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A general result on the approximation of local conservation laws by nonlocal conservation laws: The singular limit problem for exponential kernels

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Abstract. We deal with the problem of approximating a scalar conservation law by a conservation law with nonlocal flux. As convolution kernel in the nonlocal flux, we consider an exponential-type approximation of the Dirac distribution. We then obtain a total variation bound on the nonlocal term and can prove that the (unique) weak solution of the nonlocal problem converges strongly in $C(L^1_{\mathrm{loc}})$ to the entropy solution of the local conservation law. We conclude with several numerical illustrations which underline the main results and, in particular, the difference between the solution and the nonlocal term.

1. Introduction

Nonlocal conservation laws have been studied quite intensively over the last decade with a particular focus on models arising in traffic flow [6, 22, 33, 36, 41, 49, 58], supply chains [35, 43, 60], pedestrian flow/crowd dynamics [24], opinion formation [2, 56], chemical engineering [55, 62], sedimentation [7], conveyor belts [59] and more. For the underlying dynamics, existence and uniqueness [13, 16, 28, 39, 44, 47–49, 51], (optimal) control problems [5, 15, 23, 27, 37, 42], and suitable numerical schemes [1, 12, 14, 31, 57] have been analyzed.

In this work, "nonlocal" refers to the fact that the velocity $V:\mathbb{R}\to\mathbb{R}$ of the corresponding flux $f:\mathbb{R}\to\mathbb{R}$, i.e. $f(s)=sV(s)$, $s\in\mathbb{R}$, does not depend on the solution locally at a given space point but on the integral of the solution over a (spatial) neighborhood.

First, in [3] it was observed that, at least numerically, there is some hope that the solution of the nonlocal conservation law converges to the local entropy solution when the nonlocal term approaches a Dirac distribution. Positive results in this direction were obtained in [63], provided that the limit entropy solution is smooth and the convolution kernel is even, and in [46] for a large class of nonlocal conservation laws under the assumption of having monotone initial data. Under the assumption that the initial datum

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has bounded total variation, is bounded away from zero and satisfies a one-sided Lipschitz condition, a positive result was obtained in [20]. In [9], for an exponential weight in the nonlocal term, it was shown – provided that the initial datum is bounded away from zero and has bounded total variation (but without monotonicity assumptions) – that the nonlocal solutions converge (up to subsequences) to weak solutions of the corresponding local conservation law; it was also shown that the limit is the unique entropy solution under the additional assumption that V is an affine function. More recently, in [10], the result was extended to more general fluxes.

A viscous nonlocal conservation law with kernel of exponential type was considered in [17]: as the nonlocal term together with the viscosity approximation approaches zero, the sequence of solutions converges to the local entropy solution. The positive effect of viscosity in the nonlocal-to-local approximation process was previously studied in [18, 19, 21] for more general compactly supported kernels (see also [11] in the case of more regular initial data and linear velocity).

In conclusion, although some progress has been made under rather restrictive assumptions, a general theory concerning convergence is missing. Even more, [20] demonstrates via a counterexample that a total variation blow-up of the solution of the nonlocal conservation law can occur if the data is not bounded away from zero, so that the standard methods via compactness in L^1 seemed to be out of reach.

This is why, in this work, we focus instead on the corresponding nonlocal term: it turns out that this term itself satisfies a local transport equation with nonlocal source (see Lemma 3.1), and we can use this to show a uniform total variation bound (see Theorem 3.2). Thanks to the specific structure of the nonlocal term this directly implies that also the solution of the conservation law converges strongly in L^1 (see Theorem 4.1 and Corollary 4.1) (although it does not necessarily satisfy a total variation bound as discussed before).

More precisely, we consider the following setting. For a nonlocal parameter $\eta \in \mathbb{R}_{>0}$ and time horizon $T \in \mathbb{R}_{>0}$, let $q_\eta\colon (0,T)\times\mathbb{R} \to \mathbb{R}$ be the unique weak solution (weak solutions are unique in the nonlocal setup, compare the later-stated results, particularly Theorem 2.1) of the nonlocal conservation law on $\mathbb{R}$,

$$
\begin{aligned}
\partial_t q_\eta(t,x) + \partial_x\big(V(W_\eta[q_\eta](t,x))q_\eta(t,x)\big) &= 0, && (t,x)\in\Omega_T,\\
q_\eta(0,x) &= q_0(x), && x\in\mathbb{R},
\end{aligned}
$$

with $\Omega_T := (0,T)\times\mathbb{R}$, supplemented by the nonlocal term W_η with exponential weight

$$
W_\eta[q](t,x) := \tfrac{1}{\eta}\int_x^\infty \exp\big(\tfrac{x-y}{\eta}\big) q(t,y)\,\mathrm{d}y, \quad (t,x)\in\Omega_T,
$$

and let $q\colon \Omega_T \to \mathbb{R}$ be the entropy solution of the corresponding local conservation law on $\mathbb{R}$ (for the "local theory" and corresponding entropy solutions, we refer to [8, 29, 34, 40]),

$$
\begin{aligned}
\partial_t q(t,x) + \partial_x\big(V(q(t,x))q(t,x)\big) &= 0, && (t,x)\in\Omega_T, && (1)\\
q(0,x) &= q_0(x), && x\in\mathbb{R}. && (2)
\end{aligned}
$$

Then we can show

$$q_\eta \xrightarrow{\eta\to 0} q \quad \text{in } C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R})).$$

We achieve this by first analyzing the nonlocal term $W_\eta[q_\eta]$. Thanks to the relation $\eta\partial_2 W_\eta[q_\eta] \equiv W_\eta[q_\eta] - q_\eta$, the strong convergence (of subsequences) of q to a weak solution of the local conservation law follows immediately from the strong convergence of W_η, which itself is guaranteed by the stated total variation bound in Theorem 3.2. Eventually, we use [10] to obtain that the solution is indeed also entropic. Even more, we show that the nonlocal term $W_\eta[q_\eta]$ also converges to the local entropy solution.

Our "nonlocal-to-local convergence" result closes the gap between local and nonlocal modeling of phenomena governed by conservation laws; moreover, it provides a way of defining the entropy admissible solutions of local conservation laws as limits of weak solutions to nonlocal conservation laws, which usually do not require an entropy condition for uniqueness (see [26,45,48,49]). This kind of singular limit would be an alternative to the classical vanishing viscosity approach (see [8,29,40] and references therein). In the case of a nonlocal approximation, no smoothing phenomena happen and the character of the approximating equation remains somewhat "hyperbolic" (finite propagation of mass, but infinite propagation of information).

