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Simple purely infinite C^*-algebras associated with normal subshifts

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Abstract. We will introduce the notion of normal subshift. A subshift (Λ, σ) is said to be normal if it satisfies a certain synchronizing property called λ-synchronizing and is infinite as a set. There are many normal subshifts such as irreducible infinite sofic shifts, Dyck shifts, and β-shifts whose associated C^*-algebras are simple and purely infinite. Eventual conjugacy of one-sided normal subshifts and topological conjugacy of two-sided normal subshifts are characterized in terms of the associated C^*-algebras and the associated stabilized C^*-algebras with their diagonals and gauge actions, respectively.

Contents

1. Introduction

In [19] (see [33,34]), W. Krieger and the author introduced the notion of λ-synchronization for subshifts. The class of λ-synchronizing subshifts contains a lot of important and interesting subshifts such as irreducible shifts of finite type, irreducible sofic shifts, synchronizing subshifts, Dyck shifts, β-shifts, substitution minimal shifts. In this paper, we will introduce the notion of normal subshift. A subshift Λ is said to be *normal* if it is a λ-synchronizing subshift and has infinite cardinality as a set. The class of normal subshifts is closed under topological conjugacy, and consists of irreducible λ-synchronizing subshifts excluding trivial subshifts. An important property of λ-synchronization is that

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each of them has a minimal λ-graph system presentation. The notion of λ-graph systems was introduced in [24] as a generalization of finite labeled graphs. Any λ-graph system presents a subshift, conversely any subshift can be presented by a λ-graph system in a canonical way. The λ-graph system that presents a subshift in a canonical way is called the canonical λ-graph system for the subshift. Besides the canonical λ-graph system, there are in general many other λ-graph systems that present a given subshift. The canonical λ-graph system corresponds to its left Krieger cover graph. We in fact see that the canonical λ-graph system for a sofic shift is the λ-graph system associated to the left Krieger cover graph. Hence the canonical λ-graph system in general does not have certain irreducibility unless the subshift is an irreducible shift of finite type. An irreducible sofic shift has an irreducible minimal presentation as a labeled graph. The presentation is called the left (or right) Fischer cover graph. It is an irreducible ergodic component of its left Krieger cover graph. To catch the Fischer cover analogue of general subshifts, we introduced in [19] the notion of λ-synchronization of subshifts. It was shown that any λ-synchronizing subshift has a minimal presentation of λ-graph system corresponding to the Fischer cover [19]. In [26], the author introduced a C^*-algebra associated with a λ-graph system as a generalization of Cuntz–Krieger algebras. The C^*-algebra is written $\mathcal{O}_{\mathfrak{L}}$ for a λ-graph system $\mathfrak{L}$ and has a universal property subject to certain operator relations encoded by structure of the λ-graph system $\mathfrak{L}$. If a λ-graph system is the canonical λ-graph system $\mathfrak{L}^\Lambda$ for a subshift Λ, the C^*-algebra in general is far from simple, namely has nontrivial ideals, unless the subshift is a shift of finite type or special kinds of subshifts, because the canonical λ-graph system corresponds to the left Krieger cover, that is not irreducible in general.

On the other hand, if a subshift is normal, that is, λ-synchronizing, we may construct a minimal λ-graph system as its presentation called the λ-synchronizing λ-graph system written $\mathfrak{L}_\Lambda^{\min}$. It is called the minimal presentation (see [34]), so that the associated C^*-algebra are simple and purely infinite in many cases (see [33]). For a normal subshift Λ, we write the C^*-algebra as $\mathcal{O}_{\Lambda^{\min}}$. Let us denote by X_Λ the associated right one-sided subshift of a two-sided subshift Λ. As in the previous papers [19, 33], the C^*-algebra $\mathcal{O}_{\Lambda^{\min}}$ has a natural action of the circle group $\mathbb{T}$ called gauge action written ρ^Λ. The fixed point algebra $\mathcal{F}_{\Lambda^{\min}}$ of $\mathcal{O}_{\Lambda^{\min}}$ under ρ^Λ is an AF-algebra having its diagonal algebra denoted by $\mathcal{D}_{\mathfrak{L}_\Lambda^{\min}}$. The commutative C^*-algebra $C(X_\Lambda)$ of continuous functions on the right one-sided subshift X_Λ is naturally regarded as a subalgebra of $\mathcal{D}_{\mathfrak{L}_\Lambda^{\min}}$ denoted by $\mathcal{D}_\Lambda$. We know that the relative commutant $\mathcal{D}_\Lambda{}' \cap \mathcal{O}_{\Lambda^{\min}}$ of $\mathcal{D}_\Lambda$ in $\mathcal{O}_{\Lambda^{\min}}$ coincides with $\mathcal{D}_{\mathfrak{L}_\Lambda^{\min}}$ (Proposition 3.13). Hence we have a triplet $(\mathcal{O}_{\Lambda^{\min}}, \mathcal{D}_\Lambda, \rho^\Lambda)$ from a normal subshift Λ.

In the first half of the paper, we will summarize the λ-synchronization of subshifts and describe a simplicity condition of the C^*-algebras $\mathcal{O}_{\Lambda^{\min}}$ so that we have the following theorem.

Theorem 1.1. *Let Λ be a normal subshift. If Λ is λ-irreducible, then the C^*-algebra $\mathcal{O}_{\Lambda^{\min}}$ is simple. If in addition Λ satisfies λ-condition (I), then the C^*-algebra $\mathcal{O}_{\Lambda^{\min}}$ is simple and purely infinite, where λ-condition (I) is defined in* [28] *(see also Definition* 2.15 (i)*).*

As a corollary, we have the following result.

Corollary 1.2 (Proposition 4.2). *Let Λ be an irreducible sofic shift such that Λ is not of finite set. The C^*-algebra $\mathcal{O}_{\Lambda^{\min}}$ is simple, purely infinite. It is isomorphic to the Cuntz–Krieger algebra for the transition matrix of the left Fischer cover graph of the sofic shift Λ.*

We will present several examples of simple purely infinite C^*-algebras associated with normal subshifts in Section 5. They are the C^*-algebras associated with Dyck shifts, Markov–Dyck shifts, Motzkin shifts and β-shifts.

In the second half of the paper, we will study the relationship between several kinds of topological conjugacy of normal subshifts and structure of the associated C^*-algebras. Let $\mathfrak{L}_1$, $\mathfrak{L}_2$ be left-resolving λ-graph systems that present the subshifts Λ_1, Λ_2, respectively. In [37], the notion of $(\mathfrak{L}_1, \mathfrak{L}_2)$-continuous orbit equivalence between their one-sided subshifts $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ was introduced. The author then proved that $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are $(\mathfrak{L}_1, \mathfrak{L}_2)$-continuously orbit equivalent if and only if there exists an isomorphism $\Phi : \mathcal{O}_{\mathfrak{L}_1} \to \mathcal{O}_{\mathfrak{L}_2}$ of C^*-algebras such that $\Phi(\mathcal{D}_{\Lambda_1}) = \mathcal{D}_{\Lambda_2}$ where $\mathcal{D}_{\Lambda_i}$ is a canonical commutative C^*-subalgebra of $\mathcal{O}_{\mathfrak{L}_i}$ isomorphic to $C(X_{\Lambda_i})$ for $i = 1, 2$. We will see that, under the condition that $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are $(\mathfrak{L}_1, \mathfrak{L}_2)$-continuously orbit equivalent, if Λ_1 is a normal subshift and $\mathfrak{L}_1$ is its minimal presentation, then Λ_2 is a normal subshift and $\mathfrak{L}_2$ is its minimal presentation (Lemma 6.2). We then define the one-sided subshifts $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ for normal subshifts Λ_1 and Λ_2 to be *continuously orbit equivalent* if they are $(\mathfrak{L}^{\min}_{\Lambda_1}, \mathfrak{L}^{\min}_{\Lambda_2})$-continuously orbit equivalent (Definition 6.3). We then have that for normal subshifts Λ_1 and Λ_2, their one-sided subshifts $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are continuously orbit equivalent if and only if there exists an isomorphism $\Phi : \mathcal{O}_{{\Lambda_1}^{\min}} \to \mathcal{O}_{{\Lambda_2}^{\min}}$ of C^*-algebras such that $\Phi(\mathcal{D}_{\Lambda_1}) = \mathcal{D}_{\Lambda_2}$ (Proposition 6.4).

In [37], the author also introduced the notion of $(\mathfrak{L}_1, \mathfrak{L}_2)$-eventual conjugacy between their one-sided subshifts $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ and proved that $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are $(\mathfrak{L}_1, \mathfrak{L}_2)$-eventually conjugate if and only if there exists an isomorphism $\Phi : \mathcal{O}_{\mathfrak{L}_1} \to \mathcal{O}_{\mathfrak{L}_2}$ of C^*-algebras such that

$$\Phi(\mathcal{D}_{\Lambda_1}) = \mathcal{D}_{\Lambda_2} \quad \text{and} \quad \Phi \circ \rho^{\mathfrak{L}_1}_t = \rho^{\mathfrak{L}_2}_t \circ \Phi, \quad t \in \mathbb{T},$$

where $\mathcal{D}_{\Lambda_i}$ is a canonical commutative C^*-subalgebra of $\mathcal{O}_{\mathfrak{L}_i}$ isomorphic to $C(X_{\Lambda_i})$, and $\rho^{\mathfrak{L}_i}_t$ is the gauge action on $\mathcal{O}_{\mathfrak{L}_i}$ for $i = 1, 2$.

Let us denote by $\mathcal{K}$ the C^*-algebra of compact operators on the separable infinite dimensional Hilbert space $\ell^2(\mathbb{N})$ and $\mathcal{C}$ its commutative C^*-subalgebra of diagonal operators. For two-sided topological conjugacy, the notion of $(\mathfrak{L}_1, \mathfrak{L}_2)$-conjugacy between two-sided subshifts $(\Lambda_1, \sigma_{\Lambda_1})$, $(\Lambda_2, \sigma_{\Lambda_2})$ was introduced in [30, 37]. It was proved in [37] that (Λ_1, σ_1) and (Λ_2, σ_2) are $(\mathfrak{L}_1, \mathfrak{L}_2)$-conjugate if and only if there exists an isomorphism $\widetilde{\Phi} : \mathcal{O}_{\mathfrak{L}_1} \otimes \mathcal{K} \to \mathcal{O}_{\mathfrak{L}_2} \otimes \mathcal{K}$ of C^*-algebras such that

$$\widetilde{\Phi}(\mathcal{D}_{\Lambda_1} \otimes \mathcal{C}) = \mathcal{D}_{\Lambda_2} \otimes \mathcal{C}, \quad \widetilde{\Phi} \circ (\rho^{\mathfrak{L}_1}_t \otimes \mathrm{id}) = (\rho^{\mathfrak{L}_2}_t \otimes \mathrm{id}) \circ \widetilde{\Phi}, \quad t \in \mathbb{T}.$$

In [19], it was proved that λ-synchronization is invariant under topological conjugacy of two-sided subshifts. Hence if a normal subshift Λ_1 is topologically conjugate to another subshift Λ_2, then Λ_2 is normal. We will first show the following theorems concerning one-sided conjugacies.

Theorem 1.3. *Let Λ_1 and Λ_2 be normal subshifts. Assume that their one-sided subshifts $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are topologically conjugate. Then there exists an isomorphism $\Phi : \mathcal{O}_{\Lambda_1^{\min}} \to \mathcal{O}_{\Lambda_2^{\min}}$ of C^*-algebras such that $\Phi(\mathcal{D}_{\Lambda_1}) = \mathcal{D}_{\Lambda_2}$ and $\Phi \circ \rho_t^{\Lambda_1} = \rho_t^{\Lambda_2} \circ \Phi$, $t \in \mathbb{T}$.*

Theorem 1.3 is a generalization of Cuntz–Krieger's theorem [6, Proposition 2.17]. Related results are seen in [3, 4, 36], etc.

The following theorem is a generalization of the results for irreducible topological Markov shifts in [36] (cf. [3, 4]).

Theorem 1.4. *Let Λ_1 and Λ_2 be normal subshifts. Their one-sided subshifts $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are eventually conjugate if and only if there exists an isomorphism $\Phi : \mathcal{O}_{\Lambda_1^{\min}} \to \mathcal{O}_{\Lambda_2^{\min}}$ of C^*-algebras such that $\Phi(\mathcal{D}_{\Lambda_1}) = \mathcal{D}_{\Lambda_2}$ and $\Phi \circ \rho_t^{\Lambda_1} = \rho_t^{\Lambda_2} \circ \Phi$, $t \in \mathbb{T}$.*

The if part of Theorem 1.4 follows from a result in [37]. The proof of its only if part is a main body in the second half of this paper. To prove the only if part, we provide an auxiliary subshift written Λ_2' whose one-sided subshift $X_{\Lambda_2'}$ is topologically conjugate to X_{Λ_1}. We will then prove that there exists an isomorphism of C^*-algebras $\Phi_2 : \mathcal{O}_{\Lambda_2'^{\min}} \to \mathcal{O}_{\Lambda_2^{\min}}$ satisfying $\Phi_2(\mathcal{D}_{\Lambda_2'}) = \mathcal{D}_{\Lambda_2}$ and $\Phi_2 \circ \rho_t^{\Lambda_2'} = \rho_t^{\Lambda_2} \circ \Phi_2$, $t \in \mathbb{T}$, so that we will obtain Theorem 1.4 by using Theorem 1.3.

We will second show the following theorem concerning two-sided conjugacy, that is a generalization of the case of topological Markov shifts proved by Cuntz–Krieger [6] and Carlsen–Rout [5].

Theorem 1.5. *Let Λ_1 and Λ_2 be normal subshifts. The two-sided subshifts $(\Lambda_1, \sigma_{\Lambda_1})$ and $(\Lambda_2, \sigma_{\Lambda_2})$ are topologically conjugate if and only if there exists an isomorphism $\widetilde{\Phi} : \mathcal{O}_{\Lambda_1^{\min}} \otimes \mathcal{K} \to \mathcal{O}_{\Lambda_2^{\min}} \otimes \mathcal{K}$ of C^*-algebras such that $\widetilde{\Phi}(\mathcal{D}_{\Lambda_1} \otimes \mathcal{C}) = \mathcal{D}_{\Lambda_2} \otimes \mathcal{C}$ and $\widetilde{\Phi} \circ (\rho_t^{\Lambda_1} \otimes \mathrm{id}) = (\rho_t^{\Lambda_2} \otimes \mathrm{id}) \circ \widetilde{\Phi}$, $t \in \mathbb{T}$.*

The C^*-algebraic characterizations of eventual conjugacy and topological conjugacy appeared in Theorems 1.4 and 1.5 are rephrased in terms of the associated groupoids as seen in [37, Theorem 1.3] and [37, Theorem 1.4], respectively.

We may apply the above theorems to irreducible sofic shifts. Let Λ be an irreducible sofic shift such that Λ is infinite. Let G_Λ^F be its left Fischer cover graph, that is the unique left-resolving irreducible minimal finite labeled graph that presents Λ ([9], cf. [21]). Then the C^*-algebra $\mathcal{O}_{\Lambda^{\min}}$ is a simple purely infinite C^*-algebra such that $\mathcal{O}_{\Lambda^{\min}}$ is isomorphic to the Cuntz–Krieger algebra $\mathcal{O}_{\hat{A}}$ for the transition matrix $\hat{A}$ of the topological Markov shift defined by the Fischer cover G_Λ^F (Proposition 4.2). By Proposition 6.4, Theorems 1.4 and 1.5, we have the following result.

Corollary 1.6. *Let Λ_1 and Λ_2 be two irreducible sofic shifts such that Λ_i, $i = 1,2$ are infinite.*

(i) *Their one-sided sofic shifts $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are continuously orbit equivalent if and only if there exists an isomorphism $\Phi : \mathcal{O}_{\Lambda_1{}^{\min}} \to \mathcal{O}_{\Lambda_2{}^{\min}}$ of simple C^*-algebras such that $\Phi(\mathcal{D}_{\Lambda_1}) = \mathcal{D}_{\Lambda_2}$.*

(ii) *Their one-sided sofic shifts $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are eventually conjugate if and only if there exists an isomorphism $\Phi : \mathcal{O}_{\Lambda_1{}^{\min}} \to \mathcal{O}_{\Lambda_2{}^{\min}}$ of simple C^*-algebras such that $\Phi(\mathcal{D}_{\Lambda_1}) = \mathcal{D}_{\Lambda_2}$ and $\Phi \circ \rho_t^{\Lambda_1} = \rho_t^{\Lambda_2} \circ \Phi$, $t \in \mathbb{T}$.*

(iii) *Their two-sided sofic shifts $(\Lambda_1, \sigma_{\Lambda_1})$ and $(\Lambda_2, \sigma_{\Lambda_2})$ are topologically conjugate if and only if there exists an isomorphism $\widetilde{\Phi} : \mathcal{O}_{\Lambda_1{}^{\min}} \otimes \mathcal{K} \to \mathcal{O}_{\Lambda_2{}^{\min}} \otimes \mathcal{K}$ of simple C^*-algebras such that $\widetilde{\Phi}(\mathcal{D}_{\Lambda_1} \otimes \mathcal{C}) = \mathcal{D}_{\Lambda_2} \otimes \mathcal{C}$ and $\widetilde{\Phi} \circ (\rho_t^{\Lambda_1} \otimes \mathrm{id}) = (\rho_t^{\Lambda_2} \otimes \mathrm{id}) \circ \widetilde{\Phi}$, $t \in \mathbb{T}$.*

We have to remark that in a recent paper [4] by Brix–Carlsen, similar results to the present paper are seen. The C^*-algebras treated by Brix–Carlsen are different from our C^*-algebras. In fact, their C^*-algebras in [4] are not simple in many cases unless the subshifts are irreducible shifts of finite type, whereas our C^*-algebras in the present paper are simple in many cases including infinite irreducible sofic shifts.

In what follows, the set of nonnegative integers and the set of positive integers are denoted by $\mathbb{Z}_+$ and $\mathbb{N}$, respectively.

2. λ-synchronization and normal subshifts

2.1. λ-synchronization of subshifts

Let Σ be a finite set with its discrete topology. Denote by $\Sigma^{\mathbb{Z}}$ (resp. $\Sigma^{\mathbb{N}}$) the set of bi-infinite (resp. right one-sided) sequences of Σ. We endow $\Sigma^{\mathbb{Z}}$ (resp. $\Sigma^{\mathbb{N}}$) with infinite product topology, so that they are compact Hausdorff spaces. The shift homeomorphism $\sigma : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ is defined by $\sigma((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$. A continuous surjection $\sigma : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ is similarly defined. Let $\Lambda \subset \Sigma^{\mathbb{Z}}$ be a closed σ-invariant subset, that is, $\sigma(\Lambda) = \Lambda$. We denote the restriction $\sigma|_\Lambda$ of σ to Λ by σ_Λ. The topological dynamical system $(\Lambda, \sigma_\Lambda)$ is called a subshift over alphabet Σ. It is often written as Λ for short. Let X_Λ be the set of right infinite sequence $(x_n)_{n\in\mathbb{N}}$ of Σ such that $(x_n)_{n\in\mathbb{Z}} \in \Lambda$. The set X_Λ is a closed subset of $\Sigma^{\mathbb{N}}$ such that $\sigma(X_\Lambda) = X_\Lambda$. We similarly denote $\sigma|_{X_\Lambda}$ by σ_Λ. The topological dynamical system $(X_\Lambda, \sigma_\Lambda)$ is called the right one-sided subshift for Λ. For an introduction to the theory of subshifts, we refer to text books of symbolic dynamical systems [13, 21]. For $l \in \mathbb{Z}_+$, denote by $B_l(\Lambda)$ the admissible words $\{(x_1, \dots, x_l) \in \Sigma^l \mid (x_n)_{n\in\mathbb{Z}} \in \Lambda\}$ of Λ with its length l. Denote by $B_*(\Lambda)$ the set $\bigcup_{l=0}^\infty B_l(\Lambda)$ of admissible words of Λ, where $B_0(\Lambda)$ denotes the empty word. The length m of a word $\mu = (\mu_1, \dots, \mu_m)$ is denoted by $|\mu|$. For two words $\mu = (\mu_1, \ldots, \mu_m), \nu = (\nu_1, \ldots, \nu_n) \in B_*(\Lambda)$ denote by $\mu\nu$ the concatenation $(\mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_n)$. For $\mu = (\mu_1, \ldots, \mu_m) \in B_*(\Lambda)$ and $x = (x_n)_{n\in\mathbb{N}} \in X_\Lambda$, we put $\mu x = (\mu_1, \ldots, \mu_m, x_1, x_2, \ldots) \in \Sigma^{\mathbb{N}}$. For a word $\mu = (\mu_1, \ldots, \mu_m) \in B_m(\Lambda)$, the cylinder set $U_\mu \subset X_\Lambda$ is defined by

$$U_\mu = \{(x_n)_{n\in\mathbb{N}} \in X_\Lambda \mid x_1 = \mu_1, \ldots, x_m = \mu_m\}.$$

For $x = (x_n)_{n\in\mathbb{N}} \in X_\Lambda$ and $k, l \in \mathbb{N}$ with $k \le l$, we put $x_{[k,l]} = (x_k, \ldots, x_l) \in B_{l-k+1}(\Lambda)$, $x_{[k,l)} = (x_k, \ldots, x_{l-1}) \in B_{l-k}(\Lambda)$ and $x_{[k,\infty)} = (x_k, x_{k+1}, \ldots) \in X_\Lambda$.

A subshift Λ is said to be irreducible if for any $\mu, \nu \in B_*(\Lambda)$, there exists a word $\eta \in B_*(\Lambda)$ such that $\mu\eta\nu \in B_*(\Lambda)$ (cf. [21]). We note the following lemma. Although it is well-known, the author has not been able to find a suitable reference, so that the proof is given.

Lemma 2.1 (cf. [20, p. 142]). *If a subshift Λ is irreducible and the cardinality of Λ is infinite, then the subshift Λ and its right one-sided subshift X_Λ are both homeomorphic to a Cantor set.*

Proof. We will show that X_Λ does not have any isolated point. Since Λ is irreducible, one may find a point $z \in X_\Lambda$ such that its orbit $\{\sigma_\Lambda^n(z) \mid n \in \mathbb{Z}_+\}$ is dense in X_Λ. For any point $x \in X_\Lambda$ and word $\mu \in B_m(\Lambda)$ with $x \in U_\mu$, there exists $n_1 \in \mathbb{Z}_+$ such that $\sigma_\Lambda^{n_1}(z) \in U_\mu$ As $\{\sigma_\Lambda^n(\sigma_\Lambda^{n_1}(z)) \mid n \in \mathbb{N}\}$ is also dense in X_Λ, there exists $n_2 \in \mathbb{N}$ such that $\sigma_\Lambda^{n_2}(\sigma_\Lambda^{n_1}(z)) \in U_\mu$. If $\sigma_\Lambda^{n_2}(\sigma_\Lambda^{n_1}(z)) = \sigma_\Lambda^{n_1}(z)$, then $\sigma_\Lambda^{n_1}(z)$ is periodic, so that $\{\sigma_\Lambda^n(z) \mid n \in \mathbb{Z}_+\}$ is finite, and X_Λ becomes a finite set, a contradiction. Therefore $\sigma_\Lambda^{n_2+n_1}(z) \ne \sigma_\Lambda^{n_1}(z)$, and hence U_μ contains two distinct points $\sigma_\Lambda^{n_2+n_1}(z), \sigma_\Lambda^{n_1}(z)$ so that x is not isolated. As X_Λ is totally disconnected compact metric space, it is homeomorphic to a Cantor set. Similarly we can prove that Λ does not have any isolated points. ∎

We define predecessor sets and follower sets of a word $\mu \in B_m(\Lambda)$ as follows:

$$\Gamma_l^-(\mu) = \{\nu \in B_l(\Lambda) \mid \nu\mu \in B_{l+m}(\Lambda)\},$$
$$\Gamma_l^+(\mu) = \{\nu \in B_l(\Lambda) \mid \mu\nu \in B_{l+m}(\Lambda)\}$$

and $\Gamma_*^-(\mu) = \bigcup_{l=0}^\infty \Gamma_l^-(\mu)$, $\Gamma_*^+(\mu) = \bigcup_{l=0}^\infty \Gamma_l^+(\mu)$.

Following [19, 33, 34], a word $\mu \in B_*(\Lambda)$ for $l \in \mathbb{Z}_+$ is said to be *l-synchronizing* if the equality $\Gamma_l^-(\mu) = \Gamma_l^-(\mu\omega)$ holds for all $\omega \in \Gamma_*^+(\mu)$. Let us denote by $S_l(\Lambda)$ the set of l-synchronizing words of Λ, where $S_0(\Lambda) = B_*(\Lambda)$.

Definition 2.2 ([19, 33, 34]). An irreducible subshift Λ is said to be *λ-synchronizing* if for any word $\eta \in B_l(\Lambda)$ and positive integer $k \ge l$, there exists $\nu \in S_k(\Lambda)$ such that $\eta\nu \in S_{k-l}(\Lambda)$.

It is shown in [19, 33, 34] that the following subshifts are λ-synchronizing:

- irreducible shifts of finite type,
- irreducible sofic shifts,

- synchronizing systems,
- Dyck shifts,
- Motzkin shifts,
- irreducible Markov–Dyck shifts,
- primitive substitution subshifts,
- β-shifts for every $\beta > 1$, etc.

There is an example of a coded system that is not λ-synchronizing (cf. [19]).

Following [34], two admissible words $\mu, \nu \in B_*(\Lambda)$ are said to be l-past equivalent if $\Gamma_l^-(\mu) = \Gamma_l^-(\nu)$. In this case we write $\mu \sim_l \nu$.

Definition 2.3. A λ-synchronizing subshift Λ is said to be λ-*transitive* if for any two admissible words $\mu, \nu \in S_l(\Lambda)$, there exists $k_{\mu,\nu} \in \mathbb{N}$ such that for any $\eta \in S_{l+k_{\mu,\nu}}(\Lambda)$ satisfying $\nu \sim_l \eta$, there exists $\xi \in B_{k_{\mu,\nu}}(\Lambda)$ such that $\mu \sim_l \xi\eta$.

In [19], the term "synchronized irreducible" was used for the above λ-transitivity.

Definition 2.4. A subshift Λ is said to be *normal* if it is λ-synchronizing and its cardinality $|\Lambda|$ is not finite.

Hence the class of normal subshifts contains a lot of important nontrivial subshifts.

2.2. λ-graph systems

A λ-graph system $\mathfrak{L}$ over alphabet Σ consists of a quadruple (V, E, λ, ι), where (V, E, λ) is a labeled Bratteli diagram with its vertex set $V = \bigcup_{l \in \mathbb{Z}_+} V_l$, edge set $E = \bigcup_{l \in \mathbb{Z}_+} E_{l,l+1}$ and labeling map $\lambda : E \to \Sigma$. For an edge $e \in E_{l,l+1}$, denote by $s(e) \in V_l$ and $t(e) \in V_{l+1}$ its source vertex and terminal vertex, respectively. The additional object ι is a surjection $\iota(= \iota_{l,l+1}) : V_{l+1} \to V_l$ for each $l \in \mathbb{Z}_+$. The quadruple (V, E, λ, ι) is needed to satisfy the following local property. Put for $u \in V_{l-1}$ and $v \in V_{l+1}$,

$$E^{\iota}_{l,l+1}(u, v) = \{e \in E_{l,l+1} \mid t(e) = v,\ \iota(s(e)) = u\},$$
$$E^{l-1,l}_{\iota}(u, v) = \{e \in E_{l-1,l} \mid s(e) = u,\ t(e) = \iota(v)\}.$$

The local property requires a bijective correspondence preserving their labels between $E^{\iota}_{l,l+1}(u, v)$ and $E^{l-1,l}_{\iota}(u, v)$ for every pair of vertices u, v. For $k < l$, we put

$$E_{k,l} = \{(e_1, \ldots, e_{l-k}) \in E_{k,k+1} \times \cdots \times E_{l-1,l} \mid t(e_i) = s(e_{i+1}),\ i = 1, \ldots, l-k-1\}.$$

A member of $E_{k,l}$ is called a labeled path. For $\gamma = (e_1, \ldots, e_{l-k}) \in E_{k,l}$, we put $s(\gamma) := s(e) \in V_k$, $t(\gamma) := t(e_{l-k}) \in V_l$ and $\lambda(\gamma) := (\lambda(e_1), \ldots, \lambda(e_{l-k})) \in \Sigma^{l-k}$. For $v \in V_l$, we put

$$\Gamma_l^-(v) = \{(\lambda(e_1), \ldots, \lambda(e_l)) \in \Sigma^l \mid (e_1, \ldots, e_l) \in E_{0,l},\ t(e_l) = v\}. \tag{2.1}$$

For a labeled path $\gamma \in E_{k,l}$ and a vertex $v \in V_k$, if $v = s(\gamma)$, then γ is said to leave v.

A λ-graph system $\mathfrak{L}$ is said to be *predecessor-separated* if $\Gamma_l^-(v) \neq \Gamma_l^-(u)$ for every distinct pair $u, v \in V_l$. A λ-graph system $\mathfrak{L}$ is said to be *left-resolving* if $e, f \in E_{l,l+1}$ satisfy $t(e) = t(f)$, $\lambda(e) = \lambda(f)$, then $e = f$.

Let us denote by $\Lambda_{\mathfrak{L}}$ the two-sided subshift over Σ, whose admissible words $B_*(\Lambda_{\mathfrak{L}})$ are defined by the set of words appearing in the finite labeled sequences in the labeled Bratteli diagram (V, E, λ) of the λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$. We say that a subshift Λ is presented by a λ-graph system $\mathfrak{L}$ or $\mathfrak{L}$ presents Λ if $\Lambda = \Lambda_{\mathfrak{L}}$.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \lambda)$ be a predecessor-separated left-resolving finite labeled graph over alphabet Σ with finite vertex set $\mathcal{V}$, finite edge set $\mathcal{E}$ and labeling $\lambda : \mathcal{E} \to \Sigma$. It naturally gives rise to a λ-graph system $\mathfrak{L}_{\mathcal{G}}$ by setting $V_l = \mathcal{V}$, $E_{l,l+1} = \mathcal{E}$ for all $l \in \mathbb{Z}_+$ and $\iota = \mathrm{id}$. The presented subshift $\Lambda_{\mathfrak{L}_{\mathcal{G}}}$ by the λ-graph system $\mathfrak{L}_{\mathcal{G}}$ is noting but the sofic shift $\Lambda_{\mathcal{G}}$ presented by the finite labeled graph $\mathcal{G}$. A detailed study of λ-graph systems can be found in [24].

Definition 2.5 ([33]). Let $\mathfrak{L} = (V, E, \lambda, \iota)$ be a λ-graph system over Σ.

(i) $\mathfrak{L}$ is said to be *ι-irreducible* if for any two vertices $u, v \in V_l$ and a labeled path γ leaving u, there exist labeled paths η of length n and γ' such that $s(\eta) = v$, $\iota^n(t(\eta)) = u$, and $s(\gamma') = t(\eta)$, $\iota^n(t(\gamma')) = t(\gamma)$ and $\lambda(\gamma') = \lambda(\gamma)$.

(ii) $\mathfrak{L}$ is said to be *λ-irreducible* if for any ordered pair $u, v \in V_l$ of vertices, there exists $L(u, v) \in \mathbb{N}$ such that for any vertex $w \in V_{l+L(u,v)}$ satisfying

$$\iota^{L(u,v)}(w) = u,$$

there exists a labeled path γ such that $s(\gamma) = v$ and $t(\gamma) = w$.

Lemma 2.6. *Let $\mathfrak{L} = (V, E, \lambda, \iota)$ be a λ-graph system that presents a subshift Λ. Consider the following three conditions.*

(i) *$\mathfrak{L}$ is λ-irreducible.*

(ii) *$\mathfrak{L}$ is ι-irreducible.*

(iii) *Λ is irreducible.*

Then we have (i)$\Rightarrow$(ii)$\Rightarrow$(iii).

Proof. (i)$\Rightarrow$(ii): Assume that $\mathfrak{L}$ is λ-irreducible. Let $u, v \in V_l$ be two vertices and γ a labeled path leaving u, Take $L(u, v) \in \mathbb{N}$ satisfying the λ-irreducibility condition in Definition 2.5 (ii). Let k denote the length of the path γ and $u_\gamma = t(\gamma) \in V_{l+k}$. Take $u' \in V_{l+k+L(u,v)}$ such that $\iota^{L(u,v)}(u') = u_\gamma$. By the local property of λ-graph system, one may find $w \in V_{l+L(u,v)}$ and a labeled path γ' such that

$$\iota^{L(u,v)}(w) = u, \quad s(\gamma') = w, \quad t(\gamma') = u'.$$

By the λ-irreducibility, there exists a labeled path η such that $s(\eta) = v$, $t(\eta) = w$.

(ii)$\Rightarrow$(iii): The assertion comes from [34, Lemma 3.5]. ∎

Remark 2.7. (i) If $\mathfrak{L}$ is a λ-graph system $\mathfrak{L}_{\mathcal{G}}$ associated to a left-resolving finite labeled graph $\mathcal{G}$, then the presented subshift $\Lambda_{\mathfrak{L}_{\mathcal{G}}}$ by $\mathfrak{L}_{\mathcal{G}}$ is a sofic shift defined by $\mathcal{G}$. It is easy to see that for the λ-graph system $\mathfrak{L}_{\mathcal{G}}$, all of the conditions (i), (ii) and (iii) in Lemma 2.6 are mutually equivalent.

(ii) Let Λ_C be the coded system defined by the code $C = \{a^n b^n \mid n = 1, 2, \ldots\}$ for alphabet $\Sigma = \{a, b\}$ (see [2]). Then the subshift Λ_C has a synchronizing word $\omega = aba$, so that it is an irreducible synchronizing subshift. Hence Λ_C is a λ-synchronizing [19]. Let $\mathfrak{L}^{\lambda(\Lambda_C)}$ be its λ-synchronizing λ-graph system as in [34]. By [34, Lemma 3.6], irreducibility of Λ_C implies ι-irreducibility, so that $\mathfrak{L}^{\lambda(\Lambda_C)}$ is ι-irreducible. However, it is not difficult to see that $\mathfrak{L}^{\lambda(\Lambda_C)}$ is not λ-irreducible. Hence there is an example of λ-graph system such that the implication (ii)⇒(i) above does not hold.

(iii) Let Λ_{ev} be the even shift, that is defined to be a sofic shift over $\{0, 1\}$ whose admissible words are

$$1\overbrace{0\cdots0}^{\text{even}}1.$$

Let $\mathfrak{L}^{\Lambda_{\text{ev}}}$ be the canonical λ-graph system for Λ^{ev} (see [24]). The subshift Λ_{ev} is irreducible, whereas $\mathfrak{L}^{\Lambda_{\text{ev}}}$ is not ι-irreducible. Hence there is an example of λ-graph system such that the implication (iii)⇒(ii) above does not hold.

2.3. λ-synchronizing λ-graph systems

Let $\mathfrak{L} = (V, E, \lambda, \iota)$ be a λ-graph system that presents a subshift Λ. Let $v \in V_l$ and $\mu \in B_m(\Lambda)$, $m \in \mathbb{N}$. Following [33], we say that v *launches* μ if the following two conditions are both satisfied:

(i) There exists a labeled path $\gamma \in E_{l,l+m}$ such that $s(\gamma) = v$, $\lambda(\gamma) = \mu$.

(ii) The word μ does not leave any other vertex in V_l than v

The vertex v is called the launching vertex for μ.

Definition 2.8 ([33]). A λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ is said to be λ-*synchronizing* if any vertex of V is a launching vertex for some word of Λ.

A λ-synchronizing λ-graph system is ι-irreducible if and only if the presented subshift Λ is irreducible [33, Proposition 3.7]. It was shown that if $\mathfrak{L}$ is ι-irreducible and λ-synchronizing, then the presented subshift Λ is λ-synchronizing. Conversely, as in [33], one may construct a left-resolving, predecessor-separated ι-irreducible λ-synchronizing λ-graph system from a λ-synchronizing subshift Λ. We briefly review its construction. Let Λ be a λ-synchronizing subshift. Recall that $S_l(\Lambda)$ denotes the set of l-synchronizing words of Λ. Denote by $V_l^{\lambda(\Lambda)}$ the set of l-past equivalence classes of $S_l(\Lambda)$, where $V_0^{\lambda(\Lambda)} = \{v_0\}$ a singleton. Let us denote by $[\mu]_l$ the equivalence class of $\mu \in S_l(\Lambda)$. For $\nu \in S_{l+1}(\Lambda)$ and $\alpha \in \Gamma_1^-(\nu)$, an edge from $[\alpha\nu]_l \in V_l^{\lambda(\Lambda)}$ to $[\nu]_{l+1} \in V_{l+1}^{\lambda(\Lambda)}$ with its label α is defined. The set of such edges is denoted by $E_{l,l+1}^{\lambda(\Lambda)}$. The labeling map from $E_{l,l+1}^{\lambda(\Lambda)}$ to Σ is denoted by $\lambda^{\lambda(\Lambda)}$. As $S_{l+1}(\Lambda) \subset S_l(\Lambda)$, we have a natural map

$$\iota^{\lambda(\Lambda)} : [\nu]_{l+1} \in V_{l+1}^{\lambda(\Lambda)} \to [\nu]_l \in V_l^{\lambda(\Lambda)}.$$

The quadruplet $(V^{\lambda(\Lambda)}, E^{\lambda(\Lambda)}, \lambda^{\lambda(\Lambda)}, \iota^{\lambda(\Lambda)})$ defines a left-resolving, predecessor-separated, ι-irreducible λ-graph system that presents the subshift Λ [33, Proposition 3.2]. The λ-graph system was denoted by $\mathfrak{L}^{\lambda(\Lambda)}$ in [33, Proposition 3.2] and called the canonical λ-synchronizing λ-graph system for Λ. The following proposition was proved in [33, Theorem 3.9].

Proposition 2.9 ([33, Theorem 3.9]). *Let Λ be a λ-synchronizing subshift. Then there uniquely exists a left-resolving, predecessor-separated, ι-irreducible, λ-synchronizing λ-graph system that presents the subshift Λ. The unique λ-synchronizing λ-graph system is the canonical λ-synchronizing λ-graph system $\mathfrak{L}^{\lambda(\Lambda)}$ for Λ.*

Lemma 2.10. *Let Λ be a λ-synchronizing subshift.*

(i) *Λ is irreducible if and only if $\mathfrak{L}^{\lambda(\Lambda)}$ is ι-irreducible.*

(ii) *Λ is λ-transitive if and only if $\mathfrak{L}^{\lambda(\Lambda)}$ is λ-irreducible.*

Proof. (i) The assertion comes from [33, Proposition 3.7].

(ii) The equivalence between λ-transitivity of Λ and λ-irreducibility of $\mathfrak{L}^{\lambda(\Lambda)}$ is direct by definition. ∎

Definition 2.11 ([33]). A λ-graph system $\mathfrak{L}$ is said to be *minimal* if $\mathfrak{L}$ has no proper λ-graph subsystem of $\mathfrak{L}$.

It was proved that for a λ-synchronizing subshift Λ, the canonical λ-synchronizing λ-graph system $\mathfrak{L}^{\lambda(\Lambda)}$ is minimal.

