d\mu_t^k \, dt \to 0 \text{ as } k \to \infty.$$

Thus, by (5.7),

$$\begin{aligned}
&\lim_{k\to\infty} \int_{s_1}^{s_2} \int_{\Sigma_t^k \cap B_R(0)} |(F_k - X_k)^\perp|^2 \, d\mu_t^k \, dt \\
&\quad \leq 2 \lim_{k\to\infty} \int_{s_1}^{s_2} \int_{\Sigma_t^k \cap B_R(0)} |F_k^\perp|^2 \, d\mu_t^k \, dt + C(s_2 - s_1) R^2 \lim_{k\to\infty} |X_k|^2 \\
&\quad = 0.
\end{aligned}$$

Let us denote the tangent spaces of Σ_t^k at the point $F_k(x,t)$ and of Σ^∞ at the point $F^\infty(x,t)$ by $T\Sigma_t^k$ and $T\Sigma^\infty$ respectively. It is clear that

$$(F_k - X_k)^\perp = \operatorname{dist}(X_k, T\Sigma_t^k)$$

and

$$(F_\infty)^\perp = \operatorname{dist}(0, T\Sigma^\infty).$$

By Allard's compactness theorem, i.e., Theorem 5.5 (ii), we have

$$\begin{aligned}
\int_{s_1}^{s_2} \int_{\Sigma^\infty \cap B_R(0)} |(F_\infty)^\perp|^2 \, d\mu^\infty \, dt &= \int_{s_1}^{s_2} \int_{\Sigma^\infty \cap B_R(0)} |\operatorname{dist}(0, T\Sigma^\infty)|^2 \, d\mu^\infty \, dt \\
&= \lim_{k\to\infty} \int_{s_1}^{s_2} \int_{\Sigma_t^k \cap B_R(0)} |\operatorname{dist}(X_k, T\Sigma_t^k)|^2 \, d\mu_t^k \, dt \\
&= \lim_{k\to\infty} \int_{s_1}^{s_2} \int_{\Sigma_t^k \cap B_R(0)} |(F_k - X_k)^\perp|^2 \, d\mu_t^k \, dt \\
&= 0.
\end{aligned}$$

By Proposition 5.1, we know that Σ^∞ satisfies that $\mathbf{H}^\infty$ and $\mathbf{V}^\infty$ both vanish. Hence Σ^∞ is a holomorphic curve in $\mathbb{C}^2$ so that the singular locus $\mathcal{S}$ of Σ^∞ consists of isolated points. So outside $\mathcal{S}$, we have

$$\langle F_\infty, v_\alpha\rangle = 0.$$

Note that the above inner product is taken in $\mathbb{R}^4$, and differentiating in $\mathbb{R}^4$ then yields

$$0 = \langle \partial_i F_\infty, v_\alpha\rangle + \langle F_\infty, \partial_i v_\alpha\rangle = \langle F_\infty, \partial_i v_\alpha\rangle,$$

where we used the fact that $\partial_i F_\infty$ is tangential to Σ^∞. By Weingarten's equation, we observe that

$$(h_\infty)^\alpha_{ij}\langle F_\infty, e_j\rangle = 0 \quad \text{for all } \alpha, i = 1, 2.$$

So for $\alpha = 1, 2$, we have

$$\det((h_\infty)^\alpha_{ij}) = 0.$$

Since $\mathbf{H} = 0$, for $\alpha = 1, 2$ we also have

$$\operatorname{tr}((h_\infty)^\alpha_{ij}) = 0.$$

It then follows immediately that the symmetric matrix $((h_\infty)^\alpha_{ij})$ is in fact the zero matrix, for all $i, j, \alpha = 1, 2$, which obviously yields $|\mathbf{A}_\infty| \equiv 0$. By Lemma 5.2, the tangent cone consists of finitely many planes. This completes the proof of Theorem 5.7. ∎

6. Convergence of the flow near a holomorphic curve

In this section we will use the ε-regularity theorem to prove that flow (1.1) exists globally and converges to a holomorphic curve if the ambient Kähler–Einstein surface has positive scalar curvature and the initial surface is sufficiently close to a holomorphic curve.

