



Mathematical physics — *The relative power and its invariance*, by PAOLO MARIA MARIANO, communicated on 11 June 2009.

ABSTRACT. — The *relative power* of actions in Cauchy bodies suffering *mutations* due to defect evolution is introduced. It is shown that its invariance under the action of the Euclidean group over the ambient space and the material space allows one to obtain (i) the balance of standard and configurational actions and (ii) the identification of configurational ingredients from a unique source.

KEY WORDS: Relative power; Invariance; Mutant Bodies.

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1.

Actions driving the evolution of defects (such as inclusions, holes, vacancies, dislocations, cracks) in materials appear in processes changing the material structure in such a way that the natural picture involves alterations of the reference macroscopic configuration. Such a picture justifies the term ‘configurational’, attributed to Nabarro (see remarks in [9]). In a pioneering paper [10], Eshelby observed that, in simple elastic bodies undergoing large deformations, the equations obtained by means of horizontal variations of the bulk elastic energy—they are the variations generated by altering the reference place by means of appropriate diffeomorphisms—describe the balance of actions on defects with non-vanishing volume. The analysis of the irreversible evolution of these bulk defects requires also the introduction of peculiar driving forces, which are ‘models’ of the circumstances breaking the material bonds. The evolution of point, line, surface and bulk defects has been discussed largely in the subsequent literature (see [1, 31, 20, 28, 32, 33, 34, 35, 39, 2, 8, 17, 36, 37, 38, 26, 18] and references therein, just to mention a few contributions).

Various points of view have determined a debate about the nature of the balances of configurational actions, governing equilibrium and possible evolution of defects in simple and complex bodies.

- On one side it has been claimed that the local configurational balance is just the projection through the inverse motion of the Cauchy balance in terms of Piola-Kirchhoff stress, in absence of dissipative driving forces [24, 25, 27].
- On another side the fundamental independent nature of the balance of configurational forces has been supported: such a balance has been postulated a

priori in an abstract way, then its (at the beginning) unspecified ingredients (Hamilton-Eshelby stress and configurational bulk forces) have been identified in terms of energy, standard actions and driving forces by means of an invariance requirement and the second law of thermodynamics, the use of which presumes the assignment of the free energy [16, 14, 15].

In the mechanics of simple elastic bodies undergoing large deformations, essential differences between the balance of forces involving the first Piola-Kirchhoff stress (essentially the Euler-Lagrange equations arising from the ‘vertical’ variations of the elastic energy) and the balance of configurational actions (the ones accruing in this case by the ‘horizontal’ variations, according to Eshelby’s procedure) have been pointed out by the results in [12] (see also further remarks in [13] and [11], above all pages 152–153, vol. I, of the last reference).

Consider only the balances in the bulk and neglect circumstances in which point, line and surface defects occur just for the sake of simplicity. For smooth minimizers it is obviously true that, in absence of evolution governed by a driving force, the configurational balance equations can be obtained by pulling back in the reference place by means of the inverse motion the relevant balances in terms of standard Piola-Kirchhoff stress. Different is the case of irregular minimizers. They are common because existence theorems place minimizers of the elastic energy in Sobolev spaces. Sobolev maps do not admit always tangential derivatives. For this reason, in absence of regularity of minimizers, one cannot compute the balance of forces in terms of Piola-Kirchhoff stress from the first (‘vertical’) variation of the energy functional. The so-called horizontal variations are however admitted: they alter the reference place and lead to the balance of configurational forces (at least the one not accounting for driving force, the absence of which is justified in this case by the purely conservative behavior). Similar variations are also admissible on the actual place of the body: under appropriate bounds for the derivatives of the energy (or better of its polyconvex representative) one finds the weak form of the balance of forces in terms of Cauchy stress and proves also that such a stress is locally L^1 . In fact, not always natural minimizers of the elastic energy admit the inverse. Additional conditions are needed. They are expressed in [3, 6, 12]. So, even if one reduces the attention to the statics of elastic simple bodies undergoing large deformation, the use of the inverse deformation to deduce the balance of configurational forces has to be tackled perhaps with some care.

Horizontal variations have been used later to justify the use of configurational forces in various circumstances (see, e.g., [26]).

2.

In the ensuing sections, by restricting the attention to simple continuous bodies, I present a procedure based on $\mathbb{R}^3 \times SO(3)$ invariance of a certain power that I call the *relative power*. It allows one to obtain (i) the balance of both standard and configurational actions and (ii) the identification of configurational ingre-

dients from a unique source. The idea is based on the definition of two virtual velocity fields v and w acting one over the ambient space and the other over the space in which the material configuration of the body is placed. The latter field is then pushed forward on the ambient space, along the motion, and the power performed by the standard actions in the difference between v and the image of w is evaluated. Such a power is supplemented by power generated in the matter by the possible disarrangement and permutation of defects that are determined by the action of w in the material space. The sum of all these contributions is exactly the functional that I call the *relative power*. Its definition is not exotic and is not different in essence from the one of standard power. It reduces to the standard expression of the power when the reference place is fixed once and for all as it happens in standard continuum mechanics.

Neither surface, line and point defects, nor material complexity inside material elements are accounted for. They are matter of future work. Here the attention is focused only on the basic skeletal idea.

