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A Class of Approximations of Brownian Motion

Dedicated to Professor K. Itô on his 60 th birthday

By

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§ 1. Introduction

Let $B(t)=(B^1(t), B^2(t), \cdots, B^d(t))$ be a d-dimensional Brownian motion and let $\{B_n(t)=(B_n^1(t), B_n^2(t), \cdots, B_n^d(t))\}$ be a sequence of approximations to $B(t)$. We assume that the sample paths of $B_n(t)$ are continuous and piecewise smooth for each n and $B_n(t)$ converges to $B(t)$. Let $u(x)$ be a twice continuously differentiable function on R^d whose partial derivatives of order $\leqq 2$ are all bounded. In the one-dimensional case E. Wong and M. Zakai [5] showed that $\int_0^t u(B_n(s))dB_n(s)$ converges to $\int_0^t u(B(s))\circ dB(s)$ where the symbol $\circ$ denotes the symmetric stochastic integral of Stratonovich (K. Itô [2]). They also dealt with the convergence of the more general functional of $B_n(\cdot)$, ([6]). In the two-dimensional case P. Lévy [3] showed that $S(t;n)=\int_0^t (B_n^1(s)\cdot dB_n^2(s)-B_n^2(s)dB_n^1(s))/2$ converges to the stochastic integral $S(t)=\int_0^t (B^1(s)\circ dB^2(s)-B^2(s)\circ dB^1(s))/2$ if $\{B_n(t)\}$ is a sequence of polygonal approximations to $B(t)$. E. J. McShane [4], on the other hand, gave an example of the sequence $\{B_n(t)\}$ of approximations to $B(t)$ such that $S(t;n)$ converges to $S(t)+t/\pi$.

In this paper we treat systematically a class of approximations of Brownian motion including McShane's example. In Section 2 we state the main results of the paper. We consider a sequence of Stieltjes integrals of the form $I_n(u)=\int_0^t u(B_n(s))dB_n^j(s)$. First we will give some conditions under which $I_n(u)$ converges in the quadratic-mean sense. It

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is then shown that the limit of $I_n(u)$ is expressed as the sum of the symmetric stochastic integral $\int_0^t u(B(s))\circ dB^j(s)$ and a certain "*correction term*", (cf. Theorem 2. 1). In particular, we will give a criterion such that $S(t;n)$ converges to $S(t)$, (cf. Corollary 2. 1 in Section 2). We will also give a couple of examples for Theorem 2. 1 in which the correction terms really appear. Section 3 is devoted to the proof of Theorem 2. 1. Finally Section 4 concerns the convergence of the solutions of the ordinary differential equations determined by $B_n(t)$.

§ 2. Approximations of Stochastic Integral

Let Ω be the space of continuous functions defined on $[0, \infty)$ with values in R^d. The value of the function $\omega\in\Omega$ at time t will be denoted by $B(t, \omega)=(B^1(t, \omega), B^2(t, \omega), \cdots, B^d(t, \omega))$. The argument ω may be suppressed occasionally. $\mathcal{F}_t$ and $\mathcal{F}$ denote the smallest σ-algebras with respect to which $B(s, \omega)$ are measurable for $0\leqq s\leqq t$ and for $0\leqq s<\infty$ respectively. The shift operator is denoted by θ_t: that is $B(s, \theta_t\omega) =B(s+t, \omega)$, $(s\geqq 0)$. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, B(t), \theta_t, P_x)$ be the d-dimensional Brownian motion. In this paper the following class of the approximations of the Brownian motion will be considered.

Definition 2. 1. Let $\{B_\delta(t, \omega)=(B_\delta^1(t, \omega), B_\delta^2(t, \omega), \cdots, B_\delta^d(t, \omega))$; $\delta>0\}$ be a family of R^d-valued stochastic processes defined on $(\Omega, \mathcal{F}, P_x)$ and let κ be a positive constant. We say $\{B_\delta(t, \omega)\}\in\mathcal{A}(B; \kappa)$ if, for each $\delta>0$, $B_\delta(t, \omega)$ satisfies the following conditions:

(A. 1) $B_\delta(k\delta, \omega)=B(k\delta, \omega)$, for $\omega\in\Omega$ and $k=0, 1, \cdots$.

(A. 2) $B_\delta(t+k\delta, \omega)=B_\delta(t, \theta_{k\delta}\omega)$, for $\omega\in\Omega$ and $k=1, 2, \cdots$.

(A. 3) $B_\delta(t, \omega+x)=B_\delta(t, \omega)+x$, for $\omega\in\Omega$, $t>0$ and $x\in R^d$,

where $\omega+x$ is the function defined by $(\omega+x)(t)=B(t, \omega)+x$, $t\geqq 0$.

(A. 4) $B_\delta(t, \omega)$ is $\mathcal{F}_\delta$-measurable for $0\leqq t\leqq\delta$.

(A. 5) $B_\delta(t, \omega)$ is continuous and piecewise smooth in t for $\omega\in\Omega$.

(A. 6) $E_0\left[\left(\int_0^\delta |\dot{B}_\delta^i(s)|\, ds\right)^6\right]\leqq\kappa\delta^3$, for $i=1, 2, \cdots, d$, where $\dot{B}_\delta^i(s) = \frac{\partial}{\partial s} B_\delta^i(s)$, $i=1, 2, \cdots, d$, and $E_x[\cdot]$ denotes the expectation with respect to the probability measure P_x.