Such a convergence result would also give additional insights into questions related to control theory (see [5]), in the spirit of [25,32,38,52]. Showing control results for nonlocal conservation laws might be easier due to the fact that these equations are invertible in time, so that one can actually go back from a current state to the initial datum. Optimal control problems might also become mathematically more approachable as the problem with adjoint equations and shocks prohibiting differentiability in a certain local framework might be resolvable in the nonlocal theory and one might then just consider the limit controls when the nonlocal term approaches a Dirac.

2. Preliminary results on nonlocal conservation laws

In this section, we present some well-known and important results on existence and uniqueness of solutions and their properties, which will become crucial in what follows. We also state precisely the problem setup and the required assumptions.

Definition 2.1 (The nonlocal conservation law and the weak solution). Let $T \in \mathbb{R}_{>0}$ be given. For $\eta \in \mathbb{R}_{>0}$ we consider the following nonlocal conservation law in the "density" $q_\eta\colon \Omega_T \to \mathbb{R}$, $\Omega_T := (0,T)\times\mathbb{R}$,

$$\partial_t q_\eta(t,x) + \partial_x\big(V(W_\eta[q_\eta](t,x))q_\eta(t,x)\big) = 0, \qquad (t,x)\in\Omega_T, \tag{3}$$

$$q_\eta(0,x) = q_0(x), \quad x\in\mathbb{R}, \tag{4}$$

supplemented by the nonlocal term W_η,

$$W_\eta[q_\eta](t,x) := \tfrac{1}{\eta}\int_x^\infty \exp\big(\tfrac{x-y}{\eta}\big) q_\eta(t,y)\,\mathrm{d}y, \quad (t,x)\in\Omega_T. \tag{5}$$

We call $q_0\colon \mathbb{R} \to \mathbb{R}$ the *initial datum* and $W_\eta[q_\eta]\colon \Omega_T \to \mathbb{R}$ the *nonlocal impact* affecting the *velocity function* $V\colon \mathbb{R} \to \mathbb{R}$ of the nonlocal conservation law. We say that $q_\eta \in C([0,T]; L^1_{\text{loc}}(\mathbb{R}))$ is a weak solution for $q_0 \in L^1_{\text{loc}}(\mathbb{R})$ and $\eta \in \mathbb{R}_{>0}$ iff for all $\varphi \in C^1_c((-42,T)\times\mathbb{R})$ it holds that

$$\iint_{\Omega_T} \partial_t\varphi(t,x)q_\eta(t,x) + \partial_x\varphi(t,x)V(W_\eta[q_\eta](t,x))q_\eta(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ + \int_{\mathbb{R}} \varphi(0,x)q_0(x)\,\mathrm{d}x = 0. \tag{6}$$

For the analysis to follow and the well-posedness, we require the following not restrictive assumptions:

Assumption 2.1 (Assumptions on input data). The functions in Definition 2.1 satisfy

- $q_0 \in L^\infty(\mathbb{R};\mathbb{R}_{\geq 0}) \cap \mathrm{TV}(\mathbb{R})$,
- $V \in W^{1,\infty}_{\text{loc}}(\mathbb{R}) : V'(s) \leq 0\ \forall s \in (\operatorname{ess-inf}_{x\in\mathbb{R}} q_0(x), \|q_0\|_{L^\infty(\mathbb{R})})$.

Theorem 2.1 (Existence and uniqueness of weak solutions and maximum principle). *Given Assumption 2.1, there exists a unique weak solution $q \in C([0,T]; L^1_{\text{loc}}(\mathbb{R})) \cap L^\infty((0,T); L^\infty(\mathbb{R})) \cap L^\infty((0,T); \mathrm{TV}(\mathbb{R}))$ of the nonlocal conservation law in Definition 2.1 and the following maximum principle is satisfied:*

$$\operatorname*{ess-inf}_{x\in\mathbb{R}} q_0(x) \leq q(t,x) \leq \|q_0\|_{L^\infty(\mathbb{R})} \quad a.e.\ (t,x) \in \Omega_T. \tag{7}$$

Proof. See [44, Theorems 2.20, 3.2 & Corollary 4.3]. ∎

In the presented framework, we restrict ourselves to monotonically decreasing velocities and nonnegative initial datum. However, this can be extended directly to different setups and is detailed in Remark 2.1.

Remark 2.1 (Generalization of the assumptions on the velocity function V). The assumption on V being monotonically decreasing (see Assumption 2.1) can be changed to V monotonically increasing as long as one also changes the nonlocal range for $q \in C([0,T]; L^1_{\text{loc}}(\mathbb{R}))$ as

$$W_\eta[q](t,x) := \tfrac{1}{\eta}\int_{-\infty}^{x} \exp\big(\tfrac{y-x}{\eta}\big)q(t,y)\,\mathrm{d}y, \quad (t,x) \in \Omega_T.$$

Analogously, the results can be extended to hold also for nonpositive initial datum when changing the nonlocal term accordingly. We do not go into details.

Even more, when assuming that $V'(s)s$ has a sign for all $s \in \mathbb{R}$, one does not need even a maximum principle to be satisfied and thus the initial datum can be chosen arbitrarily in $L^\infty(\mathbb{R}) \cap \mathrm{TV}(\mathbb{R})$ (no sign restrictions). However, then one does not obtain convergence of q_η but of W_η, which remains essentially bounded and for which the total variation bound derived in Theorem 3.2 still holds. However, Theorem 4.2 is not directly applicable and we are left with the limit being a weak solution. Compare also Remark 3.2.

3. Total variation bound on the nonlocal term

As we will tackle the convergence first in the nonlocal term $W_\eta[q_\eta]$, we deduce a transport equation with a nonlocal source which will enable us to study $W_\eta[q_\eta]$ without q_η itself.