Use of the inverse motion is not required. No integral configurational balance is postulated. The integral balance of configurational forces and a configurational balance of torques are derived and correspond to Killing fields of the metric in the material space. The existence of a free energy density is postulated but the list of state variables entering its constitutive structure is not specified to a certain extent. No use is made of the mechanical dissipation inequality to identify the purely mechanical part of configurational forces. The procedure does not require a variational structure. A balance arising by the requirement of invariance of the relative power under changes in observers corresponds in purely conservative case to an integral version of Nöther theorem.

Differences and analogies with the two different points of view analyzed in [25] and [14] (developed in subsequent papers) are further discussed in the last section.

3.

The description of the standard kinematics of simple continuous deformable bodies is so well known that it barely needs to be retold. The setting is the classical space-time. A fit region \mathcal{B} (more simply, an open, connected set with Lipschitz boundary) of the standard ambient space \mathcal{E}^3 (the three-dimensional Euclidean point space) receives a body in its reference place. Each ensuing configuration is reached in an isomorphic copy of \mathcal{E}^3 , indicated by \mathcal{E}'^3 , by means of a transplacement, an orientation preserving diffeomorphism $x \mapsto y := y(x) \in \mathcal{E}'^3$. The set $\mathcal{B}_a := y(\mathcal{B})$ is then the actual configuration (placement) of the body. The spatial derivative of $x \mapsto y$ is indicated by $F := Dy(x) \in \text{Hom}(T_x\mathcal{B}, T_{y(x)}\mathcal{B}_a)$. The positivity of the determinant of F at each x from \mathcal{B} , i.e. $\det F > 0$, is implied by the assumption that the generic transplacement be orientation preserving. The additional requirement

$$\int_{\mathcal{B}} \tilde{f}(x, y(x)) \det Dy(x) dx \leq \int_{\mathbb{R}^3} \sup_{x \in \mathcal{B}} \tilde{f}(x, z) dz$$

for all $\tilde{f} \in C_0^\infty(\mathcal{B} \times \hat{\mathbb{R}}^3)$ is a global one-to-one condition allowing frictionless self-contact of the boundary while still preventing self-penetration (see [13]).

The forethought to put the reference place \mathcal{B} and the corresponding actual places in different (isomorphic) copies of the Euclidean space allows one to explain in a simple way the manner in which changes in observers are used here. Such a choice is also common in calculus of variations; it appears an useful point of view in determining the existence of minimizers of the energy of elastic simple bodies (see [13], above all vol. II).

In representing motions, time come into play and one has

$$(x, t) \mapsto y := y(x, t) \in \hat{\mathbb{R}}^3, \quad x \in \mathcal{B}, t \in [0, \bar{t}],$$

with a presumption of sufficient smoothness in time t , so that the velocity field is defined by

$$(x, t) \mapsto \dot{y} = \frac{d}{dt} y(x, t) \in \hat{\mathbb{R}}^3,$$

in the reference configuration.

Every subset b from \mathcal{B} with non-vanishing ‘volume’ measure and the same regularity of \mathcal{B} itself is called a *part*. The set $\mathfrak{P}(\mathcal{B})$ of all parts of \mathcal{B} is an algebra with respect to the operations of meet and join (see [5]).

Virtual velocity fields are defined over the ambient space and the reference places:

$$x \in \mathcal{B}, t \in [0, \bar{t}], \quad (x, t) \mapsto v := v(x, t) \in \hat{\mathbb{R}}^3, \quad (x, t) \mapsto w := w(x, t) \in \mathbb{R}^3.$$

They are assumed to be differentiable in space at every instant. The symbols V_v and V_w denote the function spaces containing them. Elements from V_v and V_w can be considered as virtual velocity fields over the body.

In the previous picture, the generic material element is collapsed just in a point which is the sole geometrical descriptor of its morphology. I use to call Cauchy bodies those bodies for which the minimalist approach summarized above is sufficient to represent the essential peculiarities of their morphology, the representation of actions is then conjugated in terms of power. Different is the case of complex bodies for which descriptors of the material substructure selected in a differentiable manifold are included in the representation of the morphology of the generic material element¹.

4.

*An observer is a representation of all geometrical environments that are necessary to describe the morphology of a body and its motion*². Here an observer is then a triple of atlas, one over the ambient space $\hat{\mathbb{R}}^3$, one over the material space con-

¹ See [4, 5, 7, 22] and references therein.

taining \mathcal{B} and the last one over the time interval $[0, \bar{t}]$. Changes in observers are then changes in these atlas, governed by the relevant groups of diffeomorphisms. In particular I consider synchronous isometric changes in observers. Synchronicity means that the representation of the time interval is left invariant.

The requirement that changes in observers be isometric means that one is considering the action of $\hat{\mathbb{R}}^3 \ltimes SO(3)$ over \mathcal{E}^{I3} (the symbol \ltimes indicates the semi-direct product) and the one of $\mathbb{R}^3 \ltimes SO_\diamond(3)$ over \mathcal{E}^3 . To maintain distinguished the changes in observers in \mathcal{E}^{I3} and \mathcal{E}^3 respectively, $\hat{\mathbb{R}}^3$ and \mathbb{R}^3 are considered distinct isomorphic spaces. The same relation occurs between $SO(3)$ and $SO_\diamond(3)$.

By indicating by v^* the pull-back in the frame of the first observer of the rate measured by the second observer, the action of $\hat{\mathbb{R}}^3 \ltimes SO(3)$ over \mathcal{E}^{I3} gives rise to the standard formula

$$v^* = \hat{c}(t) + \hat{q}(t) \times (y - y_0) + v,$$

where y_0 is an arbitrary point in \mathcal{E}^{I3} , $\hat{c}(t) \in \hat{\mathbb{R}}^3$ and $\hat{q}(t) \times \in so(3)$, with $so(3)$ the Lie algebra of $SO(3)$.

