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Quaternionic Speh representations

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Abstract. For a central division algebra D, we study a family of representations of $\mathrm{GL}_{k,D}$ (both locally and globally), which can be viewed as analogs of the Speh representations. At the non-Archimedean places, we show that these representations support unique models of degenerate type. Globally, we show that these representations support certain non-vanishing Fourier coefficients. We also obtain some partial results regarding unique models at the Archimedean places.

1. Introduction

The uniqueness of Whittaker models is a fundamental result in the study of automorphic representations and has many important applications. For example, it leads to the proof of the functional equations of certain automorphic L-functions via the Langlands–Shahidi method and several Rankin–Selberg integrals. Unfortunately, this important property does not hold for non-quasi-split groups. As a result, it seems rather difficult to develop a theory of L-functions for these groups. The purpose of this paper is to study a family of representations with unique models of $\mathrm{GL}_{k,D}$ (both locally and globally) for a central division algebra D over a local or global field F.

We first start with a simple example to get some basic ideas of this problem. Let D be a central division algebra over a local field F of dimension d^2. Then the only nilpotent orbit of the group $D^\times$ is the trivial orbit. As a consequence, only one-dimensional representations of $D^\times$ have unique models and most of the representations do not support unique models.

Similarly, for $\mathrm{GL}_{k,D}$, it is not difficult to see that most representations do not have unique models, either. In this article, we would like to search for representations of $\mathrm{GL}_{k,D}$ with unique models.

A natural method of constructing representations of $\mathrm{GL}_{k,D}$ is the Jacquet–Langlands correspondence. The Jacquet–Langlands correspondence (as in [12]) for discrete series representations says that there is a bijection between discrete series representations of $\mathrm{GL}_{k,D}$ and GL_{kd} satisfying a character identity. This correspondence was later extended in [4, 5] to allow unitary representations as well. For our purpose, we would like to take

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the latter version for the following two reasons. First, it does produce more representations than the discrete series version. Second, it is also local-to-global compatible and is more suitable if we are aiming for some global applications.

For an admissible representation π of a p-adic group, a result of Mœglin–Waldspurger [27] says that the dimension of a certain generalized Whittaker models for π is related to the leading coefficients of the local character expansion of π. As a result, one hope that the Jacquet–Langlands transfer of appropriate representations of GL_{kd} (with suitable unique generalized Whittaker models) might give some desired representations of $\mathrm{GL}_{k,D}$ as the Jacquet–Langlands transfer satisfies a character identity. It turns out that, at least in the non-Archimedean case, this idea works for the Speh representations of GL_{kd}.

The Speh representations were first studied in the real case and later this construction was also extended to the p-adic case. Locally, we view the construction of the Speh representations as the following:

$$\tau \mapsto \mathrm{Speh}(\tau, n)$$

where τ is an irreducible generic unitary representation of GL_k and $\mathrm{Speh}(\tau, n)$ is the "smallest" piece of a highly reducible induced representation defined using τ (see Remark 4.6). A key property of the Speh representations is that the maximal nilpotent orbit that supports generalized Whittaker models is (k^n). Moreover, the dimension of such generalized Whittaker models is 1. We now observe that if $n = dn'$ for some n', then the nilpotent orbit (k^n) is an orbit that "comes from" $\mathrm{GL}_{kn',D}$.

Our attempt is to define

$$\mathrm{Speh}_D(\tau, n) := |\mathbf{LJ}|\big(\mathrm{Speh}(\tau, nd)\big).$$

Here $|\mathbf{LJ}|$ is the Jacquet–Langlands transfer for unitary representations in [4, 5]. Note that $\mathrm{Speh}_D(\tau, n)$ is a representation of $\mathrm{GL}_{kn,D}$. This definition works both locally and globally.

Our main theorem is the following.

Theorem 1.1 (Corollary 6.15 and Theorems 4.13, 5.8, and 5.9). *For the representation* $\mathrm{Speh}_D(\tau, n)$ *of* $\mathrm{GL}_{kn,D}$*, we have the following:*

(1) *(Local vanishing result) For any nilpotent orbit $\mathcal{O}$ greater than or not comparable with $(k^n)_D$, the representation* $\mathrm{Speh}(\tau, n)$ *does not have generalized Whittaker models associated to $\mathcal{O}$.*

(2) *(Local multiplicity one result) The Archimedean case is based on certain natural hypotheses (Hypotheses 6.8 and 6.14). Then*

$$\dim \mathrm{Hom}_{N_{(k^n)_D}}\left(\mathrm{Speh}_D(\tau, n), \psi_{(k^n)_D}\right) \leq 1.$$

Here, the pair $(N_{(k^n)_D}, \psi_{(k^n)_D})$ is the unipotent subgroup and character used in the definition of the generalized Whittaker models associated to the nilpotent orbit $(k^n)_D$. Moreover, in the non-Archimedean case, this dimension is exactly 1.

(3) *(Global result) The maximal nilpotent orbit that supports a nonzero global generalized Whittaker coefficient for* $\mathrm{Speh}_D(\tau, n)$ *is* $(k^n)_D$.

In the real case, the Speh representations originally refer to $\mathrm{Speh}_{\mathbb{R}}(\tau, 2)$, where τ is a discrete series representation of $\mathrm{GL}_2(\mathbb{R})$. The representation $\mathrm{Speh}_{\mathbb{H}}(\tau, 1)$ can be viewed as quaternionic analogs of the Speh representations. We call them *quaternionic Speh representations*.

Let us now say a few words regarding the proofs. As we indicated above, the non-Archimedean case can be done using the character identity without too much trouble. The more difficult part is deal with the Archimedean and global theory.

The Archimedean version of the result of Mœglin–Waldspurger is not known at the moment. Partial results can be found in [14, 17] and these allow us to settle the vanishing part. For the multiplicity one part, we need two natural hypotheses (Hypotheses 6.8 and 6.14). The idea is as follows. The definition of Speh representations is given by induction. Thus we first use Hypothesis 6.14 to reduce to the case when τ is a discrete series. In the case of discrete series, we use a global method in [22]. In this paper, Kazhdan–Patterson used a global method to prove that the local components of the theta representations at bad primes support unique Whittaker models if the same holds for unramified places. This method can be adapted to our case, under certain natural hypothesis (Hypothesis 6.8) on the Kirillov models for representations of $\mathrm{GL}_{k,D}$. This only treats representations that can be realized as local components of global representations, but should be sufficient for applications.

Hang Xue suggested to us that in the minimal case, $\tau \mapsto \mathrm{Speh}_D(\tau, 1)$ can be realized using the theta correspondence. Thus, in this particular case, the desired unique model is a consequence of a result by Gomez–Zhu [16]. We will explain this in Section 4.9.

The global correspondence is proved using the method of the trace formula. As a result, it seems difficult to gain information regarding the Fourier coefficients. In [22], another method was used to show that theta representations in certain cases are globally generic. This is again adaptable to our case to prove a base case, and we prove the general case using an induction-by-stages argument.

We end this introduction by saying a few words regarding some potential applications of our results. Here, we assume that F is a global field and $\mathbb{A}$ is the ring of adeles of F. The first application is the twisted doubling integrals [10]. The twisted doubling method is a generalization of the doubling method. It gives a family of Rankin–Selberg integrals that represents the tensor product L-function $L(s, \pi \times \tau)$ for π of a classical group, and τ of a general linear group GL_k. A key ingredient in the construction is the use of the Speh representations, which can be viewed as

$$\tau \in \mathrm{DS}_{\mathrm{cusp}}\big(\mathrm{GL}_k(\mathbb{A})\big) \mapsto \mathrm{Speh}(\tau, n) \in \mathrm{DS}\big(\mathrm{GL}_{kn}(\mathbb{A})\big)$$

for every positive integer n. Here, $\mathrm{DS}(\mathrm{GL}_k(\mathbb{A}))$ denotes the set of discrete series representations of $\mathrm{GL}_k(\mathbb{A})$ and the subscript cusp indicated cuspidal representations. The unfolding argument and the Eulerian property rely on the fact that the representation

$\mathrm{Speh}(\tau, n)$ is supported on a sufficiently small nilpotent orbit and admits unique models of degenerate type at every local place. To extend the twisted doubling integrals to the quaternion unitary groups (see [9]), we need an analogous construction

$$\tau \in \mathrm{DS}_{\mathrm{cusp}}\big(\mathrm{GL}_k(\mathbb{A})\big) \mapsto \mathrm{Speh}_D(\tau, n) \in \mathrm{DS}\big(\mathrm{GL}_{kn,D}(\mathbb{A})\big)$$

for a positive integer n and a central division algebra D over F. We also have to prove analogous properties for these representations. This is what we are seeking for in this paper.

Another application is related to Lapid–Mao [24]. In this paper, a local version of the Rankin–Selberg convolution of two Speh representations is given and several properties are studied. It is also mentioned that a global version is possible by convolving two global Speh representations modulo some regularization problem. We would like to suggest that, one can take convolution of two representations of the form $\mathrm{Speh}_D(\tau, 1)$ and its local integrals will also be the ones studied by Lapid–Mao. It is possible for $\mathrm{Speh}_D(\tau, 1)$ to be cuspidal so that no regularization is necessary.

The rest of the paper is organized as follows. In Section 2 we recall some preliminary results. In particular, we review the classification of unitary representations of $\mathrm{GL}_{k,D}$. We review the extended Jacquet–Langlands correspondence, following [4, 5], in Section 3. In Section 4, we define the Speh representations over central division algebras locally and study some properties. In particular, the non-Archimedean part of Theorem 1.1 is proved and Section 4.8 treats the vanishing part in the Archimedean case. We start the global investigation in Section 5. Section 6 proves the global non-vanishing statement. Moreover, the uniqueness part in the Archimedean case is studied using global methods in Section 6.5. In Appendix A, we prove a result related to Kirillov models for $\mathrm{GL}_{k,D}$ in the non-Archimedean case.

2. Preliminaries

Let F be a local field of characteristic zero. Let D be a central division algebra over F of dimension d^2. For a positive integer k, set $G_k = \mathrm{GL}_k(F)$ and $G'_k = G_{k,D} = \mathrm{GL}_k(D)$. From now on we identify a smooth representation of finite length with its equivalence class, so we consider two equivalent representations as being equal.

We introduce the following notation:

- For an admissible representation π, we denote χ_π the function character of π. This is a map which is stable under conjugation and defined on the set of regular semisimple elements of G_k.

For all positive integers k, we fix the following notations:

- $\mathrm{Irr}_{k,D}$: the set of irreducible smooth representations of $G_{k,D}$
- $\mathcal{D}_{k,D}$: the subset of essentially square integrable representations in $\mathrm{Irr}_{k,D}$
- $\mathcal{D}_D = \sqcup_{k\geq 1}\mathcal{D}_{k,D}$

- $\mathcal{C}_{k,D}$: the subset of cuspidal representations in $\mathcal{D}_{k,D}$
- $\mathrm{Irr}^{\mathrm{u}}_{k,D}$ (resp. $\mathcal{D}^{\mathrm{u}}_{k,D}$, $\mathcal{C}^{\mathrm{u}}_{k,D}$): the subset of unitary representations in $\mathrm{Irr}_{k,D}$ (resp. $\mathcal{D}_{k,D}$, $\mathcal{C}_{k,D}$)
- $\mathrm{Irr}^{\mathrm{eu}}_{k,D}$: the subset of essentially unitary representations in $\mathrm{Irr}_{k,D}$
- $\mathcal{R}_{k,D}$: the Grothendieck groups of admissible representations of finite length of $G_{k,D}$
- $\nu = \nu_{k,D}$: the character of $G_{k,D}$, defined by the absolute value of the reduced norm of the determinant
- $\times$: the standard notation for normalized parabolic induction; see also [8].

Moreover, all the induced representations are normalized.

Let $\mathfrak{g}_{k,D}$ be the Lie algebra of $G_{k,D}$. The coadjoint nilpotent orbits in $\mathfrak{g}^*_{k,D}$ are classified by partitions of k. A typical orbit is denoted by $(k_1^{n_1} \cdots k_m^{n_m})_D$ with $k_1 n_1 + \cdots + k_m n_m = k$.

When $D = F$, we usually drop the subscript in the above notations. For example, (1^k) is the trivial orbit in $\mathfrak{g}^*_k$.

We introduce the following observation which may be useful when considering local-to-global questions. Let $D' = \mathrm{M}_n(D)$. Then $G_{k,D'} \simeq G_{kn,D}$. We can fix such an isomorphism and the nilpotent orbit $(k_1 \cdots k_m)_{D'}$ corresponds to the nilpotent orbit $(k_1^n \cdots k_m^n)_D$.

For nilpotent orbits $\mathcal{O} = (k_1^d \cdots k_m^d)$ of GL_{kd} and $\mathcal{O}' = (k_1 \cdots k_m)_D$ of $\mathrm{GL}_{k,D}$, we say that they correspond to each other and write $\mathcal{O} \leftrightarrow \mathcal{O}'$.

2.1. Unitary dual

In this section, we review the classification of the unitary dual of $G_{k,D}$. The case of G_k is also included and this case is originally proved in [33, 36]. The reader is referred to [4, 5] for more details.

Let $\sigma \in \mathcal{D}_{l,D}$ (meaning: essentially square-integrable modulo center). Consider $\sigma \times \nu^\alpha \sigma$ with $\alpha > 0$. There exists a smallest number $\alpha_0 > 0$ such that $\sigma \times \nu^{\alpha_0}\sigma$ is reducible.