Let us consider a differential 1-form on R^d of the following form:

$$\alpha_{ij} = (x^i dx^j - x^j dx^i)/2, \qquad (i, j=1, 2, \cdots, d),$$

and let $S_{ij}(t; \delta)$ be the integral of α_{ij} along $C_\delta[0, t]$: i.e.

$$S_{ij}(t; \delta) = \int_{C_\delta[0,t]} \alpha_{ij},$$

where $C_\delta[0, t]$ is the curve defined by $C_\delta[0, t] = \{B_\delta(s, \omega); 0 \leqq s \leqq t\}$. Then we have

$$S_{ij}(t; \delta) = \int_0^t (B_\delta^i(s) dB_\delta^j(s) - B_\delta^j(s) dB_\delta^i(s))/2. \tag{2.1}$$

Setting

$$s_{ij}(\delta) = E_0[S_{ij}(\delta; \delta)]/\delta,$$

we have

Proposition 2.1. *Suppose $\{B_\delta(t)\} \in \mathcal{A}(B; \kappa)$ for some positive constant κ. Then there exists a sequence $\{\delta_n\}$ of positive numbers such that $\lim_{n\to\infty} \delta_n = 0$ and for $1 \leqq i, j \leqq d$ the sequence $\{s_{ij}(\delta_n)\}$ has a finite limit as $n \to \infty$.*

Proof. Fix i and j. To complete the proof we need only to show that $\{s_{ij}(\delta)\}$ is bounded. By (A. 6), we have

$$\begin{aligned} |s_{ij}(\delta)| &\leqq E_0\left[\int_0^\delta |\dot{B}_\delta^i(s)| ds \int_0^\delta |\dot{B}_\delta^j(s)| ds\right]/\delta \\ &\leqq \left(E_0\left[\left(\int_0^\delta |\dot{B}_\delta^i(s)| ds\right)^6\right]\right)^{1/6} \left(E_0\left[\left(\int_0^\delta |\dot{B}_\delta^j(s)| ds\right)^6\right]\right)^{1/6}/\delta \\ &\leqq \kappa^{1/3}. \end{aligned}$$

This estimate proves the proposition.

In the remainder of the paper let $S=(s_{ij})$, $(1 \leqq i, j \leqq d)$, be a skew-symmetric $d \times d$-matrix and let $\{\delta_n\}$ be a sequence of positive numbers satisfying $\lim_{n\to\infty}\delta_n=0$. Now we will give a notation.

Definition 2. 2. Let $\{B_\delta(t)\}\in\mathcal{A}(B;\kappa)$. We say $\{B_{\delta_n}(t)\}\in\mathcal{A}(B;\kappa, S)$ if

$$\lim_{n\to\infty} s_{ij}(\delta_n)=s_{ij}, \qquad \text{for every} \quad 1\leqq i,\ j\leqq d.$$

Still some more notation is needed. Let $\mathcal{H}(R^d)$ be the space of twice continuously differentiable functions on R^d whose partial derivatives of order $\leqq 2$ are all bounded. Finally set

$$(2.2) \qquad S_{ij}(t)=\int_0^t (B^i(s)\circ dB^j(s)-B^j(s)\circ dB^i(s))/2, \quad t>0, \qquad i,j=1,2,\cdots,d.$$

The result we want to show is the following:

Theorem 2. 1. *Suppose $\{B_\delta(t)\}\in\mathcal{A}(B;\kappa)$ for some positive constant κ. Let $S=(s_{ij})$ be a skew-symmetric $d\times d$-matrix. Then the following four statements are equivalent.*

(i) $\{B_{\delta_n}(t)\}\in\mathcal{A}(B;\kappa, S)$.

(ii) $\lim_{n\to\infty} E_0[|S_{ij}(t;\delta_n)-S_{ij}(t)-s_{ij}t|^2]=0,$

for $1\leqq i,\ j\leqq d$ *and* $t>0$.

(iii) $$\lim_{n\to\infty} E_0\left[\left|\int_0^t B^i_{\delta_n}(s)\,dB^j_{\delta_n}(s)-\int_0^t B^i(s)\circ dB^j(s)-s_{ij}t\right|^2\right]=0,$$

for $1\leqq i,\ j\leqq d$ *and* $t>0$.

(iv) $$\lim_{n\to\infty} E_0\left[\left|\int_0^t u(B_{\delta_n}(s))\,dB^j_{\delta_n}(s)-\int_0^t u(B(s))\circ dB^j(s)-\sum_{i=1}^d s_{ij}\int_0^t \frac{\partial}{\partial x^i}u(B(s))\,ds\right|^2\right]=0,$$

for $u\in\mathcal{H}(R^d)$, $1\leqq j\leqq d$ *and* $t>0$.

The proof of Theorem 2. 1 will be given in the next section. Now we will define a typical subclass of approximations in $\mathcal{A}(B;\kappa, S)$.

Definition 2. 3. Let $\{B_{\delta_n}(t)\}\in\mathcal{A}(B;\kappa,S)$. We say that $\{B_{\delta_n}(t)\}$ is symmetric if each component of S is equal to 0.

The following corollary is an immediate consequence of Theorem 2. 1.

Corollary 2. 1. *Let $\{B_{\delta_n}(t)\}\in\mathcal{A}(B;\kappa,S)$. Then $\{B_{\delta_n}(t)\}$ is symmetric if and only if*

$$\lim_{n\to\infty} E_0[|S_{ij}(t;\delta_n)-S_{ij}(t)|^2]=0\,, \tag{2. 3}$$

for $1\leqq i,\ j\leqq d$ and $t>0$.