In standard approaches, it is then assumed that all observers evaluate the same \mathcal{B} . Here the assumption is removed and the independent action of $\mathbb{R}^3 \ltimes SO_\diamond(3)$ over \mathcal{E}^3 leads to

$$w^* = c(t) + q(t) \times (x - x_0) + w,$$

where w^* is the pull-back in the frame of the first observer of the counterpart w' of w measured by the second observer, x_0 is an arbitrary point in \mathcal{E}^3 , $c(t) \in \mathbb{R}^3$ and $q(t) \times \in so_\diamond(3)$, with $so_\diamond(3)$ the Lie algebra of $SO_\diamond(3)$. The transformation $w \mapsto w^*$ can be also considered as an isometric shift superposed to a generic re-labeling in the ‘material space’ with infinitesimal generator w .³

5.

Surface and bulk actions are associated with (generated by) relative changes of places between neighboring material elements: at every $x \in \mathcal{B}$ they are represented respectively by the first Piola-Kirchhoff stress $P \in Hom(T_x^* \mathcal{B}, T_{y(x)}^* \mathcal{B}_a) \simeq \mathbb{R}^3 \otimes \mathbb{R}^{3*}$ and the vector of bulk forces $b \in \hat{\mathbb{R}}^{3*}$ which includes inertial actions when they are present.

²Such a definition has non-trivial consequences above all in the mechanics of complex bodies, rather than in the one of simple bodies (see references in footnote 2).

³The point of view has been also discussed in [29] for different purposes. It is also used in [15] with strict reference to the derivation of configurational balances. In fact, a requirement of invariance of an expression of a power with respect to such changes in observers is called upon. The power selected is *different* from the one used here and involves a number of configurational actions (stresses, internal and external bulk forces and couples) in an abstract way, without having at that stage their possible expression in terms of standard actions. This point is further analyzed in the last section.

The standard power of external actions over a generic part b is defined by

$$\mathcal{P}_b^{ext}(\dot{y}) := \int_b b \cdot \dot{y} dx + \int_{\partial b} Pn \cdot \dot{y} d\mathcal{H}^2.$$

Notice that the expression of the external power is usually written by imagining that the reference place \mathcal{B} does not undergo mutations. The requirement of invariance of $\mathcal{P}_b^{ext}(\dot{y})$, under changes in observers leaving invariant \mathcal{B} and altering isometrically the ambient space, furnishes integral and then pointwise balance equations under appropriate regularity conditions [30].

In the lines above it has been stressed that bulk actions b and standard tractions Pn are co-vectors. Consequently, the products $b \cdot \dot{y}$ and $Pn \cdot \dot{y}$ have to be intended as the values that the co-vectors b and Pn take over the vector \dot{y} . No use is made here of the internal product in \mathbb{R}^3 . Of course, in the present setting such a distinction can be considered superfluous because \mathbb{R}^3 and its dual are naturally isomorphic, so that co-vectors and vectors can be identified. I maintain the distinction because it plays a non-trivial role when one would try to extend the procedure to the mechanics of complex bodies.

Here the point of view is different: the body can mutate its material structure. The world ‘mutation’ needs mechanical definition. I do not consider any specific mechanism of mutation. Rather, I account for the indirect effects of classes of mutations: energy fluxes in the material, bulk driving forces and configurational couples. All these ingredients are pictured in \mathcal{B} . They can be considered as due to the rearrangements of possible inhomogeneities, their possible evolution and/or to more general alterations of the material structure that can be pictured through mutations of the reference placement \mathcal{B} . An extended notion of power is then required. I call it the *relative power*: it is the power of standard actions evaluated on the relative velocity to the rates of mutations in the reference place, and supplemented by the energy fluxes and the power of additional bulk actions. The definition of the relative power is presented after necessary ensuing preliminaries.

A free energy density e is defined over \mathcal{B} ; it is function of the state ζ , the place x and the time t , namely

$$e := e(x, t; \zeta).$$

The state ζ of a material element is not specified here. The explicit (direct) dependence on x underlines the assumption that the material is not homogeneous. The explicit dependence on time may describe only some aspects of possible mutations like aging. For example, for elastic bodies with time-dependent moduli, the Clausius-Duhem inequality in its isothermal version implies $\partial_t e \leq 0$ which corresponds exactly to aging in time. In what follows the derivative $\partial_x e$ is considered as the ‘explicit’ derivative of e with respect to x , holding fixed the state and the time.

For the sake of simplicity, I do not consider below the explicit dependence on time, so that from now on the free energy depends on the place x and the state ζ .

Standard tractions and bulk forces arise during a generic motion. They are power-conjugated with the rate of changes of (relative) places of material elements. They contribute to the equilibrium of defects and their evolution.

In presence of evolving structural *mutations in the bulk*, annihilation and creation of material bonds occur. Moreover, microscopic viscous or plastic phenomena may cluster up to determining macroscopic structural bulk mutations which have consequent configurational effects. Bulk actions are then power-conjugated with mechanisms of annihilation and creation (or restoration) of material bonds and the other dissipative effects just mentioned. Such actions are modeled here by a force f and a couple $\tilde{\mu}$ (co-vectors over \mathcal{B}) with the proviso that

- (i) f be power-conjugated with the translational part of the material velocity field $x \mapsto w$, namely $w - \text{curl}w \times (x - x_0)$,
- (ii) $\tilde{\mu}$ be power-conjugated with the rotational part of $x \mapsto w$, namely $\text{curl}w$.