In what follows, for a λ-synchronizing subshift Λ, the canonical λ-synchronizing λ-graph system $\mathfrak{L}^{\lambda(\Lambda)}$ is denoted by $\mathfrak{L}^{\min}_{\Lambda}$. Recall that a subshift Λ is said to be *normal* if it is λ-synchronizing and its cardinality $|\Lambda|$ is not finite as a set. We call the λ-graph system $\mathfrak{L}^{\min}_{\Lambda}$ for a normal subshift Λ the *minimal presentation* of a normal subshift Λ. We often write $\mathfrak{L}^{\min}_{\Lambda} = (V^{\min}, E^{\min}, \lambda^{\min}, \iota^{\min})$ or $(V^{\Lambda^{\min}}, E^{\Lambda^{\min}}, \lambda^{\Lambda^{\min}}, \iota^{\Lambda^{\min}})$.

2.4. Condition (I) for λ-graph systems

Let $\mathfrak{L}$ be a λ-graph system over Σ and Λ the presented subshift $\Lambda_{\mathfrak{L}}$. The condition (I) for a λ-graph system was introduced in [26] that yields uniqueness of certain operator relations of canonical generators of the associated C^*-algebra $\mathcal{O}_{\mathfrak{L}}$.

Definition 2.12. A λ-graph system $\mathfrak{L}$ is said to satisfy *condition (I)* if for any vertex $v \in V_l$, the follower set $\Gamma^+_\infty(v)$ of v defined by

$$\Gamma^+_\infty(v) := \{(\lambda(e_1), \lambda(e_2), \dots) \in X_\Lambda \mid s(e_1) = v,\ e_i \in E_{l+i-1,l+i},\ t(e_i) = s(e_{i+1}),\ i = 1, 2, \dots\}$$

contains at least two distinct sequences.

In [23, Lemma 5.1], the following lemma is shown for the case of the canonical λ-graph system $\mathfrak{L}^{\Lambda}$ for Λ.

Lemma 2.13 (cf. [23, Lemma 5.1]). *Let $\mathfrak{L}$ be a left-resolving λ-graph system. Consider the following three conditions:*

(i) *$\mathfrak{L}$ satisfies condition (I).*

(ii) *For $l \in \mathbb{Z}_+$, $v \in V_l$, $(x_n)_{n\in\mathbb{N}} \in \Gamma_\infty^+(v)$ and $m \in \mathbb{N}$, there exists $(y_n)_{n\in\mathbb{N}} \in \Gamma_\infty^+(v)$ such that*

$$x_j = y_j \quad \text{for all } j = 1, 2, \dots, m \text{ and } x_N \neq y_N \text{ for some } N > m.$$

(iii) *For $k, l \in \mathbb{N}$ with $k \le l$, there exists $y(i) \in \Gamma_\infty^+(v_i^l)$ for each $i = 1, 2, \dots, m(l)$ such that*

$$\sigma_\Lambda^m(y(i)) \neq y(j) \quad \text{for all } i, j = 1, 2, \dots, m(l) \text{ and } m = 1, 2, \dots, k.$$

Then we have implications: (i)$\Leftrightarrow$(ii)$\Rightarrow$(iii). *If in particular, $\mathfrak{L}$ is the minimal λ-graph system $\mathfrak{L}_\Lambda^{\min}$ for a normal subshift Λ, then the three conditions are all equivalent.*

Proof. (i)$\Rightarrow$(ii): For $x = (\lambda(e_n))_{n\in\mathbb{N}} \in \Gamma_\infty^+(v_i^l)$, put $v_j^{l+m} = t(e_m) \in V_{l+m}$. Since $\Gamma_\infty^+(v_j^{l+m})$ contains at least two distinct sequences, one may find $y \in \Gamma_\infty^+(v_i^l)$ such that $x_j = y_j$ for all $j = 1, 2, \dots, m$ and $x_N \neq y_N$ for some $N > m$.

(ii)$\Rightarrow$(i): The assertion is clear.

(ii)$\Rightarrow$(iii): Take and fix $k \le l$. We will first see that for a vertex $v_i^l \in V_l$,

$$\text{there exists } y \in \Gamma_\infty^+(v_i^l) \text{ such that } \sigma_\Lambda^n(y) \neq y \text{ for } 1 \le n \le k. \tag{2.2}$$

Take $x \in \Gamma_\infty^+(v_i^l)$. If $\sigma_\Lambda(x) = x$, we may find $y \in \Gamma_\infty^+(v_i^l)$ such that $\sigma_\Lambda(y) \neq y$ by the assertion (ii). We may assume that $\sigma_\Lambda(x) \neq x$. Now suppose that $\sigma_\Lambda^n(x) \neq x$ for all $n \in \mathbb{N}$ with $1 \le n \le K$ for some $K \in \mathbb{N}$. We will show that

$$\text{there exists } y \in \Gamma_\infty^+(v_i^l) \text{ such that } \sigma_\Lambda^n(y) \neq y \text{ for } 1 \le n \le K+1.$$

Let $x = (x_i)_{i\in\mathbb{N}}$. As $\sigma_\Lambda^n(x) \neq x$ for all $n \in \mathbb{N}$ with $1 \le n \le K$, there exists $k_n \in \mathbb{N}$ such that $x_{k_n} \neq x_{n+k_n}$ for each $n \in \mathbb{N}$ with $1 \le n \le K$. Put

$$M = \max\{n + k_n \mid n = 1, 2, \dots, K\}$$

so that $M \ge K + 1$. Suppose that $\sigma_\Lambda^{K+1}(x) = x$. By the condition (ii) for $m = M$, there exists $y = (y_n)_{n\in\mathbb{N}} \in \Gamma_\infty^+(v_i^l)$ such that

$$x_j = y_j \quad \text{for all } j = 1, 2, \dots, M, \tag{2.3}$$

$$x_N \neq y_N \quad \text{for some } N > M. \tag{2.4}$$

As $x_{k_n} \neq x_{n+k_n}$ for each $n \in \mathbb{N}$ with $1 \le n \le K$, the equality (2.3) implies $y_{k_n} \neq y_{n+k_n}$ for all n with $1 \le n \le K$. Hence we have

$$\sigma_\Lambda^n(y) \neq y \quad \text{for } 1 \le n \le K. \tag{2.5}$$

Now $\sigma_\Lambda^{K+1}(x) = x$ so that $x_{K+1+i} = x_i$ for all $i \in \mathbb{N}$. If $\sigma_\Lambda^{K+1}(y) = y$, the equality (2.3) implies $x_j = y_j$ for all $j \in \mathbb{N}$, a contradiction to (2.4). Hence we see that $\sigma_\Lambda^{K+1}(y) \neq y$ so that by (2.5), we obtain that $\sigma_\Lambda^n(y) \neq y$ for all $n \in \mathbb{N}$ with $1 \le n \le K+1$ and thus the assertion (2.2).

We will next show the following: for $i = 1, 2, \dots, m(l)$ and $k, l \in \mathbb{N}$ with $k \le l$, there exists $y_i^l \in \Gamma_\infty^+(v_i^l)$ such that

$$\sigma_\Lambda^n(y_j^l) \neq y_i^l \quad \text{for all } i, j = 1, 2, \dots, m(l) \text{ and } n = 1, 2, \dots, k.$$

For $i = 1$, by (2.2), there exists $y_1^l \in \Gamma_\infty^+(v_1^l)$ such that $\sigma_\Lambda^n(y_1^l) \neq y_1^l$ for $1 \le n \le k$. By the condition (ii), it is easy to see that the set of $\Gamma_\infty^+(v_i^l)$ satisfying (2.2) for each $i = 1, 2, \dots, m(l)$ is infinite. We will show that for a fixed $k \le l$,

there exists $y_i^l \in \Gamma_\infty^+(v_i^l)$ for each $i = 1, 2, \dots, m \le m(l)$ such that

$$\sigma_\Lambda^n(y_j^l) \neq y_i^l \quad \text{for all } i, j = 1, 2, \dots, m \text{ and } n = 1, 2, \dots, k \tag{2.6}$$

by induction on m with $1 \le m \le m(l)$.

As in the preceding argument, (2.6) holds for $m = 1$. Now assume that (2.6) holds for all $i \le m$. We will then prove that (2.6) holds for all $i \le m+1$. It is easy to see that the set

$$Y_i = \{y \in \Gamma^+(v_i^l) \mid \sigma_\Lambda^n(y) \neq y \text{ for } 1 \le n \le k\}$$

is infinite by the above argument. In particular, Y_{m+1} is infinite. Take $y_i^l \in \Gamma_\infty^+(v_i^l)$ for $i = 1, 2, \dots, m$ such that

$$\sigma_\Lambda^n(y_j^l) \neq y_i^l \quad \text{for all } i, j = 1, 2, \dots, m \text{ and } n = 1, 2, \dots, k.$$

We may take and fix the above $y_i^l \in \Gamma_\infty^+(v_i^l)$ for $i = 1, 2, \dots, m$ by the induction hypothesis. Consider the following set for the y_i^l, $i = 1, 2, \dots, m$:

$$\begin{aligned} Z = &\{y \in \Gamma^+(v_{m+1}^l) \mid \sigma_\Lambda^n(y_j^l) = y \text{ for some } j = 1, 2, \dots, m \text{ and } n = 1, 2, \dots, k\} \\ &\cup \{y \in \Gamma^+(v_{m+1}^l) \mid \sigma_\Lambda^n(y) = y_j^l \text{ for some } j = 1, 2, \dots, m \text{ and } n = 1, 2, \dots, k\}. \end{aligned}$$

As Z is a finite set and Y_{m+1} is an infinite set, the set $Y_{m+1} \cap Z^c$ is infinite. Hence we may find an element $y_{m+1}^l \in Y_{m+1} \cap Z^c$ satisfying

$$\sigma_\Lambda^n(y_{m+1}^l) \neq y_{m+1}^l, \quad \sigma_\Lambda^n(y_j^l) \neq y_{m+1}^l, \quad \sigma_\Lambda^n(y_{m+1}^l) \neq y_j^l$$

for all $j = 1, 2, \dots, m$ and $n = 1, 2, \dots, k$. Therefore the assertion (2.6) holds for $m+1$, so that the induction completes. We thus obtain the assertion (iii).

(iii)⇒(i): Assume that $\mathfrak{L}$ is the minimal λ-graph system $\mathfrak{L}_\Lambda^{\min}$ for a normal subshift Λ. Suppose that $\mathfrak{L}$ does not satisfy condition (I), so that there exists a vertex $v_i^l \in V_l$ such that $\Gamma_\infty^+(v_i^l) = \{y\}$ a singleton for some $y \in X_\Lambda$. Now we are assuming that $\mathfrak{L}$ is minimal and hence λ-synchronizing, so that there exists $N_0 \in \mathbb{N}$ such that v_i^l launches $y_{[1,N_0]}$. Let

$v_j^{l+1} \in V_{l+1}$ be a vertex such that $\iota(v_j^{l+1}) = v_i^l$. For any $y' \in \Gamma_\infty^+(v_j^{l+1})$, the local property of λ-graph system $\mathfrak{L}$ ensures us that $y' \in \Gamma_\infty^+(v_i^l)$ and hence $y' = y$. Hence we have $\Gamma_\infty^+(v_j^{l+1}) = \Gamma_\infty^+(v_i^l)$ whenever $v_j^{l+1} \in V_{l+1}$ with $\iota(v_j^{l+1}) = v_i^l$. Since $\mathfrak{L}$ is λ-synchronizing, y never leaves any other vertex than v_j^{l+1} in V_{l+1}. Hence a vertex $v_j^{l+1} \in V_{l+1}$ satisfying $\iota(v_j^{l+1}) = v_i^l$ is unique. We may write j as $i(l+1)$, so that $\Gamma_\infty^+(v_{i(l+1)}^{l+1}) = \{y\}$. Similarly we have a unique sequence of vertices $v_{i(l+n)}^{l+n}$, $n = 1, 2, \dots$ satisfying

$$v_{i(l+n)}^{l+n} \in V_{l+n}, \quad \iota(v_{i(l+n)}^{l+n}) = v_{i(l+n-1)}^{l+n-1} \quad \text{for } n = 1, 2, \dots.$$

Now by the assumption (iii), we have $\sigma_\Lambda(y) \neq y$, and hence there exists $j_1 = 1, 2, \dots, m(l+1)$ such that $\sigma_\Lambda(y) \in \Gamma_\infty^+(v_{j_1}^{l+1})$. Hence we have $j_1 \neq i(l+1)$. As $y = y_1\sigma_\Lambda(y)$ and $\Gamma_\infty^+(v_i^l) = \{y\}$, we have $\Gamma_\infty^+(v_{j_1}^{l+1}) = \{\sigma_\Lambda(y)\}$. Together with $\Gamma_\infty^+(v_{i(l+1)}^{l+1}) = \{y\}$, we have a contradiction to the condition (iii). ∎

Proposition 2.14. *Let $\mathfrak{L}_\Lambda^{\min}$ be the minimal presentation of a normal subshift Λ. Then the λ-graph system $\mathfrak{L}_\Lambda^{\min}$ satisfies condition (I).*

Proof. By Lemma 2.1, X_Λ is homeomorphic to a Cantor set. For $v_i^l \in V_l^{\min}$, there exists an l-synchronizing word $\mu \in S_l(\Lambda)$ for which v_i^l launches μ. Hence we have $U_\mu \subset \Gamma_\infty^+(v_i^l)$ the cylinder set for the word μ. As X_Λ is homeomorphic to a Cantor set, the cylinder set U_μ contains at least two points, so that $\mathfrak{L}_\Lambda^{\min}$ satisfies condition (I). ∎

The following definition have been already introduced in previously published papers. The first one was introduced in [28], that is stronger than condition (I) for λ-graph system in Definition 2.12. The second one was introduced in [19] that was named as synchronizing condition (I) [19, (5.1)].

Definition 2.15. (i) A λ-graph system $\mathfrak{L}$ is said to satisfy λ-*condition (I)* if for any vertex $v_i^l \in V_l$, there exists a vertex $v_j^{L'} \in V_{L'}$ for some $L' > l$ such that there exist labeled paths γ_1, γ_2 in $\mathfrak{L}$ satisfying

$$s(\gamma_1) = s(\gamma_2) = v_i^l, \quad t(\gamma_1) = t(\gamma_2) = v_j^{L'}, \quad \lambda(\gamma_1) \neq \lambda(\gamma_2).$$

(ii) A normal subshift Λ is said to satisfy λ-*condition (I)* if for any $l \in \mathbb{N}$ and $\mu \in S_l(\Lambda)$, there exist $\xi_1, \xi_2 \in B_k(\Lambda)$ and $\nu \in S_{l+K}(\Lambda)$ for some $K \in \mathbb{N}$ such that

$$\xi_1, \xi_2 \in \Gamma_K^-(\nu), \quad \xi_1 \neq \xi_2, \quad [\xi_1\nu]_l = [\xi_2\nu]_l = [\mu]_l.$$

The λ-condition (I) for a normal subshift had been called synchronizing condition (I) in [19]. Hence we know the following lemma that was already shown in [19].

Lemma 2.16 ([19, Lemma 5.1]). *Let Λ be a normal subshift. Then the following two conditions are equivalent.*

(i) *Λ satisfies the λ-condition (I).*

(ii) *$\mathfrak{L}_\Lambda^{\min}$ satisfies the λ-condition (I).*

3. Structure and simplicity of $\mathcal{O}_{\Lambda^{\min}}$

3.1. The C^*-algebras associated with λ-graph systems

Following [26], let us recall the construction of the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with a left-resolving λ-graph system $\mathfrak{L}$. The C^*-algebra was first defined as a groupoid C^*-algebra $C^*(G_{\mathfrak{L}})$ of an étale amenable groupoid $G_{\mathfrak{L}}$ defined by a continuous graph $E_{\mathfrak{L}}$ in the sense of V. Deaconu (cf. [7, 8]). Let $\mathfrak{L} = (V, E, \lambda, \iota)$ be a left-resolving λ-graph system over Σ and Λ its presented subshift. The vertex set $\Omega_{\mathfrak{L}}$ of the continuous graph is defined by the compact Hausdorff space of the projective limit:

$$\Omega_{\mathfrak{L}} = \Big\{ (u^l)_{l\in\mathbb{Z}_+} \in \prod_{l\in\mathbb{Z}_+} V_l \mid \iota_{l,l+1}(u^{l+1}) = u^l,\ l \in \mathbb{Z}_+ \Big\}.$$

of the system $\iota_{l,l+1} : V_{l+1} \to V_l$, $l \in \mathbb{Z}_+$ of continuous surjections. It is endowed by its projective limit topology. We call each element of $\Omega_{\mathfrak{L}}$ a vertex or an ι-orbit. The continuous graph $E_{\mathfrak{L}}$ for $\mathfrak{L}$ is defined by the set of triplets $(u, \alpha, w) \in \Omega_{\mathfrak{L}} \times \Sigma \times \Omega_{\mathfrak{L}}$ where $u = (u^l)_{l\in\mathbb{Z}_+}$, $w = (w^l)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}}$ such that there exists an edge $e_{l,l+1} \in E_{l,l+1}$ satisfying

$$u^l = s(e_{l,l+1}), \quad w^{l+1} = t(e_{l,l+1}), \quad \text{and} \quad \alpha = \lambda(e_{l,l+1}) \quad \text{for each } l \in \mathbb{Z}_+$$

([26, Proposition 2.1], cf. [7, 8]). Let us denote by $X_{\mathfrak{L}}$ the set of one-sided paths of $E_{\mathfrak{L}}$:

$$X_{\mathfrak{L}} = \Big\{ (\alpha_i, u_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} (\Sigma \times \Omega_{\mathfrak{L}}) \mid (u_0, \alpha_1, u_1) \in E_{\mathfrak{L}} \text{ for some } u_0 \in \Omega_{\mathfrak{L}} \text{ and } (u_i, \alpha_{i+1}, u_{i+1}) \in E_{\mathfrak{L}} \text{ for all } i \in \mathbb{N} \Big\}.$$

We endow $X_{\mathfrak{L}}$ with the relative topology from the infinite product topology of $\Pi_{i\in\mathbb{N}}(\Sigma \times \Omega_{\mathfrak{L}})$, that makes $X_{\mathfrak{L}}$ a zero-dimensional compact Hausdorff space. The continuous surjection of the shift map $\sigma_{\mathfrak{L}} : (\alpha_i, u_i)_{i\in\mathbb{N}} \in X_{\mathfrak{L}} \to (\alpha_{i+1}, u_{i+1})_{i\in\mathbb{N}} \in X_{\mathfrak{L}}$ is defined on $X_{\mathfrak{L}}$. Since the λ-graph system $\mathfrak{L}$ is left-resolving, it follows that $\sigma_{\mathfrak{L}}$ is a local homeomorphism on $X_{\mathfrak{L}}$ [26, Lemma 2.2]. Let us define a factor map

$$\pi_{\mathfrak{L}} : (\alpha_i, u_i)_{i\in\mathbb{N}} \in X_{\mathfrak{L}} \to (\alpha_i)_{i\in\mathbb{N}} \in \Sigma^{\mathbb{N}}.$$

The image $\pi_{\mathfrak{L}}(X_{\mathfrak{L}})$ in $\Sigma^{\mathbb{N}}$ is the shift space X_Λ of the one-sided subshift $(X_\Lambda, \sigma_\Lambda)$ with shift transformation $\sigma_\Lambda((\alpha_i)_{i\in\mathbb{N}}) = (\alpha_{i+1})_{i\in\mathbb{N}}$. We then have $\pi_{\mathfrak{L}} \circ \sigma_{\mathfrak{L}} = \sigma_\Lambda \circ \pi_{\mathfrak{L}}$.

For the shift dynamical system $(X_{\mathfrak{L}}, \sigma_{\mathfrak{L}})$, one may construct a locally compact étale groupoid $G_{\mathfrak{L}}$, called a Deaconu–Renault groupoid as in the following way. We put

$$G_{\mathfrak{L}} = \{(x, n, z) \in X_{\mathfrak{L}} \times \mathbb{Z} \times X_{\mathfrak{L}} \mid \text{there exist } k, l \in \mathbb{Z}_+;\ \sigma_{\mathfrak{L}}^k(x) = \sigma_{\mathfrak{L}}^l(z),\ n = k - l\}$$

(cf. [7, 8, 44, 46, 47]). The unit space $G_{\mathfrak{L}}^0 = \{(x, 0, x) \in G_{\mathfrak{L}} \mid x \in X_{\mathfrak{L}}\}$ is identified with the space $X_{\mathfrak{L}}$ through the map $x \in X_{\mathfrak{L}} \to (x, 0, x) \in G_{\mathfrak{L}}^0$. The range map and the domain map of $G_{\mathfrak{L}}$ are defined by $r(x,n,z) = x$ and $d(x,n,z) = z$ for $(x,n,z) \in G_{\mathfrak{L}}$. The multiplication and the inverse operation are defined by $(x,n,z)(z,m,w) = (x,n+m,w)$ and $(x,n,z)^{-1} = (z,-n,x)$. An open neighborhood basis of $G_{\mathfrak{L}}$ is given by

$$Z(U,k,l,V) = \{(x,k-l,z) \in G_{\mathfrak{L}} \mid x \in U, z \in V, \sigma_{\mathfrak{L}}^k(x) = \sigma_{\mathfrak{L}}^l(z)\}$$

for open sets U, V of $X_{\mathfrak{L}}$ and k, l nonnegative integers such that $\sigma_{\mathfrak{L}}^k|_U$ and $\sigma_{\mathfrak{L}}^l|_V$ are homeomorphisms with the same open range. We then have an étale amenable groupoid $G_{\mathfrak{L}}$. We will describe the construction of the groupoid C^*-algebra $C^*(G_{\mathfrak{L}})$ for the groupoid $G_{\mathfrak{L}}$ as in the following way ([44, 46, 47], cf. [7, 8]). Let us denote by $C_c(G_{\mathfrak{L}})$ the set of compactly supported continuous functions on $G_{\mathfrak{L}}$ that has a natural product structure and $*$-involution of $*$-algebra given by

$$(f * g)(s) = \sum_{t_1,t_2 \in G_{\mathfrak{L}},\, s=t_1t_2} f(t_1)g(t_2) = \sum_{t \in G_{\mathfrak{L}},\, r(t)=r(s)} f(t)g(t^{-1}s),$$
$$f^*(s) = \overline{f(s^{-1})} \quad \text{for } f, g \in C_c(G_{\mathfrak{L}}),\ s \in G_{\mathfrak{L}}.$$

Let us denote by $C_0(G_{\mathfrak{L}}^0)$ the C^*-algebra of continuous functions on $G_{\mathfrak{L}}^0$ that vanish at infinity. The algebra $C_c(G_{\mathfrak{L}})$ has a structure of $C_0(G_{\mathfrak{L}}^0)$-right module with a $C_0(G_{\mathfrak{L}}^0)$-valued inner product by

$$(\eta f)(x,n,z) = \eta(x,n,z)f(z), \quad \langle \xi, \eta \rangle(z) = \sum_{(x,n,z) \in G_{\mathfrak{L}}} \overline{\xi(x,n,z)}\eta(x,n,z),$$

for $\xi, \eta \in C_c(G_{\mathfrak{L}})$, $f \in C_0(G_{\mathfrak{L}}^0)$, $(x,n,z) \in G_{\mathfrak{L}}$, $z \in X_{\mathfrak{L}}$. The completion of the inner product $C_0(G_{\mathfrak{L}}^0)$-right module $C_c(G_{\mathfrak{L}})$ is denoted by $\ell^2(G_{\mathfrak{L}})$, that is a Hilbert C^*-right module over the commutative C^*-algebra $C_0(G_{\mathfrak{L}}^0)$. Let us denote by $B(\ell^2(G_{\mathfrak{L}}))$ the C^*-algebra of all bounded adjointable $C_0(G_{\mathfrak{L}}^0)$-module maps on $\ell^2(G_{\mathfrak{L}})$. Let π be the $*$-homomorphism of $C_c(G_{\mathfrak{L}})$ into $B(\ell^2(G_{\mathfrak{L}}))$ defined by $\pi(f)\eta = f * \eta$ for $f, \eta \in C_c(G_{\mathfrak{L}})$. The (reduced) C^*-algebra of the groupoid $G_{\mathfrak{L}}$ is defined by the closure of $\pi(C_c(G_{\mathfrak{L}}))$ in $B(\ell^2(G_{\mathfrak{L}}))$, that we denote by $C_r^*(G_{\mathfrak{L}})$. General theory of C^*-algebras of groupoids says that for a Deaconu–Renault groupoid G, the reduced C^*-algebra $C_r^*(G)$ and the universal C^*-algebra $C^*(G)$ are canonically isomorphic and hence they are identified (see for instance [45, Proposition 2.4]). We denote them by $C^*(G)$.

Definition 3.1 ([26]). The C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with a left-resolving λ-graph system $\mathfrak{L}$ is defined to be the C^*-algebra $C^*(G_{\mathfrak{L}})$ of the groupoid $G_{\mathfrak{L}}$.

The vertex set V_l at level l of $\mathfrak{L}$ is denoted by $\{v_1^l, \dots, v_{m(l)}^l\}$. For $x = (\alpha_n, u_n)_{n \in \mathbb{N}} \in X_{\mathfrak{L}}$, we put $\lambda(x)_n = \alpha_n \in \Sigma$, $v(x)_n = u_n \in \Omega_{\mathfrak{L}}$ for $n \in \mathbb{N}$, respectively. The ι-orbit $v(x)_n$ is written as $v(x)_n = (v(x)_n^l)_{l \in \mathbb{Z}_+} \in \Omega_{\mathfrak{L}}$. Now $\mathfrak{L}$ is left-resolving so that there exists a unique vertex $v(x)_0 \in \Omega_{\mathfrak{L}}$ satisfying $(v(x)_0, \alpha_1, u_1) \in E_{\mathfrak{L}}$. Define $U(\alpha) \subset G_{\mathfrak{L}}$ for $\alpha \in \Sigma$, and $U(v_i^l) \subset G_{\mathfrak{L}}$ for $v_i^l \in V_l$ by

$$U(\alpha) = \{(x,1,z) \in G_{\mathfrak{L}} \mid \sigma_{\mathfrak{L}}(x) = z, \lambda(x)_1 = \alpha\},$$

and

$$U(v_i^l) = \{(x,0,x) \in G_{\mathfrak{L}} \mid v(x)_0^l = v_i^l\}$$

where $v(x)_0 = (v(x)_0^l)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}}$. They are clopen sets of $G_{\mathfrak{L}}$. We define

$$S_\alpha = \pi(\chi_{U(\alpha)}), \quad E_i^l = \pi(\chi_{U(v_i^l)}) \quad \text{in } \pi\big(C_c(G_{\mathfrak{L}})\big)$$

where $\chi_F \in C_c(G_{\mathfrak{L}})$ stands for the characteristic function of a clopen set F on the groupoid $G_{\mathfrak{L}}$.

The transition matrix system $(A_{l,l+1}, I_{l,l+1})_{l\in\mathbb{Z}_+}$ for the λ-graph system $\mathfrak{L}$ determines the structure of the λ-graph system $\mathfrak{L}$ that are defined by

$$A_{l,l+1}(i,\alpha,j) = \begin{cases} 1 & \text{if there exists } e \in E_{l,l+1};\ s(e) = v_i^l,\ \lambda(e) = \alpha,\ t(e) = v_j^{l+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$I_{l,l+1}(i,j) = \begin{cases} 1 & \text{if } \iota_{l,l+1}(v_j^{l+1}) = v_i^l, \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1,2,\dots,m(l)$, $j = 1,2,\dots,m(l+1)$, $\alpha \in \Sigma$. More generally for $v_i^l \in V_l$, $v_k^{l+n} \in V_{l+n}$ and $\nu = (\nu_1,\dots,\nu_n) \in B_n(\Lambda)$, we define

$$A_{l,l+n}(i,\nu,k) = \begin{cases} 1 & \text{if there exists } \gamma \in E_{l,l+n};\ s(\gamma) = v_i^l,\ \lambda(\gamma) = \nu,\ t(\gamma) = v_k^{l+n}, \\ 0 & \text{otherwise,} \end{cases}$$

$$I_{l,l+n}(i,k) = \begin{cases} 1 & \text{if } (\iota_{l,l+1} \circ \cdots \circ \iota_{l+n-1,l+n})(v_k^{l+n}) = v_i^l, \\ 0 & \text{otherwise} \end{cases}$$

so that

$$A_{l,l+n}(i,\nu,k) = \sum_{j_1,\dots,j_{n-1}} A_{l,l+1}(i,\nu_1,j_1)\cdots A_{l+n-1,l+n}(j_{n-1},\nu_n,k),$$

$$I_{l,l+n}(i,k) = \sum_{j_1,\dots,j_{n-1}} I_{l,l+1}(i,j_1)\cdots I_{l+n-1,l+n}(j_{n-1},k).$$

For a vertex $v_i^l \in V_l$, denote by $\Gamma_l^-(v_i^l)$ the predecessor set of v_i^l that is defined in (2.1) as the set of words in $B_l(\Lambda)$ that are realized by labeled edges in $\mathfrak{L}$ whose terminal is v_i^l. Recall that $\mathfrak{L}$ is predecessor-separated if $\Gamma_l^-(v_i^l) \neq \Gamma_l^-(v_j^l)$ for distinct $i,j = 1,2,\dots,m(l)$. We had proved the following theorem.

Proposition 3.2 ([26, Theorem 3.6, Theorem 4.3, and Proposition 5.6]). *Let $\mathfrak{L}$ be a left-resolving λ-graph system. The C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is a universal unital C^*-algebra generated by partial isometries S_α for $\alpha \in \Sigma$ and projections E_i^l for $v_i^l \in V_l$ subject to the following* *relations called* $(\mathfrak{L})$*:*

$$\sum_{\beta\in\Sigma} S_\beta S_\beta^* = \sum_{i=1}^{m(l)} E_i^l = 1, \qquad S_\alpha S_\alpha^* E_i^l = E_i^l S_\alpha S_\alpha^*$$

$$E_i^l = \sum_{j=1}^{m(l+1)} I_{l,l+1}(i,j) E_j^{l+1}, \quad S_\alpha^* E_i^l S_\alpha = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i,\alpha,j) E_j^{l+1}$$

for $\alpha \in \Sigma$, $i = 1,2,\dots,m(l)$, $l \in \mathbb{Z}_+$. *If in particular* $\mathfrak{L}$ *satisfies condition (I), then any non-zero generators satisfying the above relations* $(\mathfrak{L})$ *generate an isomorphic copy of* $\mathcal{O}_{\mathfrak{L}}$. *Hence* $\mathcal{O}_{\mathfrak{L}}$ *is a unique nuclear* C^**-algebra subject to the relations* $(\mathfrak{L})$ *and belongs to the UCT class if* $\mathfrak{L}$ *satisfies condition (I). If in addition,* $\mathfrak{L}$ *is* λ*-irreducible, the* C^**-algebra* $\mathcal{O}_{\mathfrak{L}}$ *is simple and purely infinite* [28].

Remark 3.3. (i) In [26], the notion of *irreducibility* of a left-resolving λ-graph system $\mathfrak{L}$ had been defined so that if $\mathfrak{L}$ satisfies condition (I) and is irreducible, the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is simple. The irreducibility is weaker than λ-irreducibility. In a recent paper [40], the two notions of *transitivity* and λ*-minimality* of a left-resolving λ-graph system were introduced. As a result, the following four assertions under the condition (I)

(a) The λ-graph system $\mathfrak{L}$ is irreducible.

(b) The λ-graph system $\mathfrak{L}$ is transitive.

(c) The λ-graph system $\mathfrak{L}$ is λ-minimal.

(d) The C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is simple.

were proved to be all equivalent [40, Theorem 1.1].

(ii) In [40], the notion of *locally contracting* λ-graph system was introduced. It was proved that if a left-resolving λ-graph system $\mathfrak{L}$ satisfying condition (I) is irreducible and locally contracting, then the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is simple and purely infinite [40, Theorem 1.2].

If $\mathfrak{L}$ is predecessor-separated, then the projections E_i^l are written by using the partial isometries $S_\alpha, \alpha \in \Sigma$ in the following way:

$$E_i^l = \prod_{\mu\in\Gamma_l^-(v_i^l)} S_\mu^* S_\mu \cdot \prod_{\nu\in\Gamma_l^-(v_i^l)^c\cap B_l(\Lambda)} (1 - S_\nu^* S_\nu), \quad i = 1,2,\dots,m(l) \tag{3.1}$$

where S_μ denotes $S_{\mu_1}\cdots S_{\mu_m}$ for $\mu = (\mu_1,\dots,\mu_m) \in B_*(\Lambda)$. Hence the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is generated by the finite family $S_\alpha, \alpha \in \Sigma$ of partial isometries. By the above relation $(\mathfrak{L})$, one sees that the algebra of finite linear combinations of the elements of the form

$$S_\mu E_i^l S_\nu^* \quad \text{for } \mu,\nu \in B_*(X_\Lambda),\ i = 1,\dots,m(l),\ l \in \mathbb{Z}_+$$

forms a dense $*$-subalgebra of $\mathcal{O}_{\mathfrak{L}}$. Let us denote by $\mathcal{D}_{\mathfrak{L}}$ the C^*-subalgebra of $\mathcal{O}_{\mathfrak{L}}$ generated by the projections of the form $S_\mu E_i^l S_\mu^*$, $i = 1,2,\dots,m(l)$, $l \in \mathbb{Z}_+$, $\mu \in B_*(\Lambda)$.

We also know that the algebra $\mathcal{D}_{\mathfrak{L}}$ is canonically isomorphic to the commutative C^*-algebra $C(X_{\mathfrak{L}})$ of continuous functions on $X_{\mathfrak{L}}$. The C^*-subalgebra of $\mathcal{D}_{\mathfrak{L}}$ generated by the projections of the form $S_\mu S_\mu^*$, $\mu \in B_*(\Lambda)$ is canonically isomorphic to the commutative C^*-algebra $C(X_\Lambda)$ of continuous functions on the right one-sided subshift X_Λ, that is written $\mathcal{D}_\Lambda$.

Let us define several kinds of C^*-subalgebras of $\mathcal{O}_{\mathfrak{L}}$ that will be useful in our further discussions. For a subset $F \subset \mathcal{O}_{\mathfrak{L}}$, we denote by $C^*(F)$ the C^*-subalgebra of $\mathcal{O}_{\mathfrak{L}}$ generated by all elements of F. Let $k, l \in \mathbb{Z}_+$ with $k \le l$. We define C^*-subalgebras of $\mathcal{O}_{\mathfrak{L}}$ by

$$\begin{aligned}
\mathcal{A}_l &= C^*\big(E_i^l : i = 1,2,\dots,m(l)\big),\\
\mathcal{A}_{\mathfrak{L}} &= C^*\big(E_i^l : i = 1,2,\dots,m(l),\ l \in \mathbb{Z}_+\big),\\
\mathcal{D}_{k,l} &= C^*\big(S_\mu E_i^l S_\mu^* : i = 1,2,\dots,m(l),\ \mu \in B_k(\Lambda)\big),\\
\mathcal{D}_{k,\mathfrak{L}} &= C^*\big(S_\mu E_i^l S_\mu^* : i = 1,2,\dots,m(l),\ \mu \in B_k(\Lambda),\ l \in \mathbb{Z}_+\big),\\
\mathcal{F}_{k,l} &= C^*\big(S_\mu E_i^l S_\nu^* : i = 1,2,\dots,m(l),\ \mu,\nu \in B_k(\Lambda)\big),\\
\mathcal{F}_{k,\mathfrak{L}} &= C^*\big(S_\mu E_i^l S_\nu^* : i = 1,2,\dots,m(l),\ \mu,\nu \in B_k(\Lambda),\ l \in \mathbb{Z}_+\big),\\
\mathcal{F}_{\mathfrak{L}} &= C^*\big(S_\mu E_i^l S_\nu^* : i = 1,2,\dots,m(l),\ \mu,\nu \in B_k(\Lambda),\ k,l \in \mathbb{Z}_+\big).
\end{aligned}$$

As in the papers [19, 26, 33], etc., the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ has a natural action of the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ called gauge action written $\rho^{\mathfrak{L}}$, that is defined by for $t \in \mathbb{T}$,

$$\rho_t^{\mathfrak{L}}(S_\alpha) = e^{2\pi\sqrt{-1}t} S_\alpha,\ \alpha \in \Sigma, \quad \rho_t^{\mathfrak{L}}(E_i^l) = E_i^l,\ i = 1,\dots,m(l),\ l \in \mathbb{Z}_+. \tag{3.2}$$

The fixed point algebra of $\mathcal{O}_{\mathfrak{L}}$ under $\rho^{\mathfrak{L}}$ is the AF-algebra $\mathcal{F}_{\mathfrak{L}}$ with its diagonal algebra $\mathcal{D}_{\mathfrak{L}}$. Let us define $\phi_{\mathfrak{L}} : \mathcal{D}_{\mathfrak{L}} \to \mathcal{D}_{\mathfrak{L}}$ by $\phi(X) = \sum_{\alpha\in\Sigma} S_\alpha X S_\alpha^*$, $X \in \mathcal{D}_{\mathfrak{L}}$. The restriction of $\phi_{\mathfrak{L}}$ to $\mathcal{D}_\Lambda$ is denoted by ϕ_Λ.

Lemma 3.4. *Let $\mathfrak{L}$ be a left-resolving λ-graph system. Then the following two conditions are equivalent:*

(i) *For $k, l \in \mathbb{N}$ with $k \le l$ and $i = 1,2,\dots,m(l)$, there exists $y(i) \in \Gamma_\infty^+(v_i^l)$ for each $i = 1,2,\dots,m(l)$ such that*

$$\sigma_\Lambda^n\big(y(i)\big) \ne y(j) \quad \text{for all } i,j = 1,2,\dots,m(l),\ n = 1,2,\dots,k. \tag{3.3}$$

(ii) *For $k, l \in \mathbb{N}$ with $k \le l$, there exists a projection $q_k^l \in \mathcal{D}_\Lambda$ such that*
   (1) *$q_k^l a \ne 0$ for all $0 \ne a \in \mathcal{A}_l$,*
   (2) *$q_k^l \phi_\Lambda^n(q_k^l) = 0$ for $n = 1,2,\dots,k$.*

Proof. (i)⇒(ii): By the condition (i), take $y(i) \in \Gamma_\infty^+(v_i^l)$ for each $i = 1,2,\dots,m(l)$ satisfying (3.3). Put a finite subset of X_Λ

$$Y = \big\{y(i) \mid i = 1,2,\dots,m(l)\big\} \subset X_\Lambda.$$

We then have $\sigma_\Lambda^{-n}(Y) \cap Y = \emptyset$ for all $n = 1, 2, \dots, k$. Now X_Λ is Hausdorff so that we may take a clopen set $V \subset X_\Lambda$ such that $Y \subset V$ and $\sigma_\Lambda^{-n}(V) \cap V = \emptyset$ for all $n = 1, 2, \dots, k$. Define $q_k^l = \chi_V \in C(X_\Lambda)(= \mathcal{D}_\Lambda)$ the characteristic function of V on X_Λ. Since $y(i) \in Y \subset V$ and $y(i) \in \Gamma_\infty^+(v_i^l)$ we have $q_k^l \cdot E_i^l \neq 0$. On the other hand, the condition $\sigma_\Lambda^{-n}(V) \cap V = \emptyset$ for all $n = 1, 2, \dots, k$ ensures us $q_k^l \phi_\Lambda^n(q_k^l) = 0$ for $n = 1, 2, \dots, k$. As the C^*-subalgebra $\mathcal{A}_l$ is the direct sum $\oplus_{i=1}^{m(l)} \mathbb{C} E_i^l$, we see that $q_k^l a \neq 0$ for all $0 \neq a \in \mathcal{A}_l$.