Remark 2. 1. Let $B(t)=(B^1(t),\ B^2(t))$ be a two-dimensional Brownian motion starting at 0 and let $C_\delta{}^*=\{C_\delta{}^*(s);0\leqq s\leqq t+1\}$ be the closed curve in R^2 defined by

$$C_\delta{}^*(s)=\begin{cases}(B_\delta{}^1(s),\ B_\delta{}^2(s)), & 0\leqq s\leqq t\,,\\ (t+1-s)\,(B_\delta{}^1(t),\ B_\delta{}^2(t)), & t<s\leqq t+1\,.\end{cases}$$

As mentioned in the Introduction, P. Lévy [3] proved (2. 3) in the case that $\{B_{\delta_n}(t)\}$ is a sequence of polygonal approximations to $B(t)$. In this case, we can write

$$S_{12}(t;\delta)=\int_{C_\delta{}^*}\alpha\,,$$

where $\alpha=(x^1dx^2-x^2dx^1)/2$. We may, therefore, consider $S_{12}(t)$ as a stochastically defined area enclosed by a Brownian curve up to moment t and its chord, (P. Lévy [3], pp. 262–266).

Remark 2. 2. Suppose $\{B_{\delta_n}(t)\}\in\mathcal{A}(B;\kappa,S)$. If $\{B_{\delta_n}(t)\}$ is symmetric, then $(B^i_{\delta_n}(t),S_{jk}(t;\delta_n))$, $(1\leqq i,j,k\leqq d)$, converges to the diffusion process $(B^i(t),S_{jk}(t))$, $(1\leqq i,j,k\leqq d)$, in $L^2(\Omega,P_0)$, (cf. M.B. Gaveau [1]).

Finally we will give three examples. For this purpose we introduce the following notations. Φ denotes the space of continuously differentiable functions $\phi(t)$ on $[0, 1]$ such that

$$\phi(0)=0 \qquad \text{and} \qquad \phi(1)=1\,.$$

For $\phi\in\Phi$, set $\dot{\phi}=\frac{d}{dt}\phi$. For $\delta>0$ and $k=0, 1, \cdots$, set

$$\Delta_k B^i = B^i(k\delta+\delta) - B^i(k\delta).$$

Example 2. 1. Let $\phi^k\in\Phi$, $k=1, 2, \cdots, d$. Set, for $i=1, 2, \cdots, d$,

$$B_\delta^i(t) = B^i(k\delta) + \phi^i((t-k\delta)/\delta)\Delta_k B^i, \qquad \text{if } k\delta\leqq t<k\delta+\delta,\ k=0,\ 1,\ \cdots.$$

Then $\{B_\delta(t)=(B_\delta^1(t), B_\delta^2(t), \cdots, B_\delta^d(t))\}\in\mathcal{A}(B;\ \kappa)$ for some positive constant κ. In this case, since

$$s_{ij}(\delta)=0, \qquad \text{for every } \delta>0 \text{ and } 1\leqq i,\ j\leqq d,$$

$\{B_\delta(t)\}$ is symmetric. Hence if $\{B_\delta(t)\}$ is a sequence of polygonal approximations to $B(t)$, $\{B_\delta(t)\}$ is symmetric and $S_{ij}(t;\delta)$ converges to $S_{ij}(t)$ in the quadratic-mean sense.

Example 2. 2. (E.J. McShane [4]). Let $d=2$ and let $\phi^i\in\Phi$, $i=1, 2.$ For $i=1,\ 2,$ we define

$$(2.4)\quad B_\delta^i(t) = \begin{cases} B^i(k\delta)+\phi^i((t-k\delta)/\delta)\Delta_k B^i, & \text{if } \Delta_k B^1\Delta_k B^2\geqq 0, \\ & \qquad \text{for } k\delta\leqq t<k\delta+\delta. \\ B^i(k\delta)+\phi^{3-i}((t-k\delta)/\delta)\Delta_k B^i, & \text{if } \Delta_k B^1\Delta_k B^2<0, \end{cases}$$

Then $\{B_\delta(t)=(B_\delta^1(t), B_\delta^2(t))\}\in\mathcal{A}(B;\ \kappa)$ for some positive constant κ. By (2. 1) and (2. 4) we have

$$S_{12}(\delta;\delta)=\frac{|\Delta_0 B^1\Delta_0 B^2|}{2}\left\{1-2\int_0^1 \dot{\phi}^1(s)\phi^2(s)\,ds\right\}$$
$$+[B^1(0)B^2(\delta)-B^2(0)B^1(\delta)]/2.$$

Since $E_0[|\Delta_0 B^1\Delta_0 B^2|]=2\delta/\pi$, it follows that

$$s_{12}(\delta)=\left(1-2\int_0^1 \dot{\phi}^1(s)\phi^2(s)\,ds\right)/\pi, \qquad \text{for every } \delta>0.$$

Example 2. 3. Let $\phi_j^i\in\Phi$, ($i=1, 2, \cdots, d$ and $j=1, 2$). Set, for $i=1, 2, \cdots, d$,

$$(2.5)\quad B_\delta^i(t) = \begin{cases} B^i(k\delta)+\phi_1^i((t-k\delta)/\delta)\Delta_k B^i, & \text{if } \Delta_k B^i\geqq 0, \\ & \qquad \text{for } k\delta\leqq t<k\delta+\delta. \\ B^i(k\delta)+\phi_2^i((t-k\delta)/\delta)\Delta_k B^i, & \text{if } \Delta_k B^i<0, \end{cases}$$

Then $\{B_\delta(t)=(B_\delta^1(t), B_\delta^2(t), \cdots, B_\delta^d(t))\}\in\mathcal{A}(B;\kappa)$ for some positive constant κ. By (2.1) and (2.5), we have, for every $\delta>0$ and $i\neq j$,