Definition 2.1. Let $\sigma \in \mathcal{D}_{l,D}$. Set $\nu_\sigma = \nu^{\alpha_0}$, where α_0 is the smallest real number $\alpha > 0$ such that $\sigma \times \nu^\alpha \sigma$ is reducible.

For $\sigma \in \mathcal{D}_{l,D}$ and a positive integer n, we define $u(\sigma, n)$ to be the Langlands quotient of

$$\nu_\sigma^{(n-1)/2}\sigma \times \nu_\sigma^{(n-3)/2}\sigma \times \cdots \times \nu_\sigma^{(1-n)/2}\sigma.$$

The representation $u(\sigma, n)$ is an irreducible representation of $G_{ln,D}$.

For $\sigma \in \mathcal{D}_{l,D}$, a positive integer n and a real number $\alpha \in (0, 1/2)$, we denote by $\pi(u(\sigma, n), \alpha)$ the induced representation

$$\nu_\sigma^\alpha u(\sigma, n) \times \nu_\sigma^{-\alpha} u(\sigma, n).$$

The representation $\pi(u(\sigma, n), \alpha)$ is also irreducible.

The unitary dual of $G_{k,D}$ is given as follows.

Theorem 2.2. *Let $\mathcal{U}_D$ be the set of all representations of type $u(\sigma, n)$ or $\pi(u(\sigma, n), \alpha)$ where l, n range over all positive integers, $\sigma \in \mathcal{D}^{\mathrm{u}}_{l,D}$ and $\alpha \in (0, 1/2)$. Then we have the following:*

(1) *all the representations in $\mathcal{U}_D$ are unitary;*

(2) *any product of representations in $\mathcal{U}_D$ is irreducible and unitary;*

(3) *every irreducible unitary representation π of $G_{k,D}$ is a product of representations in $\mathcal{U}_D$.*

We refer the reader to [5, Section 7] for a comprehensive history of this result.

2.2. Classification of the generic unitary dual

The generic unitary dual of G_k is given as follows.

Theorem 2.3. *Let $\mathcal{U}_{\mathrm{gen}}$ be the set of all representations of type $u(\sigma, 1)$ or $\pi(u(\sigma, 1), \alpha)$ where l range over all positive integers, $\sigma \in \mathcal{D}^{\mathrm{u}}_{l}$ and $\alpha \in (0, 1/2)$. Then we have the following:*

(1) *all the representations in $\mathcal{U}_{\mathrm{gen}}$ are unitary and generic;*

(2) *any product of representations in $\mathcal{U}_{\mathrm{gen}}$ is irreducible generic and unitary;*

(3) *every irreducible generic unitary representation of G_k is a product of representations in $\mathcal{U}_{\mathrm{gen}}$.*

We refer the reader to [5, Section 8] for more details.

2.3. The local Jacquet–Langlands correspondence

Let $g \in G_{kd}$ and $g' \in G_{k,D}$. We say that g *corresponds* to g' if both g and g' are regular semisimple and have the same characteristic polynomial. We shortly write $g \leftrightarrow g'$. Denote $G_{kd,d}$ the set of elements $g \in G_{kd}$ such that there exists $g' \in G_{k,D}$ with $g \leftrightarrow g'$. (Note that this set is defined using the characteristic polynomial, so it depends on d only.)

The following theorem is proved in [12] if the characteristic of the base field F is zero and [2] for the nonzero characteristic case.

Theorem 2.4. *There is a unique bijection $\mathbf{C} : \mathcal{D}_{kd} \to \mathcal{D}_{k,D}$ such that for all $\sigma \in \mathcal{D}_{kd}$ we have*

$$\chi_\sigma(g) = (-1)^{kd-k} \chi_{\mathbf{C}(\sigma)}(g')$$

for all $g \in G_{kd}$ and $g' \in G_{k,D}$ such that $g \leftrightarrow g'$.

We identify the centers of G_{kd} and $G_{k,D}$ via the canonical isomorphism. Thus the correspondence $\mathbf{C}$ preserves central characters. In particular, $\sigma \in \mathcal{D}^{\mathrm{u}}_{kd}$ if and only if $\mathbf{C}(\sigma) \in \mathcal{D}^{\mathrm{u}}_{k,D}$.

2.4. Classification of $\mathcal{D}_{k,D}$

In this section, we review some necessary results on the classification of the discrete series representations.

2.4.1. The case $D = F$, non-Archimedean. The classification of $\mathcal{D}_k$ is given in terms of $\mathcal{C}_l$, $l|k$.

Let l and n be two positive integers and set $k = ln$. Let $\rho \in \mathcal{C}_l$. Then the representation

$$\rho \times \nu\rho \times \cdots \times \nu^{n-1}\rho$$

has a unique irreducible quotient σ. The representation σ is an essentially square integrable representation of G_k. Notation: $\sigma = Z(\rho, n)$. Every $\sigma \in \mathcal{D}_k$ is obtained in this way and l, n and ρ are determined by σ. (See [37, Section 9].) Moreover, for $\sigma \in \mathcal{D}_k$, $\nu_\sigma = \nu$.

We also define $Z^{\mathrm{u}}(\rho, n)$ to be the unique irreducible quotient of

$$\nu^{(1-n)/2}\rho \times \nu^{(3-n)/2}\rho \times \cdots \times \nu^{(n-1)/2}\rho.$$

This is a unitary representation.

2.4.2. The case of general D, non-Archimedean. Let l be a positive integer and $\rho' \in \mathcal{C}_{l,D}$. Then $\sigma = \mathbf{C}^{-1}(\rho')$ is an essentially square integrable representation of G_{dl}. We may write $\sigma = Z(\rho, p)$ for some p and $\rho \in \mathcal{C}_{dl/p}$. Set $s(\rho') = p$. Then it is known that $\nu_{\rho'} = \nu^{s(\rho')}$.

Let n be a positive integers and set $k = ln$. Then the representation

$$\rho' \times \nu_{\rho'}\rho' \times \cdots \times \nu_{\rho'}^{n-1}\rho'$$

has a unique irreducible quotient σ'. The representation σ' is an essentially square integrable representation of $G_{k,D}$. Notation: $\sigma' = T(\rho', n)$. Every $\sigma' \in \mathcal{D}_{k,D}$ is obtained in this way and l, n and ρ' are determined by σ'. We then set $s(\sigma') = s(\rho')$. For $\sigma' \in \mathcal{D}_{k,D}$, we have $\nu_{\sigma'} = \nu^{s(\sigma')}$. (For this classification, see [34].)

2.4.3. Some notations. Let $\sigma' \in \mathcal{D}^{\mathrm{u}}_{k,D}$. For any positive integer n, recall that $u(\sigma', n)$ the Langlands quotient of the induced representation

$$\nu_{\sigma'}^{(n-1)/2}\sigma' \times \nu_{\sigma'}^{(n-3)/2}\sigma' \times \cdots \times \nu_{\sigma'}^{(1-n)/2}\sigma'.$$

We denote by $\bar{u}(\sigma', n)$ the Langlands quotient of the induced representation

$$\nu^{(n-1)/2}\sigma' \times \nu^{(n-3)/2}\sigma' \times \cdots \times \nu^{(1-n)/2}\sigma'.$$

Both $u(\sigma', k)$ and $\bar{u}(\sigma', k)$ are irreducible representations.

2.4.4. The Archimedean case. We refer the reader to [5] for a comprehensive discussion. We first discuss the real case. In this case, $\mathcal{D}_{\mathbb{R}}$ consists of the following representations:

- the unitary characters of $\mathbb{R}^\times$;
- the Langlands quotient of the induced representation

$$\chi\nu^n \times \chi\nu^{-n}$$

  for a positive integer n.

For all $\sigma \in \mathcal{D}_{\mathbb{R}}$, $\nu_\sigma = \nu$.

In the case $F = \mathbb{C}$, the set $\mathcal{D}_{\mathbb{C}}$ is the set of unitary characters of $\mathbb{C}^\times$.

Finally, if $D = \mathbb{H}$ is the unique quaterion algebra over $\mathbb{R}$, then $\mathcal{D}_{\mathbb{H}}$ consists of all the irreducible finite-dimensional representations of $\mathbb{H}^\times$. Moreover, $\nu_\sigma = \nu^2$ if $\dim \sigma = 1$; $\nu_\sigma = \nu$ if $\dim \sigma > 1$.

2.5. The involution

We need the Aubert involution in the non-Archimedean case. We will use the notation only and will not use any explicit calculation.

Aubert defined in [1] an involution of the Grothendieck group of smooth representations of finite length of a reductive group over a local non-Archimedean field. The involution sends an irreducible representation to an irreducible representation up to a sign. We specialize this involution to G_k (resp. $G_{k,D}$) and denote it i_k (resp. i'_k). We will write i and i' when the index is not relevant or it is clearly understood. With this notation we have the relation $i(\pi_1) \times i(\pi_2) = i(\pi_1 \times \pi_2)$, i.e., "the involution commutes with the parabolic induction". The same holds for i'. The reader may find all these facts in [1].

If $\pi \in \mathrm{Irr}_k$, then one of $i(\pi)$ and $-i(\pi)$ is an irreducible representation. We denote it by $|i(\pi)|$. We denote $|i|$ the involution of Irr_k defined by $\pi \mapsto |i(\pi)|$. The same facts and definitions also hold for i'.

3. Local Jacquet–Langlands correspondence

In order to define a global Jacquet–Langlands correspondence for automorphic representations, it is not sufficient to transfer only square integrable representations as in the classical theory (for example, see [12]). It would be necessary to transfer at least the local components of global discrete series. This was achieved in [4, 5]. In these two papers, the local transfer for all unitary representations is established. A global correspondence for discrete series compatible with the local transfer is also proved.

In this section, we review the Jacquet–Langlands correspondence as proved in [4, 5]. The notations are **C** (for discrete series representations), **LJ** (for the Grothendieck ring) and |**LJ**| (for all unitary representations), respectively.

3.1. The extended correspondence

The correspondence **C** can be extended in a natural way to a correspondence between the Grothendieck groups.

Theorem 3.1 ([3]). (1) *For all positive integers k, $\mathbf{LJ}_k$ is the unique map from $\mathcal{R}_{kd}$ to $\mathcal{R}_{k,D}$ such that for all $\pi \in \mathcal{R}_{kd}$, we have*

$$\chi_\pi(g) = (-1)^{kd-k} \chi_{\mathbf{LJ}_k(\pi)}(g') \tag{3.1}$$

for all $g \leftrightarrow g'$.

(2) *The maps $\mathbf{LJ}_k$ commute with the parabolic induction.*

Remark 3.2. In [3], the correspondence $\mathbf{C}^{-1}$ was extended to a correspondence **JL** from $\mathcal{R}_{k,D}$ to $\mathcal{R}_{kd}$. As a result, the notation **LJ** was used in the reverse direction. We use the same notation here.

We sometimes drop the subscript k in the notation $\mathbf{LJ}_k$ when it is clear from the context. We say that $\pi \in \mathcal{R}_{kd}$ is d-compatible if $\mathbf{LJ}_k(\pi) \neq 0$. This means that there exists $g \in G_{kd,d}$ such that $\chi_\pi(g) \neq 0$.

In this paper, we only need to understand this correspondence for unitary representations. By the classification of unitary representations of G_k, it suffices to understand **LJ** for representations of the form $u(\sigma, n)$.

Remark 3.3. The convention in [5, Section 4] (the Archimedean case) is slightly different from [4]. To unify the presentation, we also take the convention in [4] in the Archimedean case.

Moreover, in the Archimedean case, the sign in (3.1) is not explicitly given in [5]. But it is not difficult to see that the calculation can be done as in the non-Archimedean case.

3.2. Transfer of $u(\sigma, n)$: non-Archimedean case

We now collect results regarding the transfer of $u(\sigma, n)$ (see [4, Section 3.2]).

Let n, l, q be positive integers. Set $k = lnq$. Let $\rho \in \mathcal{C}^{\mathrm{u}}_q$ and $\sigma = Z^{\mathrm{u}}(\rho, l) \in \mathcal{D}^{\mathrm{u}}_{lq}$, $\tau = Z^{\mathrm{u}}(\rho, n) \in \mathcal{D}^{\mathrm{u}}_{nq}$.

Let s be the smallest positive integer such that $d \mid sq$. We first define the transfer of $u(\sigma, n)$. This question has no meaning unless $d \mid k$ (i.e., $s \mid ln$) which we shall assume.

Proposition 3.4 ([4, Proposition 3.7]). (1) *If $d \mid lq$ (i.e. $s \mid l$), then $\sigma' \in \mathbf{C}(\sigma)$ is well defined; we have $s = s(\sigma')$ and*

$$\mathbf{LJ}\big(u(\sigma, n)\big) = \bar{u}(\sigma', n).$$

(2) *If $d \mid nq$ (i.e. $s \mid n$), then $\tau' = \mathbf{C}(\tau)$ is well defined; we have $s = s(\tau')$ and*

$$\mathbf{LJ}\big(u(\sigma, n)\big) = \varepsilon \big| i'\big(\bar{u}(\tau', l)\big)\big|$$

where $\varepsilon = 1$ if s is odd and $\varepsilon = (-1)^{\frac{ln}{s}}$ if s is even.

