

## Remarks on the Limiting Gibbs States on a $(d+1)$ -Tree

By

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### §1. Introduction

In this paper we investigate limiting Gibbs states with nearest neighbour ferromagnetic potentials on a  $(d+1)$ -tree. Preston [1] has got a necessary and sufficient condition for the non-uniqueness of the Gibbs states for a given interaction (i.e. the necessary and sufficient condition for the phase transition to occur) in these models. The Gibbs states on a countable tree are studied in [1] and [2]. Our aim is to obtain several limiting Gibbs states by changing boundary conditions.

Spitzer [2] has shown that (i) every extremal Gibbs state invariant under graph isomorphisms is a “Markov chain” in the sense of his definition (see Definition 1), and (ii) there are at most three “Markov chains” among the Gibbs states for any given nearest neighbour ferromagnetic potential. In section 4 we will prove that every “Markov chain” which is Gibbsian for the given interaction is obtained as a limiting Gibbs state for the same interaction with certain boundary conditions.

In section 5 we will give examples of limiting Gibbs states such that the number of up-spins appearing in the corresponding boundary conditions is much smaller than that of down-spins on every boundary, while the probability for the spin at the origin to be up is larger than  $1/2$ .

In section 6 we will give several extremal Gibbs states using above examples.

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Section 2 and 3 are devoted to the preliminary definitions and discussions.

## §2. Definitions and Notations

Let  $T$  be a  $(d+1)$ -tree with a positive integer  $d \geq 2$ . Two points  $t, s \in T (t \neq s)$  are called to be neighbours if they are connected by one bond. Each point of  $T$  has  $d+1$  neighbours. We fix an origin  $0$  of  $T$ . We write  $t \rightarrow s$  if  $t \neq s$  and the path connecting  $0$  and  $s$  passes  $t$ . If  $t \rightarrow s$  and  $t, s$  are neighbours, we write  $t \rightarrow_1 s$ . For any finite  $V \subset T$ , the boundary  $\partial V$  of  $V$  is

$$\partial V = \{t \in T \setminus V; \text{there exists a neighbour of } t \text{ in } V\}.$$

For every  $V \subset T$ , let  $\Sigma_V = \{-1, +1\}^V$  be the space of all possible spin configurations on  $V$ . In particular we write  $\Sigma$  instead of  $\Sigma_T$ . We assign the product topology to  $\Sigma$ . For every  $V \subset T$ , define the  $\sigma$ -algebra  $\mathcal{B}_V$  by

$$\mathcal{B}_V = \text{the } \sigma\text{-algebra generated by } \{X_t, t \in V\},$$

where  $X_t(\sigma) = \sigma(t)$  for all  $t \in T, \sigma \in \Sigma$ . In particular we write  $\mathcal{B}$  instead of  $\mathcal{B}_T$ .

Let  $V$  be a finite subset of  $T, \omega \in \Sigma$  and  $\sigma \in \Sigma_V$ . We define an interaction energy on  $V$  with the inner configuration  $\sigma$  and the boundary condition  $\omega$  by

$$(1) \quad E_V^\omega(\sigma) = -H \sum_{t \in V} \sigma(t) - J \sum_{t, s \in V; t \rightarrow_1 s} \sigma(t)\sigma(s) \\ - J \sum_{t \in V, s \in \partial V; t, s \text{ are neighbours}} \sigma(t)\omega(s),$$

where  $H$  is a real number called the external magnetic field and  $J$  is a positive number called the ferromagnetic nearest neighbour potential. A finite Gibbs state  $P_V^\omega$  on  $\Sigma_V$  corresponding to  $E_V^\omega$  is defined by

$$(2) \quad P_V^\omega(\sigma) = [Z_V^\omega]^{-1} \exp[-E_V^\omega(\sigma)] \quad \sigma \in \Sigma_V,$$

where  $Z_V^\omega = \sum_{\sigma \in \Sigma_V} \exp[-E_V^\omega(\sigma)]$ . As usual,  $P_V^\omega$  can be considered as a probability measure on  $(\Sigma, \mathcal{B})$ .

For fixed  $H$  and  $J$ , if there is an increasing sequence of finite subsets  $\{V_n\}$  such that  $V_n \nearrow T$  as  $n \rightarrow \infty$  and  $P^\omega = \text{w-lim}_{n \rightarrow \infty} P_{V_n}^\omega$  (the weak convergence of measures) exists for suitably fixed  $\omega \in \Sigma$ , then  $P^\omega$  is called a limiting Gibbs state with boundary condition  $\omega$  for  $(H, J)$ . On the other hand, a Gibbs state  $P$  for  $(H, J)$  is defined as a probability measure on  $(\Sigma, \mathcal{B})$  such that for every  $A$  in  $\mathcal{B}_{V_n}$ ,

$$(3) \quad P(A|\mathcal{B}_{V_n})(\omega) = P_{V_n}^\omega(A) \quad a.s. (P).$$

It is well known that the set  $\mathcal{G}(H, J)$  of all Gibbs states for any fixed  $(H, J)$  is a non-empty, compact convex set. A limiting Gibbs state is a Gibbs state for the same  $(H, J)$ . Conversely, every extremal point of  $\mathcal{G}(H, J)$  is a limiting Gibbs state with a suitable boundary condition for the same  $(H, J)$ .