$S_{ij}^*(\delta;\delta)$

$$=\begin{cases} |\Delta_0B^i\Delta_0B^j|\left(1-2\int_0^1\dot{\phi}_1^i(s)\phi_1^j(s)\,ds\right)/2, & \text{if } \Delta_0B^i\geqq 0,\ \Delta_0B^j\geqq 0,\\ -|\Delta_0B^i\Delta_0B^j|\left(1-2\int_0^1\dot{\phi}_1^i(s)\phi_2^j(s)\,ds\right)/2, & \text{if } \Delta_0B^i\geqq 0,\ \Delta_0B^j<0,\\ -|\Delta_0B^i\Delta_0B^j|\left(1-2\int_0^1\dot{\phi}_2^i(s)\phi_1^j(s)\,ds\right)/2, & \text{if } \Delta_0B^i<0,\ \Delta_0B^j\geqq 0,\\ |\Delta_0B^i\Delta_0B^j|\left(1-2\int_0^1\dot{\phi}_2^i(s)\phi_2^j(s)\,ds\right)/2, & \text{if } \Delta_0B^i<0,\ \Delta_0B^j<0,\end{cases}$$

where $S_{ij}^*(\delta;\delta)=S_{ij}(\delta;\delta)-[B^i(0)B^j(\delta)-B^j(0)B^i(\delta)]/2$. Hence

$$s_{ij}(\delta)=-\int_0^1(\dot{\phi}_1^i-\dot{\phi}_2^i)(s)\,(\phi_1^j-\phi_2^j)(s)\,ds/2\pi\,, \tag{2.6}$$

for every $\delta>0$ and $i\neq j$.

Using (2.6) we can prove that for any skew-symmetric $d\times d$-matrix S, there exists a sequence $\{B_{\delta_n}(t)\}$ of approximations to $B(t)$ such that $\{B_{\delta_n}(t)\}\in\mathcal{A}(B;\kappa,S)$.

§ 3. Proof of Theorem 2.1

Before proceeding to the proof of Theorem 2.1 we will prepare four lemmas. Set

$$c_{ij}(\delta)=E_0\left[\int_0^\delta \dot{B}_\delta^i(s)\,(B_\delta^j(\delta)-B_\delta^j(s))\,ds\right]/\delta\,. \tag{3.1}$$

Lemma 3.1. *For* $\delta>0$,

$$c_{ii}(\delta)=1/2\,,\qquad \textit{for } 1\leqq i\leqq d, \tag{3.2}$$

$$c_{ij}(\delta)=s_{ij}(\delta),\qquad \textit{for } 1\leqq i,\ j\leqq d \textit{ and } i\neq j\,. \tag{3.3}$$

Proof. By (3.1),

$$c_{ij}(\delta)+c_{ji}(\delta)=E_0[B^i(\delta)B^j(\delta)]/\delta\,.$$

Since $E_0[B^i(\delta)B^j(\delta)]=\delta\delta_{i,j}$, we have

(3. 4) $c_{ii}(\delta)=1/2$ and $c_{ij}(\delta)=-c_{ji}(\delta)$ for $i\neq j$.

Combining this with (3. 1) we can prove that if $i\neq j$, then

$$
\begin{aligned}
c_{ij}(\delta) &= (c_{ij}(\delta)-c_{ji}(\delta))/2 \\
&= E_0\left[\int_0^\delta (B_\delta^i(s)\,dB_\delta^j(s)-B_\delta^j(s)\,dB_\delta^i(s))\right]/2\delta \\
&= s_{ij}(\delta).
\end{aligned}
$$

This completes the proof of Lemma 3. 1.

Lemma 3. 2. *For any* $\delta>0$ *and* $1\leqq i,\ j\leqq d$,

$$
\begin{aligned}
&E_x\left[\left\{\int_0^\delta \dot{B}_\delta^i(s)\,(B_\delta^j(\delta)-B_\delta^j(s))\,ds\right\}^p\right] \\
&\qquad = E_0\left[\left\{\int_0^\delta \dot{B}_\delta^i(s)\,(B_\delta^j(\delta)-B_\delta^j(s))\,ds\right\}^p\right], \qquad (3.5)
\end{aligned}
$$

for $p=1, 2$ *and* $x\in R^d$,

and

$$
E_0\left[\int_{k\delta}^{(k+1)\delta} \dot{B}_\delta^i(s)\,(B_\delta^j(k\delta+\delta)-B_\delta^j(s))\,ds/\mathcal{F}_{k\delta}\right]=\delta c_{ij}(\delta), \qquad (3.6)
$$

for $k=0, 1, \cdots$.

Proof. (3. 5) follows from (A. 3). Appealing to the Markov property, we have

$$
\begin{aligned}
&E_0\left[\int_{k\delta}^{(k+1)\delta} \dot{B}_\delta^i(s)\,(B_\delta^j(k\delta+\delta)-B_\delta^j(s))\,ds/\mathcal{F}_{k\delta}\right] \\
&\quad = E_0\left[\int_0^\delta \dot{B}_\delta^i(s,\theta_{k\delta}\omega)\,(B_\delta^j(\delta,\theta_{k\delta}\omega)-B_\delta^j(s,\theta_{k\delta}\omega))\,ds/\mathcal{F}_{k\delta}\right], \\
&\qquad\qquad \text{(by (A. 2))}, \\
&\quad = E_{B(k\delta)}\left[\int_0^\delta \dot{B}_\delta^i(s)\,(B_\delta^j(\delta)-B_\delta^j(s))\,ds\right].
\end{aligned}
$$

Combining this with (3. 5) we can complete the proof of Lemma 3. 2.

