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Helical magnetic fields and semi-classical asymptotics of the lowest eigenvalue

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Abstract. We study the three-dimensional Neumann magnetic Laplacian in the presence of a semi-classical parameter and a non-uniform magnetic field with constant intensity. We determine a sharp two term asymptotics for the lowest eigenvalue, where the second term involves a quantity related to the magnetic field and the geometry of the domain. In the special case of the unit ball and a helical magnetic field, the concentration takes place on two symmetric points of the unit sphere.

1. Main results

Let $\Omega \subset \mathbb{R}^3$ be an open and bounded set with a smooth boundary $\partial\Omega$. Let us consider a smooth magnetic field $\mathbf{B} : \bar{\Omega} \to \mathbb{R}^3$ (so $\mathbf{B}$ should be closed) which will always be assumed to satisfy

$$\forall x \in \Omega, \quad \big|\mathbf{B}(x)\big| = b, \tag{1.1}$$

where $b > 0$ is a constant. Without loss of generality, we assume from now on that $b = 1$. Let $\mathbf{A}(x)$ be a magnetic potential such that

$$\operatorname{curl} \mathbf{A} = \mathbf{B}. \tag{1.2}$$

We are interested in the analysis of the lowest eigenvalue $\lambda_1(\mathbf{A}, h)$ of the Neumann realization of the Schrödinger operator in Ω with magnetic field

$$P^h_{\mathbf{A}} := \Delta_{h,\mathbf{A}} = \sum_{j=1}^{3} \big(hD_{x_j} + A_j(x)\big)^2. \tag{1.3}$$

We introduce the following assumptions.

Assumption 1.1. *The set of boundary points where* $\mathbf{B}$ *is tangent to* $\partial\Omega$*, i.e.*

$$\Gamma := \big\{x \in \partial\Omega \mid \mathbf{B} \cdot \mathbf{N}(x) = 0\big\}, \tag{1.4}$$

is a regular submanifold of $\partial\Omega$*:*

$$\kappa_{n,\mathbf{B}}(x) := \big|d^T(\mathbf{B} \cdot \mathbf{N})(x)\big| \neq 0, \quad \forall x \in \Gamma. \tag{1.5}$$

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Here d^T is the differential defined on functions on $\partial\Omega$ and $\mathbf{N}(x)$ is the unit inward normal of Ω.

Assumption 1.2. *The set of points where $\mathbf{B}$ is tangent to Γ is finite.*

These assumptions are rather generic and for instance satisfied for ellipsoids, when $\mathbf{B}$ is constant. When $|\mathbf{B}|$ is constant, the above assumptions hold for the sphere with a helical magnetic field (see Section 3).

Let us introduce the constant $\widehat{\gamma}_{0,\mathbf{B}}$ involving the "magnetic curvature" in (1.5), which is defined by

$$\widehat{\gamma}_{0,\mathbf{B}} := \inf_{x\in\Gamma} \widetilde{\gamma}_{0,\mathbf{B}}(x), \tag{1.6}$$

where

$$\widetilde{\gamma}_{0,\mathbf{B}}(x) := 2^{-2/3}\widehat{\nu}_0\delta_0^{1/3}\big|\kappa_{n,\mathbf{B}}(x)\big|^{2/3}\Big(1-(1-\delta_0)\big|\mathbf{T}(x)\cdot\mathbf{B}(x)\big|^2\Big)^{1/3}. \tag{1.7}$$

Here $\mathbf{T}(x)$ is the oriented, unit tangent vector to Γ at the point x, $\delta_0 \in\,]0,1[$ and $\widehat{\nu}_0 > 0$ are spectral quantities relative to the de Gennes and Montgomery operators which will be introduced in (4.2) and (4.4).

When $\mathbf{B}$ is constant, the lowest eigenvalue is expected to have the following asymptotic expansion [9, 20]

$$\lambda_1^N(B) \sim \sum_{j\geq 6} c_j h^{j/6}.$$

The first terms of the foregoing expansion have been obtained by Helffer–Morame [17] and Pan [25].

Theorem 1.3 (Helffer–Morame [17]). *Let us assume that $\mathbf{B}$ is constant. Then, if Ω and $\mathbf{B}$ satisfy Assumptions 1.1 and 1.2, there exists $\eta > 0$ such that the lowest eigenvalue $\lambda_1^N(\mathbf{A},h)$ satisfies as $h\to 0$*

$$\lambda_1^N(\mathbf{A},h) = \Theta_0 h + \widehat{\gamma}_{0,\mathbf{B}} h^{\frac{4}{3}} + \mathcal{O}(h^{\frac{4}{3}+\eta}). \tag{1.8}$$

The aim of this paper is to prove that Theorem 1.3 also holds under the weaker assumption that $|\mathbf{B}|$ is constant.

Theorem 1.4. *If Assumptions 1.1 and 1.2 hold, and if $|\mathbf{B}|$ is constant, then the asymptotics in (1.8) holds for the lowest eigenvalue $\lambda_1^N(\mathbf{A},h)$.*

An interesting example of a non-constant magnetic field but with a constant intensity is the helical magnetic field occurring in the theory of liquid crystals. Up to the action of an orthogonal matrix, it can be expressed as follows [27]

$$\mathbf{B} = \operatorname{curl}\mathbf{n}_\tau = -\tau\mathbf{n}_\tau, \quad \mathbf{n}_\tau = \left(\frac{1}{\tau}\cos(\tau x_3), \frac{1}{\tau}\sin(\tau x_3), 0\right). \tag{1.9}$$

Here $\tau > 0$ is a given constant. In this situation ($\mathbf{B} = -\tau\mathbf{n}_\tau$), [27] derived an upper bound on the eigenvalue $\lambda_1^N(\mathbf{A},h)$, which is consistent with Theorem 1.4. Our contribution is valid for a more general class of magnetic fields with constant intensity and also determines the asymptotically matching lower bound of the lowest eigenvalue.

Discussion and applications

The inspection of the eigenvalue $\lambda_1^N(\mathbf{A}, h)$ is vital in understanding the transition between *superconducting* and *normal* states in the Ginzburg–Landau model [5]. In this context, the magnetic field is typically constant. Accurate estimates of the lowest eigenvalue $\lambda_1^N(\mathbf{A}, h)$ under constant magnetic fields [16, 17] led to a precise understanding of the transition between superconducting and normal states [4, 10].

Non-homogeneous magnetic fields with constant intensity are encountered in the Landau–de Gennes theory of liquid crystals, which is the analog of the Ginzburg–Landau theory of superconductivity. Here a transition between *smectic* and *nematic* phases occurs. Our main result, Theorem 1.4, yields an accurate estimate of the lowest eigenvalue $\lambda_1^N(\mathbf{A}, h)$ for magnetic fields with constant intensity, and by analogy with [4], we expect it to yield a precise description of the transition between surface smectic and nematic states (see [24]).

At the threshold of the phase transition, both superconductive and smectic states nucleate on the surface of the domain (near the curve Γ introduced in (3.3)). The paper [26] contains a nice discussion of this interesting analogy. The analysis of 3D surface superconductivity is the subject of the papers [7, 8, 25], while surface smectics are rigorously studied in [6, 18]. It would be interesting to complete this analysis by providing more accurate estimates at the threshold, where the linear analysis (such as the one in this paper) becomes handy.

The analysis in this paper concerns the lowest eigenvalue. In the presence of a constant magnetic field, and a "single well" assumption (i.e. the minimum in (1.6) is non-degenerate and attained at a unique point), accurate estimates of the low-lying eigenvalues were obtained recently in [20]. In our setting of a non-homogeneous magnetic field, the example of the ball under the helical magnetic field suggests the presence of multiple wells (see Remark 3.5).

The interaction between magnetic fields and 3D domains is interesting in other situations. In particular, for the Robin problem, we observe pure magnetic wells on the surface of the domain [11], and in the case of a constant magnetic field, strong diamagnetism does not hold for the ball [22].

Organization and outline of the proof

The proof of Theorem 1.4 is split into two parts. In the first part, we establish a lower bound of the lowest eigenvalue, by comparing the quadratic form via a simpler form related to a new model operator. Comparing with the constant magnetic field in [17], we prove that the model operator in our setting is a perturbation of the one considered in [17].

The second part of the proof is devoted to an upper bound of the lowest eigenvalue, already studied for **B** in (1.9) [27], but we revisit it since our formulation is not the same as [27]. The upper bound follows after computing the quadratic form of a suitable trial state, having the same structure as the constant magnetic field case in [17, 25]. However, there are additional terms in the computations due to the varying magnetic field, which require a careful handling.

The model operator takes into consideration two phenomena. First, after decomposing our domain into small cells and working in a small cell near the domain's boundary, we have to express the integrals in a flat geometry, which requires a careful expansion of the Riemannian metric in particular. This part is essentially the same as for the constant magnetic field case in [17].

Then, we have to express the magnetic potential in adapted coordinates, in each small cell, and apply a Taylor expansion and a gauge transformation to obtain a "normal" form, i.e. a simpler effective magnetic potential. In this part, we deviate from the constant magnetic field situation and find additional terms in the effective magnetic potential. Interestingly, we can still show that the analysis with this magnetic potential is somehow independent of those additional terms and treat the new model as a perturbation of the model with a constant magnetic field.

The paper is organized as follows. In Section 2 we introduce the adapted coordinates in a small "boundary" cell. In Section 3, we analyze the case of the unit ball with the "helical" magnetic field occurring in liquid crystals and verify that Assumptions 1.1 and 1.2 hold. Interestingly, after computing the energy in (1.6), we notice that this example shows a phenomenon of multiple "wells" induced by the "magnetic" geometry.

In Section 4, we review two standard 1D operators that we need in defining the quantities appearing in (1.6) and the statement in Theorem 1.4. Then, in Section 5, we introduce a new model, specific to our case of a varying magnetic field with a constant intensity, and analyze it through a perturbation argument.

With the model in Section 5, we can adjust the proof in [17] and prove Theorem 1.4. The first step is to localize the ground states near the boundary, which is the content of Section 6. Then, the approximation of the quadratic form and the magnetic potential are the subject of Section 7, which allows us, in the subsequent Section 8, to obtain a lower bound on the lowest eigenvalue.

Finally, Section 9 is devoted to the computation of the energy of a trial state, which yields an upper bound of the lowest eigenvalue, and thereby completes the proof of Theorem 1.4.

2. Adapted coordinates

We recall a rather standard choice of coordinates in the neighborhood of Γ, which straightens the boundary $\partial\Omega$ locally. For every $p \in \partial\Omega$, recall that $\mathbf{N}(p)$ denotes the inward normal vector to $\partial\Omega$.

2.1. Description of the coordinates

Let g_0 be the Riemannian metric on $\mathbb{R}^3$, which induces a Riemannian metric G on $\partial\Omega$. Given two vector fields $\mathbf{X}, \mathbf{Y}$ of $\mathbb{R}^3$, we denote by

$$\mathbf{X} \cdot \mathbf{Y} = \langle \mathbf{X}, \mathbf{Y} \rangle := g_0(\mathbf{X}, \mathbf{Y}).$$

Consider a moving direct frame $(\mathbf{V}(m), \mathbf{T}(m), \mathbf{N}(m))_{m \in \Gamma}$ along Γ such that

- $\mathbf{T}(m)$ is an oriented unit tangent vector of Γ;
- $\mathbf{V}(m) := \mathbf{T}(m) \times \mathbf{N}(m)$, hence determining an oriented normal to the curve Γ in the tangent space to $\partial\Omega$.

For $m \in \Gamma$, let Λ_m be the geodesic that passes through m and is normal to Γ. Let $x_0 \in \Gamma$. In some neighborhood $\mathcal{N}_{x_0} \subset \overline{\Omega}$ of x_0, we can introduce new coordinates (r, s, t) as follows:

- For $x \in \mathcal{N}_{x_0}$, $p(x) \in \partial\Omega$ is defined by $\operatorname{dist}(x, p(x)) = t(x) := \operatorname{dist}(x, \partial\Omega)$;
- For $x \in \mathcal{N}_{x_0}$, $\gamma(x) \in \Gamma$ is defined by $\operatorname{dist}_{\partial\Omega}(p(x), \gamma(x)) = \operatorname{dist}_{\partial\Omega}(p(x), \Gamma)$, where $\operatorname{dist}_{\partial\Omega}$ denotes the (geodesic) distance in $\partial\Omega$;
- Γ is parameterized by arc-length s so that $s = s_0$ defines x_0, and for $x \in \mathcal{N}_{x_0}$, $s = s(x)$ defines $\gamma(x)$;
- For $x \in \mathcal{N}_{x_0}$, the geodesic $\Lambda_{p(x)}$ passing through $p(x)$ is parameterized by arclength r, so that $r = 0$ defines $\gamma(x)$ and $r = r(x)$ defines $p(x)$.

In this way, we observe that Φ_{x_0} defined by

$$\mathcal{N}_{x_0} \ni x \mapsto \Phi_{x_0}(x) := \big(r(x), s(x), t(x)\big) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+ \tag{2.1}$$

is a local diffeomorphism which straightens $\mathcal{N}_{x_0}$. We pick a sufficiently small $\epsilon_0 > 0$ such that

$$(r, s, t) \in (-\epsilon_0, \epsilon_0) \times (-\epsilon_0 + s_0, s_0 + \epsilon_0) \times (0, \epsilon_0) \to x = \Phi_{x_0}^{-1}(r, s, t) \tag{2.2}$$

is a diffeomorphism, whose image is a neighborhood of $x_0 \in \Gamma$ parameterized by (r, s, t). Within these coordinates, $t = 0$ means that we are on $\partial\Omega$, and $r = t = 0$ means we are on the curve Γ. Moreover, $s \in (-\epsilon_0 + s_0, s_0 + \epsilon_0)$ marks a point m on Γ, $r \in (-\epsilon_0, \epsilon_0)$ marks a point $p \in \partial\Omega$ on the geodesic passing through m and orthogonal to Γ, and finally t marks a point in Ω lying on the normal to $\partial\Omega$ passing through p (see Figure 1). We can then compute

$$\big|d^T(\mathbf{B} \cdot \mathbf{N})(x)\big| = \big|\partial_r(\mathbf{B} \cdot \mathbf{N})|_{r=0}\big| \quad (x \in \Gamma). \tag{2.3}$$

It is convenient to express the magnetic field along Γ as follows

$$\mathbf{B}(x) = \sin\theta \mathbf{T}(x) + \cos\theta \mathbf{V}(x) \quad \big(x = \Phi_{x_0}^{-1}(0, s, 0) \in \Gamma\big), \tag{2.4}$$

where $\theta = \theta(s) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is the angle defined by

$$\theta = \arcsin\big(\mathbf{B}(x) \cdot \mathbf{T}(x)\big). \tag{2.5}$$

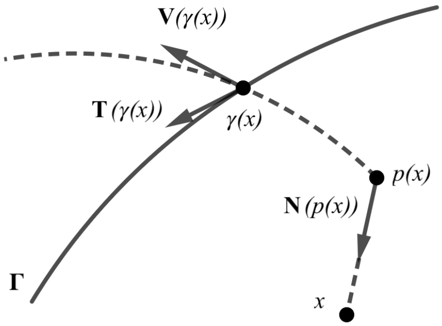

Figure 1. The point $x \in \Omega$, its projection $p(x)$ on $\partial\Omega$ and the inward normal vector $\mathbf{N}(p(x))$ of $\partial\Omega$ at $p(x)$. The point $\gamma(x)$ is the geodesic projection of $p(x)$ on Γ and $(\mathbf{T}(\gamma(x)), \mathbf{V}(\gamma(x)))$ is a direct frame in the tangent plane to $\partial\Omega$ at $\gamma(x)$. Note that, if $p(x) \in \Gamma$, then $p(x) = \gamma(x)$ and $\mathbf{N}(p(x))$ is orthogonal to $\mathbf{T}(\gamma(x)), \mathbf{V}(\gamma(x))$.

2.2. The metric in the new coordinates

Let us consider an arbitrary point $x_0 \in \Gamma$ and a neighborhood $\mathcal{N}_{x_0} \subset \overline{\Omega}$ of x_0 such that the adapted coordinates introduced in (2.1) and (2.2) are valid. Modulo a translation, we can center the coordinates at x_0 so that $(r = 0, s = 0, t = 0)$ are the coordinates of x_0 in the new frame. In the sequel, we follow closely the presentation of [17, Sec. 8] mainly following the first chapter of [2] (see also the volume two of Spivak's book [29]).

We label the new coordinates as follows

$$(y_1, y_2, y_3) = (r, s, t), \tag{2.6}$$

and the Riemannian metric g_0 becomes [17, (8.26)]

$$g_0 = dy_3 \otimes dy_3 + \sum_{1\le i,j\le 2} \big[G_{ij} - 2y_3 K_{ij} + y_3^2 L_{ij}\big] dy_i \otimes dy_j\,, \tag{2.7}$$

where

- $G := \sum_{1\le i,j\le 2} G_{ij}\, dy_i \otimes dy_j$ is the first fundamental form on $\partial\Omega$;
- $K := \sum_{1\le i,j\le 2} K_{ij}\, dy_i \otimes dy_j$ is the second fundamental form on $\partial\Omega$;
- $L := \sum_{1\le i,j\le 2} L_{ij}\, dy_i \otimes dy_j$ is the third fundamental form on $\partial\Omega$.

The matrix g of the metric g_0 takes the form

$$g := (g_{ij})_{1\le i,j\le 3} = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.8}$$

whose inverse is

$$g^{-1} = (g^{ij})_{1\le i,j\le 3} = \begin{pmatrix} g^{11} & g^{12} & 0 \\ g^{21} & g^{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We will express these matrices in a more pleasant form involving, in particular, the curvatures on the boundary. To that end, let $s \mapsto \gamma(s)$ be an arc-length parameterization of Γ near x_0, so that $|\dot\gamma(s)| = 1$, $\gamma(0) = x_0$ and $\mathbf{T}(\gamma(s)) = \dot\gamma(s)$. We can introduce the *geodesic* and *normal* curvatures at $\gamma(s)$, $\kappa_g(\gamma(s))$ and $\kappa_n(\gamma(s))$, as follows

$$\ddot\gamma(s) = -\kappa_g\big(\gamma(s)\big)\mathbf{V}\big(\gamma(s)\big) + \kappa_n\big(\gamma(s)\big)\mathbf{N}\big(\gamma(s)\big). \tag{2.9}$$

The choice of our coordinates (r, s) ensures that the metric G is diagonal on $\partial\Omega$ [17, Lem. 8.2]

$$G = dr \otimes dr + \alpha(r,s) ds \otimes ds, \tag{2.10}$$

with

$$\alpha(r,s) = 1 - 2\kappa_g\big(\gamma(s)\big)r + \mathcal{O}(r^2), \quad \alpha(0,s) = 1, \tag{2.11}$$

and

$$\frac{\partial\alpha}{\partial s}(0,s) = 0.$$

Then, with (2.6), we have for the determinant of the matrix of g (see [17, (8.29) and (8.30)]),

$$|g| = \alpha(r,s) - 2t\big[\alpha(r,s)K_{11}(r,s) + K_{22}(r,s)\big] + t^2\varepsilon_3(r,s,t), \tag{2.12}$$

and

$$(g^{ij})_{1\le i,j\le 2} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha^{-1}(r,s) \end{pmatrix} + 2t \begin{pmatrix} K_{11}(r,s) & \alpha^{-1}K_{12}(r,s) \\ \alpha^{-1}K_{21}(r,s) & \alpha^{-2}K_{22}(r,s) \end{pmatrix} + t^2 R,$$

where ε_3 and R are smooth functions.