### §3. Limiting Gibbs States

Let  $|t|$  denote the distance between 0 and  $t \in T$ , i.e.  $|t| = n$  if there exists a chain  $0 \xrightarrow{1} u_1 \xrightarrow{1} u_2 \xrightarrow{1} \dots \xrightarrow{1} u_{n-1} \xrightarrow{1} t$ . From now on we only consider the sequence of boxes  $V_n = \{t \in T; |t| \leq n\}$ ,  $n \geq 1$ . For every  $s \in T$  we define  $T_s = \{s\} \cup \{t \in T; s \rightarrow t\}$  and  $V_{n,s} = T_s \cap V_n$ ,  $n \geq 1$ . The members of  $V_1 \setminus \{0\}$  are numbered by  $1, 2, \dots, d+1$ . For every  $\omega \in \Sigma$ ,  $s \in T \setminus \{0\}$  and  $n \geq |s|$ , put

$$(4) \quad W_{n,s}^\omega(\varepsilon) = \sum_{\sigma \in \Sigma_{V_{n,s}}; \sigma(s) = \varepsilon} \exp[-E_{V_{n,s}}^\omega(\sigma) - J\varepsilon\omega(t)], \quad \varepsilon = \pm 1,$$

$$(5) \quad R_{n,s}(\omega) = W_{n,s}^\omega(+1)/W_{n,s}^\omega(-1),$$

where  $t$  is the unique point such that  $t \xrightarrow{1} s$ . Then we get

$$(6) \quad \begin{aligned} P_{V_n}^\omega(\sigma(0) = +1)/P_{V_n}^\omega(\sigma(0) = -1) \\ = e^{2H} \prod_{i=1}^{d+1} [e^J R_{n,i}(\omega) + e^{-J}] / [e^{-J} R_{n,i}(\omega) + e^J], \end{aligned}$$

$$(7) \quad W_{n,s}^\omega(\varepsilon) = e^{\varepsilon H} \prod_{u; s \xrightarrow{1} u} [e^{\varepsilon J} W_{n,u}^\omega(+1) + e^{-\varepsilon J} W_{n,u}^\omega(-1)], \quad \varepsilon = \pm 1$$

and for  $n > m$ ,  $\eta \in \Sigma_{V_m}$ ,

$$(8) \quad P_{V_n}^\omega(\{\sigma(s)=\eta(s), s \in V_m\}) \\ = \frac{\exp[-E_{V_{m-1}}^\eta(\eta)] \prod_{s \in \partial V_{m-1}} W_{n,s}^\omega(\eta(s))}{\sum_{\xi \in \Sigma_{V_m}} \exp[-E_{V_{m-1}}^\xi(\xi)] \prod_{s \in \partial V_{m-1}} W_{n,s}^\omega(\xi(s))}.$$

From the above equalities we obtain easily the following

**Lemma 1.** *Let  $\omega \in \Sigma$  be given. If there is  $N > 0$  such that  $R_{n,s}(\omega)$  converges as  $n \rightarrow \infty$  for every  $s \in T \setminus V_N$ , then  $P^\omega = \lim_{n \rightarrow \infty} P_{V_n}^\omega$  exists.*

Now, let us consider a boundary condition  $\omega$  such that  $c(\omega) = \sum_{s; t \rightarrow s} \omega(s)$  is independent of  $t \in T \setminus \{0\}$ . Then it is easy to see that  $R_n(\omega) = \bar{R}_{n,i}(\omega)$  is independent of  $i = 1, 2, \dots, d+1$ , and  $R_{n,s}(\omega) = R_{n-|s|+1}(\omega)$ . Hence we get

$$(9) \quad Y_n(\omega) = H + df(Y_{n-1}(\omega))$$

from (7), where  $Y_n(\omega) = \frac{1}{2} \log R_n(\omega)$ , and

$$(10) \quad f(x) = \text{th}^{-1}(\text{th } J \cdot \text{th } x) = \frac{1}{2} \log \left[ \frac{\text{ch}(x+J)}{\text{ch}(x-J)} \right],$$

which is a concave increasing function on  $[0, \infty)$  and  $f(-x) = -f(x)$  for all  $x$  (see [1], [3]).

The following is obtained from (9) and FKG inequality.

**Theorem (Preston [1]).** *For fixed  $(H, J)$ , the following (i)~(iii) are equivalent:*

(i) *There exist two different Gibbs states (i.e. a phase transition occurs).*

(ii) *The equation*

$$(11) \quad x = H + df(x)$$

*has more than one solutions.*

(iii)  *$J > \frac{1}{2} \log \left[ \frac{d+1}{d-1} \right]$  and  $|H| \leq df(u) - u$ ,*