For the sake of brevity, we introduce the following notations. For $\delta>0$, set

$$\begin{cases} [s]^+(\delta) = (k+1)\delta \\ [s]^-(\delta) = k\delta \end{cases}, \text{ for } k\delta \leqq s < (k+1)\delta, \ (k=0,1,2,\cdots).$$

Setting $s(\delta) = [s]^-(\delta)/\delta$, we have

Lemma 3. 3. *Let $Z_1(s,\omega)$ be a bounded $\mathcal{F}_s$-adapted process defined on $(\Omega, \mathcal{F}, P_x)$ with piecewise continuous sample paths. If $\{B_\delta(t)\} \in \mathcal{A}(B;\kappa)$, then*

$$E_0\left[\left\{\int_0^{[t]^-(\delta)} Z_1([s]^-(\delta))\,[\dot{B}_\delta^i(s)\,(B_\delta^j([s]^+(\delta)) - B_\delta^j(s)) - c_{ij}(\delta)]ds\right\}^2\right]$$

$$\leqq \kappa^{2/3}(K_1)^2[t]^-(\delta)\delta, \qquad \text{for } 1\leqq i,\ j\leqq d \text{ and } t>0,$$

where $K_1 = \sup_{s,\omega} |Z_1(s,\omega)|$.

Proof. Since

$$E_0\left[\int_{k\delta}^{(k+1)\delta} [\dot{B}_\delta^i(s)\,(B_\delta^j(k\delta+\delta) - B_\delta^j(s)) - c_{ij}(\delta)]ds/\mathcal{F}_{k\delta}\right] = 0$$

from (3. 6), it follows that

$$(3.7)\qquad \begin{aligned} &E_0\left[\left\{\int_0^{[t]^-(\delta)} Z_1([s]^-(\delta))\,[\dot{B}_\delta^i(s)\,(B_\delta^j([s]^+(\delta)) - B_\delta^j(s)) - c_{ij}(\delta)]ds\right\}^2\right] \\ &= E_0\left[\sum_{k=0}^{t(\delta)-1} Z_1(k\delta)^2 \left\{\int_{k\delta}^{(k+1)\delta} [\dot{B}_\delta^i(s)\,(B_\delta^j(k\delta+\delta) - B_\delta^j(s)) - c_{ij}(\delta)]ds\right\}^2\right]. \end{aligned}$$

Using Lemma 3. 2, we have

$$(3.8)\qquad \begin{aligned} &E_0\left[\left(\int_{k\delta}^{(k+1)\delta} [\dot{B}_\delta^i(s)\,(B_\delta^j(k\delta+\delta) - B_\delta^j(s)) - c_{ij}(\delta)]ds\right)^2/\mathcal{F}_{k\delta}\right] \\ &= E_0\left[\left(\int_0^\delta \dot{B}_\delta^i(s)\,(B_\delta^j(\delta) - B_\delta^j(s))ds\right)^2\right] - (c_{ij}(\delta)\delta)^2. \end{aligned}$$

On the other hand, by (A. 6) in Section 2,

$$(3.9)\qquad E_0\left[\left(\int_0^\delta \dot{B}_\delta^i(s)\,(B_\delta^j(\delta) - B_\delta^j(s))ds\right)^2\right]$$

$$\leqq E_0\left[\left(\int_0^\delta |\dot{B}_\delta{}^i(s)|ds\right)^2\left(\int_0^\delta |\dot{B}_\delta{}^j(s)|ds\right)^2\right]$$

$$\leqq \kappa^{2/3}\delta^2.$$

Combining (3. 7), (3. 8) and (3. 9), we have

$$E_0\left[\left\{\int_0^{[t]^-(\delta)} Z_1([s]^-(\delta))\,[\dot{B}_\delta{}^i(s)\,(B_\delta{}^j([s]^+(\delta))-B_\delta{}^j(s))-c_{ij}(\delta)]ds\right\}^2\right]$$

$$\leqq \kappa^{2/3}(K_1)^2[t]^-(\delta)\delta,$$

which completes the proof of Lemma 3. 3.

Lemma 3. 4. *Let K_2 be a positive constant and let $Z_2(s,\omega)$ be a stochastic process defined on $(\Omega,\mathcal{F},P_x)$ with piecewise continuous sample paths satisfying the following condition*:

$$(3.10)\qquad |Z_2(s)|\leqq K_2\sum_{m=1}^{d}\int_{[s]^-(\delta)}^{[s]^+(\delta)}|\dot{B}_\delta{}^m(u)|du, \qquad \textit{for } s\geqq 0.$$

If $\{B_\delta(t)\}\in\mathcal{A}(B;\kappa)$, then

$$(3.11)\qquad E_0\left[\left\{\int_0^{[t]^-(\delta)} Z_2(s)\,\dot{B}_\delta{}^i(s)\,(B_\delta{}^j([s]^+(\delta))-B_\delta{}^j(s))\,ds\right\}^2\right]$$

$$\leqq \kappa(K_2[t]^-(\delta)d)^2\delta, \qquad \textit{for } 1\leqq i,\ j\leqq d \textit{ and } t>0.$$

Proof. By (3. 10),

$$\left|\int_0^{[t]^-(\delta)} Z_2(s)\,\dot{B}_\delta{}^i(s)\,(B_\delta{}^j([s]^+(\delta))-B_\delta{}^j(s))\,ds\right|$$

$$\leqq\left|\sum_{k=0}^{t(\delta)-1}\int_{k\delta}^{(k+1)\delta} Z_2(s)\,\dot{B}_\delta{}^i(s)\,(B_\delta{}^j([s]^+(\delta))-B_\delta{}^j(s))\,ds\right|$$

$$\leqq K_2\sum_{k=0}^{t(\delta)-1}\sum_{m=1}^{d}\int_{k\delta}^{(k+1)\delta}|\dot{B}_\delta{}^m(s)|ds\int_{k\delta}^{(k+1)\delta}|\dot{B}_\delta{}^i(s)|ds\int_{k\delta}^{(k+1)\delta}|\dot{B}_\delta{}^j(s)|ds.$$

Hence by (A. 2), (A. 4) and (A. 6), the left-hand side of (3. 11) is bounded above by

$$(K_2t(\delta))^2\sum_{m=1}^{d}\sum_{k=1}^{d}E_0\left[\int_0^\delta|\dot{B}_\delta{}^m(s)|ds\int_0^\delta|\dot{B}_\delta{}^k(s)|ds\right.$$

$$\left.\times\left(\int_0^\delta|\dot{B}_\delta{}^i(s)|ds\right)^2\left(\int_0^\delta|\dot{B}_\delta{}^j(s)|ds\right)^2\right]\leqq\kappa(K_2[t]^-(\delta)d)^2\delta,$$

which completes the proof of Lemma 3. 4.