Theorem 6.1. *For any $\beta \geq 0$, there is a constant $\varepsilon_0 > 0$, depending on β, Σ_0 and M, such that if Σ_0 is a closed symplectic surface in a compact Kähler–Einstein surface with positive scalar curvature $K_0 > 0$ and $\cos\alpha > 1 - \varepsilon_0$, then flow (1.1) starting from Σ_0 exists globally and converges to a holomorphic curve at infinity.*

Proof. Fix any positive T. By definition (3.5), we see that

$$\Psi(X_0, t, t - r^2) = \int_{\Sigma_{t-r^2}} \frac{1}{\cos^p\alpha}\phi(F)\frac{1}{4\pi r^2}e^{-\frac{|F-X_0|^2}{4r^2}}\, d\mu_{t-r^2}.$$

We first estimate the quantity

$$\Psi_1(X_0, t, t - r^2) = \int_{\Sigma_{t-r^2}} \phi(F)\frac{1}{4\pi r^2}e^{-\frac{|F-X_0|^2}{4r^2}}\, d\mu_{t-r^2}.$$

Differentiating this quantity with respect to t, we get

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\Sigma_{t-r^2}}\phi(F)\frac{1}{4\pi r^2}e^{-\frac{|F-X_0|^2}{4r^2}}\,d\mu_{t-r^2} &= \int_{\Sigma_{t-r^2}}\langle D\phi,\mathbf{f}\rangle\frac{1}{4\pi r^2}e^{-\frac{|F-X_0|^2}{4r^2}}\,d\mu_{t-r^2}\\ &\quad-\int_{\Sigma_{t-r^2}}\frac{\phi}{8\pi r^4}e^{-\frac{|F-X_0|^2}{4r^2}}\langle F-X_0,\mathbf{f}\rangle\,d\mu_{t-r^2}\\ &\quad-\int_{\Sigma_{t-r^2}}\phi\frac{1}{4\pi r^2}e^{-\frac{|F-X_0|^2}{4r^2}}\langle\mathbf{f},\mathbf{H}\rangle\,d\mu_{t-r^2}.\end{aligned}$$

Integrating the above equation from r^2 to T, we get

$$\begin{aligned}&\int_{\Sigma_{T-r^2}}\phi(F)\frac{1}{4\pi r^2}e^{-\frac{|F-X_0|^2}{4r^2}}\,d\mu_{T-r^2}-\int_{\Sigma_0}\phi(F)\frac{1}{4\pi r^2}e^{-\frac{|F-X_0|^2}{4r^2}}\,d\mu_0\\ &\quad=\int_{r^2}^T\int_{\Sigma_{t-r^2}}\langle D\phi,\mathbf{f}\rangle\frac{1}{4\pi r^2}e^{-\frac{|F-X_0|^2}{4r^2}}\,d\mu_{t-r^2}\,dt\\ &\qquad-\int_{r^2}^T\int_{\Sigma_{t-r^2}}\frac{\phi}{8\pi r^4}e^{-\frac{|F-X_0|^2}{4r^2}}\langle F-X_0,\mathbf{f}\rangle\,d\mu_{t-r^2}\,dt\\ &\qquad-\int_{r^2}^T\int_{\Sigma_{t-r^2}}\phi\frac{1}{4\pi r^2}e^{-\frac{|F-X_0|^2}{4r^2}}\langle\mathbf{f},\mathbf{H}\rangle\,d\mu_{t-r^2}\,dt\\ &\quad\le\int_{r^2}^T\int_{\Sigma_{t-r^2}}|D\phi|\,|\mathbf{f}|\frac{1}{4\pi r^2}e^{-\frac{|F-X_0|^2}{4r^2}}\,d\mu_{t-r^2}\,dt\\ &\qquad+\int_{r^2}^T\int_{\Sigma_{t-r^2}}\frac{\phi}{8\pi r^3}e^{-\frac{|F-X_0|^2}{4r^2}}|\mathbf{f}|\,d\mu_{t-r^2}\,dt\\ &\qquad+\int_{r^2}^T\int_{\Sigma_{t-r^2}}\phi\frac{1}{4\pi r^2}e^{-\frac{|F-X_0|^2}{4r^2}}|\mathbf{f}|\,|\mathbf{H}|\,d\mu_{t-r^2}\,dt.\end{aligned}\tag{6.1}$$