Lemma 3.1 (Transport equation with nonlocal source satisfied by the nonlocal term). *Given the dynamics in Definition 2.1, the nonlocal term $W_\eta[q_\eta]$ as in (8) is Lipschitz continuous and satisfies the following transport equation with nonlocal source in the strong sense:*

$$\begin{aligned}
&\partial_t W_\eta(t,x) + V(W_\eta(t,x))\partial_x W_\eta(t,x)\\
&\quad = -\tfrac{1}{\eta}\int_x^\infty \exp\big(\tfrac{x-y}{\eta}\big) V'(W_\eta(t,y))\partial_y W_\eta(t,y) W_\eta(t,y)\,\mathrm{d}y, \quad (t,x)\in\Omega_T, &&(8)\\
&W_\eta(0,x) = \tfrac{1}{\eta}\int_x^\infty \exp\big(\tfrac{x-y}{\eta}\big) q_0(y)\,\mathrm{d}y, \qquad x\in\mathbb{R}. &&(9)
\end{aligned}$$

In particular, for $\eta\in\mathbb{R}_{>0}$, we have $W_\eta \in W^{1,\infty}(\Omega_T)$.

Proof. We first show that $W_\eta[q_\eta]$ is Lipschitz continuous. To this end, recall the definition in (5) and compute for $(t,x)\in\Omega_T$,

$$\begin{aligned}
\partial_x W_\eta[q_\eta](t,x) &= \partial_x \tfrac{1}{\eta}\int_x^\infty \exp\big(\tfrac{x-y}{\eta}\big) q_\eta(t,y)\,\mathrm{d}y\\
&= \tfrac{1}{\eta} W_\eta[q_\eta](t,x) - \tfrac{1}{\eta} q_\eta(t,x). \qquad (10)
\end{aligned}$$

However, as $\eta\in\mathbb{R}_{>0}$, $W_\eta[q_\eta]\in L^\infty(\Omega_T)$ and $q_\eta\in L^\infty(\Omega_T)$ thanks to Theorem 2.1, we obtain the uniform boundedness of the spatial derivative. The time derivative is slightly more tricky. Due to the lack of regularity, we use the method of characteristics analyzed in [44, Lemma 2.6] to write down the solution q_η and have on $(t,x)\in\Omega_T$,

$$\begin{aligned}
\partial_t W_\eta[q_\eta](t,x) &= \partial_t \tfrac{1}{\eta}\int_x^\infty \exp\big(\tfrac{x-y}{\eta}\big) q_\eta(t,y)\,\mathrm{d}y\\
&= \partial_t \tfrac{1}{\eta}\int_x^\infty \exp\big(\tfrac{x-y}{\eta}\big) q_0(\xi(t,y;0))\partial_2\xi(t,y;0)\,\mathrm{d}y\\
&= \partial_t \tfrac{1}{\eta}\int_{\xi(t,x;0)}^\infty \exp\big(\tfrac{x-\xi(0,z;t)}{\eta}\big) q_0(z)\,\mathrm{d}z\\
&= -\tfrac{1}{\eta^2}\int_{\xi(t,x;0)}^\infty \exp\big(\tfrac{x-\xi(0,z;t)}{\eta}\big) q_0(z)\partial_3\xi(0,z;t)\,\mathrm{d}z\\
&\quad - \tfrac{1}{\eta} q_0(\xi(t,x;0))\partial_1\xi(t,x;0). \qquad (11)
\end{aligned}$$

Recalling some nice properties of the characteristics [44, Lemma 2.6] and in particular

$$\begin{aligned}
\partial_3\xi(0,\xi(t,y;0);t) &= V(W_\eta[q_\eta](t,y)) && \forall (t,y)\in\Omega_T,\\
\partial_1\xi(t,y;0) &= -\partial_2\xi(t,y;0)V(W_\eta[q_\eta](t,y)) && \forall (t,y)\in\Omega_T,
\end{aligned}$$

we obtain, by continuing (11),

$$
\begin{aligned}
\partial_t W_\eta[q_\eta](t,x) &= -\tfrac{1}{\eta^2}\int_{\xi(t,x;0)}^{\infty} \exp\big(\tfrac{x-\xi(0,z;t)}{\eta}\big) q_0(z)\partial_3\xi(0,z;t)\,\mathrm{d}z\\
&\quad - \tfrac{1}{\eta} q_0(\xi(t,x;0))\partial_1\xi(t,x;0)\\
&= -\tfrac{1}{\eta^2}\int_x^{\infty} \exp\big(\tfrac{x-y}{\eta}\big) q_0(\xi(t,y;0))\partial_3\xi(0,\xi(t,y;0);t)\partial_2\xi(t,y;0)\,\mathrm{d}y\\
&\quad + \tfrac{1}{\eta} q_0(\xi(t,x;0))\partial_2\xi(t,x;0)V(W_\eta[q_\eta](t,x))\\
&= -\tfrac{1}{\eta^2}\int_x^{\infty} \exp\big(\tfrac{x-y}{\eta}\big) q_\eta(t,y)V(W_\eta[q_\eta](t,y))\,\mathrm{d}y\\
&\quad + \tfrac{1}{\eta} q_\eta(t,x)V(W_\eta[q_\eta](t,x)).
\end{aligned}
$$

This expression is essentially bounded for $\eta \in \mathbb{R}_{>0}$ so that we obtain the claimed Lipschitz continuity. Next, we show that the nonlocal operator indeed satisfies the Cauchy problem in (8)–(9). Using the identity computed for $\partial_t W_\eta$ above, we have for the left-hand side of (8) and $(t,x) \in \Omega_T$,

$$
\begin{aligned}
&\partial_t W_\eta[q_\eta](t,x) + V(W_\eta[q_\eta](t,x))\partial_x W_\eta[q_\eta](t,x)\\
&\quad = \tfrac{1}{\eta} q_\eta(t,x)V(W_\eta[q_\eta](t,x)) - \tfrac{1}{\eta^2}\int_x^{\infty} \exp\big(\tfrac{x-y}{\eta}\big) q_\eta(t,y)V(W_\eta[q_\eta](t,y))\,\mathrm{d}y\\
&\qquad + V(W_\eta[q_\eta](t,x))\big(\tfrac{1}{\eta}W_\eta[q_\eta](t,x) - \tfrac{1}{\eta}q_\eta(t,x)\big)\\
&\quad = V(W_\eta[q_\eta](t,x))\tfrac{1}{\eta}W_\eta[q_\eta](t,x)\\
&\qquad - \tfrac{1}{\eta^2}\int_x^{\infty} \exp\big(\tfrac{x-y}{\eta}\big)(W_\eta[q_\eta](t,y) - \eta\partial_y W_\eta[q_\eta](t,y))V(W_\eta[q_\eta](t,y))\,\mathrm{d}y\\
&\quad = V(W_\eta[q_\eta](t,x))\tfrac{1}{\eta}W_\eta[q_\eta](t,x)\\
&\qquad - \tfrac{1}{\eta^2}\int_x^{\infty} \exp\big(\tfrac{x-y}{\eta}\big) W_\eta[q_\eta](t,y)V(W_\eta[q_\eta](t,y))\,\mathrm{d}y\\
&\qquad + \tfrac{1}{\eta}\int_x^{\infty} \exp\big(\tfrac{x-y}{\eta}\big)\partial_y W_\eta[q_\eta](t,y)V(W_\eta[q_\eta](t,y))\,\mathrm{d}y\\
&\quad = -\tfrac{1}{\eta}\int_x^{\infty} \exp\big(\tfrac{x-y}{\eta}\big) V'(W_\eta[q_\eta](t,x))\partial_y W_\eta[q_\eta](t,y)W_\eta[q_\eta](t,y)\,\mathrm{d}y,
\end{aligned}
$$

