

Remarks on the diagonal embedding and strong 1-boundedness

Srivatsav Kunnawalkam Elayavalli

Abstract. We identify a large class of hyperbolic groups whose von Neumann algebras are not strongly 1-bounded: Sela’s hyperbolic towers over F_2 subgroups. We also show that any intermediate subalgebra of the diagonal embedding of $L(F_2)$ into its ultrapower does not have Property (T).

1. Introduction

Voiculescu initiated a revolutionary theory of free entropy in his paper [46]. The free entropy computes the asymptotic volume of the microstate spaces (matrix models approximating the distribution of a fixed tuple in a tracial von Neumann algebra). Voiculescu’s asymptotic freeness theorem allowed him to show that the free entropy of a tuple of freely independent semicirculars is non vanishing. On the other hand, one is able to compute the free entropy when there are algebraic constraints present in the ambient algebra, such as sufficiently many commutation relations or the existence of diffuse regular subalgebras that are hyperfinite. Combining these two ideas, Voiculescu (in [47]) showed that $L(F_2)$ admits no Cartan subalgebras, then Ge (in [19]) obtained using the same idea, that $L(F_2)$ is prime. These settled problems left open by Popa in [41] where he showed that $L(F_X)$ is prime and admits no Cartan subalgebras where X is an uncountable set.

One of the main modern threads of Voiculescu’s free entropy theory is that of *strong 1-boundedness* for von Neumann algebras, which originated with remarkable ideas of Jung in [31]. Inspired by ideas from geometric measure theory, in particular Besicovitch’s classification of metric spaces with Hausdorff measure 1, Jung developed technical tools to study the case when Voiculescu’s free entropy dimension (a Minkowski dimension type quantity for the microstate spaces) for a tuple is 1, and discovered natural conditions wherein this property passes to the von Neumann algebra generated by the tuple. In particular, if one locates such a tuple in a von Neumann algebra, one can automatically conclude non isomorphism with $L(F_2)$, as the free entropy dimension of the semicircular generating set is 2. Jung used this in [31] to prove that $L(F_2)$ cannot be generated by two amenable (more generally strongly 1-bounded) subalgebras with diffuse intersection.

More recently, by carefully analyzing ideas of Jung, Hayes in [24] extracted a numerical invariant of the von Neumann algebra, called the 1-bounded entropy. This framework has proved very robust and has been used to obtain several new rigidity results for non strongly 1-bounded von Neumann algebras (such as $L(F_2)$). For instance, Hayes showed in [24] that $L(F_2)$ does not admit even quasi regular diffuse strongly 1-bounded subalgebras, generalizing the Theorem of Voiculescu. For more recent results see [4, 9, 10, 25, 28]. For these reasons it is of great interest to identify examples of non strongly 1-bounded von Neumann algebras.

In this note we observe using 1-bounded entropy, some structural properties of intermediate subalgebras of the diagonal embedding of $L(F_2)$ into its ultrapower. By way of leveraging *existentially closed* (see Section 2) copies of F_2 in the group level, our first main result identifies a family of hyperbolic groups introduced by Sela [43] whose von Neumann algebras are not strongly 1-bounded von Neumann algebras. Some familiar examples are hyperbolic surface groups.

Theorem A. *The group von Neumann algebras of all hyperbolic towers over F_2 subgroups are not strongly 1-bounded.*

Remark 1.1. We thank D. Shlyakhtenko for pointing out to us that surface group von Neumann algebras are not strongly 1-bounded is already known through a computation of the free entropy dimension, which is an a priori stronger result (see paragraph below [8, Theorem 4.13]). Roughly speaking, one sees that the hyperbolic surface groups are decomposed as an iterated amalgamated free product over copies of \mathbb{Z} . Then, using the free entropy dimension estimate for amalgamated free products over hyperfinite subalgebras, which is the main technical result of [8, Theorem 4.4], one can identify a generating set using an iterative process, whose microstates free entropy dimension has a precise lower bound (in the case of genus g , the lower bound is $2g - 1$ which is significantly greater than 1). We would like to point out that our proof not only applies to more groups but is conceptually different and softer. Indeed, on the von Neumann algebra level we only use the fact that $L(F_2)$ is not strongly 1-bounded, and the 1-bounded entropy inequality (see Fact 5.2) whose proof as outlined in [24, Proposition 4.5] is quite elementary.