(3) *If d does not divide neither lq, nor nq (i.e. s does not divide neither l nor n), then* $\mathbf{LJ}(u(\sigma, n)) = 0$.

Remark 3.5. It is easy to see that if $q = 1$, then $s = d$.

3.3. The Archimedean case

The above discussion can be carried out for the Archimedean case. We recall the necessary results here. For more details, see [5].

Let $X_{\mathbb{R}}$ be the set of unitary characters of $\mathbb{R}$. For $\chi \in X_{\mathbb{R}}$ and a positive integer n, we define $\chi_n := \chi \circ \nu$ and $\chi'_n := \chi \circ \nu_{n,\mathbb{H}}$.

The computation of **LJ** on representations of the form $u(\sigma, n)$ is given as follows [5, Theorem 13.8]. We remind the reader that the definition of **LJ** in [5, Section 4] is slightly different from what we use here.

(1) $\mathbf{LJ}(\chi_{2n}) = \chi'_n$ and $\mathbf{LJ}(\pi(\chi_{2n}, \alpha)) = \pi(\chi'_n, \alpha)$ for all $\chi \in X_{\mathbb{R}}$ and $\alpha \in (0, 1/2)$.

(2) If $\delta \in \mathcal{D}_2^{\mathrm{u}}$ is such that $\dim \mathbf{C}(\delta) > 1$, then

$$\mathbf{LJ}\big(u(\delta, n)\big) = (-1)^n u\big(\mathbf{C}(\delta), n\big),$$
$$\mathbf{LJ}\big(\pi\big(u(\delta, k), \alpha\big)\big) = \pi\big(u(\mathbf{C}(\delta), k), \alpha\big)$$

for all $\alpha \in (0, 1/2)$.

(3) If $\delta \in \mathcal{D}_2^{\mathrm{u}}$ is such that $\mathbf{C}(\delta)$ is a one-dimensional representation χ'_1, then

- $\mathbf{LJ}(u(\delta, n)) = \pi(\chi'_{n/2}, 1/2)$ and $\mathbf{LJ}(\pi(u(\delta, n), \alpha)) = \pi(u(\chi'_{n/2}, 1/2), \alpha)$ if n is even and $\alpha \in (0, 1/2)$;
- $\mathbf{LJ}(u(\delta, n)) = (-1)^n \chi'_{(n+1)/2} \times \chi'_{(n-1)/2}$ and $\mathbf{LJ}(\pi(u(\delta, n)\alpha)) = \pi(\chi'_{(n+1)/2}, \alpha) \times \pi(\chi'_{(n-1)/2}, \alpha)$ if $n \neq 1$ is odd and $\alpha \in (0, 1/2)$;
- $\mathbf{LJ}(\delta) = \chi'_1$ and $|\mathbf{LJ}|(\pi(\delta, \alpha)) = \pi(\chi'_1, \alpha)$ for $\alpha \in (0, 1/2)$.

3.4. Transfer of unitary representations

An irreducible unitary representation π is written as a product of elements in $\mathcal{U}_F$. Note that **LJ** commutes with parabolic induction. If $\pi \in \mathrm{Irr}^{\mathrm{u}}_{kd}$, then $\mathbf{LJ}(\pi) = 0$ or $\mathbf{LJ}(\pi)$ is an irreducible unitary representation π' of $G_{k,D}$ up to a sign. We write $\pi' = |\mathbf{LJ}|(\pi)$. So we have a map $|\mathbf{LJ}|$ from the set of d-compatible irreducible unitary representations of G_{kd} to the set of unitary representations of $G_{k,D}$.

As an immediate consequence, we have the following result.

Theorem 3.6 ([4, 5]). *If π is a d-compatible irreducible unitary representation of G_{kd}, then there exists a unique irreducible unitary representation π' of $G_{k,D}$ and a unique sign $\varepsilon_\pi \in \{-1, 1\}$ such that*

$$\chi_\pi(g) = \varepsilon_\pi \chi_{\pi'}(g')$$

for all $g \in G_{nd,d}$ and $g \leftrightarrow g'$.

Remark 3.7. The signs ε_π and π' can be computed explicitly. We refer the reader to [4, Section 3.3] for more details. In this paper, we only need to calculate ε for a special class of representations.

4. Quaternionic Speh representations

The purpose of this section is to define a class of representation of $G_{k,D}$ with unique models. They are defined using the Jacquet–Langlands correspondence of the Speh representations.

4.1. The Speh representations

In this section we recall the construction of the Speh representations and discuss some of their properties.

We now consider the local situation. Let F be a local field. Let τ be an irreducible unitary generic representation of G_k. We attach a representation $\mathrm{Speh}(\tau, n)$ of G_{kn} inductively as follows.

If τ is a discrete series representation, then we define $\mathrm{Speh}(\tau, n) := u(\tau, n)$.

Let τ be an irreducible unitary generic representation of G_k. By the classification of unitary representations [33, 36], there exist discrete series representations $\tau_1, \dots, \tau_m$ and $s_1, \dots, s_d \in (-1/2, 1/2)$ such that

$$\tau = \tau_1 \nu^{s_1} \times \cdots \times \tau_m \nu^{s_m}.$$

The order of $1, \dots, m$ does not change the isomorphism class of τ. We define

$$\mathrm{Speh}(\tau, n) = \mathrm{Speh}(\tau_1, n)\nu^{s_1} \times \cdots \times \mathrm{Speh}(\tau_m, n)\nu^{s_m}.$$

By [27, Section I.11], $\mathrm{Speh}(\tau, n)$ is irreducible and thus well defined (i.e. the isomorphism class of $\mathrm{Speh}(\tau, n)$ does not depend on the order of $1, \dots, m$).

4.2. Digression on generalized Whittaker models

An important property of the Speh representations is that they are degenerate representations with unique models. We now briefly recall the theory of generalized Whittaker models and related topics. This was originally studied in [27]. In this paper we use [14] as the main reference. The reader is also referred to [18] for a more comprehensive discussion on the history of the subject.

Let G be a reductive group over F. Let $\mathfrak{g}$ denote the Lie algebra of G and $\mathfrak{g}^*$ denotes its dual space. From now on, we fix a nontrivial unitary additive character

$$\psi : F \to \mathbb{C}^1$$

such that if F is Archimedean we have

$$\psi(x) = \exp\big(2\pi i \Re(x)\big)$$

and if F is non-Archimedean, then $\mathrm{Ker}(\psi)$ is the ring of integers.

A *Whittaker pair* is an ordered pair $(S, \varphi) \in \mathfrak{g} \times \mathfrak{g}^*$ such that S is semisimple and the eigenvalues of the adjoint action $\mathrm{ad}(S)$ are rational numbers, and that $\mathrm{ad}^*(S)(\varphi) = -2\varphi$. Given such a Whittaker pair, we define a *degenerate Whittaker model* for G as follows (we refer the reader to [14, Section 2.5] for more details). Under the adjoint action $\mathrm{ad}(S)$, $\mathfrak{g}$ can be written as a direct sum of eigenspaces. Let $\mathfrak{u} \subset \mathfrak{g}$ be the sum of all eigenspaces of $\mathrm{ad}(S)$ with eigenvalues greater than or equal to 1. Then $\mathfrak{u}$ is a nilpotent subalgebra. Let $U := \mathrm{Exp}(\mathfrak{u}) \subset G$ be the corresponding unipotent subgroup. We now have the following cases:

- If $\varphi = 0$, then we set $\mathcal{W}_{S,0} := \mathrm{ind}_U^G(\mathbb{C})$;

- Assume now that $\varphi \neq 0$. Then let $\mathfrak{n}$ be the radical of the anti-symmetric form on $\mathfrak{u}$ given by
$$\omega_\varphi(X, Y) := \varphi\big([X, Y]\big)$$
and set $N := \mathrm{Exp}(\mathfrak{n})$. Set $\mathfrak{n}' := \mathfrak{n} \cap \mathrm{Ker}(\varphi)$ and let $N' := \mathrm{Exp}(\mathfrak{n}')$. Then N' is a normal subgroup of U and U/N' is a Heisenberg group whose center is N/N'. The element φ defines a character of N/N' given by
$$\psi_\varphi\big(\exp(X)\big) := \psi\big(\varphi(x)\big).$$
Let σ_φ denote the oscillator representation of U/N' with central character ψ_φ and consider it as a representation of U. We now define
$$\mathcal{W}_{S,\varphi} := \mathrm{ind}_U^G(\sigma_\varphi).$$

The element φ determines a unique nilpotent element f_φ via the Killing form pairing. If S is a neutral element for f_φ, then (S, φ) is called a *neutral pair* and $\mathcal{W}_{S,\varphi}$ is called a *neutral* model or a *generalized Whittaker model*. The generalized Whittaker model $\mathcal{W}_{S,\varphi}$ does not depend on the choice of a neutral S, and thus is always denoted $\mathcal{W}_\varphi$. Moreover, conjugate nilpotent elements give rise to isomorphic generalized Whittaker models, thus for a nilpotent coadjoint orbit $\mathcal{O}$, we also use the notation $\mathcal{W}_\mathcal{O}$.

Let $\mathcal{M}(G)$ denote the category of smooth admissible representations of G. For $\pi \in \mathcal{M}(G)$ and a nilpotent coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$, we set
$$\mathcal{W}_\mathcal{O}(\pi) := \mathrm{Hom}_G(\mathcal{W}_\mathcal{O}, \pi^*).$$
Here, π^* denotes the dual representation of π. Let $\mathrm{WO}(\pi)$ denote the set of all nilpotent coadjoint orbits $\mathcal{O}$ with $\mathcal{W}_\mathcal{O}(\pi) \neq 0$ and $\mathrm{WS}(\pi)$ denote the set of maximal orbits in $\mathrm{WO}(\pi)$ with respect to the closure ordering. We call $\mathrm{WS}(\pi)$ the *Whittaker support* of π.

The follow result explains the relation of $\mathcal{W}_\mathcal{O}(\pi)$ and the usual definition of models for representation.

Lemma 4.1 ([14, Lemma 2.5.2]). *Let $\mathfrak{l} \subset \mathfrak{u}$ be a maximal isotropic subalgebra and $L :=$* $\mathrm{Exp}(\mathfrak{l})$. *We use the same notation ψ_φ to denote its trivial extension of ψ_φ to L. Let $\pi \in$* $\mathcal{M}(G)$. *Then*
$$\mathcal{W}_\mathcal{O}(\pi) \cong \mathrm{Hom}_L(\pi, \psi_\varphi^{-1}).$$

This can also be done in the global situation using *generalized Whittaker coefficients*. We use notation $\mathcal{F}_\mathcal{O}(f)$ for an automorphic form f to denote such generalized Whittaker coefficients.

4.3. Explicit cases

In the main body of this paper, we only use the explicit definition of the generalized Whittaker models for $\mathrm{GL}_{kn,D}$ associated to the orbit $(k^n)_D$. We recall the construct it here.

Definition 4.2. The unipotent subgroup $N_{(k^n)_D}$ of $\mathrm{GL}_{kn,D}$ is defined as follows:

$$N_{(k^n)_D} = \left\{ u \mid u = \begin{pmatrix} I_n & X_{12} & \cdots & \cdots & * \\ & I_n & X_{23} & \cdots & * \\ & & \ddots & \cdots & \cdots \\ & & & I_n & X_{k-1,k} \\ & & & & I_n \end{pmatrix} \in \mathrm{GL}_{kn,D} \right\}.$$

In other words, it is the set of upper triangular unipotent $n \times n$ block matrices. We define a character $\psi_{(k^n)_D} : N_{(k^n)_D} \to \mathbb{C}^\times$ by

$$\psi_{(k^n)_D}(u) = \psi^{-1}\big(\mathrm{tr}(X_{12} + X_{23} + \cdots + X_{k-1,k})\big).$$

When $n = 1$, this gives the usual "Whittaker" coefficient. (Note that we add an extra -1 in the definition of $\psi_{(k^n)_D}$ just for ease of notations later.)

It is easy to see that

$$\mathcal{W}_{(k^n)_D} \cong \mathrm{ind}_{N_{(k^n)_D}}^{\mathrm{GL}_{kn,D}} \psi_{(k^n)_D}^{-1}, \quad \mathcal{W}_{(k^n)_D}(\pi) \cong \mathrm{Hom}_{N_{(k^n)_D}}(\pi, \psi_{(k^n)_D}).$$

If $\mathcal{W}_{(k^n)_D}(\pi) \neq 0$, we sometimes say that π supports a non-vanishing $(k, n)_D$-model.

Remark 4.3. Note that the above definition works when D is a central simple algebra as well. Sometimes it is convenient to use this notation so we discuss it here.

Let $D = \mathrm{M}_m(D')$ for a central division algebra D' over F. Then

$$\mathrm{GL}_{n,D} = \mathrm{GL}_{mn,D'} \quad \text{and} \quad (N_{(k^n)_D}, \psi_{(k^n)_D}) = (N_{(k^{mn})_{D'}}, \psi_{(k^{mn})_{D'}}).$$

4.4. The Whittaker support of the Speh representations

The Whittaker support of the Speh representations is already calculated.