(ii)$\Rightarrow$(i): Assume the condition (ii). For $k, l \in \mathbb{N}$ with $k \leq l$, there exists a projection $q_k^l \in \mathcal{D}_\Lambda$ satisfying the conditions (1) and (2). The condition (1) implies that $q_k^l E_i^l \neq 0$ for all $i = 1, 2, \dots, m(l)$. One may take a clopen set $V \subset X_\Lambda$ such that $q_k^l = \chi_V$ and hence

$$V \cap \Gamma_\infty^+(v_i^l) \neq \emptyset \text{ for } i = 1, 2, \dots, m(l) \quad \text{and} \quad V \cap \sigma_\Lambda^{-n}(V) = \emptyset \text{ for } n = 1, 2, \dots, k.$$

Take $y(i) \in V \cap \Gamma_\infty^+(v_i^l)$ for each $i = 1, 2, \dots, m(l)$, so that we have $\sigma_\Lambda^n(y(i)) \neq y(j)$ for all $i, j = 1, 2, \dots, m(l)$, $n = 1, 2, \dots, k$. Thus the assertion (i) holds. ∎

Since the condition (i) in the above lemma is the same as the condition (iii) in Lemma 2.13, the following lemma holds.

Lemma 3.5. *Let $\mathfrak{L}$ be a left-resolving λ-graph system satisfying condition (I). Then for $k, l \in \mathbb{N}$ with $k \leq l$, there exists a projection $q_k^l \in \mathcal{D}_\Lambda$ such that*

(1) $q_k^l a \neq 0$ *for all* $0 \neq a \in \mathcal{A}_l$,

(2) $q_k^l \phi_\Lambda^n(q_k^l) = 0$ *for* $n = 1, 2, \dots, k$.

Now we put $Q_k^l := \phi_\Lambda^k(q_k^l) \in \mathcal{D}_\Lambda$ a projection in $\mathcal{D}_\Lambda$. We note that each element of $\mathcal{D}_\mathfrak{L}$ commutes with elements of $\mathcal{A}_\mathfrak{L}$. As we see the identity

$$S_\mu \phi_\mathfrak{L}^j(X) = \phi_\mathfrak{L}^{j+|\mu|}(X) S_\mu \quad \text{for } X \in \mathcal{D}_\mathfrak{L},\ \mu \in B_*(\Lambda),\ j \in \mathbb{Z}_+,$$

where $|\mu|$ denotes the length of the word μ, a similar argument to [6, 2.9 Proposition] leads to the following lemma, that was seen in [26, Lemma 4.2].

Lemma 3.6. *Using the above notation, the following hold.*

(i) *The correspondence $X \in \mathcal{F}_{k,l} \to Q_k^l X Q_k^l \in Q_k^l \mathcal{F}_{k,l} Q_k^l$ extends to an isomorphism from $\mathcal{F}_{k,l}$ to $Q_k^l \mathcal{F}_{k,l} Q_k^l$.*

(ii) *For $X \in \mathcal{F}_\mathfrak{L}$, we have*

$$\|Q_k^l X - X Q_k^l\| \to 0 \quad \textit{and} \quad \|Q_k^l X\| - \|X\| \to 0 \quad \textit{as } k, l \to \infty.$$

(iii) *For $\mu \in B_*(\Lambda)$, we have $\|Q_k^l S_\mu\|$, $\|Q_k^l S_\mu^* Q_k^l\| \to 0$ as $k, l \to \infty$.*

The following lemma was seen in [37, Lemma 2.5] and [30, Lemma 6.5] without its detail proofs. We will give its detail proof here, where $\mathcal{D}'_\Lambda$ stands for the commutant of $\mathcal{D}$.

Lemma 3.7 (cf. [25, Lemma 3.1, Lemma 3.2]). *Let $\mathfrak{L}$ be a left-resolving λ-graph system satisfying condition (I).*

(i) $\mathcal{D}_\Lambda{}' \cap \mathcal{O}_{\mathfrak{L}} \subset \mathcal{F}_{\mathfrak{L}}$.

(ii) $\mathcal{D}_\Lambda{}' \cap \mathcal{F}_{\mathfrak{L}} \subset \mathcal{D}_{\mathfrak{L}}$.

Proof. (i) Let $E : \mathcal{O}_{\mathfrak{L}} \to \mathcal{F}_{\mathfrak{L}}$ be the conditional expectation defined by

$$E(X) = \int_{\mathbb{T}} \rho_t^{\mathfrak{L}}(X)dt \quad X \in \mathcal{O}_{\mathfrak{L}}$$

where dt denotes the normalized Lebesgue measure on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. For $X \in \mathcal{D}_\Lambda{}' \cap \mathcal{O}_{\mathfrak{L}}$, we put

$$X_\mu = E(S_\mu^* X), \quad X_{-\mu} = E(X S_\mu) \quad \text{for } \mu \in B_*(\Lambda).$$

We will show that $X_\mu = X_{-\mu} = 0$ for $\mu \in B_*(\Lambda)$ with $|\mu| \geq 1$. For $f \in \mathcal{D}_\Lambda$, we have

$$X_\mu S_\mu f S_\mu^* = E(S_\mu^* X S_\mu f S_\mu^*) = E(S_\mu^* S_\mu f S_\mu^* X) = E(f S_\mu^* X) = f X_\mu.$$

It follows that

$$X_\mu \phi_{\mathfrak{L}}^{|\mu|}(f) = X_\mu S_\mu S_\mu^* \sum_{\nu \in B_{|\mu|}(\Lambda)} S_\nu f S_\nu^* = X_\mu S_\mu S_\mu^* S_\mu f S_\mu^* = f X_\mu.$$

Now suppose that $X_\mu \neq 0$. For $\varepsilon > 0$, there exist $k, l \in \mathbb{Z}_+$ with $k \leq l$ and $X_{k,l} \in \mathcal{F}_{k,l}$ such that $|\mu| \leq k$ and $\|X_\mu - X_{k,l}\| < \varepsilon$. We may assume that $\|X_\mu\| = \|X_{k,l}\| = 1$. We then have for $f \in \mathcal{D}_\Lambda$,

$$\| f X_{k,l} - X_{k,l} \phi_{\mathfrak{L}}^{|\mu|}(f) \| \leq 2\|f\|\varepsilon.$$

Now $\mathfrak{L}$ satisfies condition (I), so that there exists a projection Q_k^l in $\mathcal{D}_\Lambda$ defined by $Q_k^l = \phi_{\mathfrak{L}}^k(q_k^l)$ satisfying the previous lemma. By considering $S_\xi S_\xi^* X_{k,l} S_\xi S_\xi^*$ instead of $X_{k,l}$, we may suppose that $X_{k,l}$ is of the form $S_\xi E_i^l S_\eta^*$ for some $\xi, \eta \in B_k(\Lambda)$. It then follows that

$$Q_k^l X_{k,l} = \sum_{\nu \in B_k(\Lambda)} S_\nu q_k^l S_\nu^* S_\xi E_i^l S_\eta^* = S_\xi q_k^l S_\xi^* S_\xi E_i^l S_\eta^* = S_\xi E_i^l q_k^l S_\eta^*$$

and

$$X_{k,l} Q_k^l = S_\xi E_i^l S_\eta^* \sum_{\nu \in B_k(\Lambda)} S_\nu q_k^l S_\nu^* = S_\xi E_i^l S_\eta^* S_\eta q_k^l S_\eta^* = S_\xi E_i^l q_k^l S_\eta^*$$

so that Q_k^l commutes with $X_{k,l}$. Hence we have

$$\| X_{k,l} Q_k^l - X_{k,l} \phi_{\mathfrak{L}}^{|\mu|}(Q_k^l) \| = \| Q_k^l X_{k,l} - X_{k,l} \phi_{\mathfrak{L}}^{|\mu|}(Q_k^l) \| \leq 2\|Q_k^l\|\varepsilon = 2\varepsilon. \tag{3.4}$$

As $Q_k^l \phi_{\mathfrak{L}}^{|\mu|}(Q_k^l) = \phi_{\mathfrak{L}}^k(q_k^l \phi_{\mathfrak{L}}^{|\mu|}(q_k^l)) = 0$, we have

$$\| X_{k,l} Q_k^l - X_{k,l} \phi_{\mathfrak{L}}^{|\mu|}(Q_k^l) \| = \max\{\|X_{k,l} Q_k^l\|, \|X_{k,l} \phi_{\mathfrak{L}}^{|\mu|}(Q_k^l)\|\}.$$

Since the correspondence $X \in \mathcal{F}_{k,l} \to Q_k^l X Q_k^l \in Q_k^l \mathcal{F}_{k,l} Q_k^l$ extends to an isomorphism from $\mathcal{F}_{k,l}$ to $Q_k^l \mathcal{F}_{k,l} Q_k^l$ so that $\|X_{k,l} Q_k^l\| = \|X_{k,l}\| = 1$. Hence we have

$$\|X_{k,l} Q_k^l - X_{k,l} \phi_{\mathfrak{L}}^{|\mu|}(Q_k^l)\| \geq 1$$

a contradiction to (3.4). We thus have $X_\mu = 0$ and similarly $X_{-\mu} = 0$. This means that $X = E(X) \in \mathcal{F}_{\mathfrak{L}}$.

(ii) For $\mu \in B_k(\Lambda)$, we put $P_\mu = S_\mu S_\mu^*$ and define the map $\mathcal{E}_k^l : \mathcal{F}_{k,l} \to \mathcal{D}_{k,l}$ by setting $\mathcal{E}_k^l(X) = \sum_{\mu \in B_k(\Lambda)} P_\mu X P_\mu$ for $X \in \mathcal{F}_{k,l}$. Since the restriction of $\mathcal{E}_k^{l+1}$ to $\mathcal{F}_{k,l}$ coincides with $\mathcal{E}_k^l$, the sequence $\{\mathcal{E}_k^l\}_{k \leq l}$ gives rise to an expectation $\mathcal{E}_k^{\mathfrak{L}} : \mathcal{F}_{k,\mathfrak{L}} \to \mathcal{D}_{k,\mathfrak{L}}$ for $k \in \mathbb{N}$. Similarly the above sequence $\{\mathcal{E}_k^{\mathfrak{L}}\}_{k \in \mathbb{N}}$ of expectations yields an expectation $\mathcal{E}^{\mathfrak{L}} : \mathcal{F}_{\mathfrak{L}} \to \mathcal{D}_{\mathfrak{L}}$ such that the restriction of $\mathcal{E}^{\mathfrak{L}}$ to $\mathcal{F}_{k,\mathfrak{L}}$ coincides with $\mathcal{E}_k^{\mathfrak{L}}$ for $k \in \mathbb{N}$.

For $X \in \mathcal{D}_{\mathfrak{L}}' \cap \mathcal{F}_{\mathfrak{L}}$, we know that $\mathcal{E}_k^{\mathfrak{L}}(X) = X$ for $k \in \mathbb{N}$, so that $\mathcal{E}^{\mathfrak{L}}(X) = X$. Since $\mathcal{E}^{\mathfrak{L}}(X) \in \mathcal{D}_{\mathfrak{L}}$, we have $\mathcal{E}^{\mathfrak{L}}(X) \in \mathcal{D}_{\mathfrak{L}}$. ∎

We thus have the following proposition.

Proposition 3.8 (cf. [25, Lemma 3.1, Lemma 3.2]). *Let $\mathfrak{L}$ be a left-resolving λ-graph system satisfying condition (I). Then we have*

$$\mathcal{D}_\Lambda{}' \cap \mathcal{O}_{\mathfrak{L}} = \mathcal{D}_{\mathfrak{L}}.$$

Proof. The inclusion relation $\mathcal{D}_\Lambda{}' \cap \mathcal{O}_{\mathfrak{L}} \supset \mathcal{D}_{\mathfrak{L}}$ is obvious. For $X \in \mathcal{D}_\Lambda{}' \cap \mathcal{O}_{\mathfrak{L}}$ by the assertions (i) and (ii) in Lemma 3.7, we know that X belongs to $\mathcal{F}_{\mathfrak{L}}$ and $\mathcal{D}_{\mathfrak{L}}$ so that $\mathcal{D}_\Lambda{}' \cap \mathcal{O}_{\mathfrak{L}} \subset \mathcal{D}_{\mathfrak{L}}$. ∎

3.2. The C^*-algebras associated with normal subshifts

For a normal subshift Λ, denote by $\mathfrak{L}_\Lambda^{\min}$ its minimal presentation.

Definition 3.9. The C^*-algebra $\mathcal{O}_{\Lambda^{\min}}$ associated with the normal subshift Λ is defined by the C^*-algebra $\mathcal{O}_{\mathfrak{L}_\Lambda^{\min}}$ associated with the minimal λ-graph system $\mathfrak{L}_\Lambda^{\min}$.

Let $(A_{l,l+1}^{\min}, I_{l,l+1}^{\min})_{l \in \mathbb{Z}_+}$ be the transition matrix system for the minimal λ-graph system $\mathfrak{L}_\Lambda^{\min}$ that is defined before Proposition 3.2. Then we have the following proposition.

Proposition 3.10. *The C^*-algebra $\mathcal{O}_{\Lambda^{\min}}$ is the universal concrete unique C^*-algebra generated by partial isometries S_α indexed by symbols $\alpha \in \Sigma$ and projections E_i^l indexed by vertices $v_i^l \in V_l^{\min}$ subject to the following operator relations called $(\mathfrak{L}_\Lambda^{\min})$:*

$$1 = \sum_{\alpha \in \Sigma} S_\alpha S_\alpha^* = \sum_{i=1}^{m(l)} E_i^l, \qquad S_\alpha S_\alpha^* E_i^l = E_i^l S_\alpha S_\alpha^*,$$

$$E_i^l = \sum_{j=1}^{m(l+1)} I_{l,l+1}^{\min}(i,j) E_j^{l+1}, \quad S_\alpha^* E_i^l S_\alpha = \sum_{j=1}^{m(l+1)} A_{l,l+1}^{\min}(i,\alpha,j) E_j^{l+1}$$

for $\alpha \in \Sigma$, $i = 1, 2, \ldots, m(l)$.

Proof. By Proposition 2.14, the λ-graph system $\mathfrak{L}_\Lambda^{\min}$ satisfies condition (I) so that we know that the C^*-algebra $\mathcal{O}_{\Lambda^{\min}}$ is the universal concrete unique C^*-algebra generated by partial isometries S_α indexed by symbols $\alpha \in \Sigma$ and projections E_i^l indexed by vertices $v_i^l \in V_l$ subject to the operator relations $(\mathfrak{L}_\Lambda^{\min})$. ∎

We thus have the following theorem, that was already seen in [19, 33].

Theorem 3.11. *Let Λ be a normal subshift.*

(i) *If Λ is λ-transitive, then the C^*-algebra $\mathcal{O}_{\Lambda^{\min}}$ is simple.*

(ii) *If Λ is λ-transitive and satisfies the λ-condition (I), then the C^*-algebra $\mathcal{O}_{\Lambda^{\min}}$ is simple and purely infinite.*

Proof. (i) The assertion was already seen in [19, 33].

(ii) By Lemma 2.16, the λ-graph system $\mathfrak{L}_\Lambda^{\min}$ satisfies the λ-condition (I). By [28], the C^*-algebra $\mathcal{O}_{\Lambda^{\min}}$ is simple and purely infinite. ∎

The following lemma is useful in our further discussions.

Lemma 3.12 ([33, Proposition 3.3]). *Let Λ be a normal subshift. For a vertex $v_i^l \in V_l^{\min}$ in $\mathfrak{L}_\Lambda^{\min}$, there exists $\mu \in S_l(\Lambda)$ such that $E_i^l \geq S_\mu S_\mu^*$ in $\mathcal{O}_{\Lambda^{\min}}$. That is, if v_i^l launches μ, the inequality $E_i^l \geq S_\mu S_\mu^*$ holds.*

The above algebraic property of the C^*-algebra $\mathcal{O}_{\Lambda^{\min}}$ characterizes the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ to be $\mathcal{O}_{\Lambda^{\min}}$.

We note that the minimal λ-graph system $\mathfrak{L}_\Lambda^{\min}$ is predecessor-separated, so that the projections E_i^l are written in terms of the partial isometries $S_\alpha, \alpha \in \Sigma$ as in (3.1). Hence the C^*-algebra $\mathcal{O}_{\Lambda^{\min}}$ is generated by only the finite family of the partial isometries $S_\alpha, \alpha \in \Sigma$.

We will see that irreducible sofic shifts Λ such that Λ is not finite as a set satisfy the condition (ii) in the above theorem. We will study more detail in Section 4.

Recall that the C^*-algebras $\mathcal{D}_{\mathfrak{L}_\Lambda^{\min}}$ and $\mathcal{D}_\Lambda$ are both commutative C^*-subalgebras of $\mathcal{O}_{\Lambda^{\min}}$ defined by

$$\mathcal{D}_{\mathfrak{L}_\Lambda^{\min}} = C^*\big(S_\mu E_i^l S_\mu^* : \mu \in B_*(\Lambda),\ i = 1, 2, \dots, m(l),\ l \in \mathbb{Z}_+\big),$$
$$\mathcal{D}_\Lambda = C^*\big(S_\mu S_\mu^* : \mu \in B_*(\Lambda)\big).$$

The former is isomorphic to $C(X_{\mathfrak{L}_\Lambda^{\min}})$, and the latter is isomorphic to $C(X_\Lambda)$. The natural factor map $\pi_{\mathfrak{L}} : X_{\mathfrak{L}_\Lambda^{\min}} \to X_\Lambda$ induces the inclusion

$$\mathcal{D}_\Lambda\big(= C(X_\Lambda)\big) \subset \mathcal{D}_{\mathfrak{L}_\Lambda^{\min}}\big(= C(X_{\mathfrak{L}_\Lambda^{\min}})\big).$$

Since the minimal λ-graph system $\mathfrak{L}_\Lambda^{\min}$ of a normal subshift Λ satisfies condition (I) by Proposition 2.14, we have the following proposition.

Proposition 3.13. *Let Λ be a normal subshift and $\mathfrak{L}_\Lambda^{\min}$ be its minimal presentation. Then we have*

$$\mathcal{D}_\Lambda{}' \cap \mathcal{O}_{\Lambda^{\min}} = \mathcal{D}_{\mathfrak{L}_\Lambda^{\min}}.$$

4. Irreducible sofic shifts

Let Λ be an irreducible sofic shift over alphabet Σ. An irreducible sofic shit is defined by using an irreducible finite directed labeled graph. It is realized as a factor of an irreducible shift of finite type. The class of irreducible sofic shifts includes the class of irreducible shifts of finite type (see [9,13,15,16,21,49], etc.). It is shown in [19,33,34] that irreducible sofic shifts are λ-synchronizing. Let $G_\Lambda^F = (V_\Lambda^F, E_\Lambda^F, \lambda_\Lambda^F)$ be its irreducible left-resolving predecessor-separated finite labeled graph over Σ that presents Λ, where $(V_\Lambda^F, E_\Lambda^F)$ is a finite directed graph with vertex set V_Λ^F and edge set E_Λ^F, and $\lambda_\Lambda^F : E_\Lambda^F \to \Sigma$ is a labeling map. It is well-known that such a finite labeled graph always exists for Λ. It is minimal and unique up to graph isomorphism [9, 21]. The labeled graph is called the minimal left-resolving presentation of an irreducible sofic shift, or the left Fischer cover. Let $V_\Lambda^F = \{v_1, \dots, v_N\}$ and $E_\Lambda^F = \{e_1, \dots, e_M\}$. We will first define a labeled Bratteli diagram (V, E, λ) over Σ as follows. Let $V_0 = \{v_0\}$ a singleton, and $V_l = \{v_1, \dots, v_N\}$ for $l \in \mathbb{N}$. Let $E_{0,1} = \{f_1^0, \dots, f_M^0\}$ such that

$$s(f_i^0) = v_0, \quad t(f_i^0) = t(e_i), \quad \lambda(f_i^0) = \lambda^F(e_i) \quad \text{for } i = 1, 2, \dots, M,$$

and $E_{l,l+1} = \{f_1^l, \dots, f_M^l\}$ for $l \in \mathbb{N}$ such that

$$s(f_i^l) = s(e_i), \quad t(f_i^l) = t(e_i), \quad \lambda(f_i^l) = \lambda^F(e_i) \quad \text{for } i = 1, 2, \dots, M.$$

For $v_i \in V_1$, put let $\Gamma_1^-(v_i)$ be its predecessor set for the vertex v_i, that is defined by

$$\Gamma_1^-(v_i) = \left\{\lambda(f_n^0) \in \Sigma \mid t(f_n^0) = v_i\right\}, \quad i = 1, 2, \dots, N.$$

If $\Gamma_1^-(v_i) = \Gamma_1^-(v_j)$, then the two vertices v_i and v_j are identified with each other in V_1, and we have a new vertex set written V_1^F. The sources $\{s(f_1^0), \dots, s(f_M^0)\}$ of edges $\{f_1^0, \dots, f_M^0\}$ are identified following the identification in V_1, so that we obtain a new edge set written $E_{0,1}^F$. Similarly, for $v_i, v_j \in V_2$, if $\Gamma_2^-(v_i) = \Gamma_2^-(v_j)$, then the two vertices v_i and v_j are identified in V_2, and the sources $\{s(f_1^1), \dots, s(f_M^1)\}$ of edges $\{f_1^1, \dots, f_M^1\}$ are identified following the identification in V_2, so that we obtain a new edge set written $E_{1,2}^F$. Like this way, we continue this procedure to get new vertex sets V_l^F, $l = 0, 1, 2, \dots$ and edge sets $E_{l,l+1}^F$, $l = 0, 1, 2, \dots$. Since Λ is sofic and the original labeled graph $G_\Lambda^F = (V_\Lambda^F, E_\Lambda^F, \lambda_\Lambda^F)$ is predecessor-separated, there exists $K \in \mathbb{N}$ such that $\Gamma_k^-(v_i) \neq \Gamma_k^-(v_j)$ in $B_k(\Lambda)$ for all $k \geq K$ and $i, j = 1, 2, \dots, N$ with $i \neq j$, so that we have

$$V_l^F = V_l (= V_\Lambda^F), \quad E_{l,l+1}^F = E_{l,l+1} (= E_\Lambda^F) \quad \text{for all } l \geq K.$$

We thus have a labeled Bratteli diagram $(V_l^F, E_{l,l+1}^F, \lambda_{l,l+1}^F)_{l \in \mathbb{Z}_+}$ over Σ. Let us denote by $\{v_1^l, \dots, v_{m(l)}^l\}$ the vertex set V_l^F. Since $\Gamma_{l+1}^-(v_i) = \Gamma_{l+1}^-(v_j)$ implies $\Gamma_l^-(v_i) = \Gamma_l^-(v_j)$, we have a natural surjective map $V_{l+1}^F \to V_l^F$ written $\iota_{l+1,l}^F$ for $l \leq K$. For $l \geq K$, the identity map $V_{l+1}^F \to V_l^F$ written $\iota_{l+1,l}^F$ is defined. We thus have a λ-graph system

$$\mathfrak{L}_\Lambda^F = (V^F, E^F, \lambda^F, \iota^F)$$

that presents the original sofic shift Λ. As the original labeled graph $G_\Lambda^F = (V_\Lambda^F, E_\Lambda^F, \lambda_\Lambda^F)$ is minimal, left-resolving and hence predecessor-separated, our λ-graph system $\mathfrak{L}_\Lambda^F$ is left-resolving and predecessor-separated and presents Λ. And also, every vertex v_i of the directed graph G_Λ^F has a word μ such that any directed labeled path labeled μ in G_Λ^F must leave the vertex v_i (cf. [21, Proposition 3.3.17]), so that every vertex of the λ-graph system $\mathfrak{L}_\Lambda^F$ launches some word (see [34, Section 3]). Therefore the λ-graph system $\mathfrak{L}_\Lambda^F$ is λ-synchronizing. As Λ is irreducible, $\mathfrak{L}_\Lambda^F$ is ι-irreducible by Lemma 2.10 (i). Hence $\mathfrak{L}_\Lambda^F$ is nothing but the minimal λ-graph system $\mathfrak{L}^{\min}$ of Λ. Therefore we have the following proposition.

Proposition 4.1. *For an irreducible sofic shift Λ, let $\mathfrak{L}^{\min} = (V^{\min}, E^{\min}, \lambda^{\min}, \iota^{\min})$ be the minimal λ-graph system for Λ. Let $G_\Lambda^F = (V^F, E^F, \lambda^F)$ be its Fischer cover graph for Λ. Then there exists $L \in \mathbb{N}$ such that*

$$V_l^{\min} = V_\Lambda^F, \quad E_{l,l+1}^{\min} = E_\Lambda^F, \quad \lambda^{\min} = \lambda_\Lambda^F, \quad \iota^{\min}|_{V_l^{\min}} = \mathrm{id}$$

for all $l \geq L$.

By the previous proposition, we can identify the minimal λ-graph system $\mathfrak{L}^{\min}$ of an irreducible sofic shift Λ with the left Fischer cover of Λ. Let Λ be an irreducible sofic shift such that Λ is not finite as a set, so that Λ is a normal subshift. Let $G_\Lambda^F = (V_\Lambda^F, E_\Lambda^F, \lambda_\Lambda^F)$ be its left Fischer cover graph with vertex set $V_\Lambda^F = \{v_1, \dots, v_N\}$. Consider the following matrix:

$$A(i,\alpha,j) = \begin{cases} 1 & \text{if there exists } e \in E_\Lambda^F;\ \lambda_\Lambda^F(e) = \alpha,\ s(e) = v_i,\ t(e) = v_j, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Let $S_\alpha, \alpha \in \Sigma$ and $E_i, i = 1, 2, \dots, N$ be partial isometries and projections respectively satisfying the following operator relations:

$$1 = \sum_{\alpha \in \Sigma} S_\alpha S_\alpha^* = \sum_{i=1}^N E_i, \quad S_\alpha S_\alpha^* E_i = E_i S_\alpha S_\alpha^*, \quad S_\alpha^* E_i S_\alpha = \sum_{j=1}^N A(i,\alpha,j) E_j \tag{4.2}$$

for $\alpha \in \Sigma$, $i = 1, 2, \dots, N$. Let us denote by $\mathcal{O}_{G_\Lambda^F}$ the universal C^*-algebra generated by $S_\alpha, \alpha \in \Sigma$ and $E_i, i = 1, 2, \dots, N$ satisfying the above relations. We put

$$\widehat{\Sigma} = \{(\alpha, i) \in \Sigma \times \{1, 2, \dots, N\} \mid \text{ there exists } e \in E_\Lambda^F;\ \lambda_\Lambda^F(e) = \alpha,\ t(e) = v_i\}.$$

For $(\alpha, i), (\beta, j) \in \widehat{\Sigma}$, by using the matrix A given by (4.1), we define a matrix

$$\widehat{A}\big((\alpha,i),(\beta,j)\big) = \sum_{k=1}^N A(k,\alpha,i) A(i,\beta,j). \tag{4.3}$$

Since the labeled graph G_Λ^F is left-resolving, the $(\alpha, i), (\beta, j)$-entry $\widehat{A}((\alpha,i),(\beta,j))$ of the matrix $\widehat{A}$ is one or zero. Let us denote by $\mathcal{O}_{\widehat{A}}$ the Cuntz–Krieger algebra for the matrix $\widehat{A}$. We then have the following proposition.

Proposition 4.2. *Let Λ be an irreducible sofic shift such that Λ is infinite. Then the C^*-algebra $\mathcal{O}_{\Lambda^{\min}}$ of the minimal presentation $\mathfrak{L}_\Lambda^{\min}$ of λ-graph system for the irreducible sofic shift Λ is a simple purely infinite C^*-algebra that is isomorphic to the Cuntz–Krieger algebra $\mathcal{O}_{\hat{A}}$ for $\hat{A}$ defined by (4.3) for its left Fischer cover graph $G_\Lambda^F = (V_\Lambda^F, E_\Lambda^F, \lambda_\Lambda^F)$.*

Proof. By the universality and the uniqueness of the C^*-algebra $\mathcal{O}_{\Lambda^{\min}}$ for the canonical generating partial isometries $S_\alpha, \alpha \in \Sigma$ and projections $E_i^l, i = 1, 2, \dots, m(l), l \in \mathbb{Z}_+$ subject to the relations $(\mathfrak{L}_\Lambda^{\min})$ as in Proposition 3.10, the C^*-algebra $\mathcal{O}_{\Lambda^{\min}}$ is canonically isomorphic to the above C^*-algebra $\mathcal{O}_{G_\Lambda^F}$.

We will henceforth show that $\mathcal{O}_{G_\Lambda^F}$ is isomorphic to the Cuntz–Krieger algebra $\mathcal{O}_{\hat{A}}$. Let $S_\alpha, \alpha \in \Sigma$ and $E_i, i = 1, 2, \dots, N$ be partial isometries and projections respectively satisfying the operator relations (4.2). For $(\alpha, i) \in \hat{\Sigma}$, put $S_{(\alpha,i)} = S_\alpha E_i$. We then have

$$\sum_{(\alpha,i)\in\hat{\Sigma}} S_{(\alpha,i)} S_{(\alpha,i)}^* = \sum_{\alpha\in\Sigma}\sum_{i=1}^N S_\alpha E_i S_\alpha^* = 1.$$

As $S_\alpha^* S_\alpha = \sum_{k=1}^N S_\alpha^* E_k S_\alpha = \sum_{k=1}^N \sum_{j=1}^N A(k, \alpha, j) E_j$, we have

$$S_{(\alpha,i)}^* S_{(\alpha,i)} = E_i \left(\sum_{k=1}^N \sum_{j=1}^N A(k,\alpha,j) E_j \right) E_i = \sum_{k=1}^N A(k,\alpha,i) E_i. \tag{4.4}$$

Since $S_\beta^* E_i S_\beta = \sum_{j=1}^N A(i, \beta, j) E_j$, we have

$$E_i = \sum_{\beta\in\Sigma}\sum_{j=1}^N A(i,\beta,j) S_\beta E_j S_\beta^* = \sum_{(\beta,j)\in\hat{\Sigma}} A(i,\beta,j) S_{(\beta,j)} S_{(\beta,j)}^*. \tag{4.5}$$

By (4.4) and (4.5), we thus obtain

$$\begin{aligned} S_{(\alpha,i)}^* S_{(\alpha,i)} &= \sum_{k=1}^N A(k,\alpha,i) \Big(\sum_{(\beta,j)\in\hat{\Sigma}} A(i,\beta,j) S_{(\beta,j)} S_{(\beta,j)}^* \Big) \\ &= \sum_{(\beta,j)\in\hat{\Sigma}} \sum_{k=1}^N A(k,\alpha,i) A(i,\beta,j) S_{(\beta,j)} S_{(\beta,j)}^* \\ &= \sum_{(\beta,j)\in\hat{\Sigma}} \hat{A}\big((\alpha,i),(\beta,j)\big) S_{(\beta,j)} S_{(\beta,j)}^*. \end{aligned}$$

Hence the C^*-algebra $C^*(S_{(\alpha,i)}; (\alpha,i) \in \hat{\Sigma})$ generated by $S_{(\alpha,i)}, (\alpha,i) \in \hat{\Sigma}$ is isomorphic to the Cuntz–Krieger algebra $\mathcal{O}_{\hat{A}}$ for the matrix $\hat{A}$. By (4.5), we have

$$E_i = \sum_{(\beta,j)\in\hat{\Sigma}} A(i,\beta,j) S_{(\beta,j)} S_{(\beta,j)}^*, \quad S_\alpha = \sum_{i=1}^N S_\alpha E_i = \sum_{i=1}^N S_{(\alpha,i)}$$

so that S_α, E_i are generated by $S_{(\alpha,i)}$, $(\alpha, i) \in \hat{\Sigma}$. We thus have

$$C^*(S_\alpha, E_i;\ \alpha \in \Sigma,\ i = 1, 2, \ldots, N) = C^*\big(S_{(\alpha,i)};\ (\alpha, i) \in \hat{\Sigma}\big)$$

and hence $\mathcal{O}_{G_\Lambda^F} = \mathcal{O}_{\hat{A}}$. ∎

5. Other examples of normal subshifts

In this section, other examples of normal subshifts than irreducible sofic shifts and their C^*-algebras will be presented.

5.1. Dyck shifts

For a positive integer $N > 1$, the Dyck shift D_N of order N was introduced by W. Krieger [14], related to Dyck language in formal language theory in computer science (cf. [11]). Consider an alphabet

$$\Sigma = \Sigma^+ \sqcup \Sigma^- \quad \text{where } \Sigma^- = \{\alpha_1, \ldots, \alpha_N\},\ \Sigma^+ = \{\beta_1, \ldots, \beta_N\}.$$

Following [14], the Dyck inverse monoid for Σ is the inverse monoid defined by the product relations: $\alpha_i \beta_j = \mathbf{1}$ if $i = j$, otherwise $\alpha_i \beta_j = 0$, for $i, j = 1, \ldots, N$. The symbol $\mathbf{1}$ plays a rôle of empty word such that $\alpha_i \mathbf{1} = \mathbf{1}\alpha_i = \alpha_i$, $\beta_j \mathbf{1} = \mathbf{1}\beta_j = \beta_j$. By the product structure, a word $\omega_1 \cdots \omega_n$ of Σ is defined to be admissible if the reduced word of the product $\omega_1 \cdots \omega_n$ in the monoid is not 0. The Dyck shift written D_N is defined to be the subshift over alphabet Σ whose admissible words are the admissible words in this sense. It is well-known that the subshift D_N is not sofic for every $N > 1$. It is shown in [19] that the Dyck shift D_N is λ-synchronizing and hence normal. Its minimal λ-graph system $\mathfrak{L}_{D_N}^{\min} = (V^{\min}, E^{\min}, \lambda^{\min}, \iota^{\min})$ was already studied in [18], in which the minimal λ-graph system $\mathfrak{L}_{D_N}^{\min}$ was called the Cantor horizon λ-graph system written $\mathfrak{L}^{Ch(D_N)}$. Let us briefly review its construction.

Let Λ_N be the two-sided full N-shift over $\{1, 2, \ldots, N\}$. Let

$$V_l^{\min} := \big\{\beta_{\mu_1} \cdots \beta_{\mu_l} \in (\Sigma^+)^l \mid \mu_1 \cdots \mu_l \in B_l(\Lambda_N)\big\} \tag{5.1}$$

and the mapping $\iota^{\min} : V_{l+1}^{\min} \to V_l^{\min}$ is defined by

$$\iota(\beta_{\mu_1} \cdots \beta_{\mu_l} \beta_{\mu_{l+1}}) = \beta_{\mu_1} \cdots \beta_{\mu_l} \quad \text{for } \beta_{\mu_1} \cdots \beta_{\mu_l} \beta_{\mu_{l+1}} \in V_{l+1}^{\min}.$$

Define a labeled edge labeled α_j from $\beta_{\mu_1} \cdots \beta_{\mu_l} \in V_l^{\min}$ to $\beta_{\mu_0} \beta_{\mu_1} \cdots \beta_{\mu_l} \in V_{l+1}^{\min}$ precisely if $\mu_0 = j$. Define a labeled edge labeled β_j from $\beta_j \beta_{\mu_1} \cdots \beta_{\mu_{l-1}} \in V_l^{\min}$ to $\beta_{\mu_1} \cdots \beta_{\mu_l} \beta_{\mu_{l+1}} \in V_{l+1}^{\min}$. Such edges are denoted by $E_{l,l+1}^{\min}$. We then have a λ-graph system presenting the Dyck shift D_N. It is the minimal left-resolving presentation and hence it is the minimal λ-graph system $\mathfrak{L}_{D_N}^{\min}$ (cf. [33]). Since the subshift D_N is λ-irreducible satisfying λ-condition (I), we have the following proposition.

Proposition 5.1 ([18, 29, 33]). *The C^*-algebra $\mathcal{O}_{D_N^{\min}}$ associated with the minimal λ-graph system $\mathfrak{L}_{D_N}^{\min}$ for the Dyck shift D_N is simple and purely infinite.*

The K-groups of the algebra $\mathcal{O}_{D_N^{\min}}$ was computed in the following way:

$$\mathrm{K}_0(\mathcal{O}_{D_N^{\min}}) \cong \mathbb{Z}/N\mathbb{Z} \oplus C(\mathfrak{C}, \mathbb{Z}), \quad \mathrm{K}_1(\mathcal{O}_{D_N^{\min}}) \cong 0$$

where $C(\mathfrak{C}, \mathbb{Z})$ denotes the abelian group of $\mathbb{Z}$-valued continuous functions on a Cantor set $\mathfrak{C}$ [18, 33].

5.2. Markov–Dyck shifts

The class of Markov–Dyck shifts contains the class of Dyck shifts. It is a natural generalization of Dyck shifts as the class of topological Markov shifts contains the class of full shifts. Let $A = [A(i, j)]_{i,j=1}^N$ be an $N \times N$ square matrix with entries in $\{0, 1\}$. We assume that the matrix is irreducible satisfying condition (I) in the sense of Cuntz–Krieger [6]. The Markov–Dyck shift D_A for the matrix A is defined by using the canonical generating partial isometries of the Cuntz–Krieger algebra $\mathcal{O}_A$ in the following way. Let $s_1, \ldots, s_N$ be the canonical generating partial isometries of the Cuntz–Krieger algebra $\mathcal{O}_A$ that satisfies the relations:

$$1 = \sum_{j=1}^N s_j s_j^*, \quad s_i^* s_i = \sum_{j=1}^N A(i, j) s_j s_j^*, \quad i = 1, 2, \ldots, N.$$

Similarly to the Dyck shift, we consider the alphabet

$$\Sigma = \Sigma^+ \sqcup \Sigma^- \quad \text{where } \Sigma^- = \{\alpha_1, \ldots, \alpha_N\},\ \Sigma^+ = \{\beta_1, \ldots, \beta_N\}.$$

Let $\widehat{\alpha}_i = s_i^*, \widehat{\beta}_i = s_i, i = 1, 2, \ldots, N$. We say that a word $\gamma_1 \cdots \gamma_n$ of Σ for $\gamma_1, \ldots, \gamma_n \in \Sigma$ is forbidden if $\widehat{\gamma}_1 \cdots \widehat{\gamma}_n = 0$ in the algebra $\mathcal{O}_A$. The Markov–Dyck shift D_A for the matrix A is defined by the subshift over alphabet Σ by the forbidden words. These kinds of subshifts first appeared in [17] by using certain semigroups. More general setting was studied in [10]. The above definition by using generators of C^*-algebras was seen in [32] (cf. [35]). If all entries of A are one's, then the product structure of $\widehat{\alpha}_i, \widehat{\beta}_i, i = 1, 2, \ldots, N$ goes to that of the Dyck inverse monoid, so that the Markov–Dyck shift D_A coincides with the Dyck shift D_N.