*where  $u$  is the unique positive value such that  $f'(u) = 1/d$ .*

The equation (11) has at most three solutions, say  $\alpha \leq \beta \leq \gamma$ . Preston [1] has shown that  $Y_n(\omega^+) \searrow \gamma$  and  $Y_n(\omega^-) \nearrow \alpha$  as  $n \rightarrow \infty$ , where  $\omega^+, \omega^- \in \Sigma$  are defined by

$$(12) \quad \omega^+(t) = +1, \omega^-(t) = -1 \quad \text{for all } t \in T.$$

Since (9) implies that the asymptotic behavior of  $Y_n(\omega)$  depends only on  $Y_1(\omega)$ , we get

$$(13) \quad \lim_{n \rightarrow \infty} Y_n(\omega) = \begin{cases} \alpha & \text{if } H + Jc(\omega) < \beta, \\ \beta & \text{if } H + Jc(\omega) = \beta \\ \gamma & \text{if } H + Jc(\omega) > \beta. \end{cases}$$

Combining Lemma 1 and (13), we have

$$P^\omega = \begin{cases} P^{\omega^+} & \text{if } H + Jc(\omega) > \beta, \\ P^{\omega^-} & \text{if } H + Jc(\omega) < \beta. \end{cases}$$

In general,  $c_t(\omega) = \sum_{s; t \xrightarrow{1} s} \omega(s)$  depends on  $t \in T \setminus \{0\}$ , but from FKG inequality we have

**Proposition 1.** *For a general  $\omega \in \Sigma$ ,*

$$P^\omega = \begin{cases} P^{\omega^+} & \text{if } H + Jc_t(\omega) > \beta \quad \text{for all } t \in T \setminus \{0\}, \\ P^{\omega^-} & \text{if } H + Jc_t(\omega) < \beta \quad \text{for all } t \in T \setminus \{0\}. \end{cases}$$

#### §4. Markov Chains

In this section we construct the third “Markov chain”.

**Definition 1** (Spitzer [2]). A probability measure  $P$  on  $(\Sigma, \mathcal{B})$  is called a “Markov chain” if there is a  $2 \times 2$  positive stochastic matrix  $Q$  with its invariant probability measure  $\pi$  ( $\pi Q = \pi$ ) such that for any connected finite subset  $V$  of  $T$  and for any  $\varepsilon \in \Sigma_V$ ,

$$P(\{\sigma(t) = \varepsilon(t), t \in V\}) = \pi(\varepsilon(t_0)) \prod_{\substack{t, t' \in V \\ t \xrightarrow{1} t'}} Q(\varepsilon(t), \varepsilon(t')),$$

where  $t_0$  is the point such that  $t_0 \in V$  and  $t_0 \rightarrow t$  for every  $t \in V \setminus \{t_0\}$ .

We remark that the above definition does not depend on the choice of the origin because  $(Q, \pi)$  is reversible.

Let  $\mathcal{M}$  be the set of all “Markov chains”

**Theorem** (Spitzer [2]). (i) *If  $P$  is an extremal point of  $\mathcal{G}(H, J)$  and invariant under graph isomorphisms (i.e. bijections of  $T$  preserving the relation of neighbouring), then  $P \in \mathcal{M}$ .*

(ii) *There are at most three elements in  $\mathcal{M} \cap \mathcal{G}(H, J)$  for every  $(H, J)$ .*

Obviously  $P^{\omega^+}$  and  $P^{\omega^-}$  are “Markov chains” from the above theorem (i). As for the third element of  $\mathcal{M} \cap \mathcal{G}(H, J)$ , at first we consider the case when  $H=0$  (then  $\beta=0$ ) and  $d$  is even. In this case, putting  $\omega^0$  to satisfy  $c_t(\omega^0)=0$  for all  $t \in T \setminus \{0\}$ , we have  $\lim_{n \rightarrow \infty} R_{n,t}(\omega^0)=1$  for all  $t \in T \setminus \{0\}$ . Hence from Lemma 1 and (6), we have the existence of  $P^{\omega^0}$  different from  $P^{\omega^+}$  and  $P^{\omega^-}$  when  $J > \frac{1}{2} \log [(d+1)/(d-1)]$ . In general, we have

**Theorem 1.** *Let  $\alpha \leq \beta \leq \gamma$  be the possible solutions of (11). Then we have the followings:*

(i) *If three sequences  $\{d_n\}$ ,  $\{a_n\}$ ,  $\{\tilde{a}_n\}$  of real numbers satisfy*

$$(14-a) \quad 1 \leq d_n \leq d, \quad d_n \text{ is an integer,}$$

$$(14-b) \quad a_n = H + d_n f(a_{n-1}) + (d - d_n) f(\tilde{a}_{n-1}) \geq \beta,$$

$$(14-c) \quad \tilde{a}_n = H + (d_n - 1) f(a_{n-1}) + (d - d_n + 1) f(\tilde{a}_{n-1}) < \beta,$$

*for every  $n \geq 1$ , then  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \tilde{a}_n = \beta$ .*

(ii) *Let us take the initial conditions;*

$$d_1 = \min \{k; k \text{ is an integer and } H + (2k - d)J \geq \beta\},$$

$$a_1 = H + (2d_1 - d)J, \quad \tilde{a}_1 = H + (2d_1 - 2 - d)J.$$

*Then a triple of sequences  $\{d_n\}$ ,  $\{a_n\}$ ,  $\{\tilde{a}_n\}$  can be defined inductively to satisfy (14). Moreover, there is a configuration  $\omega^0 \in \Sigma$  such that  $Y_{n,s}(\omega^0) = a_{n-|s|+1}$  or  $\tilde{a}_{n-|s|+1}$  for every  $s \in T \setminus \{0\}$ , where*

$$(15) \quad Y_{n,s}(\omega) = \frac{1}{2} \log R_{n,s}(\omega).$$

(iii) The limiting Gibbs state  $P^{\omega^0}$  exists and  $\mathcal{M} \cap \mathcal{G}(H, J) = \{P^{\omega^+}, P^{\omega^-}, P^{\omega^0}\}$  for all  $H$  and  $J$ .

