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(Non)local logistic equations with Neumann conditions

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Abstract. We consider here a problem of population dynamics modeled on a logistic equation with both classical and nonlocal diffusion, possibly in combination with a pollination term. The environment considered is a niche with zero-flux, according to a new type of Neumann condition. We discuss situations that are more favorable for the survival of the species, in terms of the first positive eigenvalue. Quite surprisingly, the eigenvalue analysis for the one-dimensional case is structurally different from the higher-dimensional setting, and it sensibly depends on the nonlocal character of the dispersal.

The mathematical framework of this problem takes into consideration the equation $-\alpha\Delta u + \beta(-\Delta)^s u = (m - \mu u)u + \tau J \star u$ in Ω, where m can change sign. This equation is endowed with a set of Neumann conditions that combines the classical normal derivative prescription and the nonlocal condition introduced in Dipierro, Ros-Oton, and Valdinoci [Rev. Mat. Iberoam. 33 (2017), 377–416]. We will establish the existence of a minimal solution for this problem and provide a thorough discussion on whether it is possible to obtain nontrivial solutions (corresponding to the survival of the population).

The investigation will rely on a quantitative analysis of the first eigenvalue of the associated problem and on precise asymptotics for large lower and upper bounds of the resource. In this, we also analyze the role played by the optimization strategy in the distribution of the resources, showing concrete examples that are unfavorable for survival, in spite of the large resources that are available in the environment.

Contents

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1. Introduction

We consider here a biological population with density u which is self-competing for the resources in a given environment Ω.

These resources are described by a function m, which is allowed to change sign: the positive values of m correspond to areas of the environment favorable for life and produce a positive birth rate, whereas the negative values model a hostile environment whose by-product is a positive death rate of linear type.

The competition for the resource is encoded by a nonnegative function μ. Resources and competitions are combined into a standard logistic equation. In addition, the population is assumed to present a combination of classical and nonlocal diffusion (the cases of purely classical and purely nonlocal diffusions are also included in our setting, and the results obtained are new also for these cases). The population is also endowed with an additional birth rate possibly provided by pollination[1] and modeled by a convolution operator (the case of no pollination is also included in our setting, and the results obtained are new also for this case).

The environment Ω describes an ecological niche and is endowed by a zero-flux condition of Neumann type. Given the possible presence of both classical and nonlocal dispersal, this Neumann condition appears to be new in the literature: when the diffusion is of purely classical type this new prescription reduces to the standard normal derivative condition along $\partial\Omega$, and when the diffusion is of purely nonlocal type it coincides with the nonlocal Neumann condition set in $\mathbb{R}^n \setminus \overline{\Omega}$ that has been recently introduced in [43] – but in the case that the population is subject to both the classical and the nonlocal dispersion processes, the Neumann condition that we introduce here takes into account the combination of both the classical and the nonlocal prescriptions (interestingly, without producing an overdetermined, or ill-posed, problem).

The main question addressed in this paper is whether or not the environmental niche is suited for the survival of the population (notice that life is not always promoted by the ambient resource, since m can attain negative values). We will investigate this question by using spectral analysis and providing a detailed quantification of favorable and unfavorable scenarios in terms of the first eigenvalue compared with the resource and pollination parameters.

More precisely, the mathematical framework in which we work goes as follows. We consider a bounded open set $\Omega \subset \mathbb{R}^n$ with boundary of class C^1: that is, we suppose that there exist $R > 0$ and $p_1, \dots, p_K \in \partial\Omega$ such that $\partial\Omega \subset B_R(p_1) \cup \dots \cup B_R(p_K)$, and, for each $i \in \{1, \dots, K\}$,

$$\text{the set } \Omega \cap B_R(p_i) \text{ is } C^1\text{-diffeomorphic to } B_1^+ := \{(x_1, \dots, x_n) \in B_1 \text{ s.t. } x_n > 0\}. \quad (1.1)$$

[1]While we use the name of pollination throughout this paper, we observe that the pollination analysis performed is not limited to vegetable species: indeed, for animal species the convolution term that we study can be seen as a birth rate of nonlocal type produced, for instance, by a mating call that attracts partners from surrounding neighbors.

Given $s \in (0,1)$, $\alpha, \beta \in [0,+\infty)$, with $\alpha+\beta>0$, $m\colon \Omega \to \mathbb{R}$, $\mu\colon \Omega \to [\underline{\mu},+\infty)$, with $\underline{\mu}>0$, $\tau \in [0,+\infty)$ and $J \in L^1(\mathbb{R}^n,[0,+\infty))$ with

$$J(x) = J(-x) \tag{1.2}$$

and

$$\int_{\mathbb{R}^n} J(x)\,dx = 1, \tag{1.3}$$

we consider the mixed-order logistic equation

$$-\alpha\Delta u + \beta(-\Delta)^s u = (m-\mu u)u + \tau J \star u \quad \text{in } \Omega, \tag{1.4}$$

where

$$J \star u(x) := \int_\Omega J(x-y)u(y)dy.$$

When $\beta = 0$, we take the additional hypothesis that

$$\Omega \text{ is connected.} \tag{1.5}$$

We observe that the operator in (1.4) is of mixed local and nonlocal type, and also of mixed fractional- and integer-order type. Interestingly, the nonlocal character of the operator is encoded both in the fractional Laplacian

$$(-\Delta)^s u(x) := \frac{1}{2}\int_{\mathbb{R}^n} \frac{2u(x)-u(x+\zeta)-u(x-\zeta)}{|\zeta|^{n+2s}}\,d\zeta$$

and in the convolution operator given by J.

The use of the convolution operator in biological models to comprise the interaction of the population with the resource at a certain range has a very consolidated tradition; see e.g. [4,6,13,19,23,35,36] and the references therein.

As for the nonlocal diffusive operator, for the sake of concreteness we stick here to the prototypical case of the fractional Laplacian, but the arguments that we develop are in fact usable in more general contexts, including various interaction kernels of singular type.

Given the presence of both the Laplacian and the fractional Laplacian, the operator in (1.4) falls within the diffusive processes of mixed orders, which have been widely addressed by several methodologies and arose from a number of different motivations; see for instance various viscosity solution approaches [5,7–9,17,33,38,48,49], the Aubry–Mather theory for pseudo differential equations [37], Cahn–Hilliard and Allen–Cahn-type equations [22,25], probability and Harnack inequalities [10,11,31,32], decay for parabolic equations [3,45], friction and dissipation effects [39], smooth approximation with suitable solutions [24], Bernstein-type regularity results [21], variational methods [16], nonlinear operators [1] and plasma physics [18].

We endow the problem in (1.4) with a set of Neumann boundary conditions that correspond to a “zero-flux” condition according to the stochastic process producing the diffusive operator in (1.4). This Neumann condition appears to be new in the literature and depends on the different ranges of α and β according to the following setting. If $\alpha = 0$, we consider the nonlocal Neumann condition introduced in [43], thus prescribing that

$$\mathcal{N}_s u(x) := \int_\Omega \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy = 0 \quad \text{for every } x \in \mathbb{R}^n \setminus \bar{\Omega}. \tag{1.6}$$

If instead $\beta = 0$, we prescribe the classical Neumann condition

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{1.7}$$

Finally, if $\alpha \neq 0$ and $\beta \neq 0$, we prescribe both the classical and the nonlocal Neumann conditions, by requiring that

$$\begin{cases} \mathcal{N}_s u = 0 & \text{in } \mathbb{R}^n \setminus \bar{\Omega}, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

We remark that the prescription in (1.8) is *not* an "overdetermined" condition (as will be confirmed by the existence result in Theorem 1.1 below).

The set of boundary/external Neumann conditions in (1.6), (1.7) and (1.8), in dependence of the different ranges of α and β, will be denoted by "(α, β)-Neumann conditions", and, with this notation and (1.4), the main question studied in this paper focuses on the problem

$$\begin{cases} -\alpha\Delta u + \beta(-\Delta)^s u = (m - \mu u)u + \tau J \star u & \text{in } \Omega, \\ \text{with } (\alpha, \beta)\text{-Neumann condition.} \end{cases} \tag{1.9}$$

In this setting, the (α, β)-Neumann conditions provide an "ecological niche" for the population with density u, making Ω a natural environment in which a given species can live and compete for a resource m, according to a competition function μ. In this setting, the parameter τ, as modulated by the interaction kernel J, describes an additional birth rate due to further intercommunication than just with the closest neighbors, as happens, for instance, in pollination.

As a matter of fact, the role of the (α, β)-Neumann conditions is precisely to make the boundary and the exterior of the niche Ω "reflective": namely, when an individual exits the niche, it is forced to immediately come back into the niche itself, following the same diffusive process; see [43, Section 2] (see also [70] for a thorough probabilistic discussion of this process).

As a technical remark, we also observe that our (α, β)-Neumann condition is structurally different (even when $\alpha = 0$ and $s = 1/2$) from the case of bounded domains with reflecting barriers presented in [63, 65], and the diffusive operator taken into account in (1.9) cannot be obtained by the spectral decomposition of the classical Laplacian in Ω (except for the special case of periodic environments; see e.g. [2, Section 2.3 and Appendix Q]).

The possible presence in (1.9) of two different diffusion operators, one of classical and the other of fractional flavor, has a clear biological interpretation, namely the population with density u can possibly alternate both short- and long-range random walks, and this could be motivated, for instance, by a superposition between local exploration of the environment and hunting strategies (see e.g. [28–30, 34, 46, 50, 58, 67, 69]). A detailed presentation of this superposition of stochastic processes will be presented in Appendix C; see also [44] for a detailed description of the local/nonlocal reflecting barrier also in terms of the population dynamics model.

The notion of a solution of (1.9) is intended here in the weak sense, as will be precisely discussed in formula (2.5). See however [16, 47] for a regularity theory for weak solutions of the equations driven by the mixed-order operators as in (1.9).

Our first result in this setting is that the problem in (1.9) admits a minimal energy solution (under very natural and mild structural assumptions). To state it, it is convenient to define

$$\begin{aligned} \underline{q} &:= \begin{cases} \dfrac{2^*}{2^*-2} & \text{if } \beta = 0 \text{ and } n > 2,\\ \dfrac{2^*_s}{2^*_s-2} & \text{if } \beta \neq 0 \text{ and } n > 2s,\\ 1 & \text{if } \beta = 0 \text{ and } n \leqslant 2, \text{ or if } \beta \neq 0 \text{ and } n \leqslant 2s, \end{cases} \\ &= \begin{cases} \dfrac{n}{2} & \text{if } \beta = 0 \text{ and } n > 2,\\ \dfrac{n}{2s} & \text{if } \beta \neq 0 \text{ and } n > 2s,\\ 1 & \text{if } \beta = 0 \text{ and } n \leqslant 2, \text{ or if } \beta \neq 0 \text{ and } n \leqslant 2s. \end{cases} \end{aligned} \tag{1.10}$$

As is customary, the exponent 2^*_s denotes the fractional Sobolev critical exponent for $n > 2s$ and it is equal to $\frac{2n}{n-2s}$. Similarly, the exponent 2^* denotes the classical Sobolev critical exponent for $n > 2$ and it is equal to $\frac{2n}{n-2}$.

We remark that $\underline{q} \geqslant n/2$, and we have the following theorem:

Theorem 1.1. *Assume that*

$$m \in L^q(\Omega) \text{ for some } q \in (\underline{q}, +\infty] \text{ and } (m+\tau)^3 \mu^{-2} \in L^1(\Omega).$$

Then there exists a nonnegative solution of (1.9) *which can be obtained as a minimum of an energy functional.*

The precise definition of energy functional used in Theorem 1.1 will be presented in (3.1): roughly speaking, the energy associated to Theorem 1.1 will be the "natural" functional for the variational methods, and its Euler–Lagrange equation will correspond to the notion of a weak solution.

While the functional analysis part of the proof of Theorem 1.1 relies on standard direct methods in the calculus of variations, the more interesting part of the argument makes use of a structural property of the nonlocal Neumann condition that will be presented in Theorem 2.1 (roughly speaking, the nonlocal Neumann condition in (1.6) will be instrumental in minimizing the Gagliardo seminorm, thus clarifying the energetic role of the nonlocal reflection introduced in [43]).

Though the result in Theorem 1.1 has an obvious interest in pure mathematics, our main analysis will focus on whether problem (1.9) does admit a *nontrivial* solution (notice indeed that $u \equiv 0$ is always a solution of (1.9)). In particular, in view of Theorem 1.1, a useful mathematical tool to detect nontrivial solutions consists in proving that the minimal energy configuration is not attained by the trivial solution (hence, in this case, the solution produced by Theorem 1.1 is nontrivial). The question of the existence of nontrivial solutions has a central importance for the mathematical model, since it corresponds to the possibility of a population to survive in the environmental condition provided by the niche. Interestingly, in our model, the survival of the population can be enhanced by the possibility of exploiting resources by long-range interactions. Indeed, we stress that the nonlocal resource m in (1.4) is not necessarily positive (hence, the natural environment can be "hostile" for the population): in this configuration, we show that the survival of the species is still possible if the "pollination" birth rate τ is sufficiently large. The quantitative result that we have is the following:

Theorem 1.2. *Assume that*

$$m \in L^q(\Omega) \text{ for some } q \in (\underline{q}, +\infty] \text{ and } (m+\tau)^3\mu^{-2} \in L^1(\Omega).$$

Then,

(i) *if m is nonpositive and $\tau = 0$, the only solution of* (1.9) *is the one identically zero;*

(ii) *if*

$$\int_\Omega (m(x) + \tau J \star 1(x))\, dx > 0 \tag{1.11}$$

and

$$\mu \in L^1(\Omega), \tag{1.12}$$

problem (1.9) *admits a nonnegative solution $u \not\equiv 0$.*

A particular case of Theorem 1.2 is when the resource m is nonnegative. In this situation, Theorem 1.2(i) gives that no survival is possible without resources and pollination, i.e. when both m and τ vanish identically (unless also μ vanishes identically, then reducing the problem to that of mixed operator harmonic functions), whereas Theorem 1.2(ii) guarantees survival if at least one between the environmental resource and the pollination is favorable to life. Precisely, one can immediately deduce from Theorem 1.2 the following result:

Corollary 1.3. *Assume that*

$$m \in L^q(\Omega) \text{ for some } q \in (\underline{q}, +\infty],\ m \text{ is nonnegative and } (m+\tau)^3\mu^{-2} \in L^1(\Omega).$$

Then

(i) *if* $m \equiv 0$ *and* $\tau = 0$, *the only solution of* (1.9) *is the one identically zero;*

(ii) *if either* $m > 0$ *or* $\tau(J \star 1) > 0$ *in a set of positive measure and* $\mu \in L^1(\Omega)$, (1.9) *admits a nonnegative solution* $u \not\equiv 0$.

Problems related to Corollary 1.3 have been studied in [23] under Dirichlet (rather than Neumann) boundary conditions.

From the biological point of view, assumption (1.11) states that the environment is "on average" favorable for the survival of the species. It is therefore natural to investigate the situation in which the environment is "mostly hostile to life". To study this phenomenon, when $m \in L^q$ with $q > n/2$, with $m^+ \not\equiv 0$ and

$$\int_\Omega m(x)\,dx < 0,$$

we denote[2] by λ_1 the first positive eigenvalue associated with the diffusive operator in (1.9). More precisely, we consider the weighted eigenvalue problem

$$\begin{cases} -\alpha\Delta u + \beta(-\Delta)^s u = \lambda m u & \text{in } \Omega, \\ \text{with } (\alpha,\beta)\text{-Neumann condition.} \end{cases} \tag{1.13}$$

As will be discussed in detail in Proposition 4.1 here and in [42], problem (1.13) admits the existence of two unbounded sequences of eigenvalues, one positive and one negative. In this setting, the smallest strictly positive eigenvalue will be denoted by λ_1. When we want to emphasize the dependence of λ_1 on the resource m, we will write it as $\lambda_1(m)$.

We also denote by e an eigenfunction corresponding to λ_1 normalized such that

$$\int_\Omega m(x)e^2(x)\,dx = 1.$$

The first eigenvalue will be an important threshold for the survival of the species, quantifying the role of the necessary pollination parameter τ in order to overcome the presence of hostile behavior on average. The precise result that we obtain is the following one:

Theorem 1.4. *Assume that* $m \in L^q(\Omega)$, *for some* $q \in (\underline{q}, +\infty]$, *and* $(m+\tau)^3\mu^{-2} \in L^1(\Omega)$.

Then,

(i) *if* $m \leqslant -\tau$, *the only solution of* (1.9) *is the one identically zero;*

[2] As is customary, in this paper we freely use the standard notation

$$m^+(x) := \max\{0, m(x)\} \quad \text{and} \quad m^-(x) := \max\{0, -m(x)\}.$$

(ii) *if* $m^+ \not\equiv 0$, $\mu \in L^1(\Omega)$,

$$\int_\Omega m(x)\,dx < 0, \tag{1.14}$$

and

$$\lambda_1 - 1 < \tau \int_\Omega (J \star e(x))e(x)\,dx, \tag{1.15}$$

(1.9) *admits a nonnegative solution* $u \not\equiv 0$.

Once again, in Theorem 1.4, the case described in (i) is the one less favorable to life, since the combination of both the resources and the pollination is on average negative, while the case in (ii) gives a lower bound of the pollination parameter τ which is needed for the survival of the species, as quantified by (1.15).

We recall that the link between the survival ability of a biological population and the analysis of the eigenvalues of a linearized problem is a classical topic in mathematical biology; see e.g. [12, 14, 15, 51, 52, 59, 60, 63, 66] (yet we believe that this is the first place in which a detailed analysis of this type is carried over to the case of mixed operators with our new type of Neumann conditions).

We also remark that condition (1.15) can be sharpened by considering, instead of (1.13), an eigenvalue problem also containing the convolution term. This observation will be expanded in Remark 4.3.

In light of (1.15), a natural question consists in quantifying the size of the first eigenvalue. Roughly speaking, from (1.15), the smaller λ_1, the smaller the threshold for the pollination guaranteeing survival, hence configurations with small first eigenvalues correspond to the ones with better chances of life.

To address this problem, since the eigenvalue $\lambda_1 = \lambda_1(m)$ depends on the resource m, it is convenient to consider an optimization problem for λ_1 in terms of three structural parameters of the resource m, namely its minimum, its maximum and its average, in order to detect under which conditions on these parameters the first eigenvalue can be made conveniently small. More precisely, given $\overline{m}, \underline{m} \in (0, +\infty)$ and $m_0 \in (-\underline{m}, 0)$ we consider the class of resources

$$\mathcal{M} = \mathcal{M}(\overline{m}, \underline{m}, m_0) := \Big\{m \in L^\infty(\Omega) \text{ s.t. } \inf_\Omega m \geqslant -\underline{m},\ \sup_\Omega m \leqslant \overline{m}, \\ \textstyle\int_\Omega m(x)\,dx = m_0|\Omega| \text{ and } m^+ \not\equiv 0\Big\}. \tag{1.16}$$

We will also consider the smallest possible first eigenvalue among all the resources in $\mathcal{M}$, namely we set

$$\underline{\lambda} := \inf_{m\in\mathcal{M}} \lambda_1(m). \tag{1.17}$$

When we want to emphasize the dependence of $\underline{\lambda}$ on the structural quantities $\overline{m}$, $\underline{m}$ and m_0 that characterize $\mathcal{M}$, we will adopt the explicit notation $\underline{\lambda}(\overline{m}, \underline{m}, m_0)$.

Our main objective will be to detect whether or not $\underline{\lambda}$ can be made arbitrarily small in a number of different regimes: we stress that the smallness of $\underline{\lambda}$ corresponds to a choice of an optimal distribution of resources that is particularly favorable for survival.

The first result that we present in this direction is a general estimate controlling $\underline{\lambda}$ with $O(\frac{1}{\overline{m}})$, provided that the maximal hostility of the environment does not prevail with respect to the maximal and average resources. In terms of survival of the species, this is a rather encouraging outcome, since it allows the existence of nontrivial solutions provided that the maximal resource is sufficiently large. The precise result that we have is the following:

Theorem 1.5. *Let*

$$\frac{\underline{m} + m_0}{\underline{m} + \overline{m}} \geqslant d_0 \tag{1.18}$$

for some $d_0 > 0$*. Then*

$$\underline{\lambda}(\overline{m}, \underline{m}, m_0) \leqslant \frac{C}{\overline{m}}$$

for some $C = C(\Omega, d_0) > 0$*.*

A direct consequence of Theorem 1.5 is that when the upper and lower bounds of the resource are the same and get arbitrarily large, then $\underline{\lambda}$ gets arbitrarily small (hence, in view of (1.15)), there exists a resource distribution which is favorable to survival). More[3] precisely, we have the following corollary:

Corollary 1.6. *We have*

$$\lim_{\mathpzc{m} \nearrow +\infty} \underline{\lambda}(\mathpzc{m}, \mathpzc{m}, m_0) = 0.$$

We now investigate the behavior of $\underline{\lambda}$ for large upper and lower bounds on the resource (maintaining the other parameters constant). Interestingly, this behavior sensibly depends on the dimension n. In this setting, we first consider the asymptotics in dimension $n \geqslant 2$: we show that large upper and lower bounds are both favorable for life for a given $m_0 < 0$, according to the following two results:

Theorem 1.7. *Let* $n \geqslant 2$*. Then*

$$\lim_{\mathpzc{m} \nearrow +\infty} \underline{\lambda}(\mathpzc{m}, \underline{m}, m_0) = 0.$$

Theorem 1.8. *Let* $n \geqslant 2$*. Then*

$$\lim_{\mathpzc{m} \nearrow +\infty} \underline{\lambda}(\overline{m}, \mathpzc{m}, m_0) = 0.$$

While Theorem 1.7 is somehow intuitive (large resources are favorable to survival), at first glance Theorem 1.8 may look unintuitive, since it seems to suggest that a largely hostile environment is also favorable to survival: but we remark that in Theorem 1.8, m_0 being given, an optimal strategy for m may well correspond to a very harmful environment

[3] To avoid notational confusion, we reserve the name m for the resource in (1.4) and we denote by $\mathpzc{m}$ a "free variable" dimensionally related to the resource.

confined to a small portion of the domain, with a positive resource allowing for the survival of the species.

Quite surprisingly, the structural analysis developed in Theorems 1.7 and 1.8 is significantly different in dimension 1. Indeed, for $n = 1$, we have that $\underline{\lambda}$ does not become infinitesimal for large upper and lower bounds on the resource, unless the diffusion is purely nonlocal with strongly nonlocal fractional parameter. Namely, we have the following two results.

Theorem 1.9. *Let $n = 1$, $\alpha > 0$ and $\beta \geqslant 0$. Then, for any $\underline{m} > 0$ and $m_0 \in (-\underline{m}, 0)$,*

$$\underline{\lambda}(\mathscr{m}, \underline{m}, m_0) \geqslant C \tag{1.19}$$

for every $\mathscr{m} > 0$, for some $C = C(\underline{m}, m_0, \alpha, \beta, \Omega) > 0$, and

$$\lim_{\mathscr{m} \searrow 0} \underline{\lambda}(\mathscr{m}, \underline{m}, m_0) = +\infty. \tag{1.20}$$

Moreover, for any $\overline{m} > 0$ and $m_0 < 0$,

$$\underline{\lambda}(\overline{m}, \mathscr{m}, m_0) \geqslant C \tag{1.21}$$

for every $\mathscr{m} > -m_0$, for some $C = C(\overline{m}, m_0, \alpha, \beta, \Omega) > 0$.

Theorem 1.10. *Let $n = 1$, $\alpha = 0$ and $\beta > 0$.*

If $s \in (1/2, 1)$, then, for any $\underline{m} > 0$ and $m_0 \in (-\underline{m}, 0)$

$$\underline{\lambda}(\mathscr{m}, \underline{m}, m_0) \geqslant C \tag{1.22}$$

for every $\mathscr{m} > 0$, for some $C = C(\underline{m}, m_0, \alpha, \beta, \Omega) > 0$, and

$$\lim_{\mathscr{m} \searrow 0} \underline{\lambda}(\mathscr{m}, \underline{m}, m_0) = +\infty. \tag{1.23}$$

Moreover, for any $\overline{m} > 0$ and $m_0 < 0$,

$$\underline{\lambda}(\overline{m}, \mathscr{m}, m_0) \geqslant C \tag{1.24}$$

for every $\mathscr{m} > -m_0$, for some $C = C(\overline{m}, m_0, \alpha, \beta, \Omega) > 0$.

If $s \in (0, 1/2]$, then

$$\lim_{\mathscr{m} \nearrow +\infty} \underline{\lambda}(\mathscr{m}, \underline{m}, m_0) = 0 \tag{1.25}$$

and

$$\lim_{\mathscr{m} \nearrow +\infty} \underline{\lambda}(\overline{m}, \mathscr{m}, m_0) = 0. \tag{1.26}$$

An interesting feature of Corollary 1.6, Theorems 1.7 and 1.8, (1.25) and (1.26) in terms of real-world applications is that their proofs are based on the explicit constructions of suitable resources: though perhaps not optimal, these resources are sufficiently well located to ensure the maximal chances of survival for the population, and their explicit representation allows one to use them concretely and to build on this specific knowledge.

We also think that the phenomenon detected in Theorems 1.9 and 1.10 reveals an important role played by the nonlocal dispersal of the species in dimension 1: indeed, in this situation, the only configurations favorable to survival are those in (1.25) and (1.26), that are induced by purely nonlocal diffusion (that is, $\alpha = 0$) with a strongly nonlocal diffusion exponent (that is, $s \leqslant 1/2$, corresponding to very long flies in the underlying stochastic process).