Potentially, there is a larger class of groups that admit an existential copy of F_2 , however in light of Sela's classification of groups that are elementarily equivalent to free groups, finding more examples could be hard. Note also that there also is the famous class of existentially closed groups (see [29]). We document a proof that the group von Neumann algebras of these groups are on the other hand McDuff II_1 factors (see Proposition 4.1).

By virtue of being hyperbolic, many rigidity results are already known in the setting of Theorem A through Ozawa's biexactness techniques and Popa's deformation rigidity (see [11, 36, 37, 42]). However, one obtains using non-strong 1-boundedness, the following stronger rigidity results below:

Corollary B. *Let N be the group von Neumann algebra of a hyperbolic tower over an F_2 subgroup. Then the following hold:*

- (1) N cannot be written as the join of two strongly 1-bounded subalgebras (see a comprehensive list in Fact 5.5) with diffuse intersection.
- (2) N contains no diffuse quasi regular strongly 1-bounded subalgebra.

Our next observation describes a surprising structural property of intermediate subalgebras of the diagonal embedding:

Theorem C. *For any ultrafilter \mathcal{U} on any set I , one cannot embed a property (T) von Neumann algebra M into $L(F_2)^{\mathcal{U}}$ such the M contains the diagonal copy of $L(F_2)$.*

We would like to point out that Theorem C works if we just replace $L(F_2)$ by any finite von Neumann algebra N with $h(N \cap M : N) = \infty$ (see the first paragraph of Section 5). Also note that any M here (regardless of whether M has property (T) or not) will also not have any diffuse quasi regular amenable subalgebra. We record below a question asked to us by J. Peterson, as a conjecture:

Conjecture 1.2. *Let N be a non Gamma II_1 factor with the Haagerup property. Then for any ultrafilter \mathcal{U} on any set I , one cannot embed a property (T) von Neumann algebra M into $N^{\mathcal{U}}$ containing the diagonal embedding.*

Note that J. Peterson's conjecture on ultrapowers (see [40, Problem U.5]) implies the above conjecture. In particular Theorem C, can be seen as some evidence for the more general J. Peterson's conjecture.

We remark that in the realm of model theory of II_1 factors (see [15–17]), N being an intermediate subalgebra of $L(F_2)$ and its ultrapower is a very well known situation. This means that N admits an *existential embedding* of $L(F_2)$. Thus one of the features of this article is to emphasize the link between model theory and strong 1-boundedness (see also [30]).

2. Theories of groups

We work in the language of groups. Here the terms are words in the variables, their inverses, and the identity element (denoted by 1). An atomic formula is two terms separated by “ $=$ ”. Combining atomic formulae, logical connectors (\vee , \wedge , \neg) and quantifiers (\forall , \exists) one describes first order formulae. When there are no free variables we call it a sentence φ : for example, $\forall x_1 \forall x_2 \exists x_3 (x_1^2 x_2^3 x_3 = 1)$. The *elementary theory* of a group is the collection of sentences that hold true in the group.

An existential sentence is a formula without free variables that can be expressed in the following way: $\sigma = \exists x_1 \cdots \exists x_n \varphi(x_1 \cdots x_n)$. The *existential theory* of a group is the collection of existential sentences that hold true in the group. Note that one can also define a universal sentence, as a formula without free variables of the form $\forall x_1 \cdots \forall x_n \varphi(x_1 \cdots x_n)$, and the *universal theory* of a group as the collection of universal sentences that hold true in the group.

Remark 2.1. Observe that if G and H have the same universal theory, then they have the same existential theory because the negation of every universal sentence is an existential sentence.

An embedding of groups $\pi: H \rightarrow G$ is said to be *elementary* if for every first-order formula ϕ with k free variables x_1, \dots, x_k in the language of groups, and for every k -tuple $h_1, \dots, h_k \in H$, the statement $\phi(h_1, \dots, h_k)$ is true in H if and only if the statement $\phi(\pi(h_1), \dots, \pi(h_k))$ is true in G . Similarly one defines an embedding to be *existential* by replacing an arbitrary first-order formula ϕ with an existential first-order formula.