*Proof.* (i) From (14) and the property of  $f(x)$ , we have  $0 < a_n - \tilde{a}_n = f(a_{n-1}) - f(\tilde{a}_{n-1}) \leq 2f((a_{n-1} - \tilde{a}_{n-1})/2)$ . Since  $f(x) \leq x$  for  $x \geq 0$  and the equality holds if and only if  $x=0$ , we get  $\lim_{n \rightarrow \infty} (a_n - \tilde{a}_n) = 0$ , which implies  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \tilde{a}_n = \beta$ .

(ii) The first part is easy to verify. To construct  $\omega^0$ , we define  $\omega_n^0, \tilde{\omega}_n^0 \in \Sigma_{\partial V_{n,1} \setminus \{0\}}$  inductively. Let  $\partial V_{1,1} \setminus \{0\} = \{s_1, \dots, s_d\}$ . We define  $\omega_1^0$  and  $\tilde{\omega}_1^0$  on  $\partial V_{1,1} \setminus \{0\}$  as  $\sum_{1 \leq v \leq d} \omega_1^0(s_v) = 2d_1 - d$ , and  $\sum_{v=1}^d \tilde{\omega}_1^0(s_v) = 2d_1 - d - 2$ . Let  $\tau_v$  be a shift operator from  $T_{s_v}$  to  $T_1$  such that  $\tau_v(s_v) = 1 \in T_1$  for every  $1 \leq v \leq d$ . Then we can define  $\omega_n^0$  and  $\tilde{\omega}_n^0$  on  $\partial V_{n,1} \setminus \{0\}$  by

$$\omega_n^0(t) = \begin{cases} \omega_{n-1}^0(\tau_v(t)) & \text{if } t \in T_{s_v} \cap \partial V_n, & 1 \leq v \leq d_n, \\ \tilde{\omega}_{n-1}^0(\tau_v(t)) & \text{if } t \in T_{s_v} \cap \partial V_n, & d_n + 1 \leq v \leq d, \end{cases}$$

$$\tilde{\omega}_n^0(t) = \begin{cases} \omega_{n-1}^0(\tau_v(t)) & \text{if } t \in T_{s_v} \cap \partial V_n, & 1 \leq v \leq d_n - 1, \\ \tilde{\omega}_{n-1}^0(\tau_v(t)) & \text{if } t \in T_{s_v} \cap \partial V_n, & d_n \leq v \leq d. \end{cases}$$

Now put  $\omega^0(t) = \omega_n^0(t)$  for  $t \in \partial V_{n,1} \setminus \{0\}, n \geq 1$  and extend it to be invariant under the rotations around the origin. Then it is easy to see that  $\omega^0$  is a required configuration.

(iii) Since  $Y_{n,s}(\omega^0)$  converges as  $n \rightarrow \infty$  for every  $s \in T \setminus \{0\}$ , Lemma 1 implies that  $P^{\omega^0}$  exists. As for the ‘‘Markov chain’’ property, we will only give here the transition matrix  $Q$  and its invariant probability measure  $\pi$ .

$$Q = \begin{pmatrix} e^{2(\beta+J)}[e^{2(\beta+J)}+1]^{-1} & [e^{2(\beta+J)}+1]^{-1} & +1 \\ e^{2(\beta-J)}[e^{2(\beta-J)}+1]^{-1} & [e^{2(\beta-J)}+1]^{-1} & -1 \\ +1 & -1 & \end{pmatrix}$$

$$\pi = \begin{pmatrix} [e^{2(\beta-J)}+e^{4\beta}]/[e^{4\beta}+2e^{2(\beta-J)}+1] & +1 \\ [e^{2(\beta-J)}+1]/[e^{4\beta}+2e^{2(\beta-J)}+1] & -1 \end{pmatrix}$$

Now we will prove the equality  $\mathcal{M} \cap \mathcal{G}(H, J) = \{P^{\omega^+}, P^{\omega^-}, P^{\omega^0}\}$  for all

$H$  and  $J$ . At first, assume that  $J \leq \frac{1}{2} \log [(d+1)/(d-1)]$ , then from Preston's theorem in §3,  $\mathcal{G}(H, J)$  consists of a single point. Hence  $P^{\omega^+} = P^{\omega^-} = P^{\omega^0}$ . Thus in this case,

$$\mathcal{M} \cap \mathcal{G}(H, J) = \{P^{\omega^+}, P^{\omega^-}, P^{\omega^0}\} (= \{P^{\omega^+}\}).$$

