Nonlinear PDE in the presence of singular randomness

English translation of the paper "EDP non linéaires en présence d'aléa singulier"
published in *La Gazette des Mathématiciens*, **160**, pp. 6–14, 2019

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This paper describes results concerning the construction of low-regularity solutions of nonlinear partial differential equations that depend on a random parameter. The motivations for this study are very varied. However, in the end, the results obtained and the methods used are conceptually very similar.

1 Multiple Fourier series and Sobolev spaces on the torus

For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we set

$$\langle x \rangle := (1 + x_1^2 + \cdots + x_d^2)^{1/2}.$$

Let $\mathbb{T}^d = (\mathbb{R}/(2\pi\mathbb{Z}))^d$ be the torus of dimension d. If $f\colon \mathbb{T}^d \to \mathbb{C}$ is a function of class C^∞, then for all $x \in \mathbb{T}^d$,

$$f(x) = \sum_{n \in \mathbb{Z}^d} \hat{f}(n) e^{in\cdot x},$$

where $\hat{f}(n)$ are the Fourier coefficients of f. For $s \in \mathbb{R}$, the Sobolev norm of f is defined by

$$\|f\|^2_{H^s(\mathbb{T}^d)} = \sum_{n \in \mathbb{Z}^d} \langle n \rangle^{2s} |\hat{f}(n)|^2. \tag{1}$$

For an integer $s \geq 0$, we have the equivalence of norms

$$\|f\|^2_{H^s(\mathbb{T}^d)} \simeq \sum_{|\alpha| \leq s} \|\partial^\alpha f\|^2_{L^2(\mathbb{T}^d)}. \tag{2}$$

In (2), ∂^α represents a partial derivative of order $\leq s$. For $s = 0$, we recover the norm of the Lebesgue space $L^2(\mathbb{T}^d)$.

The Sobolev space $H^s(\mathbb{T}^d)$ is defined as the completion of $C^\infty(\mathbb{T}^d)$ with respect to the norm (1). In contrast to the case $s \geq 0$, for $s < 0$ the elements of $H^s(\mathbb{T}^d)$ are not classical functions on the torus, but can be interpreted as Schwartz distributions. Note that the Sobolev spaces are nested: the larger s is, the more regular the elements of $H^s(\mathbb{T}^d)$ are; the intersection of all $H^s(\mathbb{T}^d)$ is $C^\infty(\mathbb{T}^d)$. On the other hand, the smaller s is, the larger $H^s(\mathbb{T}^d)$ is; the union of all the spaces $H^s(\mathbb{T}^d)$ is the Schwartz space of $(2\pi\mathbb{Z})^d$-periodic distributions on $\mathbb{R}^d$.

2 Probabilistic effects in fine questions of analysis

2.1 An almost sure improvement of the Sobolev embedding

Let $(\Omega, \mathcal{F}, p)$ be a probability space. Recall that a random variable $g\colon \Omega \to \mathbb{R}$ belongs to $\mathcal{N}(0, \sigma^2)$, with $\sigma > 0$, if the image of the measure p under g is

$$\frac{1}{\sigma\sqrt{2\pi}} \exp\Big(-\frac{x^2}{2\sigma^2}\Big) dx,$$

where dx is the Lebesgue measure on $\mathbb{R}$. The random variable g then follows the centered normal distribution with variance σ. Similarly, a random variable $g\colon \Omega \to \mathbb{C}$ belongs to $\mathcal{N}_\mathbb{C}(0, \sigma^2)$, if $g = h + il$ with $h \in \mathcal{N}(0, \sigma^2)$ and $l \in \mathcal{N}(0, \sigma^2)$ independent.

Let $u \in L^2(\mathbb{T}^2)$ be a deterministic function. There exists a sequence $(c_n)_{n \in \mathbb{Z}} \in l^2(\mathbb{Z})$ (which consists of the Fourier coefficients of u) such that

$$u(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx}.$$

Now consider a randomized version of u given by the expression

$$u_\omega(x) = \sum_{n \in \mathbb{Z}} c_n g_n(\omega) e^{inx},$$

where $(g_n(\omega))_{n \in \mathbb{Z}}$ is a sequence of independent variables of $\mathcal{N}_\mathbb{C}(0, 1)$. The randomization has no effect on the Sobolev regularity of u_ω (see, e.g., [5]). On the other hand, randomization has an important effect on regularity in Lebesgue spaces $L^p(\mathbb{T}^d)$. Using the rotation invariance of $\mathcal{N}_\mathbb{C}(0, 1)$, we obtain that $g_n(\omega)e^{inx} \in \mathcal{N}_\mathbb{C}(0, 1)$, and then the independence of g_n ensures that for a fixed x

$$u_\omega(x) \in \mathcal{N}_\mathbb{C}\Big(0, \sum_{n \in \mathbb{Z}} |c_n|^2\Big).$$