3. Ultrapowers of groups and of von Neumann algebras

Let G be a countable group and let 1_G denote the identity element of G . For an ultrafilter \mathcal{U} on a set I , we denote by $G^{\mathcal{U}}$ the algebraic ultrapower: $(\prod_I G)/N$ where

$$N = \{(g_i) : \{i \in I : g_i = 1_G\} \in \mathcal{U}\}.$$

Note that N is a normal subgroup of $\prod_I G$, hence $G^{\mathcal{U}}$ is a group. We have a natural diagonal inclusion $d_{\mathcal{U}}: G \rightarrow G^{\mathcal{U}}$ given by $g \mapsto (g_i)_{\mathcal{U}}$, where $g_i = g$, for all $i \in I$.

The following are standard results in model theory, following from work of Keisler–Shelah [33, 44]. For the sake of convenience assume the continuum hypothesis in this paper.

Fact 3.1. (1) G and H are elementarily equivalent, if and only if exists an ultrafilter \mathcal{U} on a set I such that $G^{\mathcal{U}} \cong H^{\mathcal{U}}$.

(2) $\pi: H \rightarrow G$ is an elementary embedding if and only if there exists an ultrafilter \mathcal{U} on a set I such that $\pi^{\mathcal{U}}: H^{\mathcal{U}} \rightarrow G^{\mathcal{U}}$ is an isomorphism.

(3) $\pi: H \rightarrow G$ is an existential embedding if and only if there exists an ultrafilter \mathcal{U} on a set I and an embedding $\rho: G \rightarrow H^{\mathcal{U}}$ such that $\rho \circ \pi = d_{\mathcal{U}}$.

In this paper we will be concerned with the objects (M, τ) , tracial von Neumann algebras, i.e., a pair consisting of a von Neumann algebra M and a faithful normal tracial state $\tau: M \rightarrow \mathbb{C}$. For any group G , one denotes the group von Neumann algebra (see [1, Section 1.3]) as $L(G) = \{u_g\}_{g \in G}'' \subset \mathbb{B}(\ell^2 G)$. For any subgroup $H < G$, one has the natural inclusion of von Neumann algebras $L(H) \subset L(G)$.

For an ultrafilter \mathcal{U} on a set I , we denote by $M^{\mathcal{U}}$ the tracial ultrapower: the quotient $\ell^\infty(I, M)/\mathcal{I}$ by the closed ideal $\mathcal{I} \subset \ell^\infty(I, M)$ consisting of $x = (x_n)_{\mathcal{U}}$ with $\lim_{n \rightarrow \mathcal{U}} \|x_n\|_2 = 0$. We have the canonical trace on $M^{\mathcal{U}}$ given by

$$\tau^{\mathcal{U}}((x_n)_{\mathcal{U}}) = \lim_{n \rightarrow \mathcal{U}} \tau(x_n).$$

We have a natural diagonal inclusion $D_{\mathcal{U}}: M \rightarrow M^{\mathcal{U}}$ given by $x \mapsto (x_n)_{\mathcal{U}}$, where $x_n = x$, for all $n \in I$.

Lemma 3.2. *Let G be a countable group, and \mathcal{U} be an ultrafilter on a set I . Then there exists a unital $*$ -homomorphism $\Theta: L(G^{\mathcal{U}}) \rightarrow L(G)^{\mathcal{U}}$ such that $\Theta(L(d_{\mathcal{U}}(G))) = D_{\mathcal{U}}(L(G))$.*

Proof. Let $\theta: G^{\mathcal{U}} \rightarrow L(G)^{\mathcal{U}}$ be the injective homomorphism given by

$$\theta((g_n)\mathcal{U}) = (u_{g_n})\mathcal{U}.$$

Observe that $\tau^{\mathcal{U}}((u_{g_n})\mathcal{U}) = \lim_{n \rightarrow \mathcal{U}} \tau(u_{g_n}) = 0$. Hence it follows that θ extends to a unital $*$ -homomorphism $\Theta: L(G^{\mathcal{U}}) \rightarrow L(G)^{\mathcal{U}}$ such that

$$\Theta(L(d_{\mathcal{U}}(G))) = D_{\mathcal{U}}(L(G)). \quad \blacksquare$$

4. Existentially closed groups and their von Neumann algebras

A group G is said to be *existentially closed* if every embedding of G into a group H is existential. Such groups exist, and have been studied extensively in group theory. See [29] for a survey. Recall also that a II_1 factor M is McDuff (see [34, 35]) if the central sequence algebra, namely $M' \cap M^{\mathcal{U}}$ is non abelian. The McDuff property has been very useful in the study of model theory of II_1 factors recently, particularly in identifying elementary equivalence classes (see [3, 5, 13, 18, 20, 22]).