To better visualize the results in Theorems 1.7, 1.8, 1.9 and 1.10, we summarize them in Table 1. For typographical convenience, in Table 1 we used the check symbol ✓ to denote the cases in which $\underline{\lambda}$ gets as small as we wish (cases favorable to life) and the cross symbol ✗ to mark the situations in which $\underline{\lambda}$ remains bounded away from zero (cases unfavorable to life which require stronger pollination for survival).

	Large $\overline{m}$	Large $\underline{m}$
$n \geqslant 2$	✓	✓
$n = 1$ and $\alpha > 0$	✗	✗
$n = 1, \alpha = 0$ and $s > 1/2$	✗	✗
$n = 1, \alpha = 0$ and $s \leqslant 1/2$	✓	✓

Table 1. Summarizing the results in Theorems 1.7, 1.8, 1.9 and 1.10.

We stress that the optimization of the resources plays a crucial role in the survival results provided by Corollary 1.6, Theorems 1.7 and 1.8, and formulas (1.25) and (1.26): that is, given $m_0 < 0$, very large but *badly displayed* resources may lead to nonnegligible first eigenvalues (different from the case of optimal distribution of resources discussed in Corollary 1.6, Theorems 1.7 and 1.8, and formulas (1.25) and (1.26)).

To state this phenomenon precisely, given $m_0 < 0$ and $\Lambda > -4m_0$, we let

$$\mathcal{M}^{\sharp}_{\Lambda,m_0} := \left\{m \in \mathcal{M}(2\Lambda, 2\Lambda, m_0) \text{ s.t. } \inf_\Omega m \leqslant -\tfrac{\Lambda}{2} \text{ and } \sup_\Omega m \geqslant \tfrac{\Lambda}{2}\right\}. \tag{1.27}$$

Roughly speaking, the resources m in $\mathcal{M}^{\sharp}_{\Lambda,m_0}$ have a prescribed average equal to m_0 and attain maximal positive and negative values comparable with a large parameter Λ, and a natural question in this case is whether large Λ's provide sufficient conditions for the survival of the species. The next result shows that this is not the case, namely the abundance of the resource without an optimal distribution strategy is not sufficient for prosperity:

Theorem 1.11. *Given $m_0 < 0$ and $\Lambda > -4m_0$, we have*

$$\sup_{m \in \mathcal{M}^{\sharp}_{\Lambda,m_0}} \lambda_1(m) = +\infty.$$

Interestingly, the proof of Theorem 1.11 will be "constructive", namely we will provide an explicit example of a sequence of badly displayed resources which make the first eigenvalue diverge: a telling feature of this sequence is that it is highly oscillatory, thus suggesting that a hectic and erratic alternation of highly positive resources with very harmful surroundings is potentially lethal for the development of the species.

We recall that the investigation of the roles of fragmentation and concentration for resources is a classical topic in mathematical biology, and, in this sense, our result in Theorem 1.11 confirms the main paradigm according to which concentrated resources favor survival (see e.g. [14, 15, 54]) – however, there are several circumstances in which this general paradigm is violated and fragmentation is better than concentration; see e.g. the small diffusivity regime analyzed in [53,56,62]. In any case, the analysis of fragmentation and concentration for mixed operators with our Neumann condition is, to the best of our knowledge, completely new.

We also remark that the results presented here are new even in the simpler cases in which no classical diffusion and no pollination term is present in (1.4), as well as in the cases in which the death rate and the pollination functions are constant.

The rest of this paper is organized as follows. In Section 2 we will introduce the functional framework in which we work and the notion of weak solutions, also providing a new result showing that the nonlocal Neumann condition naturally produces functions with minimal Gagliardo seminorm (this is a nonlocal phenomenon, which has no counterpart in the classical setting, and will play a pivotal role in the minimization process).

Then, in Section 3, we prove the existence results in Theorems 1.1 and 1.2. In Section 4 we study the eigenvalue problem in (1.13), and we give the proof of Theorem 1.4. Not to overburden this paper, some technical proofs related to the spectral theory of the problem are deferred to the article [42].

In Section 5, we deal with the proofs of Theorem 1.5, Theorems 1.7 and 1.8 when $n \geqslant 3$, and Theorems 1.9 and 1.10.

When $n = 2$, the proofs of Theorems 1.7 and 1.8 require some technical modification of logarithmic type, hence their proofs are deferred to Appendix A.

The proof of Theorem 1.11 is contained in Section 6.

An alternative proof of some technical lemmata is provided in Appendix B. Finally, Appendix C contains some probabilistic motivations related to the diffusive operators of mixed integer and fractional order.

2. Functional analysis setting

In this section we define the functional space in which we work. First, we recall the space H^s_Ω introduced in [43] and defined as

$$H^s_\Omega := \Big\{u\colon \mathbb{R}^n \to \mathbb{R} \text{ s.t. } u \in L^2(\Omega) \text{ and } \iint_{\mathcal{Q}} \tfrac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy < +\infty\Big\}, \tag{2.1}$$

where

$$\mathcal{Q} := \mathbb{R}^{2n} \setminus (\mathbb{R}^n \setminus \Omega)^2.$$

As is customary, by $u \in L^2(\Omega)$ in (2.1) we mean that the restriction of the function u to Ω belongs to $L^2(\Omega)$ (we stress that functions in H^s_Ω are defined in the whole of $\mathbb{R}^n$). Also, all functions considered will be implicitly assumed to be measurable.

Furthermore, we define

$$X_{\alpha,\beta} = X_{\alpha,\beta}(\Omega) := \begin{cases} H^1(\Omega) & \text{if } \beta = 0, \\ H^s_\Omega & \text{if } \alpha = 0, \\ H^1(\Omega) \cap H^s_\Omega & \text{if } \alpha\beta \neq 0. \end{cases} \tag{2.2}$$

In light of this definition, $X_{\alpha,\beta}$ is a Hilbert space with respect to the scalar product

$$\begin{aligned}(u, v)_{X_{\alpha,\beta}} &:= \int_\Omega u(x)v(x)\,dx + \alpha \int_\Omega \nabla u(x) \cdot \nabla v(x)\,dx \\ &\quad + \frac{\beta}{2} \iint_{\mathcal{Q}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}}\,dx\,dy \end{aligned} \tag{2.3}$$

for every $u, v \in X_{\alpha,\beta}$.

We also define the seminorm

$$[u]^2_{X_{\alpha,\beta}} := \frac{\alpha}{2} \int_\Omega |\nabla u(x)|^2\,dx + \frac{\beta}{4} \iint_{\mathcal{Q}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}}\,dx\,dy. \tag{2.4}$$

From the compact embeddings of the spaces $H^1(\Omega)$ and H^s_Ω (see e.g. [41, Corollary 7.2] when $\alpha = 0$), we deduce the compact embedding of $X_{\alpha,\beta}$ into $L^p(\Omega)$, for every $p \in [1, 2^*)$ if $\alpha \neq 0$, and for every $p \in [1, 2^*_s)$ if $\alpha = 0$.

We say that $u \in X_{\alpha,\beta}$ is a solution of (1.9) if

$$\begin{aligned}&\alpha \int_\Omega \nabla u(x) \cdot \nabla v(x)\,dx + \frac{\beta}{2} \iint_{\mathcal{Q}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}}\,dx\,dy \\ &\quad = \int_\Omega \big(\big(m(x) - \mu(x)u(x)\big)u(x) + \tau(x) J \star u(x)\big) v(x)\,dx \end{aligned} \tag{2.5}$$

for all functions $v \in X_{\alpha,\beta}$.

Now we show that *among all the functions in H^s_Ω, the ones minimizing the Gagliardo seminorm are those satisfying the nonlocal Neumann condition in* (1.6). This is a useful result in itself, which also clarifies the structural role of the Neumann condition introduced in [43]:

Theorem 2.1. *Let $u\colon \mathbb{R}^n \to \mathbb{R}$ with $u \in L^1(\Omega)$, and set, for all $x \in \mathbb{R}^n \setminus \overline{\Omega}$,*

$$E_u(x) := \int_\Omega \frac{u(z)}{|x - z|^{n+2s}}\,dz.$$

Then, if we define

$$\tilde{u}(x) := \begin{cases} u(x) & \text{if } x \in \Omega, \\ \dfrac{E_u(x)}{E_1(x)} & \text{if } x \in \mathbb{R}^n \setminus \overline{\Omega}, \end{cases} \tag{2.6}$$

we have

$$\iint_{\mathcal{Q}} \frac{|\tilde{u}(x) - \tilde{u}(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \leqslant \iint_{\mathcal{Q}} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy. \tag{2.7}$$

Also, the equality in (2.7) *holds if and only if u satisfies* (1.6).

Proof. We remark that the notation E_1 in (2.6) stands for E_u when $u \equiv 1$. Moreover, without loss of generality, we can suppose that

$$\iint_{\mathcal{Q}} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy < +\infty;$$

otherwise the claim in (2.7) is obviously true.

In addition,

$$\int_\Omega \int_\Omega \frac{|\tilde{u}(x) - \tilde{u}(y)|^2}{|x-y|^{n+2s}}\, dx\, dy = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy, \tag{2.8}$$

so we only need to consider the integral on $(\mathbb{R}^n \setminus \Omega) \times \Omega$ (the integral on $\Omega \times (\mathbb{R}^n \setminus \Omega)$ being the same).

Setting $\varphi(x) := u(x) - \tilde{u}(x)$, for every $y \in \mathbb{R}^n \setminus \overline{\Omega}$ we have

$$\begin{aligned} \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx &= \int_\Omega \frac{|u(x) - \tilde{u}(y) - \varphi(y)|^2}{|x-y|^{n+2s}}\, dx \\ &= \int_\Omega \frac{|u(x)-\tilde{u}(y)|^2 - 2\varphi(y)(u(x)-\tilde{u}(y)) + |\varphi(y)|^2}{|x-y|^{n+2s}}\, dx. \end{aligned} \tag{2.9}$$

Now we observe that, for every $y \in \mathbb{R}^n \setminus \overline{\Omega}$,

$$\int_\Omega \frac{u(x) - \tilde{u}(y)}{|x-y|^{n+2s}}\, dx = E_u(y) - \frac{E_u(y)}{E_1(y)} E_1(y) = 0.$$

Accordingly, (2.9) becomes

$$\int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}\, dx = \int_\Omega \frac{|\tilde{u}(x) - \tilde{u}(y)|^2 + |\varphi(y)|^2}{|x-y|^{n+2s}}\, dx \geqslant \int_\Omega \frac{|\tilde{u}(x) - \tilde{u}(y)|^2}{|x-y|^{n+2s}}\, dx$$

for every $y \in \mathbb{R}^n \setminus \overline{\Omega}$, and the equality holds if and only if $\varphi(y) = 0$. Integrating over $\mathbb{R}^n \setminus \Omega$ (or, equivalently, on $\mathbb{R}^n \setminus \overline{\Omega}$), we get

$$\int_{\mathbb{R}^n \setminus \Omega} \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \geqslant \int_{\mathbb{R}^n \setminus \Omega} \int_\Omega \frac{|\tilde{u}(x) - \tilde{u}(y)|^2}{|x-y|^{n+2s}}\, dx\, dy,$$

and the equality holds if and only if $\varphi \equiv 0$ in $\mathbb{R}^n \setminus \Omega$. From this observation and (2.8) we obtain (2.7), as desired. ∎

3. Existence results and proofs of Theorems 1.1 and 1.2

The proof of Theorem 1.1 is based on a minimization argument. More precisely, given the functional setting introduced in Section 2 (recall in particular (2.2)), in order to deal with problem (1.9), we consider the energy functional $\mathcal{E}\colon X_{\alpha,\beta} \to \mathbb{R}$ defined as

$$\begin{aligned}\mathcal{E}(u) := {} & \frac{\alpha}{2}\int_\Omega |\nabla u|^2\,dx \\ & + \frac{\beta}{4}\iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ & + \int_\Omega \Big(\frac{\mu|u|^3}{3} - \frac{mu^2}{2} - \frac{\tau u(J\star u)}{2}\Big)\,dx. \end{aligned} \tag{3.1}$$

As a technical remark, we observe that our objective here is to distinguish between trivial and nontrivial solutions, to detect appropriate conditions for the survival of the solutions, and we do not indulge in the distinction nonnegative and nontrivial versus strictly positive solutions. For the reader interested in this point, we mention however that, under appropriate conditions, one could develop a regularity theory (see e.g. [47, Theorems 3.1.11, 3.1.12]) that allows the use of a strong maximum principle for smooth solutions (see e.g. [47, Theorem 3.1.4]).

Now we prove that the functional in (3.1) is the one associated with (1.9):

Lemma 3.1. *The Euler–Lagrange equation associated to the energy functional $\mathcal{E}$ introduced in* (3.1) *at a nonnegative function u is* (1.9).

Proof. We compute the first variation of $\mathcal{E}$, and we focus on the convolution term in (3.1) (the computation for the other terms being standard; see in particular [43, Proposition 3.7] to deal with the term involving the Gagliardo seminorm, which is the one producing the nonlocal Neumann condition).

For this, we set

$$\mathcal{J}(u) := \frac{\tau}{2}\int_\Omega u(x)(J\star u(x))\,dx.$$

For any $\phi \in X_{\alpha,\beta}$ and $\varepsilon \in (-1,1)$ we have

$$\begin{aligned}\mathcal{J}(u+\varepsilon\phi) &= \frac{\tau}{2}\int_\Omega (u+\varepsilon\phi)(x)(J\star(u+\varepsilon\phi))(x)\,dx \\ &= \frac{\tau}{2}\int_\Omega u(x)(J\star u)(x) + \varepsilon\big[u(x)(J\star\phi)(x) + \phi(x)(J\star u)(x)\big] \\ &\qquad + \varepsilon^2\phi(x)(J\star\phi)(x)\,dx.\end{aligned}$$

Accordingly,

$$\frac{d\mathcal{J}}{d\varepsilon}(u+\varepsilon\phi)\Big|_{\varepsilon=0} = \frac{\tau}{2}\int_\Omega u(x)(J\star\phi)(x) + \phi(x)(J\star u)(x)\,dx. \tag{3.2}$$

Now, since J is even (recall (1.2)), we see that

$$\begin{aligned}\int_\Omega u(x)(J\star\phi)(x)\,dx &= \int_\Omega u(x)\Big(\int_\Omega J(x-y)\phi(y)\,dy\Big)\,dx\\ &= \int_\Omega \phi(y)\Big(\int_\Omega J(y-x)u(x)\,dx\Big)\,dy = \int_\Omega \phi(x)(J\star u)(x)\,dx.\end{aligned}$$

Using this in (3.2) we obtain

$$\frac{d\mathcal{J}}{d\varepsilon}(u+\varepsilon\phi)\Big|_{\varepsilon=0} = \tau\int_\Omega \phi(x)(J\star u)(x)\,dx,$$

which concludes the proof. ∎

As a consequence of Lemma 3.1, to find solutions of (1.9), we will consider the minimizing problem for the functional $\mathcal{E}$ in (3.1). First, we show the following useful inequality:

Lemma 3.2. *Let $v, w \in L^2(\Omega)$. Then*

$$\int_\Omega |v(x)|\,|(J\star w)(x)|\,dx \leqslant \|v\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}. \tag{3.3}$$

Proof. By the Cauchy–Schwarz inequality, we have

$$\int_\Omega |v(x)|\,|(J\star w)(x)|\,dx \leqslant \|v\|_{L^2(\Omega)}\|J\star w\|_{L^2(\Omega)}. \tag{3.4}$$

Now, using the Young inequality for convolutions with exponents 1 and 2 (see e.g. [71, Theorem 9.1]), we obtain

$$\|J\star w\|_{L^2(\Omega)} = \|J*(w\chi_\Omega)\|_{L^2(\mathbb{R}^n)} \leqslant \|J\|_{L^1(\mathbb{R}^n)}\|w\chi_\Omega\|_{L^2(\mathbb{R}^n)} = \|w\|_{L^2(\Omega)},$$

where (1.3) has also been used. This and (3.4) give (3.3), as desired. ∎

We are now able to provide a minimization argument for the functional in (3.1):

Proposition 3.3. *Assume that $m \in L^q(\Omega)$, for some $q \in (\underline{q}, +\infty]$, where $\underline{q}$ was introduced in* (1.10)*, and that*

$$(m+\tau)^3\mu^{-2} \in L^1(\Omega). \tag{3.5}$$

Also let

$$p := \frac{2q}{q-1}.$$

Then the functional $\mathcal{E}$ in (3.1) *attains its minimum in $X_{\alpha,\beta}$. The minimal value is the same as the one occurring among the functions $u \in L^p(\Omega)$ for which*

$$\iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy < +\infty$$

and such that $\mathcal{N}_s u = 0$ a.e. outside Ω.

Moreover, there exists a nonnegative minimizer u, and it is a solution of (1.9).

Proof. First, we notice that $p \in [2, \frac{2q}{q-1})$ and

$$\frac{2}{p} + \frac{1}{q} = 1. \tag{3.6}$$

By (3.3) we have

$$\int_\Omega \frac{\tau u(J \star u)}{2}\,dx \leqslant \frac{\tau}{2}\|u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} = \frac{\tau}{2}\int_\Omega |u(x)|^2\,dx. \tag{3.7}$$

Moreover, we use the Young inequality with exponents $3/2$ and 3 to obtain

$$\frac{(m+\tau)u^2}{2} = \frac{\mu^{\frac{2}{3}}u^2}{2^{\frac{4}{3}}} \cdot \frac{m+\tau}{2^{-\frac{1}{3}}\mu^{\frac{2}{3}}} \leqslant \frac{\mu|u|^3}{6} + \frac{2}{3}\frac{|m+\tau|^3}{\mu^2}.$$

From this and (3.7) we have

$$\begin{aligned}\int_\Omega \frac{\mu|u|^3}{6} - \frac{mu^2}{2} - \frac{\tau u(J\star u)}{2}\,dx &\geqslant \int_\Omega \frac{\mu|u|^3}{6} - \frac{mu^2}{2} - \frac{\tau u^2}{2}\,dx \\ &\geqslant -\frac{2}{3}\int_\Omega \frac{|m+\tau|^3}{\mu^2}\,dx =: -\kappa.\end{aligned} \tag{3.8}$$

We point out that the quantity κ is finite, thanks to (3.5), and it does not depend on u.

Recalling (3.1), formula (3.8) implies that

$$\mathcal{E}(u) \geqslant \frac{\alpha}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\beta}{4}\iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega \frac{\mu|u|^3}{6}\,dx - \kappa. \tag{3.9}$$

Now we take a minimizing sequence u_j, and we observe that, in light of Theorem 2.1, we can assume that

$$\mathcal{N}_s u_j = 0 \text{ in } \mathbb{R}^n \setminus \overline{\Omega} \quad \text{for every } j \in \mathbb{N}. \tag{3.10}$$

We can also suppose that

$$\begin{aligned}0 = \mathcal{E}(0) &\geqslant \mathcal{E}(u_j) \\ &\geqslant \frac{\alpha}{2}\int_\Omega |\nabla u_j|^2\,dx + \frac{\beta}{4}\iint_{\mathcal{Q}} \frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega \frac{\mu|u_j|^3}{6}\,dx - \kappa,\end{aligned}$$

where (3.9) has also been exploited. This implies that

$$\frac{\alpha}{2}\int_\Omega |\nabla u_j|^2\,dx + \frac{\beta}{4}\iint_{\mathcal{Q}} \frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega \frac{\mu|u_j|^3}{6}\,dx \leqslant \kappa.$$

As a consequence,

$$\frac{\alpha}{2}\int_\Omega |\nabla u_j|^2\,dx + \frac{\beta}{4}\iint_{\mathcal{Q}} \frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \leqslant \kappa. \tag{3.11}$$

Moreover, by the Hölder inequality with exponents $3/2$ and 3,

$$\begin{aligned}\|u_j\|^2_{L^2(\Omega)} &\leqslant \left(\int_\Omega |u_j|^3\,dx\right)^{2/3} |\Omega|^{1/3}\\ &\leqslant \left(\int_\Omega \frac{\underline{\mu}|u_j|^3}{6}\,dx\right)^{2/3} \frac{6^{2/3}|\Omega|^{1/3}}{\underline{\mu}^{2/3}}\\ &\leqslant \left(\int_\Omega \frac{\mu|u_j|^3}{6}\,dx\right)^{2/3} \frac{6^{2/3}|\Omega|^{1/3}}{\underline{\mu}^{2/3}} \leqslant \frac{6^{2/3}|\Omega|^{1/3}\kappa}{\underline{\mu}^{2/3}}.\end{aligned}$$

From this and (3.11), and using compactness arguments, we can assume, up to a subsequence, that u_j converges to some $u \in L^p(\Omega)$ (for every $p \in [1, 2^*_s)$ if $\alpha = 0$, and for every $p \in [1, 2^*)$ if $\alpha \neq 0$; see e.g. [41, Corollary 7.2]) and a.e. in Ω, and also $|u_j| \leqslant h$ for some $h \in L^p(\Omega)$ for every $j \in \mathbb{N}$ (see e.g. [20, Theorem IV.9]).

Hence, if $x \in \mathbb{R}^n \setminus \overline{\Omega}$, by the dominated convergence theorem,

$$\int_\Omega \frac{u_j(y)}{|x-y|^{n+2s}}\,dy \to \int_\Omega \frac{u(y)}{|x-y|^{n+2s}}\,dy$$

as $j \nearrow +\infty$. Accordingly, in light of (3.10), when $x \in \mathbb{R}^n \setminus \overline{\Omega}$, we have

$$u_j(x) = \frac{\int_\Omega \frac{u_j(y)}{|x-y|^{n+2s}}\,dy}{\int_\Omega \frac{dy}{|x-y|^{n+2s}}} \to \frac{\int_\Omega \frac{u(y)}{|x-y|^{n+2s}}\,dy}{\int_\Omega \frac{dy}{|x-y|^{n+2s}}} =: u(x) \tag{3.12}$$

as $j \nearrow +\infty$ (we stress that till now u has only been defined in Ω, hence the last step in (3.12) is instrumental to also define u outside Ω). As a consequence, we obtain that u_j converges a.e. in $\mathbb{R}^n$.

Now, recalling (3.6), we have

$$\begin{aligned}\limsup_{j\nearrow+\infty}\left|\int_\Omega m(u_j^2-u^2)\,dx\right| &\leqslant \limsup_{j\nearrow+\infty}\int_\Omega |m(u_j^2-u^2)|\,dx\\ &= \limsup_{j\nearrow+\infty}\int_\Omega |m(u_j-u)(u_j+u)|\,dx\\ &\leqslant \limsup_{j\nearrow+\infty} \|m\|_{L^q(\Omega)}\|u_j-u\|_{L^p(\Omega)}\|u_j+u\|_{L^p(\Omega)} = 0,\end{aligned}$$

so that

$$\lim_{j\nearrow+\infty}\int_\Omega m(u_j^2-u^2)\,dx = 0.$$

Also,

$$\begin{aligned}\int_\Omega (u_j(J\star u_j) - u(J\star u))\,dx &= \int_\Omega (u_j-u)(J\star u_j)\,dx\\ &\quad + \int_\Omega (J\star u_j - J\star u)u\,dx.\end{aligned} \tag{3.13}$$

Using (3.3) with $v := u_j - u$ and $w := u$, we obtain

$$\limsup_{j\nearrow+\infty}\int_\Omega |u_j - u|\,|J\star u_j|\,dx \leqslant \limsup_{j\nearrow+\infty}\|u_j-u\|_{L^2(\Omega)}\|u_j\|_{L^2(\Omega)} = 0. \tag{3.14}$$

Similarly, exploiting (3.3) with $v := u$ and $w := u_j - u$, we have

$$\begin{aligned}\limsup_{j\nearrow+\infty}\int_\Omega |J\star u_j - J\star u|\,|u|\,dx &= \limsup_{j\nearrow+\infty}\int_\Omega |J\star(u_j-u)|\,|u|\,dx\\ &\leqslant \limsup_{j\nearrow+\infty}\|u_j-u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} = 0.\end{aligned} \tag{3.15}$$

From (3.13), (3.14) and (3.15) we conclude that

$$\lim_{j\nearrow+\infty}\int_\Omega (u_j(J\star u_j) - u(J\star u))\,dx = 0.$$

We also have, by the Fatou lemma and the lower semicontinuity of the L^2-norm,

$$\liminf_{j\nearrow+\infty}\iint_{\mathcal{Q}}\frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \geqslant \iint_{\mathcal{Q}}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy,$$

$$\liminf_{j\nearrow+\infty}\int_\Omega |\nabla u_j|^2\,dx \geqslant \int_\Omega |\nabla u|^2\,dx$$

and

$$\liminf_{j\nearrow+\infty}\int_\Omega \frac{\mu|u_j|^3}{3}\,dx \geqslant \int_\Omega\frac{\mu|u|^3}{3}\,dx.$$

Gathering together these observations, we conclude that

$$\liminf_{j\nearrow+\infty}\mathcal{E}(u_j) \geqslant \mathcal{E}(u),$$

and therefore u is the desired minimum.

Also, since $\mathcal{E}(|u|) \leqslant \mathcal{E}(u)$, we can suppose that u is nonnegative. Finally, u is a solution of (1.9) thanks to Lemma 3.1. ∎

The claim of Theorem 1.1 follows from Proposition 3.3.

Now we provide the proof of Theorem 1.2, relying also on the existence result in Theorem 1.1:

Proof of Theorem 1.2. Thanks to Theorem 1.1, we know that there exists a nonnegative solution to (1.9).

We now prove the claim in (i). For this, we assume that m is nonpositive and $\tau = 0$, and we argue towards a contradiction, supposing that there exists a nontrivial solution u of (1.9).