For any irreducible matrix A with entries in $\{0, 1\}$ satisfying condition (I), the Markov–Dyck shift D_A is not sofic [32]. It is always λ-synchronizing and hence normal. Hence we have its minimal λ-graph system $\mathfrak{L}_{D_A}^{\min}$ for D_A. The λ-graph system was studied in [18] in which it was called the Cantor horizon λ-graph system and written $\mathfrak{L}^{Ch(D_A)}$. Let Λ_A denotes the shift space

$$\Lambda_A = \{(x_n)_{n\in\mathbb{Z}} \in \{1, \ldots, N\}^{\mathbb{Z}} \mid A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{Z}\}$$

of the two-sided topological Markov shift defined by the matrix A. We denote by $B_l(\Lambda_A)$ the set of admissible words of Λ_A with its length l. The vertex set $V_l^{\min}$ at level l of the minimal λ-graph system $\mathfrak{L}_{D_A}^{\min}$ is defined by

$$V_l^{\min} := \{\beta_{\mu_1} \cdots \beta_{\mu_l} \in (\Sigma^+)^l \mid \mu_1 \cdots \mu_l \in B_l(\Lambda_A)\}.$$

The mapping $\iota^{\min} : V_{l+1}^{\min} \to V_l^{\min}$ is similarly defined to the minimal λ-graph system $\mathfrak{L}_{D_N}^{\min}$ of the Dyck shift by deleting its rightmost symbol of words in $V_{l+1}^{\min}$. A labeled edge labeled α_j from $\beta_{\mu_1} \cdots \beta_{\mu_l} \in V_l^{\min}$ to $\beta_{\mu_0}\beta_{\mu_1} \cdots \beta_{\mu_l} \in V_{l+1}^{\min}$ is defined precisely if $\mu_0 = j$. A labeled edge labeled β_j from $\beta_j\beta_{\mu_1} \cdots \beta_{\mu_{l-1}} \in V_l^{\min}$ to $\beta_{\mu_1} \cdots \beta_{\mu_l}\beta_{\mu_{l+1}} \in V_{l+1}^{\min}$ is defined. Such edges are denoted by $E_{l,l+1}^{\min}$. We then have a λ-graph system presenting the Markov–Dyck shift D_A. It is the minimal left-resolving presentation and hence it is the minimal λ-graph system $\mathfrak{L}_{D_A}^{\min}$ (cf. [33]). Since the matrix A is irreducible and satisfies condition (I), the subshift D_A is λ-irreducible satisfying λ-condition (I), so that we have the following proposition.

Proposition 5.2 ([18, 33]). *The C^*-algebra $\mathcal{O}_{D_A^{\min}}$ associated with the minimal λ-graph system $\mathfrak{L}_{D_A^{\min}}$ for the Makov–Dyck shift D_A is simple and purely infinite.*

K-group formulas for the C^*-algebras $\mathcal{O}_{D_A^{\min}}$ were studied in [32].

5.3. Motzkin shifts

Motzkin language appears in automata theory as well as Dyck language [11]. The Motzkin shifts are non sofic subshifts associated with the Motzkin language (cf. [27]). For a positive integer $N > 1$, similarly to the Dyck shift, we consider the alphabet $\Sigma = \Sigma^+ \sqcup \Sigma^-$ where $\Sigma^- = \{\alpha_1, \dots, \alpha_N\}$, $\Sigma^+ = \{\beta_1, \dots, \beta_N\}$ and the Dyck inverse monoid for $\Sigma^+ \sqcup \Sigma^-$ as in previous paragraphs. The Dyck inverse monoid is defined by the product relations: $\alpha_i\beta_j = \mathbf{1}$ if $i = j$, otherwise $\alpha_i\beta_j = 0$, for $i, j = 1, \dots, N$. Let us consider a new alphabet set $\Sigma_{\mathbf{1}}$ defined by

$$\Sigma_{\mathbf{1}} = \Sigma^+ \cup \Sigma^- \cup \{\mathbf{1}\}.$$

The Motzkin shift M_N of order N is defined to be a subshift over $\Sigma_{\mathbf{1}}$ such that a word $\gamma_1 \cdots \gamma_n$ of $\Sigma_{\mathbf{1}}$ is forbidden precisely if $\gamma_1 \cdots \gamma_n = 0$. As seen in [27], the subshift M_N is λ-synchronizing and hence normal. Its minimal λ-graph system $\mathfrak{L}_{M_N}^{\min}$ was described as the Cantor horizon λ-graph system written $\mathfrak{L}^{Ch(M_N)}$ in [27]. Let $V_l^{\min}$ be the vertex set defined by (5.1). The mapping $\iota : V_{l+1}^{\min} \to V_l^{\min}$ is similarly defined as in the case of Dyck shifts. Labeled edges labeled symbols in Σ from $V_l^{\min}$ to $V_{l+1}^{\min}$ are defined in a similar way to Dyck shifts. In addition to the labeled edges above, an additional labeled edge labeled $\mathbf{1}$ from $\beta_{\mu_1} \cdots \beta_{\mu_l} \in V_l^{\min}$ to $\beta_{\mu_1} \cdots \beta_{\mu_l}\beta_{\mu_{l+1}} \in V_{l+1}^{\min}$ is defined for every pair $\beta_{\mu_1} \cdots \beta_{\mu_l} \in V_l^{\min}$ and $\beta_{\mu_1} \cdots \beta_{\mu_l}\beta_{\mu_{l+1}} \in V_{l+1}^{\min}$. We then have a λ-graph system that is the minimal λ-graph system $\mathfrak{L}_{M_N}^{\min}$ for the Motzkin shift M_N. Since the λ-graph system $\mathfrak{L}_{M_N}^{\min}$ contains the minimal λ-graph system $\mathfrak{L}_{D_N}^{\min}$ of the Dyck shift D_N as a subsystem, $\mathfrak{L}_{M_N}^{\min}$ is λ-irreducible and satisfies the λ-condition (I). Therefore we have the following proposition.

Proposition 5.3 ([27]). *The C^*-algebra $\mathcal{O}_{M_N^{\min}}$ associated with the minimal λ-graph system $\mathfrak{L}_{M_N}^{\min}$ for the Motzkin shift M_N is simple and purely infinite.*

The K-groups of the algebra $\mathcal{O}_{M_N^{\min}}$ was computed in [27] for the case of $N = 2$. As in the paper [27], the strategy to compute $\mathrm{K}_i(\mathcal{O}_{M_N^{\min}})$, $i = 1, 2$ works well for general $\mathcal{O}_{M_N^{\min}}$, $N = 2, 3, \dots$, so that we have:

$$\mathrm{K}_0(\mathcal{O}_{M_N^{\min}}) \cong C(\mathfrak{C}, \mathbb{Z}), \quad \mathrm{K}_1(\mathcal{O}_{M_N^{\min}}) \cong 0$$

where $C(\mathfrak{C}, \mathbb{Z})$ denotes the abelian group of $\mathbb{Z}$-valued continuous functions on a Cantor set $\mathfrak{C}$ [27].

5.4. β-shifts

The β-shift for real number $\beta > 1$ was first introduced in [43, 48]. It is an interpolation between full shifts, simultaneously one of natural generalization of full shifts. For a real number $\beta > 1$, take a natural number N such that $N - 1 < \beta \le N$. Let $f_\beta : [0, 1] \to [0, 1]$ be the mapping $f_\beta(x) = \beta x - [\beta x]$ for $x \in [0, 1]$, where $[t]$ is the integer part of $t \in \mathbb{R}$. Let $\Sigma = \{0, 1, \dots, N - 1\}$. The β-expansion of $x \in [0, 1]$ is a sequence $d_i(x, \beta)$, $i \in \mathbb{N}$ of Σ defined by

$$d_i(x, \beta) = \left[\beta f_\beta^{i-1}(x)\right], \quad i \in \mathbb{N},$$

so that we know that $x = \sum_{i=1}^{\infty} \frac{d_i(x,\beta)}{\beta^i}$. We endow $\Sigma^{\mathbb{N}}$ with the lexicographical order. Put $\zeta_\beta = \sup_{x \in [0,1)} (d_i(x, \beta))_{i \in \mathbb{N}}$. Define the one-sided subshift X_{Λ_β} by setting

$$X_{\Lambda_\beta} = \left\{\omega \in \Sigma^{\mathbb{N}} \mid \sigma^i(\omega) \le \zeta_\beta,\ i \in \mathbb{Z}_+\right\},$$

where $\sigma^i(\omega) = (\omega_{n+i})_{n \in \mathbb{N}}$ for $\omega = (\omega_n)_{n \in \mathbb{N}}$. Its two-sided extension Λ_β is defined by

$$\Lambda_\beta = \left\{(\omega_n)_{n \in \mathbb{Z}} \in \Sigma^{\mathbb{Z}} \mid (\omega_{n+k})_{n \in \mathbb{N}} \in X_{\Lambda_\beta},\ k \in \mathbb{Z}\right\}.$$

Suppose $\zeta_\beta = (\xi_1, \xi_2, \dots)$ and let

$$b_{\xi_1 \cdots \xi_k} = \beta^k - \xi_1 \beta^{k-1} - \xi_2 \beta^{k-2} - \cdots - \xi_{k-1}\beta - \xi_k.$$

It is shown in [1, Section 4] (cf. [12, Proposition 3.8]) that

(i) Λ_β is a full shift if and only if $b_{\xi_1} = 1$.

(ii) Λ_β is a shift of finite type if and only if $b_{\xi_1 \cdots \xi_k} = 1$ for some $k \ge 1$.

(iii) Λ_β is a sofic subshift if and only if $b_{\xi_1 \cdots \xi_l} = b_{\xi_1 \cdots \xi_m}$ for some $l \ne m$.

Hence Λ_β is not sofic unless β is an algebraic integer. It is shown in [19] that the β-shift Λ_β is λ-synchronizing for every β, so that it is normal. In [12], the C^*-algebra $\mathcal{O}_\beta$ of the β-shift Λ_β was studied (cf. [22]). The C^*-algebra $\mathcal{O}_\beta$ is indeed the C^*-algebra $\mathcal{O}_{\mathfrak{L}^{\min}_{\Lambda_\beta}}$ associated with the minimal λ-graph system $\mathfrak{L}^{\min}_{\Lambda_\beta}$ of the subshift Λ_β.

We will briefly review the construction of $\mathfrak{L}^{\min}_{\Lambda_\beta}$ done in [12]. For $l \in \mathbb{N}$, order the real numbers $\{b_{\xi_1}, b_{\xi_1\xi_2}, \dots, b_{\xi_1\xi_2\cdots\xi_l}\}$ by its usual order in $\mathbb{R}$. They give rise to disjoint intervals partitioned by $\{b_{\xi_1}, b_{\xi_1\xi_2}, \dots, b_{\xi_1\xi_2\cdots\xi_l}\}$ in $[0, 1]$. Let $m(l)$ be the number of the partitions in $(0, 1]$. If Λ_β is sofic, there exist L and l_0 such that $m(l) = L$ for all $l > l_0$. If Λ_β is not sofic, then $m(l) = l + 1$ for all l. Let $v_1^l, \dots, v_{m(l)}^l$ be the ordered set of the disjoint partitions of (0, 1]. The order is defined along the usual order in $\mathbb{R}$. We denote by $V_l^{\min}$ the set $\{v_1^l, \dots, v_{m(l)}^l\}$. Suppose that v_i^l corresponds to the interval $(b_{\xi_1\cdots\xi_q}, b_{\xi_1\cdots\xi_p}]$ with $b_{\xi_1\cdots\xi_q} < b_{\xi_1\cdots\xi_p}$. For $\xi_{p+1} \in \Sigma$, we define the labeled edge labeled ξ_{p+1} from v_i^l to the vertices $v_j^{l+1} \in V_{l+1}^{\min}$ corresponding to the partitions contained in the interval $(b_{\xi_1\cdots\xi_q\xi_{p+1}}, b_{\xi_1\cdots\xi_p\xi_{p+1}}]$. For $0 \le \alpha < \xi_{p+1}$ with $\alpha \in \Sigma$, we define the labeled edge labeled α from v_i^l to the vertices $v_j^{l+1} \in V_{l+1}^{\min}$ corresponding to the partitions contained in the interval $(b_{\xi_1\cdots\xi_q\alpha}, 1]$. Such edges are written $E_{l,l+1}^{\min}$. We define the map $\iota^{\min} : V_{l+1}^{\min} \to V_l^{\min}$ by setting $\iota(v_j^{l+1}) = v_i^l$ if the interval in (0, 1] corresponding to v_j^{l+1} is contained in the interval in (0, 1] corresponding to v_i^{l+1}. The resulting labeled Bratteli diagram becomes a λ-graph system. It is not difficult to see that the λ-graph system is λ-synchronizing and hence minimal (cf. [19]). The C^*-algebra $\mathcal{O}_\beta$ studied in [12] is generated by a finite family $S_0, S_1, \dots, S_{N-1}$ of partial isometries corresponding to the letters of Σ. For an admissible word $\mu \in B_*(\Lambda_\beta)$, put $a_\mu = S_\mu^* S_\mu$. It was proved in [12] that there exists a unique KMS-state written φ for gauge action on $\mathcal{O}_\beta$ (cf. [12]). It is also shown in [12] that

$$\varphi(a_{\xi_1\xi_2\cdots\xi_k}) = b_{\xi_1\xi_2\cdots\xi_k}, \quad k \in \mathbb{N}.$$

By [12, Corollary 3.2], we see for $\alpha \in \Sigma$

$$S_\alpha^* a_{\xi_1\cdots\xi_n} S_\alpha = \begin{cases} 0 & \alpha > \xi_{n+1}, \\ a_{\xi_1\cdots\xi_{n+1}} & \alpha = \xi_{n+1}, \\ 1 & \alpha < \xi_{n+1}. \end{cases} \tag{5.2}$$

Since the projections in the commutative C^*-algebra $\mathcal{A}_\beta$ generated by the projections of the form $a_\mu, \mu \in B_*(\Lambda_\beta)$ is generated by the projection of the form $E_i^l := b_{\xi_1\cdots\xi_p} - b_{\xi_1\cdots\xi_q}$, the relation (5.2) tells us that the C^*-algebra $\mathcal{O}_{\Lambda_\beta^{\min}}$ associated with the minimal λ-graph system $\mathfrak{L}_{\Lambda_\beta}^{\min}$ is canonically isomorphic to the C^*-algebra $\mathcal{O}_\beta$ studied in [12]. We therefore have the following proposition.

Proposition 5.4 ([12, Theorems 3.6 and 4.12]). *The C^*-algebra $\mathcal{O}_{\Lambda_\beta^{\min}}$ of the β-shift Λ_β is simple and purely infinite for each $1 < \beta \in \mathbb{R}$ and*

$$\mathrm{K}_0(\mathcal{O}_{\Lambda_\beta^{\min}}) = \begin{cases} \mathbb{Z}/(\eta_1 + \cdots + \eta_m - 1)\mathbb{Z} & \text{if } d(1,\beta) = \eta_1\eta_2\cdots\eta_m\dot{0}, \\ \mathbb{Z}/(\xi_1 + \cdots + \xi_k)\mathbb{Z} & \text{if } d(1,\beta) = \nu_1\cdots\nu_l\dot{\xi}_1\cdots\dot{\xi}_k, \\ \mathbb{Z} & \text{otherwise}, \end{cases}$$

$$\mathrm{K}_1(\mathcal{O}_{\Lambda_\beta^{\min}}) = \{0\} \quad \text{for any } \beta > 1,$$

where $\dot{0} = 00\cdots$, $\dot{\xi}_1\cdots\dot{\xi}_k = \xi_1\cdots\xi_k\xi_1\cdots\xi_k\cdots$ mean the recurring words.

Remark 5.5. It was shown that the KMS-state for the gauge action on $\mathcal{O}_\beta$ is unique at the inverse temperature $\log\beta$, which is the topological entropy for the β-shift Λ_β [12]. Hence two subshifts $\Lambda_\beta, \Lambda_{\beta'}$ are topologically conjugate if and only if $\beta = \beta'$.

6. Continuous orbit equivalence

In this section, we will discuss continuous orbit equivalence of normal subshifts. Let $\mathfrak{L}_1, \mathfrak{L}_2$ be left-resolving λ-graph systems and $(\Lambda_1, \sigma_{\Lambda_1})$, $(\Lambda_2, \sigma_{\Lambda_2})$ their associated two-sided subshifts, respectively. In [37], the notion of $(\mathfrak{L}_1, \mathfrak{L}_2)$-continuous orbit equivalence between their one-sided subshifts $(X_{\Lambda_1}, \sigma_{\Lambda_1})$, $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ was introduced in the following way.

Definition 6.1 ([37, Definition 4.1], [30, Section 6]). Let $\mathfrak{L}_1, \mathfrak{L}_2$ be left-resolving λ-graph systems. Then their one-sided subshifts $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are said to be $(\mathfrak{L}_1, \mathfrak{L}_2)$-*continuously orbit equivalent* if there exist two homeomorphisms $h_{\mathfrak{L}} : X_{\mathfrak{L}_1} \to X_{\mathfrak{L}_2}$ and $h_\Lambda : X_{\Lambda_1} \to X_{\Lambda_2}$ and continuous functions $k_i, l_i : X_{\mathfrak{L}_i} \to \mathbb{Z}_+$, $i = 1, 2$ such that $\pi_{\mathfrak{L}_2} \circ h_{\mathfrak{L}} = h_\Lambda \circ \pi_{\mathfrak{L}_1}$ and

$$\sigma_{\mathfrak{L}_2}^{k_1(x)}\big(h_{\mathfrak{L}}\big(\sigma_{\mathfrak{L}_1}(x)\big)\big) = \sigma_{\mathfrak{L}_2}^{l_1(x)}\big(h_{\mathfrak{L}}(x)\big), \quad x \in X_{\mathfrak{L}_1},$$
$$\sigma_{\mathfrak{L}_1}^{k_2(y)}\big(h_{\mathfrak{L}}^{-1}\big(\sigma_{\mathfrak{L}_2}(y)\big)\big) = \sigma_{\mathfrak{L}_1}^{l_2(y)}\big(h_{\mathfrak{L}}^{-1}(y)\big), \quad y \in X_{\mathfrak{L}_2}.$$

We first show the following lemma.

Lemma 6.2. *Let $\mathfrak{L}_1, \mathfrak{L}_2$ be left-resolving λ-graph systems satisfying condition (I) and $(\Lambda_1, \sigma_{\Lambda_1})$, $(\Lambda_2, \sigma_{\Lambda_2})$ their associated two-sided subshifts, respectively. Suppose that one-sided subshifts $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are $(\mathfrak{L}_1, \mathfrak{L}_2)$-continuously orbit equivalent. If Λ_1 is a normal subshift and $\mathfrak{L}_1$ is its minimal presentation of Λ_1, then Λ_2 is also normal and $\mathfrak{L}_2$ is its minimal presentation.*

Proof. Assume that $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are $(\mathfrak{L}_1, \mathfrak{L}_2)$-continuously orbit equivalent and $\mathfrak{L}_1$ is the minimal presentation of the normal subshift Λ_1. By [37, Theorem 1.2], there exists an isomorphism $\Phi : \mathcal{O}_{\mathfrak{L}_1} \to \mathcal{O}_{\mathfrak{L}_2}$ of C^*-algebras such that $\Phi(\mathcal{D}_{\Lambda_1}) = \mathcal{D}_{\Lambda_2}$. Now $\mathfrak{L}_1 = \mathfrak{L}_{\Lambda_1}^{\min}$, so that we may write $\mathcal{O}_{\mathfrak{L}_1} = \mathcal{O}_{\Lambda_1{}^{\min}}$. Let S_α^1, E_i^{1l} and S_α^2, E_i^{2l} be the canonical generators of the C^*-algebras $\mathcal{O}_{\Lambda_1{}^{\min}}$ and $\mathcal{O}_{\mathfrak{L}_2}$, respectively. By Proposition 3.8, the condition $\Phi(\mathcal{D}_{\Lambda_1}) = \mathcal{D}_{\Lambda_2}$ implies $\Phi(\mathcal{D}_{\mathfrak{L}_1}) = \mathcal{D}_{\mathfrak{L}_2}$. Hence for a vertex v_i^{2l} in $\mathfrak{L}_2$ and the corresponding projection $E_i^{2l} \in \mathcal{D}_{\mathfrak{L}_2}$, we have $\Phi^{-1}(E_i^{2l}) \in \mathcal{D}_{\mathfrak{L}_1}$. We may find a word $\nu \in B_*(\Lambda_1)$ and a vertex v_j^{1l} in $\mathfrak{L}_1$ such that

$$\Phi^{-1}(E_i^{2l}) \geq S_\nu^1 E_j^{1l} S_\nu^{1*}, \quad S_\nu^{1*} S_\nu^1 \geq E_j^{1l}.$$

Since Λ_1 is normal, there exists a word $\eta \in B_*(\Lambda_1)$ such that $E_j^{1l} \geq S_\eta^1 S_\eta^{1*}$ by [33, Proposition 3.3], so that

$$S_\nu^1 E_j^{1l} S_\nu^{1*} \geq S_\nu^1 S_\eta^1 S_\eta^{1*} S_\nu^{1*} \neq 0.$$

Hence we have

$$E_i^{2l} \geq \Phi(S_{\nu\eta}^1 S_{\nu\eta}^{1*}). \tag{6.1}$$

Since $\Phi(\mathcal{D}_{\Lambda_1}) = \mathcal{D}_{\Lambda_2}$, one may find $\mu \in B_*(\Lambda_2)$ such that

$$\Phi(S^1_{\nu\eta} S^{1*}_{\nu\eta}) \geq S^2_{\mu} S^{2*}_{\mu}. \tag{6.2}$$

By (6.1), (6.2), we have

$$E^{2l}_i \geq S^2_{\mu} S^{2*}_{\mu}.$$

This implies that the vertex v^{2l}_i in $\mathfrak{L}_2$ launches μ by [33, Proposition 3.3] so that the λ-graph system $\mathfrak{L}_2$ is λ-synchronizing. Therefore we conclude that the subshift Λ_2 is normal and $\mathfrak{L}_2$ is its minimal presentation. ∎

Now the following definition seems to be reasonable.

Definition 6.3. Let (Λ_1, σ_1) and (Λ_2, σ_2) be normal subshifts. Their one-sided subshifts $(X_{\Lambda_1}, \sigma_1)$ and $(X_{\Lambda_2}, \sigma_2)$ are said to be *continuously orbit equivalent* if they are $(\mathfrak{L}^{\min}_{\Lambda_1}, \mathfrak{L}^{\min}_{\Lambda_2})$-continuously orbit equivalent.

Therefore we know the following proposition.

Proposition 6.4 ([37, Theorem 1.2]). *Let $(\Lambda_1, \sigma_{\Lambda_1})$ and $(\Lambda_2, \sigma_{\Lambda_2})$ be normal subshifts. Then the following two assertions are equivalent:*

(i) *Their one-sided subshifts $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are continuously orbit equivalent.*

(ii) *There exists an isomorphism $\Phi : \mathcal{O}_{{\Lambda_1}^{\min}} \to \mathcal{O}_{{\Lambda_2}^{\min}}$ of C^*-algebras such that $\Phi(\mathcal{D}_{\Lambda_1}) = \mathcal{D}_{\Lambda_2}$.*

We note the following proposition.

Proposition 6.5. *Let $(\Lambda_1, \sigma_{\Lambda_1})$ and $(\Lambda_2, \sigma_{\Lambda_2})$ be normal subshifts such that their one-sided subshifts $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are continuously orbit equivalent.*

(i) *$(X_{\Lambda_1}, \sigma_{\Lambda_1})$ is a shift of finite type if and only if $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ is a shift of finite type.*

(ii) *$(X_{\Lambda_1}, \sigma_{\Lambda_1})$ is a sofic shift if and only if $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ is a sofic shift.*

Proof. The minimal presentations $\mathfrak{L}^{\min}_{\Lambda_1}, \mathfrak{L}^{\min}_{\Lambda_2}$ of Λ_1, Λ_2 are written $\mathfrak{L}_1, \mathfrak{L}_2$, respectively.

(i) It is easy to see that a normal subshift Λ is a shift of finite type if and only if $\mathcal{D}_{\Lambda} = \mathcal{D}_{\mathfrak{L}^{\min}_{\Lambda}}$. Now there exists an isomorphism $\Phi : \mathcal{O}_{{\Lambda_1}^{\min}} \to \mathcal{O}_{{\Lambda_2}^{\min}}$ of C^*-algebras such that $\Phi(\mathcal{D}_{\Lambda_1}) = \mathcal{D}_{\Lambda_2}$. Since $\mathcal{D}_{\mathfrak{L}^{\min}_{\Lambda}} = \mathcal{D}'_{\Lambda} \cap \mathcal{O}_{\Lambda^{\min}}$ for a normal subshift Λ, we know that $\mathcal{D}_{\mathfrak{L}_1} = \mathcal{D}_{\Lambda_1}$ if and only if $\mathcal{D}_{\mathfrak{L}_2} = \mathcal{D}_{\Lambda_2}$. Hence $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ is a shift of finite type if and only if $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ is a shift of finite type.

(ii) Suppose that $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ is sofic. As in Section 4, the dynamical system $(X_{\mathfrak{L}_1}, \sigma_{\mathfrak{L}_1})$ is a shift of finite type. We know that the class of shifts of finite type is preserved under continuous orbit equivalence by the above discussion (i). By definition, the shift of finite type $(X_{\mathfrak{L}_1}, \sigma_{\mathfrak{L}_1})$ is continuously orbit equivalent to $(X_{\mathfrak{L}_2}, \sigma_{\mathfrak{L}_2})$ as shifts of finite type

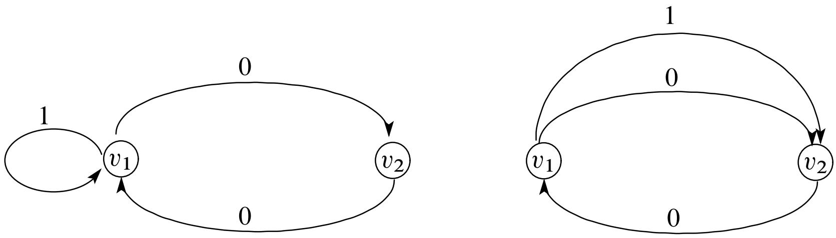

Figure 1. Left Fischer covers of Λ_0 and Λ_1.

(cf. [31]). Hence $(X_{\mathfrak{L}_2}, \sigma_{\mathfrak{L}_2})$ is a shift of finite type. As there exists a factor map

$$\pi_2 : X_{\mathfrak{L}_2} \to X_{\Lambda_2}$$

such that $\pi_2 \circ \sigma_{\mathfrak{L}_2} = \sigma_{\Lambda_2} \circ \pi_2$, we see that $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ is a sofic shift by [49]. ∎

Proposition 6.6. *Let $(\Lambda_i, \sigma_{\Lambda_i})$, $i = 1, 2$ be sofic shifts and $G^F_{\Lambda_i}$, $i = 1, 2$ be its left Fischer cover graphs. Let us denote by $\widehat{A}_i$, $i = 1, 2$ the transition matrices of the graphs $G^F_{\Lambda_i}$, $i = 1, 2$. Let $\pi_i : X_{\widehat{A}_i} \to X_{\Lambda_i}$, $i = 1, 2$ be the natural factor maps from the shifts of finite type $X_{\widehat{A}_i}$ to the sofic shifts X_{Λ_i}, $i = 1, 2$. Then the following three assertions are equivalent.*

(i) *Their one-sided sofic shifts $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are continuously orbit equivalent.*

(ii) *The shifts of finite type $(X_{\widehat{A}_1}, \sigma_{\widehat{A}_1})$ and $(X_{\widehat{A}_2}, \sigma_{\widehat{A}_1})$ are continuously orbit equivalent via a homeomorphism $h_{\widehat{A}} : X_{\widehat{A}_1} \to X_{\widehat{A}_2}$ such that there exists a homeomorphism $h_\Lambda : X_{\Lambda_1} \to X_{\Lambda_2}$ satisfying $\pi_2 \circ h_{\widehat{A}} = h_\Lambda \circ \pi_1$.*

(iii) *There exists an isomorphism $\Phi : \mathcal{O}_{\widehat{A}_1} \to \mathcal{O}_{\widehat{A}_2}$ of Cuntz–Krieger algebras such that $\Phi(C(X_{\Lambda_1})) = C(X_{\Lambda_2})$, where $C(X_{\Lambda_i})$ is embedded into $C(X_{\widehat{A}_i}) \subset \mathcal{O}_{\widehat{A}_i}$ through the factor maps $\pi_i : X_{\widehat{A}_i} \to X_{\Lambda_i}$, $i = 1, 2$.*

Proof. Since the topological dynamical systems $(X_{\mathfrak{L}_{\Lambda_i}}, \sigma_{\mathfrak{L}_{\Lambda_i}})$ are the shifts of finite type $(X_{\widehat{A}_i}, \sigma_{\widehat{A}_i})$, $i = 1, 2$, the assertions are direct from the previous discussions. ∎

We will give an example (cf. [4, Example 6.15]).

Example 6.7. Let Λ_0 and Λ_1 be the even shift over the alphabet $\{0, 1\}$ and the odd shift over the alphabet $\{0, 1\}$, respectively. Their forbidden words $\mathfrak{F}_*(\Lambda_0)$ are $\mathfrak{F}_*(\Lambda_1)$ are defined by

$$\mathfrak{F}_*(\Lambda_0) = \{10^{2n+1}1 \mid n \in \mathbb{Z}_+\}, \quad \mathfrak{F}_*(\Lambda_1) = \{10^{2n}1 \mid n \in \mathbb{Z}_+\}$$

where $10^k 1 = 1\overbrace{0\cdots0}^{k}1$ for $k = 2n+1, 2n$. It is well-known that the subshifts Λ_0, Λ_1 are both sofic shifts. Their left Fischer covers $G^F_{\Lambda_0}$, $G^F_{\Lambda_1}$ are shown in Figure 1, respectively.

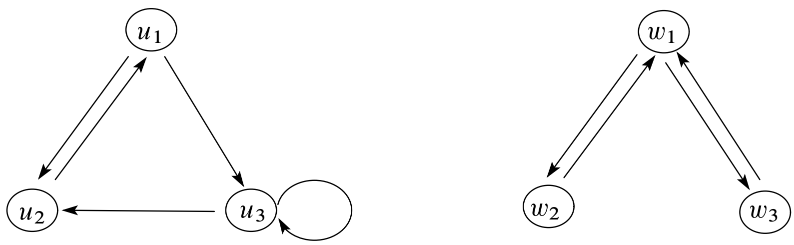

Figure 2. Transition graphs for the left Fischer covers of Λ_0 and Λ_1.

We write $\alpha = 0$, $\beta = 1$ for the alphabet $\{0, 1\}$. To describe the transition matrices for the Fischer cover graphs $G^F_{\Lambda_0}$, $G^F_{\Lambda_1}$, consider the new alphabet sets $\widehat{\Sigma}_0$, $\widehat{\Sigma}_1$ by setting

$$\widehat{\Sigma}_0 := \big\{(\alpha, v_1),\ (\alpha, v_2),\ (\beta, v_1)\big\}, \quad \widehat{\Sigma}_1 := \big\{(\alpha, v_1),\ (\alpha, v_2),\ (\beta, v_2)\big\},$$

and put

$$u_1 := (\alpha, v_1), \quad u_2 := (\alpha, v_2), \quad u_3 := (\beta, v_1) \quad \text{in } \widehat{\Sigma}_0,$$
$$w_1 := (\alpha, v_1), \quad w_2 := (\alpha, v_2), \quad w_3 := (\beta, v_2) \quad \text{in } \widehat{\Sigma}_1.$$

We then have the associated transition graphs for $G^F_{\Lambda_0}$ and $G^F_{\Lambda_1}$, respectively. They are shown in Figure 2. Their transition matrices are denoted by $\widehat{A}_0$ and $\widehat{A}_1$, respectively. They are written

$$\widehat{A}_0 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \widehat{A}_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Let s_1, s_2, s_3 and t_1, t_2, t_3 be the generating partial isometries of the Cuntz–Krieger algebras $\mathcal{O}_{\widehat{A}_0}$ and $\mathcal{O}_{\widehat{A}_1}$, respectively. They satisfy the following operator relations:

$$\sum_{i=1}^{3} s_i s_i^* = 1, \quad s_1^* s_1 = s_2 s_2^* + s_3 s_3^*, \quad s_2^* s_2 = s_1 s_1^*, \quad s_3^* s_3 = s_2 s_2^* + s_3 s_3^*,$$
$$\sum_{i=1}^{3} t_i t_i^* = 1, \quad t_1^* t_1 = t_2 t_2^* + t_3 t_3^*, \quad t_2^* t_2 = t_1 t_1^*, \quad t_3^* t_3 = t_1 t_1^*.$$

Proposition 6.8. *There exists an isomorphism $\Phi : \mathcal{O}_{\widehat{A}_0} \to \mathcal{O}_{\widehat{A}_1}$ of Cuntz–Krieger algebras such that*

$$\Phi(\mathcal{D}_{\widehat{A}_0}) = \mathcal{D}_{\widehat{A}_1}, \quad \Phi\big(C(X_{\Lambda_0})\big) = C(X_{\Lambda_1}),$$

where $\mathcal{D}_{\widehat{A}_i} = C(X_{\widehat{A}_i})$, $i = 0, 1$.

Proof. Put $s'_1 = t_1$, $s'_2 = t_2$, $s'_3 = t_3 t_1$. They are partial isometries in $\mathcal{O}_{\widehat{A}_1}$ satisfying

$$s'_1 s'^*_1 + s'_2 s'^*_2 + s'_3 s'^*_3 = 1, \qquad s'^*_1 s'_1 = s'_2 s'^*_2 + s'_3 s'^*_3,$$
$$s'^*_2 s'_2 = s'_1 s'^*_1, \quad s'^*_3 s'_3 = s'_2 s'^*_2 + s'_3 s'^*_3.$$

Since $t_3 = s_3' s_1'^*$, by putting $\Phi(s_i) = s_i'$, $i = 1, 2, 3$, Φ extends an isomorphism from $\mathcal{O}_{\widehat{A}_0}$ to $\mathcal{O}_{\widehat{A}_1}$. For an admissible word μ of the shift of finite type $(\Lambda_{\widehat{A}_0}, \sigma_{\widehat{A}_0})$ defined by the matrix $\widehat{A}_0$, denote by $\widetilde{\mu}$ an admisible word of $(\Lambda_{\widehat{A}_1}, \sigma_{\widehat{A}_1})$ defined by substituting

$$1 \to 1, \quad 2 \to 2, \quad 3 \to 31.$$

It is direct to see that the equality $\Phi(s_\mu s_\mu^*) = t_{\widetilde{\mu}} t_{\widetilde{\mu}}^*$ holds. Hence we have $\Phi(\mathcal{D}_{\widehat{A}_0}) = \mathcal{D}_{\widehat{A}_1}$.

We will next show that $\Phi(C(X_{\Lambda_0})) = C(X_{\Lambda_1})$. Define the partial isometries by setting

$$S_\alpha := s_1 + s_2, \quad S_\beta := s_3 \quad \text{in } \mathcal{O}_{\widehat{A}_0} \quad \text{and} \quad T_\alpha := t_1 + t_2, \quad T_\beta := t_3 \quad \text{in } \mathcal{O}_{\widehat{A}_1}.$$

It is easy to see that the equalities

$$\Phi(S_\alpha) = T_\alpha, \quad \Phi(S_\beta) = T_{\beta\alpha} \quad \text{and} \quad \Phi^{-1}(T_\alpha) = S_\alpha, \quad \Phi^{-1}(T_\beta) = S_\beta S_\alpha^*$$

hold. For $\xi \in B_*(\Lambda_0)$, let $\overline{\xi}$ be the admissible word of Λ_1 by substituting

$$\alpha \to \alpha, \quad \beta \to \beta\alpha$$

in ξ. Then we have $\Phi(S_\xi S_\xi^*) = T_{\overline{\xi}} T_{\overline{\xi}}^*$. As $C(X_{\Lambda_0})$ and $C(X_{\Lambda_1})$ are generated by projections $S_\xi S_\xi^*$, $\xi \in B_*(\Lambda_0)$ and $T_\eta T_\eta^*$, $\eta \in B_*(\Lambda_1)$, respectively, we know that

$$\Phi\big(C(X_{\Lambda_0})\big) = C(X_{\Lambda_1}). \qquad \blacksquare$$

Corollary 6.9. *The even shift $(X_{\Lambda_0}, \sigma_{\Lambda_0})$ and the odd shift $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ are continuously orbit equivalent to each other.*

Remark 6.10. Keep the above notation for S_α, S_β and T_α, T_β with $\alpha = 0$, $\beta = 1$. Let us denote by $C^*(S_\alpha, S_\beta)$ the C^*-subalgebra of $\mathcal{O}_{\widehat{A}_0}$ generated by the partial isometries S_α, S_β. It is easy to see that the identities

$$s_1 = S_\alpha^* S_\beta^* S_\beta S_\alpha S_\alpha, \quad s_2 = S_\alpha - S_\alpha^* S_\beta^* S_\beta S_\alpha S_\alpha, \quad s_3 = S_\beta$$

hold, so that the C^*-subalgebra $C^*(S_\alpha, S_\beta)$ coincides with $\mathcal{O}_{\widehat{A}_0}$. Similarly we know the identities

$$t_1 = T_\beta^* T_\beta T_\alpha, \quad t_2 = T_\alpha - T_\beta^* T_\beta T_\alpha, \quad t_3 = T_\beta$$

so that the C^*-subalgebra $C^*(T_\alpha, T_\beta)$ coincides with $\mathcal{O}_{\widehat{A}_1}$.

7. One-sided topological conjugacy

In what follows, a sliding bock code means a shift commuting continuous map between subshifts. Such a map is always given by a block map (see for instance [21]).

In this section, we will prove that the triplet $(\mathcal{O}_{\Lambda^{\min}}, \mathcal{D}_\Lambda, \rho^\Lambda)$ for a normal subshift Λ is invariant under topological conjugacy of one-sided subshifts, where ρ^Λ denotes the gauge action $\rho^{\mathfrak{L}_\Lambda^{\min}}$ on $\mathcal{O}_{\Lambda^{\min}}$ defined in (3.2). For a left-resolving λ-graph system $\mathfrak{L}$, let us denote by Λ the associated subshift. Recall from Section 3 that there exists a natural factor map $\pi_{\mathfrak{L}} : X_{\mathfrak{L}} \to X_\Lambda$ such that $\pi_{\mathfrak{L}} \circ \sigma_{\mathfrak{L}} = \sigma_\Lambda \circ \pi_{\mathfrak{L}}$ that is defined in Section 3.

Definition 7.1 ([30,37]). Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be left-resolving λ-graph systems with associated subshifts Λ_1 and Λ_2, respectively. The one-sided subshifts $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are said to be $(\mathfrak{L}_1, \mathfrak{L}_2)$-*conjugate* if there exist topological conjugacies

$$h_{\mathfrak{L}} : (X_{\mathfrak{L}_1}, \sigma_{\mathfrak{L}_1}) \to (X_{\mathfrak{L}_2}, \sigma_{\mathfrak{L}_2}) \quad \text{and} \quad h_{\Lambda} : (X_{\Lambda_1}, \sigma_{\Lambda_1}) \to (X_{\Lambda_2}, \sigma_{\Lambda_2})$$

such that $\pi_{\mathfrak{L}_2} \circ h_{\mathfrak{L}} = h_{\Lambda} \circ \pi_{\mathfrak{L}_1}$.

Equivalently, there exist homeomorphisms $h_{\mathfrak{L}} : X_{\mathfrak{L}_1} \to X_{\mathfrak{L}_2}$ and $h_{\Lambda} : X_{\Lambda_1} \to X_{\Lambda_2}$ such that

$$\begin{cases} h_{\mathfrak{L}}\big(\sigma_{\mathfrak{L}_1}(x)\big) = \sigma_{\mathfrak{L}_2}\big(h_{\mathfrak{L}}(x)\big), & x \in X_{\mathfrak{L}_1}, \\ h_{\mathfrak{L}}^{-1}\big(\sigma_{\mathfrak{L}_2}(y)\big) = \sigma_{\mathfrak{L}_1}\big(h_{\mathfrak{L}}^{-1}(y)\big), & y \in X_{\mathfrak{L}_2}, \end{cases} \tag{7.1}$$

and

$$\pi_{\mathfrak{L}_2} \circ h_{\mathfrak{L}} = h_{\Lambda} \circ \pi_{\mathfrak{L}_1}. \tag{7.2}$$

We remark that the equalities (7.1) and (7.2) automatically imply the equalities

$$\begin{cases} h_{\Lambda}\big(\sigma_{\Lambda_1}(a)\big) = \sigma_{\Lambda_2}\big(h_{\Lambda}(a)\big), & a \in X_{\Lambda_1}, \\ h_{\Lambda}^{-1}\big(\sigma_{\Lambda_2}(b)\big) = \sigma_{\Lambda_1}\big(h_{\Lambda}^{-1}(b)\big), & b \in X_{\Lambda_2}. \end{cases} \tag{7.3}$$

We note that if one-sided subshifts $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are $(\mathfrak{L}_1, \mathfrak{L}_2)$-conjugate, they are $(\mathfrak{L}_1, \mathfrak{L}_2)$-eventually conjugate in the sense of [37] and in the sense of the following section ((8.1)).