Theorem 4.4 (see [11] and references there). (1) *We have*

$$\mathrm{WS}\big(\mathrm{Speh}(\tau, n)\big) = \big\{(k^n)\big\}.$$

(2) $\dim \mathcal{W}_{(k^n)}(\mathrm{Speh}(\tau, n)) = 1$. *In other words, the Speh representation supports unique models of degenerate type.*

Remark 4.5. We emphasize that the first statement consists of the following two sub-statements:

- For every nilpotent coadjoint orbit $\mathcal{O}$ larger than or not comparable with (k^n),

$$\mathcal{W}_{\mathcal{O}}\big(\mathrm{Speh}(\tau, n)\big) = 0;$$

- $\mathcal{W}_{(k^n)}(\mathrm{Speh}(\tau, n)) \neq 0$.

In some papers, it is said that the nilpotent orbit attached to $\mathrm{Speh}(\tau, n)$ is (k^n).

Remark 4.6. It is expected that $\mathrm{Speh}(\tau, n)$ is the "smallest" piece of the highly reducible induced representation

$$\nu^{(n-1)/2}\tau \times \nu^{(n-3)/2}\tau \times \cdots \times \nu^{(1-n)/2}\tau. \tag{4.1}$$

For example, when F is non-Archimedean and τ is a supercuspidal representation, using the theory of derivatives [37], one can show that (k^n) is the smallest orbit that appears in the Whittaker supports of subquotients of (4.1).

4.5. The construction

Speh representations over D are defined as the Jacquet–Langlands transfer of the Speh representations.

Definition 4.7. For $\tau \in \mathrm{Irr}^{\mathrm{eu}}_{\mathrm{gen}}(G_k)$ and a central division algebra D, we define

$$\mathrm{Speh}_D(\tau, n) = |\mathbf{LJ}|\big(\mathrm{Speh}_F(\tau, nd)\big),$$

where $|\mathbf{LJ}|$ is the Jacquet–Langlands correspondence from GL_{knd} to $\mathrm{GL}_{kn,D}$.

Remark 4.8. For a positive integer m, we also define

$$\mathrm{Speh}_{\mathrm{M}_m(D)}(\tau, n) = \mathrm{Speh}_D(\tau, mn).$$

This is used in the global construction.

Remark 4.9. This construction is local-to-global compatible since the Jacquet–Langlands correspondence in [4] is local-to-global compatible.

We observe that, from Proposition 3.4 and the definition of $|\mathbf{LJ}|$, $\mathrm{Speh}_F(\tau, nd)$ is always d-compatible. Explicit constructions of $\mathrm{Speh}_D(\tau, n)$ can also be written done using the Aubert involution.

We would like to understand the Whittaker support of $\mathrm{Speh}_D(\tau, n)$. In the non-Archimedean case, this can be done using the character identity and Theorem 4.11. In the Archimedean case, we also prove some partial results. We will prove further results using global methods in Section 6.5.

4.6. Sign in the character identity

From [4, Proposition 3.9], the representations $\mathrm{Speh}_D(\tau, n)$ and $\mathrm{Speh}_F(\tau, nd)$ satisfy a character identity

$$\chi_{\mathrm{Speh}_F(\tau,nd)}(g) = \varepsilon \chi_{\mathrm{Speh}_D(\tau,n)}(g')$$

with $\varepsilon \in \{\pm 1\}$ and $g \leftrightarrow g'$. We first show that $\varepsilon = 1$.

Proposition 4.10. *We have*

$$\chi_{\mathrm{Speh}_F(\tau,nd)}(g) = \chi_{\mathrm{Speh}_D(\tau,n)}(g')$$

for all $g \in \mathrm{GL}_{nkd}$ *and* $g \in \mathrm{GL}_{nk,D}$ *such that* $g \leftrightarrow g'$.

Proof. We first treat the non-Archimedean case. Recall that ε is the product of $(-1)^{kn(d-1)}$ and the sign coming from the definition of **LJ**.

It suffices to prove the result when τ is a discrete series representation. In this case, we assume that $\tau = Z^{\mathrm{u}}(\rho, l)$ for $\rho \in \mathcal{C}_p$ where $k = pl$. We only need to use Proposition 3.4 (2) to calculate $\mathbf{LJ}(\mathrm{Speh}(\tau, nd)) = \mathbf{LJ}(u(\tau, nd))$.

Let $\sigma = Z^{\mathrm{u}}(\rho, nd)$. Let s be the smallest integer such that $d \mid sp$. Let $\sigma' = \mathbf{C}(\sigma)$. Then

$$\mathbf{LJ}\big(u(\tau, nd)\big) = \varepsilon\big|i'\big(\bar{u}(\sigma', l)\big)\big|,$$

where $\varepsilon = 1$ if s is odd and $\varepsilon = (-1)^{\frac{ndl}{s}}$ if s is even.

We now calculate the sign in the character relation. We first consider the case when s is odd. We need to show that $2 \mid kn(d-1)$. We write

$$kn(d-1) = \frac{sp}{d}\frac{dnl}{s}(d-1).$$

Note that both sp/d and d/s are integers. We now have two cases:

- $d-1$ is even: then the result is true.
- d is even: since s is odd and d is even, dnl/s must be an even integer. The result is true as well.

We now consider the case when s is even. We need to show that $2 \mid \frac{dnl}{s} - kn(d-1)$. Note that d must be even. Therefore, it suffices to show that

$$2 \mid \frac{dn}{s} - pn \quad \text{or} \quad 2 \mid \frac{(d-sp)n}{s}.$$

We now write $sp = da$. We claim that a must be odd. Otherwise, let $a = 2a'$. Then $d \mid sp/2$. This contradicts with our choice of s. Therefore, a is an odd integer and

$$\frac{(d-sp)n}{s} = \frac{d}{s}n(1-a)$$

is an even integer. This completes the proof for the non-Archimedean case.

The Archimedean case is easier. It suffices to prove the result for the discrete series representations. Recall that for $\tau \in \mathcal{D}_k$,

$$\chi_{\mathrm{Speh}(\tau,2n)}(g) = (-1)^{2kn-kn}\chi_{\mathbf{LJ}(\mathrm{Speh}(\tau,2n))}(g') = (-1)^{kn}\chi_{\mathbf{LJ}(\mathrm{Speh}(\tau,2n))}(g').$$

By Section 3.3, we have the following:

- $k = 1$: we have that $\mathbf{LJ}(\mathrm{Speh}(\tau, 2n)) = (-1)^n|\mathbf{LJ}|(\mathrm{Speh}(\tau, 2n))$.
- $k = 2$: we have that $\mathbf{LJ}(\mathrm{Speh}(\tau, 2n)) = |\mathbf{LJ}|(\mathrm{Speh}(\tau, 2n))$.

This proves that if τ is a discrete series representation of either $\mathrm{GL}_1(\mathbb{R})$ or $\mathrm{GL}_2(\mathbb{R})$, then

$$\chi_{\mathrm{Speh}(\tau,2n)}(g) = \chi_{\mathrm{Speh}_{\mathbb{H}}(\tau,n)}(g')$$

for $g \leftrightarrow g'$. ∎

4.7. Whittaker supports in non-Archimedean case

In this section, we determine $\mathrm{WS}(\mathrm{Speh}_D(\tau, n))$ and the dimension of the corresponding model in the non-Archimedean case. This is an easy consequence of the main result of [29] and Proposition 4.10. The main idea is to use the character identity and a result of Mœglin–Waldspurger [26] and Varma [35].

To proceed, we introduce another nilpotent coadjoint orbit associated to a representation. For $\pi \in \mathcal{M}(G)$, its character χ_π admits the following character expansion at identity

$$\chi_\pi\big(\exp(Y)\big) = \sum_{\mathcal{O}} c_{\mathcal{O}} \widehat{\mu}_{\mathcal{O}}(Y)$$

valid for all regular semisimple Y in the lie algebra $\mathfrak{g}$ such that Y is close enough to 0 [6, 19]. Here, the sum is over the set of nilpotent coadjoint orbits $\mathcal{O}$ in $\mathfrak{g}^*$; $\widehat{\mu}_{\mathcal{O}}$ is the function that represents the distribution that is the Fourier transform of the orbital integral $\mu_{\mathcal{O}}$ associated to $\mathcal{O}$; $c_{\mathcal{O}} = c_{\mathcal{O}}(\pi) \in \mathbb{C}$; and exp is the exponential map, or some suitable substitute. Denote by $\mathrm{WF}(\pi)$ the set of maximal elements in the set of orbits with nonzero coefficients.

Theorem 4.11 ([26, Proposition I.11, Theorem I.16, and Corollary I.17], [35]). *Assume that F is non-Archimedean and G is algebraic. Let $\pi \in \mathcal{M}(G)$. Then*

(1) $\mathrm{WF}(\pi) = \mathrm{WS}(\pi)$.

(2) *For any $\mathcal{O} \in \mathrm{WF}(\pi)$, $c_{\mathcal{O}}(\pi) = \dim \mathcal{W}_{\mathcal{O}}(\pi)$.*

In the Archimedean case, only partial results are known (see [14, Section 3.3]).

Proposition 4.12. *Suppose that $\pi \in \mathrm{Irr}_{kd}$ and $\pi' \in \mathrm{Irr}_{k,D}$. We consider that the character expansion at the identity:*

$$\chi_\pi = \sum_{\mathcal{O}} c_{\mathcal{O}} \widehat{\mu}_{\mathcal{O}} \quad \textit{and} \quad \chi_{\pi'} = \sum_{\mathcal{O}'} c_{\mathcal{O}'} \widehat{\mu}_{\mathcal{O}'}.$$

If the characters satisfy the following character identity:

$$\chi_\pi = \varepsilon_\pi \chi_{\pi'},$$

then for $\mathcal{O} \leftrightarrow \mathcal{O}'$,

$$c_{\mathcal{O}} = \varepsilon_\pi \cdot c_{\mathcal{O}'}.$$

Proof. The proof in [29] only uses the character identity and therefore applies to our case as well. ∎

We now prove the following result.

Theorem 4.13. *We have*

$$\mathrm{WS}\left(\mathrm{Speh}_D(\tau, n)\right) = \left\{(k^n)_D\right\}.$$

Moreover,

$$\dim \mathrm{Hom}_{N_{(k^n)_D}}\left(\mathrm{Speh}_D(\tau, n), \psi_{(k^n)_D}\right) = 1.$$

Proof. This is an application of Proposition 4.12 and Theorem 4.11. We first prove the vanishing part. Assume that there is an orbit $\mathcal{O}'$ that is greater than or not comparable with $(k^n)_D$ such that $\mathcal{O}' \in \mathrm{WS}(\mathrm{Speh}_D(\tau, n))$. By Theorem 4.11, $c_{\mathcal{O}'} \neq 0$. By Proposition 4.12, $c_{\mathcal{O}} \neq 0$ for $\mathcal{O} \leftrightarrow \mathcal{O}'$. Note that $\mathcal{O}$ is either greater than or not comparable with (k^{nd}). Theorem 4.11 implies that there exists $\widetilde{\mathcal{O}} \in \mathrm{WS}(\mathrm{Speh}(\tau, nd))$ such that $\widetilde{\mathcal{O}}$ is either greater than or not comparable with (k^{nd}). This contradicts with the fact that $\mathrm{WS}(\mathrm{Speh}(\tau, nd)) = \{(k^{nd})\}$.

We have proved that any element in $\mathrm{WS}(\mathrm{Speh}_D(\tau, n))$ is contained in the closure of $(k^n)_D$. Now from Theorems 4.4 and 4.11, we deduce that $c_{(k^{nd})} = 1$. This shows that $(k^n)_D \in \mathrm{WS}(\mathrm{Speh}_D(\tau, n))$ and therefore $\mathrm{WS}(\mathrm{Speh}_D(\tau, n)) = \{(k^n)_D\}$.

By Proposition 4.12, we deduce that $c_{(k^n)_D} = 1$ since $(k^{nd}) \leftrightarrow (k^n)_D$. The result now follows by applying Theorem 4.11 again. ■

Remark 4.14. The following example explains why the usual Jacquet–Langlands correspondence $\mathbf{C}$ does not work properly. Assume D is the unique non-split quaternion algebra over a local field F. Let St be the Steinberg representation. Then $\mathbf{C}(\mathrm{St}) = 1_{D^\times}$. The character relation reads

$$\chi_{\mathrm{St}}(g) = -\chi_{1_{D^\times}}(g')$$

for $g \leftrightarrow g'$. The character expansion of χ_{St} is

$$\chi_{\mathrm{St}} = \widehat{\mu}_{(2)} - \widehat{\mu}_{(1^2)}.$$

Since the Steinberg representation has a unique Whittaker model, $c_{(2)} = 1$. But this is not related to the fact that $1_{D^\times}$ being one-dimensional.

The Steinberg representation is not in the domain of $|\mathbf{LJ}|$ so this representation is irrelevant to us. Under $|\mathbf{LJ}|$, the trivial representation of $\mathrm{GL}_2(F)$ corresponds to the trivial representation of $D^\times$. In this case, the fact of 1_{GL_2} being one-dimensional is related to the nilpotent orbit of $1_{D^\times}$.

4.8. Partial results in the Archimedean case

The proof in the previous section does not work well in the Archimedean case. This is because the Archimedean version of Theorem 4.11 is unknown (see the discussion of [18]). However, one direction is shown in [17].

We first recall some basics. Let G be a reductive Lie group with maximal compact subgroup K, and let $G_{\mathbb{C}}$ and $K_{\mathbb{C}}$ be the complexification of G and K. Denote $\mathfrak{g}, \mathfrak{k}, \mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{k}_{\mathbb{C}}$ be the Lie algebra of $G, K, G_{\mathbb{C}}$ and $K_{\mathbb{C}}$, respectively. Let π be an irreducible admissible representation π of G. We briefly recall the two invariants of cycles associated with π, one defined analytically and the other algebraically.