2.3. The operator and quadratic form

We continue to work in the setting of Section 2.2. We introduce the following neighborhood of x_0

$$V_{x_0} = \Phi_{x_0}^{-1}(\widetilde V_{x_0}),$$

where (recall (2.6))

$$\widetilde V_{x_0} = \big\{(y_1,y_2,y_3) \in (-\epsilon_0,\epsilon_0)\times(-\epsilon_0,\epsilon_0)\times(0,\epsilon_0)\big\}. \tag{2.13}$$

Given a function $u : V_{x_0} \to \mathbb{C}$, we assign to it the function $\tilde u : V_{x_0} \to \mathbb{C}$ defined by

$$\tilde u(y_1,y_2,y_3) = u\big(\Phi_{x_0}^{-1}(y_1,y_2,y_3)\big). \tag{2.14}$$

By the considerations in Section 2.2 on the Riemannian metric, if $u \in L^2(V_{x_0}, dx)$, then $\tilde u \in L^2(\widetilde V_{x_0}, |g|^{1/2}dy)$ and

$$\int_{V_{x_0}} \big|u(x)\big|^2\, dx = \int_{\widetilde V_{x_0}} \big|\tilde u(y)\big|^2 |g|^{1/2} dy. \tag{2.15}$$

Moreover, assuming u supported in V_{x_0}, we have the quadratic form formula [17, (8.27)]

$$\begin{aligned} q_{\mathbf{A}}^h(u) &:= \int_{V_{x_0}} \left|(h\nabla - i\mathbf{A})u\right|^2 dx \\ &= \int_{\widetilde{V}_{x_0}} \Big[\left|(hD_{y_3} - \widetilde{A}_3)\tilde{u}\right|^2 + \sum_{1\le i,j\le 2} g^{ij}(hD_{y_i} - \widetilde{A}_i)\tilde{u}\cdot\overline{(hD_{y_j} - \widetilde{A}_j)\tilde{u}}\Big]|g|^{1/2}dy, \end{aligned} \tag{2.16}$$

where the new magnetic potential $\widetilde{\mathbf{A}} = (\widetilde{A}_1, \widetilde{A}_2, \widetilde{A}_3)$ is assigned to $\mathbf{A} = (A_1, A_2, A_3)$ by the relation

$$A_1dx_1 + A_2dx_2 + A_3dx_3 = \widetilde{A}_1dy_1 + \widetilde{A}_2dy_2 + \widetilde{A}_3dy_3, \tag{2.17}$$

and after performing a (local) gauge transformation, we may assume that

$$\widetilde{A}_3 = 0.$$

The operator $P_{\mathbf{A}}^h$ in (1.3) can be expressed in the new coordinates as follows [17, (8.28)]

$$\begin{aligned} P_{\mathbf{A}}^h = (hD_{y_3} - \widetilde{A}_3)^2 + \frac{h}{2i}|g|^{-1}\frac{\partial}{\partial y_3}|g|(hD_{y_3} - \widetilde{A}_3) \\ + |g|^{-1/2}\sum_{1\le i,j\le 2}(hD_{y_j} - \widetilde{A}_j)|g|^{1/2}g^{ij}(hD_{y_i} - \widetilde{A}_i). \end{aligned}$$

3. Helical magnetic fields

3.1. Preliminaries

Let $\tau > 0$ and consider the magnetic potential

$$\mathbf{A}(x) = \mathbf{n}_\tau(x) := \left(\frac{1}{\tau}\cos(\tau x_3), \frac{1}{\tau}\sin(\tau x_3), 0\right), \tag{3.1}$$

which generates the magnetic field

$$\mathbf{B}(x) = \operatorname{curl}\mathbf{A}(x) = -\tau\mathbf{A}(x) \tag{3.2}$$

with constant intensity

$$\left|\mathbf{B}(x)\right| = 1.$$

We will verify that Assumptions C1–C2 hold for this particular magnetic field in the case where Ω is the unit ball. In particular, with in mind that $\widehat{\gamma}_{0,\mathbf{B}}$ and $\widetilde{\gamma}_{0,\mathbf{B}}$ are introduced in (1.6) and (1.7) respectively and that $\delta_0 \in\,]0, 1[$ and $\widehat{\nu}_0 > 0$ will be introduced in (4.2) and in (4.4) (there is no need in this subsection to know more about them) we will find that

$$\widehat{\gamma}_{0,\mathbf{B}} = 2^{-2/3}\widehat{\nu}_0\delta_0^{1/3}C(\tau, \delta_0),$$

and for $\tau \leq \tau_0$, the equality,

$$\left\{x \in \Gamma \mid \widetilde{\gamma}_{0,\mathbf{B}}(x) = 2^{-2/3}\widehat{\nu}_0\delta_0^{1/3}\right\} = \left\{(0, \pm 1, 0)\right\},$$

where τ_0 is a constant and $C(\tau, \delta_0)$ is explicitly computed (see Proposition 3.4).

The inward normal of $\Omega = \{x \in \mathbb{R}^3 \mid |x| < 1\}$ along $\partial\Omega$ is

$$\mathbf{N}(x) = -x \quad \big(|x| = 1\big).$$

The restriction of the magnetic field $\mathbf{B}$ to the boundary is then tangent to $\partial\Omega$ on the following set

$$\Gamma = \left\{x \in \partial\Omega \mid x \cdot \mathbf{A}(x) = 0\right\}. \tag{3.3}$$

3.2. Γ is a regular curve

For $|x| = 1$, the equation $x \cdot \mathbf{A}(x) = 0$ reads as follows

$$x_1 \cos(\tau x_3) + x_2 \sin(\tau x_3) = 0. \tag{3.4}$$

Proposition 3.1. *The set Γ introduced in (3.3) is a C^∞ regular curve.*

Proof. The proof follows by constructing an atlas on Γ,

$$\left\{\big(\mathbf{c}_i, U := (-1, 1)\big),\ 1 \leq i \leq 4\right\}$$

which turns Γ to a C^∞ regular curve.

Let us introduce the charts $(\mathbf{c}_1, U)$ and $(\mathbf{c}_2, U)$ which cover $\Gamma \setminus \{(0, 0, \pm 1)\}$. These charts are obtained by expressing x_1 and x_2 in (3.4) in terms of $x_3 \in (-1, 1)$, provided that $(x_1, x_2, x_3) \neq (0, 0, \pm 1)$. We write for $\alpha \in]-\pi, \pi]$

$$x_1 = \sqrt{1 - x_3^2}\cos\alpha, \quad x_2 = \sqrt{1 - x_3^2}\sin\alpha.$$

Then (3.4) becomes, for $x_3^2 < 1$,

$$\cos(\tau x_3 - \alpha) = 0 \tag{3.5}$$

which in turn yields

$$\alpha = \tau x_3 - \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}.$$

In this way, we get two branches of Γ parameterized by x_3 and defined as follows

$$x_3 \in (-1, 1) \mapsto \mathbf{c}_1(x_3) := \begin{pmatrix} x_1 = \sqrt{1 - x_3^2}\sin(\tau x_3) \\ x_2 = -\sqrt{1 - x_3^2}\cos(\tau x_3) \\ x_3 \end{pmatrix},$$

$$x_3 \in (-1, 1) \mapsto \mathbf{c}_2(x_3) := \begin{pmatrix} x_1 = -\sqrt{1 - x_3^2}\sin(\tau x_3) \\ x_2 = \sqrt{1 - x_3^2}\cos(\tau x_3) \\ x_3 \end{pmatrix}.$$

Both of the foregoing branches represent regular curves. Furthermore, $\mathbf{c}_1$ and $\mathbf{c}_2$ can be extended *by continuity* to the interval $[-1,1]$, yielding a continuous representation of all Γ.

Now we introduce the charts $(\mathbf{c}_3, U)$ and $(\mathbf{c}_4, U)$ that cover the points $(0,0,\pm 1)$. In a neighborhood of $(x_1, x_2, x_3) = (0,0,\pm 1)$, we parameterize a branch of Γ with respect to $\rho := \sqrt{x_1^2 + x_2^2}$ as follows

$$x_1 = \rho \cos\alpha, \quad x_2 = \rho \sin\alpha, \quad x_3 = \sqrt{1-\rho^2}.$$

With this in hand, (3.5) continues to hold for $x_3 \neq 0$ and we can write again $\alpha = \tau x_3 - \frac{\pi}{2} + k\pi$ for some $k \in \mathbb{Z}$. Consequently, we get two regular branches of Γ defined as follows

$$\rho \in (-1,1) \mapsto \mathbf{c}_3(\rho) := \begin{pmatrix} x_1 = \rho \sin(\tau\sqrt{1-\rho^2}) \\ x_2 = -\rho\cos(\tau\sqrt{1-\rho^2}) \\ x_3 = \sqrt{1-\rho^2} \end{pmatrix},$$

$$\rho \in (-1,1) \mapsto \mathbf{c}_4(\rho) := \begin{pmatrix} x_1 = -\rho \sin(\tau\sqrt{1-\rho^2}) \\ x_2 = \rho\cos(\tau\sqrt{1-\rho^2}) \\ x_3 = \sqrt{1-\rho^2} \end{pmatrix}. \qquad \blacksquare$$

3.3. Explicit formulas in adapted coordinates

Note that $\mathbf{c} := \mathbf{c}_1$ and $\mathbf{c}_2$ parameterize all of $\Gamma \setminus \{(0,0,\pm 1)\}$. By symmetry considerations, we will compute, on $\mathbf{c}((-1,1))$ only,

$$\left|d^T(\mathbf{B}\cdot\mathbf{N})\right| = \tau\left|d^T(\mathbf{A}\cdot\mathbf{N})\right| \quad \text{and} \quad |\mathbf{B}\cdot\mathbf{T}| = \tau|\mathbf{A}\cdot\mathbf{T}|. \tag{3.6}$$

First we note that $\mathbf{N} = -x$ on $\partial\Omega$ and introduce the arc-length parameter

$$s(x_3) = \int_0^{x_3} \left|\mathbf{c}'(\tilde{x}_3)\right| d\tilde{x}_3$$

of $x_3 \mapsto \mathbf{c}(x_3)$, which satisfies

$$s'(x_3) = \left|\mathbf{c}'(x_3)\right| = \sqrt{\frac{1+\tau^2(1-x_3^2)^2}{1-x_3^2}}.$$

Clearly, $x_3 \in (-1,1)$ can be expressed in terms of the arc-length parameter as $x_3 = x_3(s)$ with

$$m(x_3) := \frac{dx_3}{ds}\big(s(x_3)\big) = \sqrt{\frac{1-x_3^2}{1+\tau^2(1-x_3^2)^2}}. \tag{3.7}$$

The arc-length parameterization is now given by

$$\gamma(s) := \mathbf{c}\big(x_3(s)\big),$$

and consequently, with $\mathbf{c} = \mathbf{c}_1$, we have

$$\mathbf{N}\big(\gamma(s)\big) = -\gamma(s) = \begin{pmatrix} -\sqrt{1-x_3^2}\sin(\tau x_3) \\ \sqrt{1-x_3^2}\cos(\tau x_3) \\ -x_3 \end{pmatrix} \quad \text{with } x_3 = x_3(s), \tag{3.8}$$

and

$$\begin{aligned}\mathbf{T}\big(\gamma(s)\big) &= \frac{d}{ds}\gamma(s) = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} \\ &= m(x_3)\begin{pmatrix} -\frac{x_3\sin(\tau x_3)}{\sqrt{1-x_3^2}} + \tau\sqrt{1-x_3^2}\cos(\tau x_3) \\ \frac{x_3\cos(\tau x_3)}{\sqrt{1-x_3^2}} + \tau\sqrt{1-x_3^2}\sin(\tau x_3) \\ 1 \end{pmatrix}.\end{aligned}$$

We also introduce the normal vector to Γ on $\gamma(s)$,

$$\begin{aligned}\mathbf{V}\big(\gamma(s)\big) &= \mathbf{T}\big(\gamma(s)\big) \times \mathbf{N}\big(\gamma(s)\big) = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \\ &= m(x_3)\begin{pmatrix} -\frac{x_3^2\cos(\tau x_3)}{\sqrt{1-x_3^2}} - \tau x_3\sqrt{1-x_3^2}\sin(\tau x_3) - \sqrt{1-x_3^2}\cos(\tau x_3) \\ -\frac{x_3^2\sin(\tau x_3)}{\sqrt{1-x_3^2}} + \tau x_3\sqrt{1-x_3^2}\cos(\tau x_3) - \sqrt{1-x_3^2}\sin(\tau x_3) \\ \tau(1-x_3^2) \end{pmatrix}.\end{aligned}$$

We are now ready to prove that our magnetic field $\mathbf{B}$ satisfies the condition in Assumption 1.2.

Proposition 3.2. *Let* $\mathbf{B}$ *be the magnetic field introduced in* (3.2)*. For all* $x \in \Gamma$*, we have*

$$\big|\mathbf{B}(x)\cdot\mathbf{T}(x)\big| = \frac{\tau(1-x_3^2)}{\sqrt{1+\tau^2(1-x_3^2)^2}}.$$

In particular, $\mathbf{B}$ *satisfies Assumption* 1.2.

Proof. It is straightforward to compute

$$\big|\mathbf{A}(x)\cdot\mathbf{T}(x)\big| = \frac{1}{\tau}\big(\big|\cos(\tau x_3)T_1 + \sin(\tau x_3)T_2\big|\big) = \frac{1-x_3^2}{\sqrt{1+\tau^2(1-x_3^2)^2}}, \tag{3.9}$$

which holds for all $-1 \le x_3 < 1$ and $x = \mathbf{c}(x_3)$. Similarly, we can compute $|\mathbf{A}(x)\cdot\mathbf{T}(x)|$ for all $x = \mathbf{c}_2(x_3) \in \Gamma$, and get that (3.9) holds globally on Γ, since Γ is a regular curve. Finally, $\mathbf{B}(x)$ is orthogonal to $\mathbf{T}(x)$ if and only if $x_3^2 = 1$, thereby Assumption 1.2 holds. ∎

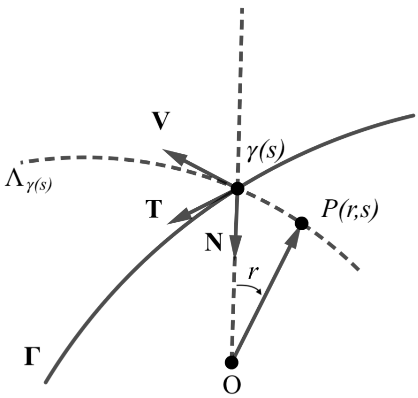

Figure 2. The curve Γ and the geodesic $\Lambda_{\gamma(s)}$ passing through $\gamma(s)$.

Our next task is to show that our magnetic field satisfies the condition in Assumption 1.1.

Proposition 3.3. *Let* **B** *be the magnetic field introduced in* (3.2)*. For all* $x \in \Gamma$*, we have*

$$\kappa_{n,\mathbf{B}}(x) = \sqrt{1 + \tau^2\big(1 - x_3(s)^2\big)^2}. \tag{3.10}$$

In particular, **B** *satisfies the condition in Assumption* 1.1.

Proof. By Proposition 3.1, Γ is a regular curve. So all we need to verify that **B** satisfies Assumption 1.1, is to derive (3.10) and observe that it yields $\kappa_{n,\mathbf{B}}(x) \neq 0$ everywhere along the curve Γ.

Consider $x = \mathbf{c}_1(x_3)$ with $x_3 = x_3(s)$, i.e., $x = \gamma(s)$. At the point $\gamma(s)$, the geodesic $\Lambda_{\gamma(s)}$ normal to the curve Γ is the great circle (of center 0 and radius 1) in the $(\mathbf{V}(\gamma(s)), \mathbf{N}(\gamma(s)))$ plane. A point $P = P(r,s)$ on $\Lambda_{\gamma(s)}$ can be described by the corresponding vector $\mathbf{p}(r,s) = \overrightarrow{OP}$ as follows

$$\mathbf{p}(r,s) = -\cos r\mathbf{N}\big(\gamma(s)\big) - \sin r\mathbf{V}\big(\gamma(s)\big),$$

where r is the angle between $\mathbf{p}$ and $-\mathbf{N}$; hence r is an arc-length parameter of $\Lambda_{\gamma(s)}$, and for $r = 0$, $p(r,s) = \gamma(s)$. Now, we can introduce the coordinates (r,s,t) in a neighborhood of $\gamma(s_0)$ as follows (see Figure 2)

$$x(r,s,t) = -(\cos r + t)\mathbf{N}\big(\gamma(s)\big) - \sin r\mathbf{V}\big(\gamma(s)\big). \tag{3.11}$$

For $x = \gamma(s)$, we would like to compute $\kappa_{n,\mathbf{B}}(x) = |d^T(\mathbf{B}\cdot\mathbf{N})|$. We will show that $\kappa_{n,\mathbf{B}}(x) = |\partial_r(\mathbf{B}\cdot\mathbf{N})|_{r=t=0}$ and end up with the computation of $|\partial_r(\mathbf{B}\cdot\mathbf{N})|_{r=t=0}$.

