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Trialitarian triples

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Abstract. Trialitarian triples are triples of central simple algebras of degree 8 with orthogonal involution that provide a convenient structure for the representation of trialitarian algebraic groups as automorphism groups. This paper explicitly describes the canonical "trialitarian" isomorphisms between the spin groups of the algebras with involution involved in a trialitarian triple, using a rationally defined shift operator that cyclically permutes the algebras. The construction relies on compositions of quadratic spaces of dimension 8, which yield all the trialitarian triples of split algebras. No restriction on the characteristic of the base field is needed.

Contents

1. Introduction

Trialitarian triples were introduced by Knus et al. [13, §42.A] to provide the groundwork for the study of algebraic groups of trialitarian type D_4 as automorphism groups of trialitarian algebras. They consist in three central simple algebras of degree 8 with orthogonal involution (A,σ_A), (B,σ_B), (C,σ_C) over a field F of characteristic different from 2 related by the property that the Clifford algebra of (A,σ_A) is isomorphic to the direct product of (B,σ_B) and (C,σ_C). Trialitarian algebras over F are defined in [13, §43] as algebras with orthogonal involution of degree 8 over a cubic étale F-algebra that are isomorphic after scalar extension of F to the direct product of the algebras in a trialitarian triple.

The main goal of this work is to elucidate the trialitarian isomorphisms that arise canonically between the spin groups of algebras with quadratic pair involved in a trialitarian triple and also between the groups of projective similitudes of these algebras, see [13, (42.5)]. The basic tool is a shift operator ∂ of period 3 on trialitarian triples, which accounts for all the trialitarian features of the theory. The cohomological approach

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in [13, (42.3)] reveals the existence of ∂ but does not provide any explicit description. By contrast, the definition of ∂ in Section 4.3 below is entirely explicit and cohomology-free, and the definition of the trialitarian isomorphisms follows easily in Section 4.4. A noteworthy feature of our discussion is that the restriction on the characteristic of the base field is made obsolete thanks to the ground-breaking paper [7] of Dolphin–Quéguiner-Mathieu, in which a canonical quadratic pair is defined on Clifford algebras (and where the cohomological approach to the definition of ∂ in arbitrary characteristic is given [7, Th. 4.11]).[1] Thus, all the structures considered in this paper are (unless explicitly mentioned) over fields of arbitrary characteristic.

To prepare for the discussion of trialitarian triples in Section 4, we found it necessary to consider first trialitarian triples of split algebras. These triples arise from compositions of quadratic spaces, which are studied in Section 3. Compositions of quadratic spaces provide a new perspective on the classical theory of composition algebras by triplicating their underlying vector space. They also demonstrate more diversity, because—in contrast with compositions arising from composition algebras—the three quadratic spaces involved in a composition need not be isometric; this accounts for the interpretation in Section 4.5 of the mod 2 cohomological invariants of $\mathbf{Spin}_8$, since compositions of quadratic spaces of dimension 8 are torsors under $\mathbf{Spin}_8$.

Compositions of three different quadratic spaces of equal dimension have been considered earlier, for instance in Knus' monograph [12, V(7.2)], in [13, (35.18)] and in the papers [3, 5.3] and [2, §3] by Alsaody–Gille and Alsaody respectively. However, the shift operator ∂ on compositions of quadratic spaces, briefly mentioned in [13, (35.18)], seems to have been mostly ignored so far. By attaching to every composition on quadratic spaces (V_1, q_1), (V_2, q_2), (V_3, q_3) two cyclic derivatives, which are compositions on (V_2, q_2), (V_3, q_3), (V_1, q_1) and on (V_3, q_3), (V_1, q_1), (V_2, q_2) respectively, the shift operator provides the model for the operator ∂ on trialitarian triples.

Compositions of quadratic spaces of dimension 8 also afford a broader view of the classical principle of triality for similitudes of the underlying vector space of an octonion algebra, as discussed by Springer–Veldkamp [19, §3.2], and also of the local version of this principle in characteristic 2 described by Elduque [8, §3, §5], see Corollaries 4.24 and 4.25. Automorphisms of the compositions of quadratic spaces arising from composition algebras are by definition the *related triples* of isometries defined in [19, §3.6], [8, §1] and [3, §3] (see Remark 3.18); they are closely related to *autotopies* of the algebra, which form the structure group defined for alternative algebras by Petersson [15], see Sections 3.5 and 4.6.