For π, one has an asymptotic expansion for the character χ_π in a neighborhood of 0 in $\mathfrak{g}$ of the form

$$\chi_\pi \sim \sum_{i=-r}^{\infty} D_i$$

with $\{D_i\}$ being a set of tempered distributions on $\mathfrak{g}$. The asymptotic support $\mathrm{AS}(\chi_\pi) \subset \mathfrak{g}^*$ is defined to be the union of the supports of the Fourier transforms $\widehat{D}_i$. It is known that $\mathrm{AS}(\chi_\pi)$ is a union of nilpotent orbits. We can view $\mathrm{AS}(\chi_\pi)$ as a subset of $\mathfrak{g}$ by identifying $\mathfrak{g}$ and $\mathfrak{g}^*$ by the Cartan–Killing form. We set

$$\mathcal{N}_{\mathrm{tr}}(\pi) := \{\mathcal{O} \in \mathcal{N} \mid \mathcal{O} \subset \mathrm{AS}(\chi_\pi)\},$$

and set $\mathcal{N}_{\mathrm{tr}}^{\max}(\pi)$ to be the subset of maximal elements.

For a smooth representation π of a real reductive group G, one can define another invariant $\mathrm{AV}(\pi)$ – the annihilator variety of π. It is sometimes called the associated variety of the annihilator of π. It is defined to be the set of zeros in $\mathfrak{g}_{\mathbb{C}}^*$ of the ideal in the symmetric algebra $S(\mathfrak{g}_{\mathbb{C}})$, which is generated by the symbols of the annihilator ideal of π in the universal enveloping algebra. A result of Kostant–Rallis [23] says that $\mathrm{AV}(\pi)$ is a finite union of nilpotent orbit $(\mathfrak{g}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}})^*$. Identifying coadjoint orbits with adjoint orbits, and using the Sekiguchi correspondence to identify the nilpotent $K_{\mathbb{C}}$-orbits in $(\mathfrak{g}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}})^*$ with the nilpotent G-orbits in $\mathfrak{g}^*$, we define the set

$$\mathcal{N}_{\mathrm{alg}}(\pi) := \{\mathcal{O} \in \mathcal{N} \mid \mathcal{O} \subset \mathrm{AV}(\pi)\},$$

and let $\mathcal{N}_{\mathrm{alg}}^{\max}(\pi)$ be the subset of maximal elements. It follows from [31] that

$$\mathcal{N}_{\mathrm{tr}}^{\max}(\pi) = \mathcal{N}_{\mathrm{alg}}^{\max}(\pi).$$

Proposition 4.15. *For $\mathcal{O}' \in \mathrm{WS}(\mathrm{Speh}_D(\tau, n))$, $\mathcal{O}'$ is a contained in the closure of $(k^n)_D$.*

Proof. Let us prove this result by contradiction. If $\mathcal{O}'$ is not contained in the closure of $(k^n)_D$, then $\mathcal{O}'$ is either greater than or not comparable with $(k^n)_D$. In either case, $\mathcal{O}'$ is of the form $(k_1 \cdots)$ with $k_1 > k$. Recall that the definition of $\mathrm{WS}(\mathrm{Speh}_D(\tau, n))$ guarantees that $\mathcal{O}'$ is a maximal nilpotent orbit that support generalized Whittaker models for $\mathrm{Speh}_D(\tau, n)$.

Corollary 4 of [25] states if $\mathcal{W}_{\mathcal{O}'}(\pi) \neq 0$, then $\mathcal{O}' \subset \mathrm{AV}(\pi)$. Without loss of generality, we may assume that $\mathcal{O}' \in \mathcal{N}_{\mathrm{alg}}^{\max}(\pi)$. Then $\mathcal{O}' \in \mathcal{N}_{\mathrm{tr}}^{\max}(\pi)$ as well and therefore $c_{\mathcal{O}'} \neq 0$.

The Archimedean analogue of Prasad also holds. As a consequence, from the character identity, we deduce that $c_{\mathcal{O}} \neq 0$ when $\mathcal{O} \leftrightarrow \mathcal{O}'$. Here, $c_{\mathcal{O}}$ are the coefficients appearing in the character expansion for the representation $\mathrm{Speh}(\tau, 2n)$. Now [17, Theorem B] or [14, Section 3.3] implies that

$$\mathcal{O} \in \mathrm{WO}\big(\mathrm{Speh}(\tau, 2n)\big),$$

contradicting Theorem 4.4. This completes the proof. ∎

Remark 4.16. The relation between the leading coefficient and the dimension of generalized Whittaker models is not known yet. Moreover, in order to prove the non-vanishing part, we need an extension of results in [17] to $\mathrm{GL}_{k,D}$.

In Section 6.5, we use global arguments to prove some partial results towards the non-vanishing and multiplicity one.

4.9. A special instance

Regarding the multiplicity one result, we now give a proof in the minimal case using the theta correspondence. The following proof follows from a suggestion by Hang Xue.

In this section only, let D be the unique non-split quaternion algebra over a local field F.

Proposition 4.17. *For an irreducible admissible representation of $D^\times$,*

$$\dim \mathrm{Hom}_{N_{(2)_D}}\left(\mathrm{Speh}_D(\tau,1), \psi_{(2)_D}\right) = 1.$$

Proof. We use the theta correspondence for the similitude pair

$$\big(\mathrm{GSp}(2), \mathrm{GSO}(5,1)\big).$$

Observe that we have the following isomorphisms:

$$\mathrm{GL}(2) = \mathrm{GSp}(2),$$
$$\mathrm{GSO}(5,1) = (\mathrm{GL}_{2,D} \times \mathrm{GL}_1)/\big\{(z\cdot \mathrm{Id}, z^{-2}) \mid z \in \mathrm{GL}(1)\big\}.$$

Via these isomorphisms, an irreducible representation of $\mathrm{GSO}(5,1)$ is of the form $\pi \boxtimes \mu$ where π is a representation of $\mathrm{GL}_{2,D}$ and μ is a square root of the central character of π.

The theta correspondence from GL_2 to $\mathrm{GSO}(5,1)$ for discrete series representations is given as follows:

$$\tau \mapsto \Theta(\tau) = Lg\big(\nu^{1/2}\mathbf{C}(\tau) \times \nu^{-1/2}\mathbf{C}(\tau)\big) \boxtimes \omega_\tau.$$

Here, Lg denotes the Langlands quotient of the induced representation considered and ω_τ is the central character of τ. Note that the above induced representation is reducible if and only if $\dim \mathbf{C}(\tau) > 1$. We have

$$\Theta(\tau) = \mathrm{Speh}_D(\tau,1) \boxtimes \omega_\tau.$$

To proceed, we use the result of Gomez–Zhu [16] to relate generalized Whittaker models of τ and $\Theta(\tau)$. The result of Gomez–Zhu says that there is an isomorphism

$$\mathcal{W}_{\mathcal{O}}(\tau) \simeq \mathcal{W}_{\mathcal{O}'}\big(\Theta(\tau)\big)$$

for two nilpotent orbits $\mathcal{O}$ and $\mathcal{O}'$ that correspond to each other under the moment map. Both GL_2 to $\mathrm{GSO}(5,1)$ have two nilpotent orbits: the trivial one and the nontrivial one. Under the moment map, the nontrivial ones correspond to each other. As a result, we have an isomorphism between the Whittaker model of τ and $\mathcal{W}_{(2)_D}(\mathrm{Speh}(\tau,1))$. Now our result follows from the uniqueness of Whittaker models for $\mathrm{GL}(2)$. ■

Remark 4.18. Prasad also pointed to us that Proposition 4.9 can be considered as a special instance of the Gross–Prasad conjecture for orthogonal groups. When $\dim \mathbb{C}(\tau) = 1$, it can be proved directly; when $\dim \mathbb{C}(\tau) = 1$, the argument can be found in [30, Consequence 4 of Conjecture 1].

5. The global Speh representations

From now on, let F be a global field and $\mathbb{A}$ be its ring of adeles. Let ψ be a nontrivial additive character of $F\backslash\mathbb{A}$. Let D be a central division algebra over F of dimension d^2. We work in this setup unless otherwise specified.

In this section, we define the global Speh representations. Denote DS_k (resp. $\mathrm{DS}_{k,D}$) the set of discrete series of $G_k(\mathbb{A})$ (resp. $G_{k,D}(\mathbb{A})$).

5.1. The residual spectrum of G_k

We now recall the construction of the Speh representations in the global setup. A theorem of Mœglin–Waldspurger [27] says that these consist of the residual spectrum of G_k.

Let m be a positive integer and $\tau \in \mathrm{DS}_m$ be a cuspidal representation. For a positive integer n, the induced representation $\prod_{i=0}^{n-1}(\nu^{\frac{n-1}{2}-i}\tau)$ has a unique constituent π which is a discrete series (i.e. $\pi \in \mathrm{DS}_{mn}$). One has $\pi_v = \mathrm{Speh}(\tau_v, n)$. As a result, we write $\pi = \mathrm{Speh}(\tau, n)$.

Discrete series of the groups $G_k(\mathbb{A})$ are all of this type, and n and ρ are determined by π. Moreover, π is cuspidal if and only if $n = 1$. (By [4, Section 5.2] and [5, Section 18], the same classification also holds for $G_{k,D}(\mathbb{A})$.)

The Speh representations admit an automorphic realization. We now recall the construction. Let τ be an irreducible cuspidal representation of $\mathrm{GL}_m(\mathbb{A})$. Let $\underline{s} = (s_1, \dots, s_n) \in \mathbb{C}^n$. We consider the following normalized induced representation

$$\mathrm{Ind}_{P_{m,n}(\mathbb{A})}^{\mathrm{GL}_{mn}(\mathbb{A})} \tau\nu^{s_1} \otimes \cdots \otimes \tau\nu^{s_n}.$$

Here, $P_{m,n}$ is the standard parabolic subgroup of GL_{mn} whose Levi part is $\mathrm{GL}_m \times \cdots \times \mathrm{GL}_m$ where GL_m appears n times. Let $f^{(\underline{s})}$ be a holomorphic section in the induced representation. Then we can form an Eisenstein series

$$E\big(f^{(\underline{s})}\big)(g) = \sum_{\gamma \in P_{m,n}(F)\backslash \mathrm{GL}_{mn}(F)} f^{(\underline{s})}(\gamma g).$$

By a result of Jacquet, the poles of this Eisenstein series occur at

$$s_1 - s_2 = s_2 - s_3 = \cdots = s_{n-1} - s_n = 1, \quad s_1 + \cdots + s_n = 0.$$

The residues of $E(f^{(\underline{s})})$ at this point give an automorphic realization of $\mathrm{Speh}(\tau, n)$.

As in the local case, given a nilpotent orbit, one can define generalized Whittaker coefficients $\mathcal{F}_{\mathcal{O}}$ of an automorphic representation π. We define $\mathrm{WS}(\pi)$ to the set of maximal nilpotent orbits that support nonzero generalized Whittaker coefficients for π. The set $\mathrm{WS}(\mathrm{Speh}(\tau, n))$ is determined by the following result.

Theorem 5.1 ([13, 21]). *We have* $\mathrm{WS}(\mathrm{Speh}(\tau, n)) = \{(k^n)\}$.

Sometimes, we also say that the nilpotent orbit attached to $\mathrm{Speh}(\tau, n)$ is (k^n).

5.2. Global Jacquet–Langlands correspondence

Recall that D is a central division algebra over F of dimension d^2. Let $m \in \mathbb{N}^*$. Set $A = \mathrm{M}_m(D)$. For each place v of F, let F_v be the completion of F at v and set $A_v = A \otimes F_v$. For each place v of F, $A_v \simeq \mathrm{M}_{r_v}(D_v)$ for some positive integer r_v and some central division algebra D_v of dimension d_v^2 over F_v such that $r_v d_v = md$. We will fix once and for all an isomorphism and identify these two algebras. We say that $\mathrm{M}_m(D)$ is split at a place v if $d_v = 1$. The set V of places where $\mathrm{M}_m(D)$ is not split is a finite. For each v, d_v divides d, and moreover d is the least common multiple of the d_v over all the places v.

Let $G_{m,D}(F)$ be the group $A^\times = \mathrm{GL}_m(D)$. For every place $v \in V$, set $(G_{m,D})_v = A'_v = \mathrm{GL}_{r_v}(D_v)$. If $v \notin V$, we have identified the group $\mathrm{GL}_{r_v}(D_v)$ and $\mathrm{GL}_{md}(F_v)$. For every finite place v of F, we set $K_v = \mathrm{GL}_{r_v}(O_v)$, where O_v is the ring of integers of D_v. We fix then a Haar measure dg_v on G'_v such that $\mathrm{vol}(K_v) = 1$. For every infinite place v, we fix an arbitrary Haar measure dg_v on G'_v.

We consider the Haar measure dg on $G'(\mathbb{A})$ which is the restricted product of the measure dg_v. We consider $G'(F)$ as a subgroup of $G'(\mathbb{A})$ via the diagonal embedding.

If π is a discrete series of $G_{kd}(\mathbb{A})$ or $G_{k,D}(\mathbb{A})$, and v is a place of F, we denote π_v the local component of π at the place v. If π is a discrete series of $G_{kd}(\mathbb{A})$, we say that π is D-compatible if for all v, π_v is d_v-compatible. Then $\mathbf{LJ}_v(\pi_v) \neq 0$ and $|\mathbf{LJ}|_v(\pi_v)$ is an irreducible representation of G'_v.