We notice that, since $u \geqslant 0$ and $\mu \geqslant \underline{\mu} > 0$ in Ω,

$$\int_\Omega \mu u^3\,dx > 0.$$

As a consequence, taking $v := u$ in (2.5) we obtain

$$0 \leqslant \alpha \int_\Omega |\nabla u|^2\,dx + \frac{\beta}{2}\iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = \int_\Omega m u^2\,dx - \int_\Omega \mu u^3\,dx < 0,$$

which is a contradiction, and therefore the claim in (i) is proved.

Now we deal with the claim in (ii). From Theorem 1.1 we know that there exists a nonnegative solution u to (1.9) which is obtained by the minimization of the functional $\mathcal{E}$ in (3.1) (recall Proposition 3.3). We claim that

$$u \text{ does not vanish identically.} \tag{3.16}$$

To prove this, we show that

$$0 \text{ is not a minimizer for } \mathcal{E}. \tag{3.17}$$

For this, we consider the constant function $v \equiv 1$ and a small parameter $\varepsilon > 0$. Then

$$\begin{aligned}\mathcal{E}(\varepsilon v) &= -\frac{\varepsilon^2}{2}\left[\int_\Omega m + \tau(J \star 1)\,dx\right] + \frac{\varepsilon^3}{3}\int_\Omega \mu\,dx\\ &\leqslant -c_1\varepsilon^2 + c_2\varepsilon^3,\end{aligned}$$

where

$$c_1 := \frac{1}{2}\int_\Omega m + \tau(J \star 1)\,dx \quad \text{and} \quad c_2 := \frac{1}{3}\|\mu\|_{L^1(\Omega)}.$$

We remark that $c_1 > 0$, thanks to (1.11), and $c_2 \in (0, +\infty)$, in light of (1.12). Then, for small ε we have $\mathcal{E}(\varepsilon v) < 0 = \mathcal{E}(0)$. This implies (3.17), which in turn proves (3.16). ∎

4. Analysis of the eigenvalue problem in (1.13) and proof of Theorem 1.4

In this section we focus on the proof of Theorem 1.4. For this, we need to exploit the analysis of the eigenvalue problem in (1.13) (some technical details are deferred to [42] for the reader's convenience).

The first result towards the proof of Theorem 1.4 concerns the existence of two unbounded sequences of eigenvalues, one positive and one negative:

Proposition 4.1. *Let*

$$m \in L^q(\Omega) \quad \textit{for some } q \in (\underline{q}, +\infty], \tag{4.1}$$

where $\underline{q}$ is given in (1.10). *Suppose that $m^+, m^- \not\equiv 0$ and that*

$$\int_\Omega m(x)\,dx \neq 0. \tag{4.2}$$

Then problem (1.13) *admits two unbounded sequences of eigenvalues:*

$$\cdots \leqslant \lambda_{-2} \leqslant \lambda_{-1} < \lambda_0 = 0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots .$$

In particular, if

$$\int_\Omega m(x)\,dx < 0,$$

then

$$\lambda_1 = \min_{u \in X_{\alpha,\beta}} \left\{ [u]^2_{X_{\alpha,\beta}} \ \text{s.t.} \int_\Omega mu^2\,dx = 1 \right\}, \tag{4.3}$$

where we use the notation in (2.4). *If instead*

$$\int_\Omega m(x)\,dx > 0,$$

then

$$\lambda_{-1} = -\min_{u \in X_{\alpha,\beta}} \left\{ [u]^2_{X_{\alpha,\beta}} \ \text{s.t.} \int_\Omega mu^2\,dx = -1 \right\}.$$

The proof of Proposition 4.1 is contained in [42].

The first positive eigenvalue λ_1, as given by Proposition 4.1, has the following properties:

Proposition 4.2. *Let $m \in L^q(\Omega)$, for some $q \in (\underline{q}, +\infty]$, where $\underline{q}$ is given in* (1.10). *Suppose that $m^+ \not\equiv 0$ and*

$$\int_\Omega m\,dx < 0.$$

Then the first positive eigenvalue λ_1 of (1.13) *is simple, and the first eigenfunction e can be taken such that $e \geqslant 0$.*

A similar statement holds for λ_{-1} if $m^- \not\equiv 0$ and

$$\int_\Omega m\,dx > 0.$$

See [42] for the proof of Proposition 4.2.

With this, we are now ready to give the proof of Theorem 1.4:

Proof of Theorem 1.4. Thanks to Theorem 1.1, we know that there exists a nonnegative solution to (1.9).

We first prove the claim in (i). For this, we assume that $m \leqslant -\tau$, and we suppose by contradiction that there exists a nontrivial solution u of (1.9).

We observe that, applying (3.3) with $v := u$ and $w := u$,

$$\tau \int_\Omega u(J \star u)\,dx \leqslant \tau \|u\|^2_{L^2(\Omega)} = \tau \int_\Omega u^2\,dx. \tag{4.4}$$

Hence, taking u as a test function in (2.5), using (4.4) and recalling that $u \geqslant 0$ and $\mu \geqslant \underline{\mu}$, we get

$$\begin{aligned}
0 &\leqslant \alpha \int_\Omega |\nabla u|^2\,dx + \frac{\beta}{2} \iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&= \int_\Omega (m - \mu u)u^2\,dx + \tau \int_\Omega (J \star u)u\,dx \\
&\leqslant -\tau \int_\Omega u^2\,dx - \underline{\mu} \int_\Omega u^3\,dx + \tau \int_\Omega u^2\,dx \\
&< 0.
\end{aligned}$$

This is a contradiction, whence the first claim is proved.

Now we show the claim in (ii). From Theorem 1.1 we know that there exists a non-negative solution u to (1.9) which is obtained by the minimization of the functional $\mathcal{E}$ in (3.1) (recall Proposition 3.3). We claim that

$$u \text{ does not vanish identically.} \tag{4.5}$$

To prove this, we show that

$$0 \text{ is not a minimizer for } \mathcal{E}. \tag{4.6}$$

For this, we take an eigenfunction e associated to the first positive eigenvalue λ_1, as given by Proposition 4.2. Namely, we take $e \in X_{\alpha,\beta}$ such that

$$\alpha \int_\Omega \nabla e \cdot \nabla v\,dx + \frac{\beta}{2} \iint_{\mathcal{Q}} \frac{(e(x)-e(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy = \lambda_1 \int_\Omega mev\,dx \tag{4.7}$$

for every $v \in X_{\alpha,\beta}$.

By taking $v := e$ in (4.7), we obtain

$$\alpha \int_\Omega |\nabla e|^2\,dx + \frac{\beta}{2} \iint_{\mathcal{Q}} \frac{|e(x)-e(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = \lambda_1 \int_\Omega me^2\,dx. \tag{4.8}$$

We also remark that, thanks to (1.14), we can use the characterization of λ_1 given in formula (4.3) of Proposition 4.1, and hence we can normalize e in such a way that

$$\int_\Omega me^2\,dx = 1. \tag{4.9}$$

By [42, Corollary 1.4], we know that

$$e \text{ is bounded.} \tag{4.10}$$

We also take $\varepsilon > 0$. Then, by (4.8) and (4.9),

$$\begin{aligned}\mathcal{E}(\varepsilon e) &= \frac{\varepsilon^2}{2}\Big[\alpha \int_\Omega |\nabla e|^2\,dx + \frac{\beta}{2}\iint_{\mathcal{Q}} \frac{|e(x)-e(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ &\qquad - \int_\Omega me^2\,dx - \int_\Omega \tau(J\star e)e\,dx\Big] + \frac{\varepsilon^3}{3}\int_\Omega \mu e^3\,dx \\ &= \frac{\varepsilon^2}{2}\Big[(\lambda_1 - 1)\int_\Omega me^2\,dx - \int_\Omega \tau(J\star e)e\,dx\Big] + \frac{\varepsilon^3}{3}\int_\Omega \mu e^3\,dx \\ &= \frac{\varepsilon^2}{2}\Big[(\lambda_1 - 1) - \int_\Omega \tau(J\star e)e\,dx\Big] + \frac{\varepsilon^3}{3}\int_\Omega \mu e^3\,dx \\ &= -\frac{c_1}{2}\varepsilon^2 + c_2\frac{\varepsilon^3}{3},\end{aligned} \tag{4.11}$$

where

$$\begin{aligned} c_1 &:= 1 - \lambda_1 + \tau\int_\Omega (J\star e)e\,dx, \\ c_2 &:= \int_\Omega \mu e^3\,dx. \end{aligned}$$

We notice that $c_1 > 0$, thanks to (1.15), and $c_2 \in \mathbb{R}$, in light of (4.10). As a consequence, for small ε we have $\mathcal{E}(\varepsilon e) < 0 = \mathcal{E}(0)$, which proves (4.6). In turn, this implies (4.5), thus completing the proof of (ii). ■

Remark 4.3. For the sake of simplicity, we focused here on the eigenvalue problem in (1.13) since it is the "natural one" associated with the diffusive character of the population. In this sense, condition (1.15) relates, in a simple and explicit manner, the survival chances of the population to the corresponding values of the diffusive eigenvalue with respect to the pollination term.

We observe however that condition (1.15) can be sharpened by considering an eigenvalue problem in which one considers altogether diffusion and pollination. More specifically, one could consider, instead of (1.13), the weighted eigenvalue problem of convolution type

$$\begin{cases} -\alpha\Delta u + \beta(-\Delta)^s u = \lambda(mu + \tau J\star u) & \text{in } \Omega, \\ \text{with } (\alpha,\beta)\text{-Neumann condition.} \end{cases} \tag{4.12}$$

One could denote by $\lambda_{1\star}$ the smallest strictly positive eigenvalue of (4.12) and by $e_\star$ the eigenfunction corresponding to $\lambda_{1\star}$ normalized such that

$$\int_\Omega m(x)e_\star^2(x)\,dx + \tau\int_\Omega (J\star e_\star(x))e_\star(x)\,dx = 1.$$

In this functional analytic setting, proceeding as in (4.11), but with e replaced by $e_\star$, one would obtain, instead of (1.15), the condition

$$\lambda_{1\star} < 1. \tag{4.13}$$

We point out that (4.13) is indeed a milder condition than (1.15), since

$$\begin{aligned}\lambda_{1\star} &\leqslant \frac{\alpha \int_\Omega |\nabla e|^2\,dx + \frac{\beta}{2}\iint_{\mathcal{Q}} \frac{|e(x)-e(y)|^2}{|x-y|^{n+2s}}\,dx\,dy}{\int_\Omega me^2\,dx + \tau\int_\Omega (J\star e)e\,dx}\\ &= \frac{\lambda_1}{1+\tau\int_\Omega (J\star e)e\,dx};\end{aligned}$$

therefore if (1.15) holds true, then so does (4.13).

5. Optimization on *m* and proofs of Theorems 1.5, 1.7, 1.8, 1.9 and 1.10

This section is devoted to the understanding of the optimal configuration of the resource m, which is based on the analysis of the minimal eigenvalue problem given in (1.17).

First of all, we will see that the optimal resource distribution attaining the minimal eigenvalue in (1.17) is of bang-bang type, namely concentrated on its minimal and maximal values $\underline{m}$ and $\overline{m}$. This property is based on the so-called "bathtub principle"; see [40, Lemma 3.3] (or [55,57]). We recall this result here for the convenience of the reader:

Lemma 5.1. *Let $f \in L^1(\Omega)$ and $\mathcal{M}$ be as in (1.16). Then the maximization problem*

$$\sup_{m\in\mathcal{M}} \int_\Omega fm\,dx$$

is attained by a suitable $m \in \mathcal{M}$ given by

$$m := \overline{m}\chi_D - \underline{m}\chi_{\Omega\setminus D}$$

for some subset $D \subset \Omega$ such that

$$|D| = \frac{\underline{m}+m_0}{\underline{m}+\overline{m}}|\Omega|. \tag{5.1}$$

We now show that, in light of Lemma 5.1, to optimize the eigenvalue λ_1 in (1.17), we have to consider $m \in \mathcal{M}$ of bang-bang type. More precisely, we define

$$\begin{aligned}\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}(\overline{m},\underline{m},m_0) := \big\{&m \in \mathcal{M} \text{ s.t. } m := \overline{m}\chi_D - \underline{m}\chi_{\Omega\setminus D},\\ &\text{for some subset } D\subset\Omega \text{ with } |D| = \tfrac{\underline{m}+m_0}{\underline{m}+\overline{m}}|\Omega|\big\},\end{aligned} \tag{5.2}$$

and we have the following result:

Proposition 5.2. *We have*

$$\underline{\lambda} = \inf_{m\in\widetilde{\mathcal{M}}} \lambda_1(m).$$

Proof. We define

$$\tilde{\lambda} := \inf_{m \in \widetilde{\mathcal{M}}} \lambda_1(m),$$

and we claim that

$$\underline{\lambda} = \tilde{\lambda}. \tag{5.3}$$

To this end, we observe that, since $\widetilde{\mathcal{M}} \subset \mathcal{M}$, we have

$$\underline{\lambda} \leqslant \tilde{\lambda}. \tag{5.4}$$

Moreover, by the definition of $\underline{\lambda}$ in (1.17), we have that for every $\varepsilon > 0$ there exists $m_\varepsilon \in \mathcal{M}$ such that $\underline{\lambda} + \varepsilon \geqslant \lambda_1(m_\varepsilon)$. Then we denote by e_ε the nonnegative eigenfunction associated to $\lambda_1(m_\varepsilon)$, and we conclude that

$$\underline{\lambda} + \varepsilon \geqslant \lambda_1(m_\varepsilon) = \frac{[e_\varepsilon]^2_{X_{\alpha,\beta}}}{\int_\Omega m_\varepsilon e_\varepsilon^2 \, dx}. \tag{5.5}$$

We also observe that, in light of Lemma 5.1,

$$\int_\Omega m_\varepsilon e_\varepsilon^2 \, dx \leqslant \int_\Omega (\overline{m}\chi_{D_\varepsilon} - \underline{m}\chi_{\Omega \setminus D_\varepsilon}) e_\varepsilon^2 \, dx$$

for a suitable $D_\varepsilon \subset \Omega$ satisfying (5.1). Plugging this information into (5.5), and letting $m_\varepsilon^\star := \overline{m}\chi_{D_\varepsilon} - \underline{m}\chi_{\Omega \setminus D_\varepsilon}$, we obtain

$$\underline{\lambda} + \varepsilon \geqslant \frac{[e_\varepsilon]^2_{X_{\alpha,\beta}}}{\int_\Omega (\overline{m}\chi_{D_\varepsilon} - \underline{m}\chi_{\Omega \setminus D_\varepsilon}) e_\varepsilon^2 \, dx} \geqslant \lambda_1(m_\varepsilon^\star) \geqslant \tilde{\lambda}.$$

Hence, taking the limit as ε goes to 0, we get that $\underline{\lambda} \geqslant \tilde{\lambda}$. This, combined with (5.4), establishes (5.3), as desired. ∎

We recall that many biological models describe optimal resources of bang-bang type; see e.g. [26, 27, 54, 57, 61, 64].

In light of Proposition 5.2, from now on, when optimizing the eigenvalue $\lambda_1(m)$ as in (1.17), we will suppose that m belongs to the set $\widetilde{\mathcal{M}}$ introduced in (5.2).

Now we provide the proof of Theorem 1.5.

Proof of Theorem 1.5. We take a ball $B \subset \Omega$ such that

$$|B| \leqslant \frac{d_0}{2} |\Omega|. \tag{5.6}$$

We can assume, up to a translation, that $\Omega \subset \{x_n > 0\}$, and, for every $\xi \geqslant 0$, we define the set

$$\Omega_\xi := B \cup (\{x_n < \xi\} \cap \Omega).$$

We observe that $|\Omega_\xi|$ is nondecreasing with respect to ξ, and we define

$$\xi^* := \sup\{\xi \geqslant 0 : |\Omega_\xi| < \tfrac{\underline{m}+m_0}{\underline{m}+\overline{m}}|\Omega|\}.$$

We claim that, for every $\underline{\xi} > 0$,

$$\lim_{\xi\to\underline{\xi}} |\Omega_\xi| = |\Omega_{\underline{\xi}}|. \tag{5.7}$$

To this end, we first show that

$$\lim_{\xi\to\underline{\xi}} \chi_{\Omega_\xi}(x) = \chi_{\Omega_{\underline{\xi}}}(x) \quad \text{for a.e. } x \in \Omega. \tag{5.8}$$

For this, we consider several cases. If $x = (x', x_n) \in \Omega_{\underline{\xi}}$, then either $x \in B$ or $x_n < \underline{\xi}$. If $x \in B$, then $x \in \Omega_\xi$ for each $\xi > 0$, and accordingly $\chi_{\Omega_\xi}(x) = 1 = \chi_{\Omega_{\underline{\xi}}}(x)$, which implies (5.8). If instead $x_n < \underline{\xi}$, then there exists $\tilde{\xi} \in (x_n, \underline{\xi})$ such that, for every $\xi \in (\tilde{\xi}, \underline{\xi})$, we have $\chi_{\Omega_\xi}(x) = 1 = \chi_{\Omega_{\underline{\xi}}}(x)$, which proves (5.8) also in this case.

On the other hand, if $x \notin \Omega_{\underline{\xi}}$, then $x \notin B$ and $x_n \geqslant \underline{\xi}$. We notice that the set $\{x_n = \underline{\xi}\}$ has zero Lebesgue measure, and therefore, in order to prove (5.8), we can assume that $x_n > \underline{\xi}$. Then there exists $\tilde{\xi} \in (\underline{\xi}, x_n)$ such that, for every $\xi \in (\underline{\xi}, \tilde{\xi})$, we have $x \notin \Omega_\xi$, and so $\chi_{\Omega_\xi}(x) = 0 = \chi_{\Omega_{\underline{\xi}}}(x)$. This completes the proof of (5.8).

By (5.8) and the dominated convergence theorem we obtain (5.7), as desired.

We also notice that if $\xi = 0$, then $\Omega_\xi = B$, and therefore, by (1.18) and (5.6),

$$|\Omega_\xi| = |B| \leqslant \frac{d_0}{2}|\Omega| \leqslant \frac{\underline{m} + m_0}{2(\underline{m} + \overline{m})}|\Omega|.$$

This and the continuity statement in (5.7) guarantee that $\xi^* > 0$.

Moreover, the continuity in (5.7) implies that

$$|\Omega_{\xi^*}| = \frac{\underline{m} + m_0}{\underline{m} + \overline{m}}|\Omega|. \tag{5.9}$$

Now we set $D := \Omega_{\xi^*}$, and we observe that D satisfies (5.1), thanks to (5.9). Also, we take $v \in C_0^\infty(B)$, with $v \not\equiv 0$. Then, recalling that $B \subset D$,

$$\begin{aligned}\underline{\lambda} &\leqslant \frac{[v]^2_{X_{\alpha,\beta}}}{\int_\Omega (\overline{m}\chi_D - \underline{m}\chi_{\Omega\setminus D})v^2\,dx}\\ &= \frac{[v]^2_{X_{\alpha,\beta}}}{\overline{m}\int_B v^2\,dx} \leqslant \frac{C}{\overline{m}}\end{aligned}$$

for some positive constant C depending on Ω and d_0. This completes the proof of Theorem 1.5. ∎

With the aid of Theorem 1.5 we now prove Corollary 1.6, by arguing as follows:

Proof of Corollary 1.6. We notice that

$$\lim_{\mathscr{m}\nearrow+\infty}\frac{\mathscr{m}+m_0}{2\mathscr{m}}=\frac{1}{2}.$$

By taking $\mathscr{m}=\underline{m}=\overline{m}$, this implies that

$$\frac{\underline{m}+m_0}{\underline{m}+\overline{m}}=\frac{\mathscr{m}+m_0}{2\mathscr{m}}\geqslant\frac{1}{4},$$

as long as $\mathscr{m}$ is large enough. This says that the assumption (1.18) in Theorem 1.5 is satisfied with $d_0:=\frac{1}{4}$, and therefore, Theorem 1.5 gives that

$$\underline{\lambda}(\mathscr{m},\mathscr{m},m_0)\leqslant\frac{C}{\mathscr{m}}$$

for some $C>0$ depending only on Ω. As a consequence,

$$\lim_{\mathscr{m}\nearrow+\infty}\underline{\lambda}(\mathscr{m},\mathscr{m},m_0)=0,$$

as desired. ∎

The next goal of this section is to prove Theorem 1.7. For concreteness, we give here the proof for $n\geqslant 3$, and we defer the case $n=2$ to Appendix A.

Without loss of generality, we suppose that

$$B_2\subset\Omega. \tag{5.10}$$

For $n\geqslant 3$ and for any $\rho\in(0,1)$, we define the function $\varphi\colon\mathbb{R}^n\to\mathbb{R}$ as

$$\varphi(x):=\begin{cases} c_\star+1 & \text{if } x\in B_\rho,\\ c_\star+\dfrac{\rho^\gamma}{1-\rho^\gamma}\left(\dfrac{1}{|x|^\gamma}-1\right) & \text{if } x\in B_1\setminus B_\rho,\\ c_\star & \text{if } x\in\mathbb{R}^n\setminus B_1,\end{cases} \tag{5.11}$$

where $\gamma>0$ and

$$c_\star:=-\frac{\underline{m}+m_0}{m_0}. \tag{5.12}$$

We observe that, since $m_0\in(-\underline{m},0)$, we have $c_\star>0$.

Also, we set

$$D:=B_\rho. \tag{5.13}$$

The idea to prove Theorem 1.7 is to use the function φ in (5.11) and the resource $m=\overline{m}\chi_D-\underline{m}\chi_{\Omega\setminus D}$, with D as in (5.13), as competitors for the minimization of $\underline{\lambda}$ in (1.17).

In this setting, we notice that, since $m\in\widetilde{\mathcal{M}}$, recalling (5.1),

$$\rho^n|B_1|=|B_\rho|=|D|=\frac{\underline{m}+m_0}{\underline{m}+\overline{m}}|\Omega|.$$

This says that

$$\text{sending } \overline{m} \nearrow +\infty \text{ is equivalent to sending } \rho \searrow 0, \tag{5.14}$$

$\underline{m}$, m_0 and $|\Omega|$ being fixed quantities in this argument.

In light of these observations, the next lemmata will be devoted to estimate in terms of ρ the quantities involving φ that appear in the minimization of $\underline{\lambda}$.

We point out that, in dimension $n = 2$, the argument to prove Theorem 1.7 will be similar, but we will need to introduce a logarithmic-type function as in (A.1) instead of a polynomial-type function as in (5.11) (as often happens when passing from dimension 2 to higher dimensions).

The first result that we have in this setting deals with the H^1-seminorm of φ:

Lemma 5.3. *Let $n \geqslant 3$ and φ be as in (5.11). Then*

$$\lim_{\rho \searrow 0} \int_\Omega |\nabla \varphi|^2 \, dx = 0.$$

Proof. By the definition of φ in (5.11), we have $\nabla\varphi \neq 0$ only if $x \in B_1 \setminus B_\rho$. Accordingly, using polar coordinates,

$$\begin{aligned} \int_\Omega |\nabla\varphi|^2 \, dx &= \gamma^2 \Big(\frac{\rho^\gamma}{1-\rho^\gamma}\Big)^2 \int_{B_1 \setminus B_\rho} \frac{1}{|x|^{2\gamma+2}} \, dx \\ &= |\partial B_1| \gamma^2 \Big(\frac{\rho^\gamma}{1-\rho^\gamma}\Big)^2 \int_\rho^1 r^{n-2\gamma-3} \, dr. \end{aligned} \tag{5.15}$$

Now we point out that

$$\int_\rho^1 r^{n-2\gamma-3} \, dr \leqslant \begin{cases} \dfrac{1}{n-2\gamma-2} & \text{if } \gamma < \dfrac{n-2}{2}, \\ -\log\rho & \text{if } \gamma = \dfrac{n-2}{2}, \\ -\dfrac{\rho^{n-2\gamma-2}}{n-2\gamma-2} & \text{if } \gamma > \dfrac{n-2}{2}. \end{cases} \tag{5.16}$$

This and (5.15) entail that, for every $\gamma > 0$,

$$\lim_{\rho \searrow 0} \int_\Omega |\nabla \varphi|^2 \, dx = 0,$$

which concludes the proof. ∎

Now we deal with the Gagliardo seminorm of φ. For this, we point out the following useful inequality:

Lemma 5.4. *Let $x, y \in \mathbb{R}^n \setminus \{0\}$ and $\gamma > 0$. Then there exists $C_\gamma > 0$ such that*

$$\left| \frac{1}{|x|^\gamma} - \frac{1}{|y|^\gamma} \right| \leqslant C_\gamma \frac{\big||x| - |y|\big|}{\min\{|x|^{\gamma+1}, |y|^{\gamma+1}\}}. \tag{5.17}$$

Proof. We can assume that $|x| \geqslant |y|$, the other case being analogous. In this way, formula (5.17) boils down to

$$\frac{1}{|y|^\gamma} - \frac{1}{|x|^\gamma} \leqslant C_\gamma \frac{|x| - |y|}{|y|^{\gamma+1}}. \tag{5.18}$$

To prove (5.18), we first claim that, for every $t \geqslant 1$,

$$1 - \frac{1}{t^\gamma} \leqslant C_\gamma (t-1) \tag{5.19}$$

for a suitable $C_\gamma > 0$. Indeed, we set

$$f(t) := Ct + \frac{1}{t^\gamma} - (C+1)$$

for some positive constant C (to be chosen in what follows), and we observe that

$$f(1) = 0, \tag{5.20}$$

and

$$f'(t) = C - \frac{\gamma}{t^{\gamma+1}} \geqslant C - \gamma$$

for any $t \geqslant 1$. As a result, taking $C := \gamma + 1$, we obtain $f'(t) > 0$. This and (5.20) give that $f(t) \geqslant 0$ for every $t \geqslant 1$, which implies (5.19).