Since Gaussian variables have finite moments of any order, we get

$$u_\omega(x) \in L^p(\Omega \times \mathbb{T}),$$

which implies that $u_\omega(x) \in L^p(\mathbb{T})$ almost surely, a remarkable improvement in the L^p regularity of u_ω compared to that of u. Note that the Sobolev embedding requires that a deterministic function be $H^{1/2}(\mathbb{T})$-regular in order to conclude that it lies in $L^p(\mathbb{T})$ for any $p < +\infty$ (and this restriction on regularity is optimal). In descriptive terms, randomization saves half a derivative compared with the Sobolev embedding. Like the Sobolev embedding, this effect has been known since the beginning of the 20th century, and it may seem surprising that the interaction between these two phenomena has not been studied more in the past.

Finally, thanks to the Khinchin inequality, in the preceding discussion one is allowed to replace Gaussian variables by a more general family of random variables (e.g., Bernoulli variables).

2.2 *Almost sure products in Sobolev spaces of negative index*

Let

$$u_\omega(x) = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{inx}, \quad \frac{1}{4} < \alpha < \frac{1}{2},$$

be a random series, with g_n as in the preceding section. It is easy to verify that almost surely $u_\omega \in H^\sigma(\mathbb{T})$ for $\sigma < \alpha - \frac{1}{2}$, but almost surely $u_\omega \notin H^{\alpha-\frac{1}{2}}(\mathbb{T})$. In the following, we fix a number σ such that $\sigma < \alpha - \frac{1}{2}$ (it should be assumed that this number is very close to $\alpha - \frac{1}{2}$). The series u_ω is therefore in a Sobolev space of negative index and it is difficult to define an object like $|u_\omega|^2$. After renormalization, it is nevertheless possible to give a meaning to $|u_\omega|^2$, and even to determine its regularity in Sobolev spaces. Let us start by considering the partial sums

$$u_{\omega,N}(x) = \sum_{|n|\le N} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{inx},$$

which are C^∞ functions. Now expand $|u_{\omega,N}(x)|^2$ as

$$|u_{\omega,N}(x)|^2 = \sum_{|n|\le N} \frac{|g_n(\omega)|^2}{\langle n\rangle^{2\alpha}} + \sum_{\substack{n_1\neq n_2 \\ |n_1|,|n_2|\le N}} \frac{g_{n_1}(\omega)\overline{g_{n_2}(\omega)}}{\langle n_1\rangle^\alpha\langle n_2\rangle^\alpha} e^{i(n_1-n_2)x}.$$

The first term of this expansion (the zero-order Fourier coefficient) contains all the singularity, while the second term has a limit almost surely in $H^{2\sigma}(\mathbb{T})$. We then set

$$c_N := \mathbb{E}\Big(\sum_{|n|\le N} \frac{|g_n(\omega)|^2}{\langle n\rangle^{2\alpha}}\Big) = \sum_{|n|\le N} \frac{2}{\langle n\rangle^{2\alpha}} \sim N^{1-2\alpha},$$

and we define the renormalized partial sum as

$$|u_{\omega,N}(x)|^2 - c_N = \sum_{|n|\le N} \frac{|g_n(\omega)|^2 - 2}{\langle n\rangle^{2\alpha}} + \sum_{\substack{n_1\neq n_2 \\ |n_1|,|n_2|\le N}} \frac{g_{n_1}(\omega)\overline{g_{n_2}(\omega)}}{\langle n_1\rangle^\alpha\langle n_2\rangle^\alpha} e^{i(n_1-n_2)x}.$$

The independence of the random variables g_n ensures that the zero-order Fourier coefficient is well-defined. More precisely, we obtain

$$\mathbb{E}\Big(\Big|\sum_{|n|\le N} \frac{|g_n(\omega)|^2 - 2}{\langle n\rangle^{2\alpha}}\Big|^2\Big) = \sum_{|n|\le N} \frac{4}{\langle n\rangle^{4\alpha}},$$

which has a limit as $N \to +\infty$ if $\alpha > 1/4$.