We now recall the global theorem. In the local setup, we have a map $|\mathbf{LJ}| : \pi \mapsto \pi'$ from the set of irreducible unitary d-compatible representations of G_{kd} to the set of irreducible unitary representations of $G_{k,D}$.

Theorem 5.2 ([4, Theorem 5.1] and [5, Theorem 18.1]). *There exists a unique map* $\mathbf{G}$ *from the set of discrete series of* $G_{k,D}(\mathbb{A})$ *into the set of discrete series of* $G_{kd}(\mathbb{A})$ *such that* $\mathbf{G}(\pi') = \pi$ *implies* $|\mathbf{LJ}|_v(\pi_v) = \pi'_v$ *for all places* $v \in V$ *and* $\pi_v = \pi'_v$ *for all* $v \notin V$. *The map* $\mathbf{G}$ *is injective and onto the set of* D*-compatible discrete series of* $G_{kd}(\mathbb{A})$.

We also have the multiplicity one and strong multiplicity one theorems for $G_{k,D}(\mathbb{A})$.

Theorem 5.3 ([5, Theorem 18.1]). *The multiplicity of every discrete series of* $G_{k,D}(\mathbb{A})$ *in the discrete spectrum is one. If two discrete series of* $G_{k,D}(\mathbb{A})$ *have isomorphic local components at almost every place, then they are equal.*

Proposition 5.4 ([5, Proposition 18.2]). *Let* $\tau \in \mathrm{DS}_k$ *be a cuspidal representation. Let* $s_{\tau,D}$ *be the least common multiple of* s_{τ_v,d_v}, $v \in V$ *(see Proposition 3.4).*

(1) $\mathrm{Speh}(\tau, n)$ *is* D*-compatible if and only if* $s_{\tau,D} \mid n$. *Moreover,* $s_{\tau,D} \mid d$.

(2) $\mathbf{G}^{-1}(\mathrm{Speh}(\tau, s_{\tau,D})) = \tau' \in \mathrm{DS}_{ms_{\tau,D}/d,D}$ *is cuspidal. In particular,* $\mathbf{G}^{-1}$ *sends cuspidal representations to cuspidal representations.*

(3) *Let* τ' *be a cuspidal representation of some* $G_{k,D}(\mathbb{A})$. *Write* $\mathbf{G}(\tau') = \mathrm{Speh}(\tau, s_{\tau,D})$ *and set* $\nu_{\tau'} = \nu^{s_{\tau,D}}$. *For every positive integer* m, *the induced representation*

$$\nu_{\tau'}^{(m-1)/2}\tau' \times \nu_{\tau'}^{(m-3)/2}\tau' \times \cdots \times \nu_{\tau'}^{(1-m)/2}\tau' \qquad (5.1)$$

has a unique irreducible quotient which we denote by $\mathrm{Speh}'(\tau', m)$. *It is a discrete series, and all discrete series are obtained for some cuspidal* τ' *in this way. If* $\mathbf{G}(\tau') = \mathrm{Speh}(\tau, s_{\tau,D})$, *then* $\mathbf{G}(\mathrm{Speh}'(\tau', m)) = \mathrm{Speh}(\tau, m s_{\tau,D})$.

As a special instance of this proposition, the Speh representation $\mathrm{Speh}(\tau, nd)$ is D-compatible for all central division algebra D over F. (This is because $\mathrm{Speh}(\tau_v, nd)$ is already d-compatible and therefore d_v-compatible.)

We would like to note that, as in the construction of $\mathrm{Speh}(\tau, n)$, the representation $\mathrm{Speh}'(\tau', n)$ can also be constructed using residues of Eisenstein series attached to (5.1) [4, Lemma A.5]. We omit the details here.

5.3. The global quaternionic Speh representations

We can now define the Speh representations for $G_{kn,D}(\mathbb{A})$.

Definition 5.5. For an irreducible cuspidal representation τ of $\mathrm{GL}_k(\mathbb{A})$ and a positive integer n, we define

$$\mathrm{Speh}_D(\tau, n) = \mathbf{G}^{-1}\big(\mathrm{Speh}(\tau, nd)\big).$$

Remark 5.6. Sometimes we understand the construction in the following way. Consider the following family of representations:

$$\big\{\mathbf{G}^{-1}\big(\mathrm{Speh}(\tau, m)\big) \mid m = 1, 2, \dots\big\}.$$

Then we have the following:

(1) the first occurrence (the index for the first nonzero member) is $s_{\tau,D}$;

(2) the first occurrence gives a cuspidal representation;

(3) $\mathbf{G}^{-1}(\mathrm{Speh}(\tau, m)) \neq 0$ if and only if $s_{\tau,D} \mid m$. These representations can be constructed from $\mathbf{G}^{-1}(\mathrm{Speh}(\tau, s_{\tau,D}))$ using residues of Eisenstein series.

In some sense, this is similar to the theta correspondence.

Remark 5.7. We have three notations related to the Speh representations. So let us summarize them here:

- $\mathrm{Speh}(\tau, n) \in \mathrm{DS}_{kn}$ where $\tau \in \mathrm{DS}_k$: this can be constructed using residues of Eisenstein series;
- $\mathrm{Speh}'(\tau', n) \in \mathrm{DS}_{kn,D}$ where $\tau' \in \mathrm{DS}_{k,D}$: this can also be constructed using residues of Eisenstein series;
- $\mathrm{Speh}_D(\tau, n) \in \mathrm{DS}_{kn,D}$ where $\tau \in \mathrm{DS}_k$: this is defined using the Jacquet–Langlands correspondence.

Observe that $\mathrm{Speh}_D(\tau, n)$ is cuspidal only if $n = 1$. The representation $\mathrm{Speh}_D(\tau, 1)$ is cuspidal if and only if $s_{\tau,D} = d$. It is easy to construct examples such that $\mathrm{Speh}_D(\tau, 1)$ is not cuspidal.

For a fixed τ, it is easy to find D such that $\mathrm{Speh}_D(\tau,1)$ is cuspidal. In fact, if there exists a place w such that τ_w is unramified and D does not split at w, then $s_{\tau,D}=d$ and therefore $\mathrm{Speh}_D(\tau,1)$ is cuspidal. (As a result, $\mathrm{Speh}_D(\tau,1)$ is cuspidal for almost all D.)

5.4. Fourier coefficients

In this section, we consider generalized Whittaker coefficients for $\mathrm{Speh}_D(\tau,n)$.

Theorem 5.8. *For any nilpotent orbit $\mathcal{O}$ greater than or not comparable with $(k^n)_D$,*

$$\mathcal{F}_{\mathcal{O}}(\phi)=0 \quad \textit{for all } \phi\in\mathrm{Speh}_D(\tau,n).$$

Proof. This is a consequence of the local vanishing result. ∎

We now state the global non-vanishing result. Here we consider the generalized Whittaker coefficient

$$\mathcal{F}_{(k^n)_D}(\phi):=\int_{N_{(k^n)_D}(F)\backslash N_{(k^n)_D}(\mathbb{A})}\phi(u)\overline{\psi}_{(k^n)_D}(u)\,du.$$

Theorem 5.9. *There exists $\phi\in\mathrm{Speh}_D(\tau,n)$ such that*

$$\mathcal{F}_{(k^n)_D}(\phi)\neq 0.$$

We will give a proof in the next section. Now we can say that the nilpotent orbit $(k^n)_D$ is the maximal nilpotent orbit which supports nonzero Fourier coefficients for $\mathrm{Speh}_D(\tau,n)$.

6. A global non-vanishing result

The purpose of this section is to prove Theorem 5.9. The strategy here is to adapt [22, proof of Theorem II.2.5] to our setting. In the global setting, for an algebraic group G, we write $[G]$ for $G(F)\backslash G(\mathbb{A})$ for ease of notations. We first make some preparations.

6.1. Some observations

We start with some simple observations. In this section, let τ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_k(\mathbb{A})$ and let D be a central division algebra over F. By Theorem 5.4, if $s_{\tau,D}=d$, then $\mathrm{Speh}_D(\tau,1)$ is a cuspidal representation.

Definition 6.1. For an irreducible automorphic representation π of $\mathrm{GL}_{k,D}(\mathbb{A})$, we write $\psi_{k,D}=\psi_{(k)_D}$ and define the $\psi_{k,D}$-Whittaker function of $\phi\in\pi$ as follows:

$$W_\phi(g)=\int_{[N_{k,D}]}\phi(ug)\overline{\psi}_{k,D}(u)\,du.$$

It is straightforward to see that

$$W_\phi(g) = \mathcal{F}_{(k)_D}\big(\pi(g)\phi\big).$$

We say that π is D-generic if

$$W_\phi(g) \neq 0$$

for some $\phi \in \pi$.

Lemma 6.2. *If π is a cuspidal representation of $\mathrm{GL}_{k,D}(\mathbb{A})$, then it is D-generic.*

Proof. As in the proof of the Fourier expansion when $D = F$ [28, 32], we can similarly prove the following Fourier expansion for φ in the cuspidal representation π:

$$\phi(g) = \sum_{\gamma \in N_{k-1,D}(F)\backslash \mathrm{GL}_{k-1,D}(F)} W_\phi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

Since $\phi \neq 0$, there exists γ and g such that $W_\phi\big(\big(\begin{smallmatrix} \gamma & 0 \\ 0 & 1 \end{smallmatrix}\big)g\big) \neq 0$. This proves the result. ∎

We now move to the second observation. Let us fix a local place v_0 of F and let D_{v_0} be a central division algebra over F_{v_0}. Let τ_{v_0} be the local component of τ at v_0.

Lemma 6.3. *For any n, the local Speh representation $\mathrm{Speh}_{D_{v_0}}(\tau_{v_0}, n)$ admits a non-vanishing $(k, n)_{D_{v_0}}$-model.*

Proof. This is already known when v_0 is non-Archimedean (see Theorem 4.13). The proof here works for all places and uses global properties.

We can choose a central division algebra $\mathbb{D}$ over F such that its localization at v_0 is $\mathrm{M}_n(D_{v_0})$ and it is nonsplit for an unramified place (this holds for almost all $\mathbb{D}$). Then $s_{\tau,\mathbb{D}} = d$ and therefore, $\mathrm{Speh}_{\mathbb{D}}(\tau, 1)$ is cuspidal. Moreover, the local component of $\mathrm{Speh}_{\mathbb{D}}(\tau, 1)$ at v_0 is $\mathrm{Speh}_{D_{v_0}}(\tau_{v_0}, n)$.

By Lemma 6.2, $\mathrm{Speh}_{\mathbb{D}}(\tau, 1)$ is $\mathbb{D}$-generic since it is cuspidal. This implies that all the local components support the local functional. In particular, $\mathrm{Speh}_{D_{v_0}}(\tau_{v_0}, n)$ admits a non-vanishing $(k, n)_{D_{v_0}}$-model. ∎

6.2. Kirillov models

We need the following two results related to the Kirillov models. We only states the results here. The proofs in the non-Archimedean case will be given in Appendix A. The Archimedean case is listed as a working hypothesis and the proofs will be considered in a forthcoming article.

Let F be a local field. Let D be a simple division algebra over F. (This gives some generality and includes degenerate cases.)

Let $P_{k,D}$ be the "mirabolic" subgroup of $\mathrm{GL}_{k,D}$ defined by

$$P_{k,D} := \big\{g \in \mathrm{GL}_{k,D} \mid (0, \ldots, 0, 1)g = (0, \ldots, 0, 1)\big\}.$$

Let $U_{k,D}$ be the subgroup

$$U_{k,D} := \left\{ u \mid u = \begin{pmatrix} I_{k-1} & x \\ 0 & 1 \end{pmatrix} \in P_{k,D} \right\}.$$

The restriction of $\psi_{k,D}$ to $U_{k,D}$ is still denoted as $\psi_{k,D}$.

Let $(D^\times)^\Delta$ be the image of the diagonal embedding $D^\times \to \mathrm{GL}_{k,D}$. Let $\widetilde{P}_{k,D} = P_{k,D} \cdot (D^\times)^\Delta$. This is the standard parabolic subgroup of $\mathrm{GL}_{k,D}$ of type $(k-1, 1)$.

Remark 6.4. When $D = F$ and $k = 1$, $P_{k,D}$ the usual mirabolic subgroup.

We first treat the non-Archimedean case.

Proposition 6.5. *For $\pi \in \mathrm{Irr}(\mathrm{GL}_{k,D})$, there is an embedding of representations of $P_{k,D}$*

$$\mathcal{J} : \mathrm{ind}_{N_{k,D}}^{P_{k,D}} \left(J_{N_{k,D},\psi_{k,D}}(\pi) \ltimes \psi_{k,D} \right) \hookrightarrow \pi \tag{6.1}$$

such that for any $\lambda \in \mathcal{W}_{(k)_D}(\pi)$ and $f \in \mathrm{ind}_{N_{k,D}}^{P_{k,D}}(J_{N_{k,D},\psi_{k,D}}(\pi) \ltimes \psi_{k,D})$,

$$\langle \lambda, \mathcal{J}(f) \rangle = \langle \mathcal{J}^*(\lambda), f(1) \rangle.$$

Here $\mathcal{J}^*(\lambda) : J_{N_{k,D},\psi_{k,D}}(\pi) \to \mathbb{C}$ is the map obtained from $\lambda : \pi \to \mathbb{C}$. Moreover, the embedding (6.1) is also $(D^\times)^\Delta$-equivariant and hence $\widetilde{P}_{k,D}$-equivariant.