Proposition 4.1. *Let G be an existentially closed group. Then, $L(G)$ is a McDuff II_1 factor.*

Proof. Firstly, observe that every conjugacy class of G is infinite. Indeed, by contradiction suppose there is a finite conjugacy class $\{g_1, \dots, g_n\}$. Then there exists an element h of G such that $h \neq g_i$ for any $i \in \{1, \dots, n\}$, and $|h| = |g_1|$, i.e., they have the same order. Indeed, let $k = |g_1|$, and consider $G \rightarrow G * \mathbb{Z}/k\mathbb{Z}$ and apply that this is an existential embedding. Then, one sees that h and g_1 are conjugate by considering the HNN extension and applying again the existential property. Secondly, consider the embedding of $\pi: G \rightarrow G \times (G * G)$ given by $g \mapsto (g, 1_{G * G})$. Since G is existentially closed, π is existential. Hence for any finite set $F \subset G$, there exists $g_{F,1}, g_{F,2} \in G$ such that $[F, g_{F,i}] = 1_G$ for all $i \in \{1, 2\}$ and $[g_{F,1}, g_{F,2}] \neq 1_G$. This implies that $L(G)$ is McDuff. Indeed fix \mathcal{U} a non principal ultrafilter on \mathbb{N} . Let F_n be an increasing family of finite subsets of G such that $\bigcup_n F_n = G$. Then see that $(g_{F_n,i})\mathcal{U} \in L(G)' \cap L(G)^{\mathcal{U}}$ for all $i \in \{1, 2\}$ and also $[(g_{F_n,1})\mathcal{U}, (g_{F_n,2})\mathcal{U}] \neq 0$. Hence we are done. \blacksquare

A separable tracial von Neumann algebra N is called *existentially closed* (see [14]) if for unital inclusion $N \subset M$ into a von Neumann separable tracial von Neumann algebra M , there exists an ultrafilter \mathcal{U} on a set I and an embedding $\pi: M \rightarrow N^{\mathcal{U}}$ such that π restricted to N coincides with $D_{\mathcal{U}}(N)$. It is well known that existentially closed separable tracial von Neumann algebra exist and are McDuff (see [21]). In the context of the above Proposition, it is natural to wonder if $L(G)$ is an existentially closed tracial von Neumann algebra, when G is an existentially closed group.

5. 1-bounded entropy

For a finite tuple X of self-adjoint operators in a tracial von Neumann algebra (M, τ) , one has the *1-bounded entropy* $h(X)$, implicit in Jung's work [31] and defined explicitly by Hayes [24]. It is the exponential growth rate of the covering numbers of Voiculescu's microstate spaces (see [46]) up to unitary conjugation. For an inclusion $N \subset M$ of tracial von Neumann algebras, the *1-bounded entropy of N in the presence of M* , denoted $h(N : M)$, is defined by modifying the definition of 1-bounded entropy to only measure the size of the space of microstates for N which have an extension to microstates for M . See [26, Sections 2.2 and 2.3] for a detailed and rigorous exposition. Note that $h(X_1 : M) = h(X_2 : M)$ if $X_1'' = X_2''$ by [24, Theorem A.9]. Hence, given a von Neumann subalgebra $N \subset M$, we unambiguously write $h(N : M)$ (and call it the *1-bounded entropy of N in the presence of M*) to be $h(X : Y)$ for some generating sets X of N and Y of M . We write $h(M) = h(M : M)$ and call it the *1-bounded entropy of M* .

For the purposes of this article we recall the following facts about h :

Fact 5.1 (See [26, Fact 2.3.3]). $h(N_1 : M_1) \leq h(N_2 : M_2)$ if $N_1 \subset N_2 \subset M_2 \subset M_1$ and N_1 is diffuse.

Fact 5.2 (See [24, Proposition 4.5]). $h(N : M) = h(N : M^{\mathcal{U}})$ if $N \subset M$ is diffuse, and \mathcal{U} is an ultrafilter on a set I . (Note that [24, Proposition 4.5] asserts this fact for free ultrafilters \mathcal{U} . The fact is trivially true also for non-free (i.e., principal) ultrafilters.)

Fact 5.3 (See [31, Lemma 3.7]). $h(N_1 * N_2) = \infty$ where (N_1, τ_1) and (N_2, τ_2) are Connes-embeddable diffuse tracial von Neumann algebras. In particular $h(L(F_2)) = \infty$.