We now turn to the proof of Theorem 2. 1.

Proof of Theorem 2. 1. The implication (iv) $\to$ (iii) is trivial. Since $s_{ij}=-s_{ji}$, certainly (iii) implies (ii).

Proof of (ii) $\to$ (i). Suppose (ii) holds. First we note that $s_{ii}(\delta)=0$ for any $\delta>0$. Fix i and j such that $i\neq j$. From (ii), $E_0[S_{ij}(1;\delta_n)]$ converges to s_{ij}. Hence, using $B_\delta(t)\in\mathcal{A}(B;\kappa)$, we can prove that $E_0[S_{ij}([1]^-(\delta_n),\delta_n)]$ converges to s_{ij}. Consequently we have

$$\lim_{n\to\infty} E_0\Big[\int_0^{[1]^-(\delta_n)}\{(B_{\delta_n}{}^i(s)-B_{\delta_n}{}^i([s]^+(\delta_n)))\dot{B}_{\delta_n}{}^j(s) - (B_{\delta_n}{}^j(s)-B_{\delta_n}{}^j([s]^+(\delta_n)))\dot{B}_{\delta_n}{}^i(s)\}ds\Big]/2=s_{ij}. \tag{3.12}$$

On the other hand, by Lemma 3. 2, the left-hand side of (3. 12) is equal to

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{k=0}^{1(\delta_n)-1} E_0\Big[\int_{k\delta_n}^{(k+1)\delta_n}(B_{\delta_n}{}^i(s)-B_{\delta_n}{}^i((k+1)\delta_n))\dot{B}_{\delta_n}{}^j(s)\,ds \\
&\qquad - \int_{k\delta_n}^{(k+1)\delta_n}(B_{\delta_n}{}^j(s)-B_{\delta_n}{}^j((k+1)\delta_n))\dot{B}_{\delta_n}{}^i(s)\,ds\Big]/2 \\
&=\lim_{n\to\infty}\sum_{k=0}^{1(\delta_n)-1} E_0\Big[\int_0^{\delta_n}(B_{\delta_n}{}^j(\delta_n)-B_{\delta_n}{}^j(s))\dot{B}_{\delta_n}{}^i(s)\,ds \\
&\qquad - \int_0^{\delta_n}(B_{\delta_n}{}^i(\delta_n)-B_{\delta_n}{}^i(s))\dot{B}_{\delta_n}{}^j(s)\,ds\Big]/2 \\
&=\lim_{n\to\infty}[1]^-(\delta_n)c_{ij}(\delta_n).
\end{aligned}$$

Hence, by (3. 12) and Lemma 3. 1,

$$\lim_{n\to\infty}[1]^-(\delta_n)s_{ij}(\delta_n)=s_{ij},$$

and (i) follows.

Proof of (i) $\to$ (iv). Suppose (i) holds. Set $u_i=\dfrac{\partial}{\partial x^i}u$ for $u\in\mathcal{H}(R^d)$ and put $c_{ij}=s_{ij}+\delta_{i,j}/2$. Since

$$\int_0^t u(B(s))\circ dB^j(s)=\int_0^t u(B(s))\,dB^j(s)+\frac{1}{2}\int_0^t u_j(B(s))\,ds,$$

we have

$$\int_0^t u(B(s))\circ dB^j(s)+\sum_{i=1}^d s_{ij}\int_0^t u_i(B(s))\,ds$$

$$=\int_0^t u(B(s))\,dB^j(s)+\sum_{i=1}^d c_{ij}\int_0^t u_i(B(s))\,ds. \tag{3.13}$$

By integration by parts, we obtain

$$\int_{k\delta}^{(k+1)\delta} u(B_\delta(s))\,dB_\delta^{\,j}(s)$$

$$=-\int_{k\delta}^{(k+1)\delta} u(B_\delta(s))\frac{d}{ds}(B_\delta^{\,j}(k\delta+\delta)-B_\delta^{\,j}(s))\,ds$$

$$=u(B_\delta(k\delta))\,(B_\delta^{\,j}(k\delta+\delta)-B_\delta^{\,j}(k\delta))$$

$$+\sum_{i=1}^d\int_{k\delta}^{(k+1)\delta} u_i(B_\delta(s))\dot{B}_\delta^{\,i}(s)\,(B_\delta^{\,j}(k\delta+\delta)-B_\delta^{\,j}(s))\,ds \tag{3.14}$$

$$=u(B(k\delta))\,(B^j(k\delta+\delta)-B^j(k\delta))$$

$$+\sum_{i=1}^d\int_{k\delta}^{(k+1)\delta} u_i(B_\delta(s))\dot{B}_\delta^{\,i}(s)\,(B_\delta^{\,j}(k\delta+\delta)-B_\delta^{\,j}(s))\,ds,$$

(by (A. 1)).