Notice that, by (3.8), we have

$$\begin{aligned} x_3(r,s,t) &= -(\cos r + t)\mathbf{N}_3\big(\gamma(s)\big) - \sin r\mathbf{V}_3\big(\gamma(s)\big) \\ &= (\cos r + t)x_3(s) - \sin r m\big(x_3(s)\big)\tau\big(1 - x_3(s)^2\big), \end{aligned}$$

and we observe that by (3.11),

$$\left.\frac{\partial x}{\partial r}\right|_{r=t=0} = -\mathbf{V}\big(\gamma(s)\big). \tag{3.12}$$

In particular we have

$$\left.\frac{\partial x_3}{\partial r}\right|_{r=t=0} = -\tau\big(1 - x_3^2(s)\big)m\big(x_3(s)\big).$$

Now, using (3.7) and (3.8), we get from (3.1) that

$$\frac{\partial \mathbf{A}}{\partial r}\cdot \mathbf{N}\Big|_{r=t=0} = -\frac{\tau(1-x_3^2)^2}{\sqrt{1+\tau^2(1-x_3^2)^2}}. \tag{3.13}$$

Moreover, by (3.12) we have

$$\frac{\partial}{\partial r}\mathbf{N}\big(x(r,s,t)\big)\big|_{r=t=0} = \mathbf{V}\big(\gamma(s)\big)$$

and

$$\begin{aligned}\mathbf{A}\cdot\frac{\partial}{\partial r}\mathbf{N}\big(x(r,s,t)\big)\big|_{r=t=0} &= \frac{1}{\tau}\cos\big(\tau x_3(s)\big)V_1 + \frac{1}{\tau}\sin\big(\tau x_3(s)\big)V_2 \\ &= -\frac{1}{\tau\sqrt{1+\tau^2(1-x_3^2)^2}}.\end{aligned} \tag{3.14}$$

Summing up, we deduce from (3.13) and (3.14) that

$$\big|\partial_r(\mathbf{A}\cdot\mathbf{N})|_{r=t=0}\big| = \frac{1}{\tau}\sqrt{1+\tau^2(1-x_3^2)^2}.$$

We also observe that $\partial_s(\mathbf{A}\cdot\mathbf{N})|_{r=t=0} = 0$ and we get

$$\big|d^T(\mathbf{A}\cdot\mathbf{N})\big|\big(\gamma(s)\big) = \frac{1}{\tau}\sqrt{1+\tau^2(1-x_3^2)^2},$$

on each branch (including the end points). Inserting this into (3.6), we get the identity in (3.10). ■

We return to the function in (1.7) and give its expression in coordinates. We deduce from (3.9) and (3.10):

$$\widetilde{\gamma}_{0,\mathbf{B}}(x) = 2^{-2/3}\widehat{\nu}_0\delta_0^{1/3}\big(1+\tau^2(1-x_3^2)^2\big)^{1/3}\left(1-(1-\delta_0)\frac{\tau(1-x_3^2)}{\sqrt{1+\tau^2(1-x_3^2)^2}}\right)^{1/3}$$

for all $x = (\pm\sqrt{1-x_3^2}\sin(\tau x_3), \mp\sqrt{1-x_3^2}\cos(\tau x_3), x_3)$ with $-1 \le x_3 \le 1$.

Consequently, we can compute the quantity appearing in the two terms asymptotics by computing $\inf_{x\in\Gamma}\widetilde{\gamma}_{0,\mathbf{B}}(x)$ and determining where the infimum is attained.

Proposition 3.4. *Let*

$$\tau_0 = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{\delta_0+\delta_0(1-\delta_0)}}-1\right)^{1/2}.$$

The following holds:

(1) *If* $0<\tau\le\tau_0$, *then*

$$\inf_{x\in\Gamma}\widetilde{\gamma}_{0,\mathbf{B}}(x) = 2^{-2/3}\widehat{\nu}_0\delta_0^{1/3}(1+\tau^2)^{1/3}\left(1-(1-\delta_0)\frac{\tau^{1/3}}{(1+\tau^2)^{1/6}}\right)$$
$$= \widetilde{\gamma}_{0,\mathbf{B}}(0,\pm 1,0).$$

(2) *If* $\tau>\tau_0$, *then*

$$\inf_{x\in\Gamma}\widetilde{\gamma}_{0,\mathbf{B}}(x) = 2^{-2/3}\widehat{\nu}_0\delta_0^{1/3}(1+\tau_0^2)^{1/3}\left(1-(1-\delta_0)\frac{\tau_0^{1/3}}{(1+\tau_0^2)^{1/6}}\right),$$

and the minimum is attained on the points

$$\left(\pm\sqrt{\frac{\tau_0}{\tau}}\sin\tau\sqrt{1-\frac{\tau_0}{\tau}},\mp\sqrt{\frac{\tau_0}{\tau}}\cos\tau\sqrt{1-\frac{\tau_0}{\tau}},\sqrt{1-\frac{\tau_0}{\tau}}\right)$$

and

$$\left(\pm\sqrt{\frac{\tau_0}{\tau}}\sin\tau\sqrt{1-\frac{\tau_0}{\tau}},\pm\sqrt{\frac{\tau_0}{\tau}}\cos\tau\sqrt{1-\frac{\tau_0}{\tau}},-\sqrt{1-\frac{\tau_0}{\tau}}\right).$$

Remark 3.5. In the case where $\Omega=B(0,1)$ is the unit ball and the magnetic field is constant, $\mathbf{B}=(0,0,1)$, we have $\Gamma=\{x_1^2+x_2^2=1,\ x_3=0\}$ and $\widetilde{\gamma}_{0,\mathbf{B}}(x)$ is constant on Γ. Proposition 3.4 shows a quite different phenomenon when only the intensity of $\mathbf{B}$ is constant, $|\mathbf{B}|=1$. In fact, $\widetilde{\gamma}_{0,\mathbf{B}}(x)$ is no more constant along Γ and may have two symmetric minimum points, $(0,\pm1,0)$, which is the signature of an interesting double well tunnel effect [19] related to the magnetic geometry of the problem.

Proof of Proposition 3.4. Let us introduce $v=\tau(1-x_3^2)\in[0,\tau]$ and $\mu_0=1-\delta_0\in(0,1)$. Then

$$\widetilde{\gamma}_{0,\mathbf{B}}(x) = 2^{-2/3}\widehat{\nu}_0\delta_0^{1/3}\big(f(v)\big)^{1/3},$$

where

$$f(v)=1+v^2-\mu_0 v\sqrt{1+v^2}.$$

We have to minimize $f(v)$ on $[0,\tau]$. Notice that

$$f'(v)=2v-\mu_0\frac{1+2v^2}{\sqrt{1+v^2}},$$

and the equation $f'(v) = 0$ has a unique positive solution, which is the solution of

$$v^4 + v^2 = \frac{\mu_0^2}{4(1-\mu_0^2)}.$$

This solution is given by

$$\tau_0 = \frac{1}{\sqrt{2}} \frac{\mu_0}{\sqrt{1+\sqrt{1-\mu_0^2}}\sqrt{1-\mu_0^2}}$$

and observe that $f'(v) < 0$ for $0 < v < \tau_0$ and $f'(v) > 0$ for $v > \tau_0$. Then, for $\tau \le \tau_0$,

$$\min_{v\in[0,\tau]} f(v) = f(\tau),$$

while for $\tau > \tau_0$,

$$\min_{v\in[0,\tau]} f(v) = f(\tau_0). \qquad \blacksquare$$

4. One-dimensional models

The aim of this section is to recall the now standard properties of two important models.

4.1. The de Gennes model

We refer to [1, 15] for the proof of these now standard properties which are presented below. For $\xi \in \mathbb{R}$, we consider the harmonic oscillator on $\mathbb{R}_+$:

$$H(\xi) := D_t^2 + (t-\xi)^2, \tag{4.1}$$

with Neumann boundary condition at 0. We denote by $\mu(\xi)$ its lowest eigenvalue. $\xi \mapsto \mu(\xi)$ admits a unique minimum at a point ξ_0 which in addition is non-degenerate. This leads to introduce the spectral constants, Θ_0 and δ_0:

$$\Theta_0 = \inf_{\xi\in\mathbb{R}} \mu(\xi) = \mu(\xi_0), \quad \delta_0 = \mu''(\xi_0), \tag{4.2}$$

where $\xi_0 = \sqrt{\Theta_0}$.

Moreover $\frac{1}{2} < \Theta_0 < 1$ and that $0 < \delta_0 < 1$. Θ_0 is called the de Gennes constant.

If $\varphi_0 \in L^2(\mathbb{R}_+)$ denotes the positive and normalized ground state of $H(\xi_0)$,

$$\int_{\mathbb{R}_+} (t-\xi_0)\big|\varphi_0(t)\big|^2 dt = 0,$$

which amounts to saying, via the Feynman–Hellmann formula, that $\mu'(\xi_0) = 0$. We also introduce the regularized resolvent $\mathcal{R}_0 \in \mathcal{L}(L^2(\mathbb{R}_+))$ as follows

$$\mathcal{R}_0 u = \begin{cases} \big(H(\xi_0) - \Theta_0\big)^{-1} u & \text{if } u \perp \varphi_0, \\ 0 & \text{if } u \parallel \varphi_0. \end{cases} \tag{4.3}$$

4.2. The Montgomery model

Here we refer to [14, 28]. In Theorem 1.3, the constant $\widehat{\nu}_0 > 0$ is related to the Montgomery model [23] whose spectral analysis has a long story including recently (see [13] and references therein). For $\rho \in \mathbb{R}$, we introduce, in $L^2(\mathbb{R})$, the operator

$$S(\rho) = D_r^2 + (r^2 - \rho)^2,$$

and denote its lowest eigenvalue by $\mu^{\mathrm{Mon}}(\rho)$. Then

$$\widehat{\nu}_0 := \inf_{\rho \in \mathbb{R}} \mu^{\mathrm{Mon}}(\rho) = \mu^{\mathrm{Mon}}(\rho_0), \tag{4.4}$$

where $\rho_0 \in \mathbb{R}$ is the unique minimum of μ^{Mon}, which has been later shown to be non degenerate [12]. Finally, the normalized positive ground state $\psi_0 \in L^2(\mathbb{R})$ of $S(\rho_0)$ belongs to the Schwartz space $S(\mathbb{R})$ and is an even function.

5. Model operator for non-uniform magnetic fields

Given real parameters η, ζ, γ and θ, we consider the operator

$$\begin{aligned} P^{h,\eta,\zeta}_{0;\gamma,\theta} &:= \big(hD_r - \sin\theta t - \cos\theta(\eta s + \zeta r)t\big)^2 \\ &\quad + \Big(hD_s + \cos\theta t - \sin\theta(\eta s + \zeta r)t + \gamma\frac{r^2}{2}\Big)^2 \\ &\quad + h^2 D_t^2, \end{aligned} \tag{5.1}$$

on $\mathbb{R}^2 \times \mathbb{R}^+$ (actually in a neighborhood of $(0,0,0)$). Let us fix a positive constant M. We assume that

$$\eta, \zeta, \gamma \in [-M, M]. \tag{5.2}$$

We note, when $\eta = \zeta = 0$, we recover the model studied in [17, Sec. 11]. Our aim is to compare this situation with that when $\eta = \zeta = 0$. Our main result on this model is Proposition 5.5 below, which is useful in our derivation of the lower bound matching with the asymptotics in Theorem 1.4. The lower bound in this proposition is uniform with respect to the various parameters appearing in (5.1) provided (5.2) holds and h is sufficiently small.

Let us look at this model more carefully. We proceed essentially like in the case $\eta = \zeta = 0$. We do the following scaling

$$r = h^{\frac{1}{3}}\hat{r}, \quad s = h^{\frac{1}{3}}\hat{s}, \quad t = h^{\frac{1}{2}}\hat{t}.$$

After division by h, this leads to (forgetting the hats)

$$\begin{aligned} P^{h,\eta,\zeta}_{1;\gamma,\theta} &:= \big(h^{\frac{1}{6}}D_r - \sin\theta t - h^{\frac{1}{3}}\cos\theta t(\eta s + \zeta r)\big)^2 \\ &\quad + \Big(h^{\frac{1}{6}}D_s + \cos\theta t + h^{\frac{1}{6}}\gamma\frac{r^2}{2} - h^{\frac{1}{3}}\sin\theta t(\eta s + \zeta r)\Big)^2 + D_t^2 \end{aligned}$$

on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$.

Hence we have

$$\sigma(P^{h,\eta,\zeta}_{0;\gamma,\theta}) = h\sigma(P^{h,\eta,\zeta}_{1;\gamma,\theta}).$$

Unlike the case where $\eta = \zeta = 0$, we can no more perform a partial Fourier transform in the s-variable. But we can rewrite this operator as in the following lemma.

Lemma 5.1. *It holds,*

$$P^{h,\eta,\zeta}_{1;\gamma,\theta} = D_t^2 + (t - h^{\frac{1}{6}} L_{1,\gamma,\theta})^2 + h^{\frac{1}{3}} (L^{h,\eta,\zeta}_{2,\gamma,\theta})^2,$$

where

$$L_{1;\gamma,\theta} = \sin\theta D_r - \cos\theta\Big(\frac{\gamma}{2} r^2 + D_s\Big),$$
$$L^{h,\eta,\zeta}_{2;\gamma,\theta} := \cos\theta D_r + \sin\theta\Big(\frac{\gamma}{2} r^2 + D_s\Big) - h^{\frac{1}{6}}(\zeta r + \eta s)t.$$

Note that to compare with the case considered in [17] ($\eta = \zeta = 0$) we can write

$$L^{h,\eta,\zeta}_{2;\gamma,\theta} = L_{2;\gamma,\theta} - h^{\frac{1}{6}}(\zeta r + \eta s)t, \tag{5.3}$$

where $L_{2;\gamma,\theta} := L^{0,0,0}_{2;\gamma,\theta}$.

Proof of Lemma 5.1. Let $P^h_{1;\gamma,\theta} := P^{h,0,0}_{1;\gamma,\theta}$. Then (see [17, (11.4)])

$$P^h_{1;\gamma,\theta} = D_t^2 + (t - h^{\frac{1}{6}} L_{1,\gamma,\theta})^2 + h^{\frac{1}{3}} (L_{2,\gamma,\theta})^2.$$

With $p = (\eta s + \zeta r)t$, we have

$$P^{h,\eta,\zeta}_{1;\gamma,\theta} = P^h_{1;\gamma,\theta} + h^{\frac{1}{3}}\big[-2(h^{\frac{1}{6}} p)L_{2;\gamma,\theta} - h^{\frac{1}{6}}\big(\cos\theta(D_r p) + \sin\theta(D_s p)\big) + (h^{\frac{1}{6}} p)^2\big].$$

Finally, we observe by (5.3),

$$(L^{h,\eta,\zeta}_{2;\gamma,\theta})^2 = (L_{2,\gamma,\theta})^2 - 2(h^{\frac{1}{6}} p)L_{2;\gamma,\theta} - h^{\frac{1}{6}}\big(\cos\theta(D_r p) + \sin\theta(D_s p)\big) + (h^{\frac{1}{6}} p)^2. \quad \blacksquare$$

When $\eta = \zeta = 0$, this is the operator studied in [17], modulo a Fourier transformation with respect to the s variable. Let us recall the following important result [17, Lem. 13.4] corresponding to the case $(\eta, \zeta) = (0, 0)$.

Proposition 5.2 (Helffer–Morame). *For any $C_0 > 0$, $\delta \in]0, \frac{1}{3}[$ and $M > 0$, there exist positive constants C and h_0 such that, for all $\theta \in \mathbb{R}$, $|\gamma| \leq M$, and $h \in]0, h_0]$, we have, for any $u \in C_0^\infty(] - C_0 h^\delta, C_0 h^\delta[\times \mathbb{R} \times \overline{\mathbb{R}_+})$*

$$\langle P^{h,0,0}_{0;\gamma,\theta} u,\, u\rangle \geq \big(h\Theta_0 + h^{\frac{4}{3}} c^{\mathrm{conj}}(\gamma,\theta) - C(h^{\frac{11}{8}} + h^{\delta + \frac{13}{12}})\big)\|u\|^2,$$

where

$$c^{\mathrm{conj}}(\gamma,\theta) := \left(\frac{1}{2}\right)^{\frac{2}{3}} \delta_0^{\frac{1}{3}} |\gamma|^{\frac{2}{3}} (\delta_0 \sin^2\theta + \cos^2\theta)^{\frac{1}{3}} \widehat{\nu}_0,$$

and $P^{h,0,0}_{0;\gamma,\theta}$ is the operator introduced in (5.1).

Remark 5.3. The underlying estimate in Proposition 5.2 is in fact

$$\langle P^{h,0,0}_{1;\gamma,\theta}u,\,u\rangle \geq \big(\Theta_0 + h^{\frac{1}{3}}c^{\mathrm{conj}}(\gamma,\theta) - C(h^{\frac{3}{8}} + h^{\delta+\frac{1}{12}})\big)\|u\|^2.$$

We can not directly compare $P^{h,\eta;\zeta}_{1;\gamma,\theta}$ and $P^{h,0,0}_{1;\gamma,\theta}$ but this can be done by introducing a small perturbation of $P^{h,0,0}_{1;\gamma,\theta}$ whose spectrum is just lifted. To achieve this goal we introduce for $\tau > 0$

$$P^h_{1;\gamma,\theta,\tau} := D_t^2 + (t - h^{\frac{1}{6}}L_{1,\gamma,\theta})^2 + (1-h^\tau)h^{\frac{1}{3}}(L_{2,\gamma,\theta})^2,$$

where we have modified the coefficient of $(L_{2,\gamma,\theta})^2$ by $\epsilon = h^{1/3+\tau}$. Heuristically this leads to a maximal shift of the bottom of the spectrum by $\mathcal{O}(h^{1/3+\tau})$. More precisely, we show by a slight variation of the argument in [17, Lem. 13.3]

Proposition 5.4. *For all $\tau \in]0,1[$, for any $C_0 > 0$, $\delta \in]0,\frac{1}{3}[$ and $M > 0$, there exist positive constants C and h_0 such that, for all $\theta \in \mathbb{R}$, $|\gamma| \leq M$, and $h \in]0,h_0]$, we have, for any $u \in C_0^\infty(]-C_0h^\delta, C_0h^\delta[\times\mathbb{R}\times\overline{\mathbb{R}_+})$*

$$\langle P^h_{1,\gamma,\theta,\tau}u,\,u\rangle \geq \big(\Theta_0 + h^{\frac{1}{3}}c^{\mathrm{conj}}(\gamma,\theta) - C(h^{\tau+\frac{1}{3}} + h^{\frac{3}{8}} + h^{\delta+\frac{1}{12}})\big)\|u\|^2. \tag{5.4}$$

Note that the estimate in Proposition 5.2 holds without constraint on the support of the function in s. This will not be the case for $(\eta,\zeta) \neq 0$.