where we have used the identity in (10) twice and integration by parts. However, the last term is indeed the right-hand side of (8). The nonlocal term W_η also satisfies the initial datum in (9), which is a direct consequence of the definition of W_η in (5) when plugging in $t = 0$ (this is possible as the solution is regular enough, i.e. $q_\eta \in C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}))$. ∎

Remark 3.1 (Fully local equation in W_η). The transport equation in W_η in (8) with nonlocal source can also be transformed into a fully local equation (as in [17]) involving higher derivatives and particularly a mixed space-time derivative:

$$\begin{aligned}
&\partial_t W_\eta(t,x) + \partial_x\big(V(W_\eta(t,x))W_\eta(t,x)\big)\\
&\qquad = \eta\partial_{tx}^2 W_\eta(t,x) + \partial_x\big(V(W_\eta(t,x))\partial_x W_\eta(t,x)\big), \quad (t,x)\in\Omega_T,\\
&W_\eta(0,x) = \tfrac{1}{\eta}\int_x^\infty \exp\big(\tfrac{x-y}{\eta}\big)q_0(y)\,\mathrm{d}y, \qquad x\in\mathbb{R}.
\end{aligned}$$

For Theorem 3.2, where we prove a total variation bound on W_η uniform in η, we require a density or stability result which enables us to smooth the solution. This result, stated below, is borrowed from [46, Theorem 4.17].

Theorem 3.1 (Stability of the nonlocal conservation law w.r.t. the initial datum). *Let Assumption 2.1 hold, and let $\mathcal{C}_1,\mathcal{C}_2\in\mathbb{R}_{\geq 0}$ be given such that*

$$Q(\mathcal{C}_1,\mathcal{C}_2) := \big\{u\in \mathrm{TV}_{\mathrm{loc}}(\mathbb{R}) : \|u\|_{L^\infty(\mathbb{R})}\leq \mathcal{C}_1 \wedge |u|_{\mathrm{TV}(\mathbb{R})}\leq \mathcal{C}_2\big\}.$$

Let $q_0\in Q(\mathcal{C}_1,\mathcal{C}_2)$ be given and denote by q the solutions to the corresponding nonlocal conservation law.

Then the solutions to the corresponding nonlocal conservation laws (denoted by q) satisfy the following $C([0,T];L^1(\mathbb{R}))$ stability estimate, i.e.

$$\begin{aligned}
&\forall \varepsilon\in\mathbb{R}_{>0}\ \exists\,\delta\in\mathbb{R}_{>0}:\\
&\qquad \forall \tilde{q}_0\in Q(\mathcal{C}_1,\mathcal{C}_2) \text{ with } \|q_0-\tilde{q}_0\|_{L^1(\mathbb{R})}\leq\delta \Rightarrow \|q-\tilde{q}\|_{C([0,T];L^1(\mathbb{R}))}\leq\varepsilon,
\end{aligned}$$

where $\tilde{q}$ is the solution to the corresponding nonlocal conservation law with initial datum $\tilde{q}_0$.

Proof. Almost the required result can be found in [46, Theorem 4.17] with the difference that the kernel of the nonlocal operator is supposed to have compact support while here we have an exponential kernel (5) with evidently noncompact support. However, the changes for this result also holding for the exponential kernel are minor, and we do not go into details. ∎

The next theorem shows that the nonlocal term has a total variation which cannot increase over time and thus presents the key ingredient for our proof of convergence later.

Theorem 3.2 (Total variation bound in the spatial component of W – uniformly in η). *The nonlocal term W_η defined in (5) but which also satisfies the identity demonstrated in Lemma 3.1 admits – uniformly in η – a total variation bound, i.e.*

$$|W_\eta(t,\cdot)|_{\mathrm{TV}(\mathbb{R})}\leq |W_\eta(0,\cdot)|_{\mathrm{TV}(\mathbb{R})}\leq |q_0|_{\mathrm{TV}(\mathbb{R})} \quad \forall\eta\in\mathbb{R}_{>0},\ \forall t\in[0,T].$$