Taking $t := \frac{|x|}{|y|}$ in (5.19), we obtain

$$1 - \frac{|y|^\gamma}{|x|^\gamma} \leqslant C_\gamma \Big(\frac{|x|}{|y|} - 1 \Big).$$

Multiplying this inequality by $\frac{1}{|y|^\gamma}$ we deduce (5.18), as desired. ∎

With this, we now estimate the Gagliardo seminorm of φ as follows:

Lemma 5.5. *Let $n \geqslant 3$ and φ be as in* (5.11)*. Then*

$$\lim_{\rho \searrow 0} \iint_{\mathcal{Q}} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{n+2s}}\, dx\, dy = 0.$$

We give here a proof of Lemma 5.5 based on direct (albeit a bit long) calculations. A shorter proof based on interpolation theory will be provided in Appendix B.

Proof of Lemma 5.5. In what follows, we will assume that $\rho \leqslant 1/4$. By the definition of φ in (5.11), it plainly follows that

$$\iint_{B_\rho \times B_\rho} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{n+2s}}\, dx\, dy = 0 \tag{5.21}$$

and

$$\iint_{(\mathbb{R}^n\setminus B_1)\times(\mathbb{R}^n\setminus B_1)} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = 0. \tag{5.22}$$

Moreover, by the change of variable $z := y - x$,

$$\begin{aligned}\iint_{B_\rho\times(\mathbb{R}^n\setminus B_1)} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy &= \iint_{B_\rho\times(\mathbb{R}^n\setminus B_1)} \frac{1}{|x-y|^{n+2s}}\,dx\,dy\\ &\leqslant \int_{B_\rho} dx \int_{\mathbb{R}^n\setminus B_{\frac12}} \frac{1}{|z|^{n+2s}}\,dz\\ &\leqslant C\int_{B_\rho} dx = C\rho^n\end{aligned}$$

for some $C > 0$. As a consequence,

$$\lim_{\rho\searrow 0}\iint_{B_\rho\times(\mathbb{R}^n\setminus B_1)} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = 0. \tag{5.23}$$

Now, if $x \in B_1 \setminus B_\rho$ and $y \in B_\rho$, from (5.11) we have

$$|\varphi(x)-\varphi(y)|^2 = \Big(\frac{\rho^\gamma}{1-\rho^\gamma}\Big)^2\Big(\frac{1}{|x|^\gamma}-\frac{1}{\rho^\gamma}\Big)^2.$$

Hence, also utilizing (5.17) (applied here with $|y| := \rho$),

$$\begin{aligned}&\iint_{(B_1\setminus B_\rho)\times B_\rho} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\ &\quad= \Big(\frac{\rho^\gamma}{1-\rho^\gamma}\Big)^2 \iint_{(B_1\setminus B_\rho)\times B_\rho}\Big(\frac{1}{|x|^\gamma}-\frac{1}{\rho^\gamma}\Big)^2 \frac{1}{|x-y|^{n+2s}}\,dx\,dy\\ &\quad\leqslant \frac{C}{\rho^{2\gamma+2}}\Big(\frac{\rho^\gamma}{1-\rho^\gamma}\Big)^2 \iint_{(B_1\setminus B_\rho)\times B_\rho}\frac{(|x|-\rho)^2}{|x-y|^{n+2s}}\,dx\,dy.\end{aligned} \tag{5.24}$$

We observe that, since $x \in B_1 \setminus B_\rho$ and $y \in B_\rho$,

$$|x| - \rho \leqslant |x| - |y| \leqslant |x - y|,$$

and therefore, plugging this information into (5.24),

$$\begin{aligned}&\iint_{(B_1\setminus B_\rho)\times B_\rho} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\ &\quad\leqslant \frac{C}{\rho^{2\gamma+2}}\Big(\frac{\rho^\gamma}{1-\rho^\gamma}\Big)^2 \iint_{(B_1\setminus B_\rho)\times B_\rho}|x-y|^{2-n-2s}\,dx\,dy\\ &\quad\leqslant \frac{C}{\rho^{2\gamma+2}}\Big(\frac{\rho^\gamma}{1-\rho^\gamma}\Big)^2 \int_{B_\rho} dy\int_{B_2}|z|^{2-2s-n}\,dz\\ &\quad\leqslant \frac{C}{\rho^{2\gamma+2}}\Big(\frac{\rho^\gamma}{1-\rho^\gamma}\Big)^2 \int_{B_\rho} dy\\ &\quad\leqslant C\rho^{n-2},\end{aligned}$$

up to renaming $C > 0$ from line to line. As a result,

$$\lim_{\rho\searrow 0}\iint_{(B_1\setminus B_\rho)\times B_\rho}\frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy=0. \tag{5.25}$$

In addition, since $\mathbb{R}^n\setminus\Omega\subset\mathbb{R}^n\setminus B_2$ (recall (5.10)), changing variable $z := y-x$ and using polar coordinates,

$$\begin{aligned}
&\iint_{(B_1\setminus B_\rho)\times(\mathbb{R}^n\setminus\Omega)}\frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\
&\quad=\left(\frac{\rho^\gamma}{1-\rho^\gamma}\right)^2\iint_{(B_1\setminus B_\rho)\times(\mathbb{R}^n\setminus\Omega)}\left(\frac{1}{|x|^\gamma}-1\right)^2\frac{1}{|x-y|^{n+2s}}\,dx\,dy\\
&\quad\leqslant\left(\frac{\rho^\gamma}{1-\rho^\gamma}\right)^2\int_{B_1\setminus B_\rho}\left(\frac{1}{|x|^\gamma}-1\right)^2dx\int_{\mathbb{R}^n\setminus B_1}\frac{1}{|z|^{n+2s}}\,dz\\
&\quad\leqslant C\left(\frac{\rho^\gamma}{1-\rho^\gamma}\right)^2\int_{B_1\setminus B_\rho}\left(\frac{1}{|x|^\gamma}-1\right)^2dx\\
&\quad=C\left(\frac{\rho^\gamma}{1-\rho^\gamma}\right)^2\int_{B_1\setminus B_\rho}\left(\frac{1}{|x|^{2\gamma}}-\frac{2}{|x|^\gamma}+1\right)dx\\
&\quad\leqslant C\left(\frac{\rho^\gamma}{1-\rho^\gamma}\right)^2\int_\rho^1\left(r^{n-2\gamma-1}+r^{n-1}\right)dr\\
&\quad\leqslant C\left(\frac{\rho^\gamma}{1-\rho^\gamma}\right)^2\left[1+\int_\rho^1 r^{n-2\gamma-1}\,dr\right],
\end{aligned}\tag{5.26}$$

possibly changing $C > 0$ from line to line. We also remark that

$$\int_\rho^1 r^{n-2\gamma-1}\,dr\leqslant\begin{cases}\dfrac{1}{n-2\gamma} & \text{if } \gamma<\dfrac{n}{2},\\[2mm] -\log\rho & \text{if } \gamma=\dfrac{n}{2},\\[2mm] -\dfrac{\rho^{n-2\gamma}}{n-2\gamma} & \text{if } \gamma>\dfrac{n}{2}.\end{cases}$$

This and (5.26) imply that

$$\lim_{\rho\searrow 0}\iint_{(B_1\setminus B_\rho)\times(\mathbb{R}^n\setminus\Omega)}\frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy=0. \tag{5.27}$$

Furthermore, recalling (5.11) and making use of (5.17), we have

$$\begin{aligned}
&\iint_{(B_1\setminus B_\rho)\times(\Omega\setminus B_1)}\frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\
&\quad=\left(\frac{\rho^\gamma}{1-\rho^\gamma}\right)^2\iint_{(B_1\setminus B_\rho)\times(\Omega\setminus B_1)}\left(\frac{1}{|x|^\gamma}-1\right)^2\frac{1}{|x-y|^{n+2s}}\,dx\,dy\\
&\quad\leqslant C\left(\frac{\rho^\gamma}{1-\rho^\gamma}\right)^2\iint_{(B_1\setminus B_\rho)\times(\Omega\setminus B_1)}\frac{(1-|x|)^2}{|x|^{2\gamma+2}|x-y|^{n+2s}}\,dx\,dy.
\end{aligned}$$

Hence, noticing that, for every $x \in B_1 \setminus B_\rho$ and every $y \in \Omega \setminus B_1$,

$$1 - |x| \leqslant |y| - |x| \leqslant |x - y|,$$

we conclude that

$$\begin{aligned}
&\iint_{(B_1\setminus B_\rho)\times(\Omega\setminus B_1)} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad\leqslant C\Big(\frac{\rho^\gamma}{1-\rho^\gamma}\Big)^2 \iint_{(B_1\setminus B_\rho)\times(\Omega\setminus B_1)} \frac{1}{|x|^{2\gamma+2}|x-y|^{n+2s-2}}\,dx\,dy \\
&\quad\leqslant C\Big(\frac{\rho^\gamma}{1-\rho^\gamma}\Big)^2 \int_{B_1\setminus B_\rho} \frac{1}{|x|^{2\gamma+2}}\,dx \\
&\quad\leqslant C\Big(\frac{\rho^\gamma}{1-\rho^\gamma}\Big)^2 \int_\rho^1 r^{n-2\gamma-3}\,dr.
\end{aligned}$$

Accordingly, recalling (5.16), we conclude that

$$\lim_{\rho\searrow 0} \iint_{(B_1\setminus B_\rho)\times(\Omega\setminus B_1)} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = 0. \tag{5.28}$$

We now claim that

$$\lim_{\rho\searrow 0} \iint_{(B_1\setminus B_\rho)\times(B_1\setminus B_\rho)} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = 0. \tag{5.29}$$

For this, we observe that by (5.11),

$$\begin{aligned}
&\iint_{(B_1\setminus B_\rho)\times(B_1\setminus B_\rho)} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad= \Big(\frac{\rho^\gamma}{1-\rho^\gamma}\Big)^2 \iint_{(B_1\setminus B_\rho)\times(B_1\setminus B_\rho)} \Big(\frac{1}{|x|^\gamma}-\frac{1}{|y|^\gamma}\Big)^2 \frac{dx\,dy}{|x-y|^{n+2s}} \\
&\quad= 2\Big(\frac{\rho^\gamma}{1-\rho^\gamma}\Big)^2 \iint_{\substack{(B_1\setminus B_\rho)\times(B_1\setminus B_\rho)\\ \{|x|\leqslant|y|\}}} \Big(\frac{1}{|x|^\gamma}-\frac{1}{|y|^\gamma}\Big)^2 \frac{dx\,dy}{|x-y|^{n+2s}}.
\end{aligned}$$

Hence, from (5.17) we get

$$\begin{aligned}
&\iint_{(B_1\setminus B_\rho)\times(B_1\setminus B_\rho)} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad\leqslant C\Big(\frac{\rho^\gamma}{1-\rho^\gamma}\Big)^2 \iint_{\substack{(B_1\setminus B_\rho)\times(B_1\setminus B_\rho)\\ \{|x|\leqslant|y|\}}} \frac{|x-y|^{2-n-2s}}{|x|^{2\gamma+2}}\,dx\,dy,
\end{aligned}$$

up to renaming $C > 0$.

Since $\rho \in (0,1)$, we can take an integer k such that

$$\frac{1}{2^{k+1}} < \rho \leqslant \frac{1}{2^k}. \tag{5.30}$$

In this way, we have

$$\begin{aligned}
&\iint_{(B_1\setminus B_\rho)\times(B_1\setminus B_\rho)} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad\leqslant C\Big(\frac{\rho^\gamma}{1-\rho^\gamma}\Big)^2 \iint_{\substack{(B_1\setminus B_{1/2^{k+1}})\times(B_1\setminus B_{1/2^{k+1}})\\ \{|x|\leqslant|y|\}}} \frac{|x-y|^{2-n-2s}}{|x|^{2\gamma+2}}\,dx\,dy \\
&\quad\leqslant C\rho^{2\gamma}\sum_{i,j=0}^{k} \iint_{\substack{(B_{1/2^i}\setminus B_{1/2^{i+1}})\times(B_{1/2^j}\setminus B_{1/2^{j+1}})\\ \{|x|\leqslant|y|\}}} \frac{|x-y|^{2-n-2s}}{|x|^{2\gamma+2}}\,dx\,dy.
\end{aligned} \tag{5.31}$$

We also observe that when $x \notin B_{1/2^{i+1}}$, $y \in B_{1/2^j}$ and $|x| \leqslant |y|$, we have

$$\frac{1}{2^{i+1}} \leqslant |x| \leqslant |y| \leqslant \frac{1}{2^j},$$

and accordingly $j \leqslant i+1$. This implies that

$$\begin{aligned}
&\iint_{\substack{(B_1\setminus B_\rho)\times(B_1\setminus B_\rho)\\ \{|x|\leqslant|y|\}}} \frac{|x-y|^{2-n-2s}}{|x|^{2\gamma+2}}\,dx\,dy \\
&\quad\leqslant \sum_{i=0}^{k}\sum_{j=0}^{i+1} \iint_{\substack{(B_{1/2^i}\setminus B_{1/2^{i+1}})\times(B_{1/2^j}\setminus B_{1/2^{j+1}})\\ \{|x|\geqslant|y|\}}} \frac{|x-y|^{2-n-2s}}{|x|^{2\gamma+2}}\,dx\,dy \\
&\quad\leqslant \sum_{i=0}^{k}\sum_{j=0}^{i+1} \iint_{(B_{1/2^i}\setminus B_{1/2^{i+1}})\times(B_{1/2^j}\setminus B_{1/2^{j+1}})} 2^{(2\gamma+2)(i+1)}|x-y|^{2-n-2s}\,dx\,dy \\
&\quad\leqslant \sum_{i=0}^{k}\sum_{j=0}^{i+1} \int_{B_{1/2^i}\setminus B_{1/2^{i+1}}} 2^{(2\gamma+2)(i+1)}\,dx \int_{B_{\frac{1}{2^i}+\frac{1}{2^j}}} |z|^{2-n-2s}\,dz \\
&\quad\leqslant C\sum_{i=0}^{k}\sum_{j=0}^{i+1} 2^{-ni+(2\gamma+2)i+(2s-2)j} \\
&\quad\leqslant C\sum_{i=0}^{k} 2^{(2\gamma+2-n)i} \\
&\quad\leqslant \begin{cases} C & \text{if } \gamma < \frac{n-2}{2}, \\ Ck & \text{if } \gamma = \frac{n-2}{2}, \\ C\,2^{(2\gamma+2-n)k} & \text{if } \gamma > \frac{n-2}{2}, \end{cases} \leqslant \begin{cases} C & \text{if } \gamma < \frac{n-2}{2}, \\ C|\log\rho| & \text{if } \gamma = \frac{n-2}{2}, \\ C\rho^{n-2\gamma-2} & \text{if } \gamma > \frac{n-2}{2}, \end{cases}
\end{aligned}$$

up to renaming $C>0$, where we used (5.30).

Plugging this information into (5.31), we obtain (5.29).

Putting together (5.21), (5.22), (5.23), (5.25), (5.27), (5.28) and (5.29), we obtain the desired result. ∎

We now estimate the weighted L^2-norm of the auxiliary function φ:

Lemma 5.6. *Let $n \geqslant 3$ and φ be as in (5.11). Then*

$$\lim_{\rho \searrow 0} \overline{m} \int_D \varphi^2 \, dx - \underline{m} \int_{\Omega \setminus D} \varphi^2 \, dx = -\frac{\underline{m}(\underline{m} + m_0)}{m_0} |\Omega| > 0.$$

Proof. Recalling (5.12), (5.13) and (5.1), we see that

$$\begin{aligned} \overline{m} \int_D \varphi^2 \, dx &= \overline{m}(c_\star + 1)^2 |D| = \overline{m}\Big(-\frac{\underline{m} + m_0}{m_0} + 1\Big)^2 \frac{\underline{m} + m_0}{\underline{m} + \overline{m}} |\Omega| \\ &= \frac{\overline{m}\underline{m}^2(\underline{m} + m_0)}{m_0^2(\underline{m} + \overline{m})} |\Omega|. \end{aligned} \tag{5.32}$$

Moreover, we observe that

$$\frac{\rho^\gamma}{1 - \rho^\gamma} \frac{1}{|x|^\gamma} \chi_{B_1 \setminus B_\rho} \leqslant \frac{1}{1 - \rho^\gamma} \leqslant 2, \tag{5.33}$$

as long as ρ is small enough. As a consequence, recalling (5.11) and (5.13), and using the dominated convergence theorem, we find that

$$\begin{aligned} \lim_{\rho \searrow 0} \underline{m} \int_{\Omega \setminus D} \varphi^2 \, dx &= \lim_{\rho \searrow 0} \underline{m} \int_{\Omega \setminus B_1} c_\star^2 \, dx + \underline{m} \int_{B_1 \setminus B_\rho} \Big[c_\star + \frac{\rho^\gamma}{1 - \rho^\gamma}\Big(\frac{1}{|x|^\gamma} - 1\Big)\Big]^2 dx \\ &= \underline{m} c_\star^2 |\Omega \setminus B_1| + \underline{m} c_\star^2 |B_1| \\ &= \underline{m} c_\star^2 |\Omega| = \underline{m}\Big(\frac{\underline{m} + m_0}{m_0}\Big)^2 |\Omega|. \end{aligned}$$

From this and (5.32), and recalling (5.14), we conclude that

$$\begin{aligned} \lim_{\rho \searrow 0} \overline{m} \int_D \varphi^2 \, dx - \underline{m} \int_{\Omega \setminus D} \varphi^2 \, dx &= \lim_{\overline{m} \nearrow +\infty} \frac{\overline{m}\underline{m}^2(\underline{m} + m_0)}{m_0^2(\underline{m} + \overline{m})} |\Omega| - \underline{m}\Big(\frac{\underline{m} + m_0}{m_0}\Big)^2 |\Omega| \\ &= \frac{\underline{m}^2(\underline{m} + m_0)}{m_0^2} |\Omega| - \underline{m}\Big(\frac{\underline{m} + m_0}{m_0}\Big)^2 |\Omega| \\ &= \frac{\underline{m}(\underline{m} + m_0)}{m_0^2} [\underline{m} - (\underline{m} + m_0)] |\Omega| \\ &= -\frac{\underline{m}(\underline{m} + m_0)}{m_0} |\Omega|, \end{aligned}$$

which is positive, since $m_0 \in (-\underline{m}, 0)$, as desired. ∎

We are now in position to give the proof of Theorem 1.7 for $n \geqslant 3$.

Proof of Theorem 1.7 when $n \geqslant 3$. The strategy of the proof is to use the auxiliary function φ as defined in (5.11) and the resource $m := \overline{m}\chi_D - \underline{m}\chi_{\Omega\setminus D}$, with D as in (5.13), as a competitor in the minimization problem (1.17). Indeed, in this way we find that

$$\underline{\lambda}(\overline{m}, \underline{m}, m_0) \leqslant \frac{\alpha \int_\Omega |\nabla\varphi|^2\,dx + \beta \iint_{\mathcal{Q}} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy}{\overline{m}\int_D |\varphi|^2\,dx - \underline{m}\int_{\Omega\setminus D}|\varphi|^2\,dx}.$$

Moreover, Lemmata 5.3 and 5.5 give

$$\lim_{\rho\searrow 0} \alpha \int_\Omega |\nabla\varphi|^2\,dx + \beta \iint_{\mathcal{Q}} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = 0.$$

This, combined with (5.14) and Lemma 5.6, gives the desired result. ∎

Now we deal with the proof of Theorem 1.8. The main strategy is similar to that of the proof of Theorem 1.7, but in this setting we introduce a different auxiliary function (and this of course impacts the technical computations needed to obtain the desired results). Namely, we define

$$\psi(x) := \begin{cases} c_\sharp - 1 & \text{if } x \in B_\rho, \\ c_\sharp - \dfrac{\rho^\gamma}{1-\rho^\gamma}\left(\dfrac{1}{|x|^\gamma} - 1\right) & \text{if } x \in B_1 \setminus B_\rho, \\ c_\sharp & \text{if } x \in \mathbb{R}^n \setminus B_1, \end{cases} \tag{5.34}$$

where

$$c_\sharp := \frac{m_0 - \overline{m}}{m_0}. \tag{5.35}$$

We point out that $c_\sharp > 0$, since $m_0 < 0 < \overline{m}$. We also set

$$D := \Omega \setminus B_\rho. \tag{5.36}$$

We remark that, in this setting, since $m \in \widetilde{\mathcal{M}}$, recalling (5.1),

$$|\Omega| - |B_\rho| = |\Omega \setminus B_\rho| = |D| = \frac{\underline{m} + m_0}{\underline{m} + \overline{m}}|\Omega|.$$

This says that

$$\text{sending } \underline{m} \nearrow +\infty \text{ is equivalent to sending } \rho \searrow 0, \tag{5.37}$$

$\overline{m}$, m_0 and $|\Omega|$ being fixed quantities in this argument. The reader may compare the setting in (5.13) and (5.14) with the one in (5.36) and (5.37) to appreciate the structural difference between the two frameworks.

Now we list some useful properties of the auxiliary function ψ. Noticing that the function ψ in (5.34) differs by a constant from the function $-\varphi$ in (5.11), we obtain the following two results directly from Lemmata 5.3 and 5.5:

Lemma 5.7. *Let $n \geqslant 3$ and ψ be as in (5.34). Then*

$$\lim_{\rho \searrow 0} \int_\Omega |\nabla \psi|^2 \, dx = 0.$$

Lemma 5.8. *Let $n \geqslant 3$ and ψ be as in (5.34). Then*

$$\lim_{\rho \searrow 0} \iint_{\mathcal{Q}} \frac{|\psi(x) - \psi(y)|^2}{|x-y|^{n+2s}} \, dx \, dy = 0.$$

We now deal with the weighted L^2-norm of the auxiliary function ψ:

Lemma 5.9. *Let $n \geqslant 3$ and ψ be as in (5.34). Then*

$$\lim_{\rho \searrow 0} \overline{m} \int_D \psi^2 \, dx - \underline{m} \int_{\Omega \setminus D} \psi^2 \, dx = -\frac{\overline{m}(\overline{m} - m_0)}{m_0} |\Omega| > 0.$$

Proof. Recalling (5.34) and (5.36), we find that

$$\overline{m} \int_D \psi^2 \, dx = \overline{m} \int_{\Omega \setminus B_1} c_\sharp^2 \, dx + \overline{m} \int_{B_1 \setminus B_\rho} \left[c_\sharp - \frac{\rho^\gamma}{1 - \rho^\gamma} \left(\frac{1}{|x|^\gamma} - 1 \right) \right]^2 dx.$$

Hence, recalling (5.33) and using the dominated convergence theorem and (5.35), we deduce that

$$\lim_{\rho \searrow 0} \overline{m} \int_D \psi^2 \, dx = \overline{m} c_\sharp^2 |\Omega \setminus B_1| + \overline{m} c_\sharp^2 |B_1| = \overline{m} c_\sharp^2 |\Omega| = \overline{m} \left(\frac{m_0 - \overline{m}}{m_0} \right)^2 |\Omega|. \quad (5.38)$$

Moreover, recalling (5.36), (5.1) and (5.35),

$$\begin{aligned} \underline{m} \int_{\Omega \setminus D} \psi^2 \, dx &= \underline{m} (c_\sharp - 1)^2 |\Omega \setminus D| \\ &= \underline{m} \left(\frac{m_0 - \overline{m}}{m_0} - 1 \right)^2 \frac{\overline{m} - m_0}{\underline{m} + \overline{m}} |\Omega| \\ &= \frac{\underline{m}\, \overline{m}^2 (\overline{m} - m_0)}{m_0^2 (\underline{m} + \overline{m})} |\Omega|. \end{aligned}$$

As a consequence of this and (5.38), and recalling (5.37), we have

$$\begin{aligned} \lim_{\rho \searrow 0} \overline{m} \int_D \psi^2 \, dx - \underline{m} \int_{\Omega \setminus D} \psi^2 \, dx &= \overline{m} \left(\frac{m_0 - \overline{m}}{m_0} \right)^2 |\Omega| - \lim_{\underline{m} \nearrow +\infty} \frac{\underline{m}\, \overline{m}^2 (\overline{m} - m_0)}{m_0^2 (\underline{m} + \overline{m})} |\Omega| \\ &= \overline{m} \left(\frac{m_0 - \overline{m}}{m_0} \right)^2 |\Omega| - \frac{\overline{m}^2 (\overline{m} - m_0)}{m_0^2} |\Omega| \\ &= \frac{\overline{m}(\overline{m} - m_0)}{m_0^2} [(\overline{m} - m_0) - \overline{m}] |\Omega| \\ &= -\frac{\overline{m}(\overline{m} - m_0)}{m_0} |\Omega|, \end{aligned}$$

which is positive, since $m_0 < 0 < \overline{m}$. ∎

Now we are ready to give the proof of Theorem 1.8 for $n \geqslant 3$.