We now compare $\langle P^{h,\eta,\zeta}_{1;\gamma,\theta}u,u\rangle$ and $\langle P^h_{1;\gamma,\theta,\tau}u,u\rangle$ when

$$u \in C_0^\infty\big(]-C_0h^{\delta-\frac{1}{3}}, C_0h^{\delta-\frac{1}{3}}[\times]-C_0h^{\delta-\frac{1}{3}}, C_0h^{\delta-\frac{1}{3}}[\times\overline{\mathbb{R}_+}\big).$$

and η,ζ satisfies (5.2).

Let us fix

$$\delta \in \Big]\frac{1}{4},\frac{1}{3}\Big[\quad\text{and}\quad \tau \in \Big]0,\frac{1}{6}\Big[. \tag{5.5}$$

The estimates below hold uniformly with respect to u, $\theta \in \mathbb{R}$ and η,ζ,γ satisfying (5.2).

Comparing $L^{h,\eta,\zeta}_{2,\gamma,\theta}$ and $L_{2,\gamma,\theta}$ in (5.3), we find[1], for all $\tau > 0$,

$$\big\langle(L^{h,\eta,\zeta}_{2,\gamma,\theta})^2u,u\big\rangle = \|L^{h,\eta,\zeta}_{2,\gamma,\theta}u\|^2 \geq (1-h^\tau)\|L_{2,\gamma,\theta}u\|^2 + (1-h^{-\tau})\big\|(L^{h,\eta,\zeta}_{2,\gamma,\theta} - L_{2,\gamma,\theta})u\big\|^2.$$

Consequently,

$$\begin{aligned}\langle P^{h,\eta,\zeta}_{1;\gamma,\theta}u,u\rangle \geq{}& \big\langle\big(D_t^2 + (t-h^{\frac{1}{6}}L_{1;\gamma,\theta})^2 + (1-h^\tau)h^{\frac{1}{3}}(L_{2;\gamma,\theta})^2\big)u,u\big\rangle\\ &- h^{\frac{1}{3}-\tau}\big\|(L^{h,\eta,\zeta}_{2;\gamma,\theta} - L_{2;\gamma,\theta})u\big\|^2.\end{aligned}$$

This implies (see (5.3) and the condition on the support of u),

$$\langle P^{h,\eta,\zeta}_{1;\gamma,\theta}u,u\rangle \geq \langle P^h_{1;\gamma,\theta,\tau}u,u\rangle - C(\eta^2+\zeta^2)h^{2\delta-\tau}\|tu\|^2, \tag{5.6}$$

[1] We use $2ab \leq \varepsilon a^2 + \varepsilon^{-1}b^2$ with $\varepsilon = h^\tau$, $a = \|L^{h,\eta,\zeta}_{2,\gamma,\theta}u\|$ and $b = \|L_{2,\gamma,\theta}u\|^2$.

where we used (see (5.3))

$$L^{h,\eta,\zeta}_{2;\gamma,\theta} - L_{2;\gamma,\theta} = h^{1/6} t\mathcal{O}\big((|s| + |r|)\big) = t\mathcal{O}(h^{\delta-\frac{1}{6}})$$

in the support of u.

By (5.4) and (5.6) we have

$$\langle P^{h,\eta,\zeta}_{1;\gamma,\theta} u, u\rangle \geq \big(\Theta_0 + c^{\mathrm{conj}}(\gamma,\theta) h^{1/3} - C(h^{\frac{3}{8}} + h^{\delta+\frac{1}{12}} + h^{\tau})\big)\|u\|^2 \\ - C(\eta^2 + \zeta^2) h^{2\delta-\tau}\|tu\|^2.$$

Note that by (5.5) we have

$$h^{\frac{3}{8}} + h^{\delta+\frac{1}{12}} + h^{\tau+\frac{1}{3}} + h^{2\delta-\tau} = \mathcal{O}(h^{\frac{1}{3}+\varsigma}),$$

for some $\varsigma = \varsigma(\delta,\tau) > 0$.

Consequently, there exist $C, \varsigma > 0$ and h_0 such that, $\forall h \in]0, h_0]$,

$$\langle P^{h,\eta,\zeta}_{1;\gamma,\theta,\tau} u, u\rangle \geq \big(\Theta_0 + c^{\mathrm{conj}}(\gamma,\theta) h^{1/3} - C h^{\frac{1}{3}+\varsigma}\big)\|u\|^2 - C h^{\frac{1}{3}+\varsigma}\|tu\|^2,$$

for any $u \in C_0^\infty(]-Ch^{\delta-\frac{1}{3}}, Ch^{\delta-\frac{1}{3}}[^2 \times \overline{\mathbb{R}_+})$.

By coming back to the initial coordinates, we get the following generalization of Proposition 5.2.

Proposition 5.5. *Let $C_0, M > 0$ and $\delta \in]\frac{1}{4}, \frac{1}{3}[$ be given. There exist positive constants C, h_0, and ς, such that, for all $h \in]0, h_0]$, $\theta \in \mathbb{R}$ and $\gamma, \eta, \zeta \in [-M, M]$, we have, for any $u \in C_0^\infty(]-C_0 h^\delta, C_0 h^\delta[^2 \times \overline{\mathbb{R}^+})$,*

$$\langle P^{h,\eta,\zeta}_{0,\gamma,\theta} u,\ u\rangle \geq \big(h\Theta_0 + h^{\frac{4}{3}} c^{\mathrm{conj}}(\gamma,\theta) - C h^{\frac{4}{3}+\varsigma}\big)\|u\|^2 - C h^{\frac{1}{3}+\varsigma}\|tu\|^2.$$

Note here that the last term will be small when considering localized states satisfying (6.5).

6. Localization of bound states

We recall that the bound states of the operator $P^h_{\mathbf{A}}$ in (1.3) are localized on the boundary near the curve where the magnetic field is tangent to the boundary $\partial\Omega$. The localization is related with the analysis of a family of model operators in the half-space [21].

Consider $\mathbb{R}^3_+ := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 > 0\}$ and the Neumann realization in $\mathbb{R}^3_+$ of the operator,

$$H(\nu) = D^2_{x_1} + D^2_{x_2} + (D_{x_3} + x_1 \cos\nu - x_2 \sin\nu)^2,$$

where $\nu \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

More precisely, $H(\nu)$ is self-adjoint in $L^2(\mathbb{R}^3_+)$ with the following domain

$$\mathrm{Dom}\,\big(H(\nu)\big) = \big\{u \in L^2(\mathbb{R}^3_+) \mid H(\nu)u \in L^2(\mathbb{R}^3_+),\ \partial_{x_1} u|_{x_1=0} = 0\big\}.$$

We denote by

$$\sigma(\nu) = \inf_{\nu \in [-\frac{\pi}{2}, \frac{\pi}{2}]} \operatorname{spec}\big(H(\nu)\big).$$

We gather some properties of the lowest eigenvalue $\sigma(\nu)$ (see [15,21], [17, Sec. 3.3]):

Proposition 6.1. *The following properties hold for the lowest eigenvalue $\sigma(\nu)$ of $H(\nu)$:*

- *For all $\nu \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\sigma(-\nu) = \sigma(\nu)$.*
- $[0, \frac{\pi}{2}] \ni \nu \mapsto \sigma(\nu)$ *is monotone increasing and* $\sigma(0) = \Theta_0$.
- $\sigma(\nu) \geq \Theta_0 \cos^2 \nu + \sin^2 \nu$.
- *As $\nu \to 0$, $\sigma(\nu) = \Theta_0 + \sqrt{\delta_0}|\nu| + \mathcal{O}(\nu^2)$.*

Here we recall that Θ_0 and δ_0 are introduced in (4.2).

Let us return to the magnetic field in (1.1). Recall that, for $x \in \Omega$, $p(x) \in \partial\Omega$ satisfies

$$\operatorname{dist}(x, \partial\Omega) = \operatorname{dist}\big(x, p(x)\big),$$

and it is uniquely defined when x is sufficiently close to the boundary. For all $x \in \overline{\Omega}$, we introduce $\nu(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ by

$$(\mathbf{B} \cdot \mathbf{N})\big(p(x)\big) = \sin \nu(x). \tag{6.1}$$

Hence $\nu(x) = 0$ implies that $\mathbf{B}(p(x))$ is tangent to $\partial\Omega$ at $p(x)$, in other words that x belongs to Γ (see (1.4)). Now we recall the following lower bound related to the operator $P_{\mathbf{A}}^h$ established in [17, Thm. 4.3]:

Proposition 6.2. *Under Assumption* (1.1)*, there exist constants $C, h_0 > 0$ such that, for all $h \in (0, h_0]$ and $u \in H^1(\Omega)$, we have*

$$\int_\Omega \big|(h\nabla - i\mathbf{A})u\big|^2\, dx \geq \int_\Omega \big(hW_h(x) - Ch^{5/4}\big)\big|u(x)\big|^2\, dx,$$

where

$$W_h(x) = \begin{cases} 1 & \text{if } \operatorname{dist}(x, \partial\Omega) \geq 2h^{3/8}, \\ \sigma\big(\nu(x)\big) & \text{if } \operatorname{dist}(x, \partial\Omega) \leq 2h^{3/8}. \end{cases}$$

If additionally $u \in H_0^1(\Omega)$, we have for some positive constant C_0 the stronger lower bound

$$\int_\Omega \big|(h\nabla - i\mathbf{A})u\big|^2\, dx \geq (h - C_0 h^{5/4}) \int_\Omega |u|^2\, dx.$$

Combining the lower bound in Proposition 6.2 with the following leading term expansion of the lowest eigenvalue (see [17, Thm. 4.4])

$$\lambda_1^N(\mathbf{A}, h) = \Theta_0 h + o(h), \tag{6.2}$$

we get decay estimates for the ground states. Let us recall these localization estimates (see [5, Sec. 9.4] for details).

Proposition 6.3. *Given $M > 0$, there exists a positive constant α such that, if u_h is a normalized bound state of $\mathcal{P}_h$ with eigenvalue $\lambda(h) \le Mh$, then as $h \to 0_+$,*

$$\int_\Omega \left(\left|u_h(x)\right|^2 + h^{-1}\left|(h\nabla - i\mathbf{A})u_h\right|^2\right) \exp\left(\frac{\alpha\, \mathrm{dist}(x, \partial\Omega)}{h^{1/2}}\right) dx = \mathcal{O}(1). \tag{6.3}$$

Furthermore, there exist constants $\alpha_1, \epsilon_0 > 0$ such that, as $h \to 0_+$,

$$\int_{\{\mathrm{dist}(x,\partial\Omega)<\epsilon_0\}} \left(\left|u_h(x)\right|^2 + h^{-1}\left|(h\nabla - i\mathbf{A})u_h\right|^2\right) \exp\left(\frac{\alpha_1\, d_\Gamma(x)}{h^{1/4}}\right) dx = \mathcal{O}(1),$$

where

$$d_\Gamma(x) = \mathrm{dist}_{\partial\Omega}\left(p(x), \Gamma\right), \tag{6.4}$$

and $\mathrm{dist}_{\partial\Omega}$ *is the geodesic distance on $\partial\Omega$.*

Hence we have two levels of localization, first a strong one near $\partial\Omega$ and then an additional but weaker one near Γ. Along the proof of Theorem 1.4, we will only use (6.3) and generalizations or consequences of it, as explained in the below remark.

Remark 6.4 (Applications of Proposition 6.3). Let u_h be a normalized ground state of $P^h_{\mathbf{A}}$.

(1) By (6.2), the hypothesis in Proposition 6.3 holds, hence the ground state u_h satisfies (6.3) and (6.4).

(2) Pick an arbitrary point $x_0 \in \Gamma$. In the coordinates introduced in (2.6), where $t(x) = \mathrm{dist}(x, \partial\Omega)$, $r(x) = d_\Gamma(x)$ and $u_h(x) = \tilde{u}_h(r, s, t)$ (see (2.14)), we deduce from (6.3) the following weaker, but quite useful estimates. For any $n \ge 0$,

$$\int_{\widetilde{V}_0} t^n |\tilde{u}_h|^2\, ds\, dr\, dt = \mathcal{O}(h^{n/2}), \tag{6.5}$$

$$\int_{\widetilde{V}_0} t^n \left|(h\nabla_{r,s,t} - \widetilde{\mathbf{A}})\tilde{u}_h\right|^2 dr\, ds\, dt = \mathcal{O}(h^{1+\frac{n}{2}}), \tag{6.6}$$

where $\widetilde{V}_0 := \widetilde{V}_{x_0}$ and $\widetilde{\mathbf{A}}$ are introduced in (2.13) and (2.17), respectively.

7. Estimating the quadratic form

7.1. A comparison estimate

We fix δ and ϵ_2 satisfying

$$\frac{5}{18} < \delta < \frac{1}{3} \quad \text{and} \quad 0 < \epsilon_2 < 1. \tag{7.1}$$

We also fix $R_0 > 0$, $h_0 > 0$, $x_0 \in \Gamma$ and introduce for $h \in (0, h_0]$ the set

$$Q_h(x_0, R_0, \delta, \epsilon_2) = \left\{x \in \Omega : \left|r(x) - r_0\right| \le R_0 h^\delta,\ |s(x) - s_0| \le R_0 h^\delta,\ 0 < t(x) < \epsilon_2\right\}, \tag{7.2}$$

where $(r(x), s(x), t(x))$ are introduced in (2.1) and, since $x_0 \in \Gamma$,

$$y^0 := (r_0, s_0, t_0) := \big(r(x_0), s(x_0), t(x_0)\big) = \big(0, s(x_0), 0\big).$$

For simplicity, we omit most of the time the reference to δ and ϵ_2.

Let $\widetilde{\mathbf{A}} = (\widetilde{A}_1^{(2)}, \widetilde{A}_2^{(2)}, \widetilde{A}_3^{(2)})$ be the magnetic potential associated with $\mathbf{A}$ via (2.17), with $y = (y_1, y_2, y_3) = (r, s, t)$ (see (2.6)). We introduce the following magnetic potential

$$\widetilde{\mathbf{A}}^{(2)}(y) = \sum_{|\beta|\leq 2} \frac{\partial^\beta \widetilde{\mathbf{A}}}{\partial y^\beta}(y^0)\frac{(y-y^0)^\beta}{\beta!}, \tag{7.3}$$

which is the quadratic Taylor expansion of $\widetilde{\mathbf{A}}$ at y^0. We introduce the quadratic form associated with the magnetic potential $\widetilde{\mathbf{A}}^{(2)}$ as follows

$$\begin{aligned} q^h_{\widetilde{\mathbf{A}}^{(2)}}(u) = \int_{\widetilde{Q}_h(x_0,R_0)} & \big(1 - r\kappa_g(x_0)\big)\Big(\big|(hD_t - \widetilde{A}_3^{(2)})u\big|^2 \\ & + \big(1 + 2r\kappa_g(x_0)\big)\big|(hD_s - \widetilde{A}_2^{(2)})u\big|^2 + \big|(hD_r - \widetilde{A}_1^{(2)})u\big|^2\Big)\, dr\, ds\, dt, \end{aligned}$$

where

$$\widetilde{Q}_h(x_0, R_0, \delta, \epsilon_2) = \big\{(r,s,t) : \max\big(|r|, |s - s_0|\big) < R_0 h^\delta,\ 0 < t < \epsilon_2\big\}, \tag{7.4}$$

and (see (2.9))

$$\kappa_g(x_0) \text{ is the geodesic curvature of } \Gamma \text{ at } x_0.$$

The next lemma compares the quadratic forms $u \mapsto q^h_{\widetilde{\mathbf{A}}^{(2)}}(u)$ and $u \mapsto q^h_{\mathbf{A}}(u)$ introduced in (2.16). The errors that will arise are controlled by the following energy

$$M_h(u) = \sum_{n=0}^{6} h^{-n/2} \int_\Omega t(x)^n \big(|u|^2 + h^{-1}\big|(h\nabla - i\mathbf{A})u\big|^2\big) dx, \tag{7.5}$$

where $t(x) = \operatorname{dist}(x, \partial\Omega)$. Notice that,

$$\int_\Omega |u|^2\, dx \leq M_h(u), \tag{7.6a}$$

$$\int_\Omega \big|(h\nabla - i\mathbf{A})u\big|^2\, dx \leq M_h(u) h, \tag{7.6b}$$

$$\int_\Omega t(x)^n \big(|u|^2 + h^{-1}\big|(h\nabla - i\mathbf{A})u\big|^2\big) dx \leq M_h(u) h^{n/2} \quad (1 \leq n \leq 6). \tag{7.6c}$$

Lemma 7.1. *There exist constants $C, h_0, \varsigma_0 > 0$ such that, for all $h \in (0, h_0]$ and $u \in H^1(\Omega)$ satisfying* $\operatorname{supp} u \subset Q_h(x_0, R_0)$*, we have*

$$\begin{aligned} & (1 - Ch^{2\delta}) q^h_{\widetilde{\mathbf{A}}^{(2)}}(u) - C M_h(u) h^{\frac{4}{3}+\varsigma_0} \\ & \quad \leq q^h_{\mathbf{A}}(u) \leq (1 + Ch^{2\delta}) q^h_{\widetilde{\mathbf{A}}^{(2)}}(u) + C M_h(u) h^{\frac{4}{3}+\varsigma_0}. \end{aligned} \tag{7.7}$$

Proof. Let us recall two useful estimates whose proof does not require that the magnetic field curl $\mathbf{A}$ is constant (see [17, Lem. 10.1]):

$$\begin{aligned} q^h_{\mathbf{A}}(u) \geq{}& (1 - Ch^{2\delta}) q^h_{\widetilde{\mathbf{A}}^{(2)}}(u) - C \left\| t^{1/2}(hD_x - \mathbf{A})u \right\|^2 \\ & - C \big(q^h_{\widetilde{\mathbf{A}}^{(2)}}(u)\big)^{1/2} \left\| (h^{3\delta} + h^{2\delta} t + h^{\delta} t^2 + t^3) u \right\| \\ & - C \left\| (h^{3\delta} + h^{2\delta} t + h^{\delta} t^2 + t^3) u \right\|^2, \end{aligned} \tag{7.8}$$

$$\begin{aligned} q^h_{\mathbf{A}}(u) \leq{}& (1 + Ch^{2\delta}) q^h_{\widetilde{\mathbf{A}}^{(2)}}(u) + C \left\| t^{1/2}(hD_x - \mathbf{A})u \right\|^2 \\ & + C \big(q^h_{\widetilde{\mathbf{A}}^{(2)}}(u)\big)^{1/2} \left\| (h^{3\delta} + h^{2\delta} t + h^{\delta} t^2 + t^3) u \right\| \\ & + C \left\| (h^{3\delta} + h^{2\delta} t + h^{\delta} t^2 + t^3) u \right\|^2. \end{aligned} \tag{7.9}$$

In the sequel we use the notation $\mathcal{O}(c_h h^{\rho+})$ in the following manner

$$f_h = \mathcal{O}(c_h h^{\rho+}) \quad \text{if and only if } \exists \epsilon > 0 \text{ s.t. } f_h = \mathcal{O}(c_h h^{\rho+\epsilon}).$$

Since we have assumed (7.1), we have

$$\min\left(6\delta,\ 2\delta + 1,\ 3\delta + \frac{1}{2},\ 2 - 2\delta\right) > \frac{4}{3}.$$

We can now estimate the error terms appearing in (7.8) and (7.9). We deduce from (7.6a) that

$$\|h^{3\delta} u\|^2 = \mathcal{O}(M_h h^{6\delta}) = \mathcal{O}(M_h h^{\frac{4}{3}+}),$$

where we write M_h instead of $M_h(u)$ for the sake of simplicity.