Proof. We take advantage of the stability result in Theorem 3.1, which tells us that when smoothing q_0 by $q_0^\varepsilon :\equiv q_0 * \varphi_\varepsilon$, with φ_ε being a standard mollifier [53, C.4 Mollifiers] with smoothing parameter $\varepsilon \in \mathbb{R}_{>0}$, the corresponding solution q_η^ε will be close in the $C([0,T];L^1(\mathbb{R}))$ topology. Additionally, as the initial datum is smooth, so is the corresponding solution (see [44, Corollary 5.3]) which we will denote by q_η^ε. We now prove the total variation bound. As q_η^ε is smooth, the total variation coincides with the L^1-norm of the derivative and we can estimate for $t \in [0,T]$ as follows:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} |\partial_x W_\eta^\varepsilon(t,x)|\,\mathrm{d}x\\
&\quad= \int_{\mathbb{R}} \operatorname{sgn}(\partial_x W_\eta^\varepsilon(t,x))\partial_{tx}^2 W_\eta^\varepsilon(t,x)\,\mathrm{d}x\\
&\quad= -\int_{\mathbb{R}} \operatorname{sgn}(\partial_x W_\eta^\varepsilon(t,x))V(W_\eta^\varepsilon(t,x))\partial_{xx}^2 W^\varepsilon(t,x)\,\mathrm{d}x\\
&\qquad -\int_{\mathbb{R}} \operatorname{sgn}(\partial_x W_\eta^\varepsilon(t,x))V'(W_\eta^\varepsilon(t,x))(\partial_x W_\eta^\varepsilon(t,x))^2\,\mathrm{d}x\\
&\qquad +\tfrac{1}{\eta}\int_{\mathbb{R}} \operatorname{sgn}(\partial_x W_\eta^\varepsilon(t,x))V'(W_\eta^\varepsilon(t,x))W_\eta^\varepsilon(t,x)\partial_x W_\eta^\varepsilon(t,x)\,\mathrm{d}x\\
&\qquad -\tfrac{1}{\eta^2}\int_{\mathbb{R}} \operatorname{sgn}(\partial_x W_\eta^\varepsilon(t,x))\int_x^\infty \exp\big(\tfrac{x-y}{\eta}\big)V'(W_\eta^\varepsilon(t,y))\partial_y W_\eta^\varepsilon(t,y)W_\eta^\varepsilon(t,y)\,\mathrm{d}y\,\mathrm{d}x\\
&\quad= \int_{\mathbb{R}} 2\delta_0(\partial_x W_\eta^\varepsilon(t,x))V(W_\eta^\varepsilon(t,x))\partial_x W_\eta^\varepsilon(t,x)\partial_{xx}^2 W_\eta^\varepsilon(t,x)\,\mathrm{d}x\\
&\qquad +\int_{\mathbb{R}} \operatorname{sgn}(\partial_x W_\eta^\varepsilon(t,x))V'(W_\eta^\varepsilon(t,x))(\partial_x W_\eta^\varepsilon(t,x))^2\,\mathrm{d}x\\
&\qquad -\int_{\mathbb{R}} \operatorname{sgn}(\partial_x W_\eta^\varepsilon(t,x))V'(W_\eta^\varepsilon(t,x))(\partial_x W_\eta^\varepsilon(t,x))^2\,\mathrm{d}x\\
&\qquad +\tfrac{1}{\eta}\int_{\mathbb{R}} \operatorname{sgn}(\partial_x W_\eta^\varepsilon(t,x))V'(W_\eta^\varepsilon(t,x))W_\eta^\varepsilon(t,x)\partial_x W_\eta^\varepsilon(t,x)\,\mathrm{d}x\\
&\qquad -\tfrac{1}{\eta^2}\int_{\mathbb{R}} \operatorname{sgn}(\partial_x W_\eta^\varepsilon(t,x))\int_x^\infty \exp\big(\tfrac{x-y}{\eta}\big)V'(W_\eta^\varepsilon(t,y))\partial_y W_\eta^\varepsilon(t,y)W_\eta^\varepsilon(t,y)\,\mathrm{d}y\,\mathrm{d}x\\
&\quad\le \tfrac{1}{\eta}\int_{\mathbb{R}} |\partial_x W_\eta^\varepsilon(t,x)|V'(W_\eta^\varepsilon(t,x))W_\eta^\varepsilon(t,x)\,\mathrm{d}x\\
&\qquad -\tfrac{1}{\eta^2}\int_{\mathbb{R}} V'(W_\eta^\varepsilon(t,y))|\partial_y W_\eta^\varepsilon(t,y)|W_\eta^\varepsilon(t,y)\int_{-\infty}^y \exp\big(\tfrac{x-y}{\eta}\big)\,\mathrm{d}x\,\mathrm{d}y \qquad (12)\\
&\quad\le \tfrac{1}{\eta}\int_{\mathbb{R}} |\partial_x W_\eta^\varepsilon(t,x)|V'(W_\eta^\varepsilon(t,x))W_\eta^\varepsilon(t,x)\,\mathrm{d}x\\
&\qquad -\tfrac{1}{\eta}\int_{\mathbb{R}} V'(W_\eta^\varepsilon(t,y))|\partial_y W_\eta^\varepsilon(t,y)|W_\eta^\varepsilon(t,y)\exp\big(\tfrac{y-y}{\eta}\big)\,\mathrm{d}y\\
&\quad= 0.
\end{aligned}
$$

We thus obtain

$$|W_\eta^\varepsilon(t,\cdot)|_{\mathrm{TV}(\mathbb{R})} \le |W_\eta^\varepsilon(0,\cdot)|_{\mathrm{TV}(\mathbb{R})} \le |q_0|_{\mathrm{TV}(\mathbb{R})}, \qquad (13)$$

where the last inequality follows from the assumption on $q_0 \in \mathrm{TV}(\mathbb{R})$ as stated in Assumption 2.1 and the definition of the initial value for W_η as in (9):

$$\begin{aligned}
\|W_\eta^\varepsilon(0,\cdot)\|_{\mathrm{TV}(\mathbb{R})} &= \sup_{\substack{\psi\in C_c^1(\mathbb{R}):\\ \|\psi\|_{L^\infty(\mathbb{R})}\leq 1}} \int_{\mathbb{R}} \psi'(x) W_\eta^\varepsilon[q_0^\varepsilon](x)\,\mathrm{d}x\\
&= \sup_{\substack{\psi\in C_c^1(\mathbb{R}):\\ \|\psi\|_{L^\infty(\mathbb{R})}\leq 1}} \int_{\mathbb{R}} \psi'(x)\tfrac{1}{\eta}\int_{\mathbb{R}_{>0}} \exp\big(\tfrac{x-y}{\eta}\big) q_0^\varepsilon(y)\,\mathrm{d}y\,\mathrm{d}x\\
&= \sup_{\substack{\psi\in C_c^1(\mathbb{R}):\\ \|\psi\|_{L^\infty(\mathbb{R})}\leq 1}} \int_{\mathbb{R}} \psi'(x)\tfrac{1}{\eta}\int_{\mathbb{R}_{<0}} \exp\big(\tfrac{z}{\eta}\big) q_0^\varepsilon(x-z)\,\mathrm{d}y\,\mathrm{d}x\\
&= \sup_{\substack{\psi\in C_c^1(\mathbb{R}):\\ \|\psi\|_{L^\infty(\mathbb{R})}\leq 1}} \sup_{z\in\mathbb{R}_{<0}} \int_{\mathbb{R}} \psi'(x+z) q_0^\varepsilon(x)\,\mathrm{d}y\\
&= \sup_{\substack{\psi\in C_c^1(\mathbb{R}):\\ \|\psi\|_{L^\infty(\mathbb{R})}\leq 1}} \int_{\mathbb{R}} \psi'(x)\int_{\mathbb{R}} \varphi_\varepsilon(x-y) q_0(x)\,\mathrm{d}x\,\mathrm{d}y\\
&\leq \sup_{y\in\mathbb{R}} \int_{\mathbb{R}} \varphi_\varepsilon(x-y) \sup_{\substack{\psi\in C_c^1(\mathbb{R}):\\ \|\psi\|_{L^\infty(\mathbb{R})}\leq 1}} \int_{\mathbb{R}} \psi'(z) q_0(z)\,\mathrm{d}z\,\mathrm{d}x\\
&\leq |q_0|_{\mathrm{TV}(\mathbb{R})}.
\end{aligned}$$