The first section reviews background information on Clifford groups and their Lie algebras, notably on extended Clifford groups, which play a central rôle in subsequent sections.

More detail on the contents of this work can be found in the introduction of each section.

[1] Prior to [7], examples of trialitarian triples in characteristic 2 were given by Knus–Villa [14].

2. Clifford groups and Lie algebras

The purpose of the first subsections of this section is to recall succinctly the Clifford groups of even-dimensional quadratic spaces and their twisted analogues (in the sense of Galois cohomology), which are defined in arbitrary characteristic through central simple algebras with quadratic pair. Most of the material is taken from [13], but we incorporate a few complements that are made possible by the definition of canonical quadratic pairs on Clifford algebras by Dolphin–Quéguiner-Mathieu [7].

A detailed discussion of the corresponding Lie algebras is given in Sections 2.4 and 2.5. For a central simple algebra with quadratic pair $\mathfrak{A}$, we emphasize the difference between the Lie algebra $\mathfrak{o}(\mathfrak{A})$ of the orthogonal group and the Lie algebra $\mathfrak{pgo}(\mathfrak{A})$ of the group of projective similitudes, which are canonically isomorphic when the characteristic is different from 2 but contain different information in characteristic 2.

The last subsection provides a major tool for the definition of homomorphisms

$$\mathfrak{C}(\mathfrak{A}) \to \mathfrak{A}'$$

from the Clifford algebra of a central simple algebra with quadratic pair $\mathfrak{A}$ to a central simple algebra with quadratic pair $\mathfrak{A}'$. These homomorphisms are shown to be uniquely determined by Lie algebra homomorphisms $\mathfrak{pgo}(\mathfrak{A}) \to \mathfrak{pgo}(\mathfrak{A}')$; see Theorem 2.21.

2.1. Quadratic forms and quadratic pairs

Let (V, q) be a (finite-dimensional) quadratic space over F. The polar form $b\colon V \times V \to F$ is defined by

$$b(x, y) = q(x + y) - q(x) - q(y) \quad \text{for } x, y \in V.$$

We only consider quadratic spaces whose polar form b is nonsingular. This restriction entails that $\dim V$ is even if $\operatorname{char} F = 2$, for b is then an alternating form. Nonsingularity of b allows us to define the adjoint involution σ_b on $\operatorname{End} V$ by the condition

$$b\big(x, a(y)\big) = b\big(\sigma_b(a)(x), y\big) \quad \text{for } a \in \operatorname{End} V \text{ and } x, y \in V.$$

Moreover, we may identify $V \otimes V$ with $\operatorname{End} V$ by mapping $x \otimes y \in V \otimes V$ to the operator $z \mapsto xb(y, z)$. Under the identification $V \otimes V = \operatorname{End} V$, the involution σ_b and the (reduced) trace Trd are given by

$$\sigma_b(x \otimes y) = y \otimes x \quad \text{and} \quad \operatorname{Trd}(x \otimes y) = b(x, y) \quad \text{for } x, y \in V,$$

see [13, §5.A]. Moreover, for $a \in \operatorname{End} V$ and $x, y \in V$ we have

$$a \circ (x \otimes y) = a(x) \otimes y \quad \text{and} \quad (x \otimes y) \circ a = x \otimes \sigma_b(a)(y).$$

The identification $V \otimes V = \operatorname{End} V$, which depends on the choice of the nonsingular polar form b, will be used repeatedly in the sequel. It will be referred to as a *standard identification*.