The items (i) and (ii) can be assumed by definition. A reasonable assumption is that f and $\tilde{\mu}$ can be each one decomposed additively in parts f^r and $\tilde{\mu}^r$ associated with the rupture of the material bonds and parts f^v and $\tilde{\mu}^v$ determined by the other dissipative effects (namely $f = f^r + f^v$ and $\tilde{\mu} = \tilde{\mu}^r + \tilde{\mu}^v$). If the material bonds are not broken in the process described by the field $x \mapsto w$, one gets $f^r = 0$ and $\tilde{\mu}^r = 0$. The material velocity can be also irrotational. In this case it can describe mutations that do not involve material couples: think for example to a planar interface evolving in an isotropic bar by maintaining itself orthogonal to the axis of the bar. When the body is not homogeneous and not isotropic, the redistribution of the material elements may generate a flow $\partial_x e$ of energy. Moreover, if the redistribution of inhomogeneities determines anisotropies, a configurational couple μ^c can also appear in principle.

The symbol μ is adopted from now on to identify in a concise way the sum $\tilde{\mu} + \mu^c$.⁴

Both f and μ are co-vectors over \mathcal{B} because they are associated with mechanisms changing \mathcal{B} itself. No configurational traction associated with a primitive configurational stress is presumed a priori: it is found later as a derived ‘object’.

Inertia is neglected here for the sake of simplicity. It can be included by considering the bulk forces decomposed additively into inertial and non-inertial parts and ‘adding’ to e the kinetic energy.

DEFINITION 1. For Cauchy bodies a functional $\mathcal{P}^{rel} : \mathfrak{F}(\mathcal{B}) \times V_v \times V_w \rightarrow \mathbb{R}$ is called a relative power when (i) it is additive over disjoint parts, (ii) is linear over the space of rates and admits the explicit expression

$$\mathcal{P}_b^{rel}(v, w) := \mathcal{P}_b^{rel-a}(v, w) + \mathcal{P}_b^{dis}(v, w)$$

with

⁴Configurational bulk couples have been introduced in [15] with the analogous meaning.

$$\begin{aligned} \mathcal{P}_b^{rel-a}(v, w) &:= \int_b b \cdot (v - Fw) dx + \int_{\partial b} Pn \cdot (v - Fw) d\mathcal{H}^2, \\ \mathcal{P}_b^{dis}(v, w) &:= \int_{\partial b} (n \cdot w)e d\mathcal{H}^2 - \int_b (\partial_x e + f) \cdot (w - curl w \times (x - x_0)) dx \\ &\quad + \int_b \mu \cdot curl w dx \end{aligned}$$

I call $\mathcal{P}_b^{rel-a}(v, w)$ the *relative power of actions* and $\mathcal{P}_b^{dis}(v, w)$ the *power due to disarrangements*.

- (1) The power of actions is said to be relative because it is developed along the difference between the actual velocity and the push forward of the material velocity w in \mathcal{B}_a .
- (2) More difficult is the interpretation of the terms in $\mathcal{P}_b^{dis}(v, w)$. Recall that the velocity field $(x, t) \mapsto v$ moves just points in space where no material elements are necessarily placed. The field $(x, t) \mapsto w$ alters the distribution of the material elements, even permuting them in a virtual way. A flux of energy through the boundary ∂b appears because the material elements can cross ∂b . Such a flux through ∂b has density $(n \cdot w)e$. The second and the third addenda in the expression of $\mathcal{P}_b^{dis}(v, w)$ are *sources* of power due to the possible rupture of material bonds, to other dissipative effects and to the redistribution of inhomogeneities. The assumption that $\partial_x e$ develops power only in the translational part of w is suggested by the circumstance that it is rather natural to imagine that, at least locally, the energy levels are parallel. The negative sign before $(\partial_x e + f)$ appears for the sake of convenience.

Take note that v may coincide with the true velocity \dot{y} at x and t . Further physical justifications of Definition 1 are presented later.

THEOREM 1. *The following two sets of assertions are equivalent. Set 1: $\mathcal{P}_b^{rel}(v, w)$ is invariant under isometric changes in observers for any choice of b . Set 2: (i) If the fields $x \mapsto b := b(x)$ and $x \mapsto P := P(x)$ are integrable over \mathcal{B} , then for every part b the following integral balances hold:*

$$\begin{aligned} \int_b b dx + \int_{\partial b} Pn d\mathcal{H}^2 &= 0, \\ \int_b (y - y_0) \times b dx + \int_{\partial b} (y - y_0) \times Pn d\mathcal{H}^2 &= 0, \\ \int_{\partial b} Pn d\mathcal{H}^2 - \int_b F^* b dx - \int_b (\partial_x e + f) dx &= 0, \\ \int_{\partial b} (x - x_0) \times Pn d\mathcal{H}^2 - \int_b (x - x_0) \times F^* b dx + \int_b \mu dx &= 0. \end{aligned}$$

where, with I the second order unit tensor,

$$\mathbb{P} := eI - F^*P.$$

(ii) If the fields $x \mapsto P$ and $x \mapsto \mathbb{P}$ are of class $C^1(\mathcal{B}) \cap C^0(\overline{\mathcal{B}})$ then

$$\begin{aligned} \text{Div } P + b &= 0, \\ \text{Skw } PF^* &= 0, \\ \text{Div } \mathbb{P} - F^*b - \partial_x e &= f, \\ 2 \text{Skw } \mathbb{P} &= \mu \times \end{aligned}$$