Similarly, the independence implies that the expectation

$$\mathbb{E}\Big(\Big\|\sum_{\substack{n_1\neq n_2 \\ |n_1|,|n_2|\le N}} \frac{g_{n_1}(\omega)\overline{g_{n_2}(\omega)}}{\langle n_1\rangle^\alpha\langle n_2\rangle^\alpha} e^{i(n_1-n_2)x}\Big\|^2_{H^{2\sigma}}\Big)$$

is bounded from above by a term of the order of

$$\sum_{n_1,n_2} \frac{\langle n_1-n_2\rangle^{4\sigma}}{\langle n_1\rangle^{2\alpha}\langle n_2\rangle^{2\alpha}}.$$

The latter sum is convergent under the condition $-4\sigma + 4\alpha > 2$, which is equivalent to our assumption $\sigma < \alpha - \frac{1}{2}$. Consequently, the sequence

$$\big(|u_{\omega,N}(x)|^2 - c_N\big)_{N\ge 1} \tag{3}$$

has a limit in $L^2(\Omega; H^{2\sigma}(\mathbb{T}))$. This limit is by definition the renormalization of $|u_\omega|^2$. We can also establish (using more sophisticated arguments) the almost sure convergence in the Sobolev space $H^{2\sigma}(\mathbb{T})$ of the sequence (3). Note that since $\sigma < 0$, the norm in $H^{2\sigma}(\mathbb{T})$ is weaker than the norm in $H^\sigma(\mathbb{T})$ (where the series $u_\omega(x)$ is defined).

To sum up in a very informal way, the squared modulus of an element of H^σ is in $H^{2\sigma}$, after renormalization. This is a remarkable probabilistic effect which lies at the heart of the study of evolutionary PDE, in the presence of randomness, in Sobolev spaces of negative index. We will further develop this topic in the remainder of this text.

3 Solving the nonlinear wave equation with low-regularity initial data

The wave equation is a typical example of a dispersive PDE. Solving dispersive PDE with low-regularity initial data has a long history, dating back to the seminal works of Ginibre and Velo and of Kato. Kenig, Ponce and Vega, Klainerman and Machedon, and most notably Bourgain have developed tools from harmonic analysis allowing to obtain solutions of very low regularity. The question of the optimality of these results then arose. It was Lebeau's work that launched a series of results on the construction of counterexamples showing the optimality of the assumption of regularity in the previous results. It was in this context that the idea of proving a kind of probabilistic well-posedness for regularities where counterexamples were constructed was introduced in [23], and then implemented in [5, 6].

3.1 Solving the linear wave equation with periodic distributions as initial data

Consider the linear wave equation

$$(\partial_t^2 - \Delta)u = 0, \quad u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \quad (4)$$

where $t \in \mathbb{R}$, $x \in \mathbb{T}^3$, $u\colon \mathbb{R}\times\mathbb{T}^3 \to \mathbb{R}$ and Δ is the Laplace operator. It is readily verified that for

$$(u_0, u_1) \in C^\infty(\mathbb{T}^3) \times C^\infty(\mathbb{T}^3)$$

the solution of (4) is given by the map $S(t)$ defined by

$$S(t)(u_0,u_1) = \sum_{n\in\mathbb{Z}^3}\Big(\cos(t|n|)\widehat{u_0}(n) + \frac{\sin(t|n|)}{|n|}\widehat{u_1}(n)\Big)e^{in\cdot x},$$

where $|n| = (n_1^2+n_2^2+n_3^2)^{1/2}$. For $n=0$, the expression $\frac{\sin(t|n|)}{|n|}$ is naturally understood as its limit t.

Since $|\cos(t|n|)| \le 1$ and $|\sin(t|n|)| \le 1$, it follows from the above definition that

$$\|S(t)(u_0,u_1)\|_{H^s} \le C(1+|t|)(\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}). \quad (5)$$

Since the map $S(t)$ is linear, we can define a unique extension of $S(t)$ for

$$(u_0,u_1) \in H^s(\mathbb{T}^3)\times H^{s-1}(\mathbb{T}^3)$$

and solve (4) with initial data in $H^s(\mathbb{T}^3)\times H^{s-1}(\mathbb{T}^3)$ for arbitrary $s\in\mathbb{R}$.

3.2 The nonlinear problem. Resolution by deterministic methods

The previous discussion makes it easy to solve (4) with singular initial data (in Sobolev spaces of arbitrary negative index). The argument is based on the a priori estimate (5) and the linear nature of the map $S(t)$ (or of equation (4)). The situation changes radically if we consider a nonlinear perturbation of (4). In this text, we restrict our attention to the case of a cubic nonlinear interaction. More precisely, we consider the problem

$$(\partial_t^2 - \Delta)u + u^3 = 0, \quad u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x). \quad (6)$$

For this problem, the crucial information (5) and the linear nature of the equation are lost. Nevertheless, equation (6) is of Hamiltonian type. Therefore, formally, the solutions of (6) satisfy the algebraic relation

$$\frac{d}{dt}\int_{\mathbb{T}^3}\Big((\partial_t u(t,x))^2 + |\nabla_x u(t,x)|^2 + \frac{1}{2}u^4(t,x)\Big)dx = 0. \quad (7)$$

This relation implies that the Sobolev space $H^1(\mathbb{T}^3)$ is one of the natural settings for the study of problem (6). The starting point of this study is given by the following classical result.