Proof of Theorem 1.8. The strategy of the proof is to use the auxiliary function ψ as defined in (5.34) and the resource $m := \overline{m}\chi_D - \underline{m}\chi_{\Omega\setminus D}$, with D as in (5.36), as a competitor in the minimization problem (1.17). Indeed, in this way we find that

$$\underline{\lambda}(\overline{m}, \underline{m}, m_0) \leqslant \frac{\alpha \int_\Omega |\nabla\psi|^2\,dx + \beta \iint_{\mathcal{Q}} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy}{\overline{m}\int_D |\psi|^2\,dx - \underline{m}\int_{\Omega\setminus D} |\psi|^2\,dx}.$$

Moreover, from Lemmata 5.7 and 5.8 we have

$$\lim_{\rho\searrow 0} \alpha \int_\Omega |\nabla\psi|^2\,dx + \beta \iint_{\mathcal{Q}} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = 0.$$

This, combined with (5.37) and Lemma 5.9, implies the desired result. ∎

Having completed the cases $n \geqslant 3$ and deferred the case $n = 2$ to Appendix A, we now focus on the case $n = 1$, by providing the proofs of Theorems 1.9 and 1.10.

For this, when $n = 1$ we first establish the following lower bound for $\underline{\lambda}$ (as defined in (1.17)):

Lemma 5.10. *Let $n = 1$ and $\alpha > 0$. Then*

$$\underline{\lambda}(\overline{m}, \underline{m}, m_0) \geqslant -\frac{C m_0^3(\underline{m}+\overline{m})^4}{\overline{m}\underline{m}^3(\overline{m}-m_0)^2(\overline{m}(2\underline{m}+m_0)-\underline{m}m_0)}, \tag{5.39}$$

for some $C = C(\alpha, \beta, \Omega) > 0$.

Proof. Without loss of generality, we can set $\alpha = 2$. We take an arbitrary resource m in the set $\widetilde{\mathcal{M}}$ defined in (5.2). Moreover, we denote by e an eigenfunction associated to the first eigenvalue of problem (1.13), that is,

$$\lambda_1(m) = \frac{\int_\Omega |e'|^2\,dx + \frac{\beta}{4}\iint_{\mathcal{Q}} \frac{|e(x)-e(y)|^2}{|x-y|^{n+2s}}\,dx\,dy}{\overline{m}\int_D e^2\,dx - \underline{m}\int_{\Omega\setminus D} e^2\,dx}. \tag{5.40}$$

In light of Proposition 4.2 here and [42, Corollary 1.4], up to a sign change, we know that e is nonnegative and bounded, and therefore we set

$$a := \inf_\Omega e \quad \text{and} \quad b := \sup_\Omega e.$$

By construction, we have $a \in [0, b]$, and we can also normalize e such that $b = 1$; in this way

$$e(x) \leqslant 1 \quad \text{for each } x \in \Omega. \tag{5.41}$$

We also take $x_k, y_k \in \Omega$ such that

$$e(x_k) \to a \quad \text{and} \quad e(y_k) \to 1$$

as $k \nearrow +\infty$.

We observe that

if there exist $\bar{x}$ and $\bar{y}$ such that $|e(\bar{x}) - e(\bar{y})| \geqslant \frac{1-a}{10}$ which belong to the same connected component of Ω, then

$$(1-a)^2 \leqslant C \int_{\Omega} |e'|^2 \, dx \quad \text{for some } C > 0. \tag{5.42}$$

Indeed, for $\bar{x}$ and $\bar{y}$ as in the assumption of (5.42) we have

$$(e(\bar{y}) - e(\bar{x}))^2 = \left(\int_{\bar{x}}^{\bar{y}} e'(t) \, dt \right)^2 \leqslant C \int_{\Omega} |e'(t)|^2 \, dt$$

for some positive C. Accordingly, we obtain the desired result in (5.42).

Now we claim that

$$(1-a)^4 \leqslant C \left(\int_{\Omega} |e'|^2 \, dx + \beta \iint_{\mathcal{Q}} \frac{|e(x) - e(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \right) \tag{5.43}$$

for a suitable $C > 1$.

To prove this claim, we need to consider different possibilities according to the possible lack of connectedness of Ω. For this, we first remark that, with no loss of generality, we can suppose that

$$a < 1, \tag{5.44}$$

otherwise (5.43) is obviously satisfied.

Furthermore, Ω being a bounded set with C^1 boundary, necessarily it can have at most a finite number of connected components (otherwise, there would be accumulating components, violating the assumption in (1.1)). Hence, if Ω is not connected, we can define d_0 to be the smallest distance between the different connected components of Ω. We also let d_1 be the diameter of Ω and d_2 the smallest diameter of all the connected components of Ω (of course, d_0, d_1 and d_2 are structural constants, and the other constants are allowed to depend on them, but we will write d_0, d_1 and d_2 explicitly in the forthcoming computations whenever we need to emphasize their roles). To prove (5.43), we distinguish two cases: the first case is when

Ω has one connected component, or it has more than one connected component, with

$$\sup_{\substack{x,y \in \Omega \\ |e(x)-e(y)| \geqslant \frac{1-a}{10}}} \frac{|e(x) - e(y)|}{|x-y|} \geqslant \frac{4}{d_0}, \tag{5.45}$$

and the second case is when

Ω has more than one connected component, with

$$\sup_{\substack{x,y \in \Omega \\ |e(x)-e(y)| \geqslant \frac{1-a}{10}}} \frac{|e(x) - e(y)|}{|x-y|} < \frac{4}{d_0}. \tag{5.46}$$

Let us first discuss case (5.45). If Ω has one connected component, then we can exploit (5.42) with $\bar{x} := x_k$ and $\bar{y} := y_k$ with k sufficiently large, and the claim in (5.43) plainly follows. Thus, to complete the study of (5.45), we suppose that Ω is not connected and, in the setting of (5.45), we find $\bar{x}, \bar{y} \in \Omega$ with

$$|e(\bar{x}) - e(\bar{y})| \geqslant \frac{1-a}{10} \quad \text{and} \quad \frac{|e(\bar{x}) - e(\bar{y})|}{|\bar{x} - \bar{y}|} \geqslant \frac{3}{d_0}. \tag{5.47}$$

In this framework, we have

$$\bar{x} \text{ and } \bar{y} \text{ belong to the same connected component.} \tag{5.48}$$

Indeed, if not, we would have $|\bar{x} - \bar{y}| \geqslant d_0$, and thus, by (5.41),

$$\frac{|e(\bar{x}) - e(\bar{y})|}{|\bar{x} - \bar{y}|} \leqslant \frac{|e(\bar{x})| + |e(\bar{y})|}{d_0} \leqslant \frac{2}{d_0},$$

which is in contradiction with (5.47), thus proving (5.48).

Then, by (5.48), we can exploit (5.42), from which one deduces (5.43) in this case.

Having completed the analysis of case (5.45), we now focus on the setting provided by case (5.46) and we define

$$r := \frac{1}{100(1 + \frac{4}{d_0})} \min\{1 - a, d_2\}. \tag{5.49}$$

We observe that, $r > 0$, due to (5.44), and, if $\vartheta \in \Omega$ with $|\vartheta - x_k| \leqslant r$, then

$$|e(\vartheta) - e(x_k)| \leqslant \frac{1-a}{10}. \tag{5.50}$$

Indeed, suppose not. Then the assumption in (5.46) guarantees that

$$\frac{4}{d_0} \geqslant \frac{|e(\vartheta) - e(x_k)|}{|\vartheta - x_k|} \geqslant \frac{|e(\vartheta) - e(x_k)|}{r},$$

and therefore

$$|e(\vartheta) - e(x_k)| \leqslant \frac{4r}{d_0} \leqslant \frac{\frac{4}{d_0}}{100(1 + \frac{4}{d_0})}(1 - a) \leqslant \frac{1-a}{100},$$

against the contradiction assumption.

This proves (5.50) and similarly one can show that if $\tau \in \Omega$ with $|\tau - y_k| \leqslant r$, then

$$|e(\vartheta) - e(y_k)| \leqslant \frac{1-a}{10}. \tag{5.51}$$

Combining (5.50) and (5.51), we find that, for all $\vartheta, \tau \in \Omega$ with $|\vartheta - x_k| \leqslant r$ and $|\tau - y_k| \leqslant r$,

$$\begin{aligned} |e(\vartheta) - e(\tau)| &\geqslant |e(x_k) - e(y_k)| - |e(\vartheta) - e(x_k)| - |e(\tau) - e(y_k)| \\ &\geqslant \frac{1-a}{2} - \frac{1-a}{5} \geqslant \frac{1-a}{4}. \end{aligned}$$

For this reason, letting $\mathcal{S}_1 := B_r(x_k) \cap \Omega$ and $\mathcal{S}_2 := B_r(y_k) \cap \Omega$, we have

$$\begin{aligned}\iint_{\mathcal{Q}} \frac{|e(x)-e(y)|^2}{|x-y|^{n+2s}}\,dx\,dy &\geqslant \iint_{\Omega\times\Omega} \frac{|e(x)-e(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ &\geqslant \iint_{\mathcal{S}_1\times\mathcal{S}_2} \frac{|e(\vartheta)-e(\tau)|^2}{|\vartheta-\tau|^{n+2s}}\,d\vartheta\,d\tau \\ &\geqslant \iint_{\mathcal{S}_1\times\mathcal{S}_2} \frac{(1-a)^2}{16d_1^{n+2s}}\,d\vartheta\,d\tau \\ &= \frac{(1-a)^2|\mathcal{S}_1|\,|\mathcal{S}_2|}{16d_1^{n+2s}} \geqslant \frac{(1-a)^2 r^2}{64d_1^{n+2s}}. \end{aligned} \tag{5.52}$$

We also recall that in the case (5.46), we have that Ω is not connected and consequently $\beta \neq 0$, due to (1.5). From this and (5.52), up to renaming constants, we deduce that

$$\int_\Omega |e'|^2\,dx + \beta \iint_{\mathcal{Q}} \frac{|e(x)-e(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \geqslant \frac{(1-a)^2r^2}{C},$$

which, together with (5.49), proves (5.43), as desired.

We also remark that, testing the weak formulation of (1.13) against a constant function, one sees that

$$\int_\Omega me\,dx = 0,$$

and therefore, recalling (5.41),

$$\overline{m}|D| \geqslant \overline{m}\int_D e\,dx = \underline{m}\int_{\Omega\setminus D} e\,dx \geqslant a\underline{m}|\Omega\setminus D|. \tag{5.53}$$

Recalling (5.1), we see that

$$|\Omega\setminus D| = \frac{\overline{m}-m_0}{\underline{m}+\overline{m}}|\Omega|, \tag{5.54}$$

and therefore

$$\begin{aligned}\overline{m}\int_D e^2\,dx - \underline{m}\int_{\Omega\setminus D} e^2\,dx &\leqslant \overline{m}|D| - a^2\underline{m}|\Omega\setminus D| \\ &= \overline{m}\frac{\underline{m}+m_0}{\underline{m}+\overline{m}}|\Omega| - a^2\underline{m}\frac{\overline{m}-m_0}{\underline{m}+\overline{m}}|\Omega| \\ &= \frac{\overline{m}\underline{m}(1-a^2)+m_0(\overline{m}+a^2\underline{m})}{\underline{m}+\overline{m}}|\Omega| \\ &< \frac{\overline{m}\underline{m}}{\underline{m}+\overline{m}}(1-a^2)|\Omega|, \end{aligned} \tag{5.55}$$

since $m_0 < 0$ and $\overline{m}, \underline{m} > 0$. Using this inequality and (5.43) in (5.40), we conclude that

$$C\lambda_1(m) \geqslant \frac{\underline{m}+\overline{m}}{\overline{m}\underline{m}}\cdot\frac{(1-a)^4}{1-a^2} = \frac{\underline{m}+\overline{m}}{\overline{m}\underline{m}}\cdot\frac{(1-a)^3}{1+a}, \tag{5.56}$$

up to renaming $C > 0$.

Furthermore, from (5.53) we know that

$$a \leqslant \frac{\overline{m}}{\underline{m}} \cdot \frac{\underline{m} + m_0}{\overline{m} - m_0}.$$

Consequently, since the map $[0, 1] \ni t := \frac{(1-t)^3}{1+t}$ is decreasing, we discover that

$$\frac{(1-a)^3}{1+a} \geqslant \frac{(1 - \frac{\overline{m}}{\underline{m}} \cdot \frac{\underline{m}+m_0}{\overline{m}-m_0})^3}{1 + \frac{\overline{m}}{\underline{m}} \cdot \frac{\underline{m}+m_0}{\overline{m}-m_0}} = -\frac{m_0^3(\underline{m} + \overline{m})^3}{\underline{m}^2(\overline{m} - m_0)^2} \cdot \frac{1}{2\overline{m}\underline{m} + m_0(\overline{m} - \underline{m})}.$$

Combining this information and (5.56), we deduce that

$$\begin{aligned} C\lambda_1(m) &\geqslant -\frac{\underline{m} + \overline{m}}{\overline{m}\underline{m}} \cdot \frac{m_0^3(\underline{m} + \overline{m})^3}{\underline{m}^2(\overline{m} - m_0)^2} \cdot \frac{1}{2\overline{m}\underline{m} + m_0(\overline{m} - \underline{m})} \\ &= -\frac{m_0^3(\underline{m} + \overline{m})^4}{\overline{m}\underline{m}^3(\overline{m} - m_0)^2(2\overline{m}\underline{m} + m_0(\overline{m} - \underline{m}))}. \end{aligned}$$

Taking the infimum of this expression, we find the desired result. ■

With this, we are in position to give the proof of Theorem 1.9.

Proof of Theorem 1.9. For any $\underline{m} > 0$ and any $m_0 \in (-\underline{m}, 0)$, we define the function $g_{\underline{m},m_0}\colon (0, +\infty) \to (0, +\infty)$ as

$$g_{\underline{m},m_0}(\mathscr{m}) := -\frac{m_0^3(\underline{m} + \mathscr{m})^4}{\mathscr{m}\underline{m}^3(\mathscr{m} - m_0)^2(\mathscr{m}(2\underline{m} + m_0) - \underline{m}m_0)}.$$

We observe that

$$\lim_{\mathscr{m}\nearrow+\infty} g_{\underline{m},m_0}(\mathscr{m}) = -\frac{m_0^3}{\underline{m}^3(2\underline{m} + m_0)} > 0,$$

and

$$\lim_{\mathscr{m}\searrow 0} g_{\underline{m},m_0}(\mathscr{m}) = +\infty. \tag{5.57}$$

In particular,

$$\inf_{\mathscr{m}\in(0,+\infty)} g_{\underline{m},m_0}(\mathscr{m}) > 0. \tag{5.58}$$

Now, by Lemma 5.10, we know that

$$\underline{\lambda}(\mathscr{m}, \underline{m}, m_0) \geqslant C g_{\underline{m},m_0}(\mathscr{m}). \tag{5.59}$$

As a result, (1.19) follows from (5.58) and (5.59). Moreover, from (5.57) and (5.59) we obtain (1.20).

To prove (1.21), for any $\overline{m} > 0$ and $m_0 < 0$, we define the function $\tilde{g}_{\overline{m},m_0}\colon (-m_0, +\infty) \to (0, +\infty)$ as

$$\tilde{g}_{\overline{m},m_0}(\mathscr{m}) := -\frac{m_0^3(\mathscr{m} + \overline{m})^4}{\overline{m}\mathscr{m}^3(\overline{m} - m_0)^2(\overline{m}(2\mathscr{m} + m_0) - \mathscr{m}m_0)}.$$

We notice that

$$\lim_{\mathscr{m}\searrow -m_0} \tilde{g}_{\overline{m},m_0}(\mathscr{m}) = -\frac{\overline{m}-m_0}{\overline{m}m_0} > 0$$

and that

$$\lim_{\mathscr{m}\nearrow +\infty} \tilde{g}_{\overline{m},m_0}(\mathscr{m}) = -\frac{m_0^3}{\overline{m}(\overline{m}-m_0)^2(2\overline{m}-m_0)} > 0.$$

Accordingly,

$$\inf_{\mathscr{m}\in(-m_0,+\infty)} \tilde{g}_{\overline{m},m_0}(\mathscr{m}) > 0.$$

Using this and the fact that, by Lemma 5.10,

$$\underline{\lambda}(\overline{m},\mathscr{m},m_0) \geqslant C\,\tilde{g}_{\overline{m},m_0}(\mathscr{m}),$$

we obtain the desired result in (1.21). ∎

Having established Theorem 1.9, we now deal with the case in which $\alpha = 0$, namely when only the nonlocal dispersal is active. This case is considered in Theorem 1.10, according to two different ranges of the fractional parameter s. For this, we divide the proof of Theorem 1.10 into two parts.

Proof of Theorem 1.10 *when* $s \in (1/2, 1)$. We denote by e the eigenfunction associated to the first eigenvalue of problem (1.13), normalized such that

$$a := \inf_{\Omega} e \geqslant 0 \quad \text{and} \quad \sup_{\Omega} e = 1. \tag{5.60}$$

We recall (5.53), (5.54) and (5.55) to write

$$a \leqslant \frac{\overline{m}}{\underline{m}} \cdot \frac{\underline{m}+m_0}{\overline{m}-m_0} \tag{5.61}$$

and

$$\overline{m}\int_D e^2\,dx - \underline{m}\int_{\Omega\setminus D} e^2\,dx \leqslant \frac{\overline{m}\underline{m}}{\underline{m}+\overline{m}}(1-a^2)|\Omega|. \tag{5.62}$$

We stress that, in view of (5.61),

$$\delta_0 := 1 - a \geqslant 1 - \frac{\overline{m}}{\underline{m}} \cdot \frac{\underline{m}+m_0}{\overline{m}-m_0} = -\frac{m_0(\underline{m}+\overline{m})}{\underline{m}(\overline{m}-m_0)} > 0. \tag{5.63}$$

In particular, by (5.60), we can find $\bar{x}$ and $\bar{y}$ in Ω such that

$$e(\bar{x}) \leqslant a + \frac{\delta_0}{100} \quad \text{and} \quad e(\bar{y}) \geqslant 1 - \frac{\delta_0}{100}. \tag{5.64}$$

Now we claim that

$$\iint_{\mathcal{Q}} \frac{(e(x)-e(y))^2}{|x-y|^{1+2s}}\,dx\,dy \geqslant c(1-a)^{\frac{4s+2}{2s-1}} \tag{5.65}$$

for some $c \in (0, 1)$ depending only on s and Ω (in particular, this c is independent of m). Indeed, if the left-hand side of (5.65) is larger than 1, we are done; therefore we can suppose, without loss of generality, that

$$\iint_{\mathcal{Q}} \frac{(e(x) - e(y))^2}{|x - y|^{1+2s}}\, dx\, dy \leqslant 1.$$

As a result,

$$\iint_{\Omega\times\Omega} \frac{(e(x) - e(y))^2}{|x - y|^{1+2s}}\, dx\, dy + \|e\|^2_{L^2(\Omega)} \leqslant 1 + |\Omega|,$$

and consequently we can exploit [41, Theorem 8.2] and conclude that $\|e\|_{C^{\frac{2s-1}{2}}(\Omega)} \leqslant C_0$, for some $C_0 > 0$ depending only on s and Ω.

We let d_1 be the diameter of Ω and d_2 be the smallest diameter of all the connected components of Ω. We also define

$$r_0 := \min\left\{\left(\frac{\delta_0}{100C_0}\right)^{\frac{2}{2s-1}}, \frac{d_2}{100}\right\}, \tag{5.66}$$

and we observe that, for each $x \in \Omega \cap B_{r_0}(\bar{x})$,

$$\begin{aligned} e(x) &\leqslant e(\bar{x}) + |e(x) - e(\bar{x})| \leqslant e(\bar{x}) + C_0|x - \bar{x}|^{\frac{2s-1}{2}} \\ &\leqslant a + \frac{\delta_0}{100} + C_0 r_0^{\frac{2s-1}{2}} \leqslant a + \frac{\delta_0}{50}, \end{aligned}$$

thanks to (5.64).

Similarly, for each $y \in \Omega \cap B_{r_0}(\bar{y})$,

$$e(y) \geqslant 1 - \frac{\delta_0}{50},$$

and consequently

$$\begin{aligned} &\iint_{(\Omega\cap B_{r_0}(\bar{x}))\times(\Omega\cap B_{r_0}(\bar{y}))} \frac{(e(x) - e(y))^2}{|x - y|^{1+2s}}\, dx\, dy \\ &\geqslant \left(1 - a - \frac{\delta_0}{25}\right)^2 \iint_{(\Omega\cap B_{r_0}(\bar{x}))\times(\Omega\cap B_{r_0}(\bar{y}))} \frac{dx\, dy}{|x - y|^{1+2s}} \\ &\geqslant \left(1 - a - \frac{\delta_0}{25}\right)^2 \frac{|\Omega \cap B_{r_0}(\bar{x})|\, |\Omega \cap B_{r_0}(\bar{y})|}{d_1^{1+2s}} \\ &\geqslant \frac{(1 - a)^2}{4d_1^{2+2s}} |\Omega \cap B_{r_0}(\bar{x})|\, |\Omega \cap B_{r_0}(\bar{y})| \\ &\geqslant \frac{(1 - a)^2 r_0^2}{4d_1^{2+2s}}. \end{aligned}$$

From this and (5.66), noticing that $\frac{4s+2}{2s-1} > 2$, we obtain (5.65), as desired.

Gathering (5.65) and (5.62) we find that

$$\lambda_1(m) = \frac{\frac{\beta}{4}\iint_{\mathcal{Q}} \frac{|e(x)-e(y)|^2}{|x-y|^{n+2s}}\,dx\,dy}{\bar{m}\int_D e^2\,dx - \underline{m}\int_{\Omega\setminus D} e^2\,dx} \geqslant \frac{\underline{m}+\bar{m}}{\bar{m}\underline{m}}\cdot\frac{C(1-a)^{\frac{4s+2}{2s-1}}}{1-a^2} \tag{5.67}$$

for some $C > 0$.

We also notice that, in view of (5.63),

$$\begin{aligned}
\frac{(1-a)^{\frac{4s+2}{2s-1}}}{1-a^2} &= \frac{(1-a)^{\frac{4s+2}{2s-1}}}{(1-a)(1+a)}\\
&= \frac{(1-a)^{\frac{2s+3}{2s-1}}}{1+a}\\
&\geqslant \frac{1}{2}(1-a)^{\frac{2s+3}{2s-1}}\\
&\geqslant \frac{1}{2}\Big(-\frac{m_0(\underline{m}+\bar{m})}{\underline{m}(\bar{m}-m_0)}\Big)^{\frac{2s+3}{2s-1}}.
\end{aligned}$$

By inserting this inequality into (5.67), we conclude that

$$\lambda_1(m) \geqslant \frac{C(\underline{m}+\bar{m})}{\bar{m}\underline{m}}\cdot\Big(-\frac{m_0(\underline{m}+\bar{m})}{\underline{m}(\bar{m}-m_0)}\Big)^{\frac{2s+3}{2s-1}}, \tag{5.68}$$

up to renaming C.

Now, for any $m_0 < 0$ and any $\underline{m} > -m_0$, we define the function $\bar{g}_{\underline{m},m_0}\colon (0,+\infty) \to (0,+\infty)$ given by

$$\bar{g}_{\underline{m},m_0}(\mathscr{m}) := \frac{\underline{m}+\mathscr{m}}{\mathscr{m}\underline{m}}\cdot\Big(-\frac{m_0(\underline{m}+\mathscr{m})}{\underline{m}(\mathscr{m}-m_0)}\Big)^{\frac{2s+3}{2s-1}}.$$

We remark that

$$\lim_{\mathscr{m}\searrow 0}\bar{g}_{\underline{m},m_0}(\mathscr{m}) = +\infty \quad\text{and}\quad \lim_{\mathscr{m}\nearrow+\infty}\bar{g}_{\underline{m},m_0}(\mathscr{m}) = \frac{1}{\underline{m}}\cdot\Big(-\frac{m_0}{\underline{m}}\Big)^{\frac{2s+3}{2s-1}} > 0,$$

and consequently

$$\inf_{\mathscr{m}\in(0,+\infty)}\bar{g}_{\underline{m},m_0}(\mathscr{m}) > 0. \tag{5.69}$$

Also, by (5.68), and making use of (5.2) and Proposition 5.2, we find that

$$\underline{\lambda}(\mathscr{m},\underline{m},m_0) = \inf_{m\in\widetilde{\mathcal{M}}(\mathscr{m},\underline{m},m_0)}\lambda_1(m) \geqslant C\,\bar{g}_{\underline{m},m_0}(\mathscr{m}).$$

In particular,

$$\underline{\lambda}(\mathscr{m},\underline{m},m_0) \geqslant C\inf_{\mathscr{m}\in(0,+\infty)}\bar{g}_{\underline{m},m_0}(\mathscr{m}),$$

which, combined with (5.69), proves (1.22).

Similarly, we see that

$$\lim_{\mathfrak{m}\searrow 0} \underline{\lambda}(\mathfrak{m}, \underline{m}, m_0) \geqslant C \lim_{\mathfrak{m}\searrow 0} \bar{g}_{\underline{m},m_0}(\mathfrak{m}) = +\infty,$$

which establishes (1.23).