To estimate the space-time integral of $|\mathbf{f}|$ and $|\mathbf{H}|$, we note that $\int_\Sigma\cos\alpha\,d\mu=[\omega][\Sigma]$ is a constant in the homological class of Σ. Furthermore, applying (3.7) with $f(x)=x$, we obtain

$$\begin{aligned}&\Big(\frac{\partial}{\partial t}-\Delta\Big)\frac{1}{\cos\alpha}\\ &\quad=-\frac{\cos\alpha}{\cos^2\alpha+\beta\sin^2\alpha}|\mathbf{H}|^2-\frac{2(\cos^4\alpha-\sin^4\alpha)}{\cos\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}\mathbf{H}\cdot\mathbf{V}\\ &\qquad-\frac{2\beta^2\sin^6\alpha+(\beta^2+2\beta)\sin^4\alpha\cos^2\alpha+(\beta+2)\sin^2\alpha\cos^4\alpha+2\cos^6\alpha}{\cos^3\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}|\mathbf{V}|^2\\ &\qquad-\frac{K_0\sin^2\alpha\cos\alpha}{4(\cos^2\alpha+\beta\sin^2\alpha)}.\end{aligned}$$

Therefore, we can compute

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{\Sigma_t}\frac{\sin^2\alpha}{\cos\alpha}\,d\mu_t=\frac{\partial}{\partial t}\int_{\Sigma_t}\frac{1}{\cos\alpha}\,d\mu_t\\
&=\int_{\Sigma_t}\Big(\frac{\partial}{\partial t}-\Delta\Big)\frac{1}{\cos\alpha}\,d\mu_t-\int_{\Sigma_t}\frac{\mathbf{f}\cdot\mathbf{H}}{\cos\alpha}\,d\mu_t\\
&=-\int_{\Sigma_t}\frac{\cos\alpha}{\cos^2\alpha+\beta\sin^2\alpha}|\mathbf{H}|^2\,d\mu_t-\int_{\Sigma_t}\frac{2(\cos^4\alpha-\beta\sin^4\alpha)}{\cos\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}\mathbf{H}\cdot\mathbf{V}\,d\mu_t\\
&\quad-\int_{\Sigma_t}\frac{2\beta^2\sin^6\alpha+(\beta^2+2\beta)\sin^4\alpha\cos^2\alpha+(\beta+2)\sin^2\alpha\cos^4\alpha+2\cos^6\alpha}{\cos^3\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}|\mathbf{V}|^2\,d\mu_t\\
&\quad-\int_{\Sigma_t}\frac{K_0\cos\alpha\sin^2\alpha}{4(\cos^2\alpha+\beta\sin^2\alpha)}\,d\mu_t-\int_{\Sigma_t}\frac{\cos^2\alpha|\mathbf{H}|^2-\beta\sin^2\alpha\mathbf{V}\cdot\mathbf{H}}{\cos\alpha(\cos^2\alpha+\beta\sin^2\alpha)}\,d\mu_t\\
&=-\int_{\Sigma_t}\frac{2\cos\alpha}{\cos^2\alpha+\beta\sin^2\alpha}|\mathbf{H}|^2\,d\mu_t\\
&\quad-\int_{\Sigma_t}\frac{2\cos^4\alpha-\beta\sin^2\alpha\cos^2\alpha-\beta(\beta+2)\sin^4\alpha}{\cos\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}\mathbf{H}\cdot\mathbf{V}\,d\mu_t\\
&\quad-\int_{\Sigma_t}\frac{2\beta^2\sin^6\alpha+(\beta^2+2\beta)\sin^4\alpha\cos^2\alpha+(\beta+2)\sin^2\alpha\cos^4\alpha+2\cos^6\alpha}{\cos^3\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}|\mathbf{V}|^2\,d\mu_t\\
&\quad-\int_{\Sigma_t}\frac{K_0\cos\alpha\sin^2\alpha}{4(\cos^2\alpha+\beta\sin^2\alpha)}\,d\mu_t. \qquad (6.2)
\end{aligned}$$

Since $0<\delta\le\cos\alpha\le 1$ and $K_0>0$, we have

$$\frac{K_0\cos\alpha\sin^2\alpha}{4(\cos^2\alpha+\beta\sin^2\alpha)}=\frac{\sin^2\alpha}{\cos\alpha}\frac{K_0\cos^2\alpha}{4(\cos^2\alpha+\beta\sin^2\alpha)}\ge\frac{K_0\delta^2}{4(1+\beta)}\frac{\sin^2\alpha}{\cos\alpha}.$$