Now we put

$$I_1(\delta)=\int_{[t]^-(\delta)}^t u(B_\delta(s))\,dB_\delta^{\,j}(s)-\int_{[t]^-(\delta)}^t u(B(s))\,dB^j(s)$$

$$-\sum_{i=1}^d c_{ij}\int_{[t]^-(\delta)}^t u_i(B(s))\,ds,$$

$$I_2(\delta)=\int_0^{[t]^-(\delta)}(u(B([s]^-(\delta)))-u(B(s)))\,dB^j(s),$$

$$I_3(\delta)=\sum_{i=1}^d\int_0^{[t]^-(\delta)} u_i(B([s]^-(\delta)))\,[\dot{B}_\delta^{\,i}(s)\,(B_\delta^{\,j}([s]^+(\delta))-B_\delta^{\,j}(s))$$

$$-c_{ij}(\delta)]\,ds,$$

$$I_4(\delta)=\sum_{i=1}^d\int_0^{[t]^-(\delta)}[u_i(B_\delta(s))-u_i(B([s]^-(\delta)))]\dot{B}_\delta^{\,i}(s)\,(B_\delta^{\,j}([s]^+(\delta))$$

$$-B_\delta^{\,j}(s))\,ds,$$

$$I_5(\delta)=\sum_{i=1}^d\int_0^{[t]^-(\delta)}[u_i(B([s]^-(\delta)))-u_i(B(s))]\,dsc_{ij},$$

$$I_6(\delta) = \sum_{i=1}^{d} \int_0^{[t]^-(\delta)} u_i(B([s]^-(\delta)))ds(c_{ij}(\delta) - c_{ij})$$

Combining (3. 13) with (3. 14), we have

$$(3.15) \quad \int_0^t u(B_\delta(s))dB_\delta^j(s) - \int_0^t u(B(s)) \circ dB^j(s) - \sum_{i=1}^{d} s_{ij} \int_0^t u_i(B(s))ds = \sum_{i=1}^{6} I_i(\delta).$$

It is obvious that

$$(3.16) \quad \lim_{\delta\to 0} E_0[\{I_1(\delta) + I_2(\delta) + I_5(\delta)\}^2] = 0.$$

Applying Lemmas 3. 3 and 3. 4 to $I_3(\delta)$ and $I_4(\delta)$ respectively, we have

$$(3.17) \quad \lim_{\delta\to 0} E_0[\{I_3(\delta) + I_4(\delta)\}^2] = 0.$$

It is also clear that (i) implies

$$(3.18) \quad \lim_{n\to\infty} E_0[(I_6(\delta_n))^2] = 0.$$

Combining (3. 15), (3. 16), (3. 17) and (3. 18), we can see that (iv) follows from (i).

§ 4. Stochastic Differential Equations and Related Ordinary Differential Equations

Let $\sigma(x) = (\sigma_j^\alpha(x))$, $(1 \leqq \alpha, j \leqq d)$ be a $d\times d$-matrix valued function defined on R^d. We assume that each component of $\sigma(x)$ is a bounded twice continuously differentiable function whose partial derivatives of order $\leqq 2$ are all bounded. We will consider a sequence $\{B_\delta(t)\}$ of approximations to $B(t)$ such that $\{B_{\delta_n}(t)\} \in \mathcal{A}(B; \kappa, S)$ for some skew-symmetric $d\times d$-matrix $S = (s_{ij})$. Let $X_\delta(t) = (X_\delta^1(t), X_\delta^2(t), \cdots, X_\delta^d(t))$ be the unique solution of the following ordinary differential equation:

$$(4.1) \quad \begin{cases} dX_\delta(t) = \sigma(X_\delta(t))dB_\delta(t), \\ X_\delta(0) = x_0 \in R^d. \end{cases}$$

Let $X(t) = (X^1(t), X^2(t), \cdots, X^d(t))$ be the unique solution of the following stochastic differential equation:

$$
(4.2) \qquad \begin{cases} dX^\alpha(t) = \sum_{j=1}^d \sigma_j^\alpha(X(t)) \circ dB^j(t) \\ \qquad + \sum_{i,j=1}^d \sum_{\beta=1}^d s_{ij}\left(\sigma_i^\beta \frac{\partial}{\partial x^\beta}\sigma_j^\alpha\right)(X(t))dt, \quad \text{for } 1 \leqq \alpha \leqq d, \\ X(0) = x_0 \in R^d. \end{cases}
$$

The result we want to show is the following:

Theorem 4.1. *If* $\{B_{\delta_n}\} \in \mathcal{A}(B; \kappa, S)$, *then*

$$
(4.3) \qquad \lim_{n\to\infty} E_0[\|X_{\delta_n}(t) - X(t)\|^2] = 0, \qquad \textit{for } t \geqq 0.
$$

Proof. The proof uses the same lemmas as in the proof of Theorem 2.1. First we note that for every $\delta > 0$ and $s \geqq 0$,

$$
(4.4) \qquad \|X_\delta(s) - X_\delta([s]^-(\delta))\| \leqq K_3 \sum_{m=1}^d \int_{[s]^-(\delta)}^{[s]^+(\delta)} |\dot{B}_\delta^m(u)| du,
$$

where K_3 is a positive constant depending only on σ. By integration by parts, we have

$$
(4.5) \qquad \begin{aligned} X_\delta^\alpha([t]^-(\delta)) - X_\delta^\alpha(0) &= \sum_{j=1}^d \int_0^{[t]^-(\delta)} \sigma_j^\alpha(X_\delta([s]^-(\delta)))dB^j(s) \\ &+ \sum_{i,j,\beta=1}^d \int_0^{[t]^-(\delta)} \sigma_i^\beta \frac{\partial}{\partial x^\beta}\sigma_j^\alpha(X_\delta(s))\dot{B}_\delta^i(s)(B_\delta^j([s]^+(\delta)) \\ &- B_\delta^j(s))ds, \qquad \text{for } \alpha = 1, 2, \cdots, d. \end{aligned}
$$