Let us fix  $J > \frac{1}{2} \log [(d+1)/(d-1)]$  arbitrarily. If  $|H| > df(u) - u$ , where  $u$  is the positive solution of  $f'(u) = 1/d$ , then from the same reason as above,  $\mathcal{M} \cap \mathcal{G}(H, J) = \{P^{\omega^+}, P^{\omega^-}, P^{\omega^0}\} (= \{P^{\omega^+}\})$ . If  $|H| < df(u) - u$ , then the equation (11) has exactly three solutions  $\alpha < \beta < \gamma$ , so  $P^{\omega^+}, P^{\omega^-}, P^{\omega^0}$  are distinct. From Spitzer's theorem  $\mathcal{M} \cap \mathcal{G}(H, J)$  has at most three elements, hence  $\mathcal{M} \cap \mathcal{G}(H, J) = \{P^{\omega^+}, P^{\omega^-}, P^{\omega^0}\}$ . If  $H = df(u) - u$ , then  $\alpha = \beta \neq \gamma$ , i.e.  $P^{\omega^+} \neq P^{\omega^0} = P^{\omega^-}$ . If  $H = u - df(u)$ , then  $\alpha \neq \beta = \gamma$ , i.e.  $P^{\omega^+} = P^{\omega^0} \neq P^{\omega^-}$ . Anyway, if  $|H| = df(u) - u$ ,  $\{P^{\omega^+}, P^{\omega^-}, P^{\omega^0}\} = \{P^{\omega^+}, P^{\omega^-}\} =$  the two point set. But we must note that Spitzer has shown in his paper [2] that the number of elements of  $\mathcal{M} \cap \mathcal{G}(H, J)$  is equal to the number of  $t$ -solutions in  $[0, 1]$  of the equation

$$(*) \quad tx^2 + 1 - t = \left(\frac{1-t}{t}\right)^{1/d} x(xy)^{-1/d}$$

with  $x = e^{-2J}, y = e^{-2H}$ . When  $H$  varies monotonously from  $-\infty$  to  $\infty$ , only the value of  $y$  in (\*) changes and it changes from 0 to  $\infty$  monotonously. Hence the number of solutions of (\*) changes  $1 \rightarrow 2 \rightarrow 3 \rightarrow 2 \rightarrow 1$  as  $H$  increases from  $-\infty$  to  $\infty$ . This means that the number of elements of  $\mathcal{M} \cap \mathcal{G}(H, J)$  is equal to 2 if  $|H| = df(u) - u$ . Thus we have  $\mathcal{M} \cap \mathcal{G}(H, J) = \{P^{\omega^+}, P^{\omega^-}, P^{\omega^0}\}$  for all  $H$  and  $J$ . Q.E.D.

*Remark.* Spitzer [2] has claimed that a "Markov chain" is an extremal Gibbs state, but it is not true. In fact T. Kamae [4] has shown that if  $H=0$ , then  $\mathcal{B}_\infty = \bigcap_{V; \text{finite}} \mathcal{B}_{V^c}$  is not trivial with respect to  $P^{\omega^0}$ . We will briefly quote it here. Put  $f_n(\sigma) = \sum_{t \in \partial V_n} \sigma(t)$ . Then  $f_n(\sigma)$  is  $\mathcal{B}_{V_n^c}$ -measurable and  $\|f_n\|_{2, \omega^0}^2 = E^{\omega^0}[f_n^2] \leq (d+1)d^{n-1} \sum_{k=0}^n [d(2p-1)^2]^k, \langle f_n \rangle_{\omega^0} = E^{\omega^0}(f_n) = 0, \langle f_n, X_0 \rangle_{\omega^0} = E^{\omega^0}[f_n X_0] = (d+1)d^{n-1}(2p-1)^n,$  where  $p = Q(+1, +1) = Q(-1, -1) = e^{2J}[e^{2J} + 1]^{-1}$ . Hence if  $d(2p-1)^2 > 1$ ,

$$(**) \quad \langle f_n / \|f_n\|_{2, \omega^0}, X_0 \rangle_{\omega^0} \geq C(d, p) > 0 \quad \text{for all } n \geq 1,$$

where  $C(d, p)$  is a positive constant depending only on  $d$  and  $p$ . But (\*\*) implies that

$$\lim_{n \rightarrow \infty} \sup_{\substack{f; \mathcal{B}_{V_n^0} \text{ m'ble} \\ \|f\|_2, \omega^0 \leq 1}} |\langle f, X_0 \rangle_{\omega^0} - \langle f \rangle_{\omega^0} \langle X_0 \rangle_{\omega^0}| \neq 0,$$

which means that  $\mathcal{B}_\infty$  is non-trivial with respect to  $P^{\omega^0}$ .