Furthermore,

$$\begin{aligned}
&\frac{2\cos\alpha}{\cos^2\alpha+\beta\sin^2\alpha}|\mathbf{H}|^2+\frac{2\cos^4\alpha-\beta\sin^2\alpha\cos^2\alpha-\beta(\beta+2)\sin^4\alpha}{\cos\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}\mathbf{H}\cdot\mathbf{V}\\
&\qquad+\frac{2\beta^2\sin^6\alpha+(\beta^2+2\beta)\sin^4\alpha\cos^2\alpha+(\beta+2)\sin^2\alpha\cos^4\alpha+2\cos^6\alpha}{\cos^3\alpha(\cos^2\alpha+\beta\sin^2\alpha)^2}|\mathbf{V}|^2\\
&\quad=\frac{\cos\alpha}{\cos^2\alpha+\beta\sin^2\alpha}|\mathbf{H}|^2\\
&\qquad+\frac{\cos\alpha}{\cos^2\alpha+\beta\sin^2\alpha}\Big|\mathbf{H}+\frac{2\cos^4\alpha-\beta\sin^2\alpha\cos^2\alpha-\beta(\beta+2)\sin^4\alpha}{2\cos^2\alpha(\cos^2\alpha+\beta\sin^2\alpha)}\mathbf{V}\Big|\\
&\qquad+\Big(\frac{4\cos^8\alpha+8(\beta+1)\cos^6\alpha\sin^2\alpha+(11\beta^2+8\beta)\cos^4\alpha\sin^4\alpha}{4\cos^3\alpha(\cos^2\alpha+\beta\sin^2\alpha)^3}\\
&\qquad\quad-\frac{\beta^2(\beta-2)^2\sin^8\alpha}{4\cos^3\alpha(\cos^2\alpha+\beta\sin^2\alpha)^3}\Big)|\mathbf{V}|^2.
\end{aligned}$$

Note that for fixed β, we can choose $\varepsilon_1 > 0$ depending on β such that if $\cos\alpha \geq \delta \geq 1-\varepsilon_1$, then the right-hand side of the above equation is bounded from below by

$$\frac{1}{2}(|\mathbf{H}|^2 + |\mathbf{V}|^2).$$

Then we obtain from (6.2) that

$$\frac{\partial}{\partial t}\int_{\Sigma_t}\frac{\sin^2\alpha}{\cos\alpha}\,d\mu_t \leq -\frac{1}{2}\int_{\Sigma_t}(|\mathbf{H}|^2+|\mathbf{V}|^2)\,d\mu_t - \frac{K_0\delta^2}{4(1+\beta)}\int_{\Sigma_t}\frac{\sin^2\alpha}{\cos\alpha}\,d\mu_t. \tag{6.3}$$

At first, we obtain from (6.3) that

$$\frac{\partial}{\partial t}\int_{\Sigma_t}\frac{\sin^2\alpha}{\cos\alpha}\,d\mu_t \leq -\frac{K_0\delta^2}{4(1+\beta)}\int_{\Sigma_t}\frac{\sin^2\alpha}{\cos\alpha}\,d\mu_t,$$

so that

$$\int_{\Sigma_t}\frac{\sin^2\alpha}{\cos\alpha}\,d\mu_t \leq e^{-\frac{K_0\delta^2}{4(1+\beta)}t}\int_{\Sigma_0}\frac{\sin^2\alpha}{\cos\alpha}\,d\mu_0 := C_0 e^{-\frac{K_0\delta^2}{4(1+\beta)}t}, \tag{6.4}$$

where $C_0 = \int_{\Sigma_0}\frac{\sin^2\alpha}{\cos\alpha}\,d\mu_0$. Then we obtain from (6.3) that

$$\frac{1}{2}\int_{\Sigma_t}(|\mathbf{H}|^2+|\mathbf{V}|^2)\,d\mu_t \leq -\frac{\partial}{\partial t}\int_{\Sigma_t}\frac{\sin^2\alpha}{\cos\alpha}\,d\mu_t.$$

Integrating the above inequality from t to $t+1$, we obtain

$$\begin{aligned}\frac{1}{2}\int_t^{t+1}\int_{\Sigma_t}(|\mathbf{H}|^2+|\mathbf{V}|^2)\,d\mu_t\,dt &\leq \int_{\Sigma_t}\frac{\sin^2\alpha}{\cos\alpha}\,d\mu_t - \int_{\Sigma_{t+1}}\frac{\sin^2\alpha}{\cos\alpha}\,d\mu_{t+1}\\ &\leq \int_{\Sigma_t}\frac{\sin^2\alpha}{\cos\alpha}\,d\mu_t \leq C_0 e^{-\frac{K_0\delta^2}{4(1+\beta)}t}.\end{aligned} \tag{6.5}$$