(iii) If the material is homogeneous, no driving force is present, and $\mu = 0$, then \mathbb{P} is symmetric and, in absence of body forces,

$$\int_{\partial \mathfrak{b}} \mathbb{P} n d\mathcal{H}^2 = 0$$

(iv) An extended version of the virtual power principle holds:

$$\mathcal{P}_b^{\text{rel}}(v, w) = \mathcal{P}_b^{\text{rel-inn}}(v, w),$$

where

$$\begin{aligned} \mathcal{P}_b^{\text{rel-inn}}(v, w) := & \int_b (P \cdot \nabla v + \mathbb{P} \cdot \nabla w + (x - x_0) \\ & \otimes (\partial_x e - f) \cdot \text{Skw } \nabla w + \mu \cdot \text{curl } w) dx; \end{aligned}$$

it reduces to

$$\begin{aligned} & \int_b b \cdot (v - Fw) dx + \int_{\partial \mathfrak{b}} Pn \cdot (v - Fw) d\mathcal{H}^2 - \int_b (\partial_x e + f) \cdot w dx \\ & = \int_b P \cdot (\nabla v - F\nabla w) dx - \int_b \nabla e \cdot w dx. \end{aligned}$$

PROOF. Here only *Set 1* \Rightarrow *Set 2* is proven because the converse is almost immediate. The axiom of invariance and some elementary algebra impose that

$$\begin{aligned} & \hat{c} \cdot \left(\int_b b dx + \int_{\partial \mathfrak{b}} Pn d\mathcal{H}^2 \right) + \hat{q} \cdot \left(\int_b (y - y_0) \times b dx + \int_{\partial \mathfrak{b}} (y - y_0) \times Pn d\mathcal{H}^2 \right) \\ & + c \cdot \left(\int_{\partial \mathfrak{b}} (eI - F^*P)n d\mathcal{H}^2 - \int_b F^*b dx + \int_b (\partial_x e - f) dx \right) \\ & + q \cdot \left(\int_{\partial \mathfrak{b}} (x - x_0) \times \mathbb{P}n d\mathcal{H}^2 - \int_b (x - x_0) \times F^*b dx + \int_b \mu dx \right) = 0. \end{aligned}$$

The arbitrariness of c , q , \hat{c} and \hat{q} implies the integral balances in Theorem 1, once one defines $\mathbb{P} := eI - F^*P$. The pointwise balances follow by the application of

Gauss theorem. They imply the equality between $\mathcal{P}_b^{rel}(v, w)$ and $\mathcal{P}_b^{rel-imm}(v, w)$. The remaining statements follow straight away by direct calculation. \square

The tensor \mathbb{P} , representing in Theorem 1 the sum $eI - F^*P$ in a concise way, is called the *Hamilton-Eshelby stress*. As recalled in the preamble, the word ‘configurational’ is attributed to balances involving it. Besides its immediateness, the earlier theorem has some stringent theoretical consequences, as anticipated in the preamble.

- (1) To obtain the balance of configurational forces it is not necessary to make use of the procedure exploiting the inverse motion.
- (2) The Hamilton-Eshelby stress $\mathbb{P} := eI - F^*P$ and the bulk actions $-F^*b$ and $-\partial_x e$ are not introduced a priori as unknown objects and then identified with a procedure discussed further in the last section.
- (3) A version of the principle of virtual power different from usual arises, it includes the standard one when the reference place is considered invariant, invariance intended in the sense of absence of evolving defects.

The result can be extended to the case of complex bodies and to the case in which structured discontinuity surfaces and line defects are present. In the process, appropriate additional measures of interactions need to be introduced as objects power conjugated with the rate of change of the morphological descriptors of the substructure in the material elements (in the case of complex bodies) and/or deformations and evolution of surface and line defects. Even in that cases the procedure avoids the specification of the constitutive structure of the local state and the use to the mechanical dissipation inequality to identify the expression of the configurational forces in terms of standard measures of interaction.

Actually, all observers ‘measure’ the same value of the power which is a scalar. The statement in the Set 1 in the theorem above can be assumed as axiom. In this case it would be not different in its intrinsic meaning from the axiom of invariance of the standard power [30]. Differences in the expression of the power are dictated only by the situation under scrutiny.

6.

To explain further on the nature of the relative power, one may notice that in purely conservative case the equation

$$\mathcal{P}_b^{rel}(v, w) = \mathcal{P}_b^{rel-imm}(v, w)$$

reduces to an integral version of the pointwise balance appearing in Nöther theorem.

To prove such a statement consider a non-linear elastic inhomogeneous body with total energy given by

$$e(x, F) + u(y),$$

with e the elastic energy density—a function which is polyconvex in the gradient of deformation—and u the potential of body forces. Both e and u are assumed to be differentiable with respect to their arguments. The essential ingredients preparing Nöther theorem need also to be recalled briefly.