Using again (7.6c) with $n = 1$, $n = 2$, $n = 4$ and $n = 6$, we get

$$\begin{aligned} \left\| t^{1/2}(hD_x - \mathbf{A})u \right\|^2 &= \mathcal{O}(M_h h^{\frac{5}{4}}), \\ \|h^{2\delta} t u\|^2 &= \mathcal{O}(M_h h^{4\delta+1}), \\ \|h^{\delta} t^2 u\|^2 &= \mathcal{O}(M_h h^{2\delta+2}), \\ \|t^3 u\|^2 &= \mathcal{O}(M_h h^3). \end{aligned}$$

Consequently,

$$\left\| t^{1/2}(hD_x - \mathbf{A})u \right\|^2 + \left\| (h^{3\delta} + h^{2\delta} t + h^{\delta} t^2 + t^3) u \right\|^2 = \mathcal{O}(M_h h^{\frac{4}{3}+}). \tag{7.10}$$

Notice that $|\widetilde{\mathbf{A}} - \widetilde{\mathbf{A}}^{(2)}| = \mathcal{O}(h^{3\delta}) + \mathcal{O}(t^3)$ in $\widetilde{Q}_h(x_0, R_0)$. By the triangle inequality and (2.16)

$$q^h_{\widetilde{\mathbf{A}}^{(2)}}(u) \leq C\big(q^h_{\mathbf{A}}(u) + \|t^3 u\|^2\big).$$

So by using (7.6) we get

$$q^h_{\widetilde{\mathbf{A}}^{(2)}}(u) = \mathcal{O}(M_h h).$$

Consequently, the foregoing estimate and (7.10) yield,

$$\big(q^h_{\widetilde{\mathbf{A}}^{(2)}}(u)\big)^{1/2} \left\| (h^{3\delta} + h^{2\delta} t + h^{\delta} t^2 + t^3) u \right\| = \mathcal{O}(M_h h).$$

This finishes the proof of (7.7). ∎

7.2. Normal form

Recall that we have fixed an arbitrary point $x_0 \in \Gamma$ and denoted its coordinates, in the (r,s,t)-frame, by $(0,s_0,0)$. Let us also recall that the magnetic field $\mathbf{B}(x_0)$ can be expressed by (2.4).

Performing an appropriate gauge transformation on the set $\widetilde{Q}_h(x_0,R_0)$ introduced in (7.4), will yield a convenient normal form of the magnetic potential $\widetilde{\mathbf{A}}^{(2)}$ introduced in (7.3).

Lemma 7.2. *There exist positive constants C and $\widehat{C}$, and for all $x_0 \in \Gamma$, there exist $\check{\kappa}, \zeta \in [-\widehat{C}, \widehat{C}]$ and a smooth function $\check{p}$ on a neighborhood of $\widetilde{Q}_h(x_0,R_0,\delta,\epsilon_2)$, such that,*

$$\big|\widetilde{\mathbf{A}}^{(2)}(r,s,t) - \mathbf{A}^{00}(r,s,t) + \nabla \check{p}(r,s,t)\big| \le C\big(r^3 + t^2 + |s-s_0|^3\big),$$

where

$$\mathbf{A}^{00}(r,s,t) = \Big(ta_1(r,s), ta_2(r,s) + \frac{1}{2}\kappa_{n,\mathbf{B}}(x_0)r^2, 0\Big),$$

$\kappa_{n,\mathbf{B}}(x_0)$ is introduced in (1.5), and

$$a_1(r,s) = \sin\theta(s_0) + \big(\zeta r + \check{\kappa}(s-s_0)\big)\cos\theta(s_0),$$
$$a_2(r,s) = -\cos\theta(s_0) + r\kappa_g(x_0)\cos\theta(s_0) + \big(\zeta r + \check{\kappa}(s-s_0)\big)\sin\theta(s_0).$$

Here $\theta(s_0)$ is the angle introduced in (2.5) with $x = x_0$.

This lemma is an extension of Lemma 9.1 in [17] to the case when the magnetic field is not necessarily constant. In the constant magnetic field case we have $\zeta = 0$ and $\check{\kappa} = \kappa_g(x_0)$, where κ_g is the geodesic curvature introduced in (2.9). Note that we do not try at the moment to explicitly compute $\check{\kappa}$ and ζ in the general case. We plan indeed to show that the result on the lowest eigenvalue is independent of $\check{\kappa}$ and ζ.

Proof of Lemma 7.2. Our goal is to determine the Taylor expansion up to order 1 of the magnetic field vector and corresponding magnetic field 2-form in the variables (r,s,t), the Taylor expansion being computed at $t = r = 0$ and $s = s_0$. Up to a translation, we assume that $s_0 = 0$.

Writing the magnetic vector field in (1.2) as

$$\mathbf{B} = \tilde{b}_1\partial_r + \tilde{b}_2\partial_s + \tilde{b}_3\partial_t,$$

the Taylor expansion of order 1 at $(0,0,0)$ takes the form

$$\tilde{b}_1(r,s,t) = \cos\theta + \gamma_1 r + \delta_1 s + \sigma_1 t + \mathcal{O}(r^2+s^2+t^2),$$
$$\tilde{b}_2(r,s,t) = \sin\theta + \gamma_2 r + \delta_2 s + \sigma_2 t + \mathcal{O}(r^2+s^2+t^2),$$
$$\tilde{b}_3(r,s,t) = \gamma_3 r + \sigma_3 t + \mathcal{O}(r^2+s^2+t^2),$$

where $\theta = \theta(s_0)$ and where we used (2.3)–(2.4). Here we have used that by definition of the coordinate r, the function $(r,s) \mapsto \tilde{b}_3(r,s,0)$ vanishes exactly at order 1 on $r=0$. Note that γ_3 is $\kappa_{n,\mathbf{B}}(x_0)$, introduced in (1.5).

We now express that on $t=0$ the norm of $\mathbf{B}$ should be one. In fact

$$|\mathbf{B}|^2 = \sum_{1\le i,j\le 1} g_{ij}\tilde{b}_i\tilde{b}_j + \tilde{b}_3^2, \tag{7.11}$$

where the coefficients g_{ij} can be computed by (2.7), (2.8) and (2.10).

For $t=0$, this reads

$$\big(\tilde{b}_1(r,s,0)\big)^2 + \alpha(r,s)\big(\tilde{b}_2(r,s,0)\big)^2 + \big(\tilde{b}_3(r,s,0)\big)^2 = 1,$$

where $\alpha(r,s)$ is introduced in (2.10) and satisfies (2.11). We expand the last formula around $t=r=s=0$. This leads, by taking $t=0$ and considering the coefficients of r and s, to the two identities

$$\gamma_1\cos\theta + \gamma_2\sin\theta - \kappa_g(x_0)\sin^2\theta = 0,$$
$$\delta_1\cos\theta + \delta_2\sin\theta = 0.$$

So it is natural to introduce the new parameters $\hat{\kappa}$ and ζ as follows

$$\check{\kappa} = -\delta_1\sin\theta + \delta_2\cos\theta, \quad \zeta = -\gamma_1\sin\theta + \big(\gamma_2 - \kappa_g(x_0)\sin\theta\big)\cos\theta. \tag{7.12}$$

So we observe that

$$\delta_1 = -\check{\kappa}\sin\theta, \quad \delta_2 = \check{\kappa}\cos\theta,$$
$$\gamma_1 = -\zeta\sin\theta, \quad \gamma_2 = \zeta\cos\theta + \kappa_g(x_0)\sin\theta.$$

Hence our "normal" form becomes

$$\tilde{b}_j(r,s,t) = \tilde{b}_j^0(r,s,t) + \mathcal{O}(r^2+s^2+t^2)$$

with

$$\tilde{b}_1^0(r,s,t) = \cos\theta - (\zeta r + \check{\kappa}s)\sin\theta + \sigma_1 t,$$
$$\tilde{b}_2(r,s,t) = \sin\theta + (\zeta r + \check{\kappa}s)\cos\theta + \kappa_g(x_0)r\sin\theta + \sigma_2 t,$$
$$\tilde{b}_3(r,s,t) \equiv \gamma_3 r + \sigma_3 t,$$

with

$$\gamma_3 = \kappa_{n,\mathbf{B}}(x_0) = \partial_r\langle \mathbf{B} \mid N\rangle.$$

Now consider $\widetilde{\mathbf{B}} = \operatorname{curl}_{(r,s,t)}\widetilde{\mathbf{A}}$. We have $\widetilde{\mathbf{B}} = |g|^{1/2}(\tilde{b}_1,\tilde{b}_2,\tilde{b}_3)$ (see [17, (5.13)]), where g is introduced in (2.8). So we obtain by (2.12),

$$\widetilde{\mathbf{B}}_{ij}(r,s,t) = \widetilde{\mathbf{B}}_{ij}^0(r,s,t) + \mathcal{O}(r^2+s^2+t^2)$$

with

$$\widetilde{\mathbf{B}}_{23}^0(r,s,t) = \big(1-\kappa_g(x_0)r\big)\cos\theta - (\zeta r + \check{\kappa}s)\sin\theta + \sigma_1 t,$$

$$\widetilde{\mathbf{B}}^0_{31}(r,s,t) = \sin\theta + (\zeta r + \check\kappa s)\cos\theta + \sigma_2 t,$$
$$\widetilde{\mathbf{B}}^0_{12}(r,s,t) \equiv \gamma_3 r + \sigma_3 t.$$

Notice that the condition $\operatorname{div}_{(r,s,t)} \widetilde{\mathbf{B}} = 0$ reads (at $r = t = 0$ and $s = 0$) as follows

$$\sigma_3 = \big(\kappa_g(x_0) - \check\kappa\big)\cos\theta + \zeta\sin\theta.$$

We have now to choose a suitable corresponding magnetic potential to $\widetilde{\mathbf{B}}^0$. We find

$$\widetilde{\mathbf{A}}^{00}(r,s,t) = \begin{pmatrix} \widetilde{A}^{00}_1 \\ \widetilde{A}^{00}_2 \\ \widetilde{A}^{00}_3 \end{pmatrix} = \begin{pmatrix} t a_1(r,s) + \frac{\sigma_2}{2} t^2 \\ t a_2(r,s) + \frac{1}{2}\gamma_3 r^2 - \frac{\sigma_1}{2} t^2 \\ 0 \end{pmatrix} = \mathbf{A}^{00}(r,s,t) + \mathcal{O}(t^2),$$

with

$$a_1(r,s) = \sin\theta + (\zeta r + \check\kappa s)\cos\theta,$$
$$a_2(r,s) = -\big(1 - \kappa_g(x_0) r\big)\cos\theta + (\zeta r + \check\kappa s)\sin\theta.$$

Moreover $\operatorname{curl} \widetilde{\mathbf{A}}^{(2)} = \widetilde{\mathbf{B}}^0$ in the simply connected domain $\widetilde{Q}_h(x_0, R_0, \delta, \epsilon_2)$, so we can find a function $\check p$ such that $\widetilde{\mathbf{A}}^{(2)} = \widetilde{\mathbf{A}}^{00} - \nabla \check p$.

Finally,

$$\gamma_j(s) := \frac{\partial \tilde b_j}{\partial r}(0,s,0) \quad\text{and}\quad \delta_j(s) := \frac{\partial \tilde b_j}{\partial s}(0,s,0)$$

are bounded functions. Setting

$$M_j = \sup\big(|\gamma_j(s)| + |\delta_j(s)|\big) \quad\text{and}\quad M = \max(M_1, M_2),$$

we get from (7.12) that

$$|\check\kappa| \le 2M \quad\text{and}\quad |\zeta| \le 2M + \|\kappa_g\|_\infty. \qquad ■$$

7.3. A second comparison estimate

We use the magnetic potential in Lemma 7.2 to approximate the quadratic form, as we did in Lemma 7.1. In particular, we approximate the metric by a flat one. Let us introduce the quadratic form corresponding to the magnetic potential in Lemma 7.1 (see [17, Lem. 10.2]):

$$q^h_{\mathbf{A}^{00}}(v) = \int_{\widetilde{Q}_h(x_0,R_0)} \Big(|hD_t v|^2 + \big(1 + 2r\kappa_g(x_0)\big)\big|(hD_s - A^{00}_2)v\big|^2 + \big|(hD_r - A^{00}_1)v\big|^2\Big)\, dr\, ds\, dt, \tag{7.13}$$

where $v \in H^1(\widetilde{Q}_h(x_0, R_0))$ and $\widetilde{Q}_h(x_0, R_0) = \widetilde{Q}_h(x_0, R_0, \delta, \epsilon_2)$ is the set introduced in (7.4).

We can obtain a further approximation of the quadratic form for functions obeying the conditions in (7.6).

Lemma 7.3 (Helffer–Morame). *There exist positive constants C, h_0, ς_0 such that, for all $h \in (0, h_0]$ and $u \in H^1(\Omega)$ s.t.* $\operatorname{supp} u \subset Q_h(x_0, R_0, \delta, \epsilon_2)$*, we have*

$$q^h_{\mathbf{A}00}(\tilde{u}) - CM_h(u)h^{\frac{4}{3}+\varsigma_0} \le q^h_{\mathbf{A}}(u) \le q^h_{\mathbf{A}00}(\tilde{u}) + CM_h(u)h^{\frac{4}{3}+\varsigma_0},$$

where $M_h(u)$ is introduced in (7.5) and

$$\tilde{u} = \big(1 - r\kappa_g(x_0)\big)^{1/2} u e^{-i\check{p}/h}.$$

Proof. We have the following two estimates from [17, Lem. 10.2] (whose proof does not require that the magnetic field curl $\mathbf{A}$ is constant)

$$\begin{aligned} q^h_{\mathbf{A}}(u) \ge\ & q^h_{\mathbf{A}00}(\tilde{u}) - C\left\|t^{1/2}(hD_x - \mathbf{A})u\right\|^2 - C\big(q^h_{\mathbf{A}00}(\tilde{u})\big)^{1/2}\left\|(h^{3\delta} + h + h^{2\delta}t + t^2)u\right\| \\ & - C\left\|(h^{3\delta} + h + h^{2\delta}t + t^2)u\right\|^2, \\ q^h_{\mathbf{A}}(u) \le\ & q^h_{\mathbf{A}00}(\tilde{u}) + C\left\|t^{1/2}(hD_x - \mathbf{A})u\right\|^2 + C\big(q^h_{\mathbf{A}00}(\tilde{u})\big)^{1/2}\left\|(h^{3\delta} + h + h^{2\delta}t + t^2)u\right\| \\ & + C\left\|(h^{3\delta} + h + h^{2\delta}t + t^2)u\right\|^2. \end{aligned}$$

We can then estimate the remainder terms, using (7.6), as we did in the proof of Lemma 7.1. The only term that was not present satisfies

$$\|t^2 u\|^2 \le M_h(u)h^2,$$

where we used (7.6c) with $n = 4$. ■

7.4. An estimate away from the curve Γ

Let us now look at the quadratic form, $q^h_{\mathbf{A}}(u)$, when u is supported away from Γ. We start with a rough lower bound.

Lemma 7.4. *Given $c > 0$, $\epsilon_2 \in (0, 1)$ and $\rho \in (0, \frac{1}{4})$, there exist positive constants h_0, $\tilde{c}$ such that, if $u \in H^1(\Omega)$ satisfies*

$$\operatorname{supp} u \subset \big\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \epsilon_2,\ d_\Gamma(x) \ge ch^\rho\big\},$$

where $d_\Gamma(x) = \operatorname{dist}_{\partial\Omega}(p(x), \Gamma)$ is introduced in (6.4), then

$$q^h_{\mathbf{A}}(u) \ge (\Theta_0 + \tilde{c}h^\rho)h \int_\Omega |u|^2\, dx.$$

Proof. If we verify that, for a given constant $c > 0$,

$$d_\Gamma(x) \ge ch^\rho \implies \exists c' > 0,\ \big|\nu(x)\big| \ge c'h^\rho, \tag{7.14}$$

then the proof follows from Proposition 6.2, by using that $h^{5/4} = o(h^{1+\rho})$ and the lower bound from Proposition 6.1,

$$\sigma(\nu) \ge \Theta_0 + \frac{\sqrt{\delta_0}}{2}|\nu|,$$

in a neighborhood of 0.

Let us denote by $m_* = \min_{x\in\Gamma} \kappa_{n,\mathbf{B}}(x)$, then $m_* > 0$ by Assumption 1.1, and (7.14) holds with $c' = m_* c/2$. In fact, if $|\nu(x)| \le c' h^\rho$, we get by (6.1)

$$\big|\mathbf{B}\cdot\mathbf{N}\big(p(x)\big)\big| \le c' h^\rho,$$

and it follows from (2.4) that (recall that $d_\Gamma(x) = |r|$, see Section 2)

$$m_* d_\Gamma(x) \le c' h^\rho = m_* \frac{c}{2} h^\rho. \qquad \blacksquare$$

The next proposition is an improvement of Proposition 7.4 since it allows for the support of u to be closer to the curve Γ.

Proposition 7.5. *Given $c > 0$, $\epsilon_2 \in (0,1)$ and $\delta \in [\frac14, \frac13)$, there exist positive constants h_0, c_*, C, ς_0 such that, if $u \in H^1(\Omega)$ satisfies*

$$\operatorname{supp} u \subset \big\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \epsilon_2,\ d_\Gamma(x) \ge c h^\delta\big\}, \tag{7.15}$$

where $d_\Gamma(x) = \operatorname{dist}_{\partial\Omega}(p(x), \Gamma)$ is introduced in (6.4), then

$$q_{\mathbf{A}}^h(u) \ge (\Theta_0 + c_* h^\delta) h \int_\Omega |u|^2\,dx - C M_h(u) h^{\frac43 + \varsigma_0},$$

where $M_h(u)$ is introduced in (7.5).