We need the fact that $\mathrm{ind}_{P_{k-1,D}U_{k,D}}^{P_{k,D}}(\Phi^-(\pi) \ltimes \psi_{k,D})$ is "cuspidal". For the partition (a, b) of k, we have a standard parabolic subgroup $Q'_{a,b} = M'_{a,b}U'_{a,b}$ of $G_{k,D}$. The restriction of $\psi_{k,D}$ to $U'_{a,b}$ is denoted $\psi_{k,D}$ as well. Let $P'_{a,b}$ be the stabilizer of $\psi_{k,D}$ in $M'_{a,b}$.

Definition 6.6. Let π be an admissible representation of $P_{k,D}$. We say that π is D-cuspidal if the Jacquet module

$$J_{U'_{a,b}}(\pi) = 0$$

for all $k = a + b$ where $a, b > 0$.

Proposition 6.7. *The representation $\mathrm{ind}_{N_{k,D}}^{P_{k,D}}(J_{N_{k,D},\psi_{k,D}}(\pi) \ltimes \psi_{k,D})$ is D-cuspidal.*

In the Archimedean case, we only consider unitarizable representations. We need the following hypothesis.

Hypothesis 6.8. *For $\pi \in \mathrm{Irr}^{\mathrm{u}}(\mathrm{GL}_{k,D})$, there exist a vector space π_1, an embedding*

$$\mathcal{J} : \mathrm{ind}_{N_{k,D}}^{P_{k,D}}(\pi_1 \ltimes \psi_{k,D}) \hookrightarrow \pi,$$

and an isomorphism

$$\mathcal{J}^* : \mathcal{W}_{(k)_D}(\pi) \to \pi_1^\vee$$

such that

$$\langle \lambda, \mathcal{J}(f) \rangle = \langle \mathcal{J}^*(\lambda), f(1) \rangle$$

for all $\lambda \in \mathcal{W}_{(k)_D}(\pi)$ and $f \in \mathrm{ind}_{N_{k,D}}^{P_{k,D}}(\pi_1 \ltimes \psi_{k,D})$.

In fact, we only need a much weaker result, as can be seen from later sections. When $D = \mathbb{R}$, this result follows from the proof of [20, Proposition 3.8].

6.3. Proof of Theorem 5.9: Case $n = 1$

We first consider the case $n = 1$. This proof here is inspired by Piatetski-Shapiro's proof of the strong multiplicity one theorem. The proof presented here is adapted from [22, proof of Theorem II.2.5].

We show that $\mathrm{Speh}_D(\tau, 1)$ is D-generic. We know that $\tau' := |\mathbf{LJ}|(\mathrm{Speh}(\tau, s_{\tau,D}))$ is a cuspidal representation. If $s_{\tau,D} = d$, then this representation is D-generic by Lemma 6.2. We only have to treat the case $s_{\tau,D} < d$.

The representation $\mathrm{Speh}_D(\tau, 1)$ can be constructed from τ' using residues of Eisenstein series. With this automorphic realization, it comes with a $G_{kn,D}(F)$-invariant functional

$$\ell : \mathrm{Speh}_D(\tau, 1) \to \mathbb{C}.$$

For simplicity, we temporarily write $\pi = \mathrm{Speh}_D(\tau, 1)$. We now fix a local non-Archimedean place v_0. From Proposition 6.7, the restriction $\mathrm{Speh}_D(\tau, 1)|_{P_{k,D_{v_0}}}$ contains a D_{v_0}-cuspidal representation $\mathcal{K}_{v_0}$. We construct the following $P_{k,D}(\mathbb{A})$-subspace

$$T := \mathcal{K}_{v_0} \otimes (\otimes'_{v \neq v_0} \pi_v) \subset \pi_{v_0} \otimes (\otimes'_{v \neq v_0} \pi_v).$$

By our construction, any $\phi \in T$ the function

$$\widetilde{P}_{k,D}(\mathbb{A}) \to \mathbb{C}, \quad p \mapsto \ell\big(\pi(p)\phi\big)$$

is cuspidal with respect to any unipotent subgroup of $P_{k,D}$. Therefore, it has a Fourier expansion

$$\ell\big(\pi(p)\phi\big) = \sum_{\gamma \in N_{k-1,D}(F)\backslash \mathrm{GL}_{k-1,D}(F)} W_\phi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} p\right)$$

for $p \in \widetilde{P}_{k,D}(\mathbb{A})$.

Since π is irreducible, we see that π is generated by the subspaces

$$\pi(g)T \quad \big(g \in G_{k,D}(\mathbb{A})\big).$$

From this we know that there exists $g \in G_{k,D}(\mathbb{A})$ such that $\ell|_{\pi(g)T} \neq 0$. In other words, there exists $g \in G_{k,D}(\mathbb{A})$ and $\phi \in T$ such that $\ell(\pi(g)\phi) \neq 0$. Let us fix such a ϕ. By the strong approximation theorem, $[\widetilde{P}_{k,D}]$ is a dense subset of $[\mathrm{GL}_{k,D}]$. Therefore, $\ell(\pi(p)\phi) \neq 0$ for some $p \in \widetilde{P}_{k,D}$. From the Fourier expansion of $\ell(\pi(p)\phi)$, we deduce that

$$W_\phi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} p\right) \neq 0$$

for some γ. This completes the proof.

Remark 6.9. A similar argument shows that, for an automorphic representation of GL_n, genericity at a local non-Archimedean place implies global genericity.

6.4. Proof of Theorem 5.9: General case

We now treat the case of general n. This is based on the so-called induction-by-stages argument and the result when $n=1$. We recall that $\mathrm{Speh}_D(\tau,n)$ can be constructed from τ' using residues of Eisenstein series and $\mathrm{Speh}_D(\tau,n)=\mathrm{Speh}'(\tau',nd/s_{\tau,D})$.

Let $P_{(k^n)_D}=MV$ be the standard parabolic subgroup of $\mathrm{GL}_{kn,D}$ with Levi part

$$\mathrm{GL}_{k,D}\times\cdots\times\mathrm{GL}_{k,D},$$

where $\mathrm{GL}_{k,D}$ appears n times. We have the following result concerning the constant terms of $\mathrm{Speh}_D(\tau,n)$.

Lemma 6.10. *For $\phi\in\mathrm{Speh}_D(\tau,n)$, there is a section*

$$f\in\mathrm{Ind}_{P_{(k^n)_D}(\mathbb{A})}^{\mathrm{GL}_{kn,D}(\mathbb{A})}\big(\nu_\tau^{\frac{d(1-n)}{2s_{\tau,D}}}\,\mathrm{Speh}_D(\tau,1)\otimes\nu_\tau^{\frac{d(3-n)}{2s_{\tau,D}}}\,\mathrm{Speh}_D(\tau,1)\otimes\cdots\otimes\nu_\tau^{\frac{d(n-1)}{2s_{\tau,D}}}\,\mathrm{Speh}_D(\tau,1)\big) \tag{6.2}$$

such that the constant term ϕ_V of ϕ along V is

$$\phi_V(g)=f(g)(I_{k,D}\times\cdots\times I_{k,D}).$$

Moreover, for any

$$\varphi\in\nu_\tau^{\frac{d(1-n)}{2s_{\tau,D}}}\,\mathrm{Speh}_D(\tau,1)\otimes\nu_\tau^{\frac{d(3-n)}{2s_{\tau,D}}}\,\mathrm{Speh}_D(\tau,1)\otimes\cdots\otimes\nu_\tau^{\frac{d(n-1)}{2s_{\tau,D}}}\,\mathrm{Speh}_D(\tau,1),$$

there is $\phi\in\mathrm{Speh}_D(\tau,n)$ such that

$$\phi_V(I_{kn,D})=\varphi.$$

Proof. The proof of [21, Lemma 4.1] can be easily adapted to this case. The last part is straightforward. ∎

We want to show that $\mathrm{Speh}_D(\tau,n)$ has a non-vanishing $(k^n)_D$ Fourier coefficient. We now introduce a slight different Fourier coefficient.

We define the following character $\psi'_{(k^n)_D}$ on $N_{kn,D}$ as follows. We first write $u\in N_{kn,D}$ as the product of

$$u'=\mathrm{diag}(u_1,\dots,u_n)\in M=\mathrm{GL}_{k,D}\times\cdots\times\mathrm{GL}_{k,D},\quad u_i\in N_{k,D}$$

and $u''\in V$. We define

$$\psi'_{(k^n)_D}(u)=\psi_{(k)_D}(u_1+\cdots+u_n).$$

In other words, $\psi'_{(k^n)_D}$ is the extension of the $(k)_D\times\cdots\times(k)_D$-coefficients of the Levi part to $N_{kn,D}$. We then set

$$\mathcal{F}'_{(k^n)_D}(\phi)=\int_{[N_{kn,D}]}\phi(u)\overline{\psi}'_{(k^n)_D}(u)\,du.$$

Lemma 6.11. *The Fourier coefficient* $\mathcal{F}_{(k^n)_D}(\phi) \neq 0$ *for some* $\phi \in \mathrm{Speh}_D(\tau, n)$ *if and only if* $\mathcal{F}'_{(k^n)_D}(\phi) \neq 0$ *for some* $\phi \in \mathrm{Speh}_D(\tau, n)$.

Proof. This is a special case of [15, Theorem 8.2.1]. ∎

We now show that $\mathcal{F}'_{(k^n)_D}(\phi) \neq 0$ for some $\phi \in \mathrm{Speh}_D(\tau, n)$. Note that this Fourier coefficient can be written as the composition of a constant term along V and a Fourier coefficient for a Levi subgroup M. In other words,

$$\mathcal{F}'_{(k^n)_D}(\phi) = \int_{[N_{kn,D} \cap M]} \int_{[V]} \phi(u''u')\, du'' \overline{\psi}'_{kn,D}(u')\, du'.$$

By Lemma 6.10, it is enough to show that

$$\int_{[N_{kn,D} \cap M]} f(u')(I_{k,D} \times \cdots \times I_{k,D}) \overline{\psi}'_{kn,D}(u')\, du'$$

is nonzero for some f in (6.2). This is a $(k)_D \times \cdots \times (k)_D$ Fourier coefficient for the representation

$$\nu_\tau^{\frac{d(1-n)}{2s_{\tau,D}}} \mathrm{Speh}_D(\tau, 1) \otimes {}^{\frac{d(3-n)}{2s_{\tau,D}}} \mathrm{Speh}_D(\tau, 1) \otimes \cdots \otimes {}^{\frac{d(n-1)}{2s_{\tau,D}}} \mathrm{Speh}_D(\tau, 1).$$

We already know that this is nonzero for some choice of φ from the base case $n = 1$. This completes the proof of Theorem 5.9.

6.5. An Archimedean result

In this section, we treat the Archimedean case. We first prove that $\mathrm{Speh}_D(\tau, n)$ has a unique $(k, n)_D$-model when τ appears as the local component of a global cuspidal representation. This proof here is also inspired by Piatetski-Shapiro's proof of the strong multiplicity one theorem. See also [22, proof of Theorem II.2.5].

Theorem 6.12. *Let* τ_∞ *be an irreducible unitary generic representation of* $\mathrm{GL}_k(\mathbb{R})$*, which can be realized as the local component of an irreducible cuspidal representation of* $\mathrm{GL}_k(\mathbb{A})$*. Then under Hypothesis 6.8,*

$$\dim \mathrm{Hom}_{N_{(k^n)_{\mathbb{H}}}}\left(\mathrm{Speh}_{\mathbb{H}}(\tau_\infty, n), \psi_{(k^n)_{\mathbb{H}}}\right) = 1.$$

Proof. Let τ be a cuspidal representation of $\mathrm{GL}_k(\mathbb{A})$ which has τ_∞ as its local component τ_{v_1} at a real place. We first choose a central division D of dimension $(2n)^2$ over F such that $D_{v_1} = \mathrm{M}_n(\mathbb{H})$ and is non-split (at least) at another non-Archimedean place where τ is unramified. We also assume that D splits over all other Archimedean place other than v_1. With this choice, $\mathrm{Speh}_D(\tau, 1)$ is a cuspidal representation of $\mathrm{GL}_{k,D}(\mathbb{A})$ and $\mathrm{Speh}_{\mathbb{H}}(\tau_\infty, n)$ appears as the local component of $\mathrm{Speh}_D(\tau, 1)$ at v_1.

We now factor

$$\mathrm{GL}_{k,D}(\mathbb{A}) = \mathrm{GL}_{k,D}(F_{v_1}) \times \mathrm{GL}_{k,D}(\mathbb{A}_{v \neq v_1}).$$

Again, for ease of notations, let us write $\pi = \mathrm{Speh}_D(\tau, 1)$. We can decompose the representation $\mathrm{Speh}_D(\tau, 1)$ as

$$\pi = \mathrm{Speh}_D(\tau, 1) = \pi_{v_1} \otimes \pi_{v \neq v_1}.$$

Since $\mathrm{Speh}_D(\tau, 1)$ is a cuspidal representation, we now have the Fourier expansion:

$$\ell(x \otimes y) = \sum_{\gamma \in N_{k-1,D}(F) \backslash \mathrm{GL}_{k-1,D}(F)} W_{x \otimes y}\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \right),$$

where $x \otimes y \in \pi_{v_1} \otimes \pi_{v \neq v_1}$.