Consider smooth curves $s \mapsto \mathbf{f}_s$ on the group of diffeomorphisms $\text{Diff}(\hat{\mathbb{R}}^3, \hat{\mathbb{R}}^3)$ such that $\mathbf{f}_0 = \text{identity}$ and at every point in $\hat{\mathbb{R}}^3$ one gets $v = \frac{d}{ds} \mathbf{f}_s|_{s=0}$, where the field $y \mapsto v(y)$ coincides with the virtual velocity field introduced above over the ambient space.

The usual relabeling of the reference place is accounted for in \mathbb{R}^3 . From a physical point of view it reduces just to the permutation of inhomogeneities over \mathcal{B} . The relabeling is induced by the action of the special group of diffeomorphisms $\text{SDiff}(\mathbb{R}^3, \mathbb{R}^3)$, a group on which one selects smooth curves $s_1 \mapsto \mathbf{f}_{s_1}^1$ such that $\mathbf{f}_0^1 = \text{identity}$ and at every x one gets $w = \frac{d}{ds_1} \mathbf{f}_{s_1}^1|_{s_1=0}$, where the field $x \mapsto w(x)$ is a special case of the virtual velocity field w introduced earlier, special in the sense that it is isochoric.

Balance equations are obtained by requiring the minimum of the overall energy

$$\mathcal{E}(y) := \int_{\mathcal{B}} (e(x, F) + u(y)) \, dx$$

over an appropriate Sobolev space (commonly such a space is some $W^{1,p}$ or, more precisely, the space of weak diffeomorphisms discussed in [12, 13]). In the case of C^1 minimizers, pointwise Euler-Lagrange equations can be derived in standard way. They read

$$b + \text{Div } P = 0,$$

where now $b := -\partial_y u \in \hat{\mathbb{R}}^{3*}$ and $P := \partial_{F_e} \in \text{Hom}(T_x^* \mathcal{B}, T_{y(x)}^* \mathcal{B}_a)$. In the case of irregular minimizers one cannot obtain the previous equation because Sobolev maps do not admit in general tangential derivatives. The balance of configurational forces (discussed previously) arises as a consequence of the evaluation of the horizontal variations—the ones generated by altering \mathcal{B} through $s_1 \mapsto \mathbf{f}_{s_1}^1$. Moreover, one may obtain in distributional sense the balance of forces in terms of Cauchy stress $\sigma := (\det F)^{-1} \partial_{F_e} F^{-*} \in \hat{\mathbb{R}}^{3*} \otimes \hat{\mathbb{R}}^{3*}$ and may prove also that $y \mapsto \sigma$ belongs to L^1_{loc} (see [13]), as mentioned previously.

By focusing the attention for the sake of simplicity on the Euler-Lagrange equations above, if one defines the vector density

$$\mathfrak{F} := (e + u)w + \partial_{F_e} e^*(v - Fw),$$

if the total energy is equivariant with respect to the action of $\text{Diff}(\hat{\mathbb{R}}^3, \hat{\mathbb{R}}^3)$ and $\text{SDiff}(\mathbb{R}^3, \mathbb{R}^3)$, then (Nöther theorem, see e.g. [23] and [19] for a rather different point of view)

$$\text{Div } \mathfrak{F} = 0.$$

The requirement that the total energy be equivariant means that

$$e(x, F) + u(y) = e(\mathbf{f}_{s_1}^1(x), (\text{grad } \mathbf{f}_s(y))F(\nabla \mathbf{f}_{s_1}^1(x))^{-1}) + u(\mathbf{f}_s(y)),$$

where grad is the gradient with respect to y . The previous relation is verified when (Nöther conditions)

$$\frac{d}{ds} (e(\mathbf{f}_{s_1}^1(x), (\text{grad } \mathbf{f}_s(y))F(\nabla \mathbf{f}_{s_1}^1(x))^{-1}) + u(\mathbf{f}_s(y)))|_{s=0} = 0,$$

$$\frac{d}{ds_1} (e(\mathbf{f}_{s_1}^1(x), (\text{grad } \mathbf{f}_s(y))F(\nabla \mathbf{f}_{s_1}^1(x))^{-1}) + u(\mathbf{f}_s(y)))|_{s_1=0} = 0.$$

Such conditions read explicitly

$$\partial_y u \cdot v + \partial_F e \cdot \nabla v = 0,$$

$$\partial_x e \cdot w - \partial_F e \cdot F \nabla w = 0.$$

By taking into account that (i) in this case the field $x \mapsto w(x)$ is isochoric, namely $\text{div } w = 0$, and (ii) absence of dissipative effects implies $f = 0$, and considering also (iii) the explicit form of Nöther conditions, one realizes (after some algebra) that the last relation in Theorem 1 reduces to the integral version of Nöther theorem, namely

$$\int_{\mathfrak{b}} \text{Div } \mathfrak{F} \, dx = 0$$

on some arbitrary part \mathfrak{b} of \mathcal{B} .

Conversely, one can say that the presence of a principle of relative power is hidden in Nöther theorem.

I have already introduced in earlier papers, namely [7, 21, 22], a version of the relative power including constitutive issues and arising directly from Nöther theorem, at least formally. I have used it in the description of surfaces and lines defects, without being conscious at that time of its generality in a full non-conservative setting. The extension to such a setting (at least with reference to bulk actions) is the main result here.

7.

Other comparisons with the existing descriptions of the origin and the nature of configurational forces may clarify further the point of view discussed here.

The approach to configurational forces presented in [24, 25, 26] relies on constitutive assumptions. They are called upon only partially in this paper: the sole existence of the energy and the assumption $e := e(x, t; \zeta)$ are invoked without specifying the nature of the state ζ .