Lemma 7.2. *Let $\mathfrak{L}_{\Lambda_1}^{\min}$ and $\mathfrak{L}_{\Lambda_2}^{\min}$ be the minimal λ-graph systems for normal subshifts Λ_1 and Λ_2, respectively. Assume that there exists a topological conjugacy $h : X_{\Lambda_1} \to X_{\Lambda_2}$. Then there exists $L \in \mathbb{N}$ such that for any $l \in \mathbb{N}$ and a word $\mu \in S_l(\Lambda_1)$, there exists a word $\widetilde{\mu} \in S_l(\Lambda_2)$ with $|\widetilde{\mu}| = |\mu| + L$ such that*

- (i) *for $\eta \in \Gamma_l^-(\widetilde{\mu})$ and $y \in \Gamma_\infty^+(\widetilde{\mu})$, the equality $h^{-1}(\eta\widetilde{\mu}y)_{[l+1,l+|\mu|]} = \mu$ holds,*
- (ii) *there exists $\gamma \in \Gamma_{2L}^+(\mu)$ such that for $\xi \in \Gamma_l^-(\mu)$, $x \in \Gamma_\infty^+(\xi\mu\gamma)$, the equality $h(\xi\mu\gamma x)_{[l+1,l+|\mu|+L]} = \widetilde{\mu}$ holds, so that $h(\mu\gamma x)_{[1,|\mu|+L]} = \widetilde{\mu}$.*

Proof. Let Λ_1, Λ_2 be normal subshifts over alphabets Σ_1, Σ_2, respectively. Since $h : X_{\Lambda_1} \to X_{\Lambda_2}$ is a topological conjugacy, there exist $L \in \mathbb{N}$ and block maps

$$\varphi : B_{L+1}(\Lambda_1) \to \Sigma_2, \quad \phi : B_{L+1}(\Lambda_2) \to \Sigma_1,$$

such that

$$h = \varphi_\infty^{[0,L]} : X_{\Lambda_1} \to X_{\Lambda_2} \quad \text{and} \quad h^{-1} = \phi_\infty^{[0,L]} : X_{\Lambda_2} \to X_{\Lambda_1}$$

where $\varphi_\infty^{[0,L]}((x_n)_{n\in\mathbb{N}}) = (\varphi(x_n, \dots, x_{L+n}))_{n\in\mathbb{N}}$ and $\phi_\infty^{[0,L]}$ is similarly defined (see [21]). Let $\mu \in S_l(\Lambda_1)$ with $\mu = (\mu_1, \dots, \mu_m)$. Since $h^{-1} : X_{\Lambda_2} \to X_{\Lambda_1}$ is a sliding block code, there exists an admissible word $\widetilde{\mu} = (\widetilde{\mu}_1, \dots, \widetilde{\mu}_{|\mu|+L}) \in B_{|\mu|+L}(\Lambda_2)$ such that

$$\mu_n = \phi(\widetilde{\mu}_n, \dots, \widetilde{\mu}_{n+L}), \quad n = 1, 2, \dots, m.$$

Suppose that $y, y' \in \Gamma_\infty^+(\widetilde{\mu})$ and $\eta \in \Gamma_l^-(\widetilde{\mu}y)$. Hence we have $h^{-1}(\eta\widetilde{\mu}y) \in X_{\Lambda_1}$ such that $h^{-1}(\eta\widetilde{\mu}y)_{[l+1,l+|\mu|]} = \mu$. Take $\xi \in B_l(\Lambda_1)$ such that $h^{-1}(\eta\widetilde{\mu}y) = \xi h^{-1}(\widetilde{\mu}y)$. Since $h^{-1}(\widetilde{\mu}y) = \mu z$ for some $z \in \Gamma_\infty^+(\mu)$, we have $h^{-1}(\eta\widetilde{\mu}y) = \xi\mu z$, so that we have $\xi \in \Gamma_l^-(\mu z)$.

Let $h^{-1}(\widetilde{\mu}y') = \mu z'$ for some $z' \in \Gamma_\infty^+(\mu)$. As $\mu \in S_l(\Lambda_1)$, the condition $\xi \in \Gamma_l^-(\mu z)$ implies $\xi \in \Gamma_l^-(\mu z')$, so that $\xi\mu z' \in X_{\Lambda_1}$. As $h(\xi\mu z) = \eta\widetilde{\mu}y$, we see $h(\xi\mu)_{[1,l]} = \eta$. Now $\widetilde{\mu}y' = h(\mu z')$ so that we have

$$h(\xi\mu z') = h(\xi\mu)_{[1,l]}h(\mu z') = \eta\widetilde{\mu}y'.$$

Hence we have $\eta \in \Gamma_l^-(\widetilde{\mu}y')$. This implies that $\Gamma_l^-(\widetilde{\mu}y) = \Gamma_l^-(\widetilde{\mu}y')$ so that we conclude that $\widetilde{\mu} \in S_l(\Lambda_2)$. One may find $\gamma = (\gamma_1, \dots, \gamma_{2L}) \in \Gamma_{2L}^+(\mu)$ such that $h(\mu\gamma x)_{[1,|\mu|+L]} = \widetilde{\mu}$ for any $x \in \Gamma_\infty^+(\mu\gamma)$. Hence $h(\xi\mu\gamma x)_{[l+1,l+|\mu|+L]} = \widetilde{\mu}$ holds for $\xi \in \Gamma_l^-(\mu)$, $x \in \Gamma_\infty^+(\xi\mu\gamma)$. Since $h \circ \sigma_{\Lambda_1} = \sigma_{\Lambda_2} \circ h$, we know that $h(\mu\gamma x)_{[1,|\mu|+L]} = \widetilde{\mu}$. ■

Lemma 7.3. *For $\mu \in S_l(\Lambda_1)$, let $\widetilde{\mu} \in S_l(\Lambda_2)$ be as above. For $\gamma' \in \Gamma_{2L}^+(\mu)$, put $\widetilde{\mu}' := h(\xi\mu\gamma' x')_{[l+1,l+|\mu|+L]} \in S_l(\Lambda_2)$ for some $\xi \in \Gamma_l^-(\mu)$, $x' \in \Gamma_\infty^+(\xi\mu\gamma')$. Then $\widetilde{\mu} \sim_l \widetilde{\mu}'$ in $S_l(\Lambda_2)$. Hence the l-past equivalence class of $\widetilde{\mu}$ does not depend on the choice of γ and x as long as $\xi\mu\gamma x \in X_{\Lambda_1}$.*

Proof. We first note that $\widetilde{\mu}' = h(\mu\gamma' x')_{[1,|\mu|+L]}$. For $\eta \in \Gamma_l^-(\widetilde{\mu})$, take $y \in \Gamma_\infty^+(\widetilde{\mu})$ such that $\eta\widetilde{\mu}y \in X_{\Lambda_2}$. Hence $h^{-1}(\eta\widetilde{\mu}y) = \xi\mu z \in X_{\Lambda_1}$. As $\gamma' \in \Gamma_{2L}^+(\mu)$ and hence $\xi\mu\gamma' \in B_*(\Lambda_1)$, we have $\xi\mu\gamma' x' \in X_{\Lambda_1}$ for any $x' \in \Gamma_\infty^+(\xi\mu\gamma')$. We then have

$$h(\xi\mu\gamma' x')_{[1,|\mu|+L]} = \eta h(\xi\mu\gamma' x')_{[l+1,|\mu|+L]} = \eta\widetilde{\mu}'$$

so that $\eta \in \Gamma_l^-(\widetilde{\mu}')$ and hence $\Gamma_l^-(\widetilde{\mu}) \subset \Gamma_l^-(\widetilde{\mu}')$. Similarly we have $\Gamma_l^-(\widetilde{\mu}') \subset \Gamma_l^-(\widetilde{\mu})$ so that $\Gamma_l^-(\widetilde{\mu}) = \Gamma_l^-(\widetilde{\mu}')$. ■

Lemma 7.4. *Suppose $\nu \in B_*(\Lambda_1)$ with $|\nu| \geq L$ and $\nu\gamma \in S_l(\Lambda_1)$ and $\nu\delta \in S_{l+1}(\Lambda_1)$ for some $\gamma, \delta \in \Gamma_*^+(\nu)$ such that $\nu\gamma \sim_l \nu\delta$. Then we have $\widetilde{\nu\gamma} \in S_l(\Lambda_2)$ and $\widetilde{\nu\delta} \in S_{l+1}(\Lambda_2)$ such that $\widetilde{\nu\gamma} \sim_l \widetilde{\nu\delta}$.*

Proof. By the previous lemma, we know that $\widetilde{\nu\gamma} \in S_l(\Lambda_2)$ and $\widetilde{\nu\delta} \in S_{l+1}(\Lambda_2)$. It suffices to show that $\widetilde{\nu\gamma} \sim_l \widetilde{\nu\delta}$. For $\eta \in \Gamma_l^-(\widetilde{\nu\gamma})$, we have $\eta\widetilde{\nu\gamma}y \in X_{\Lambda_2}$ for some $y \in X_{\Lambda_2}$. Hence we have $h^{-1}(\eta\widetilde{\nu\gamma}y) = \xi\nu\gamma z$ for some $\xi \in \Gamma_l^-(\nu\gamma)$, $z \in \Gamma_l^+(\nu\gamma)$. As $\nu\gamma \sim_l \nu\delta$ and hence $\xi\nu\delta \in B_*(\Lambda_1)$, we see $\xi\nu\delta z' \in X_{\Lambda_1}$ for some $z' \in X_{\Lambda_1}$. Since $\eta\widetilde{\nu\gamma}y = h(\xi\nu\gamma z)$, we have

$$h(\xi\nu\delta z') = \eta h(\nu\delta z')_{[l+1,\infty)} = \eta\widetilde{\nu\delta}y' \quad \text{for some } y' \in \Gamma_\infty^+(\widetilde{\nu\delta}).$$

Hence we have $\eta \in \Gamma_l^-(\widetilde{\nu\delta})$ so that $\Gamma_l^-(\widetilde{\nu\gamma}) \subset \Gamma_l^-(\widetilde{\nu\delta})$. Similarly we have $\Gamma_l^-(\widetilde{\nu\delta}) \subset \Gamma_l^-(\widetilde{\nu\gamma})$ so that $\Gamma_l^-(\widetilde{\nu\gamma}) = \Gamma_l^-(\widetilde{\nu\delta})$. ■

Proposition 7.5. *Let $\mathfrak{L}_{\Lambda_1}^{\min}$ and $\mathfrak{L}_{\Lambda_2}^{\min}$ be the minimal λ-graph systems for normal subshifts Λ_1 and Λ_2, respectively. Assume that the one-sided subshifts $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are topologically conjugate. Then they are $(\mathfrak{L}_{\Lambda_1}^{\min}, \mathfrak{L}_{\Lambda_2}^{\min})$-conjugate.*

Proof. Let $h : X_{\Lambda_1} \to X_{\Lambda_2}$ be a topological conjugacy. Keep the notation as in the previous lemmas. For $(\alpha_i, u_i)_{i\in\mathbb{N}} \in X_{\mathfrak{L}^{\min}_{\Lambda_1}}$ where $u_i = (u_i^l)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}^{\min}_{\Lambda_1}}$, $i \in \mathbb{N}$. Put $(\widetilde{\alpha}_i)_{i\in\mathbb{N}} := h((\alpha_i)_{i\in\mathbb{N}}) \in X_{\Lambda_2}$. Fix $i \in \mathbb{N}$ and $l \in \mathbb{N}$. Take $\gamma \in B_*(\Lambda_1)$ such that u_{i+L}^{l+L} launches γ, and $\delta \in B_*(\Lambda_1)$ such that u_{i+L}^{l+L+1} launches δ. We then have the following diagram:

$$
\begin{array}{ccccccccccc}
u_i^l & \xrightarrow{\alpha_{i+1}} & u_{i+1}^{l+1} & \xrightarrow{\alpha_{i+2}} & u_{i+2}^{l+2} & \longrightarrow & \cdots & \xrightarrow{\alpha_{i+L}} & u_{i+L}^{l+L} & \xrightarrow{\gamma} & \\
{\scriptstyle\iota}\uparrow & & {\scriptstyle\iota}\uparrow & & {\scriptstyle\iota}\uparrow & & & & {\scriptstyle\iota}\uparrow & & \\
u_i^{l+1} & \xrightarrow{\alpha_{i+1}} & u_{i+1}^{l+2} & \xrightarrow{\alpha_{i+2}} & u_{i+2}^{l+3} & \longrightarrow & \cdots & \xrightarrow{\alpha_{i+L}} & u_{i+L}^{l+L+1} & \xrightarrow{\delta} &
\end{array}
$$

Put $\nu = (\alpha_{i+1}, \alpha_{i+2}, \dots, \alpha_{i+L}) \in B_L(\Lambda_1)$ with $|\nu| = L$. Hence we see that $u_i^l = [\nu\gamma]_l$ the l-past equivalence class of $\nu\gamma \in S_l(\Lambda_1)$, and $u_i^{l+1} = [\nu\delta]_{l+1}$ the $l+1$-past equivalence class of $\nu\delta \in S_{l+1}(\Lambda_1)$. By the preceding lemma, we know that $\widetilde{\nu\gamma} \in S_l(\Lambda_2)$, $\widetilde{\nu\delta} \in S_{l+1}(\Lambda_2)$ and $\widetilde{\nu\gamma} \sim_l \widetilde{\nu\delta}$. Define $\tilde{u}_i^l := [\widetilde{\nu\gamma}]_l$ the l-past equivalence class of $\widetilde{\nu\gamma} \in S_l(\Lambda_2)$, and $\tilde{u}_i^{l+1} := [\widetilde{\nu\delta}]_{l+1}$ the $l+1$-past equivalence class of $\widetilde{\nu\delta} \in S_{l+1}(\Lambda_2)$. It follows from Lemma 7.4 that the equivalence classes $[\widetilde{\nu\gamma}]_l$ and $[\widetilde{\nu\delta}]_{l+1}$ do not depend on the choice of γ and δ. We then have that

$$\tilde{u}_i^l \in V_l^{\Lambda_2^{\min}} \quad \text{and} \quad \tilde{u}_i^{l+1} \in V_{l+1}^{\Lambda_2^{\min}}.$$

Since $\widetilde{\nu\gamma} \sim_l \widetilde{\nu\delta}$, we have $\iota(\tilde{u}_i^{l+1}) = \tilde{u}_i^l$ so that we have an ι-orbit

$$\tilde{u}_i = (\tilde{u}_i^l)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}^{\min}_{\Lambda_2}} \quad \text{for each } i \in \mathbb{N}.$$

By its construction, we have for some $x \in X_{\Lambda_1}$

$$
\begin{aligned}
h\big((\alpha_1, \dots, \alpha_i)\nu\gamma x\big) &= h\big((\alpha_1, \dots, \alpha_i)(\alpha_{i+1}, \dots, \alpha_{i+L})\gamma x\big)_{[1,i]} h(\nu\gamma x) \\
&= (\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_i) h(\nu\gamma x)_{[1,|\nu\gamma|]} h(x)_{[1,\infty)} \\
&= (\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_i)\widetilde{\nu\gamma} h(x).
\end{aligned}
$$

Hence we have $(\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_i) \in \Gamma_i^-(\tilde{u}_i^l)$. It is easy to see that $(\tilde{u}_{i-1}, \alpha_i, \tilde{u}_i) \in E_{\mathfrak{L}^{\min}_{\Lambda_2}}$ so that we have a sequence $(\widetilde{\alpha}_i, \tilde{u}_i)_{i\in\mathbb{N}} \in X_{\mathfrak{L}^{\min}_{\Lambda_2}}$. Consequently we get a map

$$\varphi : (\alpha_i, u_i)_{i\in\mathbb{N}} \in X_{\mathfrak{L}^{\min}_{\Lambda_1}} \to (\widetilde{\alpha}_i, \tilde{u}_i)_{i\in\mathbb{N}} \in X_{\mathfrak{L}^{\min}_{\Lambda_2}}$$

that is continuous by its construction. Since $h((\alpha_i)_{i\in\mathbb{N}}) = (\widetilde{\alpha}_i)_{i\in\mathbb{N}}$, it satisfies $h \circ \pi_{\mathfrak{L}_1} = \pi_{\mathfrak{L}_2} \circ \varphi$. Similarly we get a map

$$\phi : (\beta_i, w_i)_{i\in\mathbb{N}} \in X_{\mathfrak{L}^{\min}_{\Lambda_2}} \to (\widetilde{\beta}_i, \tilde{w}_i)_{i\in\mathbb{N}} \in X_{\mathfrak{L}^{\min}_{\Lambda_1}}$$

satisfying $\varphi \circ \phi = \mathrm{id}_{X_{\mathfrak{L}^{\min}_{\Lambda_2}}}$ and $\phi \circ \varphi = \mathrm{id}_{X_{\mathfrak{L}^{\min}_{\Lambda_1}}}$. By putting $h_{\mathfrak{L}} = \varphi$, we have a desired topological conjugacy

$$h_{\mathfrak{L}} : X_{\mathfrak{L}^{\min}_{\Lambda_1}} \to X_{\mathfrak{L}^{\min}_{\Lambda_2}}.$$

∎

Theorem 7.6. *Let Λ_1 and Λ_2 be normal subshifts. Assume that their one-sided subshifts $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are topologically conjugate. Then there exists an isomorphism $\Phi : \mathcal{O}_{\Lambda_1^{\min}} \to \mathcal{O}_{\Lambda_2^{\min}}$ of C^*-algebras such that $\Phi(\mathcal{D}_{\Lambda_1}) = \mathcal{D}_{\Lambda_2}$ and $\Phi \circ \rho_t^{\Lambda_1} = \rho_t^{\Lambda_2} \circ \Phi$, $t \in \mathbb{T}$.*

Proof. By Proposition 7.5, $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are $(\mathfrak{L}_{\Lambda_1}^{\min}, \mathfrak{L}_{\Lambda_2}^{\min})$-conjugate, so that they are $(\mathfrak{L}_{\Lambda_1}^{\min}, \mathfrak{L}_{\Lambda_2}^{\min})$-eventually conjugate in the sense of [37] and in the sense of the following section ((8.1)). By [37, Theorem 1.3], we have a desired isomorphism $\Phi : \mathcal{O}_{\Lambda_1^{\min}} \to \mathcal{O}_{\Lambda_2^{\min}}$ of C^*-algebras. ∎

Remark 7.7. (i) Brix–Carlsen in [3] gave an example of a pair (X_A, σ_A) and (X_B, σ_B) of irreducible shifts of finite type such that the converse of Theorem 7.6 does not hold. They found two irreducible matrices A, B with entries in $\{0, 1\}$ such that there exists an isomorphism $\Phi : \mathcal{O}_A \to \mathcal{O}_B$ of the Cuntz–Krieger algebras such that $\Phi(\mathcal{D}_A) = \mathcal{D}_B$ and $\Phi \circ \rho_t^A = \rho_t^B \circ \Phi$, but the one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are not topologically conjugate.

(ii) After the submission of the paper, Theorem 7.6 was strengthened in [38] in the following way: Let Λ_1 and Λ_2 be normal subshifts. Then their one-sided subshifts $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are topologically conjugate if and only if there exists an isomorphism $\Phi : \mathcal{O}_{\Lambda_1^{\min}} \to \mathcal{O}_{\Lambda_2^{\min}}$ of C^*-algebras such that $\Phi(\mathcal{D}_{\Lambda_1}) = \mathcal{D}_{\Lambda_2}$ and

$$\Phi \circ \rho_t^{\Lambda_1, f} = \rho_t^{\Lambda_2, \Phi(f)} \circ \Phi \quad \text{for all } f \in C(X_{\Lambda_1}, \mathbb{Z}),\ t \in \mathbb{T},$$

where $\rho_t^{\Lambda_1, f}$, $\rho_t^{\Lambda_2, \Phi(f)}$ are generalized gauge actions with potential functions f, $\Phi(f)$, respectively (see [38] for details, see also [39]). In the proof of the result, Theorem 7.6 was used.

8. One-sided eventual conjugacy

In this section, we will prove that a slightly weaker equivalence relation than one-sided topological conjugacy in one-sided normal subshifts $X_{\Lambda_1}, X_{\Lambda_2}$, called eventual conjugacy, is equivalent to the condition that there exists an isomorphism $\Phi : \mathcal{O}_{\Lambda_1^{\min}} \to \mathcal{O}_{\Lambda_2^{\min}}$ of C^*-algebras satisfying $\Phi(\mathcal{D}_{\Lambda_1}) = \mathcal{D}_{\Lambda_2}$ and $\Phi \circ \rho_t^{\Lambda_1} = \rho_t^{\Lambda_2} \circ \Phi$, $t \in \mathbb{T}$. A part of its proof will need Theorem 7.6.

Let Λ_1 and Λ_2 be subshifts. Suppose that their one-sided subshifts $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are *eventually conjugate*. This means that there exist a homeomorphism $h : X_{\Lambda_1} \to X_{\Lambda_2}$ and an integer $K \in \mathbb{Z}_+$ such that

$$\begin{cases} \sigma_{\Lambda_2}^K\big(h(\sigma_{\Lambda_1}(x))\big) = \sigma_{\Lambda_2}^{K+1}\big(h(x)\big), & x \in X_{\Lambda_1}, \\ \sigma_{\Lambda_1}^K\big(h^{-1}(\sigma_{\Lambda_2}(y))\big) = \sigma_{\Lambda_1}^{K+1}\big(h^{-1}(y)\big), & y \in X_{\Lambda_2}. \end{cases} \tag{8.1}$$

Let $h_{[1,K]} : X_{\Lambda_1} \to B_K(\Lambda_2)$ and $h_1 : X_{\Lambda_1} \to X_{\Lambda_2}$ be continuous maps defined by setting

$$h_{[1,K]}(x) := h(x)_{[1,K]}, \quad h_1(x) := \sigma_{\Lambda_2}^K\big(h(x)\big), \quad x \in X_{\Lambda_1}.$$

We then have

$$h(x) = h_{[1,K]}(x)h_1(x), \quad x \in X_{\Lambda_1}.$$

Since $h_{[1,K]} : X_{\Lambda_1} \to B_K(\Lambda_2)$ is continuous, for $\xi_i \in \{\xi_1, \dots, \xi_m\} = B_K(\Lambda_2)$, $h_{[1,K]}^{-1}(\xi_i)$ is a finite union of cylinder sets, so that there exist $M_1 \in \mathbb{N}$ and a block map $\varphi_1 : B_{M_1}(\Lambda_1) \to B_K(\Lambda_2)$ such that

$$h_{[1,K]}(x) = \varphi_1(x_1, \dots, x_{M_1}) \quad \text{for } x = (x_i)_{i\in\mathbb{N}} \in X_{\Lambda_1}.$$

Hence we have

$$h(x) = \varphi_1(x_{[1,M_1]})h_1(x), \quad x \in X_{\Lambda_1}.$$

By (8.1), we have the equality

$$h_1\big(\sigma_{\Lambda_1}(x)\big) = \sigma_{\Lambda_2}\big(h_1(x)\big), \quad x \in X_{\Lambda_1},$$

so that $h_1 : X_{\Lambda_1} \to X_{\Lambda_2}$ is a sliding block code (cf. [21]).

Similarly there exist $M_2 \in \mathbb{N}$, a block map $\varphi_2 : B_{M_2}(\Lambda_2) \to B_K(\Lambda_1)$ and a continuous map $h_2 : X_{\Lambda_2} \to X_{\Lambda_1}$ such that

$$h^{-1}(y) = \varphi_2(y_{[1,M_2]})h_2(y), \quad h_2\big(\sigma_{\Lambda_2}(y)\big) = \sigma_{\Lambda_1}\big(h_2(y)\big)$$

for $y = (y_i)_{i\in\mathbb{N}} \in X_{\Lambda_2}$. We may assume that $M_1 = M_2$ written M such that $M \geq K$. It then follows that

$$\begin{aligned} x &= h^{-1}\big(h(x)\big) \\ &= \varphi_2\big(h(x)_{[1,M]}\big)h_2\big(h(x)\big) \\ &= \varphi_2\big(\varphi_1(x_{[1,M]})h_1(x)_{[1,M-K]}\big)h_2\big(\varphi_1(x_{[1,M]})h_1(x)\big). \end{aligned}$$

This implies

$$x_{[1,K]} = \varphi_2\big(\varphi_1(x_{[1,M]})h_1(x)_{[1,M-K]}\big), \tag{8.2}$$

$$x_{[K+1,\infty)} = h_2\big(\varphi_1(x_{[1,M]})h_1(x)\big), \tag{8.3}$$

and hence

$$x_{[2K+1,\infty)} = \sigma_{\Lambda_1}^K(x_{[K+1,\infty)}) = h_2\big(\sigma_{\Lambda_2}^K\big(\varphi_1(x_{[1,M]})h_1(x)\big)\big) = h_2\big(h_1(x)\big)$$

for $x \in X_{\Lambda_1}$. Similarly we have

$$\begin{aligned} y_{[1,K]} &= \varphi_1\big(\varphi_2(y_{[1,M]})h_2(y)_{[1,M-K]}\big), \\ y_{[K+1,\infty)} &= h_1\big(\varphi_2(y_{[1,M]})h_2(y)\big), \end{aligned}$$

and

$$y_{[2K+1,\infty)} = h_1\big(h_2(y)\big), \quad y \in X_{\Lambda_2}.$$

For $\xi = (\xi(1), \dots, \xi(K)) \in B_K(\Lambda_2)$ and $y = (y_n)_{n\in\mathbb{N}} \in X_{\Lambda_2}$ with $y \in \Gamma^+_\infty(\xi)$, we write $(\xi, y) := (\xi(1), \dots, \xi(K), y_1, y_2, \dots) \in X_{\Lambda_2}$. Now suppose that $(\xi, y) \in X_{\Lambda_2}$ such that $\xi = \varphi_1(x_{[1,M]})$, $y = h_1(x)$ for some $x = (x_n)_{n\in\mathbb{N}} \in X_{\Lambda_1}$. Define

$$\tau(\xi, y) := \big(\varphi_1(x_{[2,M+1]}), h\big(\sigma_{\Lambda_1}(x)\big)\big).$$

Under the identification $(\xi, y) = (\varphi_1(x_{[1,M]}), h_1(x)) = h(x)$, we have

$$\tau\big(h(x)\big) = h\big(\sigma_{\Lambda_1}(x)\big) \quad \text{for } x \in X_{\Lambda_1}.$$

Hence we have a continuous surjection $\tau : X_{\Lambda_2} \to X_{\Lambda_2}$ such that

$$\tau = h \circ \sigma_{\Lambda_1} \circ h^{-1}.$$

By the relations (8.2) and (8.3), we know that

$$x_{[2,M+1]} = x_{[2,K]}x_{[K+1,M+1]} = \varphi_2(\xi y_{[1,M-K]})_{[2,K]}h_2(\xi y)_{[1,M-K+1]}$$

so that

$$\varphi_1(x_{[2,M+1]}) = \varphi_1\big(\varphi_2(\xi y_{[1,M-K]})_{[2,K]}h_2(\xi y)_{[1,M-K+1]}\big)$$

and

$$\tau(\xi, y) = \big(\varphi_1\big(\varphi_2(\xi y_{[1,M-K]})_{[2,K]}h_2(\xi y)_{[1,M-K+1]}\big), \sigma_{\Lambda_2}(y)\big) \in B_K(\Lambda_2) \times X_{\Lambda_2},$$

for $(\xi, y) \in X_{\Lambda_2}$. As $h_1 : X_{\Lambda_1} \to X_{\Lambda_2}$ and $h_2 : X_{\Lambda_2} \to X_{\Lambda_1}$ are both sliding block codes, one may take integers $N_1, N_2 \in \mathbb{N}$ and block maps $\phi_1 : B_{N_1}(\Lambda_1) \to \Sigma_2$, $\phi_2 : B_{N_2}(\Lambda_2) \to \Sigma_1$ such that

$$h_1(x) = \phi_1(x_{[i,N_1+i]})_{i\in\mathbb{N}} \quad \text{for } x \in X_{\Lambda_1} \quad \text{and} \quad h_2(y) = \phi_2(y_{[i,N_2+i]})_{i\in\mathbb{N}}$$

for $y \in X_{\Lambda_2}$. We may assume that $N_1, N_2 \geq K$. We then have that $h_2(\xi y)_{[1,M-K+1]} = \phi_2(\xi y_{[1,M-2K+1+N_2]})$. We put $L = M - 2K + 1 + N_2$ and

$$\tau^{\varphi_1}(\xi, y) = \varphi_1\big(\varphi_2(\xi y_{[1,M-K]})_{[2,K]}\phi_2(\xi y_{[1,L]})\big)$$

so that

$$\tau(\xi, y) = \big(\tau^{\varphi_1}(\xi, y), \sigma_{\Lambda_2}(y)\big) \in B_K(\Lambda_2) \times X_{\Lambda_2}, \quad (\xi, y) \in X_{\Lambda_2}.$$

As $N_2 \geq K$, we note that $L \geq M - K$. Hence the word $\tau^{\varphi_1}(\xi, y) \in B_K(\Lambda_2)$ is determined by only $\xi \in B_K(\Lambda_2)$ and $y_{[1,L]}$, so that we may write

$$\tau^{\varphi_1}(\xi, y) = \tau^{\varphi_1}(\xi, y_{[1,L]}).$$

Let $X_{\Lambda_2^{[L]}}$ be the right one-sided subshift of the Lth higher block shift $\Lambda_2^{[L]}$ of Λ_2 (see for instance [21]). Define sliding block codes:

$$g_1 : x \in X_{\Lambda_1} \to \big(\varphi_1(x_{[n,M+n-1]})\big)_{n\in\mathbb{N}} \in B_K(\Lambda_2)^{\mathbb{N}},$$
$$h_1^L : x \in X_{\Lambda_1} \to \big(h_1(x)_{[n,L+n-1]}\big)_{n\in\mathbb{N}} \in X_{\Lambda_2^{[L]}}$$

and put

$$g_1(x)_n = \varphi_1(x_{[n,M+n-1]}) \in B_K(\Lambda_2),$$
$$h_1^L(x)_n = h_1(x)_{[n,L+n-1]} \in B_L(\Lambda_2)$$

so that $g_1(x) = (g_1(x)_n)_{n\in\mathbb{N}}$, $h_1^L(x) = (h_1^L(x)_n)_{n\in\mathbb{N}}$. Since $h_1 \circ \sigma_{\Lambda_1} = \sigma_{\Lambda_2} \circ h_1$, we have $h_1^L \circ \sigma_{\Lambda_1} = \sigma_{{\Lambda_2}^{[L]}} \circ h_1^L$. Define $\hat{h}^L : X_{\Lambda_1} \to (B_K(\Lambda_2) \times B_L(\Lambda_2))^{\mathbb{N}}$ by setting

$$\hat{h}^L(x) = \big(g_1(x), h_1^L(x)\big) = \big(\varphi_1(x_{[n,M+n-1]}), h_1(x)_{[n,L+n-1]}\big)_{n\in\mathbb{N}}.$$

Lemma 8.1. *Define*

$$X_{\Lambda_2^{\varphi_1}\times\Lambda_2^{[L]}} = \big\{\big(g_1(x), h_1^L(x)\big) \in \big(B_K(\Lambda_2) \times B_L(\Lambda_2)\big)^{\mathbb{N}} \mid x \in X_{\Lambda_1}\big\},$$

and the map $\sigma_{\Lambda_2^{\varphi_1}\times\Lambda_2^{[L]}} : X_{\Lambda_2^{\varphi_1}\times\Lambda_2^{[L]}} \to X_{\Lambda_2^{\varphi_1}\times\Lambda_2^{[L]}}$ *by setting*

$$\sigma_{\Lambda_2^{\varphi_1}\times\Lambda_2^{[L]}}\big(\big(g_1(x)_n, h_1^L(x)_n\big)_{n\in\mathbb{N}}\big) = \big(g_1(x)_{n+1}, h_1^L(x)_{n+1}\big)_{n\in\mathbb{N}}.$$

Then $(X_{\Lambda_2^{\varphi_1}\times\Lambda_2^{[L]}}, \sigma_{\Lambda_2^{\varphi_1}\times\Lambda_2^{[L]}})$ *is a subshift over* $B_K(\Lambda_2) \times B_L(\Lambda_2)$ *that is topologically conjugate to* $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ *via*

$$\hat{h}^L : x \in X_{\Lambda_1} \to \big(g_1(x)_n, h_1^L(x)_n\big)_{n\in\mathbb{N}} \in X_{\Lambda_2^{\varphi_1}\times\Lambda_2^{[L]}}.$$

Proof. Since

$$\hat{h}^L : X_{\Lambda_1} \to X_{\Lambda_2^{\varphi_1}\times\Lambda_2^{[L]}}$$

is a sliding block code, the pair $(X_{\Lambda_2^{\varphi_1}\times\Lambda_2^{[L]}}, \sigma_{\Lambda_2^{\varphi_1}\times\Lambda_2^{[L]}})$ gives rise to a subshift over $B_K(\Lambda_2) \times B_L(\Lambda_2)$. As

$$\hat{h}^L \circ \sigma_{\Lambda_1} = \sigma_{\Lambda_2^{\varphi}\times{\Lambda_2}^{[L]}} \circ \hat{h}^L,$$

it remains to show that $\hat{h}^L$ is injective. Suppose that $\hat{h}^L(x) = \hat{h}^L(z)$ for some $x = (x_n)_{n\in\mathbb{N}}$, $z = (z_n)_{n\in\mathbb{N}}$. Hence we have

$$\varphi_1(x_{[1,M_1]}) = \varphi_1(z_{[1,M_1]}), \quad h_1^L(x) = h_1^L(z) \quad \text{and hence} \quad h_1(x) = h_1(z)$$

so that $h(x) = h(z)$ proving $x = z$. ∎

Define

$$\Sigma_2' = \big\{(\xi, y_{[1,L]}) \in B_K(\Lambda_2) \times B_L(\Lambda_2) \mid \xi \in \Gamma_K^-(y_{[1,L]})\big\}$$

and a subshift $(\Lambda'_2, \sigma_{\Lambda'_2})$ over Σ'_2 by its right one-sided subshift

$$X_{\Lambda'_2} = \{(\xi_n, y_{[n,L+n-1]})_{n\in\mathbb{N}} \in \big(B_K(\Lambda_2) \times B_L(\Lambda_2)\big)^{\mathbb{N}} \mid \xi_{n+1} = \tau^{\varphi_1}(\xi_n, y_{[n,L+n-1]}),\ n \in \mathbb{N},\ \big(\xi_1, (y_n)_{n\in\mathbb{N}}\big) \in X_{\Lambda_2}\}$$

and $\sigma_{\Lambda'_2} : X_{\Lambda'_2} \to X_{\Lambda'_2}$ by

$$\sigma_{\Lambda'_2}\big((\xi_n, y_{[n,L+n-1]})_{n\in\mathbb{N}}\big) = (\xi_{n+1}, y_{[n+1,L+n]})_{n\in\mathbb{N}}.$$

We then have the following lemma.

Lemma 8.2. $(X_{\Lambda_2^{\varphi_1}\times\Lambda_2^{[L]}}, \sigma_{\Lambda_2^{\varphi_1}\times\Lambda_2^{[L]}}) = (X_{\Lambda'_2}, \sigma_{\Lambda'_2})$ *so that* $(X_{\Lambda'_2}, \sigma_{\Lambda'_2})$ *is topologically conjugate to* $(X_{\Lambda_1}, \sigma_{\Lambda_1})$. *Hence the subshift* $(\Lambda'_2, \sigma_{\Lambda'_2})$ *is normal if* (Λ_1, σ_1) *is normal.*

Proof. Take an arbitrary element $(\xi_n, y_{[n,L+n-1]})_{n\in\mathbb{N}} \in X_{\Lambda'_2}$, so that $(\xi_1, y_1, y_2, \dots) \in X_{\Lambda_2}$. Put $x = h^{-1}(\xi_1, y_1, y_2, \dots) \in X_{\Lambda_1}$. We then have that $\xi_n = g_1(x)_n$ and $y_{[n,L+n-1]} = h_1^L(x)_n$ for all $n \in \mathbb{N}$. Hence we may identify $(g_1(x), h_1^L(x))$ with $(\xi_n, y_{[n,L+n-1]})_{n\in\mathbb{N}}$. The identification between $(g_1(x), h_1^L(x))$ and $(\xi_n, y_{[n,L+n-1]})_{n\in\mathbb{N}}$ yields the identification between the subshifts $(X_{\Lambda_2^{\varphi_1}\times\Lambda_2^{[L]}}, \sigma_{\Lambda_2^{\varphi_1}\times\Lambda_2^{[L]}})$ and $(X_{\Lambda'_2}, \sigma_{\Lambda'_2})$. This implies that $(X_{\Lambda'_2}, \sigma_{\Lambda'_2})$ is topologically conjugate to $(X_{\Lambda_1}, \sigma_{\Lambda_1})$. ∎

In what follows, we assume that the subshifts $(\Lambda_1, \sigma_{\Lambda_1})$ and $(\Lambda_2, \sigma_{\Lambda_2})$ are both normal. Since the one-sided subshift $(X_{\Lambda'_2}, \sigma_{\Lambda'_2})$ is topologically conjugate to $(X_{\Lambda_1}, \sigma_{\Lambda_1})$, Theorem 7.6 ensures us that there exists an isomorphism $\Phi_1 : \mathcal{O}_{\Lambda_1^{\min}} \to \mathcal{O}_{\Lambda_2'^{\min}}$ of C^*-algebras such that

$$\Phi_1(\mathcal{D}_{\Lambda_1}) = \mathcal{D}_{\Lambda'_2} \quad \text{and} \quad \Phi_1 \circ \rho_t^{\Lambda_1} = \rho_t^{\Lambda'_2} \circ \Phi_1, \quad t \in \mathbb{T}.$$

We will henceforth prove that there exists an isomorphism $\Phi_2 : \mathcal{O}_{\Lambda_2'^{\min}} \to \mathcal{O}_{\Lambda_2^{\min}}$ of C^*-algebras such that

$$\Phi_2(\mathcal{D}_{\Lambda'_2}) = \mathcal{D}_{\Lambda_2} \quad \text{and} \quad \Phi_2 \circ \rho_t^{\Lambda'_2} = \rho_t^{\Lambda_2} \circ \Phi_2, \quad t \in \mathbb{T}.$$

Let $(V', E', \lambda', \iota')$ be the minimal λ-graph system $\mathfrak{L}^{\min}_{\Lambda'_2}$ of Λ'_2. The vertex set V'_l is denote by $\{v_1'^l, \dots, v_{m'(l)}'^l\}$. Since $\mathfrak{L}^{\min}_{\Lambda'_2}$ is predecessor-separated, the projections of the form $E_i'^l$ in the C^*-algebra $\mathcal{O}_{\Lambda_2'^{\min}}$ corresponding to the vertex $v_i'^l \in V'_l$ of $\mathfrak{L}^{\min}_{\Lambda'_2}$ is written in terms of the generating partial isometries $S'_{(\xi, y_{[1,L]})}$, $(\xi, y_{[1,L]}) \in \Sigma'_2$ by the formula (3.1).