Fact 5.4 (See [24, Proposition A.16]). N is strongly 1-bounded in the sense of Jung [31] if and only if $h(N) < \infty$.

The following are examples of strongly 1-bounded von Neumann algebras:

Fact 5.5. (1) Diffuse amenable tracial von Neumann algebras (see [31]).

(2) Tensor products of two diffuse tracial von Neumann algebras (see [31]).

(3) Tracial von Neumann algebras that have a diffuse hyperfinite quasi regular subalgebra (see [24, Theorem 3.8]).

(4) Tracial von Neumann algebras that have a Kazhdan set (see [26]). This includes all property (T) II_1 factors, and property (T) group von Neumann algebras.

(5) Group von Neumann algebras of groups that have vanishing first L^2 Betti number and are sofic and finitely presented (see [27, 32, 45]).

(6) Free orthogonal quantum group von Neumann algebras (see [6]).

(7) The non Gamma factors constructed in [10] and their ultrapowers.

6. Existential embeddings of F_2

A sequence of homomorphisms $\{\pi_n : G \rightarrow H\}$, $n \in \mathbb{N}$, is *eventually faithful* if for every $g \neq 1_G$, there exists an $N \in \mathbb{N}$ such that for all $n > N$, $\pi_n(g) \neq 1_H$.

Lemma 6.1 (See also [2, Lemma 3.1]). *Let G be a finitely generated group and let H be a finitely generated subgroup of G . If there is a sequence of eventually faithful homomorphisms $\{\pi_n: G \rightarrow H\}$, such that $\pi_n(h) = h$ for all $h \in H$, $n \in \mathbb{N}$, then there exists a non principal ultrafilter \mathcal{U} on \mathbb{N} and an embedding $\pi: G \rightarrow H^{\mathcal{U}}$ such that $\pi|_H = d_{\mathcal{U}}(H)$.*

Proof. Fix a non principal ultrafilter \mathcal{U} on \mathbb{N} . Define the map $\pi: G \rightarrow H^{\mathcal{U}}$ as follows:

$$\pi(g) = (\pi_n(g)).$$

Clearly, $\pi(h) = (h)_{\mathcal{U}}$ and hence, $\pi|_H = d_{\mathcal{U}}(H)$. Moreover, the fact that π is a homomorphism is checked easily since π_n is a homomorphism for all $n \in \mathbb{N}$. Also, π is injective because for all $g \in G$, $\{\pi_n(g) \neq 1_H\} \in \mathcal{U}$. ■

One can construct by hand such an eventually faithful sequence of homomorphisms in the case that G is a hyperbolic surface group and H is a particular non abelian free subgroup.

Lemma 6.2 (See [7, Corollary 2.2]). *Let G be the fundamental group of a surface of an orientable surface of genus $2r$. Consider a presentation of G as follows:*

$$G = \left\langle a_i, a'_i, b_i, b'_i, 1 \leq i \leq r \mid \prod_{i=1}^r [a_i, a'_i] \prod_{j=1}^r [b_j, b'_j] = 1_G \right\rangle.$$

Then consider the automorphism σ of G that leaves a_i and a'_i fixed for all $1 \leq i \leq r$ and sends every b_i to $gb_i g^{-1}$ and every b'_i to $gb'_i g^{-1}$ where $g = \prod_{i=1}^r [a_i, a'_i]$. Abuse notation and let

$$\pi: G \rightarrow \langle a_1, \dots, a_r, a'_1, \dots, a'_r \rangle$$

given by $\pi(a_i) = \pi(b_i) = a_i$ and $\pi(a'_i) = \pi(b'_i) = a'_i$. Then the sequence of homomorphisms $\{\pi \circ \sigma^n\}_{n \in \mathbb{N}}$ is eventually faithful.