The comparison with the approach proposed in [16, 14] (see also [15]) requires a preliminary description. That approach is based on two steps: (1) The balance of configurational forces is postulated first. Such a postulate can be expressed through the statement of an independent integral balance (like in [14]) or by requiring the invariance of a certain power (a power which is different from the one used here) with respect to the transformation $w \mapsto w^*$ (like in [15]; see also [33]). In [14] and [15], independently of its origin, the balance of configurational forces involves a configurational stress, say \mathbb{P} , and external and internal configurational bulk forces, say $\tilde{\mathbf{g}}$ and $\tilde{\mathbf{e}}$ respectively. They are assumed to perform work only *after* a Galilean change in observer (i.e. $w \mapsto w + c$) at a first glance (see [15], page 36). The point of view is then changed ([15], page 39) by saying that only $\tilde{\mathbf{g}}$ does not perform power under time-dependent changes in the reference place. Whether \mathbb{P} , $\tilde{\mathbf{g}}$ and $\tilde{\mathbf{e}}$ can be expressed in terms of standard actions and energy is not known at this stage. The identification of \mathbb{P} , $\tilde{\mathbf{g}}$ and $\tilde{\mathbf{e}}$ is matter of the second step. (2) It is essentially based on the exploitation of the second law of thermodynamics written in terms of a mechanical dissipation inequality in which only the power of the configurational traction $\mathbb{P}n$ is added to the one of standard actions (actually the power of $\tilde{\mathbf{g}}$ and $\tilde{\mathbf{e}}$ does not appear). The mechanical dissipation inequality is written with respect to control volumes with boundaries evolving in time. Invariance with respect to the reparametrization of such boundaries (which is the key ingredient in the procedure presented in [14]) leads to the identification $\mathbb{P} := eI - F^*P$. Notice that in the mechanical dissipation inequality the energy is introduced. It is assumed also that e is differentiable with respect to time. The identification of $\tilde{\mathbf{e}}$ with $-F^*b$ follows directly from the insertion of the expression $eI - F^*P$ in the balance of configurational forces. The mechanical dissipation inequality has to be exploited to recognize that $\tilde{\mathbf{g}}$ coincides with $-\nabla e + P : \nabla F$. The additional specification of the constitutive structure of the energy shows how $\tilde{\mathbf{g}}$ reduces finally to $-\partial_x e$ (see [14], page 78).

Comparison of the results in [14] with the point of view in the previous sections has to be done at the end of the identification procedure (so that after step 2) because not only Theorem 1 collects the balance of configurational forces but also it *includes* the results of the identification recalled above, at least in the setting discussed here.

In the theorem above the configurational actions f and μ are left unspecified. However, if there is absence of effects due to dissipative stresses and rupture of the material bonds, $f = 0$ and $\tilde{\mu} = 0$. The configurational couple μ^c remains still unspecified. Moreover, if no rupture of the material bonds occur and the stress P has a non-zero dissipative part P^v (so it is the sum of non-zero conservative and dissipative components), the use of the configurational balance in the bulk implies (after the explicit calculation of $\text{Div } \mathbb{P}$) that $f = -P^v : \nabla F$, where the double dots indicate that the third-rank tensor ∇F acts over P^v by saturation of both its two indices.

To obtain Theorem 1—I stress once more—no use is made of the mechanical dissipation inequality. No use is made of an additional requirement of invariance of the power with respect to the reparametrization of ∂b . Although the energy is also introduced here, no assumption of differentiability in time is necessary.

The state here is not specified: for example it can include F , the history of deformation, a number of internal variables conjugated with affinities, their histories and gradients. The sole restriction is that the state be compatible with the relative power of actions. In fact, higher-order Cauchy bodies (like, e.g., second-grade elastic bodies) or complex bodies require expressions of \mathcal{P}^{rel-a} involving hyperstresses or microstresses and self-actions respectively.

I do not claim that the treatment proposed here is better than the ones discussed in this section. My approach is just parallel in some sense and is also rather concise. The reader interested in foundational issues can find by himself/herself the right position of this thin note, written by using elementary mathematics.