Let $S_\alpha, \alpha \in \Sigma_2$ be the generating partial isometries of the C^*-algebra $\mathcal{O}_{\Lambda_2^{\min}}$. For $(\xi, y) \in B_K(\Lambda_2) \times X_{\Lambda_2}$ with $\xi \in \Gamma_K^-(y)$, let us define a sequence $(\xi_n)_{n\in\mathbb{N}}$ of words of $B_K(\Lambda_2)$ by

$$\xi_1 := \xi, \quad \xi_{n+1} = \tau^{\varphi_1}(\xi_n, y_{[n,L+n-1]}), \quad n \in \mathbb{N}. \tag{8.4}$$

For a word $w = (w_1, \dots, w_k) \in B_k(\Lambda_2)$, we write the partial isometry $S_{w_1} \cdots S_{w_k} \in \mathcal{O}_{\Lambda_2{}^{\min}}$ as S_w in $\mathcal{O}_{\Lambda_2{}^{\min}}$. For $(\xi, y_{[1,L]}) \in \Sigma_2'$, we define a partial isometry $\widehat{S}_{(\xi, y_{[1,L]})}$ in $\mathcal{O}_{\Lambda_2{}^{\min}}$ by setting

$$\widehat{S}_{(\xi, y_{[1,L]})} := S_{\xi_1 y_{[1,L]}} S^*_{\xi_2 y_{[2,L]}} \in \mathcal{O}_{\Lambda_2{}^{\min}} \quad \text{where } \xi_1 = \xi,\ \xi_2 = \tau^{\varphi_1}(\xi, y_{[1,L]}).$$

We also write for $\mu = (\mu_1, \dots, \mu_m) \in B_m(\Lambda_2')$

$$\widehat{S}_\mu := \widehat{S}_{\mu_1} \cdots \widehat{S}_{\mu_m} \in \mathcal{O}_{\Lambda_2{}^{\min}}.$$

We write $\mathfrak{L}^{\min}_{\Lambda_2} = (V^{\Lambda_2^{\min}}, E^{\Lambda_2^{\min}}, \lambda^{\Lambda_2^{\min}}, \iota^{\Lambda_2^{\min}})$. The transition matrix system of $\mathfrak{L}^{\min}_{\Lambda_2}$ is denoted by $(A^{\min}_{l,l+1}, I^{\min}_{l,l+1})_{l \in \mathbb{Z}_+}$. For $w = (w_1, \dots, w_l) \in B_l(\Lambda_2)$ and $v^l_j \in V_l^{\Lambda_2^{\min}}$, we define a matrix component $A^{\min}_{0,l}(0, w, j)$ by

$$A^{\min}_{0,l}(0, w, j) = \begin{cases} 1 & \text{if there exists } \gamma \in E_{0,l}^{\Lambda_2^{\min}};\ \lambda(\gamma) = w,\ t(\gamma) = v^l_j, \\ 0 & \text{otherwise,} \end{cases}$$

where the top vertex $V_0^{\Lambda_2^{\min}} = \{v_0\}$ a singleton. We note the following lemma.

Lemma 8.3. *For $(\xi, y) \in B_K(\Lambda_2) \times X_{\Lambda_2}$ with $\xi \in \Gamma^-_K(y)$, let $(\xi_n)_{n \in \mathbb{N}}$ be the sequence of $B_K(\Lambda_2)$ defined by (8.4). We then have $S^*_{\xi_1 y_{[1,L+1]}} S_{\xi_1 y_{[1,L+1]}} \le S^*_{\xi_2 y_{[2,L+1]}} S_{\xi_2 y_{[2,L+1]}}$ and hence $S_{\xi_1 y_{[1,L+1]}} S^*_{\xi_2 y_{[2,L+1]}} S_{\xi_2 y_{[2,L+1]}} = S_{\xi_1 y_{[1,L+1]}}$. More generally we have*

$$S^*_{\xi_n y_{[n,L+n]}} S_{\xi_n y_{[n,L+n]}} \le S^*_{\xi_{n+1} y_{[n+1,L+n]}} S_{\xi_{n+1} y_{[n+1,L+n]}} \tag{8.5}$$

and

$$S_{\xi_n y_{[n,L+n]}} S^*_{\xi_{n+1} y_{[n+1,L+n]}} S_{\xi_{n+1} y_{[n+1,L+n]}} = S_{\xi_n y_{[n,L+n]}}, \quad n \in \mathbb{N}. \tag{8.6}$$

Proof. For $z = (z_n)_{n \in \mathbb{N}} \in X_{\Lambda_2}$ with $z \in \Gamma^+_\infty(\xi_1 y_{[1,L+1]})$, we put $x = h^{-1}(\xi_1 y_{[1,L+1]} z) \in X_{\Lambda_1}$. Let $y'_n = y_n$ for $n = 1, 2, \dots, L+1$ and $y'_{L+n+1} = z_n$ for $n \in \mathbb{N}$, and hence $(y'_n)_{n \in \mathbb{N}} = y_{[1,L+1]} z \in X_{\Lambda_2}$. Put $\xi'_1 = \xi_1$ and $\xi'_{n+1} = \tau^\varphi(\xi'_n, y'_{[n,L+n-1]})$, $n \in \mathbb{N}$. Hence

$$\sigma_{\Lambda_2^{\varphi_1} \times \Lambda_2^{[L]}}\big(\hat{h}^L(x)\big) = \big((\xi'_2, y'_{[2,L+1]}), (\xi'_3, y'_{[3,L+2]}), \dots\big).$$

Since $\xi'_2 y'_{[2,\infty)} \in X_{\Lambda_2}$ and $y'_{[L+2,\infty)} = z$, we have

$$\xi'_2 y'_{[2,L+1]} z \in X_{\Lambda_2}.$$

As $\xi'_2 = \tau^\varphi(\xi'_1, y'_{[1,L]}) = \tau^\varphi(\xi_1, y_{[1,L]}) = \xi_2$ and $y'_{[1,L+1]} = y_{[1,L+1]}$, we know

$$\xi_2 y_{[2,L+1]} z \in X_{\Lambda_2}.$$

Hence we see

$$\Gamma^+_\infty(\xi_1 y_{[1,L+1]}) \subset \Gamma^+_\infty(\xi_2 y_{[2,L+1]}) \quad \text{in } X_{\Lambda_2}$$

and hence

$$\Gamma_*^+(\xi_1 y_{[1,L+1]}) \subset \Gamma_*^+(\xi_2 y_{[2,L+1]}) \quad \text{in } B_*(\Lambda_2). \tag{8.7}$$

Consider the λ-graph system $\mathfrak{L}_{\Lambda_2}^{\min}$. Suppose that $A_{0,K+L+1}^{\min}(0, \xi_1 y_{[1,L+1]}, j) = 1$ for some $j \in \{1, 2, \dots, m(K + L + 1)\}$. Since $\mathfrak{L}_{\Lambda_2}^{\min}$ is a λ-synchronizing λ-graph system, the vertex v_j^{K+L+1} in $V_{K+L+1}^{\Lambda_2^{\min}}$ is written as $v_j^{K+L+1} = [w]_{K+L+1}$ for some $w \in S_{K+L+1}(\Lambda_2)$. Hence $w \in \Gamma_*^+(\xi_1 y_{[1,L+1]})$. By (8.7), $w \in \Gamma_*^+(\xi_2 y_{[2,L+1]})$ so that $\xi_2 y_{[2,L+1]} w \in S_1(\Lambda_2)$ and there exists a labeled edge labeled $\xi_2 y_{[2,L+1]}$ from the top vertex v_0 to $[w]_{K+L} \in V_{K+L}^{\Lambda_2^{\min}}$ in $\mathfrak{L}_{\Lambda_2}^{\min}$. Since $[w]_{K+L} = \iota^{\min}([w]_{K+L+1})$, by putting $v_{j'}^{K+L} = [w]_{K+L}$, we have

$$A_{0,K+L}^{\min}(0, \xi_2 y_{[2,L+1]}, j') = 1, \quad I_{K+L,K+L+1}(j', j) = 1$$

so that

$$E_j^{K+L+1} \le E_{j'}^{K+L} \quad \text{in } \mathcal{O}_{\Lambda_2{}^{\min}}.$$

As

$$S_{\xi_1 y_{[1,L+1]}}^* S_{\xi_1 y_{[1,L+1]}} = \sum_{j=1}^{m(K+L+1)} A_{0,K+L+1}^{\min}(0, \xi_1 y_{[1,L+1]}, j) E_j^{K+L+1},$$

$$S_{\xi_2 y_{[2,L+1]}}^* S_{\xi_2 y_{[2,L+1]}} = \sum_{j'=1}^{m(K+L)} A_{0,K+L}^{\min}(0, \xi_2 y_{[2,L+1]}, j') E_{j'}^{K+L},$$

we have

$$S_{\xi_1 y_{[1,L+1]}}^* S_{\xi_1 y_{[1,L+1]}} \le S_{\xi_2 y_{[2,L+1]}}^* S_{\xi_2 y_{[2,L+1]}}$$

so that

$$S_{\xi_1 y_{[1,L+1]}} S_{\xi_2 y_{[2,L+1]}}^* S_{\xi_2 y_{[2,L+1]}} = S_{\xi_1 y_{[1,L+1]}}.$$

Similarly we have the inequality (8.5) and the equality (8.6). ∎

By using the above lemma, we see that the following lemma holds.

Lemma 8.4. *For $(\xi, y) \in B_K(\Lambda_2) \times X_{\Lambda_2}$ with $\xi \in \Gamma_K^-(y)$, let $(\xi_n)_{n \in \mathbb{N}}$ be the sequence of $B_K(\Lambda_2)$ defined by (8.4). We then have*

$$\widehat{S}_{(\xi_1, y_{[1,L]})} \widehat{S}_{(\xi_2, y_{[2,L+1]})} \cdots \widehat{S}_{(\xi_n, y_{[n,L+n-1]})} = S_{\xi_1 y_{[1,L+n-1]}} S_{\xi_{n+1} y_{[n+1,L+n-1]}}^*$$

for $n \in \mathbb{N}$.

Proof. The following equalities hold:

$$\begin{aligned}
&\widehat{S}_{(\xi_1, y_{[1,L]})} \widehat{S}_{(\xi_2, y_{[2,L+1]})} \cdots \widehat{S}_{(\xi_n, y_{[n,L+n-1]})} \\
&\quad = S_{\xi_1 y_{[1,L]}} S_{\xi_2 y_{[2,L]}}^* S_{\xi_2 y_{[2,L+1]}} S_{\xi_3 y_{[3,L+1]}}^* \cdots \widehat{S}_{(\xi_n, y_{[n,L+n-1]})} \\
&\quad = S_{\xi_1 y_{[1,L]}} S_{\xi_2 y_{[2,L]}}^* S_{\xi_2 y_{[2,L]}} S_{y_{L+1}} S_{y_{L+1}}^* S_{y_{L+1}} S_{\xi_3 y_{[3,L+1]}}^* \cdots \widehat{S}_{(\xi_n, y_{[n,L+n-1]})} \\
&\quad = S_{\xi_1 y_{[1,L]}} S_{y_{L+1}} S_{y_{L+1}}^* S_{\xi_2 y_{[2,L]}}^* S_{\xi_2 y_{[2,L]}} S_{y_{L+1}} S_{\xi_3 y_{[3,L+1]}}^* \cdots \widehat{S}_{(\xi_n, y_{[n,L+n-1]})} \\
&\quad = S_{\xi_1 y_{[1,L+1]}} S_{\xi_2 y_{[2,L+1]}}^* S_{\xi_2 y_{[2,L+1]}} S_{\xi_3 y_{[3,L+1]}}^* \cdots \widehat{S}_{(\xi_n, y_{[n,L+n-1]})}.
\end{aligned}$$

By Lemma 8.3, we see that

$$S_{\xi_1 y_{[1,L+1]}} S^*_{\xi_2 y_{[2,L+1]}} S_{\xi_2 y_{[2,L+1]}} = S_{\xi_1 y_{[1,L+1]}}.$$

We thus have

$$\widehat{S}_{(\xi_1, y_{[1,L]})} \widehat{S}_{(\xi_2, y_{[2,L+1]})} \cdots \widehat{S}_{(\xi_n, y_{[n,L+n-1]})} = S_{\xi_1 y_{[1,L+1]}} S^*_{\xi_3 y_{[3,L+1]}} \cdots \widehat{S}_{(\xi_n, y_{[n,L+n-1]})},$$

so that inductively we have the desired equality. ∎

The following lemma directly follows from Lemma 8.4.

Lemma 8.5. *For $(\xi, y) \in B_K(\Lambda_2) \times X_{\Lambda_2}$ with $\xi \in \Gamma^-_K(y)$, let $(\xi_n)_{n\in\mathbb{N}}$ be the sequence of $B_K(\Lambda_2)$ defined by (8.4). We have the following three equalities:*

$$(\widehat{S}_{(\xi_1, y_{[1,L]})} \widehat{S}_{(\xi_2, y_{[2,L+1]})} \cdots \widehat{S}_{(\xi_n, y_{[n,L+n-1]})})^* \cdot (\widehat{S}_{(\xi_1, y_{[1,L]})} \widehat{S}_{(\xi_2, y_{[2,L+1]})} \cdots \widehat{S}_{(\xi_n, y_{[n,L+n-1]})})$$
$$= S_{\xi_{n+1} y_{[n+1,L+n-1]}} S^*_{\xi_1 y_{[1,L+n-1]}} S_{\xi_1 y_{[1,L+n-1]}} S^*_{\xi_{n+1} y_{[n+1,L+n-1]}}, \quad n \in \mathbb{N}, \tag{8.8}$$

$$(\widehat{S}_{(\xi_1, y_{[1,L]})} \widehat{S}_{(\xi_2, y_{[2,L+1]})} \cdots \widehat{S}_{(\xi_n, y_{[n,L+n-1]})}) \cdot (\widehat{S}_{(\xi_1, y_{[1,L]})} \widehat{S}_{(\xi_2, y_{[2,L+1]})} \cdots \widehat{S}_{(\xi_n, y_{[n,L+n-1]})})^*$$
$$= S_{\xi_1 y_{[1,L+n-1]}} S^*_{\xi_1 y_{[1,L+n-1]}}, \quad n \in \mathbb{N}, \tag{8.9}$$

$$\sum_{(\xi_1, y_{[1,L]}) \in \Sigma'_2} \widehat{S}_{(\xi_1, y_{[1,L]})} \widehat{S}^*_{(\xi_1, y_{[1,L]})} = 1. \tag{8.10}$$

The following lemma directly follows from the above lemma.

Lemma 8.6. *For $\nu, \mu \in B_*(\Lambda'_2)$, we have*

$$\widehat{S}^*_\nu \widehat{S}_\nu \widehat{S}_\mu \widehat{S}^*_\mu = \widehat{S}_\mu \widehat{S}^*_\mu \widehat{S}^*_\nu \widehat{S}_\nu.$$

Proof. Let

$$\nu = \big((\xi_1, y_{[1,L]}), \ldots, (\xi_n, y_{[n,L+n-1]})\big),$$
$$\mu = \big((\eta_1, w_{[1,L]}), \ldots, (\eta_m, w_{[m,L+m-1]})\big).$$

Since

$$\widehat{S}^*_\nu \widehat{S}_\nu = S_{\xi_{n+1} y_{[n+1,L+n-1]}} S^*_{\xi_1 y_{[1,L+n-1]}} S_{\xi_1 y_{[1,L+n-1]}} S^*_{\xi_{n+1} y_{[n+1,L+n-1]}}$$

and

$$\widehat{S}_\mu \widehat{S}^*_\mu = S_{\eta_1 w_{[1,L+m-1]}} S^*_{\eta_1 w_{[1,L+m-1]}},$$

both $\widehat{S}^*_\nu \widehat{S}_\nu$ and $\widehat{S}_\mu \widehat{S}^*_\mu$ are contained in the commutative C^*-algebra $\mathcal{D}_{\mathfrak{L}^{\min}_{\Lambda_2}}$. ∎

Let $\widehat{\mathcal{O}}_{{\Lambda'_2}^{\min}}$ be the C^*-subalgebra of $\mathcal{O}_{{\Lambda_2}^{\min}}$ generated by the partial isometries

$$\widehat{S}_{(\xi_1, y_{[1,L]})}, \quad (\xi_1, y_{[1,L]}) \in \Sigma'_2.$$

The C^*-subalgebra $\widehat{\mathcal{D}}_{\mathfrak{L}^{\min}_{\Lambda'_2}}$ of $\widehat{\mathcal{O}}_{\Lambda'_2{}^{\min}}$ is defined by the C^*-algebra generated by elements of the form:

$$\widehat{S}_\mu \widehat{S}_\nu^* \widehat{S}_\nu \widehat{S}_\mu^*, \quad \mu, \nu \in B_*(\Lambda'_2)$$

and the C^*-subalgebra $\widehat{\mathcal{D}}_{\Lambda'_2}$ of $\widehat{\mathcal{O}}_{\Lambda'_2{}^{\min}}$ is defined by the C^*-algebra generated by elements of the form:

$$\widehat{S}_\mu \widehat{S}_\mu^*, \quad \mu \in B_*(\Lambda'_2).$$

Lemma 8.7. *Let $\nu, \mu \in B_*(\Lambda'_2)$ be $\nu = ((\xi_1, y_{[1,L]}), \dots, (\xi_n, y_{[n,L+n-1]}))$ and $\mu = ((\eta_1, w_{[1,L]}), \dots, (\eta_m, w_{[m,L+m-1]}))$. We then have*

$$\widehat{S}_\nu \widehat{S}_\mu^* \widehat{S}_\mu \widehat{S}_\nu^* = \begin{cases} S_{\xi_1 y_{[1,L+n-1]}} S^*_{\eta_1 w_{[1,L+m-1]}} S_{\eta_1 w_{[1,L+m-1]}} S^*_{\xi_1 y_{[1,L+n-1]}} \\ \quad \text{if } \xi_{n+1} = \eta_{m+1},\ y_{[n+1,L+n-1]} = w_{[m+1,L+m-1]}, \\ 0 \quad \text{otherwise}, \end{cases} \tag{8.11}$$

where $\xi_{n+1} = \tau^{\varphi_1}(\xi_n, y_{[n,L+n-1]})$, $\eta_{m+1} = \tau^{\varphi_1}(\eta_m, w_{[m,L+m-1]})$.

Proof. We have

$$\begin{aligned} \widehat{S}_\nu \widehat{S}_\mu^* \widehat{S}_\mu \widehat{S}_\nu^* = {} & S_{\xi_1 y_{[1,L+n-1]}} S^*_{\xi_{n+1} y_{[n+1,L+n-1]}} \cdot S_{\eta_{m+1} w_{[m+1,L+m-1]}} S^*_{\eta_1 w_{[1,L+m-1]}} \\ & \cdot S_{\eta_1 w_{[1,L+m-1]}} S^*_{\eta_{m+1} w_{[m+1,L+m-1]}} \cdot S_{\xi_{n+1} y_{[n+1,L+n-1]}} S^*_{\xi_1 y_{[1,L+n-1]}}. \end{aligned}$$

Similarly to (8.7), we know $\Gamma^+_*(\xi_1 y_{[1,L+n-1]}) \subset \Gamma^+_*(\xi_{n+1} y_{[n+1,L+n-1]})$, so that the equality

$$S_{\xi_1 y_{[1,L+n-1]}} S^*_{\xi_{n+1} y_{[n+1,L+n-1]}} \cdot S_{\eta_{m+1} w_{[m+1,L+m-1]}} = S_{\xi_1 y_{[1,L+n-1]}}$$

holds if and only if $\xi_{n+1} y_{[n+1,L+n-1]} = \eta_{m+1} w_{[m+1,L+m-1]}$, otherwise

$$S_{\xi_1 y_{[1,L+n-1]}} S^*_{\xi_{n+1} y_{[n+1,L+n-1]}} \cdot S_{\eta_{m+1} w_{[m+1,L+m-1]}} = 0.$$

Hence we have the equality (8.11). ∎

Recall that $S'_{(\xi, y_{[1,L]})}$ for $(\xi, y_{[1,L]}) \in \Sigma'_2$ and E'^l_i for $v'^l_i \in V'_l$ stand for the canonical generating partial isometries and projections in $\mathcal{O}_{\Lambda'_2{}^{\min}}$, respectively.

Lemma 8.8. *For $\nu, \mu \in B_*(\Lambda'_2)$, we have*

(i) $S'^*_\mu S'_\mu \ge S'_\nu S'^*_\nu$ *in* $\mathcal{O}_{\Lambda'_2{}^{\min}}$ *if and only if* $\widehat{S}^*_\mu \widehat{S}_\mu \ge \widehat{S}_\nu \widehat{S}^*_\nu$ *in* $\widehat{\mathcal{O}}_{\Lambda'_2{}^{\min}}$.

(ii) $1 - S'^*_\mu S'_\mu \ge S'_\nu S'^*_\nu$ *in* $\mathcal{O}_{\Lambda'_2{}^{\min}}$ *if and only if* $1 - \widehat{S}^*_\mu \widehat{S}_\mu \ge \widehat{S}_\nu \widehat{S}^*_\nu$ *in* $\widehat{\mathcal{O}}_{\Lambda'_2{}^{\min}}$.

Proof. Let $\nu, \mu \in B_*(\Lambda'_2)$ be $\nu = ((\xi_1, y_{[1,L]}), \dots, (\xi_n, y_{[n,L+n-1]}))$ and

$$\mu = \big((\eta_1, w_{[1,L]}), \dots, (\eta_m, w_{[m,L+m-1]})\big).$$

(i) Assume that $S'^*_\mu S'_\mu \ge S'_\nu S'^*_\nu$. Put $\eta_{m+1} = \tau^{\varphi_1}(\eta_m, w_{[m,L+m-1]})$. Since

$$\mu\nu = \big((\eta_1, w_{[1,L]}), \dots, (\eta_m, w_{[m,L+m-1]}), (\xi_1, y_{[1,L]}), \dots, (\xi_n, y_{[n,L+n-1]})\big)$$

is admissible in Λ'_2, we see that $\xi_1 = \eta_{m+1}, w_{[m+1,L+m-1]} = y_{[1,L-1]}$. Hence we have

$$\begin{aligned}
\widehat{S}^*_\mu \widehat{S}_\mu \cdot \widehat{S}_\nu \widehat{S}^*_\nu &= S_{\eta_{m+1} w_{[m+1,L+m-1]}} S^*_{\eta_1 w_{[1,L+m-1]}} S_{\eta_1 w_{[1,L+m-1]}} S^*_{\eta_{m+1} w_{[m+1,L+m-1]}} \\
&\quad \cdot S_{\xi_1 y_{[1,L+n-1]}} S^*_{\xi_1 y_{[1,L+n-1]}} \\
&= S_{\xi_1 y_{[1,L-1]}} S^*_{\eta_1 w_{[1,L+m-1]}} S_{\eta_1 w_{[1,L+m-1]}} S^*_{\xi_1 y_{[1,L-1]}} \cdot S_{\xi_1 y_{[1,L+n-1]}} S^*_{\xi_1 y_{[1,L+n-1]}} \\
&= S_{\xi_1 y_{[1,L+n-1]}} S^*_{\xi_1 y_{[1,L+n-1]}} S_{\xi_1 y_{[1,L-1]}} S^*_{\eta_1 w_{[1,L+m-1]}} \\
&\quad \cdot S_{\eta_1 w_{[1,L+m-1]}} S^*_{\xi_1 y_{[1,L-1]}} S_{\xi_1 y_{[1,L+n-1]}} S^*_{\xi_1 y_{[1,L+n-1]}} \\
&= S_{\xi_1 y_{[1,L+n-1]}} S^*_{y_{[L,L+n-1]}} S^*_{\xi_1 y_{[1,L-1]}} S_{\xi_1 y_{[1,L-1]}} S^*_{\eta_1 w_{[1,L+m-1]}} \\
&\quad \cdot S_{\eta_1 w_{[1,L+m-1]}} S^*_{\xi_1 y_{[1,L-1]}} S_{\xi_1 y_{[1,L-1]}} S_{y_{[L,L+n-1]}} S^*_{\xi_1 y_{[1,L+n-1]}}.
\end{aligned}$$

Now the equality

$$\begin{aligned}
S_{\eta_1 w_{[1,L+m-1]}} S^*_{\xi_1 y_{[1,L-1]}} S_{\xi_1 y_{[1,L-1]}} &= S_{\eta_1 w_{[1,L+m-1]}} S^*_{\eta_{m+1} w_{[m+1,L+m-1]}} S_{\eta_{m+1} w_{[m+1,L+m-1]}} \\
&= S_{\eta_1 w_{[1,L+m-1]}}
\end{aligned}$$

holds because the last equality may be shown in a similar way to (8.6). Hence we have

$$\begin{aligned}
\widehat{S}^*_\mu \widehat{S}_\mu \cdot \widehat{S}_\nu \widehat{S}^*_\nu &= S_{\xi_1 y_{[1,L+n-1]}} S^*_{y_{[L,L+n-1]}} S^*_{\eta_1 w_{[1,L+m-1]}} \cdot S_{\eta_1 w_{[1,L+m-1]}} S_{y_{[L,L+n-1]}} S^*_{\xi_1 y_{[1,L+n-1]}} \\
&= S_{\xi_1 y_{[1,L+n-1]}} S^*_{\eta_1 w_{[1,L+m-1]} y_{[L,L+n-1]}} S_{\eta_1 w_{[1,L+m-1]} y_{[L,L+n-1]}} S^*_{\xi_1 y_{[1,L+n-1]}}.
\end{aligned}$$

Since for

$$\gamma = \big((\zeta_1, z_{[1,L]}), (\zeta_2, z_{[2,L+1]}), \dots, (\zeta_k, z_{[k,L+k-1]})\big) \in \Gamma^+_*(\nu), \tag{8.12}$$

with

$$\begin{aligned}
\nu\gamma = \big(&(\xi_1, y_{[1,L]}), \dots, (\xi_n, y_{[n,L+n-1]}), \\
&(\zeta_1, z_{[1,L]}), (\zeta_2, z_{[2,L+1]}), \dots, (\zeta_k, z_{[k,L+k-1]})\big) \in B_*(\Lambda'_2),
\end{aligned}$$

the condition $S'^*_\mu S'_\mu \ge S'_\nu S'^*_\nu$ implies

$$\begin{aligned}
\mu\nu\gamma = \big(&(\eta_1, w_{[1,L]}), \dots, (\eta_m, w_{[m,L+m-1]}), (\xi_1, y_{[1,L]}), \dots, (\xi_n, y_{[n,L+n-1]}), \\
&(\zeta_1, z_{[1,L]}), (\zeta_2, z_{[2,L+1]}), \dots, (\zeta_k, z_{[k,L+k-1]})\big) \in B_*(\Lambda'_2).
\end{aligned} \tag{8.13}$$

Hence in addition to $\xi_1 = \eta_{m+1}, w_{[m+1,L+m-1]} = y_{[1,L-1]}$, we have the inequality

$$\begin{aligned}
&S_{\xi_1 y_{[1,L+n-1]}} S^*_{\eta_1 w_{[1,L+m-1]} y_{[L,L+n-1]}} S_{\eta_1 w_{[1,L+m-1]} y_{[L,L+n-1]}} S^*_{\xi_1 y_{[1,L+n-1]}} \\
&\ge S_{\xi_1 y_{[1,L+n-1]}} S^*_{\xi_1 y_{[1,L+n-1]}},
\end{aligned} \tag{8.14}$$

proving $\widehat{S}^*_\mu \widehat{S}_\mu \cdot \widehat{S}_\nu \widehat{S}^*_\nu \ge \widehat{S}_\nu \widehat{S}^*_\nu$ and hence

$$\widehat{S}^*_\mu \widehat{S}_\mu \cdot \widehat{S}_\nu \widehat{S}^*_\nu = \widehat{S}_\nu \widehat{S}^*_\nu.$$

Conversely suppose that the inequality $\widehat{S}_\mu^* \widehat{S}_\mu \geq \widehat{S}_\nu \widehat{S}_\nu^*$ in $\widehat{\mathcal{O}}_{\Lambda_2'^{\min}}$ holds. The inequality is equivalent to the equality $\widehat{S}_\mu^* \widehat{S}_\mu \cdot \widehat{S}_\nu \widehat{S}_\nu^* = \widehat{S}_\nu \widehat{S}_\nu^*$ that is also equivalent to the inequality (8.14) because of the preceding equality

$$\widehat{S}_\mu^* \widehat{S}_\mu \cdot \widehat{S}_\nu \widehat{S}_\nu^* = S_{\xi_1 y_{[1,L+n-1]}} S^*_{\eta_1 w_{[1,L+m-1]} y_{[L,L+n-1]}} S_{\eta_1 w_{[1,L+m-1]} y_{[L,L+n-1]}} S^*_{\xi_1 y_{[1,L+n-1]}}.$$

For $\gamma \in \Gamma_*^+(\nu)$ as in (8.12), the inequality (8.14) together with (8.13) implies $\mu\nu\gamma \in B_*(\Lambda_2')$ and hence $\nu\gamma \in \Gamma_*^+(\mu)$. In the identity

$$S_\nu' S_\nu'^* = \sum_{k=1}^{m'(|\mu|+|\nu|)} S_\nu' E_k'^{|\mu|+|\nu|} S_\nu'^* \quad \text{in } \mathcal{O}_{\Lambda_2'^{\min}}, \tag{8.15}$$

take $k \in \{1, 2, \ldots, m'(|\mu| + |\nu|)\}$ such that $S_\nu' E_k'^{|\mu|+|\nu|} S_\nu'^* \neq 0$. As $S_\nu'^* S_\nu' \geq E_k'^{|\mu|+|\nu|}$, we see that

$$A'_{|\mu|,|\mu|+|\nu|}(j, \nu, k) = 1 \quad \text{for some } j \in \big\{1, 2, \ldots, m'\big(|\mu|\big)\big\}$$

where $A'_{|\mu|,|\mu|+|\nu|}(j, \nu, k)$ is a matrix component defined by the transition matrix system $(A'_{l,l+1}, I'_{l,l+1})_{l\in\mathbb{Z}_+}$ of $\mathfrak{L}^{\min}_{\Lambda_2'}$. Since

$$S_\nu' S_\nu'^* E_j'^{|\mu|} S_\nu' S_\nu'^* = \sum_{k=1}^{m'(|\mu|+|\nu|)} A'_{|\mu|,|\mu|+|\nu|}(j, \nu, k) S_\nu' E_k'^{|\mu|+|\nu|} S_\nu'^*,$$

we have

$$E_j'^{|\mu|} \geq S_\nu' E_k'^{|\mu|+|\nu|} S_\nu'^*. \tag{8.16}$$

Take $\delta \in E'_{|\mu|,|\mu|+|\nu|}$ such that

$$\lambda(\delta) = \nu, \quad s(\delta) = v_j'^{|\mu|} \in V'_{|\mu|}, \quad t(\delta) = v_k'^{|\mu|+|\nu|} \in V'_{|\mu|+|\nu|}$$

in the λ-graph system $\mathfrak{L}^{\min}_{\Lambda_2'}$. There exists a word $\gamma \in B_*(\Lambda_2')$ such that $v_k'^{|\mu|+|\nu|}$ launches γ. Take $\delta' \in E'_{|\mu|+|\nu|,|\mu|+|\nu|+|\gamma|}$ such that

$$\lambda(\delta') = \gamma, \quad s(\delta') = v_k'^{|\mu|+|\nu|} \in V'_{|\mu|+|\nu|} \quad \text{in } \mathfrak{L}^{\min}_{\Lambda_2'}.$$

The labeled path $\delta\delta'$ is the unique path labeled $\nu\gamma$ in $E'_{|\mu|+|\nu|,|\mu|+|\nu|+|\gamma|}$. Since $\gamma \in \Gamma_*^+(\nu)$ implies $\nu\gamma \in \Gamma_*^+(\mu)$, we know that $\mu \in \Gamma^-_{|\mu|}(v_j'^{|\mu|})$. Hence there exists a labeled path $\delta'' \in E'_{0,|\mu|}$ labeled μ such that $t(\delta'') = v_j'^{|\mu|}$. This implies that

$$S_\mu'^* S_\mu' \geq E_j'^{|\mu|}$$

so that by (8.16) we see $S_\mu'^* S_\mu' \geq S_\nu' E_k'^{|\mu|+|\nu|} S_\nu'^*$. By the identity (8.15), we conclude the inequality $S_\mu'^* S_\mu' \geq S_\nu' S_\nu'^*$ in $\mathcal{O}_{\Lambda_2'^{\min}}$.

(ii) Assume that $1 - S'^*_\mu S'_\mu \ge S'_\nu S'^*_\nu$ in $\mathcal{O}_{\Lambda_2'^{\min}}$. Now suppose that

$$\widehat{S}^*_\mu \widehat{S}_\mu \cdot \widehat{S}_\nu \widehat{S}^*_\nu \ne 0 \quad \text{in } \widehat{\mathcal{O}}_{\Lambda_2'^{\min}}.$$

By Lemma 8.5, we have

$$\begin{aligned}
0 &\ne \widehat{S}^*_\mu \widehat{S}_\mu \cdot \widehat{S}_\nu \widehat{S}^*_\nu \\
&= S_{\eta_{m+1} w_{[m+1,L+m-1]}} S^*_{\eta_1 w_{[1,L+m-1]}} S_{\eta_1 w_{[1,L+m-1]}} S^*_{\eta_{m+1} w_{[m+1,L+m-1]}} \\
&\quad \cdot S_{\xi_1 y_{[1,L+n-1]}} S^*_{\xi_1 y_{[1,L+n-1]}}.
\end{aligned}$$

Since $S^*_{\eta_{m+1} w_{[m+1,L+m-1]}} S_{\xi_1 y_{[1,L+n-1]}} \ne 0$, we have $\xi_1 = \eta_{m+1}$, $w_{[m+1,L+m-1]} = y_{[1,L-1]}$. We thus have

$$\begin{aligned}
&S_{\xi_1 y_{[1,L+n-1]}} S^*_{\xi_1 y_{[1,L+n-1]}} S_{\eta_{m+1} w_{[m+1,L+m-1]}} \\
&\quad = S_{\xi_1 y_{[1,L+n-1]}} S^*_{\xi_1 y_{[1,L+n-1]}} S_{\xi_1 y_{[1,L-1]}} = S_{\xi_1 y_{[1,L+n-1]}} S^*_{y_{[L,L+n-1]}}
\end{aligned}$$

and

$$S^*_{\eta_{m+1} w_{[m+1,L+m-1]}} S_{\xi_1 y_{[1,L+n-1]}} S^*_{\xi_1 y_{[1,L+n-1]}} = S_{y_{[L,L+n-1]}} S^*_{\xi_1 y_{[1,L+n-1]}}$$

so that

$$\begin{aligned}
\widehat{S}^*_\mu \widehat{S}_\mu \cdot \widehat{S}_\nu \widehat{S}^*_\nu &= \widehat{S}_\nu \widehat{S}^*_\nu \cdot \widehat{S}^*_\mu \widehat{S}_\mu \cdot \widehat{S}_\nu \widehat{S}^*_\nu \\
&= S_{\xi_1 y_{[1,L+n-1]}} S^*_{y_{[L,L+n-1]}} \cdot S^*_{\eta_1 w_{[1,L+m-1]}} S_{\eta_1 w_{[1,L+m-1]}} \\
&\quad \cdot S_{y_{[L,L+n-1]}} S^*_{\xi_1 y_{[1,L+n-1]}} \\
&= S_{\xi_1 y_{[1,L+n-1]}} S^*_{\eta_1 w_{[1,L+m-1]} y_{[L,L+n-1]}} S_{\eta_1 w_{[1,L+m-1]} y_{[L,L+n-1]}} S^*_{\xi_1 y_{[1,L+n-1]}}.
\end{aligned}$$

Since $|\eta_1 w_{[1,L+m-1]} y_{[L,L+n-1]}| = K + L + m - 1 + n$, there exists

$$j \in \{1, 2, \dots, m(K + L + m + n - 1)\}$$

such that

$$0 \ne S_{\xi_1 y_{[1,L+n-1]}} E_j^{K+L+m+n-1} S^*_{\xi_1 y_{[1,L+n-1]}} \quad \text{in } \mathcal{O}_{\Lambda_2^{\min}},$$

and hence

$$\xi_1 y_{[1,L+n-1]} \in \Gamma^-_{K+L+n-1}(v_j^{K+L+m+n-1}) \quad \text{in } \mathfrak{L}_{\Lambda_2^{\min}}.$$

Take $z_{[1,k]} \in S_{K+L+m+n-1}(\Lambda_2)$ such that $k > L$ and $v_j^{K+L+m+n-1}$ launches $z_{[1,k]}$. By putting

$$\zeta_1 = \tau^{\varphi_1}(\xi_n, y_{[n,L+n-1]}),\ \zeta_2 = \tau^{\varphi_1}(\zeta_1, z_{[1,L]}), \dots,\ \zeta_{k-L} = \tau^{\varphi_1}(\zeta_{k-L-1}, z_{[k-L+1,k]}),$$

the word

$$\begin{aligned}
&\big((\eta_1, w_{[1,L]}), \dots, (\eta_m, w_{[m,L+m-1]}), (\xi_1, y_{[1,L]}), \dots, (\xi_n, y_{[n,L+n-1]}), \\
&(\zeta_1, z_{[1,L]}), \dots, (\zeta_{k-L}, z_{[k-L+1,k]})\big)
\end{aligned}$$

belongs to $B_{m+n+k-L}(\Lambda_2')$. Put

$$\gamma = \big((\zeta_1, z_{[1,L]}), (\zeta_2, z_{[2,L+1]}), \ldots, (\zeta_{k-L}, z_{[k-L+1,k]})\big).$$

Since $z_{[1,k]} \in S_{K+L+m+n-1}(\Lambda_2)$, we have $\gamma \in S_{m+n}(\Lambda_2')$ so that we have

$$S_\mu'^* S_\mu' \cdot S_\nu' S_\nu'^* \geq S_{\nu\gamma}' S_{\nu\gamma}'^* \neq 0 \quad \text{in } \mathcal{O}_{\Lambda_2'^{\min}},$$

a contradiction.