§5. Small Order Perturbations

Let us fix  $k, 1 \leq k \leq d-1$ , and a connected subgraph  $\tilde{T} \subset T$  such that (i)  $\tilde{T} \supset V_1$  and (ii) every  $t \in \tilde{T} \setminus \{0\}$  has exactly  $k+1$  neighbours in  $\tilde{T}$ . Then it is easy to see that  $\lim_{n \rightarrow \infty} \#(\tilde{T} \cap \partial V_n) / \#(\partial V_n) = 0$ . Define  $\omega^k \in \Sigma$  by

$$\omega^k(t) = \begin{cases} +1 & \text{if } t \in \tilde{T}, \\ -1 & \text{if } t \in T \setminus \tilde{T}. \end{cases}$$

**Lemma 2.**  $P^{\omega^k} = \text{w-lim}_{n \rightarrow \infty} P_{V_n}^{\omega^k}$  exists.

*Proof.* Note that

$$Y_{n,i}(\omega^k) = kf(Y_{n-1,i}(\omega^k)) + (d-k)f(Y_{n-1,i}(\omega^-)) + H$$

for every  $1 \leq i \leq d+1$  and  $n \geq 2$ , where  $f(x)$  and  $Y_{n,i}(\omega)$  are defined by (10) and (15) respectively. Moreover, we have

$$(16) \quad Y_{n,s}(\omega^k) = \begin{cases} Y_{n-|s|+1,1}(\omega^k) & \text{if } s \in \tilde{T} \setminus \{0\}, \\ Y_{n-|s|+1}(\omega^-) & \text{if } s \in T \setminus \tilde{T}. \end{cases}$$

Combining Lemma 1 and (16), we have only to show that  $Y_{n,1}(\omega^k)$  converges as  $n \rightarrow \infty$ .

Let us define a function  $F_\lambda^H(x)$  by

$$F_\lambda^H(x) = kf(x) - (d-k)f(\lambda) + H.$$

Since  $Y_n(\omega^-) \nearrow \alpha$  as  $n \rightarrow \infty$ , for any given  $\varepsilon > 0$  there is  $n_\varepsilon > 0$  such that

$$(17) \quad F_\alpha^H(Y_{n-1,1}(\omega^k)) > Y_{n,1}(\omega^k) > F_\alpha^H(Y_{n-1,1}(\omega^k)) - \varepsilon$$

for all  $n \geq n_\varepsilon$ . The equation  $F_\lambda^H(x) = x$  has at least one and at most

three solutions. We will only consider here the case when the equation  $F_\alpha^H(x) = x$  has three solutions, say  $\delta_1 < \delta_2 < \delta_3$ . (In other cases, the proof will be easier.) Choose  $\varepsilon > 0$  so small that the equation  $F_\alpha^H(x) - \varepsilon = x$  has also three solutions  $\delta_1^\varepsilon < \delta_2^\varepsilon < \delta_3^\varepsilon$ . Note that  $\delta_1^\varepsilon \nearrow \delta_1$ ,  $\delta_2^\varepsilon \searrow \delta_2$ ,  $\delta_3^\varepsilon \nearrow \delta_3$  as  $\varepsilon \rightarrow 0$ . We will prove that (i) if  $\liminf_{n \rightarrow \infty} Y_{n,1}(\omega^k) < \delta_2$  then  $\lim_{n \rightarrow \infty} Y_{n,1}(\omega^k) = \delta_1$ , (ii) if  $\limsup_{n \rightarrow \infty} Y_{n,1}(\omega^k) > \delta_2$  then  $\lim_{n \rightarrow \infty} Y_{n,1}(\omega^k) = \delta_3$ . Assume first that  $\liminf_{n \rightarrow \infty} Y_{n,1}(\omega^k) < \delta_2$ . Then there exists  $n > n_\varepsilon$  such that  $Y_{n,1}(\omega^k) < \delta_2$ . Using (17) we get  $\delta_1^\varepsilon \leq \liminf_{n \rightarrow \infty} Y_{n,1}(\omega^k) \leq \limsup_{n \rightarrow \infty} Y_{n,1}(\omega^k) \leq \delta_1$ . Letting  $\varepsilon \searrow 0$  we obtain (i). Next, we assume  $\limsup_{n \rightarrow \infty} Y_{n,1}(\omega^k) > \delta_2 + \theta$  for some  $\theta > 0$ . Choosing  $\varepsilon > 0$  so small that  $\delta_2^\varepsilon < \delta_2 + \theta$ , we can conclude from (17) that  $\delta_3^\varepsilon \leq \limsup_{n \rightarrow \infty} Y_{n,1}(\omega^k) \leq \liminf_{n \rightarrow \infty} Y_{n,1}(\omega^k) \leq \delta_3$ . This means (ii). Q. E. D.

**Theorem 2.** *If  $H = 0$ ,  $k > d/2 + 1$  and if the equation*

$$(18) \quad F_{dJ}^0(x) = x$$

*has more than one solutions, then*

$$P^{\omega^k}(\sigma(0) = 1) = \frac{1}{1 + [\text{ch}(J - \delta_3) / \text{ch}(J + \delta_3)]^{d+1}} > 1/2.$$

*Remark.* If  $J \geq \frac{k \log 2}{2(2k - d + 1)}$ , then  $kf(J) - (d - k)J \geq J$ . Since  $|f(x)| < J$

for all  $x$ , we have  $F_{dJ}^0(J) > kf(J) - (d - k)J \geq J$ . This implies that (17) has three solutions.