As (13) is uniform in $(\varepsilon,\eta)\in\mathbb{R}_{>0}^2$, we are done. ∎

Remark 3.2 (Total variation bound and the required assumptions on the velocity V). The key step in the proof of the total variation bound stated in Theorem 3.2 can be located in the estimate around (12). Reconnecting to Remark 2.1, it is enough to assume the velocity satisfies $V'(s)s \leq 0$ for all $s \in \mathbb{R}$ to obtain the uniform total variation bound without any sign restriction on the initial datum.

4. Convergence nonlocal to local

Using the results in Section 3, we can show next that the set of nonlocal terms is compact in the canonical $C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}))$ topology.

Theorem 4.1 (Compactness of W_η in $C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}))$). *The set $(W_\eta)_{\eta\in\mathbb{R}_{>0}} \subseteq C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}))$ of solutions to (8)–(9) is compactly embedded into $C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}))$, i.e.*

$$\big\{W_\eta \in C([0,T];L^1_{\mathrm{loc}}(\mathbb{R})) : W_\eta \textit{ satisfies } (8)\text{–}(9),\ \eta\in\mathbb{R}_{>0}\big\} \overset{c}{\hookrightarrow} C([0,T];L^1_{\mathrm{loc}}(\mathbb{R})).$$

Proof. The proof consists of applying the Ascoli theorem in [61, Lemma 1]. We state the details in the following.

Let B be a Banach space. Then [61, Lemma 1] states that a set $F \subset C([0,T];B)$ is relatively compact in $C([0,T];B)$ iff

- $F(t) := \{f(t) \in B : f \in F\}$ is relatively compact in B for all $t \in [0,T]$;
- F is uniformly equicontinuous, i.e.

$$\begin{aligned}&\forall \sigma \in \mathbb{R}_{>0}, \exists \delta \in \mathbb{R}_{>0}, \forall f \in F, \forall (t_1,t_2) \in [0,T]^2 \text{ with } |t_1 - t_2| \le \delta :\\ &\qquad \|f(t_1) - f(t_2)\|_B \le \sigma.\end{aligned}$$

We start with setting $B = L^1_{\mathrm{loc}}(\mathbb{R})$ and $F(t) := \{W_\eta(t,\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}) : \eta \in \mathbb{R}_{>0}\}$. Thanks to Theorem 3.2, we know that $W_\eta(t,\cdot)$ has a uniform total variation bound and by [53, Theorem 13.35], the set $F(t)$ is compact in $L^1_{\mathrm{loc}}(\mathbb{R})$, i.e.

$$F(t) \overset{\mathrm{c}}{\subseteq} L^1_{\mathrm{loc}}(\mathbb{R}) \quad \forall t \in [0,T].$$

It remains to show the second point, the uniform equicontinuity. To this end, we again smooth the initial datum q_0 by a q_0^ε for $\varepsilon \in \mathbb{R}_{>0}$ as in the proof of Theorem 3.2 and call the corresponding smooth nonlocal term W_η^ε for an $\eta \in \mathbb{R}_{>0}$. Then we can estimate

$$\|W_\eta^\varepsilon(t_1,\cdot) - W_\eta^\varepsilon(t_2,\cdot)\|_{L^1(\mathbb{R})} = \left\| \int_{t_2}^{t_1} \partial_t W_\eta^\varepsilon(s,\cdot)\,\mathrm{d}s \right\|_{L^1(\mathbb{R})},$$

plugging in (8) and using the triangle inequality,

$$\begin{aligned}&\le \left\| \int_{t_2}^{t_1} V(W_\eta^\varepsilon(s,\cdot))\partial_2 W_\eta^\varepsilon(s,\cdot)\,\mathrm{d}s \right\|_{L^1(\mathbb{R})}\\ &\quad + \left\| \int_{t_2}^{t_1} \tfrac{1}{\eta} \int_*^\infty \exp\!\left(\tfrac{*-y}{\eta}\right) V'(W_\eta^\varepsilon(s,y))\partial_y W_\eta^\varepsilon(s,y) W_\eta^\varepsilon(s,y)\,\mathrm{d}y\,\mathrm{d}s \right\|_{L^1(\mathbb{R})},\end{aligned}$$

applying (7),

$$\begin{aligned}&\le \|V\|_{L^\infty((0,\|q_0\|_{L^\infty(\mathbb{R})}))} |W_\eta^\varepsilon|_{L^\infty((0,T);\mathrm{TV}(\mathbb{R}))} |t_1 - t_2|\\ &\quad + \|V'\|_{L^\infty((0,\|q_0\|_{L^\infty(\mathbb{R})}))} \|W_\eta^\varepsilon\|_{L^\infty((0,T);L^\infty(\mathbb{R}))} |W_\eta^\varepsilon|_{L^\infty((0,T);\mathrm{TV}(\mathbb{R}))} |t_1 - t_2|\end{aligned}$$

and finally Theorem 3.2 and (7),

$$\le \left(\|V\|_{L^\infty((0,\|q_0\|_{L^\infty(\mathbb{R})}))} + \|V'\|_{L^\infty((0,\|q_0\|_{L^\infty(\mathbb{R})}))} \|q_0\|_{L^\infty(\mathbb{R})}\right) |q_0|_{\mathrm{TV}(\mathbb{R})} |t_1 - t_2|.$$

As this is a uniform bound in $\eta \in \mathbb{R}_{>0}$ and $\varepsilon \in \mathbb{R}_{>0}$, we have the uniform equicontinuity so that we indeed obtain the claimed compactness. ∎

As a direct result, from the strong convergence of W_η we have also the strong convergence of q_η to a weak solution of the local conservation law as the following corollary states:

Corollary 4.1 (Limit of q_η and W_η are weak solution to the local equation). *For every sequence $(\eta_k)_{k\in\mathbb{N}_{\geq 1}} \subset \mathbb{R}_{>0}$ with $\lim_{k\rightarrow\infty}\eta_k = 0$, there exists a subsequence (for reasons of convenience again denoted by η_k) and a function $q^* \in C([0,T]; L^1_{\text{loc}}(\mathbb{R}))$ so that the solution $q_{\eta_k} \in C([0,T]; L^1_{\text{loc}}(\mathbb{R}))$ of the nonlocal conservation law as given in Definition 2.1 converges in $C([0,T]; L^1_{\text{loc}}(\mathbb{R}))$ to the limit point q^* and so does the nonlocal term W_{η_k} as given in (5). Additionally, q^* is a weak solution of the local conservation law (1)–(2). In equations,*

$$\lim_{\eta\rightarrow 0}\|q_\eta - q^*\|_{C([0,T];L^1_{\text{loc}}(\mathbb{R}))} = 0 \wedge \lim_{\eta\rightarrow 0}\|W_\eta - q^*\|_{C([0,T];L^1_{\text{loc}}(\mathbb{R}))} = 0,$$

where q^* satisfies for all $\varphi \in C^1_{\text{c}}((-42,T)\times\mathbb{R})$,

$$\begin{aligned}\iint_{\Omega_T} &\partial_t\varphi(t,x)q^*(t,x) + \partial_x\varphi(t,x)V(q^*(t,x))q^*(t,x)\,\mathrm{d}x\,\mathrm{d}t\\ &+ \int_{\mathbb{R}}\varphi(0,x)q_0(x)\,\mathrm{d}x = 0.\end{aligned} \tag{14}$$

Proof. Thanks to Theorem 4.1, $\mathcal{W} := \{W_{\eta_k}; k\in\mathbb{N}_{\geq 1}\} \overset{\text{c}}{\subset} C([0,T]; L^1_{\text{loc}}(\mathbb{R}))$, i.e. the set $\mathcal{W}$ is compact in $C([0,T]; L^1_{\text{loc}}(\mathbb{R}))$ and there exists a limit point $q^* \in C([0,T]; L^1_{\text{loc}}(\mathbb{R}))$ so that we obtain

$$\lim_{k\rightarrow\infty}\|W_{\eta_k} - q^*\|_{C([0,T];L^1_{\text{loc}}(\mathbb{R}))} = 0.$$

The identity in (10) directly implies

$$\|W_{\eta_k}(t,\cdot) - q_{\eta_k}(t,\cdot)\|_{L^1(\mathbb{R})} = \eta_k|W_{\eta_k}(t,\cdot)|_{\text{TV}(\mathbb{R})} \leq \eta_k|q_0|_{\text{TV}(\mathbb{R})}$$

and thus we also obtain

$$\lim_{k\rightarrow\infty}\|q_{\eta_k} - q^*\|_{C([0,T];L^1_{\text{loc}}(\mathbb{R}))} = 0.$$

It remains to be shown that q^* is indeed a weak solution. This directly follows from the strong convergence of q_{η_k} to q^* in $C([0,T]; L^1_{\text{loc}}(\mathbb{R}))$ and due to the essential and uniform bound on q_η as given in Theorem 2.1 in (7). ∎

However, the previous result can actually be strengthened, and indeed we obtain that the limit q^* is unique (in particular, every subsequence converges) and that this limit is the weak *entropy* solution of the corresponding local conservation law.

Theorem 4.2 (Convergence to the entropy solution). *Given Assumption 2.1, and assuming that the flux $s \mapsto sV(s)$ is strictly convex/strictly concave on* $[\text{ess-inf}_{x\in\mathbb{R}}\, q_0(x), \|q_0\|_{L^\infty(\mathbb{R})}]$*, the nonlocal term $W_\eta[q_\eta]$ and the corresponding nonlocal solution $q_\eta \in C([0,T]; L^1_{\text{loc}}(\mathbb{R}))$ of the nonlocal conservation law in Definition 2.1 converge in $C([0,T]; L^1_{\text{loc}}(\mathbb{R}))$ to the entropy solution of the corresponding local conservation law (see (1)–(2)).*

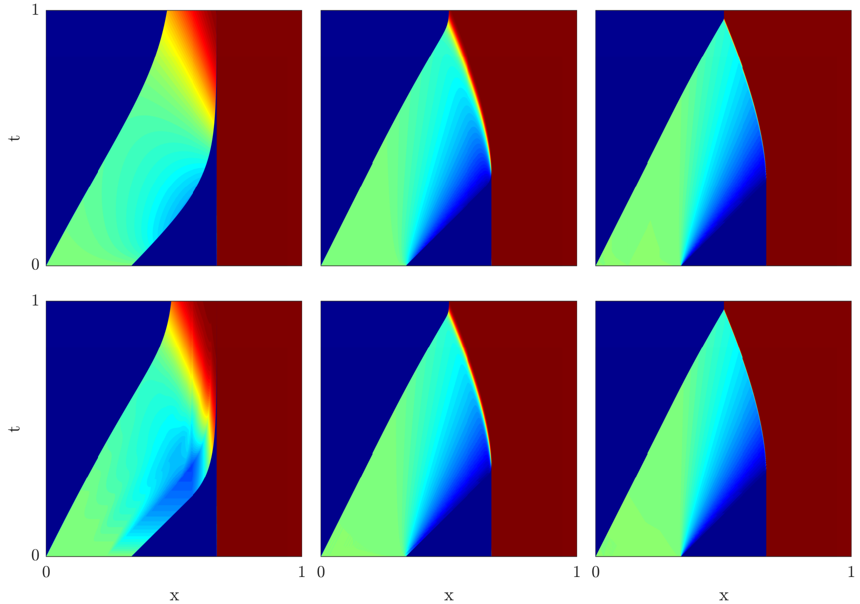

Figure 1. Solution of the nonlocal balance law with exponential kernel (*top*, (5)) and constant kernel (*bottom*, (15)) supplemented by the piecewise constant initial datum stated in (16) plotted in the space-time domain. From left to right η is decreasing, $\eta \in \{10^{-1}, 10^{-2}, 10^{-3}\}$. The rightmost figure is "by eye" not distinguishable from the corresponding local solution. Color bar: 0 1