Conversely, assume $1 - \widehat{S}_\mu^* \widehat{S}_\mu \geq \widehat{S}_\nu \widehat{S}_\nu^*$ in $\widehat{\mathcal{O}}_{\Lambda_2'^{\min}}$. Now suppose that $S_\mu'^* S_\mu' \cdot S_\nu' S_\nu'^* \neq 0$ in $\mathcal{O}_{\Lambda_2'^{\min}}$. Since

$$S_\mu'^* S_\mu' = \sum_{j=1}^{m'(|\mu|)} A_{0,|\mu|}'(0, \mu, j) E_j'^{|\mu|} \quad \text{in } \mathcal{O}_{\Lambda_2'^{\min}},$$

take $j \in \{1, 2, \ldots, m'(|\mu|)\}$ such that $A_{0,|\mu|}'(0, \mu, j) = 1$ and $E_j'^{|\mu|} \cdot S_\nu' S_\nu'^* \neq 0$ so that $S_\nu'^* E_j'^{|\mu|} \cdot S_\nu' \neq 0$. As

$$S_\nu'^* E_j'^{|\mu|} S_\nu' = \sum_{k=1}^{m'(|\mu|+|\nu|)} A_{|\mu|,|\mu|+|\nu|}'(j, \nu, k) E_k'^{|\mu|+|\nu|},$$

there exists $k \in \{1, 2, \ldots, m'(|\mu| + |\nu|)\}$ such that $A_{|\mu|,|\mu|+|\nu|}'(j, \nu, k) = 1$ and hence

$$S_\nu'^* E_j'^{|\mu|} S_\nu' \geq E_k'^{|\mu|+|\nu|}.$$

One may take an admissible word $\gamma = ((\zeta_1, z_{[1,L]}), \ldots, (\zeta_p, z_{[p,L+p-1]})) \in B_p(\Lambda_2')$ such that $v_k'^{|\mu|+|\nu|}$ launches γ so that $E_k'^{|\mu|+|\nu|} \geq S_\gamma' S_\gamma'^*$. Hence we have

$$E_j'^{|\mu|} \geq S_{\nu\gamma}' S_{\nu\gamma}'^* \quad \text{so that } S_\mu'^* S_\mu' \geq S_{\nu\gamma}' S_{\nu\gamma}'^*.$$

By (i) we have $\widehat{S}_\mu^* \widehat{S}_\mu \geq \widehat{S}_{\nu\gamma} \widehat{S}_{\nu\gamma}^*$. Since

$$\widehat{S}_\mu^* \widehat{S}_\mu \cdot \widehat{S}_\nu \widehat{S}_\nu^* \cdot \widehat{S}_{\nu\gamma} \widehat{S}_{\nu\gamma}^* = \widehat{S}_{\nu\gamma} \widehat{S}_{\nu\gamma}^* \neq 0,$$

we get $\widehat{S}_\mu^* \widehat{S}_\mu \cdot \widehat{S}_\nu \widehat{S}_\nu^* \neq 0$, a contradiction to $1 - \widehat{S}_\mu^* \widehat{S}_\mu \geq \widehat{S}_\nu \widehat{S}_\nu^*$. ■

In the minimal λ-graph system $\mathfrak{L}_{\Lambda_2'}^{\min}$, recall that $\{v_1'^l, \ldots, v_{m'(l)}'^l\}$ denote the vertex set V_l' of $\mathfrak{L}_{\Lambda_2'}^{\min}$ of the normal subshift Λ_2'. For a vertex $v_i'^l \in V_l'$, define a function $f_i^l : B_l(\Lambda_2') \to \{0, 1\}$ by setting for $\mu \in B_l(\Lambda_2')$

$$f_i^l(\mu) = \begin{cases} 1 & \text{if } \mu \in \Gamma_l^-(v_i'^l), \\ -1 & \text{if } \mu \notin \Gamma_l^-(v_i'^l). \end{cases}$$

Recall that $S'_\alpha, \alpha \in \Sigma'_2$ and $E'^{l}_i, v'^{l}_i \in V'_l$ denote the canonical generating partial isometries and projections of the C^*-algebra $\mathcal{O}_{\Lambda'_2{}^{\min}}$. We then have by (3.1)

$$E'^{l}_i = \prod_{\mu \in B_l(\Lambda'_2)} S'^*_\mu S'^{f^l_i(\mu)}_\mu \quad \text{in } \mathcal{O}_{\Lambda'_2{}^{\min}} \tag{8.17}$$

where for $\mu \in B_l(\Lambda'_2)$

$$S'^*_\mu S'^{f^l_i(\mu)}_\mu = \begin{cases} S'^*_\mu S'_\mu & \text{if } f^l_i(\mu) = 1, \\ 1 - S'^*_\mu S'_\mu & \text{if } f^l_i(\mu) = -1. \end{cases}$$

In the C^*-algebra $\widehat{\mathcal{O}}_{\Lambda'_2{}^{\min}}$, we define a projection for each $v'^{l}_i \in V'_l$ by setting

$$\widehat{E}^l_i := \prod_{\mu \in B_l(\Lambda'_2)} \widehat{S}^*_\mu \widehat{S}^{f^l_i(\mu)}_\mu \quad \text{in } \widehat{\mathcal{O}}_{\Lambda'_2{}^{\min}} \tag{8.18}$$

where for $\mu \in B_l(\Lambda'_2)$

$$\widehat{S}^*_\mu \widehat{S}^{f^l_i(\mu)}_\mu = \begin{cases} \widehat{S}^*_\mu \widehat{S}_\mu & \text{if } f^l_i(\mu) = 1, \\ 1 - \widehat{S}^*_\mu \widehat{S}_\mu & \text{if } f^l_i(\mu) = -1. \end{cases}$$

Let $(A'_{l,l+1}, I'_{l,l+1})_{l \in \mathbb{Z}_+}$ be the transition matrix system for $\mathfrak{L}^{\min}_{\Lambda'_2}$.

Lemma 8.9. *For each $v'^{l}_i \in V'_l$, we have the identities*

$$\sum_{i=1}^{m'(l)} \widehat{E}^l_i = 1, \quad \widehat{E}^l_i = \sum_{j=1}^{m'(l+1)} I'_{l,l+1}(i,j) \widehat{E}^{l+1}_j.$$

Proof. We will first show $\widehat{E}^l_i \geq \widehat{E}^{l+1}_j$ for $i = 1, 2, \dots, m'(l)$, $j = 1, 2, \dots, m'(l+1)$ with $I'_{l,l+1}(i,j) = 1$. Assume that $I'_{l,l+1}(i,j) = 1$. Hence we have $\Gamma^-_l(v'^{l+1}_j) = \Gamma^-_l(v'^{l}_i)$ in $\mathfrak{L}^{\min}_{\Lambda'_2}$. For $\nu \in \Gamma^-_l(v'^{l}_i)$ and $\beta \in \Gamma^-_1(\nu)$, we have $\widehat{S}^*_\nu \widehat{S}^*_\beta \widehat{S}_\beta \widehat{S}_\nu = \widehat{S}^*_{\beta\nu} \widehat{S}_{\beta\nu}$. Hence

$$\widehat{S}^*_\nu \widehat{S}_\nu \geq \widehat{S}^*_{\beta\nu} \widehat{S}_{\beta\nu} \quad \text{for } \beta \in \Gamma^-_1(\nu).$$

For $\nu \notin \Gamma^-_l(v'^{l}_i)$ and $\beta_1, \beta_2 \in \Gamma^-_1(\nu)$, we have

$$(1 - \widehat{S}^*_{\beta_1\nu} \widehat{S}_{\beta_1\nu})(1 - \widehat{S}^*_{\beta_2\nu} \widehat{S}_{\beta_2\nu}) = 1 - \widehat{S}^*_\nu (\widehat{S}^*_{\beta_1} \widehat{S}_{\beta_1} + \widehat{S}^*_{\beta_2} \widehat{S}_{\beta_2} - \widehat{S}^*_{\beta_1} \widehat{S}_{\beta_1} \widehat{S}^*_{\beta_2} \widehat{S}_{\beta_2}) \widehat{S}_\nu.$$

Similarly we see that

$$\prod_{\beta \in \Gamma^-_1(\nu)} (1 - \widehat{S}^*_{\beta\nu} \widehat{S}_{\beta\nu}) = 1 - \widehat{S}^*_\nu \Big(\bigvee_{\beta \in \Gamma^-_1(\nu)} \widehat{S}^*_\beta \widehat{S}_\beta \Big) \widehat{S}_\nu \tag{8.19}$$

where

$$\bigvee_{\beta\in\Gamma_1^-(\nu)} \hat{S}_\beta^*\hat{S}_\beta = \sum_{\beta\in\Gamma_1^-(\nu)} \hat{S}_\beta^*\hat{S}_\beta - \sum_{\beta 1\neq\beta_2\in\Gamma_1^-(\nu)} \hat{S}_{\beta_1}^*\hat{S}_{\beta_1}\hat{S}_{\beta_2}^*\hat{S}_{\beta_2} + \cdots - (-1)^{|\Gamma_1^-(\nu)|} \prod_{\beta\in\Gamma_1^-(\nu)} \hat{S}_\beta^*\hat{S}_\beta$$

the projection spanned by $\hat{S}_\beta^*\hat{S}_\beta, \beta \in \Gamma_1^-(\nu)$. Now $\bigvee_{\beta\in\Sigma_2'} \hat{S}_\beta^*\hat{S}_\beta = 1$ so that we have

$$\hat{S}_\nu^*\hat{S}_\nu = \hat{S}_\nu^*\Big(\bigvee_{\beta\in\Sigma_2'} \hat{S}_\beta^*\hat{S}_\beta\Big)\hat{S}_\nu = \hat{S}_\nu^*\Big(\bigvee_{\beta\in\Gamma_1^-(\nu)} \hat{S}_\beta^*\hat{S}_\beta\Big)\hat{S}_\nu. \tag{8.20}$$

By (8.19) and (8.20), we have

$$\prod_{\beta\in\Gamma_1^-(\nu)} (1 - \hat{S}_{\beta\nu}^*\hat{S}_{\beta\nu}) = 1 - \hat{S}_\nu^*\hat{S}_\nu \tag{8.21}$$

For $\nu \in \Gamma_l^-(v_i'^l)(= \Gamma_l^-(v_j'^{l+1}))$ and $\beta \in \Gamma_1^-(\nu)$ with $\mu = \beta\nu \in \Gamma_{l+1}^-(v_j'^{l+1})$, we have

$$\hat{S}_\nu^*\hat{S}_\nu \geq \hat{S}_\mu^*\hat{S}_\mu.$$

For $\nu \notin \Gamma_l^-(v_i'^l)(= \Gamma_l^-(v_j'^{l+1}))$ and $\beta \in \Gamma_1^-(\nu)$ with $\mu = \beta\nu \notin \Gamma_{l+1}^-(v_j'^{l+1})$, we have by (8.21)

$$\prod_{\beta\in\Gamma_1^-(\nu)} (1 - \hat{S}_{\beta\nu}^*\hat{S}_{\beta\nu}) = 1 - \hat{S}_\nu^*\hat{S}_\nu.$$

Hence we have

$$\begin{aligned}\hat{E}_i^l &= \prod_{\nu\in\Gamma_l^-(v_i'^l)} \hat{S}_\nu^*\hat{S}_\nu \cdot \prod_{\nu\notin\Gamma_l^-(v_i'^l)} (1 - \hat{S}_\nu^*\hat{S}_\nu) \\ &\geq \prod_{\mu\in\Gamma_{l+1}^-(v_j'^{l+1})} \hat{S}_\mu^*\hat{S}_\mu \cdot \prod_{\mu\notin\Gamma_{l+1}^-(v_j'^{l+1})} (1 - \hat{S}_\mu^*\hat{S}_\mu) \\ &\geq \hat{E}_j^{l+1}.\end{aligned}$$

We will next see that $\hat{E}_j^{l+1} \cdot \hat{E}_{j'}^{l+1} = 0$ for $j \neq j'$. As $v_j'^{l+1} \neq v_{j'}'^{l+1}$ in V_{l+1}', we know $\Gamma_{l+1}^-(v_j'^{l+1}) \neq \Gamma_{l+1}^-(v_{j'}'^{l+1})$ because $\mathfrak{L}_{\Lambda_2'}^{\min}$ is predecessor-separated. Hence we have two cases:

Case (1): There exists $\mu \in \Gamma_{l+1}^-(v_j'^{l+1})$ such that $\mu \notin \Gamma_{l+1}^-(v_{j'}'^{l+1})$.

Case (2): There exists $\mu \in \Gamma_{l+1}^-(v_{j'}'^{l+1})$ such that $\mu \notin \Gamma_{l+1}^-(v_j'^{l+1})$.

In both the cases, it is easy to see that $\hat{E}_j^{l+1} \cdot \hat{E}_{j'}^{l+1} = 0$ by its definition (8.18).

Since $\hat{E}_i^l \geq \hat{E}_j^{l+1}$ for $i = 1,2,\ldots,m'(l)$, $j = 1,2,\ldots,m'(l+1)$ with $I_{l,l+1}'(i,j) = 1$, and $\hat{E}_j^{l+1} \cdot \hat{E}_{j'}^{l+1} = 0$ for $j \neq j'$, we have the inequality

$$\hat{E}_i^l \geq \sum_{j=1}^{m'(l+1)} I_{l,l+1}'(i,j)\hat{E}_j^{l+1}. \tag{8.22}$$

We will next show that $\sum_{i=1}^{m'(l)} \widehat{E}_i^l = 1$. Denote by $\{1,-1\}^{B_l(\Lambda'_2)}$ the set of functions

$$f : B_l(\Lambda'_2) \to \{1,-1\}.$$

For $f \in \{1,-1\}^{B_l(\Lambda'_2)}$, we set

$$\widehat{S}_\mu^* \widehat{S}_\mu^{f(\mu)} = \begin{cases} \widehat{S}_\mu^* \widehat{S}_\mu & \text{if } f(\mu) = 1, \\ 1 - \widehat{S}_\mu^* \widehat{S}_\mu & \text{if } f(\mu) = -1 \end{cases}$$

and put

$$\widehat{E}_f = \prod_{\mu \in B_l(\Lambda'_2)} \widehat{S}_\mu^* \widehat{S}_\mu^{f(\mu)} \quad \text{in } \widehat{\mathcal{O}}_{\Lambda_2'^{\min}}.$$

For $\mu \in B_l(\Lambda'_2)$, the identity $1 = \widehat{S}_\mu^* \widehat{S}_\mu + (1 - \widehat{S}_\mu^* \widehat{S}_\mu)$ implies

$$1 = \prod_{\mu \in B_l(\Lambda'_2)} \big(\widehat{S}_\mu^* \widehat{S}_\mu + (1 - \widehat{S}_\mu^* \widehat{S}_\mu)\big) = \sum_{f \in \{1,-1\}^{B_l(\Lambda'_2)}} \widehat{E}_f.$$

For $\mu = ((\eta_1, w_{[1,L]}), \dots, (\eta_l, w_{[l,L+l-1]})) \in B_l(\Lambda'_2)$, put $\eta_{l+1} = \tau^{\varphi_1}(\eta_l, w_{[l,L+l-1]}) \in B_K(\Lambda'_2)$ as usual. Let us consider a subset $S(\mu) \subset V_{K+L+l-1}^{\Lambda_2^{\min}}$ defined by

$$S(\mu) = \big\{v = s(\gamma) \in V_{K+L+l-1}^{\Lambda_2^{\min}} \mid \text{there exists } \gamma \in E_{l,K+L+l-1}^{\Lambda_2^{\min}}; \\ \eta_1 w_{[1,L+l-1]} \in \Gamma_{K+L+l-1}^-(v),\ \lambda(\gamma) = \eta_{l+1} w_{[l+1,L+l-1]} \in \Gamma_{K+L-1}^-(v)\big\}.$$

Define two subsets of $V_{K+L+l-1}^{\Lambda_2^{\min}}$ for the function f by

$$S_f^1 = \bigcap \big\{S(\mu) \mid \mu \in B_l(\Lambda'_2) \text{ with } f(\mu) = 1\big\},$$
$$S_f^{-1} = \bigcap \big\{S(\nu)^c \cap V_{K+L+l-1}^{\Lambda_2^{\min}} \mid \nu \in B_l(\Lambda'_2) \text{ with } f(\nu) = -1\big\}.$$

Now suppose that $\widehat{E}_f \neq 0$. Since

$$\widehat{S}_\mu^* \widehat{S}_\mu = S_{\eta_{l+1} w_{[l+1,L+l-1]}} S_{\eta_1 w_{[1,L+l-1]}}^* S_{\eta_1 w_{[1,L+l-1]}} S_{\eta_{l+1} w_{[l+1,L+l-1]}}^* \quad \text{in } \mathcal{O}_{\Lambda_2{}^{\min}},$$

the condition $\widehat{E}_f \neq 0$ ensures us that $S_f^1 \cap S_f^{-1} \neq \emptyset$. One may take a vertex $v_{j_0}^{K+L+l-1} \in S_f^1 \cap S_f^{-1}$ and $\mu = ((\eta_1, w_{[1,L]}), \dots, (\eta_l, w_{[l,L+l-1]})) \in B_l(\Lambda'_2)$ such that $v_{j_0}^{K+L+l-1} \in S(\mu)$. As $S_{\eta_1 w_{[1,L+l-1]}}^* S_{\eta_1 w_{[1,L+l-1]}} \geq E_{j_0}^{K+L+l-1}$, we have

$$\widehat{S}_\mu^* \widehat{S}_\mu \geq S_{\eta_{l+1} w_{[l+1,L+l-1]}} E_{j_0}^{K+L+l-1} S_{\eta_{l+1} w_{[l+1,L+l-1]}}^*.$$

For $\mu' = ((\eta'_1, w'_{[1,L]}), \dots, (\eta'_l, w'_{[l,L+l-1]})) \in B_l(\Lambda'_2)$ with $f(\mu') = 1$, we have $\widehat{S}_{\mu'}^* \widehat{S}_{\mu'} \cdot \widehat{S}_\mu^* \widehat{S}_\mu \geq \widehat{E}_f \neq 0$ so that

$$S_{\eta'_{l+1} w'_{[l+1,L+l-1]}}^* S_{\eta_{l+1} w_{[l+1,L+l-1]}} \neq 0$$

and hence $\eta'_{l+1} = \eta_{l+1}$, $w'_{[l+1,L+l-1]} = w_{[l+1,L+l-1]}$. We thus have

$$\widehat{S}^*_{\mu'}\widehat{S}_{\mu'} \ge S_{\eta_{l+1}w_{[l+1,L+l-1]}}E^{K+L+l-1}_{j_0}S^*_{\eta_{l+1}w_{[l+1,L+l-1]}}. \tag{8.23}$$

For $\mu'' = ((\eta''_1, w''_{[1,L]}), \dots, (\eta''_l, w''_{[l,L+l-1]})) \in B_l(\Lambda'_2)$ satisfying $f(\mu'') = -1$, we know that $v^{K+L+l-1}_{j_0} \in S(\mu'')^c$ so that

$$\widehat{S}^*_{\mu''}\widehat{S}_{\mu''} \perp S_{\eta_{l+1}w_{[l+1,L+l-1]}}E^{K+L+l-1}_{j_0}S^*_{\eta_{l+1}w_{[l+1,L+l-1]}}.$$

Hence we have

$$1 - \widehat{S}^*_{\mu''}\widehat{S}_{\mu''} \ge S_{\eta_{l+1}w_{[l+1,L+l-1]}}E^{K+L+l-1}_{j_0}S^*_{\eta_{l+1}w_{[l+1,L+l-1]}}. \tag{8.24}$$

By (8.23) and (8.24), we obtain

$$\widehat{E}_f \ge S_{\eta_{l+1}w_{[l+1,L+l-1]}}E^{K+L+l-1}_{j_0}S^*_{\eta_{l+1}w_{[l+1,L+l-1]}}.$$

As $\mathfrak{L}^{\min}_{\Lambda_2}$ is λ-synchronizing, for the word $v^{K+L+m-1}_{j_0} \in V^{\Lambda_2^{\min}}_{K+L+m-1}$ there exists an admissible word $(b_1, \dots, b_p) \in B_p(\Lambda_2)$ such that $v^{K+L+m-1}_{j_0}$ launches $(b_1, \dots, b_p)$. This implies that the inequalities

$$E^{K+L+m-1}_{j_0} \ge S_{b_1\cdots b_p}S^*_{b_1\cdots b_p} \quad \text{in } \mathcal{O}_{\Lambda_2^{\min}}$$

and hence

$$\widehat{E}_f \ge S_{\eta_{l+1}w_{[l+1,L+l-1]}}S_{b_1\cdots b_p}S^*_{b_1\cdots b_p}S^*_{\eta_{l+1}w_{[l+1,L+l-1]}} \tag{8.25}$$

hold. Put

$$\begin{aligned} &\xi_1 = \eta_{l+1} \in B_K(\Lambda_2), \quad y_{[1,L+p]} = w_{[l+1,L+l-1]}b_1\cdots b_p \in B_{L+p}(\Lambda_2),\\ &\xi_{i+1} = \tau^{\varphi_1}(\xi_i, y_{[i,L+i-1]}), \quad i = 1, 2, \dots, p-1. \end{aligned}$$

Define the word

$$\nu = \big((\xi_1, y_{[1,L]}), \dots, (\xi_p, y_{[p,L+p-1]})\big) \in B_p(\Lambda'_2).$$

It follows from (8.9) that

$$\widehat{S}_\nu\widehat{S}^*_\nu = S_{\xi_1 y_{[1,L+p-1]}}S^*_{\xi_1 y_{[1,L+p-1]}} = S_{\eta_{l+1}w_{[l+1,L+l-1]}b_1\cdots b_p}S^*_{\eta_{l+1}w_{[l+1,L+l-1]}b_1\cdots b_p}.$$

By (8.25), we have

$$\widehat{E}_f \ge \widehat{S}_\nu\widehat{S}^*_\nu \ne 0 \quad \text{in } \widehat{\mathcal{O}}_{\Lambda_2'^{\min}}. \tag{8.26}$$

We put

$$E'_f = \prod_{\mu \in B_l(\Lambda'_2)} S'^*_\mu S'^{f(\mu)}_\mu \quad \text{in } \mathcal{O}_{\Lambda_2'^{\min}}.$$

By applying Lemma 8.8 for (8.26), we have

$$E'_f \geq S'_\nu S'^*_\nu \quad \text{in } \mathcal{O}_{\Lambda'_2{}^{\min}}$$

and hence $E'_f \neq 0$. Since

$$1 = \sum_{f \in \{1,-1\}^{B_l(\Lambda'_2)}} E'_f = \sum_{i=1}^{m'(l)} E'^l_i$$

and $E'_{f^l_i} = E'^l_i$, we know that $E'_f \neq 0$ if and only if $f = f^l_i$ for some $v'^l_i \in V'_l$. Hence the condition $E'_f \neq 0$ implies that $f = f^l_i$ for some $v'^l_i \in V'_l$. Therefore we see that $\widehat{E}_f = \widehat{E}_{f^l_i}$ for some $v'^l_i \in V'_l$. Since $\widehat{E}_{f^l_i} = \widehat{E}^l_i$ and $1 = \sum_{f \in \{1,-1\}^{B_l(\Lambda'_2)}} \widehat{E}_f$, we conclude that

$$1 = \sum_{i=1}^{m'(l)} \widehat{E}^l_i \quad \text{in } \widehat{\mathcal{O}}_{\Lambda'_2{}^{\min}}. \tag{8.27}$$

Since for each $j = 1, 2, \ldots, m'(l+1)$, there exists a unique $i = 1, 2, \ldots, m'(l)$ such that $I'_{l,l+1}(i, j) = 1$, we have $\widehat{E}^{l+1}_j = \sum_{i=1}^{m'(l)} I'_{l,l+1}(i, j)\widehat{E}^{l+1}_j$. As the identity (8.27) holds for all $l \in \mathbb{Z}_+$, we have

$$1 = \sum_{j=1}^{m'(l+1)} \widehat{E}^{l+1}_j = \sum_{i=1}^{m'(l)} \sum_{j=1}^{m'(l+1)} I'_{l,l+1}(i, j)\widehat{E}^{l+1}_j$$

so that

$$1 = \sum_{i=1}^{m'(l)} \widehat{E}^l_i = \sum_{i=1}^{m'(l)} \sum_{j=1}^{m'(l+1)} I'_{l,l+1}(i, j)\widehat{E}^{l+1}_j.$$

By the inequality (8.22), we conclude that

$$\widehat{E}^l_i = \sum_{j=1}^{m'(l+1)} I'_{l,l+1}(i, j)\widehat{E}^{l+1}_j. \qquad \blacksquare$$

Define the commutative C^*-subalgebras:

$$\mathcal{A}_{\mathfrak{L}^{\min}_{\Lambda'_2}} = C^*\big(S'^*_\mu S'_\mu : \mu \in B_*(\Lambda'_2)\big) \subset \mathcal{O}_{\Lambda'_2{}^{\min}},$$
$$\widehat{\mathcal{A}}_{\mathfrak{L}^{\min}_{\Lambda'_2}} = C^*\big(\widehat{S}^*_\mu \widehat{S}_\mu : \mu \in B_*(\Lambda'_2)\big) \subset \widehat{\mathcal{O}}_{\Lambda'_2{}^{\min}}.$$

Lemma 8.10. *Using the above notation, the following hold.*

(i) *The commutative C^*-subalgebras $\mathcal{A}_{\mathfrak{L}^{\min}_{\Lambda'_2}}$ and $\widehat{\mathcal{A}}_{\mathfrak{L}^{\min}_{\Lambda'_2}}$ satisfy that*

$$\mathcal{A}_{\mathfrak{L}^{\min}_{\Lambda'_2}} = C^*\big(E'^l_i : i = 1, 2, \ldots, m'(l),\ l \in \mathbb{Z}_+\big),$$
$$\widehat{\mathcal{A}}_{\mathfrak{L}^{\min}_{\Lambda'_2}} = C^*\big(\widehat{E}^l_i : i = 1, 2, \ldots, m'(l),\ l \in \mathbb{Z}_+\big).$$

(ii) *There exists an isomorphism* $\Phi_{\mathcal{A}} : \mathcal{A}_{\mathfrak{L}^{\min}_{\Lambda'_2}} \to \widehat{\mathcal{A}}_{\mathfrak{L}^{\min}_{\Lambda'_2}}$ *of* C^**-algebras such that*

$$\widehat{S}_\alpha^* \Phi_{\mathcal{A}}(X) \widehat{S}_\alpha = \Phi_{\mathcal{A}}(S'^*_\alpha X S'_\alpha), \quad X \in \mathcal{A}_{\mathfrak{L}^{\min}_{\Lambda'_2}}, \ \alpha \in \Sigma'_2. \tag{8.28}$$

Proof. (i) By the identity (8.17), E'^l_i is written in terms of $S'^*_\mu S'^{f^l_i(\mu)}_\mu$. Conversely for any word $\mu \in B_l(\Lambda'_2)$ we set

$$J'(\mu, i) = \begin{cases} 1 & \text{if } \mu \in \Gamma^-_l(v'^l_i), \\ 0 & \text{otherwise.} \end{cases}$$

By the formula $1 = \sum_{i=1}^{m'(l)} E'^l_i$, we have

$$S'^*_\mu S'_\mu = \sum_{i=1}^{m'(l)} S'^*_\mu S'_\mu E'^l_i = \sum_{i=1}^{m'(l)} J'(\mu, i) E'^l_i,$$

so that $S'^*_\mu S'^{f^l_i(\mu)}_\mu$ is written in terms of E'^l_i. Hence we have

$$\mathcal{A}_{\mathfrak{L}^{\min}_{\Lambda'_2}} = C^*\big(E'^l_i : i = 1, 2, \dots, m'(l), \ l \in \mathbb{Z}_+\big).$$

The other equality

$$\widehat{\mathcal{A}}_{\mathfrak{L}^{\min}_{\Lambda'_2}} = C^*\big(\widehat{E}^l_i : i = 1, 2, \dots, m'(l), \ l \in \mathbb{Z}_+\big).$$

is similarly proved.

(ii) The identities

$$E'^l_i = \sum_{j=1}^{m'(l+1)} I'_{l,l+1}(i,j) E'^{l+1}_j, \quad 1 = \sum_{i=1}^{m'(l)} E'^l_i \quad \text{in } \mathcal{A}_{\mathfrak{L}^{\min}_{\Lambda'_2}},$$
$$\widehat{E}^l_i = \sum_{j=1}^{m'(l+1)} I'_{l,l+1}(i,j) \widehat{E}^{l+1}_j, \quad 1 = \sum_{i=1}^{m'(l)} \widehat{E}^l_i \quad \text{in } \widehat{\mathcal{A}}_{\mathfrak{L}^{\min}_{\Lambda'_2}}$$

hold. Since the projections E'^l_i, $\widehat{E}^l_i$ are all nonzero, the correspondence $E'^l_i \to \widehat{E}^l_i$ extends to an isomorphism $\Phi_{\mathcal{A}} : \mathcal{A}_{\mathfrak{L}^{\min}_{\Lambda'_2}} \to \widehat{\mathcal{A}}_{\mathfrak{L}^{\min}_{\Lambda'_2}}$ of C^*-algebras such that $\Phi_{\mathcal{A}}(E'^l_i) = \widehat{E}^l_i$ and hence $\Phi_{\mathcal{A}}(S'^*_\mu S'_\mu) = \widehat{S}^*_\mu \widehat{S}_\mu$ for $\mu \in B_*(\Lambda'_2)$. We then have

$$\widehat{S}^*_\alpha \Phi_{\mathcal{A}}(S'^*_\mu S'_\mu) \widehat{S}_\alpha = \widehat{S}^*_\alpha \widehat{S}^*_\mu \widehat{S}_\mu \widehat{S}_\alpha = \widehat{S}^*_{\mu\alpha} \widehat{S}_{\mu\alpha} = \Phi_{\mathcal{A}}(S'^*_{\mu\alpha} S'_{\mu\alpha}) = \Phi_{\mathcal{A}}(S'^*_\alpha S'^*_\mu S'_\mu S'_\alpha)$$

proving the identity (8.28). ■

Recall that $A'_{l,l+1}(i, \alpha, j)$ denotes a matrix component of the transition matrix system $(A'_{l,l+1}, I'_{l,l+1})_{l \in \mathbb{Z}_+}$ of $\mathfrak{L}^{\min}_{\Lambda'_2}$.

Lemma 8.11. *The following identity holds.*

$$\widehat{S}_\alpha^* \widehat{E}_i^l \widehat{S}_\alpha = \sum_{j=1}^{m'(l+1)} A'_{l,l+1}(i,\alpha,j)\widehat{E}_j^{l+1} \quad \textit{for } \alpha = (\xi_1, y_{[1,L]}) \in \Sigma'_2.$$

Proof. We note that the identity

$$S'^*_\alpha E'^l_i S'_\alpha = \sum_{j=1}^{m'(l+1)} A'_{l,l+1}(i,\alpha,j)E'^{l+1}_j \quad \text{for } \alpha = (\xi_1, y_{[1,L]}) \in \Sigma'_2$$

holds. By using the preceding lemma, we have

$$\begin{aligned}\widehat{S}_\alpha^* \widehat{E}_i^l \widehat{S}_\alpha &= \widehat{S}_\alpha^* \Phi_{\mathcal{A}}(E'^l_i)\widehat{S}_\alpha = \Phi(S'^*_\alpha E'^l_i S'_\alpha) \\ &= \Phi\Bigg(\sum_{j=1}^{m'(l+1)} A'_{l,l+1}(i,\alpha,j)E'^{l+1}_j\Bigg) \\ &= \sum_{j=1}^{m'(l+1)} A'_{l,l+1}(i,\alpha,j)\widehat{E}_j^{l+1}.\end{aligned}$$

∎

Recall that

$$\begin{aligned}\widehat{\mathcal{O}}_{\Lambda'_2{}^{\min}} &= C^*\big(\widehat{S}_{(\xi_1, y_{[1,L]})} : (\xi_1, y_{[1,L]}) \in \Sigma'_2\big), \\ \widehat{\mathcal{D}}_{\mathfrak{L}^{\min}_{\Lambda'_2}} &= C^*\big(\widehat{S}_\mu \widehat{S}_\nu^* \widehat{S}_\nu \widehat{S}_\mu^*, : \mu,\nu \in B_*(\Lambda'_2)\big), \\ \widehat{\mathcal{D}}_{\Lambda'_2} &= C^*\big(\widehat{S}_\mu \widehat{S}_\mu^* : \mu \in B_*(\Lambda'_2)\big).\end{aligned}$$

Then the inclusion relations

$$\widehat{\mathcal{O}}_{\Lambda'_2{}^{\min}} \subset \mathcal{O}_{\Lambda_2{}^{\min}}, \quad \widehat{\mathcal{D}}_{\mathfrak{L}^{\min}_{\Lambda'_2}} \subset \mathcal{D}_{\mathfrak{L}^{\min}_{\Lambda_2}}, \quad \widehat{\mathcal{D}}_{\Lambda'_2} \subset \mathcal{D}_{\Lambda_2}$$

are obvious. Let $\widehat{\rho}_t^{\Lambda'_2}$ be the restriction of the gauge action $\rho_t^{\Lambda_2}$ on $\mathcal{O}_{\Lambda_2{}^{\min}}$ to the subalgebra $\widehat{\mathcal{O}}_{\Lambda'_2{}^{\min}}$. The gauge action on $\mathcal{O}_{\Lambda'_2{}^{\min}}$ is denoted by $\rho_t^{\Lambda'_2}$.

Lemma 8.12. *Keep the above notation. There exists an isomorphism* $\Phi : \mathcal{O}_{\Lambda'_2{}^{\min}} \to \widehat{\mathcal{O}}_{\Lambda'_2{}^{\min}}$ *of* C^**-algebras such that*

$$\Phi(\mathcal{D}_{\mathfrak{L}^{\min}_{\Lambda'_2}}) = \widehat{\mathcal{D}}_{\mathfrak{L}^{\min}_{\Lambda'_2}}, \quad \Phi(\mathcal{D}_{\Lambda'_2}) = \widehat{\mathcal{D}}_{\Lambda'_2}, \quad \Phi \circ \rho_t^{\Lambda'_2} = \widehat{\rho}_t^{\Lambda'_2} \circ \Phi.$$

Proof. By the universal property and its uniqueness of the C^*-algebra $\mathcal{O}_{\Lambda'_2{}^{\min}}$, the correspondence

$$\Phi : S'_\alpha,\ E'^l_i \in \mathcal{O}_{\Lambda'_2{}^{\min}} \to \widehat{S}_\alpha,\ \widehat{E}_i^l \in \widehat{\mathcal{O}}_{\Lambda'_2{}^{\min}} \subset \mathcal{O}_{\Lambda_2{}^{\min}}$$

yields an isomorphism $\Phi : \mathcal{O}_{\Lambda'_2{}^{\min}} \to \widehat{\mathcal{O}}_{\Lambda'_2{}^{\min}}$ of C^*-algebras such that

$$\Phi(\mathcal{D}_{\mathfrak{L}^{\min}_{\Lambda'_2}}) = \widehat{\mathcal{D}}_{\mathfrak{L}^{\min}_{\Lambda'_2}}, \quad \Phi(\mathcal{D}_{\Lambda'_2}) = \widehat{\mathcal{D}}_{\Lambda'_2}.$$

Since

$$(\widehat{\rho}_t^{\Lambda'_2} \circ \Phi)(S'_\alpha) = \widehat{\rho}_t^{\Lambda'_2}(\widehat{S}_\alpha) = \rho_t^{\Lambda_2}(S_{\xi_1 y_{[1,L]}} S^*_{\xi_2 y_{[2,L]}})$$
$$= e^{2\pi\sqrt{-1}t} S_{\xi_1 y_{[1,L]}} S^*_{\xi_2 y_{[2,L]}} = e^{2\pi\sqrt{-1}t}\widehat{S}_\alpha = (\Phi \circ \rho_t^{\Lambda'_2})(S'_\alpha),$$

the equality $\Phi \circ \rho_t^{\Lambda'_2} = \widehat{\rho}_t^{\Lambda'_2} \circ \Phi$ holds. ∎

We will finally prove that the C^*-subalgebra $\widehat{\mathcal{O}}_{\Lambda'_2{}^{\min}}$ of $\mathcal{O}_{\Lambda_2{}^{\min}}$ actually coincides with the ambient algebra $\mathcal{O}_{\Lambda_2{}^{\min}}$. This is the final step proving Theorem 1.4.

Let $\mathcal{F}_{\Lambda_2{}^{\min}}$ be the canonical AF algebra of $\mathcal{O}_{\Lambda_2{}^{\min}}$ that is realized as the fixed point subalgebra of $\mathcal{O}_{\Lambda_2{}^{\min}}$ under the gauge action $\rho_t^{\Lambda_2}$, $t \in \mathbb{T}$ of $\mathcal{O}_{\Lambda_2{}^{\min}}$. Let $\widehat{\mathcal{F}}_{\Lambda'_2{}^{\min}}$ be the C^*-subalgebra of $\widehat{\mathcal{O}}_{\Lambda'_2{}^{\min}}$ generated by elements of the form:

$$\widehat{S}_\nu \widehat{S}^*_\mu \widehat{S}_\mu \widehat{S}^*_\gamma, \quad \mu,\nu,\gamma \in B_*(\Lambda'_2) \text{ with } |\nu| = |\gamma|.$$

The subalgebra $\widehat{\mathcal{F}}_{\Lambda'_2{}^{\min}}$ is nothing but the C^*-subalgebra of $\widehat{\mathcal{O}}_{\Lambda'_2{}^{\min}}$ generated by elements of the form:

$$\widehat{S}_\nu \widehat{E}^l_i \widehat{S}^*_\gamma, \quad i = 1,2,\dots,m'(l),\ l \in \mathbb{Z}_+,\ \nu,\gamma \in B_*(\Lambda'_2) \text{ with } |\nu| = |\gamma|.$$

Lemma 8.13. $\widehat{\mathcal{F}}_{\Lambda'_2{}^{\min}} = \mathcal{F}_{\Lambda_2{}^{\min}}$.

Proof. Let

$$\nu = \big((\xi_1, y_{[1,L]}),\dots,(\xi_n, y_{[n,L+n-1]})\big), \quad \mu = \big((\eta_1, w_{[1,L]}),\dots,(\eta_m, w_{[m,L+m-1]})\big)$$

and $\gamma = ((\zeta_1, z_{[1,L]}),\dots,(\zeta_k, z_{[k,L+k-1]})) \in B_*(\Lambda'_2)$ with $k = n$. Put

$$\xi_{n+1} = \tau^{\varphi_1}(\xi_n, y_{[n,L+n-1]}),\ \eta_{m+1} = \tau^{\varphi_1}(\eta_m, w_{[m,L+m-1]}),\ \zeta_{k+1} = \tau^{\varphi_1}(\zeta_k, z_{[k,L+k-1]}).$$

By definition we know $\widehat{S}_{(\xi_1,y_{[1,L]})} = S_{\xi_1 y_{[1,L]}} S^*_{\xi_2 y_{[2,L]}} \in \mathcal{O}_{\Lambda_2{}^{\min}}$, so that we have

$$\begin{aligned}
&\widehat{S}_\nu \widehat{S}^*_\mu \widehat{S}_\mu \widehat{S}^*_\gamma \\
&= \widehat{S}_{(\xi_1,y_{[1,L]})}\cdots\widehat{S}_{(\xi_n,y_{[n,L+n-1]})} \cdot (\widehat{S}_{(\eta_1,w_{[1,L]})}\cdots\widehat{S}_{(\eta_m,w_{[m,L+m-1]})})^* \\
&\quad \cdot (\widehat{S}_{(\eta_1,w_{[1,L]})}\cdots\widehat{S}_{(\eta_m,w_{[m,L+m-1]})}) \cdot (\widehat{S}_{(\zeta_1,z_{[1,L]})}\cdots\widehat{S}_{(\zeta_n,z_{[n,L+n-1]})})^* \\
&= S_{\xi_1 y_{[1,L+n-1]}} S^*_{\xi_{n+1} y_{[n+1,L+n-1]}} \cdot S_{\eta_{m+1} w_{[m+1,L+m-1]}} S^*_{\eta_1 w_{[1,L+m-1]}} \\
&\quad \cdot S_{\eta_1 w_{[1,L+m-1]}} S^*_{\eta_{m+1} w_{[m+1,L+m-1]}} \cdot S_{\zeta_{n+1} z_{[n+1,L+n-1]}} S^*_{\zeta_1 z_{[1,L+n-1]}} \\
&= \begin{cases} S_{\xi_1 y_{[1,L+n-1]}} S^*_{\eta_1 w_{[1,L+m-1]}} \cdot S_{\eta_1 w_{[1,L+m-1]}} S^*_{\zeta_1 z_{[1,L+n-1]}} \\ \text{if } \xi_{n+1} = \eta_{m+1} = \zeta_{n+1},\ y_{[n+1,L+n-1]} = w_{[m+1,L+m-1]} = z_{[n+1,L+n-1]}, \\ 0 \quad \text{otherwise.} \end{cases}
\end{aligned}$$

Hence $\widehat{S}_\nu \widehat{S}^*_\mu \widehat{S}_\mu \widehat{S}^*_\gamma$ belongs to $\mathcal{F}_{\Lambda_2{}^{\min}}$, so that $\widehat{\mathcal{F}}_{\Lambda'_2{}^{\min}} \subset \mathcal{F}_{\Lambda_2{}^{\min}}$.