In addition, given $m_0 < 0$ and $\overline{m} > 0$, if we consider the auxiliary function $g^{\star}_{\overline{m},m_0}$: $(-m_0, +\infty) \to (0, +\infty)$ defined by

$$g^{\star}_{\overline{m},m_0}(\mathfrak{m}) := \frac{\mathfrak{m} + \overline{m}}{\overline{m}\mathfrak{m}} \cdot \left(-\frac{m_0(\mathfrak{m} + \overline{m})}{\mathfrak{m}(\overline{m} - m_0)}\right)^{\frac{2s+3}{2s-1}},$$

we see that

$$\lim_{\mathfrak{m}\searrow -m_0} g^{\star}_{\overline{m},m_0}(\mathfrak{m}) = -\frac{-m_0 + \overline{m}}{\overline{m}m_0} > 0$$

and

$$\lim_{\mathfrak{m}\nearrow +\infty} g^{\star}_{\overline{m},m_0}(\mathfrak{m}) = \frac{1}{\overline{m}} \cdot \left(-\frac{m_0}{\overline{m}}\right)^{\frac{2s+3}{2s-1}} > 0,$$

and these observations allow us to conclude that

$$\inf_{\mathfrak{m}\in(-m_0,+\infty)} g^{\star}_{\overline{m},m_0}(\mathfrak{m}) > 0. \tag{5.70}$$

Moreover, we deduce from Proposition 5.2, (5.2) and (5.68) that

$$\underline{\lambda}(\overline{m}, \mathfrak{m}, m_0) = \inf_{m\in\widetilde{\mathcal{M}}(\overline{m},\mathfrak{m},m_0)} \lambda_1(m) \geqslant C g^{\star}_{\overline{m},m_0}(\mathfrak{m}).$$

Therefore,

$$\underline{\lambda}(\overline{m}, \mathfrak{m}, m_0) \geqslant C \inf_{\mathfrak{m}\in(-m_0,+\infty)} g^{\star}_{\overline{m},m_0}(\mathfrak{m}),$$

which, together with (5.70), proves (1.24). ∎

Now we prove Theorem 1.10 in the case $s \in (0, 1/2]$. This case is somehow conceptually related to the case $n \geqslant 2$, since the problem boils down to a subcritical situation.

We suppose without loss of generality that

$$(-2, 2) = B_2 \subset \Omega, \tag{5.71}$$

and we define the function

$$\varphi(x) := \begin{cases} c_\star + 1 & \text{if } x \in B_\rho, \\ c_\star + \dfrac{\log|x|}{\log \rho} & \text{if } x \in B_1 \setminus B_\rho, \\ c_\star & \text{if } x \in \mathbb{R} \setminus B_1. \end{cases} \tag{5.72}$$

Here, $c_\star > 0$ is the constant introduced in (5.12), and we set

$$D := B_\rho. \tag{5.73}$$

For our purposes, we recall the following basic inequality:

Lemma 5.11. *For every* $x, y \in \mathbb{R}^n \setminus \{0\}$ *we have*

$$\big|\log|x| - \log|y|\big| \leqslant \frac{\big||x| - |y|\big|}{\min\{|x|, |y|\}}. \tag{5.74}$$

Proof. Without loss of generality, we assume that $|y| \leqslant |x|$. To check (5.74), we take $t := \frac{|x|}{|y|} - 1$, and we see that

$$\big|\log|x| - \log|y|\big| = \log|x| - \log|y| = \log\frac{|x|}{|y|} = \log(1+t) \leqslant t = \frac{|x|}{|y|} - 1 = \frac{|x| - |y|}{|y|},$$

as desired. ∎

With this, we now list some properties of the auxiliary function φ in (5.72).

Lemma 5.12. *Let* $n = 1$, $s \in (0, 1/2]$ *and* φ *be as in* (5.72). *Then*

$$\lim_{\rho\searrow 0} \iint_{\mathcal{Q}} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{1+2s}}\,dx\,dy = 0.$$

Proof. Without loss of generality, we suppose that $\rho < 1/4$. We observe that

$$\iint_{B_\rho \times B_\rho} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{1+2s}}\,dx\,dy = 0 \tag{5.75}$$

and

$$\iint_{(\mathbb{R}^n\setminus B_1)\times(\mathbb{R}^n\setminus B_1)} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{1+2s}}\,dx\,dy = 0. \tag{5.76}$$

Moreover,

$$\begin{aligned}
\lim_{\rho\searrow 0} \iint_{B_\rho\times(\mathbb{R}^n\setminus B_1)} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{1+2s}}\,dx\,dy &= \lim_{\rho\searrow 0} \iint_{B_\rho\times(\mathbb{R}^n\setminus B_1)} \frac{dx\,dy}{|x-y|^{1+2s}} \\
&\leqslant \lim_{\rho\searrow 0} \int_{B_\rho} dx \int_{\mathbb{R}^n\setminus B_{\frac12}} \frac{1}{|z|^{1+2s}}\,dz \\
&\leqslant \lim_{\rho\searrow 0} C\rho = 0. \qquad (5.77)
\end{aligned}$$

Now we observe that if $x \in B_1 \setminus B_\rho$ and $y \in B_\rho$, then

$$|\varphi(x) - \varphi(y)|^2 = \frac{1}{(\log\rho)^2}\big|\log|x| - \log\rho\big|^2.$$

As a consequence, changing variables $X := x/\rho$ and $Y := y/\rho$, and taking $k \in \mathbb{N}$ such that $2^{k-1} \leqslant 1/\rho \leqslant 2^k$, we see that

$$\begin{aligned}
&\iint_{(B_1\setminus B_{2\rho})\times B_\rho} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{1+2s}}\,dx\,dy \\
&\quad = \frac{1}{(\log\rho)^2} \iint_{(B_1\setminus B_{2\rho})\times B_\rho} \frac{\big|\log|x| - \log\rho\big|^2}{|x-y|^{1+2s}}\,dx\,dy
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\rho^{1-2s}}{(\log \rho)^2} \iint_{(B_{1/\rho}\setminus B_2)\times B_1} \frac{\big|\log|X|\big|^2}{|X-Y|^{1+2s}}\,dX\,dY \\
&\leqslant \frac{\rho^{1-2s}}{(\log \rho)^2} \sum_{j=2}^{k} \iint_{(B_{2^j}\setminus B_{2^{j-1}})\times B_1} \frac{|\log(2^j)|^2}{(2^{j-1}-1)^{1+2s}}\,dX\,dY \\
&\leqslant \frac{C\rho^{1-2s}}{(\log \rho)^2} \sum_{j=2}^{k} \frac{2^j j^2}{2^{j(1+2s)}} \\
&\leqslant \frac{C\rho^{1-2s}}{(\log \rho)^2} \sum_{j=1}^{k} \frac{j^2}{2^{2sj}} \\
&\leqslant \frac{C\rho^{1-2s}}{(\log \rho)^2},
\end{aligned}
\tag{5.78}
$$

up to renaming $C > 0$.

In addition, using (5.74) (with $|y| := \rho$) and changing variable $z := x - y$,

$$
\begin{aligned}
\iint_{(B_{2\rho}\setminus B_\rho)\times B_\rho} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{1+2s}}\,dx\,dy &= \frac{1}{(\log \rho)^2} \iint_{(B_{2\rho}\setminus B_\rho)\times B_\rho} \frac{\big|\log|x| - \log\rho\big|^2}{|x-y|^{1+2s}}\,dx\,dy \\
&\leqslant \frac{1}{(\log \rho)^2} \iint_{(B_{2\rho}\setminus B_\rho)\times B_\rho} \frac{(|x|-\rho)^2}{\rho^2|x-y|^{1+2s}}\,dx\,dy \\
&\leqslant \frac{1}{(\log \rho)^2} \iint_{(B_{2\rho}\setminus B_\rho)\times B_\rho} \frac{|x-y|^{1-2s}}{\rho^2}\,dx\,dy \\
&\leqslant \frac{1}{(\log \rho)^2} \iint_{B_{3\rho}\times B_\rho} \frac{|z|^{1-2s}}{\rho^2}\,dz\,dy \\
&= \frac{C\rho^{1-2s}}{(\log \rho)^2},
\end{aligned}
$$

for some $C > 0$.

From this and (5.78), we deduce that

$$
\lim_{\rho\searrow 0} \iint_{(B_1\setminus B_\rho)\times B_\rho} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{1+2s}}\,dx\,dy = 0. \tag{5.79}
$$

Moreover, recalling (5.71),

$$
\begin{aligned}
&\iint_{(B_1\setminus B_\rho)\times(\mathbb{R}^n\setminus\Omega)} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{1+2s}}\,dx\,dy \\
&= \frac{1}{(\log \rho)^2} \iint_{(B_1\setminus B_\rho)\times(\mathbb{R}^n\setminus\Omega)} \frac{\big|\log|x|\big|^2}{|x-y|^{1+2s}}\,dx\,dy \\
&\leqslant \frac{1}{(\log \rho)^2} \int_{B_1\setminus B_\rho} \big|\log|x|\big|^2\,dx \int_{\mathbb{R}^n\setminus B_1} \frac{dz}{|z|^{1+2s}} \leqslant \frac{C}{(\log \rho)^2}
\end{aligned}
$$

for some $C > 0$. This implies

$$\lim_{\rho \searrow 0} \iint_{(B_1 \setminus B_\rho) \times (\mathbb{R}^n \setminus \Omega)} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{1+2s}} \, dx \, dy = 0. \tag{5.80}$$

Now we observe that

$$\begin{aligned}
&\iint_{(B_{1/2} \setminus B_\rho) \times (\Omega \setminus B_1)} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{1+2s}} \, dx \, dy \\
&\quad = \frac{1}{(\log \rho)^2} \iint_{(B_{1/2} \setminus B_\rho) \times (\Omega \setminus B_1)} \frac{\big|\log|x|\big|^2}{|x-y|^{1+2s}} \, dx \, dy \\
&\quad \leqslant \frac{1}{(\log \rho)^2} \int_{B_{1/2} \setminus B_\rho} \big|\log|x|\big|^2 \, dx \int_{\Omega \setminus B_1} \frac{1}{|z|^{1+2s}} \, dz \leqslant \frac{C}{(\log \rho)^2}
\end{aligned} \tag{5.81}$$

for a suitable $C > 0$.

Furthermore, taking $R > 0$ such that $\Omega \subset B_R$,

$$\begin{aligned}
&\iint_{(B_1 \setminus B_{1/2}) \times (\Omega \setminus B_1)} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{1+2s}} \, dx \, dy \\
&\quad = \frac{1}{(\log \rho)^2} \iint_{(B_1 \setminus B_{1/2}) \times (\Omega \setminus B_1)} \frac{\big|\log|x|\big|^2}{|x-y|^{1+2s}} \, dx \, dy \\
&\quad \leqslant \frac{4(\log 2)^2}{(\log \rho)^2} \int_{\frac12}^1 \int_1^R (y-x)^{-1-2s} \, dx \, dy \\
&\quad = \frac{2(\log 2)^2}{s(\log \rho)^2} \int_{\frac12}^1 [(1-x)^{-2s} - (R-x)^{-2s}] \, dx \\
&\quad = \frac{2(\log 2)^2}{s(1-2s)(\log \rho)^2} \Big[(R-1)^{1-2s} + \Big(\frac12\Big)^{1-2s} - \Big(R - \frac12\Big)^{1-2s} \Big].
\end{aligned}$$

This and (5.81) give

$$\lim_{\rho \searrow 0} \iint_{(B_1 \setminus B_\rho) \times (\Omega \setminus B_1)} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{1+2s}} \, dx \, dy = 0. \tag{5.82}$$

Now we take $k \in \mathbb{N}$ such that

$$\frac{1}{2^{k+1}} < \rho \leqslant \frac{1}{2^k}, \tag{5.83}$$

and we observe that

$$\begin{aligned}
&\iint_{(B_1 \setminus B_\rho) \times (B_1 \setminus B_\rho)} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{1+2s}} \, dx \, dy \\
&\quad = \frac{1}{(\log \rho)^2} \iint_{(B_1 \setminus B_\rho) \times (B_1 \setminus B_\rho)} \frac{\big|\log|x| - \log|y|\big|^2}{|x-y|^{1+2s}} \, dx \, dy
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{(\log\rho)^2} \iint_{\substack{(B_1\setminus B_\rho)\times(B_1\setminus B_\rho)\\ \{|x|\geqslant|y|\}}} \frac{\big|\log|x|-\log|y|\big|^2}{|x-y|^{1+2s}}\,dx\,dy\\
&\leqslant \frac{2}{(\log\rho)^2} \iint_{\substack{(B_1\setminus B_{1/2^{k+1}})\times(B_1\setminus B_{1/2^{k+1}})\\ \{|x|\geqslant|y|\}}} \frac{\big|\log|x|-\log|y|\big|^2}{|x-y|^{1+2s}}\,dx\,dy\\
&= \frac{2}{(\log\rho)^2} \sum_{i,j=0}^{k} \iint_{\substack{(B_{1/2^i}\setminus B_{1/2^{i+1}})\times(B_{1/2^j}\setminus B_{1/2^{j+1}})\\ \{|x|\geqslant|y|\}}} \frac{\big|\log|x|-\log|y|\big|^2}{|x-y|^{1+2s}}\,dx\,dy. \qquad (5.84)
\end{aligned}
$$

Moreover, we remark that if $x\in B_{1/2^i}$, $y\notin B_{1/2^{j+1}}$ and $|x|\geqslant|y|$, we have

$$
\frac{1}{2^i}\geqslant|x|\geqslant|y|\geqslant\frac{1}{2^{j+1}},
$$

and accordingly $i\leqslant j+1$.

This observation and (5.84) yield

$$
\begin{aligned}
&\frac{(\log\rho)^2}{2}\iint_{(B_1\setminus B_\rho)\times(B_1\setminus B_\rho)} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{1+2s}}\,dx\,dy\\
&\qquad\leqslant \sum_{j=0}^{k}\sum_{i=0}^{j+1} \iint_{\substack{(B_{1/2^i}\setminus B_{1/2^{i+1}})\times(B_{1/2^j}\setminus B_{1/2^{j+1}})\\ \{|x|\geqslant|y|\}}} \frac{\big|\log|x|-\log|y|\big|^2}{|x-y|^{1+2s}}\,dx\,dy\\
&\qquad\leqslant I_1+I_2, \qquad (5.85)
\end{aligned}
$$

where

$$
I_1 := \sum_{j=0}^{k}\sum_{j-4\leqslant i\leqslant j+1} \iint_{\substack{(B_{1/2^i}\setminus B_{1/2^{i+1}})\times(B_{1/2^j}\setminus B_{1/2^{j+1}})\\ \{|x|\geqslant|y|\}}} \frac{\big|\log|x|-\log|y|\big|^2}{|x-y|^{1+2s}}\,dx\,dy
$$

and

$$
I_2 := \sum_{j=0}^{k}\sum_{0\leqslant i\leqslant j-4} \iint_{\substack{(B_{1/2^i}\setminus B_{1/2^{i+1}})\times(B_{1/2^j}\setminus B_{1/2^{j+1}})\\ \{|x|\geqslant|y|\}}} \frac{\big|\log|x|-\log|y|\big|^2}{|x-y|^{1+2s}}\,dx\,dy.
$$

We point out that, if $x\in B_{1/2^i}$ and $|y|\leqslant|x|$, then

$$
|x-y|\leqslant|x|+|y|\leqslant 2|x|\leqslant\frac{1}{2^{i-1}}.
$$

In light of this fact and (5.74), we have

$$
\begin{aligned}
I_1 &\leqslant \sum_{j=0}^{k}\sum_{j-4\leqslant i\leqslant j+1} \iint_{\substack{(B_{1/2^i}\setminus B_{1/2^{i+1}})\times(B_{1/2^j}\setminus B_{1/2^{j+1}})\\ \{|x|\geqslant|y|\}}} \frac{|x-y|^{1-2s}}{|y|^2}\,dx\,dy\\
&\leqslant \sum_{j=0}^{k}\sum_{j-4\leqslant i\leqslant j+1} \iint_{(B_{1/2^i}\setminus B_{1/2^{i+1}})\times(B_{1/2^j}\setminus B_{1/2^{j+1}})} \frac{2^{(1-i)(1-2s)}}{2^{-2(j+1)}}\,dx\,dy
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=0}^{k} \sum_{j-4\leqslant i\leqslant j+1} \frac{2^{-i}2^{-j}2^{(1-i)(1-2s)}}{2^{-2(j+1)}} \\
&= 2^{3-2s} \sum_{j=0}^{k} \sum_{j-4\leqslant i\leqslant j+1} 2^{-2i(1-s)}2^{j} \\
&\leqslant 3\cdot 2^{4-2s} \sum_{j=0}^{k} 2^{-2(j-4)(1-s)}2^{j} \\
&= 3\cdot 2^{12-10s} \sum_{j=0}^{k} 2^{(2s-1)j}. \qquad (5.86)
\end{aligned}
$$

In addition, if $x \in B_{1/2^i}$ and $y \notin B_{1/2^{j+1}}$ and $|x| \geqslant |y|$,

$$
\big|\log|x| - \log|y|\big| = \log|x| - \log|y| \leqslant \log\frac{1}{2^i} - \log\frac{1}{2^{j+1}} = (j-i+1)\log 2.
$$

As a result,

$$
I_2 \leqslant \log^2 2 \sum_{j=0}^{k} \sum_{0\leqslant i\leqslant j-4} \iint_{(B_{1/2^i}\setminus B_{1/2^{i+1}})\times(B_{1/2^j}\setminus B_{1/2^{j+1}})} \frac{(j-i+1)^2}{|x-y|^{1+2s}}\,dx\,dy. \qquad (5.87)
$$

Furthermore, if $x \in B_{1/2^i} \setminus B_{1/2^{i+1}}$ and $y \in B_{1/2^j}$, with $i \leqslant j-4$, we see that

$$
|x-y| \geqslant |x| - |y| \geqslant \frac{1}{2^{i+1}} - \frac{1}{2^j} = \frac{1}{2^{i+1}}\Big(1 - \frac{1}{2^{j-i-1}}\Big) \geqslant \frac{1}{2^{i+2}} \geqslant \frac{|x|}{4}.
$$

Then we insert this information into (5.87) and we conclude that

$$
\begin{aligned}
I_2 &\leqslant 4^{1+2s}\log^2 2 \sum_{j=0}^{k} \sum_{0\leqslant i\leqslant j-4} \iint_{(B_{1/2^i}\setminus B_{1/2^{i+1}})\times(B_{1/2^j}\setminus B_{1/2^{j+1}})} \frac{(j-i+1)^2}{|x|^{1+2s}}\,dx\,dy \\
&\leqslant 4^{1+2s}\log^2 2 \sum_{j=0}^{k} \sum_{0\leqslant i\leqslant j-4} \iint_{(B_{1/2^i}\setminus B_{1/2^{i+1}})\times(B_{1/2^j}\setminus B_{1/2^{j+1}})} \frac{(j-i+1)^2}{2^{-(i+1)(1+2s)}}\,dx\,dy \\
&= 4^{1+2s}\log^2 2 \sum_{j=0}^{k} \sum_{0\leqslant i\leqslant j-4} \frac{2^{-i}2^{-j}(j-i+1)^2}{2^{-(i+1)(1+2s)}} \\
&= 2^{3(1+2s)}\log^2 2 \sum_{j=0}^{k} \sum_{0\leqslant i\leqslant j-4} 2^{2si}2^{-j}(j-i+1)^2 \\
&= 2^{3(1+2s)+1}\log^2 2 \sum_{j=0}^{k} \sum_{0\leqslant i\leqslant j-4} 2^{(2s-1)i}2^{i-j-1}(j-i+1)^2.
\end{aligned}
$$

Hence, changing the index of summation by posing $\ell := j - i + 1$,

$$
\begin{aligned}
I_2 &\leqslant 2^{3(1+2s)+1} \log^2 2 \sum_{i=0}^{k} \sum_{\ell \geqslant 5} 2^{(2s-1)i} 2^{-\ell} \ell^2 \\
&\leqslant \bar{C} \sum_{i=0}^{k} 2^{(2s-1)i},
\end{aligned}
$$

where

$$
\bar{C} := 2^{3(1+2s)+1} \log^2 2 \sum_{\ell=0}^{+\infty} 2^{-\ell} \ell^2.
$$

We plug this information and (5.86) into (5.85) and we find that

$$
(\log \rho)^2 \iint_{(B_1 \setminus B_\rho) \times (B_1 \setminus B_\rho)} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{1+2s}} \, dx \, dy \leqslant C_\star \sum_{m=0}^{k} 2^{(2s-1)m}, \tag{5.88}
$$

where $C_\star := 2(\bar{C} + 3 \cdot 2^{12-10s})$.

We observe that

$$
\sum_{m=0}^{k} 2^{(2s-1)m} = k \quad \text{if } s = \frac{1}{2},
$$

while

$$
\sum_{m=0}^{k} 2^{(2s-1)m} \leqslant \sum_{m=0}^{+\infty} 2^{(2s-1)m} =: C_\sharp \quad \text{if } s \in \left(0, \frac{1}{2}\right),
$$

and consequently, by (5.88),

$$
(\log \rho)^2 \iint_{(B_1 \setminus B_\rho) \times (B_1 \setminus B_\rho)} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{1+2s}} \, dx \, dy \leqslant \begin{cases} C_\star k & \text{if } s = \frac{1}{2}, \\ C_\star C_\sharp & \text{if } s \in (0, \frac{1}{2}). \end{cases}
$$

From this and (5.83), it follows that

$$
(\log \rho)^2 \iint_{(B_1 \setminus B_\rho) \times (B_1 \setminus B_\rho)} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{1+2s}} \, dx \, dy \leqslant \begin{cases} C_\star \dfrac{|\log \rho|}{\log 2} & \text{if } s = \frac{1}{2}, \\ C_\star C_\sharp & \text{if } s \in (0, \frac{1}{2}). \end{cases}
$$

This implies that

$$
\lim_{\rho \searrow 0} \iint_{(B_1 \setminus B_\rho) \times (B_1 \setminus B_\rho)} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{1+2s}} \, dx \, dy = 0.
$$

From this, (5.75), (5.76), (5.77), (5.79), (5.80) and (5.82) we obtain the desired result. ∎

An additional useful property of the function φ defined in (5.72) is the following:

Lemma 5.13. *Let $n = 1$, $s \in (0, 1/2]$ and φ be as in (5.72). Then*

$$\lim_{\rho \searrow 0} \overline{m} \int_D \varphi^2 \, dx - \underline{m} \int_{\Omega \setminus D} \varphi^2 \, dx = -\frac{\underline{m}(\underline{m} + m_0)}{m_0} |\Omega| > 0.$$

Proof. From (5.72) and (5.73), and exploiting (5.1) and (5.12), we see that

$$\overline{m} \int_D \varphi^2 \, dx = \overline{m}(c_\star + 1)^2 |D| = \overline{m}(c_\star + 1)^2 \frac{\underline{m} + m_0}{\underline{m} + \overline{m}} |\Omega| = \frac{\overline{m}\underline{m}^2(\underline{m} + m_0)}{m_0^2(\underline{m} + \overline{m})} |\Omega|$$

and that

$$\begin{aligned} \underline{m} \int_{\Omega \setminus D} \varphi^2 \, dx &= \underline{m} \int_{B_1 \setminus B_\rho} \varphi^2 \, dx + \underline{m} \int_{\Omega \setminus B_1} \varphi^2 \, dx \\ &= \underline{m} \int_{B_1 \setminus B_\rho} \left(c_\star + \frac{\log|x|}{\log \rho} \right)^2 dx + \underline{m} \int_{\Omega \setminus B_1} c_\star^2 \, dx. \end{aligned}$$

We also remark that

$$\left(\frac{\log|x|}{\log \rho} \right)^2 \chi_{B_1 \setminus B_\rho} \leqslant 1. \tag{5.89}$$

Therefore, by the dominated convergence theorem, and recalling (5.14),

$$\begin{aligned} &\lim_{\rho \searrow 0} \overline{m} \int_D \varphi^2 \, dx - \underline{m} \int_{\Omega \setminus D} \varphi^2 \, dx \\ &= \lim_{\overline{m} \nearrow +\infty} \frac{\overline{m}\underline{m}^2(\underline{m} + m_0)}{m_0^2(\underline{m} + \overline{m})} |\Omega| \\ &\quad - \lim_{\rho \searrow 0} \left(\underline{m} \int_{B_1 \setminus B_\rho} \left(c_\star + \frac{\log|x|}{\log \rho} \right)^2 dx + \underline{m} \int_{\Omega \setminus B_1} c_\star^2 \, dx \right) \\ &= \frac{\underline{m}^2(\underline{m} + m_0)}{m_0^2} |\Omega| - \underline{m} c_\star^2 |\Omega| \\ &= \frac{\underline{m}^2(\underline{m} + m_0)}{m_0^2} |\Omega| - \underline{m} \left(\frac{\underline{m} + m_0}{m_0} \right)^2 |\Omega| \\ &= \frac{\underline{m}(\underline{m} + m_0)}{m_0^2} (\underline{m} - (\underline{m} + m_0)) |\Omega| \\ &= -\frac{\underline{m}(\underline{m} + m_0)}{m_0} |\Omega|, \end{aligned}$$

which is positive, since $m_0 \in (-\underline{m}, 0)$, as desired. ∎

In the case $s \in (0, \frac{1}{2}]$, it is also convenient to introduce the function

$$\psi(x) := \begin{cases} c_\sharp - 1 & \text{if } x \in B_\rho, \\ c_\sharp - \dfrac{\log|x|}{\log \rho} & \text{if } x \in B_1 \setminus B_\rho, \\ c_\sharp & \text{if } x \in \mathbb{R}^2 \setminus B_1, \end{cases} \tag{5.90}$$

where $c_\sharp$ is defined in (5.35). We also set

$$D := \Omega \setminus B_\rho, \tag{5.91}$$

and we study the main properties of the auxiliary function ψ.