Proof.

Step 1. Let us fix constants $c, R_0 > 0$, $\epsilon_2 \in (0,1)$, $\delta \in [\frac14, \frac13)$ and $\rho \in (0, \frac14)$. We assume that $\operatorname{supp} u \subset Q_h(x_0^*, R_0, \delta, \epsilon_2)$ where $x_0^* \in \partial\Omega$ with boundary coordinates $(r_0, s_0, t_0 = 0)$ satisfies (for h small enough) $c h^\delta \le |r_0| = d_\Gamma(x_0^*) \le 2 c h^\rho$ and $Q_h(x_0^*, R_0, \delta, \epsilon_2)$ is introduced in (7.2).

We denote by

$$\widetilde{Q}_h(x_0^*) = \widetilde{Q}_h(x_0^*, R_0, \delta, \epsilon_2)$$

the neighborhood associated with $Q_h(x_0^*, R_0, \delta, \epsilon_2)$ by (7.4). By a translation, we may assume that $s_0 = 0$.

Consider the magnetic potential $\widetilde{\mathbf{A}}^{(2)}$ introduced in (7.3). We modify the coordinates (r, s, t) so that, locally near $(r_0, 0, 0)$, the metric G in (2.10) is diagonal[2] with

$$\alpha(r_0, s) = 1 \quad \text{and} \quad \frac{\partial\alpha}{\partial r}(r_0, s) = -2\kappa_g\big(\gamma(s)\big) + \mathcal{O}(h^\rho). \tag{7.16}$$

By Taylor's formula

$$\alpha(r, s) = 1 - 2\kappa_g\big(\gamma(s)\big)(r - r_0) + \mathcal{O}\big(h^\rho (r - r_0)\big) + \mathcal{O}\big((r - r_0)^2\big).$$

[2] We consider the curve Γ_h defined by $s \mapsto \Phi_{x_0}^{-1}(r_0, s, 0)$, where $x_0 = \gamma(x_0^*)$ and Φ_{x_0} is the coordinate transformation introduced in (2.1). We parameterization Γ_h by arc-length $s \mapsto \gamma_h(s)$ and define the adapted coordinates by considering the normal geodesic to Γ_h passing through x_0^*.

In $\widetilde{Q}_h(x_0^*)$, we write

$$\begin{aligned}
&\left|\kappa_g\big(\gamma(s)\big) - \kappa_g(x_0^*)\right| \le C h^{\delta},\\
&\alpha(r,s) = 1 - 2\kappa_g(x_0^*)(r-r_0) - C h^{\delta+\rho},\\
&h D_y - \widetilde{\mathbf{A}} = (h D_y - \widetilde{\mathbf{A}}^{(2)}) - (\widetilde{\mathbf{A}} - \widetilde{\mathbf{A}}^{(2)}).
\end{aligned}$$

So we get, as in Lemma 7.1, the existence of $C', \varsigma_0 > 0$ such that

$$q_{\mathbf{A}}^h(u) \ge (1 - C h^{\delta+\rho}) q_{\widetilde{\mathbf{A}}^{(2)}}^{h,x_0^*}(u) - C' M_h(u) h^{\frac{4}{3}+\varsigma_0},$$

where

$$\begin{aligned}
q_{\widetilde{\mathbf{A}}^{(2)}}^{h,x_0^*}(u) = \int_{\widetilde{Q}_h(x_0^*)} &\big(1 - (r-r_0)\kappa_g(x_0^*)\big)\Big(\big|(hD_t - \widetilde{A}_3^{(2)})u\big|^2\\
&+ \big(1 + 2(r-r_0)\kappa_g(x_0^*)\big)\big|(hD_s - \widetilde{A}_2^{(2)})u\big|^2 + \big|(hD_r - \widetilde{A}_1^{(2)})u\big|^2\Big)\, dr\, ds\, dt.
\end{aligned}$$

Performing a change of variables

$$(r,s) \mapsto \big((r-r_0)\cos\omega - s\sin\omega, (r-r_0)\sin\omega + s\cos\omega\big)$$

which amounts to a rotation in the (r,s)-plane (centered at $(r_0, 0)$), we may assume that the second component of $\widetilde{\mathbf{B}} = \operatorname{curl}_{(r,s,t)} \widetilde{\mathbf{A}} = (\widetilde{B}_{23}, \widetilde{B}_{31}, \widetilde{B}_{12})$ vanishes at $(r_0, 0, 0)$, by choosing ω so that

$$\widetilde{B}_{31}(x_0^*)\cos\omega + \widetilde{B}_{23}(x_0^*)\sin\omega = 0.$$

At the same time, this rotation leaves $|\mathbf{B}|$ and the measure $dr\, ds$ invariant. Then performing a gauge transformation (see [17, Sec. 16.3]), we may assume that

$$\widetilde{\mathbf{A}}^{(2)}(r,s,t) = \widetilde{\mathbf{A}}^{(2,0)}(r,s,t) + \mathcal{O}\big(|r-r_0|t + |s|t + t^2\big),$$

where

$$\widetilde{\mathbf{A}}^{(2,0)}(r,s,t) := \begin{pmatrix} \tilde{c}_1^0 s^2 \\ \widetilde{B}_{23}^{(0)} t + \widetilde{B}_{12}^{(0)}(r-r_0) + \tilde{c}_2^0 (r-r_0)^2 \\ 0 \end{pmatrix}.$$

Here

$$\widetilde{\mathbf{B}}^{(0)} := \widetilde{\mathbf{B}}(r_0, 0, 0) = (\widetilde{B}_{23}^{(0)}, \widetilde{B}_{31}^{(0)} = 0, \widetilde{B}_{12}^{(0)})$$

and $\tilde{c}_1^0, \tilde{c}_2^0$ are constants.

Similarly to the proof of Lemma 7.1, by writing

$$\begin{aligned}
&h D_y - \widetilde{\mathbf{A}}^{(2)} = h D_y - \widetilde{\mathbf{A}}^{(2,0)} - (\widetilde{\mathbf{A}}^{(2)} - \widetilde{\mathbf{A}}^{(2,0)}),\\
&\big\|(hD_y - \widetilde{\mathbf{A}}^{(2,0)})u\big\| \le \big\|(hD_y - \widetilde{\mathbf{A}})u\big\| + \big\|(\widetilde{\mathbf{A}} - \widetilde{\mathbf{A}}^{(2)})u\big\| + \big\|(\widetilde{\mathbf{A}}^{(2)} - \widetilde{\mathbf{A}}^{(2,0)})u\big\|,
\end{aligned}$$

we get

$$q_{\widetilde{\mathbf{A}}^{(2)}}^{h,x_0^*}(u) \ge (1 - C h^{2\delta}) q_{\widetilde{\mathbf{A}}^{(2,0)}}^{h,x_0^*}(u) - C'' M_h(u) h^{\frac{4}{3}+\varsigma_0}.$$

Thus we are left with finding a lower bound of $q_{\widetilde{\mathbf{A}}^{(2,0)}}^{h,x_0^*}(u)$.

Note that, since $|\mathbf{B}| = 1$ and by (7.16), the metric satisfies $|g| = 1$ on x_0^*, we have by (7.11), $|\widetilde{B}_{23}^{(0)}|^2 + |\widetilde{B}_{12}^{(0)}|^2 = 1$.

Moreover, since $\mathbf{B} \cdot \mathbf{N}$ vanishes linearly on $\Gamma = \{r = 0\}$, there exist $C_1 > 0$ and $C_2 > 0$ such that

$$\frac{1}{C_1}|r_0| \leq |\widetilde{B}_{12}^{(0)}|^2 \leq C_2|r_0|, \quad \big||\widetilde{B}_{23}^{(0)}| - 1\big| \leq C_2 r_0^2, \quad |\tilde{c}_1^0| + |\tilde{c}_2^0| \leq C_2.$$

The previous estimates yield a lower bound of $q^h_{\widetilde{\mathbf{A}}^{(2,0)}}(u)$ by comparing with a model operator (after rescaling the variables $\tilde{r} = h^{1/3}(r - r_0)$, $\tilde{s} = h^{1/3}s$ and $\tilde{t} = h^{1/2}t$). In fact, by [17, Lem. 16.1], there exists $c_1 > 0$ such that,

$$q^{h,x_0^*}_{\widetilde{\mathbf{A}}^{(2,0)}}(u) \geq \big(\Theta_0 + c_1|r_0|\big)h \int_\Omega |u|^2\,dx.$$

Note that, we can use Lemma 16.1 of [17] under our assumptions on the support of u.

Step 2. We can reduce to the setting of Step 1 and Lemma 7.4 by means of a partition of unity. In fact, consider an h-dependent partition of unity $\chi_1^2 + \chi_2^2 = 1$ on $\{\mathrm{dist}(x, \partial\Omega) < \epsilon_2\}$ such that

$$\operatorname{supp}\chi_1 \subset \Big\{d_\Gamma(x) \geq \frac{c}{2}h^\rho\Big\}, \quad \operatorname{supp}\chi_2 \subset \big\{d_\Gamma(x) \leq ch^\rho\big\}, \quad \sum_{i=1}^2 |\nabla\chi_i|^2 = \mathcal{O}(h^{-2\rho}).$$

If $u \in H^1(\Omega)$ satisfies (7.15), then

$$q^h_{\mathbf{A}}(u) = \sum_{i=1}^2 \big(q^h_{\mathbf{A}}(\chi_i u) - h^2\big\||\nabla\chi_i|u\big\|^2\big),$$

where

$$q^h_{\mathbf{A}}(\chi_1 u) \geq (\Theta_0 + \tilde{c}h^\rho)h \int_\Omega |\chi_1 u|^2\,dx \quad \text{by Proposition 7.4,}$$

$$q^h_{\mathbf{A}}(\chi_2 u) \geq (1 - Ch^{\delta+\rho})(\Theta_0 + c_1 h^\delta)h \int_\Omega |\chi_2 u|^2\,dx - M_h(u)h^{\frac{4}{3}+\varsigma_0} \quad \text{by Step 1,}$$

$$\sum_{i=1}^2 h^2\big\||\nabla\chi_i|u\big\|^2 = \mathcal{O}(h^{2-2\rho}) = o(h^{1+\delta}),$$

where in the last step we used that $0 < \rho < \frac{1}{4}$ and $\frac{1}{4} < \delta < \frac{1}{3}$. ■

8. Lower bound

8.1. Another model

The model in (5.1) corresponds to the quadratic form in (7.13) when $\kappa_g(x_0) = 0$. However, when $\kappa_g(x_0) \neq 0$, the situation is similar to [17, Sec. 15]. The model compatible with (7.13) can still be reduced to the one in (5.1) with appropriate choices of the parameters η, ζ, γ (see (8.10)).

8.1.1. A new model quadratic form. Let us fix a boundary point $x_0 \in \Gamma$ and denote the model quadratic form near x_0 by

$$u \mapsto q_m(u) := q_{\mathbf{A}^{00}}(u), \tag{8.1}$$

where $q_{\mathbf{A}^{00}}$ is given in (7.13), $u \in H^1(\widetilde{Q}_h(x_0, R_0))$ and $\widetilde{Q}_h(x_0, R_0) = \widetilde{Q}_h(x_0, R_0, \delta, \epsilon_2)$ is the set introduced in (7.4). Furthermore, we assume that the metric is flat at x_0 and the coordinates of x_0 in the (r, s, t) frame are $(0, s_0 = 0, 0)$, after performing a translation with respect to the s variable.

Following the proof of [17, Lem. 15.1], we are led to the analysis of the model quadratic form (see Lemma 8.1)

$$q^h_{m,0}(u) = \int_{\widetilde{Q}_h(x_0,R_0)} \big(h^2|D_t u|^2 + |tu - L^h_1 u|^2 + |L^h_2 u|^2\big) dr\, ds\, dt, \tag{8.2}$$

where

$$\begin{aligned} L^h_1 &= a_1 h D_r + a^0_2 h D_s - \frac{1}{2}\cos\theta \kappa_{n,\mathbf{B}}(x_0) r^2, \\ L^h_2 &= a^1_2 h D_r + a^1_1 h D_s + \frac{1}{2}\sin\theta \kappa_{n,\mathbf{B}}(x_0) r^2, \end{aligned} \tag{8.3}$$

and, with $\theta = \theta(s_0)$ the angle defined by (2.5), we introduce the following functions

$$\begin{aligned} a_1(r,s) &= \sin\theta + \cos\theta(\zeta r + \check{\kappa} s), \\ a_2(r,s) &= -\cos\theta + \kappa_g(x_0)\cos\theta r + \sin\theta(\zeta r + \check{\kappa} s), \\ a^0_2(r,s) &= -\cos\theta - \kappa_g(x_0)\cos\theta r + \sin\theta(\zeta r + \check{\kappa} s), \\ a^1_2(r,s) &= \cos\theta - \sin\theta(\zeta r + \check{\kappa} s), \\ a^1_1(r,s) &= \sin\theta + \sin\theta\kappa_g(x_0) r + \cos\theta(\zeta r + \check{\kappa} s), \\ \alpha(r) &= 1 + 2\kappa_g(x_0) r. \end{aligned} \tag{8.4}$$

We will consider the form $q_{m,0}$ on the following class of functions

$$\mathscr{D}_0 = \big\{u \in H^1(\Omega^h) : u|_{(\partial\mathscr{Q}^h)\times]0,h^\delta[} = 0,\ u|_{\mathscr{Q}^h\times\{h^\delta\}} = 0\big\},$$

where

$$\Omega_h = \mathscr{Q}^h \times]0, h^\delta[, \quad \mathscr{Q}^h =]-R_0 h^\delta, R_0 h^\delta[^2.$$

The precise relation between the model quadratic forms in (8.1) and (8.2) is given in the following lemma.

Lemma 8.1. *For any $\delta \in (\frac{5}{18}, \frac{1}{3})$ and $\tau_1 > 0$, there exists $C > 0$ such that, for any $u \in \mathscr{D}_0$ and $h \in (0,1)$,*

$$(1 + Ch^{2\delta}) q^h_m(u) \geq (1 - Ch^{\tau_1}) q^h_{m,0}(u) - C\big(\big\|(h^{2\delta} + h^{\tau_1}) t u\big\|^2 + h^{6\delta-\tau_1}\|u\|^2\big).$$

Proof. The proof follows that of Lemma 15.1 in [17] with some adjustments in the formulas (15.9), (15.16) and (15.17) in [17].

We have indeed

$$\left|1-(a_1)^2-\alpha(a_2)^2\right| \le Ch^{2\delta},$$

where we used that

$$\alpha(r)^{1/2} = 1+\kappa_g(x_0)r + \mathcal{O}(h^{2\delta})$$

on the support of u, which follows by (8.4).

We also observe that:

$$|\alpha a_2 - a_2^0| + |\alpha^{1/2}a_2 + a_2^1| + |\alpha^{1/2}a_1 - a_1^1| \le C(r^2+s^2),$$
$$|\alpha^{1/2}a_1 - \sin\theta| + |\alpha a_2 + \cos\theta| \le C(r^2+s^2)^{1/2}. \quad \blacksquare$$

Later on, we will choose δ and τ_1 in a convenient way (see Remark 8.3).

8.1.2. Linearizing change of variable. In order to reduce to the case $\kappa_g = 0$ and eliminate the slightly variable coefficients of D_r and D_s in (7.13), we argue as [17, Sec. 15.2] by performing a change of variables. The argument does not work in our case in the same way as [17, Sec. 15.2], but it leads to the fact that for our lower bound the only relevant parameters are $\eta := \check{\kappa} - \kappa_g$ and ζ (see (7.13)).

The below computations are essentially the same as in [17, Sec. 15.2] but we have to do them carefully in order to capture the correct η and ζ appearing in (5.1).

Let us follow, what this change of variable was doing. We introduce

$$\kappa := \kappa_g(x_0). \tag{8.5}$$

Let us make the change of variables $(r,s) = \Phi_\kappa(p,q)$ with

$$\begin{aligned} r &= \sin\theta p + \cos\theta q - \frac{\kappa}{2}[-\cos\theta p + \sin\theta q]^2,\\ s &= -\cos\theta p + \sin\theta q - \frac{\kappa}{2}\big[\sin(2\theta)(p^2-q^2) + 2\cos(2\theta)pq\big], \end{aligned} \tag{8.6}$$

where $\theta = \theta(s_0)$ is the angle defined by (2.5).

The map Φ_κ is a perturbation of a rotation and, by the local inversion theorem, it is easily seen as a local diffeomorphism sending a fixed neighborhood of $(0,0)$ onto another neighborhood of $(0,0)$.

Then, for h small enough, $\mathcal{Q}^h :=]-R_0h^\delta, R_0h^\delta[^2$ is transformed by Φ_κ^{-1} to the set $\mathcal{Q}_0^h$ satisfying:

$$\mathcal{Q}_0^h = \Phi_\kappa^{-1}(\mathcal{Q}^h) \subset]-R_0'h^\delta, R_0'h^\delta[\times]-R_0'h^\delta, R_0'h^\delta[.$$

Let us write

$$D_p = c_{11}D_r + c_{12}D_s, \quad D_q = c_{21}D_r + c_{22}D_s.$$

We can express the functions c_{ij} in terms of the (p,q) variables, by using (8.6). In fact, we introduce $c_{ij}(r,s) = \check{c}_{ij}(p,q)$, and observe that

$$\begin{aligned}
\check{c}_{11}(p,q) &= \frac{\partial r}{\partial p} = \sin\theta + \kappa\cos\theta(-\cos\theta p + \sin\theta q);\\
\check{c}_{12}(p,q) &= \frac{\partial s}{\partial p} = -\cos\theta - \kappa\big(\sin(2\theta)p + \cos(2\theta)q\big);\\
\check{c}_{21}(p,q) &= \frac{\partial r}{\partial q} = \cos\theta - \kappa\sin\theta(-\cos\theta p + \sin\theta q);\\
\check{c}_{22}(p,q) &= \frac{\partial s}{\partial q} = \sin\theta - \kappa\big(-\sin(2\theta)q + \cos(2\theta)p\big).
\end{aligned}$$

Then we return back to the (r,s) variables, by using (8.6). Noticing that, as $(p,q)\to(0,0)$,

$$r = \sin\theta p + \cos\theta q + \mathcal{O}(p^2+q^2), \quad s = -\cos\theta p + \sin\theta q + \mathcal{O}(p^2+q^2), \tag{8.7}$$

we get

$$\begin{aligned}
c_{11}(r,s) &= \sin\theta + \kappa\cos\theta s + \mathcal{O}(r^2+s^2);\\
c_{12}(r,s) &= -\cos\theta - \kappa(\cos\theta r - \sin\theta s) + \mathcal{O}(r^2+s^2);\\
c_{21}(r,s) &= \cos\theta - \kappa\sin\theta s + \mathcal{O}(r^2+s^2);\\
c_{22}(r,s) &= \sin\theta + \kappa(\sin\theta r + \cos\theta s) + \mathcal{O}(r^2+s^2).
\end{aligned}$$

Let us now control the measure in the change of variable. By an easy computation, we get:

$$dr\,ds = \check{\alpha}_1 dpdq, \quad \check{\alpha}_1(p,q) = 1 + \kappa(\sin\theta p + \cos\theta q) + \mathcal{O}(p^2+q^2).$$

By using (8.7), $\alpha_1(r,s) = \check{\alpha}_1(p,q)$ satisfies

$$|\alpha_1 - 1 - \kappa r| \le C(r^2+s^2),$$

where $r = r(p,q)$ is defined in (8.6).