Next we would like to estimate the space-time integrals of $|\mathbf{H}|$ and $|\mathbf{V}|$. First note that since flow (1.1) is a negative gradient flow of the functional L_β, we see that we have the uniform bound for the area due to (3.9). Then by the Hölder inequality, we compute for any $T > 0$ that

$$\begin{aligned}\int_0^T\int_{\Sigma_t}|\mathbf{H}|\,d\mu_t\,dt &= \sum_{k=0}^{T-1}\int_k^{k+1}\int_{\Sigma_t}|\mathbf{H}|\,d\mu_t\,dt\\ &\leq \sum_{k=0}^{T-1}\Big(\int_k^{k+1}\int_{\Sigma_t}|\mathbf{H}|^2\,d\mu_t\,dt\Big)^{\frac{1}{2}}\Big(\int_k^{k+1}\mathrm{Area}(\Sigma_t)\,dt\Big)^{\frac{1}{2}}\\ &\leq L_\beta^{\frac{1}{2}}(\Sigma_0)\sum_{k=0}^{T-1}\Big(\int_k^{k+1}\int_{\Sigma_t}|\mathbf{H}|^2\,d\mu_t\,dt\Big)^{\frac{1}{2}}\\ &\leq (2C_0)^{\frac{1}{2}}L_\beta^{\frac{1}{2}}(\Sigma_0)\sum_{k=0}^{T-1}e^{-\frac{K_0\delta^2}{8(1+\beta)}k} \leq \frac{(2C_0)^{\frac{1}{2}}L_\beta^{\frac{1}{2}}(\Sigma_0)}{1-e^{-\frac{K_0\delta^2}{8(1+\beta)}}}.\end{aligned}$$

Similarly, we have

$$\int_0^T \int_{\Sigma_t} |\mathbf{V}|\, d\mu_t\, dt \le \frac{(2C_0)^{\frac{1}{2}} L_\beta^{\frac{1}{2}}(\Sigma_0)}{1 - e^{-\frac{K_0\delta^2}{8(1+\beta)}}}.$$

By the definition of $\mathbf{f}$, we see that

$$|\mathbf{f}| = \left| \frac{\cos^2\alpha \mathbf{H} - \beta \sin^2\alpha \mathbf{V}}{\cos^2\alpha + \beta \sin^2\alpha} \right| \le |\mathbf{H}| + |\mathbf{V}|,$$

so that

$$\int_0^T \int_{\Sigma_t} |\mathbf{f}|\, d\mu_t\, dt \le \frac{2(2C_0)^{\frac{1}{2}} L_\beta^{\frac{1}{2}}(\Sigma_0)}{1 - e^{-\frac{K_0\delta^2}{8(1+\beta)}}}. \tag{6.6}$$

We also have from (6.5) that

$$V \int_0^T \int_{\Sigma_t} |\mathbf{f}|\, |\mathbf{H}|\, d\mu_t\, dt \le \frac{1}{2} \int_0^T \int_{\Sigma_t} (|\mathbf{f}|^2 + |\mathbf{H}|^2)\, d\mu_t\, dt \tag{6.7}$$

$$V \le \frac{1}{2} \int_0^T \int_{\Sigma_t} (2|\mathbf{V}|^2 + 3|\mathbf{H}|^2)\, d\mu_t\, dt \tag{6.8}$$

$$\le 4C_0 \sum_{k=0}^{T-1} e^{-\frac{K_0\delta^2}{4(1+\beta)}k} \le \frac{4C_0}{1 - e^{-\frac{K_0\delta^2}{4(1+\beta)}}}. \tag{6.9}$$

Now we return to (6.1). Using (6.6), (6.9) and the fact that $|D\phi| \le C$, we compute

$$\begin{aligned}
\int_{\Sigma_{T-r^2}} \phi(F) \frac{1}{4\pi r^2} e^{-\frac{|F-X_0|^2}{4r^2}}\, d\mu_{T-r^2} \le{}& \int_{\Sigma_0} \phi(F) \frac{1}{4\pi r^2} e^{-\frac{|F-X_0|^2}{4r^2}}\, d\mu_0 \\
&+ \frac{C}{4\pi r^2} \int_{r^2}^T \int_{\Sigma_{t-r^2}} |\mathbf{f}|\, d\mu_{t-r^2}\, dt \\
&+ \frac{1}{8\pi r^3} \int_{r^2}^T \int_{\Sigma_{t-r^2}} |\mathbf{f}|\, d\mu_{t-r^2}\, dt \\
&+ \frac{1}{4\pi r^2} \int_{r^2}^T \int_{\Sigma_{t-r^2}} |\mathbf{f}|\, |\mathbf{H}|\, d\mu_{t-r^2}\, dt \\
\le{}& \int_{\Sigma_0} \phi(F) \frac{1}{4\pi r^2} e^{-\frac{|F-X_0|^2}{4r^2}}\, d\mu_0 \\
&+ \left(\frac{C}{4\pi r^2} + \frac{1}{8\pi r^3} \right) \frac{2(2C_0)^{\frac{1}{2}} L_\beta^{\frac{1}{2}}(\Sigma_0)}{1 - e^{-\frac{K_0\delta^2}{8(1+\beta)}}} \\
&+ \frac{1}{4\pi r^2} \frac{4C_0}{1 - e^{-\frac{K_0\delta^2}{4(1+\beta)}}}.
\end{aligned}$$