More generally than this discussion one can describe a class of groups that admit an elementary inclusion of F_2 , which in particular satisfies the conclusion of Lemma 6.1. For the definition of this family, we direct the reader to [38, 39]. See Section A.1 and the Proof of Corollary A.2 (of [23]), where an eventually faithful sequence of homomorphisms in the case that G is a hyperbolic tower over an F_2 subgroup is constructed:

Theorem 6.3 ([23]). *Let G be a hyperbolic tower (in the sense of Sela [43]) over an F_2 subgroup. Then this inclusion of F_2 in G satisfies the conclusion of Lemma 6.1.*

7. Proofs

Proof of Theorem A. Let G be a hyperbolic tower over $H = F_2$. Then from Theorem 6.3 and Lemma 6.1, we see that there exists a non principal ultrafilter \mathcal{U} on \mathbb{N} , $H < G$ and an embedding $\pi: G \rightarrow H^{\mathcal{U}}$ such that $\pi|_H = d_{\mathcal{U}}(H)$. Now applying Lemma 3.2, we see

that there exists a unital $*$ -homomorphism $\Theta: L(G) \rightarrow L(H)^{\mathfrak{U}}$ such that

$$\Theta(L(d_{\mathfrak{U}}(H))) = D_{\mathfrak{U}}(L(H)).$$

Now applying Facts 5.1 and 5.2 we see the following inequality:

$$\infty = h(L(F_2)) = h(L(H)) = h(L(H) : L(H)^{\mathfrak{U}}) \leq h(L(G) : L(H)^{\mathfrak{U}}) \leq h(L(G)).$$

Finally, applying Fact 5.4 we are done. ■

We record the following proposition which is contained in the above proof:

Proposition 7.1. *If there exists $F_2 < G$ an existential embedding for some group G , then $L(G)$ is not strongly 1-bounded. More generally, if N is not strongly 1-bounded, and $N \subset M$ is an existential embedding, then M is not strongly 1-bounded.*

Proof of Theorem C. From the same inequality as above, we have

$$\infty = h(L(F_2)) = h(L(F_2) : L(F_2)^{\mathfrak{U}}) \leq h(M : L(F_2)^{\mathfrak{U}}) \leq h(M).$$

However, since $L(F_2)$ does not have property Gamma, it follows that M is a factor, and hence admits a Kazhdan set (see [12, Proposition 1]). By Fact 5.5 (4), we get a contradiction. ■

Remark 7.2. From Theorem C, one actually obtains for every inclusion of $L(F_2) \subset M$, where M has property (T), a model theoretic sentence ϕ with n free variables such that $\phi_M(x_1 \cdots x_n) = 0$ and $\phi_{L(F_2)}(x_1 \cdots x_n) > 0$ for some tuple $x_i \in L(F_2)$, $i = 1, \dots, n$.

Acknowledgments. The results in this article emerged from a discussion with J. Peterson. It is our pleasure to thank him and Vanderbilt University for their warm hospitality. We also sincerely thank S. André for his great assistance with some references, and guiding us through the theory of Sela's hyperbolic towers. We thank I. Chifan and T. Sinclair for reading our draft carefully.

Funding. The author was supported by a Simons Postdoctoral Fellowship.

References

- [1] C. Anantharaman and S. Popa, An introduction to II_1 factors. <https://www.math.ucla.edu/~popa/Books/IIun.pdf>
- [2] S. André, [Acylindrical hyperbolicity and existential closedness](#). *Proc. Amer. Math. Soc.* **150** (2022), no. 3, 909–918 Zbl 1511.20169 MR 4375691
- [3] S. Atkinson, I. Goldbring, and S. Kunnawalkam Elayavalli, [Factorial relative commutants and the generalized Jung property for \$\text{II}_1\$ factors](#). *Adv. Math.* **396** (2022), article no. 108107 Zbl 1487.46068 MR 4370469
- [4] S. Belinschi and M. Capitaine, Strong convergence of tensor products of independent G.U.E. matrices. 2022, arXiv:2205.07695