Comparing (5.72) with (5.90), we observe that $|\psi(x) - \psi(y)| = |\varphi(x) - \varphi(y)|$, and therefore, from Lemma 5.12, we obtain the following lemma:

Lemma 5.14. *Let $n = 1$, $s \in (0, 1/2]$ and ψ be as in (5.90). Then*

$$\lim_{\rho \searrow 0} \iint_{\mathcal{Q}} \frac{|\psi(x) - \psi(y)|^2}{|x - y|^{1+2s}}\, dx\, dy = 0.$$

We also have the following result:

Lemma 5.15. *Let $n = 1$, $s \in (0, 1/2]$ and ψ be as in (5.90). Then*

$$\lim_{\rho \searrow 0} \int_D \psi^2\, dx - \underline{m} \int_{\Omega\setminus D} \psi^2\, dx = -\frac{\overline{m}(\overline{m} - m_0)}{m_0}|\Omega| > 0.$$

Proof. In view of (5.90), (5.91), (5.35) and (5.1),

$$\begin{aligned}
\underline{m} \int_{\Omega\setminus D} \psi^2\, dx = \underline{m}(c_\sharp - 1)^2 |\Omega \setminus D| &= \underline{m}\Big(\frac{m_0 - \overline{m}}{m_0} - 1\Big)^2 \frac{\overline{m} - m_0}{\underline{m} + \overline{m}}|\Omega| \\
&= \frac{\underline{m}\,\overline{m}^2(\overline{m} - m_0)}{m_0^2(\underline{m} + \overline{m})}|\Omega|
\end{aligned}$$

and

$$\overline{m} \int_D \psi^2\, dx = \overline{m} \int_{B_1\setminus B_\rho} \Big(c_\sharp - \frac{\log|x|}{\log \rho}\Big)^2 dx + \overline{m} \int_{\Omega\setminus B_1} c_\sharp^2\, dx.$$

Hence, recalling (5.37) and (5.89), and using the dominated convergence theorem,

$$\begin{aligned}
&\lim_{\rho \searrow 0} \overline{m} \int_D \psi^2\, dx - \underline{m} \int_{\Omega\setminus D} \psi^2\, dx \\
&\quad = \lim_{\rho \searrow 0} \overline{m} \int_{B_1\setminus B_\rho} \Big(c_\sharp - \frac{\log|x|}{\log \rho}\Big)^2 dx \\
&\qquad + \overline{m} \int_{\Omega\setminus B_1} c_\sharp^2\, dx - \lim_{\underline{m} \nearrow +\infty} \frac{\underline{m}\,\overline{m}^2(\overline{m} - m_0)}{m_0^2(\underline{m} + \overline{m})}|\Omega| \\
&\quad = \overline{m} c_\sharp^2 |\Omega| - \frac{\overline{m}^2(\overline{m} - m_0)}{m_0^2}|\Omega| \\
&\quad = \frac{\overline{m}(m_0 - \overline{m})^2}{m_0^2}|\Omega| - \frac{\overline{m}^2(\overline{m} - m_0)}{m_0^2}|\Omega| \\
&\quad = \frac{\overline{m}(\overline{m} - m_0)}{m_0^2}((\overline{m} - m_0) - \overline{m})|\Omega| \\
&\quad = -\frac{\overline{m}(\overline{m} - m_0)}{m_0}|\Omega|,
\end{aligned}$$

which is positive since $m_0 < 0 < \overline{m}$. ∎

We are now ready to complete the proof of Theorem 1.10 in the case $s \in (0, 1/2]$.

Proof of Theorem 1.10 when $s \in (0, 1/2]$. To prove (1.25), we exploit the auxiliary function φ introduced in (5.72) and the choice of the resource $m \in \widetilde{\mathcal{M}}$, as defined in (5.2), with D as in (5.73).

In this way, in light of (1.17),

$$\underline{\lambda}(\overline{m}, \underline{m}, m_0) \leqslant \frac{\frac{\beta}{4} \iint_{\mathcal{Q}} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{1+2s}}\, dx\, dy}{\overline{m} \int_D \varphi^2\, dx - \underline{m} \int_{\Omega \setminus D} \varphi^2\, dx}.$$

Hence, taking $\mathscr{m} := \overline{m}$ and utilizing Lemmata 5.12 and 5.13, we find that

$$\lim_{\mathscr{m} \nearrow +\infty} \underline{\lambda}(\mathscr{m}, \underline{m}, m_0) \leqslant \lim_{\rho \searrow 0} \frac{\frac{\beta}{4} \iint_{\mathcal{Q}} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{1+2s}}\, dx\, dy}{\mathscr{m} \int_D \varphi^2\, dx - \underline{m} \int_{\Omega \setminus D} \varphi^2\, dx} = 0.$$

This proves (1.25), and we now focus on the proof of (1.26). To this end, we exploit the auxiliary function ψ introduced in (5.90) and the choice of the resource $m \in \widetilde{\mathcal{M}}$, as defined in (5.2), with D as in (5.91).

In this framework, in light of (1.17),

$$\underline{\lambda}(\overline{m}, \underline{m}, m_0) \leqslant \frac{\frac{\beta}{4} \iint_{\mathcal{Q}} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{1+2s}}\, dx\, dy}{\overline{m} \int_D \psi^2\, dx - \underline{m} \int_{\Omega \setminus D} \psi^2\, dx}.$$

As a result, taking $\mathscr{m} := \underline{m}$ and utilizing Lemmata 5.14 and 5.15, we conclude that

$$\lim_{\mathscr{m} \nearrow +\infty} \underline{\lambda}(\overline{m}, \mathscr{m}, m_0) \leqslant \lim_{\rho \searrow 0} \frac{\frac{\beta}{4} \iint_{\mathcal{Q}} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{1+2s}}\, dx\, dy}{\overline{m} \int_D \psi^2\, dx - \mathscr{m} \int_{\Omega \setminus D} \psi^2\, dx} = 0,$$

thus establishing (1.26). ∎

6. Badly displayed resources, hectic oscillations and proof of Theorem 1.11

This section contains the proof of Theorem 1.11, relying on an explicit example of a sequence of highly oscillating resources which make the first eigenvalue diverge. The technical details go as follows.

Proof of Theorem 1.11. We suppose that $B_4 \subset \Omega$ and we consider $\eta \in C_0^\infty(B_{3/2}, [0, 1])$ with $\eta = 1$ in B_1 and $\|\eta\|_{C^1(\mathbb{R}^n)} \leqslant 8$. We let

$$m_\omega := m_0 - \frac{\Lambda}{|\Omega|} \int_\Omega \eta(x) \sin(\omega x_1)\, dx$$

and

$$m(x) := m_\omega + \Lambda\eta(x)\sin(\omega x_1),$$

with $\omega > 0$ to be taken arbitrarily large in what follows.

We remark that

$$\frac{1}{|\Omega|}\int_\Omega m(x)\,dx = m_\omega + \frac{\Lambda}{|\Omega|}\int_\Omega \eta(x)\sin(\omega x_1)\,dx = m_0. \tag{6.1}$$

Moreover, integrating by parts,

$$\begin{aligned}|m_\omega - m_0| = \frac{\Lambda}{|\Omega|}\left|\int_\Omega \eta(x)\sin(\omega x_1)\,dx\right| &= \frac{\Lambda}{|\Omega|\omega}\left|\int_\Omega \eta(x)\frac{d}{dx_1}\cos(\omega x_1)\,dx\right| \\ &= \frac{\Lambda}{|\Omega|\omega}\left|\int_\Omega \partial_1\eta(x)\cos(\omega x_1)\,dx\right| \leqslant \frac{8\Lambda}{\omega},\end{aligned}$$

which is arbitrarily small provided that ω is large enough: in particular, we can suppose that

$$2m_0 \leqslant m_\omega \leqslant \frac{m_0}{2}. \tag{6.2}$$

Also, for every $x \in \Omega$,

$$|m(x)| \leqslant |m_\omega| + \Lambda \leqslant 2|m_0| + \Lambda \leqslant 2\Lambda. \tag{6.3}$$

Furthermore, for large ω we have

$$p^\pm := \left(\pm\frac{\pi}{2\omega}, 0, \dots, 0\right) \in B_1 \subset \Omega \cap \{\eta = 1\}.$$

Therefore,

$$\begin{aligned}&\sup_\Omega m \geqslant m(p^+) = m_\omega + \Lambda \geqslant \Lambda - 2|m_0| \geqslant \frac{\Lambda}{2},\\ &\inf_\Omega m \leqslant m(p^-) = m_\omega - \Lambda \leqslant -\Lambda.\end{aligned} \tag{6.4}$$

In view of (1.27), (6.1), (6.3) and (6.4), we obtain

$$m \in \mathcal{M}^\sharp_{\Lambda,m_0}. \tag{6.5}$$

Now we take into account a function $\varphi \in X_{\alpha,\beta}$ such that

$$\int_\Omega m(x)\varphi^2(x)\,dx = 1.$$

Then, integrating by parts, we see that

$$\begin{aligned}1 &= \int_\Omega (m_\omega + \Lambda\eta(x)\sin(\omega x_1))\varphi^2(x)\,dx\\ &= m_\omega\int_\Omega \varphi^2(x)\,dx + \Lambda\int_{\mathbb{R}^n} \eta(x)\sin(\omega x_1)\varphi^2(x)\,dx\end{aligned}$$

$$
\begin{aligned}
&= m_\omega \int_\Omega \varphi^2(x)\,dx - \frac{\Lambda}{\omega}\int_{\mathbb{R}^n} \eta(x)\frac{d}{dx_1}\cos(\omega x_1)\varphi^2(x)\,dx\\
&= m_\omega \int_\Omega \varphi^2(x)\,dx + \frac{\Lambda}{\omega}\int_{\mathbb{R}^n} \partial_1\eta(x)\cos(\omega x_1)\varphi^2(x)\,dx\\
&\quad + \frac{2\Lambda}{\omega}\int_{\mathbb{R}^n} \eta(x)\cos(\omega x_1)\varphi(x)\partial_1\varphi(x)\,dx\\
&\leqslant m_\omega \int_\Omega \varphi^2(x)\,dx + \frac{8\Lambda}{\omega}\int_{\mathbb{R}^n}\varphi^2(x)\,dx + \frac{2\Lambda}{\omega}\int_{\mathbb{R}^n}\varphi(x)|\nabla\varphi(x)|\,dx\\
&\leqslant m_\omega \int_\Omega \varphi^2(x)\,dx + \frac{9\Lambda}{\omega}\int_{\mathbb{R}^n}\varphi^2(x)\,dx + \frac{\Lambda}{\omega}\int_{\mathbb{R}^n}|\nabla\varphi(x)|^2\,dx.
\end{aligned}
$$

As a consequence, if $\frac{9\Lambda}{\omega} \leqslant -m_\omega$ (which is the case for ω large, in view of (6.2)),

$$
1 \leqslant \frac{\Lambda}{\omega}\int_{\mathbb{R}^n}|\nabla\varphi(x)|^2\,dx,
$$

and accordingly

$$
\frac{\frac{\alpha}{2}\int_\Omega |\nabla\varphi(x)|^2\,dx}{\int_\Omega m(x)\varphi^2(x)\,dx} \geqslant \frac{\alpha\omega}{2\Lambda}. \tag{6.6}
$$

Now let $\zeta \in (-1,1)$ and $E' \in \mathbb{R}^{n-1}$ with $|E'| \leqslant |\zeta|$ and $E := (\zeta, E') \in \mathbb{R}^n$. We use the trigonometric identity

$$
\frac{\cos y\cos\zeta - \cos(y+\zeta)}{\sin\zeta} = \sin y \quad \text{for all } \zeta \in \mathbb{R}\setminus(\pi\mathbb{Z}) \text{ and } y \in \mathbb{R},
$$

together with the notation $\Phi := \eta\varphi^2$ and the change of variable

$$
X := x + \frac{1}{\omega}(\zeta, E') = x + \frac{E}{\omega},
$$

to write

$$
\begin{aligned}
1 &= \int_\Omega (m_\omega + \Lambda\eta(x)\sin(\omega x_1))\varphi^2(x)\,dx\\
&= m_\omega\int_\Omega \varphi^2(x)\,dx + \Lambda\int_{\mathbb{R}^n}\sin(\omega x_1)\Phi(x)\,dx\\
&= m_\omega\int_\Omega \varphi^2(x)\,dx + \frac{\Lambda}{\sin\zeta}\int_{\mathbb{R}^n}(\cos(\omega x_1)\cos\zeta - \cos(\omega x_1+\zeta))\Phi(x)\,dx\\
&= m_\omega\int_\Omega \varphi^2(x)\,dx\\
&\quad + \frac{\Lambda}{\sin\zeta}\left[\int_{\mathbb{R}^n}\cos(\omega x_1)\cos\zeta\,\Phi(x)\,dx - \int_{\mathbb{R}^n}\cos(\omega X_1)\Phi\Big(X - \frac{E}{\omega}\Big)\,dX\right]\\
&= m_\omega\int_\Omega \varphi^2(x)\,dx\\
&\quad + \frac{\Lambda}{\sin\zeta}\int_{\mathbb{R}^n}\cos(\omega x_1)\Big[\cos\zeta\,\Phi(x) - \Phi\Big(x - \frac{E}{\omega}\Big)\Big]\,dx
\end{aligned}
$$

$$= m_\omega \int_\Omega \varphi^2(x)\,dx + \frac{\Lambda}{\sin\zeta}\int_{\mathbb{R}^n} \cos(\omega x_1)(\cos\zeta - 1)\Phi(x)\,dx$$
$$+ \frac{\Lambda}{\sin\zeta}\int_{\mathbb{R}^n} \cos(\omega x_1)\Big(\Phi(x) - \Phi\Big(x - \frac{E}{\omega}\Big)\Big)\,dx.$$

Since

$$\Big|\Phi(x) - \Phi\Big(x - \frac{E}{\omega}\Big)\Big| \leqslant \varphi^2(x)\Big|\eta(x) - \eta\Big(x - \frac{E}{\omega}\Big)\Big| + \eta\Big(x - \frac{E}{\omega}\Big)\Big|\varphi^2(x) - \varphi^2\Big(x - \frac{E}{\omega}\Big)\Big|$$
$$\leqslant \frac{8\varphi^2(x)|E|}{\omega} + \Big|\varphi^2(x) - \varphi^2\Big(x - \frac{E}{\omega}\Big)\Big|,$$

we thereby discover that, if $E \in B_1$ and $\omega \geqslant 2$,

$$\begin{aligned}
1 \leqslant{}& \Big(m_\omega + \frac{8\Lambda|E|}{\omega|\sin\zeta|}\Big)\int_\Omega \varphi^2(x)\,dx \\
&+ \frac{\Lambda}{|\sin\zeta|}\int_\Omega \Big(1 - \cos\zeta\Big)\varphi^2(x)\,dx \\
&+ \frac{\Lambda}{|\sin\zeta|}\int_{B_2} \Big|\varphi^2(x) - \varphi^2\Big(x - \frac{E}{\omega}\Big)\Big|\,dx \\
\leqslant{}& \Big(m_\omega + \frac{C\Lambda}{\omega}\Big)\int_\Omega \varphi^2(x)\,dx + C\Lambda|\zeta|\int_\Omega \varphi^2(x)\,dx \\
&+ \frac{C\Lambda}{|\zeta|}\int_{B_2} \Big|\varphi^2(x) - \varphi^2\Big(x - \frac{E}{\omega}\Big)\Big|\,dx
\end{aligned}$$

for some $C > 0$.

In particular, recalling (6.2) also, it follows that there exists $r_0 \in (0, 1)$, possibly depending on m_0, Λ and n, such that, if $\zeta \in (-r_0, r_0)$ and ω is sufficiently large,

$$1 \leqslant \frac{m_\omega}{2}\int_\Omega \varphi^2(x)\,dx + \frac{C\Lambda}{|\zeta|}\int_{B_2} \Big|\varphi^2(x) - \varphi^2\Big(x - \frac{E}{\omega}\Big)\Big|\,dx. \tag{6.7}$$

We also observe that, given an additional parameter $\kappa > 0$, to be taken conveniently small in what follows,

$$\begin{aligned}
2\Big|\varphi^2(x) - \varphi^2\Big(x - \frac{E}{\omega}\Big)\Big| ={}& 2\Big|\varphi(x) + \varphi\Big(x - \frac{E}{\omega}\Big)\Big|\,\Big|\varphi(x) - \varphi\Big(x - \frac{E}{\omega}\Big)\Big| \\
\leqslant{}& \kappa|\zeta|^{n+2s-1}\Big|\varphi(x) + \varphi\Big(x - \frac{E}{\omega}\Big)\Big|^2 \\
&+ \kappa^{-1}|\zeta|^{1-n-2s}\Big|\varphi(x) - \varphi\Big(x - \frac{E}{\omega}\Big)\Big|^2 \\
\leqslant{}& 4\kappa|\zeta|^{n+2s-1}\varphi^2(x) + 4\kappa|\zeta|^{n+2s-1}\varphi^2\Big(x - \frac{E}{\omega}\Big) \\
&+ \kappa^{-1}|\zeta|^{1-n-2s}\Big|\varphi(x) - \varphi\Big(x - \frac{E}{\omega}\Big)\Big|^2.
\end{aligned}$$

Then we plug this information into (6.7) and we conclude that, if r_0 is small enough and ω is large enough,

$$\begin{aligned} 1 \leqslant \frac{m_\omega}{2}\int_\Omega \varphi^2(x)\,dx + \frac{C\Lambda}{|\zeta|}\Big[\kappa\int_{B_2}|\zeta|^{n+2s-1}\varphi^2(x)\,dx \\ + \kappa\int_{B_2}|\zeta|^{n+2s-1}\varphi^2\Big(x-\frac{E}{\omega}\Big)\,dx \\ + \frac{1}{\kappa}\int_{B_2}|\zeta|^{1-n-2s}\Big|\varphi(x)-\varphi\Big(x-\frac{E}{\omega}\Big)\Big|^2\,dx\Big] \\ \leqslant \frac{m_\omega}{2}\int_\Omega \varphi^2(x)\,dx + \frac{C\Lambda}{|\zeta|}\Big[2\kappa|\zeta|^{n+2s-1}\int_{B_4}\varphi^2(x)\,dx \\ + \frac{1}{\kappa}\int_{B_2}|\zeta|^{1-n-2s}\Big|\varphi(x)-\varphi\Big(x-\frac{E}{\omega}\Big)\Big|^2\,dx\Big]. \end{aligned} \tag{6.8}$$

We also remark that, in our notation, $E_1 = \zeta$, and accordingly,

$$\begin{aligned} \iint_{B_4\times(B_{r_0}\cap\{|E'|\leqslant|E_1|\})}|\zeta|^{n+2s-2}\varphi^2(x)\,dx\,dE \leqslant \iint_{B_4\times B_{r_0}}|E_1|^{n+2s-2}\varphi^2(x)\,dx\,dE \\ \leqslant \widetilde{C}r_0^{2n+2s-2}\int_{B_4}\varphi^2(x)\,dx \end{aligned}$$

for some $\widetilde{C} > 0$.

For this reason, letting ι be the measure of the set $\{x = (x_1, x') \in B_1 \text{ s.t. } |x'| \leqslant |x_1|\}$, we obtain

$$\begin{aligned} \frac{m_\omega}{2}\iint_{\Omega\times(B_{r_0}\cap\{|E'|\leqslant|E_1|\})}\varphi^2(x)\,dx\,dE \\ + 2C\kappa\Lambda\iint_{B_4\times(B_{r_0}\cap\{|E'|\leqslant|E_1|\})}|\zeta|^{n+2s-2}\varphi^2(x)\,dx\,dE \\ \leqslant \frac{m_\omega\iota r_0^n}{2}\int_\Omega\varphi^2(x)\,dx + 2C\widetilde{C}\kappa\Lambda r_0^{2n+2s-2}\int_{B_4}\varphi^2(x)\,dx \leqslant 0, \end{aligned}$$

by choosing

$$\kappa := \min\Big\{1, -\frac{m_\omega\iota}{C\widetilde{C}\Lambda r_0^{n+2s-2}}\Big\}.$$

Therefore, we can integrate (6.8) over $E \in B_{r_0} \cap \{|E'| \leqslant |E_1|\}$ and up to renaming constants we find that

$$\begin{aligned} r_0^n \leqslant C\Lambda\iint_{B_2\times(B_{r_0}\cap\{|E'|\leqslant|E_1|\})}|E_1|^{-n-2s}\Big|\varphi(x)-\varphi\Big(x-\frac{E}{\omega}\Big)\Big|^2\,dx\,dE \\ \leqslant C\Lambda\iint_{B_2\times(B_{r_0}\cap\{|E'|\leqslant|E_1|\})}|E|^{-n-2s}\Big|\varphi(x)-\varphi\Big(x-\frac{E}{\omega}\Big)\Big|^2\,dx\,dE \end{aligned}$$

$$
\begin{aligned}
&= C\Lambda\omega^{-2s} \iint_{B_2\times(B_{r_0/\omega}\cap\{|z'|\leqslant|z_1|\})} |z|^{-n-2s}|\varphi(x)-\varphi(x-z)|^2\,dx\,dz \\
&\leqslant C\Lambda\omega^{-2s} \iint_{B_2\times\mathbb{R}^n} \frac{|\varphi(x)-\varphi(x-z)|^2}{|z|^{n+2s}}\,dx\,dz \\
&\leqslant C\Lambda\omega^{-2s} \iint_{\mathcal{Q}} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy,
\end{aligned}
$$

and consequently,

$$
\frac{\frac{\beta}{4}\iint_{\mathcal{Q}} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy}{\int_\Omega m(x)\varphi^2(x)\,dx} \geqslant \frac{\beta r_0^n\omega^{2s}}{C\Lambda},
$$

up to renaming $C > 0$.

This and (6.6), recalling (6.5), give

$$
\lambda_1(m) \geqslant \frac{\alpha\omega}{2\Lambda} + \frac{\beta r_0^n\omega^{2s}}{C\Lambda},
$$

which, taking ω as large as we wish, yields the desired result. ■

A. Proofs of Theorems 1.7 and 1.8 when $n = 2$

The main strategy followed in this part is similar to the case $n \geqslant 3$, but when $n = 2$ we have to define different auxiliary functions. We start with the proof of Theorem 1.7. For this, we recall the setting in (5.10) and (5.12), and we define

$$
\varphi(x) := \begin{cases} c_\star + 1 & \text{if } x \in B_\rho, \\ c_\star + \dfrac{\log|x|}{\log\rho} & \text{if } x \in B_1 \setminus B_\rho, \\ c_\star & \text{if } x \in \mathbb{R}^2 \setminus B_1. \end{cases} \tag{A.1}
$$

We set

$$
D := B_\rho \tag{A.2}
$$

and we list below some interesting properties of φ:

Lemma A.1. *Let $n = 2$ and φ be as in (A.1). Then*

$$
\lim_{\rho\searrow 0}\int_\Omega |\nabla\varphi|^2 = 0.
$$

Proof. We compute

$$
\frac{1}{(\log\rho)^2}\int_{B_1\setminus B_\rho} \frac{1}{|x|^2}\,dx = \frac{2\pi}{(\log\rho)^2}\int_\rho^1 \frac{1}{r}\,dr = -\frac{2\pi}{\log\rho} \to 0
$$

as $\rho \searrow 0$. ■

Lemma A.2. *Let $n = 2$ and φ be as in (A.1). Then*

$$\lim_{\rho \searrow 0} \iint_{\mathcal{Q}} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{2+2s}} \, dx \, dy = 0.$$

As for Lemma 5.5, we provide here a proof based on direct calculations. A proof based on interpolation theory will be provided in Appendix B.

Proof of Lemma A.2. As for the proof of Lemma 5.5, we have to consider several integral contributions (given the different expressions of the competitors, the technical computations here are different from those in Lemma 5.5). First of all, we have

$$\iint_{B_\rho \times B_\rho} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{2+2s}} \, dx \, dy = 0$$

and

$$\iint_{(\mathbb{R}^2 \setminus B_1) \times (\mathbb{R}^2 \setminus B_1)} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{2+2s}} \, dx \, dy = 0.$$

Moreover, assuming $\rho < 1/4$,

$$\begin{aligned} \iint_{B_\rho \times (\mathbb{R}^2 \setminus B_1)} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{2+2s}} \, dx \, dy &= \iint_{B_\rho \times (\mathbb{R}^2 \setminus B_1)} \frac{1}{|x - y|^{2+2s}} \, dx \, dy \\ &\leqslant \int_{B_\rho} dx \int_{\mathbb{R}^2 \setminus B_{\frac{1}{2}}} \frac{1}{|z|^{2+2s}} \, dz \leqslant C |B_1| \rho^2 \to 0 \end{aligned}$$

as $\rho \searrow 0$.

Furthermore, if $(x, y) \in (B_1 \setminus B_\rho) \times B_\rho$ we have

$$|\varphi(x) - \varphi(y)|^2 = \frac{1}{(\log \rho)^2} \big| \log|x| - \log \rho \big|^2.$$

Consequently, from (5.74) (used here with $|y| = \rho$),

$$\begin{aligned} \iint_{(B_1 \setminus B_\rho) \times B_\rho} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{2+2s}} \, dx \, dy &= \frac{1}{(\log \rho)^2} \iint_{(B_1 \setminus B_\rho) \times B_\rho} \frac{\big| \log|x| - \log \rho \big|^2}{|x - y|^{2+2s}} \, dx \, dy \\ &\leqslant \frac{1}{\rho^2 (\log \rho)^2} \iint_{(B_1 \setminus B_\rho) \times B_\rho} \frac{(|x| - \rho)^2}{|x - y|^{2+2s}} \, dx \, dy \\ &\leqslant \frac{1}{\rho^2 (\log \rho)^2} \iint_{(B_1 \setminus B_\rho) \times B_\rho} |x - y|^{-2s} \, dx \, dy \\ &\leqslant \frac{1}{\rho^2 (\log \rho)^2} \int_{B_\rho} dy \int_{B_2} |z|^{-2s} \, dz \\ &\leqslant \frac{C}{(\log \rho)^2} \to 0 \end{aligned}$$

as $\rho \searrow 0$.