Theorem 3.1. *For any pair $(u_0,u_1)\in H^1(\mathbb{T}^3)\times L^2(\mathbb{T}^3)$ (real valued) there exists a unique global in time solution of* (6) *in the class*

$$(u,\partial_t u) \in C^0(\mathbb{R}; H^1(\mathbb{T}^3)\times L^2(\mathbb{T}^3)).$$

If, in addition, $(u_0,u_1)\in H^s(\mathbb{T}^3)\times H^{s-1}(\mathbb{T}^3)$, for a given $s\ge 1$, then

$$(u,\partial_t u) \in C^0(\mathbb{R}; H^s(\mathbb{T}^3)\times H^{s-1}(\mathbb{T}^3)). \quad (8)$$

Finally, the dependence on the initial data is continuous.

Using compactness methods (going back to the work of Leray) we can exploit (7) and obtain a much weaker version of Theorem 3.1, without uniqueness and without the propagation of regularity (8). In Theorem 3.1, uniqueness results from the Sobolev embedding $H^1(\mathbb{T}^3)\hookrightarrow L^6(\mathbb{T}^3)$. The L^6-norm appears here naturally when we study the L^2-norm of the nonlinear term u^3. As for the propagation of the regularity, it derives from the estimate

$$\|u^3\|_{H^s(\mathbb{T}^3)} \le C\,\|(1-\Delta)^{s/2}u\|_{L^6(\mathbb{T}^3)}\,\|u\|^2_{L^6(\mathbb{T}^3)}. \quad (9)$$

The details of the proof of (9) can be found in [1], where estimates of the type (9) are called tame. The key point in estimate (9) is that the s derivatives acting on the expression u^3 are redistributed in such a way that, at the end, the right-hand side of (9) depends only linearly on the strong norm (the one that contains derivatives).

In view of the discussion of the linear problem (4), it is now natural to ask whether Theorem 3.1 generalizes to initial data in $H^s\times H^{s-1}$ for a given $s<1$. As we shall see below, such a generalization is possible for some s, but not for all. By using Strichartz estimates instead of the Sobolev embedding $H^1(\mathbb{T}^3)\hookrightarrow L^6(\mathbb{T}^3)$, part of Theorem 3.1 generalizes to

$$(u_0,u_1)\in H^s(\mathbb{T}^3)\times H^{s-1}(\mathbb{T}^3), \quad s\ge 1/2. \quad (10)$$

More precisely, the local well-posedness of (6) can be established under the assumption (10). A more detailed description of Strichartz's estimates would go beyond our objectives in this text. We merely point out that Strichartz estimates can be seen as improvements almost everywhere in time of Sobolev embeddings, when, instead of considering an arbitrary function, we consider a function that satisfies a dispersive PDE. We refer to [24] for more details on Strichartz estimates and the generalization of Theorem 3.1 under assumption (10). It can be conjectured that the global in time part of Theorem 3.1 remains true under assumption (10). The most advanced results towards the resolution of this conjecture are in [8, 21].

3.3 The limitations of deterministic methods

The restriction (10) is optimal with respect to the well-posedness in the sense Hadamard of the problem (6) with initial data in $H^s(\mathbb{T}^3)\times H^{s-1}(\mathbb{T}^3)$. More precisely, we have the following result.

Theorem 3.2. *Let $s \in (0,1/2)$ and $(u_0,u_1) \in H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$. There exists a sequence*

$$u_N(t,x) \in C^0(\mathbb{R}; C^\infty(\mathbb{T}^3)), \quad N = 1,2,\dots$$

such that

$$(\partial_t^2 - \Delta)u_N + u_N^3 = 0,$$

with

$$\lim_{N\to+\infty} \|u_N(0) - u_0, \partial_t u_N(0) - u_1\|_{H^s(\mathbb{T}^3)\times H^{s-1}(\mathbb{T}^3)} = 0$$

but for all $T > 0$

$$\lim_{N\to+\infty} \|u_N(t), \partial_t u_N(t)\|_{L^\infty([0,T];H^s(\mathbb{T}^3)\times H^{s-1}(\mathbb{T}^3))} = +\infty.$$

Well-posedness in the sense of Hadamard requires existence, uniqueness and continuous dependence on the initial data. Theorem 3.2 shows that continuous dependence on initial data fails.