*Proof of Theorem 2.* Let  $\alpha < \beta < \gamma$  be the solutions of (11) as before. Since  $H = 0$  we have  $\beta = 0$  and  $\alpha = -\gamma$ . Noting  $\gamma = df(\gamma) < dJ$ , we have  $F_\gamma^0(x) > F_{dJ}^0(x)$  for all  $x \in (-\infty, \infty)$ . Hence the assumption of the theorem implies that the equation  $F_\gamma^0(x) = x$  has three solutions  $\delta_1 < \delta_2 < \delta_3$ . Since  $\delta_3 < \gamma$  and  $F_\gamma^0(x) < kf(x) - (d - k)f(x) = (2k - d)f(x)$  for  $0 < x < \gamma$ , the equation  $(2k - d)f(x) = x$  has three solutions  $\alpha' < \beta' < \gamma'$ , and  $\delta_3 < \gamma' = (2k - d)f(\gamma') < (2k - d)J = Y_{1,1}(\omega^k)$ .

On the other hand we have

$$(19) \quad F_{dJ}^0(Y_{n-1,1}(\omega^k)) \leq Y_{n,1}(\omega^k) \leq F_\gamma^0(Y_{n-1,1}(\omega^k))$$

for all  $n \geq 2$  from (17) and  $Y_{n-1}(\omega^-) \geq -dJ$ . From Lemma 2,  $\lim_{n \rightarrow \infty} Y_{n,1}(\omega^k)$

exists and is a solution of the equation  $F_{\gamma}^0(x)=x$ . But from (19) and  $Y_{1,1}(\omega^k) > \delta_3$  we have  $\delta_2 < \delta'_3 \leq \lim_{n \rightarrow \infty} Y_{n,1}(\omega^k) \leq \delta_3$ , where  $\delta'_1 \leq \delta'_2 \leq \delta'_3$  are the solutions of the equation (18). Hence  $\lim_{n \rightarrow \infty} Y_{n,1}(\omega^k) = \delta_3 > 0$ . From this and (6), we obtain the conclusion of the theorem. Q. E. D.

**§6. Extremal Gibbs States**

In this section, we will give several extremal points of  $\mathcal{G}(H, J)$  as the limiting Gibbs states. In order to do so, we need the following theorem which is stated under a more general framework in [5].

**Theorem** (Miyamoto [5]). *Let  $P \in \mathcal{G}(H, J)$ , then  $P$  is an extremal point of  $\mathcal{G}(H, J)$  iff*

$$(20) \quad P^\omega = w\text{-}\lim_{n \rightarrow \infty} P_{\mathcal{V}_n}^\varphi = P \quad a. s. \quad (P)$$

**Lemma 3.** *The condition (20) for  $P \in \mathcal{G}(H, J)$  is equivalent to*

$$(21) \quad \lim_{n \rightarrow \infty} R_{n,s}(\omega) = r_s \quad a. s. \quad (P)$$

for each  $s \in T$  with  $|s| \geq N$  for some  $N > 0$ , where  $r_s$  is a constant depending only on  $s$ .

*Proof.* Assume (20). Then for every pair  $t, s \in T$  with  $t \rightarrow_1 s$ , we have

$$\lim_{n \rightarrow \infty} P_{\mathcal{V}_n}^\varphi(\sigma(s) = +1 | \sigma(t) = +1) = P(\sigma(s) = +1 | \sigma(t) = +1) \quad a. s. \quad (P).$$

On the other hand we have

$$P_{\mathcal{V}_n}^\varphi(\sigma(s) = +1 | \sigma(t) = +1) = e^{2J} R_{n,s}(\omega) / (e^{2J} R_{n,s}(\omega) + 1).$$

Hence we get (21) for all  $s \in T \setminus \{0\}$ . Conversely assuming (21) we get (20) from (7), (8) and (3). Q. E. D.

**Corollary.** *If  $P \in \mathcal{G}(H, J)$ , then the limit  $r_s(\omega) = \lim_{n \rightarrow \infty} R_{n,s}(\omega)$  exists almost surely with respect to  $P$  for every  $s \in T \setminus \{0\}$ .*

**Theorem 3.** *Under the assumptions of Theorem 2,  $P^{\omega^k}$  is an extremal point of  $\mathcal{G}(0, J)$ .*

*Proof.* 1° For any  $s \in T \setminus \tilde{T}$  and  $t$  with  $t \xrightarrow{1} s$ , we have  $P^{\omega^k}(A|\sigma(t) = \varepsilon) = P^{\omega^-}(A|\sigma(t) = \varepsilon)$  for all  $A \in \mathcal{B}_{T_s}$ , where  $\varepsilon = +1$  or  $-1$ . Hence we get

$$\begin{aligned} P^{\omega^k}(\lim_{n \rightarrow \infty} R_{n,s}(\sigma) = e^{2\alpha}) &= P^{\omega^k}(\sigma(t) = +1)P^{\omega^-}(\lim_{n \rightarrow \infty} R_{n,s}(\sigma) = e^{2\alpha} | \sigma(t) = +1) \\ &\quad + P^{\omega^k}(\sigma(t) = -1)P^{\omega^-}(\lim_{n \rightarrow \infty} R_{n,s}(\sigma) = e^{2\alpha} | \sigma(t) = -1) \\ &= 1. \end{aligned}$$