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REFERENCES

- [1] ABEYARATNE R. - KNOWLES J. K. (1990), On the driving traction acting on a surface of strain discontinuity in a continuum, *J. Mech. Phys. Solids*, **48**, 1–27.
- [2] ABEYARATNE R. - VEDANTAM S. (2003), A lattice-based model of the kinetics of twin boundary motion, *J. Mech. Phys. Solids*, **51**, 1675–1700.
- [3] BALL, J. M. (1981), Global invertibility of Sobolev functions and the interpenetration of matter, *Proc. Royal Soc. Edinburgh*, **88A**, 315–328.
- [4] CAPRIZ G. (1989), *Continua with microstructure*, Springer Verlag, Berlin.
- [5] CAPRIZ G. - VIRGA E. G. (1990), Interactions in general continua with microstructure, *Arch. Rational Mech. Anal.*, **109**, 323–342.
- [6] CIARLET, P. - NEČAS, J. (1987), Injectivity and self-contact in nonlinear elasticity, *Arch. Rational Mech. Anal.*, **97**, 171–188.
- [7] DE FABRITIIS C. - MARIANO P. M. (2005), Geometry of interactions in complex bodies, *J. Geom. Phys.*, **54**, 301–323.
- [8] EPSTEIN M. - MAUGIN G. A. (1995), Thermoelastic material forces: definitions and geometric aspects, *C. R. Acad. Sci. Paris Sér. II*, **320**, 63–68.
- [9] ERICKSEN J. L. (1998), On nonlinear elasticity theory of crystal defects, *Int. J. Plasticity*, **14**, 9–24.
- [10] ESHELBY J. D. (1975), The elastic energy momentum tensor, *J. Elasticity*, **5**, 321–335.
- [11] GIAQUINTA, M. - HILDEBRANDT, S., *Calculus of variations, vol. I The Lagrangian formalism, vol. II The Hamiltonian formalism*, Springer-Verlag, Berlin, 1996.
- [12] GIAQUINTA M. - MODICA G. - SOUČEK J. (1989), Cartesian currents, weak diffeomorphisms and existence theorems in nonlinear elasticity, *Arch. Rational Mech. Anal.*, **106**, 97–159. Erratum and addendum, *Arch. Rational Mech. Anal.*, (1990) **109**, 385–392.
- [13] GIAQUINTA M. - MODICA G. - SOUČEK J. (1998), *Cartesian currents in the calculus of variations*, vols. I–II, Springer-Verlag, Berlin.
- [14] GURTIN M. E. (1995), The nature of configurational forces, *Arch. Rational. Mech. Anal.*, **131**, 67–100.
- [15] GURTIN M. E. (2000), *Configurational forces as basic concepts of continuum physics*, Springer Verlag, New York.

- [16] GURTIN M. E. - STRUTHERS A. (1990), Multiphase thermomechanics with interfacial structure. III. Evolving phase boundaries in the presence of bulk deformation, *Arch. Rational Mech. Anal.*, **112**, 97–160.
- [17] KALPAKIDES V. - DASCALU C. (2002), On the configurational force balance in thermomechanics, *Proc. Royal Soc. London A*, **458**, 3023–3039.
- [18] KIENZLER R. - HERMANN G. (2000), *Mechanics in material space*, Springer Verlag, Berlin.
- [19] KNOWLES J. K. - STERNBERG E. (1972), On a class of conservation laws in linearized and finite elastostatics, *Arch. Rational Mech. Anal.*, **44**, 187–211.
- [20] JAMES R. D. (2002), Configurational forces in magnetism with application to the dynamics of a small-scale ferromagnetic shape memory cantilever, *Contin. Mech. Thermodyn.*, **14**, 55–86.
- [21] MARIANO P. M. (2006), Mechanics of quasi-periodic alloys, *J. Nonlinear Sci.*, **16**, 45–77.
- [22] MARIANO P. M. (2008), Cracks in complex bodies: covariance of tip balances, *J. Nonlinear Sci.*, **18**, 99–141.
- [23] MARSDEN J. E. - HUGHES T. R. J. (1994), *Mathematical foundations of elasticity*, Dover, New York.
- [24] MAUGIN G. A. (1993), *Material inhomogeneities in elasticity*, Chapman & Hall, London.
- [25] MAUGIN G. A. (1995), Material forces: concepts and applications, *Appl. Mech. Rev.*, **48**, 213–245.
- [26] MAUGIN G. A. - TRIMARCO C. (1992), Pseudomomentum and material forces in nonlinear elasticity: variational formulations and application to brittle fracture, *Acta Mech.*, **94**, 1–28.
- [27] MAUGIN G. A. - TRIMARCO C. (1995), On material and physical forces in liquid crystals, *Int. J. Engng Sci.*, **33**, 1663–1678.
- [28] MENZEL A. - STEINMANN P. (2007), On configurational forces in multiplicative elastoplasticity, *Int. J. Solids Structures*, **44**, 4442–4471.
- [29] MURDOCH A. I. (2006), Some primitive concepts in continuum mechanics regarded in terms of objective space-time molecular averaging: the key rôle played by inertial observers, *J. Elasticity*, **84**, 69–97.
- [30] NOLL W. (1973), Lectures on the foundations of continuum mechanics and thermodynamics, *Arch. Rational Mech. Anal.*, **52**, 62–69.
- [31] ROGULA D. (1977), Forces in material space, *Arch. Mech.*, **29**, 705–713.
- [32] SAMOHÝL I. - PABST W. (1997), The Eshelby relation in mixtures, *Int. J. Non-Linear Mech.*, **32**, 227–233.
- [33] SEGEV R. (1996), On smoothly growing bodies and the Eshelby tensor, *Meccanica*, **31**, 507–518.
- [34] ŠILHAVÝ M. (1997), *The mechanics and thermodynamics of continuous media*, Springer Verlag, Berlin.
- [35] SIMHA N. K. - BHATTACHARYA K. (2000), Kinetics of phase boundaries with edges and junctions in a three-dimensional multi-phase body, *J. Mech. Phys. Solids*, **48**, 2619–2641.
- [36] STEINMANN P. (2002), On spatial and material settings of thermo-hyperelastodynamics, *J. Elasticity*, **66**, 109–157.
- [37] STEINMANN P. (2002), On spatial and material settings of hyperelastostatic crystal defects, *J. Mech. Phys. Solids*, **50**, 1743–1766.

- [38] STEINMANN P. (2008), On boundary potential energies in deformational and configurational mechanics, *J. Mech. Phys. Solids*, **56**, 772–800.
- [39] TRUSKINOVSKY L. - VAINCHTEIN A. (2005), Kinetics of martensitic phase transitions: lattice model, *SIAM J. Appl. Math.*, **66**, 533–553.

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