Now put $c_{ij} = s_{ij} + \delta_{i,j}/2$. Then, by (4.2),

$$
\begin{aligned} X^\alpha(t) - X^\alpha(0) &= \sum_{j=1}^d \int_0^t \sigma_j^\alpha(X(s))dB^j(s) \\ &+ \sum_{i,j,\beta=1}^d c_{ij} \int_0^t \left(\sigma_i^\beta \frac{\partial}{\partial x^\beta}\alpha_j^\alpha\right)(X(s))ds, \qquad \alpha = 1, 2, \cdots, d. \end{aligned}
$$

Combining this with (4.5), we have

$$
(4.6) \qquad X_\delta^\alpha(t) - X^\alpha(t) = \sum_{j=1}^6 I_j^\alpha(t; \delta), \qquad \alpha = 1, 2, \cdots, d,
$$

where

$$
I_1^\alpha(t; \delta) = X_\delta^\alpha(t) - X_\delta^\alpha([t]^-(\delta)) - X^\alpha(t) + X^\alpha([t]^-(\delta)),
$$

$$
I_2^\alpha(t; \delta) = \sum_{j=1}^d \int_0^{[t]^-(\delta)} [\sigma_j^\alpha(X_\delta([s]^-(\delta))) - \sigma_j^\alpha(X(s))]dB^j(s),
$$

$$I_3^\alpha(t;\delta)=\sum_{i,j,\beta=1}^{d}\int_0^{[t]^-(\delta)}\sigma_i^\beta\frac{\partial}{\partial x^\beta}\sigma_j^\alpha(X_\delta([s]^-(\delta)))$$

$$\times[\dot{B}_\delta^i(s)(B_\delta^j([s]^+(\delta))-B_\delta^j(s))-c_{ij}(\delta)]ds,$$

$$I_4^\alpha(t;\delta)=\sum_{i,j,\beta=1}^{d}\int_0^{[t]^-(\delta)}\left[\sigma_i^\beta\frac{\partial}{\partial x^\beta}\sigma_j^\alpha(X_\delta(s))-\sigma_i^\beta\frac{\partial}{\partial x^\beta}\sigma_j^\alpha(X_\delta([s]^-(\delta)))\right]$$

$$\times\dot{B}_\delta^i(s)(B_\delta^j([s]^+(\delta))-B_\delta^j(s))ds,$$

$$I_5^\alpha(t;\delta)=\sum_{i,j,\beta=1}^{d}\int_0^{[t]^-(\delta)}\left[\sigma_i^\beta\frac{\partial}{\partial x^\beta}\sigma_j^\alpha(X_\delta([s]^-(\delta)))\right.$$

$$\left.-\sigma_i^\beta\frac{\partial}{\partial x^\beta}\sigma_j^\alpha(X(s))\right]dsc_{ij},$$

$$I_6^\alpha(t;\delta)=\sum_{i,j,\beta=1}^{d}\int_0^{[t]^-(\delta)}\sigma_i^\beta\frac{\partial}{\partial x^\beta}\sigma_j^\alpha(X_\delta([s]^-(\delta)))ds[c_{ij}(\delta)-c_{ij}].$$

Now fix $T>0$. Set

$$Z_1(s,\omega)=\sum_{\beta=1}^{d}\left(\sigma_i^\beta\frac{\partial}{\partial x^\beta}\sigma_j^\alpha\right)(X_\delta([s]^-(\delta))).$$

Then $Z_1(s,\omega)$ is a bounded $\mathcal{F}_s$-adapted process with piecewise continuous sample paths. Next set

$$Z_2(s,\omega)=\sum_{\beta=1}^{d}\left[\sigma_i^\beta\frac{\partial}{\partial x^\beta}\sigma_j^\alpha(X_\delta(s))-\sigma_i^\beta\frac{\partial}{\partial x^\beta}\sigma_j^\alpha(X_\delta([s]^-(\delta)))\right].$$

Then, by (4. 4), $Z_2(s,\omega)$ satisfies (3. 10) in Lemma 3. 4. Hence we can apply Lemma 3. 3 and Lemma 3. 4 to $I_3^\alpha(t;\delta)$ and $I_4^\alpha(t;\delta)$ respectively. Hence, using (4. 4) and $\{B_{\delta_n}\}\in\mathcal{A}(B;\kappa,S)$, we obtain

$$E_0[\|X_{\delta_n}(t)-X(t)\|^2]\leqq K_4\int_0^t E_0[\|X_{\delta_n}(s)-X(s)\|^2]ds+\varepsilon_n, \quad \text{for } t\leqq T, \tag{4.7}$$

where K_4 is a positive constant depending only on σ, κ and T and $\{\varepsilon_n\}$ is a sequence of positive numbers with $\lim_{n\to\infty}\varepsilon_n=0$ depending only on σ, κ and T. By (4. 7), we have

$$E_0[\|X_{\delta_n}(t)-X(t)\|^2]\leqq\varepsilon_n\exp(K_4t), \qquad \text{for } t\leqq T,$$

which implies (4. 3).

References

[1] Gaveau, M. B., Solutions fondamentales, représentations, et estimées sous-elliptiques pour les groupes nilpotents d'ordre 2, *C. R. Acad. Sc. Paris,* **282** (1976), 563-566.
[2] Itô, K., Stochastic differentials, *Appl. Math. Optimization,* **1** (1975), 374-381.
[3] Lévy, P., *Processus stochastiques et mouvement brownien*, Gauthier-Villars, Paris, 1948.
[4] McShane, E. J., Stochastic differential equations and models of random processes, *Proc. 6-th Berkeley Symp. on Math. Statist. and Prob.*, **3** (1970), 263-294.
[5] Wong, E. and Zakai, M., On the convergence of ordinary integrals to stochastic integrals, *Ann. Math. Statist.*, **36** (1965), 1560-1564.
[6] Wong, E. and Zakai, M., Riemann-Stieltjes approximations of stochastic integrals, *Z. Wahrscheinlichkeitstheorie verw. Geb.*, **12** (1969), 87-97.