The proof of Theorem 3.2 is based on an idea of Lebeau (see for example [15]): if the initial data are localized at high frequency, then for small times a good approximation of the solution of (6) is given by the solution of

$$\partial_t^2 u + u^3 = 0, \quad u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x) \qquad (11)$$

which is obtained from (6) by neglecting the effect of the Laplacian. In other words, under the hypothesis of Theorem 3.2, nonlinear effects dominate in the regime described above. The solution of (11) manifests the phenomenon of amplification described by Theorem 3.2 and this property propagates to the solutions of (6) by a perturbative, highly non-trivial argument. A detailed proof of Theorem 3.2 can be found in [24].

3.4 Resolution by probabilistic methods beyond the limitations of deterministic theory

Despite the result of Theorem 3.2, we can ask whether a form of the well-posedness of (6) remains true for initial data in

$$H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3), \quad s < 1/2. \qquad (12)$$

The answer to this question is positive if we endow the space (12) with a non-degenerate probability measure such that we have existence, uniqueness and (a form of) continuous dependence almost surely with respect to this measure.

We will choose the initial data for (6) from the realizations of the following random series:

$$u_0^\omega(x) = \sum_{n\in\mathbb{Z}^3} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{in\cdot x}, \quad u_1^\omega(x) = \sum_{n\in\mathbb{Z}^3} \frac{h_n(\omega)}{\langle n\rangle^{\alpha-1}} e^{in\cdot x}. \qquad (13)$$

Here $(g_n)_{n\in\mathbb{Z}^3}$ and $(h_n)_{n\in\mathbb{Z}^3}$ are two families of independent random variables conditioned by $g_n = \overline{g_{-n}}$ and $h_n = \overline{h_{-n}}$, so that u_0^ω and u_1^ω are real valued. Furthermore, it is assumed that for $n \neq 0$, g_n and h_n are complex Gaussians with distribution $\mathcal{N}_\mathbb{C}(0,1)$, while g_0 and h_0 are real Gaussians with distribution $\mathcal{N}(0,1)$.

The partial sums associated with (13) are Cauchy sequences in $L^2(\Omega; H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3))$ for all $s < \alpha - \frac{3}{2}$. Therefore, the initial data (13) belong almost surely to $H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$ for $s < \alpha - \frac{3}{2}$. Furthermore, the probability of the event

$$(u_0^\omega, u_1^\omega) \in H^{\alpha-\frac{3}{2}}(\mathbb{T}^3) \times H^{\alpha-\frac{5}{2}}(\mathbb{T}^3)$$

is zero. It follows that for $\alpha > 5/2$ we can apply Theorem 3.1 to the data (u_0,u_1) described by (13). For $\alpha > 2$, we can apply refined deterministic results (based on Strichartz estimates). Finally, for $\alpha \in (3/2,2)$, Theorem 3.2 applies and we get:

Theorem 3.3 (pathological approximations). *Let $\alpha \in (3/2,2)$ and $0 < s < \alpha - 3/2$. For almost every ω, there exists a sequence*

$$u_N^\omega(t,x) \in C^0(\mathbb{R}; C^\infty(\mathbb{T}^3)), \quad N = 1,2,\dots$$

such that

$$(\partial_t^2 - \Delta)u_N^\omega + (u_N^\omega)^3 = 0,$$

with

$$\lim_{N\to+\infty} \|u_N^\omega(0) - u_0^\omega, \partial_t u_N^\omega(0) - u_1^\omega\|_{H^s(\mathbb{T}^3)\times H^{s-1}(\mathbb{T}^3)} = 0,$$

but for all $T > 0$

$$\lim_{N\to+\infty} \|u_N^\omega(t), \partial_t u_N^\omega(t)\|_{L^\infty([0,T];H^s(\mathbb{T}^3)\times H^{s-1}(\mathbb{T}^3))} = +\infty.$$

However, the following result also holds.

Theorem 3.4 (probabilistic well-posedness). *Let $\alpha \in (3/2,2)$ and $0 < s < \alpha - 3/2$. Using Theorem 3.1, define the sequence $(u_N^\omega)_{N\geq 1}$ of solutions of problem (6) with regular initial conditions given by*

$$u_{0,N}^\omega(x) = \sum_{|n|\leq N} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{in\cdot x}, \quad u_{1,N}^\omega(x) = \sum_{|n|\leq N} \frac{h_n(\omega)}{\langle n\rangle^{\alpha-1}} e^{in\cdot x}. \quad (14)$$

Then there exists a set $\Sigma \subset \Omega$ of probability 1 such that for every $\omega \in \Sigma$ the sequence $(u_N^\omega)_{N\geq 1}$ converges when $N \to +\infty$ in $C^0(\mathbb{R}; H^s(\mathbb{T}^3))$ to a (unique) limit that satisfies (6) in the sense of distributions.