2° Defining  $G(x)$  by  $G(x) = (1 + e^{2J}x)/(e^{2J} + x)$ , we have from (7) and 1°),

$$\begin{aligned} (22) \quad r_t(\omega) &= \lim_{n \rightarrow \infty} R_{n,t}(\omega) = \prod_{\substack{u \in \tilde{T} \\ t \xrightarrow{1} u}} G(r_u(\omega)) \prod_{\substack{v \in T \setminus \tilde{T} \\ t \xrightarrow{1} v}} G(r_v(\omega)) \\ &= \prod_{u \in \tilde{T}} G(r_u(\omega)) G(e^{2\alpha})^{d-k} \\ &\leq G(e^{2\gamma})^{2k-d} \end{aligned}$$

almost surely with respect to  $P^{\omega^k}$ , for all  $t \in \tilde{T} \setminus \{0\}$ . (Note that  $r_u(\omega) \leq r_u(\omega^+) = e^{2\gamma}$  for all  $\omega \in \Sigma$ ,  $u \in T \setminus \{0\}$ .) Hence we have

$$(23) \quad y_t(\omega) = \lim_{n \rightarrow \infty} Y_{n,t}(\omega) = \sum_{\substack{u \in \tilde{T} \\ t \xrightarrow{1} u}} f(y_u(\omega)) - (d-k)f(\gamma)$$

almost surely with respect to  $P^{\omega^k}$ , and

$$(24) \quad y_t(\omega) \leq (2k-d)f(\gamma) = F_\gamma^0(\gamma) \quad a. s. \quad (P^{\omega^k}),$$

for every  $t \in \tilde{T} \setminus \{0\}$ . Using (23) and (24) we get

$$y_t(\omega) \leq kf(F_\gamma^0(\gamma)) - (d-k)f(\gamma) = F_\gamma^0(F_\gamma^0(\gamma)) \quad a. s. \quad (P^{\omega^k}),$$

and inductively,

$$y_t(\omega) \leq \overbrace{F_\gamma^0(F_\gamma^0(\dots(F_\gamma^0(\gamma))\dots))}^k \quad a. s. \quad (P^{\omega^k})$$

for every  $t \in T \setminus \{0\}$  and  $n \geq 1$ . This implies

$$(25) \quad y_t(\omega) \leq \delta_3 \quad a. s. \quad (P^{\omega^k})$$

for every  $t \leq T \setminus \{0\}$  because  $\delta_3 < \gamma$ , where  $\delta_3$  is the largest solution of  $F_\gamma^0(x) = x$ .

3°) Define  $A_t = \{\omega \in \Sigma; y_t(\omega) < \delta_3\}$  for  $t \in \tilde{T} \setminus \{0\}$ . If  $t, t' \in \tilde{T}$  are such that  $t \rightarrow t'$ , (23) and (24) imply  $A_{t'} \subset A_t$ . Hence  $A_t \subset A_{t'}$  if  $t \rightarrow t'$ . Since

$$\begin{aligned} P^{\omega^k}(\sigma(0) = +1 | \bigcap_n \mathcal{B}_{V_n^c})(\omega) &= \lim_{n \rightarrow \infty} P_{V_n}^{\omega^k}(\sigma(0) = +1) \\ &= \left[ \prod_{i=1}^{d+1} G(r_i(\omega)) \right] / \left[ 1 + \prod_{i=1}^{d+1} G(r_i(\omega)) \right] \leq G(e^{2\delta_3})^{d+1} / [1 + G(e^{2\delta_3})^{d+1}] \\ &= P^{\omega^k}(\sigma(0) = +1) \quad \text{a. s. } (P^{\omega^k}), \end{aligned}$$

we have

$$P^{\omega^k}(y_i(\sigma) = \delta_3, 1 \leq i \leq d+1) = 1,$$

which means  $P^{\omega^k}(A_t) = 0$  and so  $P^{\omega^k}(A_t) = 0$ , i. e.  $\lim_{n \rightarrow \infty} R_{n,t}(\omega) = e^{2\delta_3}$  a. s.  $(P^{\omega^k})$  for all  $t \in \tilde{T}$ . Combining this with 1°), Lemma 3 proves the theorem. Q. E. D.

Let us fix any  $n \geq 1$  and write the points of  $\partial V_n$  by  $u_1, u_2, \dots, u_{m(n)}$  where  $m(n) = (1+d)d^n$ . Then we have

**Theorem 4.** *Assume that for some  $n \geq 1$ ,  $P^{\omega_1}, P^{\omega_2}, \dots, P^{\omega_{m(n)}}$  are extremal points of  $\mathcal{G}(H, J)$ . If we define a new boundary condition  $\tilde{\omega}$  by*

$$\tilde{\omega}(t) = \omega_j(t) \quad \text{if } t \in T_{u_j} \quad 1 \leq j \leq m(n)$$

then  $P^{\tilde{\omega}}$  is also an extremal point of  $\mathcal{G}(H, J)$ .

*Proof.* Using the analogous argument as in the proof of the preceeding theorem, we obtain that if  $u_j \rightarrow t$  then  $r_t(\omega) = r_t(\omega_j)$  a. s.  $(P^{\tilde{\omega}})$ . Hence from Lemma 3, we get the conclusion. Q. E. D.

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