For ease of notations, we temporarily write V to be representation space of $\mathrm{Speh}_D(\tau, 1)$ and decompose V as $V = V_{v_1} \otimes V_{v \neq v_1}$. We already know that the $(k, 1)_{D_v}$-models for V_v is unique for every place $v \neq v_1$. Thus we can choose $\lambda_{v \neq v_1}$ to be a generator of $\mathcal{W}_{(k)_D}(V_{v \neq v_1})$. Then there exists a $\lambda_{v_1} \in \mathcal{W}_{(k)_D}(V_{v_1})$ such that for all $x' \in V_{v_1}$ and $y' \in V_{v \neq v_1}$ one has

$$W_{x' \otimes y'}(1) = \lambda_{v_1}(x') \cdot \lambda_{v \neq v_1}(y').$$

We now apply to Hypothesis 6.8 to π_{v_1}. Then there is a representation π_1 of the trivial group with a $P_{k,D_{v_1}}$-homomorphism

$$\mathcal{J} : \mathrm{ind}_{N_{k,D_{v_1}}}^{P_{k,D_{v_1}}} (\pi_1 \ltimes \psi_{k,D_{v_1}}) \to V_{v_1}$$

and an isomorphism

$$\mathcal{J}^* : \mathcal{W}_{(k)_{D_{v_1}}}(V_{v_1}) \to \pi_1^\vee$$

such that

$$\langle \mathcal{J}(f), f^* \rangle = \langle f(1), \mathcal{J}^*(f^*) \rangle$$

for all $f \in \mathrm{ind}_{N_{k,D_{v_1}}}^{P_{k,D_{v_1}}} (\pi_1 \ltimes \psi_{k,D_{v_1}})$ and $f^* \in \mathcal{W}_{(k)_{D_{v_1}}}(V_{v_1})$.

Assume that

$$\dim \mathcal{W}_{(k)_D}(V_{v_1}) = \dim \pi_1^\vee > 1.$$

Considering the kernel of $\mathcal{J}^*(\lambda_{v_1})$, we see that there exists a $P_{k,D}(F_{v_1})$-subspace U_{v_1} of V_{v_1} such that

$$\lambda_{v_1}(u) = 0, \quad \text{for all } u \in U_{v_1}.$$

We now set $U_{\mathbb{A}} := V_{v \neq v_1} \otimes U_{v_1}$. This is a $P_{k,D}(F_{v_1}) \times \mathrm{GL}_{k,D}(\mathbb{A}_{v \neq v_1})$-subspace. From the Fourier expansion we also have

$$\ell(u) = 0, \quad \text{for all } u \in U_{\mathbb{A}}.$$

For $u \in U_{\mathbb{A}}$, we consider the following function on $\mathrm{GL}_{k,D}(F) \backslash \mathrm{GL}_{k,D}(\mathbb{A})$:

$$g \mapsto \ell\big(\pi(g)u\big).$$

Let us fix u and consider its set of zeros Y. Then we know the following:

- $P_{k,D}(F_{v_1}) \times \mathrm{GL}_{k,D}(\mathbb{A}_{v \neq v_1}) \subset Y$;
- Y is left-invariant under $\mathrm{GL}_{k,D}(F)$ and the center $Z(\mathrm{GL}_{k,D}(\mathbb{A}))$.

By the strong approximation theorem, we see that Y contains $\mathrm{SL}_{k,D}(\mathbb{A})$ as it contains $\mathrm{SL}_{k,D}(F) \cdot \mathrm{SL}_{k,D}(F_S)$ as a dense subset for a sufficiently large set of places S.

We can now deduce that Y contains $P_{k,D}(F_{v_1})\,\mathrm{SL}_{k,D}(F_{v_1}) \times \mathrm{GL}_{k,D}(\mathbb{A}_{v \neq v_1})$. Now it is easy to see that

$$Z\big(\mathrm{GL}_{k,D}(F_{v_1})\big) P_{k,D}(F_{v_1})\,\mathrm{SL}_{k,D}(F_{v_1}) = \mathrm{GL}_{k,D}(F_{v_1}).$$

This implies that $Y = \mathrm{GL}_{k,D}(\mathbb{A})$.

From this, we deduce that for $u \in U_{\mathbb{A}}$, one has $\ell(\pi(g)u) = 0$ for all $g \in \mathrm{GL}_{k,D}(\mathbb{A})$. Thus $U_{\mathbb{A}}$ generates a proper $\mathrm{GL}_{k,D}(\mathbb{A})$-subspace of V. This is impossible since V is irreducible. This completes the proof. ■

Remark 6.13. It is easy to see that the same argument proves the following statement: if the Speh representation has unique $(k,n)_{D_v}$-functional at every non-Archimedean local place, then so does every Archimedean place.

We now state another working hypothesis.

Hypothesis 6.14. *For $i = 1, 2$, let π_i be an irreducible representation of $\mathrm{GL}_{k_i n,\mathbb{H}}$ such that $\mathrm{WS}(\pi_i) = \{(k_i^n)_{\mathbb{H}}\}$ and*

$$\dim \mathcal{W}_{(k_i^n)_{\mathbb{H}}}(\pi_i) \leq 1.$$

Set $k = k_1 + k_2$. Then

$$\dim \mathcal{W}_{(k^n)_{\mathbb{H}}}(\pi_1 \times \pi_2) \leq 1.$$

As a corollary, we have the following.

Corollary 6.15. *Assuming Hypotheses 6.8 and 6.14, we have that*

$$\dim \mathcal{W}_{(k^n)_{\mathbb{H}}}\big(\mathrm{Speh}_{\mathbb{H}}(\tau, n)\big) \leq 1.$$

A. Kirillov models

In this section, we prove some results regarding the representation theory of the local groups. Recall that an important result in the representation theory of $\mathrm{GL}_k(F)$ is that every generic representation admits a Kirillov model [8,20]. In this section, we would like to prove similar results for representations of $G_{k,D}$.

A.1. Basic setup

Let F be a local field. To include the "degenerate" case, here we allow D to be a central simple algebra (instead of a central division algebra) over F.

Let $P_{k,D}$ be the “mirabolic” subgroup of $G_{k,D}$ defined by

$$P_{k,D} := \{g \in G_{k,D} \mid (0, \ldots, 0, 1)g = (0, \ldots, 0, 1)\}.$$

Let $U_{k,D}$ be the subgroup

$$U_{k,D} := \left\{u \mid u = \begin{pmatrix} I_{k-1} & x \\ 0 & 1 \end{pmatrix} \in P_{k,D}\right\}.$$

The restriction of $\psi_{k,D}$ to $U_{k,D}$ is still denoted as $\psi_{k,D}$.

A.2. The non-Archimedean case

We first treat the non-Archimedean case since the argument is much easier. We introduce several functors. For an algebraic representation π of $P_{k,D}$, we define a functor using the twisted Jacquet module

$$\Phi^- : \mathrm{Alg}(P_{k,D}) \to \mathrm{Alg}(P_{k-1,D}), \quad \pi \mapsto J_{U_{k,D},\psi_{k,D}}(\pi).$$

For an algebraic representation π of $P_{k-1,D}$, we define

$$\Phi^+ : \mathrm{Alg}(P_{k-1,D}) \to \mathrm{Alg}(P_{k,D}), \quad \pi \mapsto \mathrm{ind}_{P_{k-1,D}U_{k,D}}^{P_{k,D}}(\pi \ltimes \psi_{k,D}).$$

Lemma A.1. *For an algebraic representation π of $P_{k,D}$, there is a natural homomorphism*

$$j : \Phi^+\Phi^-(\pi) \to \pi.$$

Proof. The argument in the proof of [7, Proposition 5.12 (b)] works here as well. We describe it briefly here.

The representation $\pi|_{U_{k,D}}$ can be viewed as a representation of the Hecke algebra $(C_c^\infty(U_{k,D}), *)$, where $*$ is the convolution. Using the Fourier transform, $\pi|_{U_{k,D}}$ becomes a representation of $(C_c^\infty(\widehat{U}_{k,D}), \cdot)$, where $\cdot$ denotes the pointwise multiplication. As a result, we view π as an l-sheaf $\mathcal{F}^\pi$ on $\widehat{U}_{k,D}$.

The action of $G_{k-1,D}$ acts on $\widehat{U}_{k,D}$ with only one open dense orbit, and $\psi_{k,D}$ is a representative for this orbit. We now restrict the sheaf $\mathcal{F}^\pi$ to the open orbit. Since the stabilizer of $\psi_{k,D}$ in $P_{k,D}$ is $P_{k-1,D}U_{k,D}$, the restriction sheaf corresponds to an induced representation from $P_{k-1,D}U_{k,D}$ to $P_{k,D}$. The inducing data is given by the stalk of this sheaf at $\psi_{k,D}$, which is $\Phi^-(\pi)$. This completes the proof. ∎

Lemma A.2. *Any $\lambda \in \mathcal{W}_{(k)_D}(\pi)$ factors through $\overline{\lambda} : \Phi^-(\pi) \to \mathbb{C}$. Moreover,*

$$\langle \lambda, j(f)\rangle = \langle \overline{\lambda}, f(1)\rangle$$

for any $f \in \mathrm{ind}_{P_{k-1,D}U_{k,D}}^{P_{k,D}}(\Phi^-(\pi) \ltimes \psi_{k,D})$.

Proof. This is a consequence of Lemma A.1. ∎

Similarly, any $\lambda \in \mathcal{W}_{(k)_D}(\pi)$ factors through

$$\mathcal{J}^*(\lambda) : J_{N_{k,D},\psi_{k,D}}(\pi) \to \mathbb{C}.$$

We have the following result.

Corollary A.3. *We have a natural homomorphism*

$$\mathcal{J} : \mathrm{ind}_{N_{k,D}}^{P_{k,D}} \big(J_{N_{k,D},\psi_{k,D}}(\pi) \ltimes \psi_{k,D}\big) \to \pi$$

such that

$$\big\langle \lambda, \mathcal{J}(f)\big\rangle = \big\langle \mathcal{J}^*(\lambda), f(1)\big\rangle$$

for all $f \in \mathrm{ind}_{N_{k,D}}^{P_{k,D}}(J_{N_{k,D},\psi_{k,D}}(\pi) \ltimes \psi_{k,D})$.

Proof. This is a consequence of Lemma A.2 and induction. ∎

Finally, we show that $\mathrm{ind}_{P_{k-1,D}U_{k,D}}^{P_{k,D}}(\Phi^-(\pi) \ltimes \psi_{k,D})$ is "cuspidal". For the partition (a,b) of k, we have a standard parabolic subgroup $Q'_{a,b} = M'_{a,b}U'_{a,b}$ of $G_{k,D}$. The restriction of $\psi_{k,D}$ to $U'_{a,b}$ is denoted $\psi_{k,D}$ as well. Let $P'_{a,b}$ be the stabilizer of $\psi_{k,D}$ in $M'_{a,b}$.

Proposition A.4. *Let τ be a smooth representation of $P'_{a,b}$. Then* $\mathrm{ind}_{P'_{a,b}U_{a,b}}^{P_{k,D}}(\tau \ltimes \psi_{k,D})$ *is D-cuspidal, in the following sense: for any partition (a,b) of k, the Jacquet module*

$$J_{U'_{a,b}}\big(\mathrm{ind}_{P'_{a,b}U'_{a,b}}^{P_{k,D}}(\tau \ltimes \psi_{k,D})\big) = 0.$$

Proof. Let $X = P'_{a,b}U'_{a,b}\backslash P_{k,D}$. Then the representation $\mathrm{ind}_{P'_{a,b}U'_{a,b}}^{P_{k,D}}(\tau \ltimes \psi_{k,D})$ corresponds to an l-sheaf $\mathcal{F}$ on X. To prove the result, it suffices to show there is no $U'_{a,b}$-equivariant functional $\mathcal{F}(X) \to \mathbb{C}$.

We apply Bernstein's localization principle to prove this statement. Note that the action of $U'_{a,b}$ on X is trivial. Thus it is enough to show that there is no $U'_{a,b}$-equivariant functional on each stalk of $\mathcal{F}$. Notice that the action of $U'_{a,b}$ on the stalk $\mathcal{F}_x$ is through a conjugation of $\psi_{k,D}$, which is nontrivial. Thus, it is impossible for the stalks to have $U'_{a,b}$-equivariant functionals. This proves the result. ∎

Corollary A.5. *The representation* $\mathrm{ind}_{N_{k,D}}^{P_{k,D}}(J_{N_{k,D},\psi_{k,D}}(\pi) \ltimes \psi_{k,D})$ *is D-cuspidal.*

Proof. By induction-by-stages, we can write

$$\mathrm{ind}_{N_{k,D}}^{P_{k,D}}\big(J_{N_{k,D},\psi_{k,D}}(\pi) \ltimes \psi_{k,D}\big) = \mathrm{ind}_{P'_{a,b}U_{a,b}}^{P_{k,D}}\big(\big(\mathrm{ind}_{N_{k,D}}^{P'_{a,b}} J_{N_{k,D},\psi_{k,D}}(\pi) \ltimes \psi_{k,D}\big) \ltimes \psi_{k,D}\big).$$

Now the result follows from Proposition A.4. ∎

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