Theorems 3.3 and 3.4 show that the type of approximation of the initial data is crucial when establishing the probabilistic well-posedness.

Using compactness methods (à la Leray), we can hope to obtain convergence of a subsequence of $(u_N^\omega)_{N\geq 1}$. The convergence of the whole sequence $(u_N^\omega)_{N\geq 1}$ is beyond the reach of weak-solutions techniques. The fact that the whole sequence converges already contains a form of uniqueness. In [6], one can find a form of uniqueness that can be formulated in a suitable functional framework.

In [6], we also obtain a probabilistic continuous dependence on the initial data, the proof of which makes use of conditioned large deviation properties which seem to be of independent interest.

We can prove the result of Theorem 3.4 for more general randomizations than (13). For example, Gaussian variables can be replaced by Bernoulli variables and the deterministic coefficients $\langle n\rangle^{-\alpha}$ by other coefficients with "similar" behaviour for $|n| \gg 1$ (see [6] for more details).

Theorem 3.4 provides a nice dense set Σ of initial data such that for good approximations we get nice global solutions (but for bad approximations we get divergent sequences, as shown by Theorem 3.3!). On the other hand, due to [22], we also have a dense set of bad initial data, even for the natural approximations by Fourier truncation (or convolution).

Theorem 3.5 (pathological initial data). *Let $0 < s < \frac{1}{2}$. Then there is a dense set $S \subset H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$ such that for every $(f, g) \in S$, the sequence $(u_N)_{N \geq 1}$ of (smooth) solutions of*

$$(\partial_t^2 - \Delta)u + u^3 = 0$$

with data

$$u_0(x) = \sum_{|n| \leq N} \hat{f}(n) e^{in \cdot x}, \qquad u_1(x) = \sum_{|n| \leq N} \hat{g}(n) e^{in \cdot x}$$

does not converge. More precisely, for every $T > 0$,

$$\lim_{N \to \infty} \|u_N(t)\|_{L^\infty([0,T];H^s(\mathbb{T}^3))} = +\infty.$$

One may even prove that the pathological set S contains a dense G_δ set and consequently the good-data set is not generic in the sense of Baire.

3.5 Going even further

For $\alpha < 3/2$, u_0^ω is no longer a classical function. In this case, it can be interpreted as a distribution belonging to a Sobolev space with negative index. We cannot expect a result like that of Theorem 3.4 for $\alpha < 3/2$. A renormalization is necessary, as shown by the following result established in [19].

Theorem 3.6. *Let $\alpha \in (\frac{5}{4}, \frac{3}{2})$ and $s < \alpha - 3/2$. There exists positive constants γ, c, C, T_0 and a divergent sequence $(c_N)_{N \geq 1}$ such that for any $T \in (0, T_0)$, there exists a set Ω_T such that the probability of its complement is smaller than $C \exp(-c/T^\gamma)$ and such that if we denote by $(u_N^\omega)_{N \geq 1}$ the solution of*

$$(\partial_t^2 - \Delta)u_N^\omega - c_N u_N^\omega + (u_N^\omega)^3 = 0$$

with initial data given by (14), then for any $\omega \in \Omega_T$, the sequence $(u_N)_{N \geq 1}$ converges for $N \to +\infty$ in $C^0([-T, T]; H^s(\mathbb{T}^3))$. In particular, for almost every ω there exists $T_\omega > 0$ such that $(u_N^\omega)_{N \geq 1}$ converges in $C^0([-T_\omega, T_\omega]; H^s(\mathbb{T}^3))$.

Theorem 3.6 was a first step in the study of problem (6) in Sobolev spaces of negative index. In the remarkable recent work [4] the result of Theorem 3.6 was extended to the range $\alpha > 1$.

4 Invariant measures for the nonlinear Schrödinger equation

Let us now consider the nonlinear Schrödinger equation, posed on the torus of dimension two:

$$(i\partial_t + \Delta)u - |u|^2 u = 0, \quad u(0, x) = u_0(x), \quad x \in \mathbb{T}^2. \qquad (15)$$

Here the solution u is complex valued, but the equation is of first order in time. We have the following analogue of Theorem 3.1 in the context of (15).