- [5] R. Boutonnet, I. Chifan, and A. Ioana, II_1 factors with nonisomorphic ultrapowers. *Duke Math. J.* **166** (2017), no. 11, 2023–2051 Zbl 1384.46040 MR 3694564
- [6] M. Brannan and R. Vergnioux, Orthogonal free quantum group factors are strongly 1-bounded. *Adv. Math.* **329** (2018), 133–156 Zbl 1396.46055 MR 3783410
- [7] E. Breuillard, T. Gelander, J. Souto, and P. Storm, Dense embeddings of surface groups. *Geom. Topol.* **10** (2006), 1373–1389 Zbl 1132.22011 MR 2255501
- [8] N. P. Brown, K. J. Dykema, and K. Jung, Free entropy dimension in amalgamated free products. *Proc. Lond. Math. Soc. (3)* **97** (2008), no. 2, 339–367 Zbl 1158.46045 MR 2439665
- [9] I. Charlesworth, R. de Santiago, B. Hayes, D. Jekel, S. Kunnawalkam Elayavalli, and B. Nelson, Strong 1-boundedness, L^2 -Betti numbers, algebraic soficity, and graph products. 2023, arXiv:2305.19463
- [10] I. Chifan, A. Ioana, and S. Kunnawalkam Elayavalli, An exotic II_1 factor without property Gamma. *Geom. Funct. Anal.* (2023), DOI 10.1007/s00039-023-00649-4
- [11] I. Chifan and T. Sinclair, On the structural theory of II_1 factors of negatively curved groups. *Ann. Sci. Éc. Norm. Supér. (4)* **46** (2013), no. 1, 1–33 Zbl 1290.46053 MR 3087388
- [12] A. Connes and V. Jones, Property T for von Neumann algebras. *Bull. London Math. Soc.* **17** (1985), no. 1, 57–62 Zbl 1190.46047 MR 766450
- [13] J. Fang, L. Ge, and W. Li, Central sequence algebras of von Neumann algebras. *Taiwanese J. Math.* **10** (2006), no. 1, 187–200 Zbl 1105.46042 MR 2186173
- [14] I. Farah, I. Goldbring, B. Hart, and D. Sherman, Existentially closed II_1 factors. *Fund. Math.* **233** (2016), no. 2, 173–196 Zbl 1436.03213 MR 3466004
- [15] I. Farah, B. Hart, and D. Sherman, Model theory of operator algebras I: stability. *Bull. Lond. Math. Soc.* **45** (2013), no. 4, 825–838 Zbl 1295.03019 MR 3081550
- [16] I. Farah, B. Hart, and D. Sherman, Model theory of operator algebras II: model theory. *Israel J. Math.* **201** (2014), no. 1, 477–505 Zbl 1301.03037 MR 3265292
- [17] I. Farah, B. Hart, and D. Sherman, Model theory of operator algebras III: elementary equivalence and II_1 factors. *Bull. Lond. Math. Soc.* **46** (2014), no. 3, 609–628 Zbl 1303.46049 MR 3210717
- [18] I. Farah, B. Hart, and D. Sherman, Model theory of operator algebras III: elementary equivalence and II_1 factors. *Bull. Lond. Math. Soc.* **46** (2014), no. 3, 609–628 Zbl 1303.46049 MR 3210717
- [19] L. Ge, Applications of free entropy to finite von Neumann algebras. II. *Ann. of Math. (2)* **147** (1998), no. 1, 143–157 Zbl 0924.46050 MR 1609522
- [20] I. Goldbring and B. Hart, On the theories of McDuff’s II_1 factors. *Int. Math. Res. Not. IMRN* **2017** (2017), no. 18, 5609–5628 Zbl 1404.03038 MR 3704741
- [21] I. Goldbring, B. Hart, and T. Sinclair, The theory of tracial von Neumann algebras does not have a model companion. *J. Symbolic Logic* **78** (2013), no. 3, 1000–1004 Zbl 1316.03019 MR 3135510
- [22] I. Goldbring, D. Jekel, S. Kunnawalkam Elayavalli, and J. Pi, Uniformly Super McDuff II_1 Factors. 2023, arXiv:2303.02809
- [23] V. Guirardel, G. Levitt, and R. Sklinos, Towers and the first-order theory of hyperbolic groups. 2020, arXiv:2007.14148
- [24] B. Hayes, 1-bounded entropy and regularity problems in von Neumann algebras. *Int. Math. Res. Not. IMRN* **2018** (2018), no. 1, 57–137 Zbl 1415.46039 MR 3801429
- [25] B. Hayes, A random matrix approach to the Peterson–Thom conjecture. *Indiana Univ. Math. J.* **71** (2022), no. 3, 1243–1297 Zbl 1503.46049 MR 4448584