Conversely, for admissible words $a, b, c \in B_*(\Lambda_2)$ with $|a| = |c|$, by considering the identity

$$S_a S_b^* S_b S_c^* = \sum_{\delta \in B_N(\Lambda_2)} S_{a\delta} S_{b\delta}^* S_{b\delta} S_{c\delta}^* \tag{8.29}$$

for any $N \in \mathbb{N}$, one may assume that $|a|(= |c|)$, $|b| > K$, where K is the integer given in (8.1). For $\rho \in \Gamma_\infty^+(a) \cap \Gamma_\infty^+(b) \cap \Gamma_\infty^+(c)$, we define $(y_i)_{i\in\mathbb{N}}, (w_i)_{i\in\mathbb{N}}, (z_i)_{i\in\mathbb{N}} \in X_{\Lambda_2}$ by setting

$$\begin{aligned}
y_1 &= a_{K+1}, \dots, y_{|a|-K} = a_{|a|}, \quad y_{|a|-K+i} = \rho_i, \quad i = 1, 2, \dots, \\
w_1 &= b_{K+1}, \dots, w_{|b|-K} = b_{|b|}, \quad w_{|b|-K+i} = \rho_i, \quad i = 1, 2, \dots, \\
z_1 &= c_{K+1}, \dots, z_{|b|-K} = c_{|c|}, \quad z_{|c|-K+i} = \rho_i, \quad i = 1, 2, \dots .
\end{aligned}$$

Define sequences $(\xi_i)_{i\in\mathbb{N}}, (\eta_i)_{i\in\mathbb{N}}, (\zeta_i)_{i\in\mathbb{N}}$ of $B_K(\Lambda_2)$ by setting:

$$\begin{aligned}
\xi_1 &= a_{[1,K]}, \quad \xi_{i+1} = \tau^{\varphi_1}(\xi_i, y_{[i,L+i-1]}), \quad i = 1, 2, \dots, \\
\eta_1 &= b_{[1,K]}, \quad \eta_{i+1} = \tau^{\varphi_1}(\eta_i, w_{[i,L+i-1]}), \quad i = 1, 2, \dots, \\
\zeta_1 &= c_{[1,K]}, \quad \zeta_{i+1} = \tau^{\varphi_1}(\zeta_i, z_{[i,L+i-1]}), \quad i = 1, 2, \dots .
\end{aligned}$$

Define elements $x = (x_i)_{i\in\mathbb{N}}, x' = (x'_i)_{i\in\mathbb{N}}, x'' = (x''_i)_{i\in\mathbb{N}} \in X_{\Lambda_1}$ by setting

$$x = h^{-1}(\xi_1 y), \quad x' = h^{-1}(\eta_1 w), \quad x'' = h^{-1}(\zeta_1 z).$$

By the previous discussions, we know that

$$\xi_i = \varphi_1(x_{[i,M+i-1]}), \quad \eta_i = \varphi_1(x'_{[i,M+i-1]}), \quad \zeta_i = \varphi_1(x''_{[i,M+i-1]}), \quad i \in \mathbb{N}.$$

Put $p = |a| - |b| \in \mathbb{Z}$. Since $\sigma_{\Lambda_1}^K \circ h^{-1} : X_{\Lambda_1} \to X_{\Lambda_1}$ is a sliding block code, there exists $N_1, N_2 \in \mathbb{N}$ such that $w_{j+p} = y_j (= z_j)$ for all $j \ge N_2$ implies $x'_{[i+p,M+i+p-1]} = x_{[i,M+i-1]} (= x''_{[i,M+i-1]})$ for all $i \ge N_1$. Hence we have

$$\eta_i = \xi_{i+p} = \zeta_{i+p} \quad \text{for } i \ge N_1.$$

Let $n = \max\{N_1, N_2\}$, $m = n + p$. By putting

$$\begin{aligned}
\nu &= \big((\xi_1, y_{[1,L]}), \dots, (\xi_n, y_{[n,L+n-1]})\big) \in B_n(\Lambda'_2), \\
\mu &= \big((\eta_1, w_{[1,L]}), \dots, (\eta_m, w_{[m,L+m-1]})\big) \in B_m(\Lambda'_2), \\
\gamma &= \big((\zeta_1, z_{[1,L]}), \dots, (\zeta_n, z_{[k,L+n-1]})\big) \in B_n(\Lambda'_2),
\end{aligned}$$

we have

$$\eta_{m+1} = \xi_{n+1} = \zeta_{n+1}, \quad w_{[m,L+m-1]} = y_{[n,L+n-1]} = z_{[k,L+n-1]}.$$

Let $\delta = (\rho_1, \dots, \rho_{K+L+n-|a|-1}) \in B_*(\Lambda_2)$. We then have

$$\begin{aligned}
S_{a\delta} S_{b\delta}^* S_{b\delta} S_{c\delta}^* &= S_{\xi_1 y_{[1,L+n-1]}} S^*_{\eta_1 w_{[1,L+m-1]}} \cdot S_{\eta_1 w_{[1,L+m-1]}} S^*_{\zeta_1 z_{[1,L+n+1]}} \\
&= \widehat{S}_\nu \widehat{S}_\mu^* \widehat{S}_\mu \widehat{S}_\gamma^* .
\end{aligned}$$

By the formula (8.29) for $N = K + L + n - |a| - 1$, we know that $S_a S_b^* S_b S_c^*$ with $|a| = |c|$ belongs to the AF-algebra $\widehat{\mathcal{F}}_{\Lambda_2'^{\min}}$, so that we have $\mathcal{F}_{\Lambda_2^{\min}} \subset \widehat{\mathcal{F}}_{\Lambda_2'^{\min}}$. ∎

Lemma 8.14. $\widehat{\mathcal{D}}_{\Lambda_2'} = \mathcal{D}_{\Lambda_2}$ *and* $\widehat{\mathcal{O}}_{\Lambda_2'^{\min}} = \mathcal{O}_{\Lambda_2^{\min}}$.

Proof. The equality $\widehat{\mathcal{D}}_{\Lambda_2'} = \mathcal{D}_{\Lambda_2}$ is easily obtained by (8.9).

The inclusion relation $\widehat{\mathcal{O}}_{\Lambda_2'^{\min}} \subset \mathcal{O}_{\Lambda_2^{\min}}$ is obvious. To prove $\widehat{\mathcal{O}}_{\Lambda_2'^{\min}} = \mathcal{O}_{\Lambda_2^{\min}}$, it suffices to show that for any $\alpha \in \Sigma_2 = B_1(\Lambda_2)$, the partial isometry S_α belongs to $\widehat{\mathcal{O}}_{\Lambda_2'^{\min}}$. For $(\xi_1, y_{[1,L]}) \in \Sigma_2'$, we have

$$\widehat{S}_{(\xi_1, y_{[1,L]})} = S_{\xi_1 y_{[1,L]}} S^*_{\xi_2 y_{[2,L]}} \in \mathcal{O}_{\Lambda_2^{\min}},$$

so that for $t \in \mathbb{T}$

$$\begin{aligned} \rho_t^{\Lambda_2}(\widehat{S}^*_{(\xi_1, y_{[1,L]})} S_\alpha) &= \rho_t^{\Lambda_2}(S_{\xi_2 y_{[2,L]}} S^*_{\xi_1 y_{[1,L]}} S_\alpha) \\ &= S_{\xi_2 y_{[2,L]}} S^*_{\xi_1 y_{[1,L]}} S_\alpha = \widehat{S}^*_{(\xi_1, y_{[1,L]})} S_\alpha. \end{aligned}$$

This implies that $\widehat{S}^*_{(\xi_1, y_{[1,L]})} S_\alpha \in \mathcal{F}_{\Lambda_2^{\min}}$. By Lemma 8.13, we see that $\widehat{S}^*_{(\xi_1, y_{[1,L]})} S_\alpha$ belongs to $\widehat{\mathcal{F}}_{\Lambda_2'^{\min}}$ for any $(\xi_1, y_{[1,L]}) \in \Sigma_2'$. By the identity

$$S_\alpha = \sum_{(\xi_1, y_{[1,L]}) \in \Sigma_2'} \widehat{S}_{(\xi_1, y_{[1,L]})} \cdot \widehat{S}^*_{(\xi_1, y_{[1,L]})} S_\alpha,$$

we obtain that S_α belongs to $\widehat{\mathcal{O}}_{\Lambda_2'^{\min}}$, and hence $\mathcal{O}_{\Lambda_2^{\min}} \subset \widehat{\mathcal{O}}_{\Lambda_2'^{\min}}$. ∎

We thus have the following proposition.

Proposition 8.15. *There exists an isomorphism* $\Phi_2 : \mathcal{O}_{\Lambda_2'^{\min}} \to \mathcal{O}_{\Lambda_2^{\min}}$ *of* C^**-algebras such that*

$$\Phi_2(\mathcal{D}_{\Lambda_2'}) = \mathcal{D}_{\Lambda_2}, \quad \Phi_2 \circ \rho_t^{\Lambda_2'} = \rho_t^{\Lambda_2} \circ \Phi_2.$$

Proof. The assertion follows from Lemmas 8.12 and 8.14. ∎

Therefore we reach the following theorem.

Theorem 8.16. *Let* Λ_1 *and* Λ_2 *be normal subshifts. If their one-sided subshifts* $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ *and* $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ *are eventually conjugate, then there exists an isomorphism* $\Phi : \mathcal{O}_{\Lambda_1^{\min}} \to \mathcal{O}_{\Lambda_2^{\min}}$ *of* C^**-algebras such that* $\Phi(\mathcal{D}_{\Lambda_1}) = \mathcal{D}_{\Lambda_2}$ *and* $\Phi \circ \rho_t^{\Lambda_1} = \rho_t^{\Lambda_2} \circ \Phi$, $t \in \mathbb{T}$.

Proof. By Lemma 8.2, the one-sided subshifts $(X_{\Lambda_2'}, \sigma_{\Lambda_2'})$ and $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ are topologically conjugate, so that by Theorem 7.6 there exists an isomorphism $\Phi_1 : \mathcal{O}_{\Lambda_1^{\min}} \to \mathcal{O}_{\Lambda_2'^{\min}}$ of C^*-algebras such that $\Phi_1(\mathcal{D}_{\Lambda_1}) = \mathcal{D}_{\Lambda_2'}$ and $\Phi_1 \circ \rho_t^{\Lambda_1} = \rho_t^{\Lambda_2'} \circ \Phi_1$, $t \in \mathbb{T}$. By Proposition 8.15, there exists an isomorphism $\Phi_2 : \mathcal{O}_{\Lambda_2'^{\min}} \to \mathcal{O}_{\Lambda_2^{\min}}$ of C^*-algebras such that

$$\Phi_2(\mathcal{D}_{\Lambda_2'}) = \mathcal{D}_{\Lambda_2}, \quad \Phi_2 \circ \rho_t^{\Lambda_2'} = \rho_t^{\Lambda_2} \circ \Phi_2.$$

Therefore we have a desired isomorphism of C^*-algebras between $\mathcal{O}_{\Lambda_1^{\min}}$ and $\mathcal{O}_{\Lambda_2^{\min}}$. ∎

Let $\mathfrak{L}_1$, $\mathfrak{L}_2$ be left-resolving λ-graph systems that present subshifts Λ_1, Λ_2, respectively. In [37], the author introduced the notion of $(\mathfrak{L}_1, \mathfrak{L}_2)$-eventually conjugacy between one-sided subshifts $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$.

Definition 8.17 ([37, Definition 5.1]). Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be left-resolving λ-graph systems that present subshifts Λ_1 and Λ_2, respectively. Their one-sided subshifts $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are said to be $(\mathfrak{L}_1, \mathfrak{L}_2)$-*eventually conjugate* if there exist homeomorphisms $h_{\mathfrak{L}} : X_{\mathfrak{L}_1} \to X_{\mathfrak{L}_2}$, $h_\Lambda : X_{\Lambda_1} \to X_{\Lambda_2}$ and an integer $K \in \mathbb{Z}_+$ such that

$$\begin{cases} \sigma_{\mathfrak{L}_2}^K\big(h_{\mathfrak{L}}(\sigma_{\mathfrak{L}_1}(x))\big) = \sigma_{\mathfrak{L}_2}^{K+1}\big(h_{\mathfrak{L}}(x)\big), & x \in X_{\mathfrak{L}_1}, \\ \sigma_{\mathfrak{L}_1}^K\big(h_{\mathfrak{L}}^{-1}(\sigma_{\mathfrak{L}_2}(y))\big) = \sigma_{\mathfrak{L}_1}^{K+1}\big(h_{\mathfrak{L}}^{-1}(y)\big), & y \in X_{\mathfrak{L}_2}, \end{cases} \tag{8.30}$$

and

$$\pi_{\mathfrak{L}_2} \circ h_{\mathfrak{L}} = h_\Lambda \circ \pi_{\mathfrak{L}_1}. \tag{8.31}$$

We remark that the equalities (8.30) and (8.31) automatically imply the equalities

$$\begin{cases} \sigma_{\Lambda_2}^K\big(h_\Lambda(\sigma_{\Lambda_1}(a))\big) = \sigma_{\Lambda_2}^{K+1}\big(h_\Lambda(a)\big), & a \in X_{\Lambda_1}, \\ \sigma_{\Lambda_1}^K\big(h_\Lambda^{-1}(\sigma_{\Lambda_2}(b))\big) = \sigma_{\Lambda_1}^{K+1}\big(h_\Lambda^{-1}(b)\big), & b \in X_{\Lambda_2}. \end{cases}$$

In [37], the following proposition was proved.

Proposition 8.18 ([37, Theorem 1.3]). *Suppose that two left-resolving λ-graph systems $\mathfrak{L}_1$, $\mathfrak{L}_2$ satisfy condition (I). Then $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are $(\mathfrak{L}_1, \mathfrak{L}_2)$-eventually conjugate if and only if there exists an isomorphism $\Phi : \mathcal{O}_{\mathfrak{L}_1} \to \mathcal{O}_{\mathfrak{L}_2}$ of C^*-algebras such that*

$$\Phi(\mathcal{D}_{\Lambda_1}) = \mathcal{D}_{\Lambda_2} \quad \textit{and} \quad \Phi \circ \rho_t^{\mathfrak{L}_1} = \rho_t^{\mathfrak{L}_2} \circ \Phi, \quad t \in \mathbb{T}.$$

Proof of Theorem 1.4. Let Λ_1, Λ_2 be two normal subshifts. Assume that $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are eventually conjugate. By Theorem 8.16, there exists an isomorphism $\Phi : \mathcal{O}_{\Lambda_1^{\min}} \to \mathcal{O}_{\Lambda_2^{\min}}$ of C^*-algebras such that $\Phi(\mathcal{D}_{\Lambda_1}) = \mathcal{D}_{\Lambda_2}$ and $\Phi \circ \rho_t^{\Lambda_1} = \rho_t^{\Lambda_2} \circ \Phi$, $t \in \mathbb{T}$.

Conversely, suppose that there exists an isomorphism $\Phi : \mathcal{O}_{\Lambda_1^{\min}} \to \mathcal{O}_{\Lambda_2^{\min}}$ of C^*-algebras such that $\Phi(\mathcal{D}_{\Lambda_1}) = \mathcal{D}_{\Lambda_2}$ and $\Phi \circ \rho_t^{\Lambda_1} = \rho_t^{\Lambda_2} \circ \Phi$, $t \in \mathbb{T}$. Let $\mathfrak{L}_1$, $\mathfrak{L}_2$ be their minimal λ-graph systems $\mathfrak{L}_{\Lambda_1}^{\min}$, $\mathfrak{L}_{\Lambda_2}^{\min}$, respectively. By virtue of Proposition 2.14, the λ-graph systems $\mathfrak{L}_{\Lambda_1}^{\min}$, $\mathfrak{L}_{\Lambda_2}^{\min}$ both satisfy condition (I). The associated C^*-algebras $\mathcal{O}_{\mathfrak{L}_{\Lambda_1}^{\min}}$, $\mathcal{O}_{\mathfrak{L}_{\Lambda_2}^{\min}}$ are nothing but the C^*-algebras $\mathcal{O}_{\Lambda_1^{\min}}$, $\mathcal{O}_{\Lambda_2^{\min}}$, respectively. By Proposition 8.18, we know that $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are $(\mathfrak{L}_1, \mathfrak{L}_2)$-eventually conjugate, in particular, $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are eventually conjugate. ■

9. Two-sided topological conjugacy

In this section, we study two-sided topological conjugacy of normal subshifts in terms of the associated stabilized C^*-algebras with their diagonals and gauge actions. Let $\mathfrak{L}$ be a left-resolving λ-graph system over Σ that presents a subshift Λ. Following [37], we will consider the compact Hausdorff space

$$\bar{X}_{\mathfrak{L}} = \left\{ (\alpha_i, u_i)_{i\in\mathbb{Z}} \in \prod_{i\in\mathbb{Z}} (\Sigma \times \Omega_{\mathfrak{L}}) \mid (\alpha_{i+k}, u_{i+k})_{i\in\mathbb{Z}} \in X_{\mathfrak{L}} \text{ for all } k \in \mathbb{Z} \right\}$$

with the shift homeomorphism $\bar{\sigma}_{\mathfrak{L}}$

$$\bar{\sigma}_{\mathfrak{L}}\big((\alpha_i, u_i)_{i\in\mathbb{Z}}\big) = (\alpha_{i+1}, u_{i+1})_{i\in\mathbb{Z}}, \quad (\alpha_i, u_i)_{i\in\mathbb{Z}} \in \bar{X}_{\mathfrak{L}}$$

on $\bar{X}_{\mathfrak{L}}$, where $\bar{X}_{\mathfrak{L}}$ is endowed with the relative topology from the infinite product topology of $\prod_{i\in\mathbb{Z}}(\Sigma \times \Omega_{\mathfrak{L}})$. For $x = (\alpha_i, u_i)_{i\in\mathbb{Z}} \in \bar{X}_{\mathfrak{L}}$, $\alpha = (\alpha_i)_{i\in\mathbb{Z}} \in \Lambda$ and $k \in \mathbb{Z}$, we set

$$x_{[k,\infty)} = (\alpha_i, u_i)_{i=k}^{\infty}, \quad \alpha_{[k,\infty)} = (\alpha_i)_{i=k}^{\infty}.$$

Definition 9.1 ([37, Definition 7.1]). The topological dynamical systems $(\bar{X}_{\mathfrak{L}_1}, \bar{\sigma}_{\mathfrak{L}_1})$ and $(\bar{X}_{\mathfrak{L}_2}, \bar{\sigma}_{\mathfrak{L}_2})$ are said to be *right asymptotically conjugate* if there exists a homeomorphism $\psi : \bar{X}_{\mathfrak{L}_1} \to \bar{X}_{\mathfrak{L}_2}$ such that $\psi \circ \bar{\sigma}_{\mathfrak{L}_1} = \bar{\sigma}_{\mathfrak{L}_2} \circ \psi$ and

(i) for $m \in \mathbb{Z}$, there exists $M \in \mathbb{Z}$ such that $x_{[M,\infty)} = z_{[M,\infty)}$ implies $\psi(x)_{[m,\infty)} = \psi(z)_{[m,\infty)}$ for $x, z \in \bar{X}_{\mathfrak{L}_1}$,

(ii) for $n \in \mathbb{Z}$, there exists $N \in \mathbb{Z}$ such that $y_{[N,\infty)} = w_{[N,\infty)}$ implies $\psi^{-1}(y)_{[n,\infty)} = \psi^{-1}(w)_{[n,\infty)}$ for $y, w \in \bar{X}_{\mathfrak{L}_2}$.

We call $\psi : \bar{X}_{\mathfrak{L}_1} \to \bar{X}_{\mathfrak{L}_2}$ *a right asymptotic conjugacy.*

Let us denote by $\bar{\pi}_i : \bar{X}_{\mathfrak{L}_i} \to \Lambda_i$ the factor map defined by

$$\bar{\pi}_i((\alpha_i, u_i)_{i\in\mathbb{Z}}) = (\alpha_i)_{i\in\mathbb{Z}} \in \Lambda_i \quad \text{for } i = 1, 2.$$

Definition 9.2 ([37, Definition 7.2]). Two subshifts Λ_1 and Λ_2 are said to be $(\mathfrak{L}_1, \mathfrak{L}_2)$-*conjugate* if there exists a right asymptotic conjugacy $\psi_{\mathfrak{L}} : \bar{X}_{\mathfrak{L}_1} \to \bar{X}_{\mathfrak{L}_2}$ and a topological conjugacy $\psi_\Lambda : \Lambda_1 \to \Lambda_2$ such that $\bar{\pi}_2 \circ \psi_{\mathfrak{L}} = \psi_\Lambda \circ \bar{\pi}_1$.

Proposition 9.3. *Let Λ_1, Λ_2 be normal subshifts and $\mathfrak{L}_1$, $\mathfrak{L}_2$ be their minimal λ-graph systems, respectively. Suppose that Λ_1, Λ_2 are topologically conjugate, then they are $(\mathfrak{L}_1, \mathfrak{L}_2)$-conjugate.*

Proof. We may assume that Λ_1 and Λ_2 are bipartitely related by a bipartite subshift $\widehat{\Lambda}$ over alphabet $\Sigma = C \sqcup D$ (see [41,42]). Hence there exist specifications $\kappa_1 : \Sigma_1 \to C \cdot D$ and $\kappa_2 : \Sigma_2 \to D \cdot C$ such that the 2-higher block shift $\widehat{\Lambda}^{[2]}$ of $\widehat{\Lambda}$ is decomposed into two disjoint subshifts $\widehat{\Lambda}^{[2]} = \widehat{\Lambda}^{CD} \sqcup \widehat{\Lambda}^{DC}$, where

$$\begin{aligned}
\widehat{\Lambda}^{[2]} &= \{(x_i x_{i+1})_{i\in\mathbb{Z}} \mid (x_i)_{i\in\mathbb{Z}} \in \widehat{\Lambda}\}, \\
\widehat{\Lambda}^{CD} &= \{(c_i d_i)_{i\in\mathbb{Z}} \in \widehat{\Lambda}^{[2]} \mid c_i \in C,\ d_i \in D,\ i \in \mathbb{Z}\}, \\
\widehat{\Lambda}^{DC} &= \{(d_i c_{i+1})_{i\in\mathbb{Z}} \in \widehat{\Lambda}^{[2]} \mid d_i \in D,\ c_{i+1} \in C,\ i \in \mathbb{Z}\},
\end{aligned}$$

and specifications $\kappa_1 : \Sigma_1 \to C \cdot D$, $\kappa_2 : \Sigma_2 \to D \cdot C$ mean injective maps. The notion that two subshifts Λ_1, Λ_2 are bipartitely related means that Λ_1, Λ_2 are identified with $\widehat{\Lambda}^{CD}$, $\widehat{\Lambda}^{DC}$ through κ_1, κ_2, respectively.

The specifications κ_1 and κ_2 naturally extend to the maps $B_*(\Lambda_1) \to B_*(\widehat{\Lambda}^{CD})$ and $B_*(\Lambda_2) \to B_*(\widehat{\Lambda}^{DC})$, respectively. We still denote them by κ_1 and κ_2, respectively. We write $\mathfrak{L}_i = (V^i, E^i, \lambda^i, \iota^i)$, $i = 1, 2$. Let $(\alpha_i, u_i)_{i\in\mathbb{Z}} \in \bar{X}_{\mathfrak{L}_1}$. In the λ-graph system $\mathfrak{L}_1$, take a vertex $u_i^l \in V_l^1$ such that $(u_i^l)_{l\in\mathbb{N}} = u_i \in \Omega_{\mathfrak{L}_1}$, $i \in \mathbb{Z}_+$. There exists an l-synchronizing word $\mu_i^l \in S_l(\Lambda_1)$, $i \in \mathbb{N}$ such that $u_i^l = [\mu_i^l]_l \in S_l(\Lambda_1)/\sim_l$. Let $\kappa_1(\alpha_i) = c_i d_i$ for $c_i \in C$, $d_i \in D$. As $\kappa_1(\mu_i^l) \in B_*(\widehat{\Lambda}^{CD})$, take $c_i^l \in C$ such that $\kappa_1(\mu_i^l)c_i^l \in B_*(\widehat{\Lambda})$. Put $\nu_i^l = \kappa_2^{-1}(d_i\kappa_1(\mu_i^{l+1})c_i^{l+1}) \in V_l^2$ and $\beta_i = \kappa_2^{-1}(d_{i-1}c_i) \in \Sigma_2$. We then have $\beta_i\nu_i^l \sim_{l-1} \nu_{i-1}^{l-1}$ and $\nu_i^l \sim_{l-1} \nu_{i-1}^{l-1}$. Define $w_i^l = [\nu_i^l]_l \in S_l(\Lambda_2)/\sim$ so that $w_i^l \in V_l^2$. Since $\iota(w_i^{l+1}) = w_i^l$ for $l \in \mathbb{Z}_+$, we have $w_i = (w_i^l)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}_2}$ for $i \in \mathbb{Z}$ and $(\beta_i, w_i)_{i\in\mathbb{Z}} \in \bar{X}_{\mathfrak{L}_2}$. Under the identification between $\widehat{\Lambda}^{DC}$ and Λ_2, we know that the correspondence

$$(\alpha_i, u_i)_{i\in\mathbb{Z}} \in \bar{X}_{\mathfrak{L}_1} \to (\beta_i, w_i)_{i\in\mathbb{Z}} \in \bar{X}_{\mathfrak{L}_2}$$

written $\psi : \bar{X}_{\mathfrak{L}_1} \to \bar{X}_{\mathfrak{L}_2}$ gives rise to a topological conjugacy between $(\bar{X}_{\mathfrak{L}_1}, \bar{\sigma}_{\mathfrak{L}_1})$ and $(\bar{X}_{\mathfrak{L}_2}, \bar{\sigma}_{\mathfrak{L}_2})$ such that $\psi : \bar{X}_{\mathfrak{L}_1} \to \bar{X}_{\mathfrak{L}_2}$ is a right asymptotic conjugacy and there exists a topological conjugacy $\psi_\Lambda : \Lambda_1 \to \Lambda_2$ such that $\pi_2 \circ \psi = \psi_\Lambda \circ \pi_1$. Therefore the two-sided subshifts $(\Lambda_1, \sigma_{\Lambda_1})$ and $(\Lambda_2, \sigma_{\Lambda_2})$ are $(\mathfrak{L}_1, \mathfrak{L}_2)$-conjugate. ∎

Therefore we have the following proposition.

Proposition 9.4. *Let Λ_1, Λ_2 be normal subshifts and $\mathfrak{L}_1$, $\mathfrak{L}_2$ be their minimal λ-graph systems, respectively. Then the following two conditions are equivalent.*

- (i) *The two-sided subshifts $(\Lambda_1, \sigma_{\Lambda_1})$ and $(\Lambda_2, \sigma_{\Lambda_2})$ are $(\mathfrak{L}_1, \mathfrak{L}_2)$-conjugate.*
- (ii) *$(\Lambda_1, \sigma_{\Lambda_1})$ and $(\Lambda_2, \sigma_{\Lambda_2})$ are topologically conjugate.*

Let us recall that $\mathcal{K}$ denotes the C^*-algebra of compact operators on the separable infinite dimensional Hilbert space $\ell^2(\mathbb{N})$ and $\mathcal{C}$ denotes its commutative C^*-subalgebra of diagonal operators.

Proof of Theorem 1.5. Let Λ_1, Λ_2 be two normal subshifts. Suppose that the two-sided subshifts $(\Lambda_1, \sigma_{\Lambda_1})$ and $(\Lambda_2, \sigma_{\Lambda_2})$ are topologically conjugate. By Proposition 9.4, they are $(\mathfrak{L}_1, \mathfrak{L}_2)$-conjugate, so that [37, Theorem 1.4] ensures us that there exists an isomorphism $\widetilde{\Phi} : \mathcal{O}_{\Lambda_1^{\min}} \otimes \mathcal{K} \to \mathcal{O}_{\Lambda_2^{\min}} \otimes \mathcal{K}$ of C^*-algebras such that $\widetilde{\Phi}(\mathcal{D}_{\Lambda_1} \otimes \mathcal{C}) = \mathcal{D}_{\Lambda_2} \otimes \mathcal{C}$ and $\widetilde{\Phi} \circ (\rho_t^{\Lambda_1} \otimes \mathrm{id}) = (\rho_t^{\Lambda_2} \otimes \mathrm{id}) \circ \widetilde{\Phi}$, $t \in \mathbb{T}$.

Conversely suppose that there exists an isomorphism $\widetilde{\Phi} : \mathcal{O}_{\Lambda_1^{\min}} \otimes \mathcal{K} \to \mathcal{O}_{\Lambda_2^{\min}} \otimes \mathcal{K}$ of C^*-algebras such that $\widetilde{\Phi}(\mathcal{D}_{\Lambda_1} \otimes \mathcal{C}) = \mathcal{D}_{\Lambda_2} \otimes \mathcal{C}$ and $\widetilde{\Phi} \circ (\rho_t^{\Lambda_1} \otimes \mathrm{id}) = (\rho_t^{\Lambda_2} \otimes \mathrm{id}) \circ \widetilde{\Phi}$, $t \in \mathbb{T}$. By [37, Theorem 1.4] the two-sided subshifts $(\Lambda_1, \sigma_{\Lambda_1})$ and $(\Lambda_2, \sigma_{\Lambda_2})$ are $(\mathfrak{L}_1, \mathfrak{L}_2)$-conjugate, and hence they are topologically conjugate. ∎

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References

[1] F. Blanchard, β-expansions and symbolic dynamics. *Theoret. Comput. Sci.* **65** (1989), no. 2, 131–141 Zbl 0682.68081 MR 1020481

[2] F. Blanchard and G. Hansel, Systèmes codés. *Theoret. Comput. Sci.* **44** (1986), no. 1, 17–49 Zbl 0601.68056 MR 858689

[3] K. A. Brix and T. M. Carlsen, Cuntz–Krieger algebras and one-sided conjugacy of shifts of finite type and their groupoids. *J. Aust. Math. Soc.* **109** (2020), no. 3, 289–298 Zbl 1467.46067 MR 4190081

[4] K. A. Brix and T. M. Carlsen, C*-algebras, groupoids and covers of shift spaces. *Trans. Amer. Math. Soc. Ser. B* **7** (2020), 134–185 Zbl 1466.46057 MR 4168660

[5] T. M. Carlsen and J. Rout, Diagonal-preserving gauge-invariant isomorphisms of graph C^*-algebras. *J. Funct. Anal.* **273** (2017), no. 9, 2981–2993 Zbl 1383.46050 MR 3692327

[6] J. Cuntz and W. Krieger, A class of C^*-algebras and topological Markov chains. *Invent. Math.* **56** (1980), no. 3, 251–268 Zbl 0434.46045 MR 561974

[7] V. Deaconu, Groupoids associated with endomorphisms. *Trans. Amer. Math. Soc.* **347** (1995), no. 5, 1779–1786 Zbl 0826.46058 MR 1233967

[8] V. Deaconu, Generalized Cuntz–Krieger algebras. *Proc. Amer. Math. Soc.* **124** (1996), no. 11, 3427–3435 Zbl 0864.46033 MR 1343688

[9] R. Fischer, Sofic systems and graphs. *Monatsh. Math.* **80** (1975), no. 3, 179–186 Zbl 0314.54043 MR 407235

[10] T. Hamachi, K. Inoue, and W. Krieger, Subsystems of finite type and semigroup invariants of subshifts. *J. Reine Angew. Math.* **632** (2009), 37–61 Zbl 1177.37019 MR 2544142

[11] J. E. Hopcroft and J. D. Ullman, *Introduction to automata theory, languages, and computation*. Addison-Wesley Ser. Comput. Sci., Addison-Wesley, Reading, MA, 1979 Zbl 0980.68066 MR 645539

[12] Y. Katayama, K. Matsumoto, and Y. Watatani, Simple C^*-algebras arising from β-expansion of real numbers. *Ergodic Theory Dynam. Systems* **18** (1998), no. 4, 937–962 Zbl 1032.37005 MR 1645334

[13] B. P. Kitchens, *Symbolic dynamics*. Universitext, Springer, Berlin, 1998 Zbl 0892.58020 MR 1484730

[14] W. Krieger, On the uniqueness of the equilibrium state. *Math. Systems Theory* **8** (1974/75), no. 2, 97–104 Zbl 0302.28011 MR 399412

[15] W. Krieger, On sofic systems. I. *Israel J. Math.* **48** (1984), no. 4, 305–330 Zbl 0573.54032 MR 776312

[16] W. Krieger, On sofic systems. II. *Israel J. Math.* **60** (1987), no. 2, 167–176 Zbl 0638.54034 MR 931874

[17] W. Krieger, On subshifts and semigroups. *Bull. London Math. Soc.* **38** (2006), no. 4, 617–624 Zbl 1097.37006 MR 2250754

[18] W. Krieger and K. Matsumoto, A lambda-graph system for the Dyck shift and its K-groups. *Doc. Math.* **8** (2003), 79–96 Zbl 1029.37005 MR 2029162

[19] W. Krieger and K. Matsumoto, A notion of synchronization of symbolic dynamics and a class of C^*-algebras. *Acta Appl. Math.* **126** (2013), 263–275 Zbl 1329.37010 MR 3077952

[20] P. Kůrka, *Topological and symbolic dynamics*. Cours Spéc. 11, Société Mathématique de France, Paris, 2003 Zbl 1038.37011 MR 2041676

[21] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*. Cambridge University Press, Cambridge, 1995 Zbl 1106.37301 MR 1369092

[22] K. Matsumoto, On C^*-algebras associated with subshifts. *Internat. J. Math.* **8** (1997), no. 3, 357–374 Zbl 0885.46048 MR 1454478

[23] K. Matsumoto, Dimension groups for subshifts and simplicity of the associated C^*-algebras. *J. Math. Soc. Japan* **51** (1999), no. 3, 679–698 Zbl 1126.46309 MR 1691469

[24] K. Matsumoto, Presentations of subshifts and their topological conjugacy invariants. *Doc. Math.* **4** (1999), 285–340 Zbl 0926.37002 MR 1710375

[25] K. Matsumoto, On automorphisms of C^*-algebras associated with subshifts. *J. Operator Theory* **44** (2000), no. 1, 91–112 Zbl 0994.46021 MR 1774695

[26] K. Matsumoto, C^*-algebras associated with presentations of subshifts. *Doc. Math.* **7** (2002), 1–30 Zbl 1007.46051 MR 1911208

[27] K. Matsumoto, A simple purely infinite C^*-algebra associated with a lambda-graph system of the Motzkin shift. *Math. Z.* **248** (2004), no. 2, 369–394 Zbl 1065.46052 MR 2088934

[28] K. Matsumoto, Construction and pure infiniteness of C^*-algebras associated with lambda-graph systems. *Math. Scand.* **97** (2005), no. 1, 73–88 Zbl 1094.46041 MR 2179590

[29] K. Matsumoto, On the simple C^*-algebras arising from Dyck systems. *J. Operator Theory* **58** (2007), no. 1, 205–226 Zbl 1120.46054 MR 2336051

[30] K. Matsumoto, Orbit equivalence of one-sided subshifts and the associated C^*-algebras. *Yokohama Math. J.* **56** (2010), no. 1-2, 59–85 Zbl 1234.46054 MR 2791143

[31] K. Matsumoto, Orbit equivalence of topological Markov shifts and Cuntz–Krieger algebras. *Pacific J. Math.* **246** (2010), no. 1, 199–225 Zbl 1198.46049 MR 2645883

[32] K. Matsumoto, C^*-algebras arising from Dyck systems of topological Markov chains. *Math. Scand.* **109** (2011), no. 1, 31–54 Zbl 1250.37008 MR 2831146

[33] K. Matsumoto, C^*-algebras associated with lambda-synchronizing subshifts and flow equivalence. *J. Aust. Math. Soc.* **95** (2013), no. 2, 241–265 Zbl 1303.46047 MR 3142358

[34] K. Matsumoto, A certain synchronizing property of subshifts and flow equivalence. *Israel J. Math.* **196** (2013), no. 1, 235–272 Zbl 1327.37007 MR 3096590

[35] K. Matsumoto, On the Markov–Dyck shifts of vertex type. *Discrete Contin. Dyn. Syst.* **36** (2016), no. 1, 403–422 Zbl 1371.37085 MR 3369228

[36] K. Matsumoto, Uniformly continuous orbit equivalence of Markov shifts and gauge actions on Cuntz–Krieger algebras. *Proc. Amer. Math. Soc.* **145** (2017), no. 3, 1131–1140 Zbl 1369.37005 MR 3589313

[37] K. Matsumoto, A groupoid approach to C^*-algebras associated with λ-graph systems and continuous orbit equivalence of subshifts. *Dyn. Syst.* **35** (2020), no. 3, 398–429 Zbl 1476.46069 MR 4149088

[38] K. Matsumoto, One-sided topological conjugacy of normal subshifts and gauge actions on the associated C^*-algebras. *Dyn. Syst.* **36** (2021), no. 4, 586–607 Zbl 1484.37011 MR 4338820

[39] K. Matsumoto, On one-sided topological conjugacy of topological Markov shifts and gauge actions on Cuntz–Krieger algebras. *Ergodic Theory Dynam. Systems* **42** (2022), no. 8, 2575–2582 Zbl 1503.37020 MR 4448399

[40] K. Matsumoto, On simplicity of the C^*-algebras associated with λ-graph systems. *J. Math. Anal. Appl.* **515** (2022), no. 2, Paper No. 126441 Zbl 1502.46041 MR 4445682

[41] M. Nasu, Topological conjugacy for sofic systems. *Ergodic Theory Dynam. Systems* **6** (1986), no. 2, 265–280 Zbl 0607.54026 MR 857201

[42] M. Nasu, Textile systems for endomorphisms and automorphisms of the shift. *Mem. Amer. Math. Soc.* **114** (1995), no. 546, viii+215 Zbl 0845.54031 MR 1234883

[43] W. Parry, On the β-expansions of real numbers. *Acta Math. Acad. Sci. Hungar.* **11** (1960), 401–416 Zbl 0099.28103 MR 142719

[44] J. Renault, *A groupoid approach to C^*-algebras*. Lecture Notes in Math. 793, Springer, Berlin, 1980 Zbl 0433.46049 MR 584266

[45] J. Renault, Cuntz-like algebras. In *Operator theoretical methods (Timişoara, 1998)*, pp. 371–386, Theta Found., Bucharest, 2000 Zbl 1032.46535 MR 1770333

[46] J. Renault, Cartan subalgebras in C^*-algebras. *Irish Math. Soc. Bull.* **61** (2008), 29–63 Zbl 1175.46050 MR 2460017

[47] J. Renault, Examples of masas in C^*-algebras. In *Operator structures and dynamical systems*, pp. 259–265, Contemp. Math. 503, American Mathematical Society, Providence, RI, 2009 Zbl 1194.46105 MR 2590626

[48] A. Rényi, Representations for real numbers and their ergodic properties. *Acta Math. Acad. Sci. Hungar.* **8** (1957), 477–493 Zbl 0079.08901 MR 97374

[49] B. Weiss, Subshifts of finite type and sofic systems. *Monatsh. Math.* **77** (1973), 462–474 Zbl 0285.28021 MR 340556

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