Similarly to Lemma 8.1 we get also that one can go from the control of $q^h_{m,0}(u)$ to the control of the new quadratic form[3]

$$q^h_{m,1}(u) = \int_{\Omega^h_0} \big(h^2|D_t u|^2 + |tu - M^h_1 u|^2 + |M^h_2 u|^2\big)\check{\alpha}_1\, dp\, dq\, dt,$$

with

$$\begin{aligned}
&\Omega^h_0 := \mathcal{Q}^h_0 \times]0, h^\delta[,\\
&M^h_1 = hD_p + h\big((\check{\kappa}-\kappa)s + \zeta r\big)D_q - \frac{1}{2}\cos\theta\kappa_{n,\mathbf{B}}(x_0)(\sin\theta p + \cos\theta q)^2,\\
&M^h_2 = hD_q - h\big((\check{\kappa}-\kappa)s + \zeta r\big)D_p + \frac{1}{2}\sin\theta\kappa_{n,\mathbf{B}}(x_0)(\sin\theta p + \cos\theta q)^2,
\end{aligned}$$

where $(r,s) = (\sin\theta p + \cos\theta q, -\cos\theta p + \sin\theta q)$.

[3] We express L^h_1 and L^h_2 (see (8.3)) in terms of the (p,q) variables introduced in (8.6) and neglect the terms of order $\mathcal{O}(r^2+s^2) = \mathcal{O}(p^2+q^2)$.

More precisely, we have the following comparison lemma (see Lemma 8.1 and [17, Lem. 15.4]).

Lemma 8.2. *For any $\tau_1 > 0$, there exists $C > 0$ such that, for any $u \in \mathcal{D}_0$,*

$$(1 + Ch^{2\delta})q^h_{m,0}(u) \geq (1 - Ch^{\tau_1})q^h_{m,1}(\tilde{u}) - C\big(\big\|(h^{2\delta} + h^{\tau_1})tu\big\|^2 + h^{6\delta-\tau_1}\|u\|^2\big),$$

where $\tilde{u} = u \circ \Phi_\kappa^{-1}$ is associated with u by the transformation Φ_κ.

By a unitary transformation, and after control of a commutator, we can reduce to a flat measure ($dp\, dq$ instead of $\check{\alpha}_1\, dp\, dq$) and obtain the new quadratic form defined as follows

$$q^h_{m,2}(v) = \int_{\Omega_0^h} \big[h^2|D_t v|^2 + |tv - M_1^h v|^2 + |M_2^h v|^2\big]\, dp\, dq\, dt, \tag{8.8}$$

with v associated to u by $v = \check{\alpha}_1^{1/2}\tilde{u}$. In fact, we have [17, (15.29)]

$$(1 + Ch^{1/2})q_{m,1}(\tilde{u}) + Ch^{3/2}\|u\|^2 \geq q^h_{m,2}(v). \tag{8.9}$$

Let us consider the new model associated with the quadratic form in (8.8). We first observe that the result depends only on $\check{\kappa} - \kappa$ and on ζ. The proof is moreover uniform with respect to these parameters. As a consequence, if $\Phi = \Phi_\kappa$ was the transformation introduced in (8.5), the inverse (for $\kappa = 0$) Φ_0^{-1}, more explicitly the transformation $(p,q) \mapsto (\tilde{r} = \sin\theta p + \cos\theta q,\ \tilde{s} = -\cos\theta p + \sin\theta q)$ will bring us (in the new variables $(\tilde{r}, \tilde{s}, t)$) to the initial model with κ_g replaced by 0, and $\check{\kappa}$ replaced by $\check{\kappa} - \kappa_g(x_0)$. This can also be done by explicit computations.

Doing the transformations backwards, we are led to a magnetic Laplacian computed with a trivial metric $\kappa_g = 0$ but with a new magnetic potential

$$\begin{aligned} a_1(r,s)^{\text{new}} &= \sin\theta + \cos\theta\big(\zeta r + \big(\check{\kappa} - \kappa_g(x_0)\big)s\big),\\ a_2(r,s)^{\text{new}} &= -\cos\theta + \sin\theta\big(\zeta r + \big(\check{\kappa} - \kappa_g(x_0)\big)s\big). \end{aligned}$$

So the new model is not as simple as in the uniform magnetic field case (where $\check{\kappa} = \kappa_g$) but it is the model in (5.1), which we have studied in the previous section with

$$\eta = \check{\kappa} - \kappa_g(x_0), \quad \gamma = \kappa_{n,\mathbf{B}}(x_0). \tag{8.10}$$

In fact, since v is supported in Ω_0^h, we have,

$$q^h_{m,2}(v) = \langle P^{h,\eta,\zeta}_{0;\gamma,\theta} v, v\rangle, \tag{8.11}$$

where $P^{h,\eta,\zeta}_{0;\gamma,\theta}$ is the operator in (5.1).

Remark 8.3. We will choose τ_1 in such a manner that $\frac{1}{3} < \tau_1 < 6\delta - \frac{4}{3}$. This choice is possible when δ satisfies $\frac{5}{18} < \delta < \frac{1}{3}$.

8.1.3. Conclusion. We can now write a lower bound for the quadratic form $q^h_{\mathbf{A}^{00}}(u)$ in (7.13), assuming that $u \in H^1(\widetilde{Q}_h(x_0, R_0))$ and $\widetilde{Q}_h(x_0, R_0)$ is the set introduced in (7.4). Let $\frac{5}{18} < \delta < \frac{1}{3}$ and $\frac{1}{3} < \tau_1 < 6\delta - \frac{4}{3}$. Collecting Lemmas 8.1, 8.2, (8.9), (8.11) and Proposition 5.5, we get the existence of positive constants C and ς_0, such that

$$q^h_{\mathbf{A}^{00}}(u) \geq \big(h\Theta_0 + h^{\frac{4}{3}} c^{\mathrm{conj}}\big(\theta, \kappa_{n,\mathbf{B}}(x_0)\big) - Ch^{\frac{4}{3}+\varsigma_0}\big)\|u\|^2 - Ch^{\frac{1}{3}+\varsigma_0}\|tu\|^2 - C\big\|(h^{2\delta} + h^{\tau_1})tu\big\|^2 \tag{8.12}$$

where $c^{\mathrm{conj}}(\gamma, \theta)$ is introduced in Proposition 5.2 with $\theta = \theta(s_0)$ the angle in (2.5).

8.2. The general case

We return now to the proof of the asymptotics of the lowest eigenvalue, $\lambda_1^N(\mathbf{A}, h)$, of the operator $P^h_{\mathbf{A}}$ in (1.3). Under Assumptions 1.1 and 1.2, we will prove the following lower bound:

$$\lambda_1^N(\mathbf{A}, h) \geq \Theta_0 h + \widehat{\gamma}_{0,\mathbf{B}} h^{\frac{4}{3}} + \mathcal{O}(h^{\frac{4}{3}+\eta_*}), \tag{8.13}$$

for some constant $\eta_* > 0$, where $\widehat{\gamma}_{0,\mathbf{B}}$ is introduced in (1.6).

Let u_h be a normalized ground state of $P^h_{\mathbf{A}}$, i.e.

$$\lambda_1^N(\mathbf{A}, h) = q^h_{\mathbf{A}}(u_h) = \big\|(h\nabla - i\mathbf{A})u_h\big\|^2.$$

Consider $\frac{5}{18} < \delta < \frac{1}{3}$ and the following neighborhood of the curve Γ,

$$\Gamma^h_\delta = \big\{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < h^\delta,\ \mathrm{dist}_{\partial\Omega}(x, \Gamma) < h^{\delta/2}\big\}.$$

In terms of the (r, s, t) coordinates introduced in Section 2.1,

$$\Gamma^h_\delta = \{0 < t < h^\delta,\ h^{\delta/2} < r < h^{\delta/2}\}.$$

Let $\chi_h \in C_c^\infty(\Gamma^h_\delta; [0, 1])$ be a smooth function such that

$$\chi_h = 1 \quad \text{on } \Gamma^h_{\delta,0} = \left\{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \frac{1}{2}h^\delta,\ \mathrm{dist}_{\partial\Omega}(x, \Gamma) < \frac{1}{2}h^{\delta/2}\right\}$$

and

$$|\nabla\chi_h| = \mathcal{O}(h^{-\delta/2}).$$

We introduce the function

$$w_h = \chi_h u_h.$$

By Proposition 6.3, the eigenfunction u_h is exponentially small outside Γ^h_δ, since by our choice of δ we have $h^{\delta/2} \gg h^{1/4}$ and $h^\delta \gg h^{1/2}$. So we have

$$\lambda_1^N(\mathbf{A}, h) = q^h_{\mathbf{A}}(u_h) = q^h_{\mathbf{A}}(w_h) + \mathcal{O}(h^\infty), \quad \|u_h\| = \|w_h\| + \mathcal{O}(h^\infty). \tag{8.14}$$

Consider now a partition of unity of $\mathbb{R}^3$

$$\sum_{j\in\mathbb{Z}^3}|\chi_j|^2=1,\quad \sum_{j\in\mathbb{Z}^3}|\nabla\chi_j|^2<\infty,\quad \operatorname{supp}\chi_j\subset j+[-1,1]^3,$$

and introduce the following functions

$$w_{h,j}=\chi_{j,\delta}(x)w_h(x),\quad \chi_{j,\delta}(x)=\chi_j(h^{-\delta}x).$$

We can decompose the quadratic form $q^h_{\mathbf{A}}(w_h)$ as follows

$$q^h_{\mathbf{A}}(w_h)=\sum_{j\in\mathcal{J}_h}q^h_{\mathbf{A}}(w_{h,j})+\mathcal{O}(h^{2-2\delta}),\tag{8.15}$$

where

$$\mathcal{J}_h=\{j\in\mathbb{Z}^3:\operatorname{supp}\chi_{j,\delta}\cap\Omega\neq\emptyset\}.$$

Let $C_1>0$ be a fixed constant that we will choose later to be sufficiently large. We will estimate the energy $q^h_{\mathbf{A}}(w_{h,j})$ when the support of $w_{h,j}$ is near the curve Γ, or away from Γ, independently. So we introduce the sets of indices

$$\mathcal{J}^1_h=\{j\in\mathcal{J}_h:\operatorname{dist}(\operatorname{supp}\chi_{\gamma,\delta},\Gamma)\le C_1h^\delta\},$$
$$\mathcal{J}^2_h=\{j\in\mathcal{J}_h:\operatorname{dist}(\operatorname{supp}\chi_{\gamma,\delta},\Gamma)\ge C_1h^\delta\}.$$

By Proposition 7.5,

$$\sum_{j\in\mathcal{J}^2_h}q^h_{\mathbf{A}}(w_{h,j})\ge\sum_{j\in\mathcal{J}^2_h}\Big((\Theta_0h+c_*h^{1+\delta})\|w_{h,j}\|^2-Ch^{\frac43+\varsigma_0}M_h(w_{j,h})\Big),\tag{8.16}$$

where $M_h(w_{j,h})$ is introduced in (7.5). Notice that

$$M_h(w_{j,h})\le\sum_{n=0}^{6}h^{-n/2}\int_\Omega t(x)^n\Big(|\chi_{j,h}u_h|^2+2h^{-1}\big|\chi_{j,h}(h\nabla-i\mathbf{A})u_h\big|^2$$
$$+2h\big|\nabla(\chi_h\chi_{j,h})\big|^2|u_h|^2\Big)dx.$$

Since $\sum|\chi_{j,h}|^2\le1$ and $\sum|\nabla(\chi_{j,h}\chi_h)|^2=\mathcal{O}(h^{-2\delta})$, Proposition 6.3 together with (6.5) and (6.6) yield

$$\sum_{j\in\mathcal{J}_h}M_h(w_{j,h})=\mathcal{O}(1).$$

Consequently, we infer from (8.16),

$$\sum_{j\in\mathcal{J}^2_h}q^h_{\mathbf{A}}(w_{h,j})\ge(\Theta_0h+c_*h^{1+\delta})\Big(\sum_{j\in\mathcal{J}^2_h}\|w_{h,j}\|^2\Big)-C'h^{\frac43+\varsigma_0}.\tag{8.17}$$

For $j\in\mathcal{J}^1_h$, we estimate $q^h_{\mathbf{A}}(w_{h,j})$ by collecting (8.12) and the estimates in Lemmas 7.1 and 7.3. We start by picking $R_0>0$ and $x^j_0\in\Gamma$, so that

$$\operatorname{supp}w_{h,j}\subset Q_h(x^j_0)$$

where $Q_h(x_0^j)$ is introduced in (7.2). Eventually, we find

$$\sum_{j\in\mathcal{J}_h^1} q_{\mathbf{A}}^h(w_{h,j}) \geq \sum_{j\in\mathcal{J}_h^1} \big(\Theta_0 h + h^{4/3} c^{\mathrm{conj}}\big(\theta_j, \kappa_{n,\mathbf{B}}(x_j^0)\big)\big)\|w_{h,j}\|^2 - Ch^{\frac{4}{3}+\varsigma_*},$$

for some constant $\varsigma_* > 0$, where

$$\theta_j = \theta(s_0^j)$$

and $(0, s_0^j, 0)$ denote the coordinates of x_0^j in the (r,s,t)-frame (see Section 2 and equation (2.2)). Note that we used Proposition 6.3 to control the term $\sum_{j\in\mathcal{J}_h^1}\|tw_{h,j}\|^2$ appearing in (8.12); in fact $\sum_{j\in\mathcal{J}_h^1}\|tw_{h,j}\|^2 = \mathcal{O}(h)$.

Since $c^{\mathrm{conj}}(\theta_j, \kappa_{n,\mathbf{B}}(x_j^0))$ is bounded from below by $\hat{\gamma}_{0,\mathbf{B}}$ (see (1.6)), we get

$$\sum_{j\in\mathcal{J}_h^1} q_{\mathbf{A}}^h(w_{h,j}) \geq (\Theta_0 h + \hat{\gamma}_{0,\mathbf{B}} h^{4/3}) \sum_{j\in\mathcal{J}_h^1} \|w_{h,j}\|^2 - Ch^{\frac{4}{3}+\varsigma_*}. \tag{8.18}$$

Inserting (8.17) and (8.18) into (8.15), and using (8.14), we deduce the lower bound in (8.13), since $\frac{5}{18} < \delta < \frac{1}{3}$.

9. Upper bound

Fortunately, the same quasi-mode constructed in [17, Sec. 12] (see also [27] for a different formulation) yields an upper bound of the lowest eigenvalue $\lambda_1(\mathbf{A}, h)$ matching with the asymptotics in Theorem 1.4. More precisely, under Assumptions 1.1 and 1.2, we will prove that:

$$\lambda_1^N(\mathbf{A}, h) \leq \Theta_0 h + \hat{\gamma}_{0,\mathbf{B}} h^{\frac{4}{3}} + \mathcal{O}(h^{\frac{4}{3}+\eta^*}), \tag{9.1}$$

for some constant $\eta^* > 0$, where $\hat{\gamma}_{0,\mathbf{B}}$ is introduced in (1.6).

However, while computing the energy of the quasi-mode, we observe additional terms (not present in [17]) due to the non-homogeneity of the magnetic field. These terms are treated in Section 9.2.

9.1. The quasi-mode

The construction of the quasi-mode in [17] is quite lengthy and involves many auxiliary functions related to the de Gennes and Montgomery models (see (4.1) and (4.4)). We present here the definition of the quasi-mode along with a useful result from [17, Sec. 12].

9.1.1. Geometry and normal form. Select a point $x_0 \in \partial\Omega$ such that the function in (1.7) satisfies

$$\tilde{\gamma}_{0,\mathbf{B}}(x_0) = \hat{\gamma}_{0,\mathbf{B}}.$$

Let us assume that the coordinates of x_0 in the (r,s,t)-frame are $(0, s_0 = 0, t_0)$. The normal form of the effective magnetic potential in Lemma 7.2 now becomes

$$\mathbf{A}^{00} = \begin{pmatrix} A_1^{00} \\ A_2^{00} \\ A_3^{00} \end{pmatrix} = \begin{pmatrix} t\sin\theta + t(\zeta r + \check{\kappa} s)\cos\theta \\ -t\cos\theta + rt\kappa\cos + t(\zeta r + \check{\kappa} s)\sin\theta + \frac{1}{2}\gamma r^2 \\ 0 \end{pmatrix}, \tag{9.2}$$

where

$$\theta = \theta(s_0), \quad \kappa = \kappa_g(s_0), \quad \gamma = \kappa_{n,\mathbf{B}}(x_0). \tag{9.3}$$

9.1.2. Structure of the quasi-mode. Consider two positive constants C_0 and δ such that $\frac{5}{18} < \delta < \frac{1}{3}$. Let χ be a smooth *even* function, valued in $[0, 1]$, equal to 1 on $[-\frac{1}{4}, \frac{1}{4}]$ and supported in $[-\frac{1}{2}, \frac{1}{2}]$. We set

$$\chi_h(s) = c_1 h^{-\delta/2}\chi(C_0^{-1}h^{-\delta}s),$$

where $c_1 = C_0^{-1/2}(\int_{\mathbb{R}} \chi(\sigma)^2 d\sigma)^{1/2}$, so that χ_h is normalized as follows,

$$\int_{\mathbb{R}} |\chi_h(s)|^2 ds = 1.$$

Our quasi-mode, u, is supported in the set $Q_h(x_0, R_0, \delta, \epsilon_2)$ introduced in (7.2) and is of the form

$$u = e^{i\check{p}/h}(1 - r\kappa)^{-1/2}\tilde{u}, \tag{9.4}$$

where $(r, s, t) \mapsto \check{p}(r, s, t)$ is the function from Lemma 7.2 and the function $\tilde{u}$ is of the form

$$\tilde{u}(r, s, t) = \exp\Big(-\frac{i\rho\gamma s}{h^{1/3}}\Big)\exp\Big(i\frac{r\sin\theta - s\cos\theta}{h^{1/2}}\xi_0\Big)\chi_h(s)v(r, t), \tag{9.5}$$

where $\xi_0 = \sqrt{\Theta_0}$ is given by (4.2), θ and κ are introduced in (9.3).