Proof. This is a direct consequence of the convergence of W_η, q_η to a weak solution of the local conservation laws in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}))$, Corollary 4.1 and of [10]. Therein, by taking advantage of the minimal entropy condition in [30, 54], it is shown that a solution q_η of the nonlocal conservation law in Definition 2.1 with uniform TV bound converges to the entropy solution of the local problem, given that the flux is strictly convex or concave. However, when checking the proof carefully, it turns out that it suffices to assume that the solution q_η converges strongly to a weak solution q^*, which is the case. The uniqueness follows as every limit point is, by the previous argument, an entropy solution and the entropy solution is unique, thus each subsequence converges to the same limit point and thus, for every sequence $(\eta_k)_{k \in \mathbb{N}_{\geq 1}} \subset \mathbb{R}_{>0}$ with $\lim_{k \to \infty} \eta_k = 0$ we have $q_{\eta_k} \to q^*$ (for $k \to \infty$) in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}))$. ∎

5. Numerical illustrations

Some numerical results concerning the convergence can be found in [46]. We rely on a solver based on characteristics [50] which is nondissipative. On the basis of a simple example, we want to shed more light on the difference between the total variation of q_η

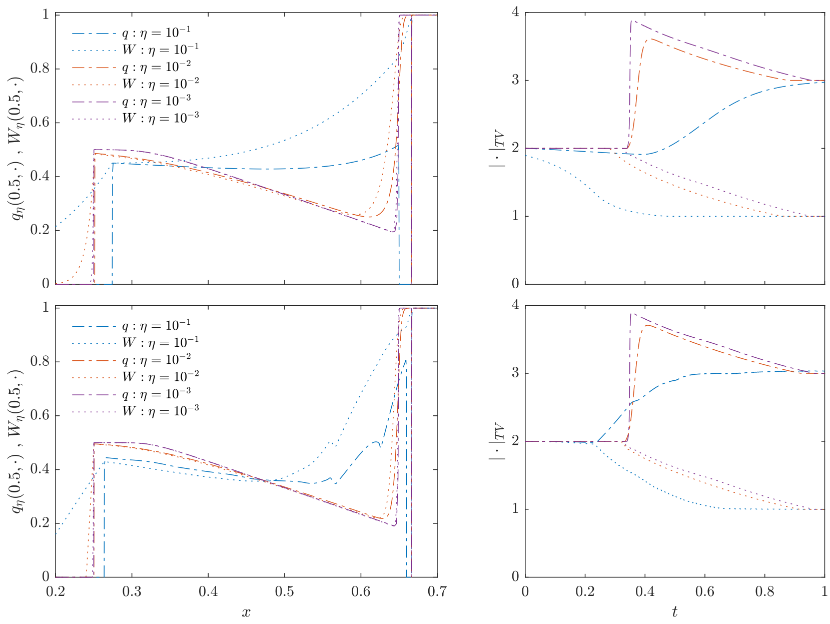

Figure 2. *Left:* Solution of the nonlocal balance law with exponential kernel (*top*, (5)) and constant kernel (*bottom*, (15)) supplemented by the piecewise constant initial datum stated in (16) and its corresponding nonlocal term plotted for $t = 0.5$ and $\eta \in \{10^{-1}, 10^{-2}, 10^{-3}\}$. *Right:* Evolution of the corresponding total variations showing a monotone decreasing nature in terms of the nonlocal term (dotted lines) which is also the case for the local counterpart. In terms of the total variation of the solution itself (dashed dotted lines), the total variation approaches 3. This is because the zero in the initial datum ($x \in (\frac{1}{3}, \frac{2}{3})$) moves and shrinks but does not vanish for all $\eta \in \mathbb{R}_{>0}$ and $t \in (0, T]$, resulting in an additional total variation of 2, compared to the total variation of the solution to the local equation being 1 for all $t \in (1, T)$.

and the nonlocal counterpart $W_\eta[q_\eta]$ (see Figure 1, upper row). We further demonstrate that the result should still hold for general nonlocal kernels by using as "worst case" a constant kernel, i.e. for $q \in C([0, T]; L^1_{\mathrm{loc}}(\mathbb{R})) \cap L^\infty((0, T); L^\infty(\mathbb{R}))$,

$$W_\eta[q](t, x) := \tfrac{1}{\eta} \int_x^{x+\eta} q(t, y)\,\mathrm{d}y, \quad x \in \mathbb{R}. \tag{15}$$

This is illustrated in the lower row of Figure 1. The examples rely on the following initial datum:

$$q_0 \equiv \tfrac{1}{2}\chi_{(0,\frac{1}{3})} + \chi_{\mathbb{R}_{>\frac{2}{3}}}. \tag{16}$$

It seems to be true that a total variation bound on the nonlocal term holds also for the "extreme case" of a constant kernel and that also the solution still converges to the local entropy solution.

The crucial points of the chosen initial datum are the roots for $x \in (\frac{1}{3}, \frac{2}{3})$. These roots are moving but kept in the nonlocal solution q_η for all times (see Figure 2). This results in an increase of the total variation. In the nonlocal term W there are by construction of the initial datum, as well as the exponential kernel, no roots, and the solution is smoothed resulting in an – as proven – nonincreasing total variation.

6. Future work

The presented results open many possibilities for future research. We detail some of them:

(1) Is it possible to obtain the same results for different kernels still satisfying the required monotonicity assumption for the solution to satisfy a maximum principle (see for this particularly Section 5 and Figure 1, lower row)? The considered exponential kernel provides a nice structure, which is crucial in our analysis for showing the stated results. However, from a numerical point of view, it seems that, as long as the kernel is monotonically decreasing, the convergence should hold (see again Figure 1).

(2) What happens in the case of a fully symmetric nonlocal kernel which is sensitive to both propagating directions? However, such a kernel immediately implies that the solutions cannot satisfy a maximum principle (for an illustration see for instance [46, Example 7.3, Figure 9]). Then, recalling [20], it is also apparent that one cannot expect the solution to converge in a strong or weak sense to the entropy solution, but there is hope – compare particularly the numerics in [46, Example 7.3] – for convergence in a measure-valued sense.

(3) Do we also obtain convergence in the case of the initial boundary value problem? In [49], we introduced the corresponding initial boundary value problem where the right-hand side boundary is located in the nonlocal term. The natural question is then whether the nonlocal conservation law on a bounded domain converges to the local entropy solution with boundary datum in the sense of [4].

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