Theorem 4.1. *For any $u_0 \in H^1(\mathbb{T}^2)$ there exists a unique global solution of (15) in the class $C^0(\mathbb{R}; H^1(\mathbb{T}^2))$. If moreover $u_0 \in H^s(\mathbb{T}^2)$ for some $s \geq 1$, then $u \in C^0(\mathbb{R}; H^s(\mathbb{T}^2))$. The dependence on the initial data is also continuous.*

Equation (15) is again Hamiltonian in nature. This implies that the functional

$$E(u) = \int_{\mathbb{T}^2} \Big(|\nabla_x u(t,x)|^2 + |u(t,x)|^2 + \frac{1}{2}|u(t,x)|^4 \Big) dx \qquad (16)$$

is preserved by (15). The Gibbs measure associated with (16) is a "renormalization" of the completely formal object

$$\exp(-E(u))du. \qquad (17)$$

This renormalization is a classic procedure in quantum field theory, which would be impossible to present in this short text. Let us just say that the measure obtained by this renormalization is absolutely continuous with respect to the Gaussian measure induced by the random series

$$u_0^\omega(x) = \sum_{n \in \mathbb{Z}^2} \frac{g_n(\omega)}{\langle n\rangle} e^{in \cdot x}, \qquad (18)$$

where $(g_n)_{n \in \mathbb{Z}^2}$ is a family of independent complex Gaussians with distribution $\mathcal{N}_{\mathbb{C}}(0, 1)$. Once the measure (17) has been rigorously defined, the natural question is whether we can define a dynamics related to (15) that leaves this measure invariant. The answer to this question is given by the work of Bourgain [3]. The difficulty lies in the fact that (18) does not define a classical function. The object defined by (18) almost surely belongs to the Sobolev space $H^s(\mathbb{T}^2)$ for all $s < 0$. Such a regularity implies that Theorem 4.1

cannot be applied in the context of an initial datum given by (18). This regularity is also beyond the reach of the most sophisticated deterministic techniques. Nevertheless, the following statement can be deduced from [3].

Theorem 4.2. *Using Theorem* 4.1*, define the sequence* $(u_N^\omega)_{N\geq 1}$ *of solutions of* (15) *for the initial conditions of class* C^∞ *given by*

$$u_{0,N}^\omega(x) = \sum_{|n|\leq N} \frac{g_n(\omega)}{\langle n\rangle} e^{in\cdot x}. \tag{19}$$

Then for any $s<0$ *the sequence*

$$\Big(\exp\Big(\frac{it}{2\pi^2}\|u_N^\omega(t)\|_{L^2}^2\Big)u_N^\omega(t)\Big)_{N\geq 1}$$

converges almost surely in $C^0(\mathbb{R};H^s(\mathbb{T}^2))$ *to a limit which satisfies (in the sense of distributions) a renormalized version of problem* (15)*.*

There is a similarity between Theorems 3.4 and 4.2. One notable difference is the need to renormalize the sequence $(u_N^\omega)_{N\geq 1}$ of Theorem 4.2 in order to obtain a limit. This renormalization is linked to the construction of the measure from the formal object (17) mentioned above.

The result of Theorem 4.2 can be formulated in the spirit of Theorem 3.6. More precisely, we can establish the convergence of the solutions of the problem

$$(i\partial_t+\Delta)u_N + c_N u_N - |u_N|^2 u_N = 0$$

with initial data (19), where $(c_N(\omega))_{N\geq 1}$ is a sequence of real numbers almost surely divergent to $+\infty$.

5 Singular stochastic PDEs

The issues considered in the previous sections are very close to the analysis of PDE in the presence of a singular random source (noise). This topic has received a lot of attention in recent years (see, for example, [7, 9–12, 14]). The closest equation to those in the previous sections is the nonlinear heat equation. More precisely, we consider the problem

$$\partial_t u - \Delta u + u^3 = \xi, \quad u(0,x)=0, \quad x\in\mathbb{T}^3. \tag{20}$$

In this equation, ξ is the space-time white noise on $[0,+\infty[\times\mathbb{T}^3$. The unknown u is a real valued function. There are many physical motivations for considering a PDE perturbed by white noise. A serious discussion of these motivations is beyond the scope of this paper.

The source term ξ represents the singular randomness in (20), while in (6) and (15), the initial datum is the source of the singular randomness. A little experience with the analysis of evolutionary PDE is enough to know that the two situations are very similar and even that, in some cases, for reasons of convenience, we can easily transform the problem with initial data into a problem with a source term and zero as initial data.

A representation in the spirit of (13) and (18) of white noise on $[0,+\infty[\times\mathbb{T}^3$ is given by the formula

$$\xi = \sum_{n\in\mathbb{Z}^3} \dot\beta_n(t)e^{in\cdot x}, \tag{21}$$

where β_n are independent Brownian motions, conditioned by $\beta_n=\overline{\beta_{-n}}$ (β_0 is real and for $n\neq 0$, β_n is complex valued). The derivative with respect to t of β_n in (21) is in the sense of distributions.