Since Σ_0 is smooth, it is well known that

$$\lim_{r \to 0} \int_{\Sigma_0} \phi(F) \frac{1}{4\pi r^2} e^{-\frac{|F-X_0|^2}{4r^2}}\, d\mu_0 = 1.$$

Thus we can choose $r_0 > 0$ sufficiently small such that

$$\int_{\Sigma_0} \phi(F) \frac{1}{4\pi r_0^2} e^{-\frac{|F-X_0|^2}{4r_0^2}} \, d\mu_0 < 1 + \frac{\varepsilon}{8},$$

where ε is given by Corollary 4.2. Notice that if $\cos\alpha \geq \delta > 0$ holds for $t = 0$, then it remains so for $t > 0$. Hence we have

$$C_0 = \int_{\Sigma_0} \frac{\sin^2\alpha}{\cos\alpha} \, d\mu_0 \leq \frac{1-\delta^2}{\delta} \operatorname{Area}(\Sigma_0),$$

and

$$\begin{aligned}
\Psi(X_0, T, T - r_0^2) &= \int_{\Sigma_{T-r^2}} \frac{1}{\cos^p\alpha} \phi(F) \frac{1}{4\pi r^2} e^{-\frac{|F-X_0|^2}{4r^2}} \, d\mu_{T-r^2} \\
&\leq \frac{1}{\delta^p} \int_{\Sigma_{T-r^2}} \phi(F) \frac{1}{4\pi r^2} e^{-\frac{|F-X_0|^2}{4r^2}} \, d\mu_{T-r^2} \\
&\leq \frac{1}{\delta^p} \Bigg[1 + \frac{\varepsilon}{8} + \left(\frac{C}{4\pi r_0^2} + \frac{1}{8\pi r_0^3} \right) \frac{2(2C_0)^{\frac{1}{2}} L_\beta^{\frac{1}{2}}(\Sigma_0)}{1 - e^{-\frac{K_0\delta^2}{8(1+\beta)}}} \\
&\qquad + \frac{1}{4\pi r_0^2} \frac{4C_0}{1 - e^{-\frac{K_0\delta^2}{4(1+\beta)}}} \Bigg] \\
&\leq \frac{1}{\delta^p} \Bigg[1 + \frac{\varepsilon}{8} + \left(\frac{C}{4\pi r_0^2} + \frac{1}{8\pi r_0^3} \right) \frac{2(2\frac{1-\delta^2}{\delta} \operatorname{Area}(\Sigma_0))^{\frac{1}{2}} L_\beta^{\frac{1}{2}}(\Sigma_0)}{1 - e^{-\frac{K_0\delta^2}{8(1+\beta)}}} \\
&\qquad + \frac{1}{4\pi r_0^2} \frac{4\frac{1-\delta^2}{\delta} \operatorname{Area}(\Sigma_0)}{1 - e^{-\frac{K_0\delta^2}{4(1+\beta)}}} \Bigg]
\end{aligned}$$

Therefore, we see that there exists a constant $\varepsilon_0 \in (0, \varepsilon_1)$, such that if $\delta \geq 1 - \varepsilon_0$, we have

$$\Psi(X_0, T, T - r_0^2) \leq 1 + \varepsilon.$$

The ε-regularity theorem (Corollary 4.2) implies that the second fundamental is uniformly bounded along flow (1.1). By Theorem 3.1, the flow exists globally and converges to a limit surface Σ_∞. By letting $t \to \infty$ in (6.4), we see that

$$\int_{\Sigma_\infty} \frac{\sin^2\alpha}{\cos\alpha} \, d\mu_\infty = 0.$$

It follows that $\cos\alpha_\infty \equiv 1$, that is, Σ_∞ is a holomorphic curve. This proves the theorem.

∎

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