The choice of ρ and v will be specified later[4] so that, for some constants $C, \varsigma_* > 0$, we have [17, (12.8)]

$$q^h_{M^{00}}(v) \le \big(\Theta_0 h + \hat{\gamma}_{0,\mathbf{B}} h^{\frac{4}{3}} + C h^{\frac{4}{3}+\varsigma_*}\big)\|v\|^2_{L^2(\mathbb{R}\times\mathbb{R}_+)}. \tag{9.6}$$

Here $q_{M^{00}}(v)$ arises while computing the quadratic form of the quasi-mode in (9.4). It is defined as follows [17, (12.9)],

$$q^h_{M^{00}}(v) = \int_{\mathbb{R}\times\mathbb{R}_+} \big(|(hD_r - M_1^{00})v|^2 + |M_2^{00}v|^2 + |hD_t v|^2\big)\, dr\, dt, \tag{9.7}$$

where

$$\begin{aligned} M_1^{00}(r,t) &= \sin\theta(t - h^{1/2}\xi_0), \\ M_2^{00}(r,t) &= (1+2\kappa r)^{1/2}\Big(-\cos\theta(t - h^{1/2}\xi_0) + \kappa\cos\theta rt - b\frac{\gamma}{2}(r^2 - h^{2/3}\rho)\Big). \end{aligned} \tag{9.8}$$

Notice that, by our normalization of χ_h, we have

$$\int_{\mathbb{R}^2\times\mathbb{R}_+} |\tilde{u}(r,s,t)|^2 dr\, ds\, dt = \|v\|^2_{L^2(\mathbb{R}\times\mathbb{R}_+)}. \tag{9.9}$$

[4] ρ is defined in (9.2). For the definition of v, see (9.10), (9.11), and (9.12).

9.1.3. Definition of the auxiliary objects. Let us recall the definition of the function v and the parameter ρ given in [17, Sec. 12]. The function v depends on h and is selected in the following form (see [17, (12.14)])

$$v(r,t) = h^{-5/12} v_0(\hat{r},\hat{t}), \tag{9.10}$$

where

$$(\hat{r},\hat{t}) = (h^{-1/3}r, h^{-1/2}t).$$

The function v_0 is selected as in [17, (12.28)]:

$$v_0(\hat{r},\hat{t}) = \chi(C_0^{-1}h^{-\delta+\frac{1}{3}}\hat{r})\chi(C_0^{-1}h^{-\delta+\frac{1}{2}}\hat{t})w_h(\hat{r},\hat{t}). \tag{9.11}$$

In the sequel, we skip the hats from the notation. The function w_h is defined as follows[5] [17, (12.22)]

$$\begin{aligned} w_h(r,t) &= \varphi_0(t)\psi(r) + h^{1/6}\varphi_1(t)L_1^0(r,D_r)\psi(r) \\ &\quad + h^{1/3}\varphi_2(t)\big(L_1^0(r,D_r)\big)^2\psi(r), \end{aligned} \tag{9.12}$$

where φ_0 is the positive normalized ground state of the harmonic oscillator in (4.1),

$$\varphi_1(t) = 2\mathscr{R}_0\big((t-\xi_0)\varphi_0\big), \quad \varphi_2(t) = 2\mathscr{R}_0\big((t-\xi_0)\varphi_1 - \langle (t-\xi_0)\varphi_1,\varphi_0\rangle\varphi_0\big)$$

and $\mathscr{R}_0$ is the regularized resolvent introduced in (4.3). Notice that φ_0, φ_1 and φ_2 are Schwartz functions (i.e. in $\mathcal{S}(\mathbb{R}_+)$, see [3, App. A]). The definition of w_h involves the differential operator

$$L_1^0(r,D_r) = \sin\theta D_r - \frac{1}{2}\cos\theta\gamma(r^2-\rho) \tag{9.13}$$

and a function $\psi \in \mathcal{S}(\mathbb{R})$ defined via the ground state ψ_0 of the Montgomery model in (4.4) and the following phase function

$$\varphi(r) = \gamma\alpha(\theta)\Big(\frac{r^3}{6} + \frac{\rho r}{2}\Big),$$

where

$$\alpha(\theta) = \frac{\sin\theta\cos\theta(1-\delta_0)}{\delta_0\sin^2\theta + \cos^2\theta},$$

and δ_0 the constant introduced in (4.2). We define now the function $\psi(r)$ as follows

$$\psi(r) = \Big(\frac{c}{d}\Big)^{-1/12}\exp\big(i\varphi(r)\big)\psi_0\Big(\Big(\frac{c}{d}\Big)^{-1/6}r\Big),$$

[5] For the convenience of the reader, we will recall the heuristics behind the construction of w_h in Section 9.1.4.

where

$$c=\cos^2\theta+\delta_0\sin^2\theta,\quad d=\frac{\delta_0^2\gamma^2}{\delta_0\sin^2\theta+\cos^2\theta},$$

and we choose (see (4.4))

$$\rho=\Big(\frac{c}{d}\Big)^{1/3}\rho_0.$$

We conclude by mentioning some estimates which follow easily from the definitions of v and v_0 in (9.10) and (9.11):

$$\begin{aligned}
&\|v\|^2_{L^2(\mathbb{R}\times\mathbb{R}_+)}=1+\mathcal{O}(h^{1/6}),\\
&\int_{\mathbb{R}\times\mathbb{R}_+} r^k t^n|v|^2\,dr\,dt=\mathcal{O}(h^{\frac{k}{3}+\frac{n}{2}})\quad(k,n\geq 0),\\
&\int_{\mathbb{R}\times\mathbb{R}_+}|hD_rv|^2\,dr\,dt=\mathcal{O}(h^{5/3}),\quad \int_{\mathbb{R}\times\mathbb{R}_+}|hD_tv|^2=\mathcal{O}(h).
\end{aligned}\tag{9.14}$$

9.1.4. Heuristics on the construction of w_h. Starting from the definition of the function v in (9.11), the quadratic form in (9.6) becomes (after neglecting error terms in the magnetic potential)

$$q^h_{M^{00}}(v)\approx h\tilde{q}^h(w_h),$$

where

$$\tilde{q}^h(w_h):=\int_{\mathbb{R}^2_+}\Big(|D_tw_h|^2+\big|\big(t-\xi_0-h^{1/6}L_1^0(r,D_r)\big)w_h\big|^2+h^{1/3}\big|L_2^0(h,D_r)w_h\big|^2\Big)\,dr\,dt,$$

$L_1^0(r,D_r)$ is introduced in (9.13) and

$$L_2^0=\cos\theta D_r+\frac{1}{2}\sin\theta\gamma(r^2-\rho).$$

The construction of w_h is based on minimizing

$$\int_{\mathbb{R}}\bigg(\int_{\mathbb{R}_+}\Big(|D_tw_h|^2+\big|\big(t-\xi_0-h^{1/6}L_1^0(r,D_r)\big)w_h\big|^2\Big)\,dt\bigg)\,dr,$$

which amounts to finding the lowest eigenvalue of the operator

$$\mathcal{T}_h:=D_t^2+\big(t-\xi_0-h^{1/6}L_1^0(r,D_r)\big)^2.$$

Writing

$$\mathcal{T}_h=D_t^2+(t-\xi_0)^2-2h^{1/6}(t-\xi_0)^2L_1^0(r,D_r)+h^{1/3}\big(L_1^0(r,D_r)\big)^2,$$

it is natural to search for w_h in the form in (9.12) and satisfying

$$\mathcal{T}_hw_h-\Big(\mu_0+\mu_1h^{1/6}L_1^0(r,D_r)+\mu_2^{1/3}\big(L_1^0(r,D_r)\big)^2\Big)w_h\approx 0$$

in the following sense (after taking the coefficients of $h^{i/6}$ to be 0, for $i = 0, 1, 2$)

$$\begin{aligned}
&\big(D_t^2 + (t-\xi_0)^2 - \mu_0\big)\varphi_0 = 0,\\
&\big(D_t^2 + (t-\xi_0)^2 - \mu_0\big)\varphi_1 = \mu_1\varphi_0,\\
&\big(D_t^2 + (t-\xi_0)^2 - \mu_0\big)\varphi_2 = \mu_2\varphi_0 + \mu_1\varphi_1.
\end{aligned}$$

Eventually, this leads to $\mu_0 = \Theta_0$, $\mu_1 = 0$, $\mu_2 = \frac{1}{2}\mu''(\xi_0)$ and $\varphi_0, \varphi_1, \varphi_2$ as in (9.12).

9.2. Energy estimates

We will estimate the following energy arising from Lemma 7.3:

$$q^h_{\mathbf{A}^{00}}(\tilde u) = \int_{\mathbb{R}^2\times\mathbb{R}_+} \big(\big|(hD_r - A_1^{00})\tilde u\big|^2 + (1+2\kappa r)\big|(hD_s - A_2^{00})\tilde u\big|^2 + |hD_t\tilde u|^2\big)\,dr\,ds\,dt,$$

where A_1^{00}, A_2^{00} are introduced in (9.2).

Actually, $q^h_{\mathbf{A}^{00}}(\tilde u)$ is bounded from above by $q_{M^{00}}(v)$ modulo error terms, where $q_{M^{00}}(v)$ and v are introduced in (9.7) and (9.5) respectively. Due to the non-homogeneity of the magnetic field, the error terms involve a quantity[6] introduced in (9.16) which has to be controlled carefully.

Due to the phase terms in the definition of $\tilde u$ in (9.5), we have

$$q^h_{\mathbf{A}^{00}}(\tilde u) = \int_{\mathbb{R}^2\times\mathbb{R}_+} \big(|hD_t\tilde u|^2 + \big|(hD_s - A^{00}_{2,\mathrm{new}})\tilde u\big|^2 + \big|(hD_r - A^{00}_{1,\mathrm{new}})\tilde u\big|^2\big)\,dr\,ds\,dt,$$

where

$$\begin{pmatrix} A^{00}_{1,\mathrm{new}} \\ A^{00}_{2,\mathrm{new}} \end{pmatrix} = \begin{pmatrix} M^{00}_{1,\zeta} \\ M^{00}_{2,\zeta} \end{pmatrix} + \begin{pmatrix} \check\kappa\cos\theta st \\ \check\kappa\sin\theta st \end{pmatrix},$$

and

$$\begin{aligned}
M^{00}_{1,\zeta}(r,t) &= \sin\theta(t - h^{1/2}\xi_0) + \zeta\cos\theta rt,\\
M^{00}_{2,\zeta}(r,t) &= (1+2\kappa r)^{1/2}\Big(-\cos\theta(t - h^{1/2}\xi_0)\\
&\qquad + (\kappa\cos\theta + \zeta\sin\theta)rt - \frac{\gamma}{2}(r^2 - h^{2/3}\rho)\Big).
\end{aligned}$$

Since the function $s\mapsto \chi_h(s)$ is even, we have

$$\big\langle (hD_s - \check\kappa\sin\theta st)u, M^{00}_{2,\zeta}u\big\rangle_{L^2(\mathbb{R}^2\times\mathbb{R}_+)} = 0$$

and

$$\big\langle \check\kappa\cos\theta stu, (hD_r - M^{00}_{1,\zeta})u\big\rangle_{L^2(\mathbb{R}^2\times\mathbb{R}_+)} = 0.$$

[6]This is $A(v) + B(v)$ appearing in (9.16), which would vanish if the magnetic field were constant.

Moreover, we have the estimates

$$\begin{aligned}\big\|(hD_s-\check{\kappa}\sin\theta st)\tilde u\big\|_{L^2(\mathbb{R}^2\times\mathbb{R}_+)} &\le C\int_{\mathbb{R}\times\mathbb{R}_+}(h^{2-2\delta}+h^{2\delta}t^2)|v|^2\,dr\,dt\\ &=\mathcal{O}(h^{2-2\delta}+h^{2\delta+1}),\\ \|\check{\kappa}\cos\theta stu\|_{L^2(\mathbb{R}^2\times\mathbb{R}_+)} &\le Ch^{2\delta}\int_{\mathbb{R}\times\mathbb{R}_+}t^2|v|^2\,dr\,dt=\mathcal{O}(h^{2\delta+1}).\end{aligned}$$

Notice that we used (9.14) and also that $|s|\le C_0h^\delta$ in the support of $\tilde u$. Consequently, we get

$$\begin{aligned}q^h_{\mathbf{A}^{00}}(\tilde u)\le&\int_{\mathbb{R}\times\mathbb{R}_+}\Big(\big|(hD_r-M^{00}_{1,\zeta})v\big|^2+|M^{00}_{2,\zeta}v|^2+|hD_tv|^2\Big)\,dr\,dt\\ &+\mathcal{O}(h^{2-2\delta}+h^{2\delta+1}).\end{aligned}\tag{9.15}$$

Let us now reduce the computations to the potentials M_1^{00} and M_2^{00} in (9.8) which amount to $M^{00}_{1,\zeta}$ and $M^{00}_{2,\zeta}$ with $\zeta=0$. A straightforward computation yields,

$$\begin{aligned}&\big\|(hD_r-M^{00}_{1,\zeta})v\big\|^2_{L^2(\mathbb{R}\times\mathbb{R}_+)}+\|M^{00}_{2,\zeta}v\|^2_{L^2(\mathbb{R}\times\mathbb{R}_+)}\\ &\quad=\big\|(hD_r-M^{00}_1)v\big\|^2_{L^2(\mathbb{R}\times\mathbb{R}_+)}+\|M^{00}_2v\|^2_{L^2(\mathbb{R}\times\mathbb{R}_+)}+\zeta\big(A(v)+B(v)\big),\end{aligned}\tag{9.16}$$

where

$$\begin{aligned}A(v)&:=\zeta\cos^2\theta\|rtv\|^2_{L^2(\mathbb{R}\times\mathbb{R}_+)}-2\cos\theta\,\mathrm{Re}\big\langle(hD_r-M^{00}_1)v,\,rtv\big\rangle_{L^2(\mathbb{R}\times\mathbb{R}_+)},\\ B(v)&:=\zeta\sin^2\theta\|rtv\|^2_{L^2(\mathbb{R}\times\mathbb{R}_+)}+2\sin\theta\,\mathrm{Re}\langle M^{00}_2v,\,rtv\rangle_{L^2(\mathbb{R}\times\mathbb{R}_+)}\end{aligned}$$

and by (9.14)

$$\|rtv\|^2_{L^2(\mathbb{R}\times\mathbb{R}_+)}=\mathcal{O}(h^{5/3}),\quad\langle hD_rv,\,rtv\rangle_{L^2(\mathbb{R}\times\mathbb{R}_+)}=\mathcal{O}(h^{5/3}).$$

So, we end up with estimating

$$F(v):=\big\langle(\cos\theta M^{00}_1+\sin\theta M^{00}_2)v,\,rtv\big\rangle_{L^2(\mathbb{R}\times\mathbb{R}_+)}.$$

Notice that

$$\begin{aligned}\cos\theta M^{00}_1(r,t)+\sin\theta M^{00}_2(r,t)=&\cos\theta\sin\theta\big(1-(1+2\kappa r)^2\big)(t-h^{1/2}\xi_0)\\ &+(1+2\kappa r)^2\cos\theta\Big(\kappa\cos\theta rt-\frac{\gamma}{2}(r^2-h^{2/3}\rho)\Big).\end{aligned}$$

By expanding

$$(1+2\kappa r)^{1/2}=1+\kappa r+\mathcal{O}(r^2)\quad(r\to0),$$

we observe that, for $|r|\le r_0$ and r_0 sufficiently small,

$$\big|\cos\theta M^{00}_1(r,t)+\sin\theta M^{00}_2(r,t)\big|\le C(r^2+t^2+h^{2/3}),$$

so we get by (9.14) and the Cauchy–Schwarz inequality that

$$F(v) = \mathcal{O}(h^{3/2}).$$

Therefore, $A(v) + B(v) = \mathcal{O}(h^{3/2})$ and we deduce from (9.16) and (9.15) that

$$q^h_{\mathbf{A}^{00}}(\tilde{u}) \leq q^h_{M^{00}}(v) + \mathcal{O}\big(h^{2-2\delta} + h^{2\delta+1} + h^{3/2}\big), \tag{9.17}$$

where $q^h_{M^{00}}(v)$ is introduced in (9.7).

9.3. Conclusion

Collecting (9.17) and (9.6), we get

$$q^h_{\mathbf{A}^{00}}(\tilde{u}) \leq \big(\Theta_0 h + \hat{\gamma}_{0,\mathbf{B}} h^{\frac{4}{3}} + C h^{\frac{4}{3}+\eta}\big)\|v\|^2_{L^2(\mathbb{R}\times\mathbb{R}_+)} + R_h(v),$$

where

$$R_h(v) = \mathcal{O}\big(h^{2-2\delta} + h^{2\delta+1} + h^{3/2}\big) = \mathcal{O}(h^{\frac{4}{3}+\hat{\eta}})$$

for some $\hat{\eta} > 0$, thanks to the condition $\frac{5}{18} < \delta < \frac{1}{3}$.

We insert this into Lemma 7.3 with u given in (9.4). Notice that u satisfies (7.6) with $M_h(u) = \mathcal{O}(1)$. So by Lemma 7.3 and (9.9), we get for some $\eta_* > 0$

$$q^h_{\mathbf{A}}(u) \leq \big(\Theta_0 h + \hat{\gamma}_{0,\mathbf{B}} h^{\frac{4}{3}} + C h^{\frac{4}{3}+\eta_*}\big)\|v\|^2_{L^2(\mathbb{R}\times\mathbb{R}_+)}.$$

Comparing (9.9) and (9.4), we get by (2.15),

$$\|u\|^2 = \big(1 + \mathcal{O}(h^{2\delta})\big)\|v\|^2_{L^2(\mathbb{R}\times\mathbb{R}_+)}.$$

Applying the min-max principle, and noticing that $1 + 2\delta > \frac{4}{3}$ for $\frac{5}{18} < \delta < \frac{1}{3}$, we finish the proof of (9.1).

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