If $\xi\in C^\infty([0,\infty[\times\mathbb{T}^3)$, equation (20) can be solved by deterministic methods. This is the analogue of Theorems 3.1 or 4.1 in the context of (20). For $N\gg 1$, an approximation of ξ given by (21) by smooth functions is defined by

$$\xi_N(t,x) = \rho_N \star \xi,$$

where $\rho_N(t,x)=N^5\rho(N^2t,Nx)$, with ρ a test function with integral 1 on $[0,\infty[\times\mathbb{T}^3$. This is a convolution regularization, very similar to the regularizations used in (14) and (19). The following statement can be deduced from [12, 17].

Theorem 5.1. *There exists a sequence* $(c_N)_{N\geq 1}$ *of positive numbers, divergent when* $N\to\infty$*, such that if we denote by* u_N *the solution of the problem*

$$\partial_t u_N - \Delta u_N - c_N u_N + u_N^3 = \xi_N, \quad u_N(0,x)=0, \quad x\in\mathbb{T}^3,$$

then $(u_N)_{N\geq 1}$ *converges in law when* $N\to\infty$*.*

It is also possible to have almost sure convergence in suitable Hölder spaces. The initial datum $u(0,x)$ can be non-zero: it just has to belong to a well-chosen function space (see [17]).

The complete analogue of (13) and (18) in the context of problem (20) would be the white noise on $\mathbb{T}\times\mathbb{T}^3$ defined by

$$\xi = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}^3} g_{m,n}(\omega)e^{imt}e^{in\cdot x}, \tag{22}$$

where $(g_{m,n})_{(m,n)\in\mathbb{Z}^4}$ is a family of independent standard Gaussian variables conditioned so that ξ is real valued. The result of Theorem 5.1 remains true for a noise ξ of the form (22).

There are other parabolic PDEs for which a result in the spirit of Theorem 5.1 can be obtained, perhaps the most famous example being the KPZ equation (see [11]).

6 Final discussion

The statements of Theorems 3.4, 3.6, 4.2 and 5.1 are similar. Their proofs also follow the same pattern. First, local in time solutions are constructed. Then we use global information, which is either an invariant measure or an energy estimate, to move towards global in time solutions.

To construct local solutions, we look for solutions in the form

$$u = u_1 + u_2,$$

where u_1 contains the singular part of the solution.

By probabilistic arguments, very close to the considerations in Section 2, u_1 and some maps related to it have better properties than those given by deterministic methods. The whole probabilistic machinery is to be found in this part of the analysis. In the proof of Theorem 3.4, we use almost sure improvements of the Sobolev embedding, while in the proofs of Theorems 3.6, 4.2 and 5.1, we construct almost sure products in Sobolev spaces of negative index.

Next, we solve the problem for u_2 using deterministic arguments. Here the nature of the equation becomes even more important. In Theorem 5.1, the basic tool is elliptic regularity, whereas in Theorems 3.4, 3.6 and 4.2 we make crucial use of oscillations in time (captured by Bourgain spaces, for example).

The transition to global in time solutions in Theorem 4.2 uses an invariant measure as a global control over the solutions. In Theorem 3.4 the globalization of solutions is done by an argument based on energy estimates. It is remarkable that, in the context of Theorem 5.1, we can also use these two methods to globalize local solutions: in [13] the globalization is done using a control coming from an invariant measure, whereas the work [17] uses the (much more flexible) method of energy estimates.

Oh's work [18] establishes the analogue of Theorem 3.2 in the context of Theorem 4.2. To the best of my knowledge, no such amplification result for particular approximations is known in the context of Theorem 5.1.

We have already mentioned that, in Theorem 3.4, we allow more general randomizations compared to Theorem 4.2. This has made it possible to consider randomizations for functions of Sobolev spaces on the whole space $\mathbb{R}^d$ and to prove results in the spirit of Theorem 3.4 for problems posed on the whole space (instead of on the torus). For work in this direction, see [2, 16].

Theorem 3.4 allows more general randomizations than Theorem 4.2, but it says nothing about the transport by the flow of the measure defining the initial data set (whereas the proof of Theorem 4.2 tells us that the initial Gaussian measure is quasi-invariant under the flow). We still do not know the nature of the measure transported by the flow in the context of Theorem 3.4 (see [20] for recent progress on this interesting problem).

The list of references below is far from complete. This is a very active field. For a description of other results directly related to what we have just described, we refer the reader to [9, 11, 24].

Acknowledgements. The *EMS Magazine* thanks *La Gazette des Mathématiciens* for authorization to republish this text, which is an English translation of the paper entitled "EDP non linéaires en présence d'aléa singulier" and published in *La Gazette des Mathématiciens*, **160**, pp. 6–14, 2019.

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