Also, exploiting (5.10), we have

$$\begin{aligned}
&\iint_{(B_1\setminus B_\rho)\times(\mathbb{R}^2\setminus\Omega)} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{2+2s}}\,dx\,dy \\
&\qquad = \frac{1}{(\log\rho)^2}\iint_{(B_1\setminus B_\rho)\times(\mathbb{R}^2\setminus\Omega)} \frac{(\log|x|)^2}{|x-y|^{2+2s}}\,dx\,dy \\
&\qquad \leqslant \frac{1}{(\log\rho)^2}\int_{B_1\setminus B_\rho}(\log|x|)^2\,dx\int_{\mathbb{R}^2\setminus B_1}\frac{1}{|z|^{2+2s}}\,dz \\
&\qquad \leqslant \frac{C}{(\log\rho)^2}\int_{B_1\setminus B_\rho}(\log|x|)^2\,dx \\
&\qquad \leqslant \frac{C}{(\log\rho)^2}\to 0
\end{aligned}$$

as $\rho\searrow 0$.

Now from (5.74), used here with $|y|=1$, we have

$$\begin{aligned}
&\iint_{(B_1\setminus B_\rho)\times(\Omega\setminus B_1)} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{2+2s}}\,dx\,dy \\
&\qquad = \frac{1}{(\log\rho)^2}\iint_{(B_1\setminus B_\rho)\times(\Omega\setminus B_1)} \frac{(\log|x|)^2}{|x-y|^{2+2s}}\,dx\,dy \\
&\qquad \leqslant \frac{1}{(\log\rho)^2}\iint_{(B_1\setminus B_\rho)\times(\Omega\setminus B_1)} \frac{(1-|x|)^2}{|x|^2|x-y|^{2+2s}}\,dx\,dy \\
&\qquad \leqslant \frac{1}{(\log\rho)^2}\int_{B_1\setminus B_\rho}\frac{1}{|x|^2}\,dx\int_{B_{R+1}}|z|^{-2s}\,dz \\
&\qquad \leqslant \frac{C}{(\log\rho)^2}\int_{B_1\setminus B_\rho}\frac{1}{|x|^2}\,dx = -\frac{C}{\log\rho}\to 0
\end{aligned}$$

as $\rho\searrow 0$, where we took $R>2$ sufficiently large such that $\Omega\subset B_R$.

In addition, utilizing (5.74) again, we first notice that

$$\begin{aligned}
&\iint_{(B_1\setminus B_\rho)\times(B_1\setminus B_\rho)} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{2+2s}}\,dx\,dy \\
&\qquad = \frac{1}{(\log\rho)^2}\iint_{(B_1\setminus B_\rho)\times(B_1\setminus B_\rho)} \frac{(\log|x|-\log|y|)^2}{|x-y|^{2+2s}}\,dx\,dy \\
&\qquad = \frac{2}{(\log\rho)^2}\iint_{\substack{(B_1\setminus B_\rho)\times(B_1\setminus B_\rho)\\ |x|\leqslant|y|}} \frac{(\log|x|-\log|y|)^2}{|x-y|^{2+2s}}\,dx\,dy. \\
&\qquad \leqslant \frac{2}{(\log\rho)^2}\iint_{\substack{(B_1\setminus B_\rho)\times(B_1\setminus B_\rho)\\ |x|\leqslant|y|}} \frac{|x-y|^2}{|x|^2|x-y|^{2+2s}}\,dx\,dy
\end{aligned}$$

$$\leqslant \frac{2}{(\log\rho)^2}\int_{B_1\setminus B_\rho}\frac{1}{|x|^2}\,dx\int_{B_2}|z|^{-2s}\,dz$$
$$\leqslant \frac{C}{(\log\rho)^2}\int_{B_1\setminus B_\rho}\frac{1}{|x|^2}\,dx = -\frac{C}{\log\rho}\to 0$$

as $\rho \searrow 0$, which concludes the proof. ∎

Lemma A.3. *Let $n = 2$ and φ be as in (A.1). Then*

$$\lim_{\rho\searrow 0}\overline{m}\int_D \varphi^2\,dx - \underline{m}\int_{\Omega\setminus D}\varphi^2\,dx = -\frac{\underline{m}(\underline{m}+m_0)|\Omega|}{m_0} > 0.$$

Proof. By (A.1) and (A.2), and recalling (5.1), we have

$$\overline{m}\int_D \varphi^2\,dx = \overline{m}(c_\star+1)^2|B_\rho| = \overline{m}(c_\star+1)^2\frac{\underline{m}+m_0}{\underline{m}+\overline{m}}|\Omega|.$$

Hence, in light of (5.12) and (5.14),

$$\lim_{\rho\searrow 0}\overline{m}\int_D \varphi^2\,dx = \lim_{\overline{m}\nearrow+\infty}\overline{m}\Big(-\frac{\underline{m}+m_0}{m_0}+1\Big)^2\frac{\underline{m}+m_0}{\underline{m}+\overline{m}}|\Omega|$$
$$= \Big(-\frac{\underline{m}+m_0}{m_0}+1\Big)^2(\underline{m}+m_0)|\Omega|.$$

Moreover, by (5.12) and the dominated convergence theorem,

$$\underline{m}\int_{\Omega\setminus D}\varphi^2\,dx = \underline{m}\int_{B_1\setminus B_\rho}\Big(c_\star+\frac{\log|x|}{\log\rho}\Big)^2dx + \underline{m}c_\star^2|\Omega\setminus B_1|$$
$$\to \underline{m}\int_{B_1}\Big(-\frac{\underline{m}+m_0}{m_0}\Big)^2dx + \underline{m}\Big(-\frac{\underline{m}+m_0}{m_0}\Big)^2|\Omega\setminus B_1|$$
$$= \underline{m}\Big(-\frac{\underline{m}+m_0}{m_0}\Big)^2|\Omega|$$

as $\rho\searrow 0$.

As a result,

$$\lim_{\rho\searrow 0}\overline{m}\int_D \varphi^2\,dx - \underline{m}\int_{\Omega\setminus D}\varphi^2\,dx$$
$$= \Big[\Big(-\frac{\underline{m}+m_0}{m_0}+1\Big)^2(\underline{m}+m_0) - \underline{m}\Big(-\frac{\underline{m}+m_0}{m_0}\Big)^2\Big]|\Omega|$$
$$= [\underline{m}^2(\underline{m}+m_0) - (\underline{m}+m_0)^2\underline{m}]\frac{|\Omega|}{m_0^2}$$
$$= [\underline{m}-(\underline{m}+m_0)]\frac{\underline{m}(\underline{m}+m_0)|\Omega|}{m_0^2}$$
$$= -\frac{\underline{m}(\underline{m}+m_0)|\Omega|}{m_0},$$

which is positive, since $m_0 \in (-\underline{m}, 0)$. ∎

With this preliminary work, we can complete the proof of Theorem 1.7 in dimension $n = 2$, by arguing as follows:

Proof of Theorem 1.7 when $n = 2$. We use the function φ in (A.1) and the resource $m := \overline{m}\chi_D - \underline{m}\chi_{\Omega\setminus D}$, with D as in (A.2), as competitors in the minimization problem in (1.17). In this way, we find that

$$\underline{\lambda}(\overline{m}, \underline{m}, m_0) \leqslant \frac{\alpha \int_\Omega |\nabla\varphi|^2\, dx + \beta \iint_{\mathcal{Q}} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{2+2s}}\, dx\, dy}{\overline{m}\int_D \varphi^2\, dx - \underline{m}\int_{\Omega\setminus D}\varphi^2\, dx}. \tag{A.3}$$

From Lemmata A.1 and A.2 we have

$$\lim_{\rho\searrow 0} \alpha \int_\Omega |\nabla\varphi|^2\, dx + \beta \iint_{\mathcal{Q}} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{2+2s}}\, dx\, dy = 0.$$

Combining this with Lemma A.3 and (5.14), we obtain the desired result in Theorem 1.7. ∎

We now focus on the proof of Theorem 1.8 when $n = 2$. For this, we introduce the function

$$\psi(x) := \begin{cases} c_\sharp - 1 & \text{if } x \in B_\rho, \\ c_\sharp - \dfrac{\log|x|}{\log\rho} & \text{if } x \in B_1 \setminus B_\rho, \\ c_\sharp & \text{if } x \in \mathbb{R}^2 \setminus B_1, \end{cases} \tag{A.4}$$

where $c_\sharp$ is the constant introduced in (5.35). Moreover, we set

$$D := \Omega \setminus B_\rho. \tag{A.5}$$

The proofs of the next two results follow directly from Lemmata A.1 and A.2, since, comparing (A.1) and (A.4),

$$|\nabla\varphi|^2 = |\nabla\psi|^2 \quad \text{and} \quad |\varphi(x)-\varphi(y)|^2 = |\psi(x)-\psi(y)|^2.$$

Lemma A.4. *Let $n = 2$ and ψ be as in (A.4). Then*

$$\lim_{\rho\searrow 0} \int_\Omega |\nabla\psi|^2 = 0.$$

Lemma A.5. *Let $n = 2$ and ψ be as in (A.4). Then*

$$\lim_{\rho\searrow 0} \iint_{\mathcal{Q}} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{2+2s}}\, dx\, dy = 0.$$

Lemma A.6. *Let $n = 2$ and ψ be as in (A.4). Then*

$$\lim_{\rho\searrow 0} \overline{m}\int_D \psi^2\, dx - \underline{m}\int_{\Omega\setminus D}\psi^2\, dx = \frac{\overline{m}(m_0-\overline{m})}{m_0}|\Omega| > 0.$$

Proof. By (A.4), (A.5) and (5.35), we see that

$$\underline{m}\int_{\Omega\setminus D}\psi^2\,dx=\underline{m}(c_\sharp-1)^2|\Omega\setminus D|=\underline{m}\Big(\frac{m_0-\overline{m}}{m_0}-1\Big)^2|\Omega\setminus D|. \tag{A.6}$$

Also, recalling (5.1), we see that

$$|\Omega\setminus D|=\frac{\overline{m}-m_0}{\underline{m}+\overline{m}}|\Omega|.$$

Plugging this information into (A.6), we conclude that

$$\underline{m}\int_{\Omega\setminus D}\psi^2\,dx=\underline{m}\Big(\frac{m_0-\overline{m}}{m_0}-1\Big)^2\frac{\overline{m}-m_0}{\underline{m}+\overline{m}}|\Omega|=\frac{\underline{m}\overline{m}^2(\overline{m}-m_0)}{m_0^2(\underline{m}+\overline{m})}|\Omega|. \tag{A.7}$$

In addition, by the dominated convergence theorem and (5.37),

$$\begin{aligned}\lim_{\rho\searrow 0}\int_D\psi^2\,dx&=\lim_{\rho\searrow 0}\int_{B_1\setminus B_\rho}\Big(c_\sharp-\frac{\log|x|}{\log\rho}\Big)^2\,dx+\int_{\Omega\setminus B_1}c_\sharp^2\,dx\\&=c_\sharp^2|B_1|+c_\sharp^2|\Omega\setminus B_1|\\&=c_\sharp^2|\Omega|\\&=\frac{(m_0-\overline{m})^2}{m_0^2}|\Omega|.\end{aligned}$$

This and (A.7) give

$$\begin{aligned}&\lim_{\rho\searrow 0}\overline{m}\int_D\psi^2\,dx-\underline{m}\int_{\Omega\setminus D}\psi^2\,dx\\&=\frac{\overline{m}(m_0-\overline{m})^2}{m_0^2}|\Omega|-\lim_{\underline{m}\nearrow+\infty}\frac{\underline{m}\overline{m}^2(\overline{m}-m_0)}{m_0^2(\underline{m}+\overline{m})}|\Omega|\\&=\frac{\overline{m}(m_0-\overline{m})^2}{m_0^2}|\Omega|-\frac{\overline{m}^2(\overline{m}-m_0)}{m_0^2}|\Omega|\\&=\frac{\overline{m}(\overline{m}-m_0)}{m_0^2}((\overline{m}-m_0)-\overline{m})|\Omega|\\&=-\frac{\overline{m}(\overline{m}-m_0)}{m_0}|\Omega|,\end{aligned}$$

which is positive since $m_0<0<\overline{m}$. ■

With this preliminary work, we are in the position of completing the proof of Theorem 1.8 in the case $n=2$.

Proof of Theorem 1.8 *when* $n=2$. We use the function ψ in (A.4) and the resource $m:=\overline{m}\chi_D-\underline{m}\chi_{\Omega\setminus D}$, with D as in (A.5), as a competitor in the minimization problem in (1.17), thus obtaining

$$\underline{\lambda}(\overline{m},\underline{m},m_0)\leqslant\frac{\alpha\int_\Omega|\nabla\psi|^2\,dx+\beta\iint_{\mathcal{Q}}\frac{|\psi(x)-\psi(y)|^2}{|x-y|^{2+2s}}\,dx\,dy}{\overline{m}\int_D\psi^2\,dx-\underline{m}\int_{\Omega\setminus D}\psi^2\,dx}. \tag{A.8}$$

From Lemmata A.4 and A.5, and recalling (5.37), we have

$$\lim_{\underline{m}\nearrow+\infty} \alpha \int_{\Omega} |\nabla\psi|^2\,dx + \beta \iint_{\mathcal{Q}} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{2+2s}}\,dx\,dy = 0.$$

This, together with Lemma A.6, gives the desired result. ∎

B. Another proof of Lemmata 5.5 and A.2 based on interpolation theory

We give here a different proof of Lemmata 5.5 and A.2, relying on the following argument of interpolation type. Given $f \in C_0^\infty(\mathbb{R}^n)$, we use the Hölder inequality with exponents $\frac{1}{s}$ and $\frac{1}{1-s}$ to see that

$$\begin{aligned}\int_{\mathbb{R}^n} |\xi|^{2s}|\hat{f}(\xi)|^2\,d\xi &= \int_{\mathbb{R}^n} (|\xi|\,|\hat{f}(\xi)|)^{2s}|\hat{f}(\xi)|^{2(1-s)}\,d\xi\\ c &\leqslant \left(\int_{\mathbb{R}^n} |\xi|^2|\hat{f}(\xi)|^2\,d\xi\right)^s \left(\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2\,d\xi\right)^{1-s}.\end{aligned}$$

Here, as is customary, we have utilized the notation $\hat{f}$ to denote the Fourier transform of f. Thus, from the latter equation and Plancherel's theorem we arrive at

$$\int_{\mathbb{R}^n} |\xi|^{2s}|\hat{f}(\xi)|^2\,d\xi \leqslant C_1\left(\int_{\mathbb{R}^n} |\nabla f(x)|^2\,dx\right)^s \left(\int_{\mathbb{R}^n} |f(x)|^2\,dx\right)^{1-s}$$

for some $C_1 > 0$.

As a result, using the equivalence of various fractional norms (see e.g. [41, Proposition 3.4]), we find that

$$\iint_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \leqslant C_2\left(\int_{\mathbb{R}^n} |\nabla f(x)|^2\,dx\right)^s \left(\int_{\mathbb{R}^n} |f(x)|^2\,dx\right)^{1-s}$$

for some $C_2 > 0$.

By density, this inequality holds true for all $f \in H^1(\mathbb{R}^n)$ and thus, choosing $f := \varphi - c_\star$,

$$\begin{aligned}&\iint_{\mathcal{Q}} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\ &\quad= \iint_{\mathcal{Q}} \frac{|(\varphi-c_\star)(x)-(\varphi-c_\star)(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\ &\quad= \iint_{\mathbb{R}^n} \frac{|(\varphi-c_\star)(x)-(\varphi-c_\star)(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\end{aligned}$$

$$\leqslant C_2\Big(\int_{\mathbb{R}^n}|\nabla(\varphi-c_\star)(x)|^2\,dx\Big)^s\Big(\int_{\mathbb{R}^n}|(\varphi-c_\star)(x)|^2\,dx\Big)^{1-s}$$
$$= C_2\Big(\int_{\Omega}|\nabla\varphi(x)|^2\,dx\Big)^s\Big(\int_{\Omega}|(\varphi-c_\star)(x)|^2\,dx\Big)^{1-s}.$$

Consequently, since, by either (5.11) or (A.1),

$$|(\varphi-c_\star)(x)|\leqslant 1,$$

we conclude that

$$\iint_{\mathcal{Q}}\frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\leqslant C_3\Big(\int_{\Omega}|\nabla\varphi(x)|^2\,dx\Big)^s.$$

From this and Lemma 5.3 we obtain the desired claim in Lemma 5.5. Instead, using Lemma A.1, one obtains the claim in Lemma A.2.

C. Probabilistic motivations for the superposition of elliptic operators with different orders

The goal of this appendix is to provide a natural framework in which sums of local/nonlocal operators naturally arise. Though the argument provided can be extended to more general superpositions of operators, for the sake of concreteness we limit ourselves to the operator in (1.9).

For this, extending a presentation in [68], we consider a discrete stochastic process on the lattice $h\mathbb{Z}^n$, with time increment τ. The space scale $h>0$ and the time step $\tau>0$ will be conveniently chosen to be infinitesimal in what follows.

We denote by $\mathcal{B}_0:=\{e_1,\dots,e_n\}$ the standard Euclidean basis of $\mathbb{R}^n$, we let $\mathcal{B}:=\mathcal{B}_0\cup(-\mathcal{B}_0)=\{e_1,\dots,e_n,-e_1,\dots,-e_n\}$ and we suppose that a particle moves on $h\mathbb{Z}^n$ and, given $p\in[0,1]$, $\lambda\in\mathbb{N}$ and $s\in(0,1)$, its probability of jumping from a point hk to $h\tilde{k}$ (with $k,\tilde{k}\in\mathbb{Z}^n$) is given by

$$\mathcal{P}(k,\tilde{k}):=\frac{p}{c|k-\tilde{k}|^{n+2s}}+\frac{(1-p)\mathcal{U}(k-\tilde{k})}{2n}, \tag{C.1}$$

where

$$c:=\sum_{j\in\mathbb{Z}^n\setminus\{0\}}\frac{1}{|j|^{n+2s}}$$

and

$$\mathcal{U}(j):=\begin{cases}1 & \text{if } j\in\lambda\mathcal{B}=\{\lambda e_1,\dots,\lambda e_n,-\lambda e_1,\dots,-\lambda e_n\},\\ 0 & \text{otherwise.}\end{cases}$$

We point out that $\mathscr{P}(\tilde{k},k) = \mathscr{P}(k,\tilde{k}) = \mathscr{P}(k-\tilde{k},0) = \mathscr{P}(\tilde{k}-k,0)$, and that

$$\sum_{j\in\mathbb{Z}^n\setminus\{0\}} \mathscr{P}(j,0) = \sum_{j\in\mathbb{Z}^n\setminus\{0\}} \Big(\frac{p}{c|j|^{n+2s}} + \frac{(1-p)\mathcal{U}(j)}{2n}\Big) = p + \sum_{j\in\lambda\mathcal{B}} \frac{(1-p)}{2n} = p + (1-p) = 1. \tag{C.2}$$

The heuristic interpretation of the probability described in (C.1) is that, at any time step, the particle has a probability p of following a jump process, and a probability $1-p$ of following a classical random walk. Indeed, with probability p, the particle experiences a jump governed by the power law $\frac{1}{c|j|^{n+2s}}$, while with probability $1-p$ it walks to one of the closest neighbors scaled by the additional parameter λ (all closest neighbors being equally probable, and the probability of the particle of not moving at all being equal to zero).

Therefore, given $x \in h\mathbb{Z}^n$ and $t \in \tau\mathbb{N}$, we define $u(x,t)$ to be the probability density of the particle being at point x at time t, and we write that the probability of being somewhere, say at x, at the subsequent time step is equal to the superposition of the probabilities of being at another point of the lattice, say $x+hj$, at the previous time step times the probability of going from $x+hj$ to x, namely, letting $k := \frac{x}{h} \in \mathbb{Z}^n$,

$$u(x,t+\tau) = \sum_{j\in\mathbb{Z}^n\setminus\{0\}} u(x+hj,t)\mathscr{P}(k,k+j) = \sum_{j\in\mathbb{Z}^n\setminus\{0\}} u(x+hj,t)\mathscr{P}(j,0).$$

As a result, in view of (C.2),

$$\begin{aligned}
&u(x,t+\tau) - u(x,t)\\
&\quad= \sum_{j\in\mathbb{Z}^n\setminus\{0\}} (u(x+hj,t)-u(x,t))\mathscr{P}(j,0)\\
&\quad= \sum_{j\in\mathbb{Z}^n\setminus\{0\}} (u(x+hj,t)-u(x,t))\Big(\frac{p}{c|j|^{n+2s}} + \frac{(1-p)\mathcal{U}(j)}{2n}\Big)\\
&\quad= \frac{p}{c}\sum_{j\in\mathbb{Z}^n\setminus\{0\}} \frac{u(x+hj,t)-u(x,t)}{|j|^{n+2s}} + \frac{1-p}{2n}\sum_{j\in\lambda\mathcal{B}} (u(x+hj,t)-u(x,t))\\
&\quad= \frac{p}{2c}\sum_{j\in\mathbb{Z}^n\setminus\{0\}} \frac{u(x+hj,t)+u(x-hj,t)-2u(x,t)}{|j|^{n+2s}}\\
&\qquad+ \frac{1-p}{4n}\sum_{j\in\lambda\mathcal{B}} (u(x+hj,t)+u(x-hj,t)-2u(x,t)).
\end{aligned} \tag{C.3}$$

Now we consider two specific situations, namely the one in which

$$\tau := h^{2s}, \quad \lambda := h^{s-1} \in \mathbb{N} \tag{C.4}$$

and p is independent of the time step, and the one in which

$$\tau = h^2, \quad p := \alpha h^{2-2s} \quad \text{and} \quad \lambda := 1, \tag{C.5}$$

for a given $\alpha > 0$, independent of the time step.

We observe that the case in (C.4) corresponds to having the closest neighborhood walk scaled by a suitably large factor (for small h), while the case in (C.5) corresponds to having the usual notion of closest neighborhood random walk, with the probability $1 - p$ that the particle follows it being large (for small h).

In case (C.4), we consider $N \in \mathbb{N}$ and define $h := N^{\frac{1}{s-1}}$. In this way, taking $N \nearrow +\infty$, one has that $h \searrow 0$, and thus we deduce from (C.3) that

$$\begin{aligned}\frac{u(x,t+\tau) - u(x,t)}{\tau} \\ &= \frac{ph^n}{2c} \sum_{j \in \mathbb{Z}^n \setminus \{0\}} \frac{u(x+hj,t) + u(x-hj,t) - 2u(x,t)}{|hj|^{n+2s}} \\ &\quad + \frac{1-p}{4n} \sum_{j \in h^{s-1}\mathcal{B}} \frac{u(x+hj,t) + u(x-hj,t) - 2u(x,t)}{h^{2s}}.\end{aligned} \tag{C.6}$$

With a formal Taylor expansion, we observe that

$$u(x+hj,t) + u(x-hj,t) - 2u(x,t) = h^2 D_x^2 u(x,t) j \cdot j + O(h^3);$$

therefore the latter sum in (C.6) can be written as

$$\begin{aligned}&\sum_{j \in h^{s-1}\mathcal{B}} h^{2(1-s)} D_x^2 u(x,t) j \cdot j + O(h^{3-2s}) \\ &\qquad = \sum_{j \in N\mathcal{B}} D_x^2 u(x,t) \frac{j}{N} \cdot \frac{j}{N} + O\Big(\frac{1}{N^{\frac{3-2s}{1-s}}}\Big) \\ &\qquad = \sum_{i \in \mathcal{B}} D_x^2 u(x,t) i \cdot i + o(1) = 2\Delta u(x,t) + o(1)\end{aligned}$$

as $N \nearrow +\infty$ (i.e. as $h \searrow 0$).

Hence, recognizing a Riemann sum in the first term of the right-hand side of (C.6), taking the limit as $h \searrow 0$ (that is $\tau \searrow 0$), we formally conclude that

$$\partial_t u(x,t) = \frac{p}{2c} \int_{\mathbb{R}^n} \frac{u(x+y,t) + u(x-y,t) - 2u(x,t)}{|y|^{n+2s}} \, dy + \frac{1-p}{2n} \Delta u(x,t),$$

which is precisely the heat equation associated to the operator in (1.9) (up to defining the structural constants correctly).

A similar argument can be carried out in case (C.5). Indeed, in this situation one deduces from (C.3) that

$$\begin{aligned}\frac{u(x,t+\tau)-u(x,t)}{\tau} = \frac{\alpha h^n}{2c} \sum_{j\in\mathbb{Z}^n\setminus\{0\}} \frac{u(x+hj,t)+u(x-hj,t)-2u(x,t)}{|hj|^{n+2s}} \\ + \frac{1-\alpha h^{2-2s}}{4n} \sum_{j\in\mathcal{B}} \frac{u(x+hj,t)+u(x-hj,t)-2u(x,t)}{h^2}.\end{aligned}$$

Hence, since

$$\begin{aligned}\sum_{j\in\mathcal{B}} \frac{u(x+hj,t)+u(x-hj,t)-2u(x,t)}{h^2} &= \sum_{j\in\mathcal{B}} D_x^2 u(x+hj,t)j\cdot j + O(h) \\ &= 2\Delta u(x,t) + o(1)\end{aligned}$$

as $h \searrow 0$, we conclude that in this case

$$\partial_t u(x,t) = \frac{\alpha}{2c}\int_{\mathbb{R}^n} \frac{u(x+y,t)+u(x-y,t)-2u(x,t)}{|y|^{n+2s}}\,dy + \frac{1}{2n}\Delta u(x,t),$$

which, once again, constitutes the parabolic equation associated to the operator in (1.9) (up to renaming the structural constants).

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