- [26] B. Hayes, D. Jekel, and S. Kunnawalkam Elayavalli, Property (T) and strong 1-boundedness for von Neumann algebras. 2021, arXiv:2107.03278
- [27] B. Hayes, D. Jekel, and S. Kunnawalkam Elayavalli, Vanishing first cohomology and strong 1-boundedness for von Neumann algebras. 2021, arXiv:2110.12324
- [28] B. Hayes, D. Jekel, B. Nelson, and T. Sinclair, [A random matrix approach to absorption in free products](#). *Int. Math. Res. Not. IMRN* **2021** (2021), 1919–1979 Zbl 1481.46059 MR 4206601
- [29] G. Higman and E. Scott, *Existentially closed groups*. London Math. Soc. Monogr. (N.S.) 3, Oxford University Press, New York, 1988 Zbl 0646.20001 MR 960689
- [30] D. Jekel, [Covering entropy for types in tracial \$W^*\$ -algebras](#). *J. Log. Anal.* **15** (2023), 2–69 MR 4600403
- [31] K. Jung, [Strongly 1-bounded von Neumann algebras](#). *Geom. Funct. Anal.* **17** (2007), no. 4, 1180–1200 Zbl 1146.46034 MR 2373014
- [32] K. Jung, The rank theorem and L^2 -invariants in free entropy: Global upper bounds. 2016, arXiv:1602.04726
- [33] H. J. Keisler, [Ultraproducts and saturated models](#). *Nederl. Akad. Wetensch. Proc. Ser. A* **67. Indag. Math.** **26** (1964), 178–186 Zbl 0199.01101 MR 0168483
- [34] D. McDuff, [Uncountably many \$II_1\$ factors](#). *Ann. of Math. (2)* **90** (1969), 372–377 Zbl 0184.16902 MR 259625
- [35] D. McDuff, [Central sequences and the hyperfinite factor](#). *Proc. London Math. Soc. (3)* **21** (1970), 443–461 Zbl 0204.14902 MR 281018
- [36] N. Ozawa, [Solid von Neumann algebras](#). *Acta Math.* **192** (2004), no. 1, 111–117 Zbl 1072.46040 MR 2079600
- [37] N. Ozawa and S. Popa, [On a class of \$II_1\$ factors with at most one Cartan subalgebra](#). *Ann. of Math. (2)* **172** (2010), no. 1, 713–749 Zbl 1201.46054 MR 2680430
- [38] C. Perin, *Elementary embeddings in torsion-free hyperbolic groups*. Ph.D. thesis, Université de Caen Normandie, 2008
- [39] C. Perin, [Elementary embeddings in torsion-free hyperbolic groups](#). *Ann. Sci. Éc. Norm. Supér. (4)* **44** (2011), no. 4, 631–681 Zbl 1245.20052 MR 2919979
- [40] J. Peterson, Open problems in operator algebras. <https://math.vanderbilt.edu/peters10/problems.html>
- [41] S. Popa, Orthogonal pairs of $*$ -subalgebras in finite von Neumann algebras. *J. Operator Theory* **9** (1983), no. 2, 253–268 Zbl 0521.46048 MR 703810
- [42] S. Popa and S. Vaes, [Unique Cartan decomposition for \$II_1\$ factors arising from arbitrary actions of hyperbolic groups](#). *J. Reine Angew. Math.* **694** (2014), 215–239 Zbl 1307.46047 MR 3259044
- [43] Z. Sela, [Diophantine geometry over groups. I. Makanin–Razborov diagrams](#). *Publ. Math. Inst. Hautes Études Sci.* (2001), no. 93, 31–105 Zbl 1018.20034 MR 1863735
- [44] S. Shelah, [Every two elementarily equivalent models have isomorphic ultrapowers](#). *Israel J. Math.* **10** (1971), 224–233 Zbl 0224.02045 MR 297554
- [45] D. Shlyakhtenko, [Von Neumann algebras of sofic groups with \$\beta_1^{\(2\)} = 0\$ are strongly 1-bounded](#). *J. Operator Theory* **85** (2021), no. 1, 217–228 Zbl 07606458 MR 4198970
- [46] D. Voiculescu, [The analogues of entropy and of Fisher’s information measure in free probability theory. II](#). *Invent. Math.* **118** (1994), no. 3, 411–440 Zbl 0820.60001 MR 1296352
- [47] D. Voiculescu, [The analogues of entropy and of Fisher’s information measure in free probability theory. III. The absence of Cartan subalgebras](#). *Geom. Funct. Anal.* **6** (1996), no. 1, 172–199 Zbl 0856.60012 MR 1371236

Communicated by Roland Speicher

Received 24 October 2022; revised 16 March 2023.

Srivatsav Kunnawalkam Elayavalli

Institute of Pure and Applied Mathematics, UCLA, 460 Portola Plaza, Los Angeles, CA 90095,
USA; srivatsav.kunnawalkam.elayavalli@vanderbilt.edu