Let

$$\mathrm{Sym}(\sigma_b) = \{a \in \operatorname{End} V \mid \sigma_b(a) = a\}.$$

To the quadratic form q on V we further associate a linear form $\mathfrak{f}_q$ on $\mathrm{Sym}(\sigma_b)$ defined by the condition

$$\mathfrak{f}_q(x \otimes x) = q(x) \quad \text{for } x \in V,$$

see [13, (5.11)]. Linearizing this condition yields $\mathfrak{f}_q(x \otimes y + \sigma_b(x \otimes y)) = b(x, y)$ for $x, y \in V$, hence

$$\mathfrak{f}_q\big(a + \sigma_b(a)\big) = \mathrm{Trd}(a) \quad \text{for } a \in \operatorname{End} V.$$

The pair $(\sigma_b, \mathfrak{f}_q)$ determines the quadratic form q up to a scalar factor by [13, (5.11)], which is sufficient to define the orthogonal group $\mathrm{O}(q)$ of isometries of (V, q), as well as the group of similitudes $\mathrm{GO}(q)$ and the group of projective similitudes $\mathrm{PGO}(q)$, as follows:

$$\begin{aligned}
\mathrm{O}(q) &= \{a \in \operatorname{End} V \mid q\big(a(x)\big) = q(x) \text{ for all } x \in V\} \\
&= \{a \in \operatorname{End} V \mid \sigma_b(a)a = 1 \text{ and } \mathfrak{f}_q(asa^{-1}) = \mathfrak{f}_q(s) \text{ for all } s \in \mathrm{Sym}(\sigma_b)\}, \\
\mathrm{GO}(q) &= \{a \in \operatorname{End} V \mid \text{ there exists } \mu \in F^\times \text{ such that } q\big(a(x)\big) = \mu q(x) \text{ for all } x \in V\} \\
&= \{a \in \operatorname{End} V \mid \sigma_b(a)a \in F^\times \text{ and } \mathfrak{f}_q(asa^{-1}) = \mathfrak{f}_q(s) \text{ for all } s \in \mathrm{Sym}(\sigma_b)\}, \\
\mathrm{PGO}(q) &= \mathrm{GO}(q)/F^\times.
\end{aligned}$$

In the equivalent definitions of $\mathrm{GO}(q)$, the scalar μ such that $q(a(x)) = \mu q(x)$ for all $x \in V$ is $\sigma_b(a)a$. It is called the *multiplier* of the similitude a.

Isometries and similitudes are also defined between different quadratic spaces: if (V, q) and $(\widetilde{V}, \tilde{q})$ are quadratic spaces over a field F, a *similitude* $u\colon (V, q) \to (\widetilde{V}, \tilde{q})$ is a linear bijection $V \to \widetilde{V}$ for which there exists a scalar $\mu \in F^\times$ such that $\tilde{q}(u(x)) = \mu q(x)$ for all $x \in V$. The scalar μ is called the *multiplier* of the similitude, and similitudes with multiplier 1 are called *isometries*. Abusing notation, for every linear bijection $u\colon V \to \widetilde{V}$ we write

$$\mathrm{Int}(u)\colon \operatorname{End} V \to \operatorname{End} \widetilde{V} \quad \text{for the map } a \mapsto u \circ a \circ u^{-1}.$$

It is readily verified that for every similitude u the isomorphism $\mathrm{Int}(u)$ restricts to group isomorphisms

$$\mathrm{O}(q) \xrightarrow{\sim} \mathrm{O}(\tilde{q}), \quad \mathrm{GO}(q) \xrightarrow{\sim} \mathrm{GO}(\tilde{q}), \quad \mathrm{PGO}(q) \xrightarrow{\sim} \mathrm{PGO}(\tilde{q}).$$

The groups $\mathrm{O}(q)$, $\mathrm{GO}(q)$ and $\mathrm{PGO}(q)$ are groups of rational points of algebraic groups (i.e., smooth affine algebraic group schemes) which are denoted respectively by $\mathbf{O}(q)$, $\mathbf{GO}(q)$ and $\mathbf{PGO}(q)$, see [13, §23]. As pointed out in [13], twisted forms (in the sense of Galois cohomology) of these groups can be defined through a notion of quadratic pair on central simple algebras, which is recalled next.

Let A be a central simple algebra over an arbitrary field F. An F-linear involution σ on A is said to be *orthogonal* (resp. *symplectic*) if after scalar extension to a splitting field of A it is adjoint to a symmetric nonalternating (resp. to an alternating) bilinear form. For any involution σ on A we write

$$\mathrm{Sym}(\sigma) = \{a \in A \mid \sigma(a) = a\}.$$

Definition 2.1. A *quadratic pair* $(\sigma, \mathfrak{f})$ on a central simple algebra A consists of an involution σ on A and a linear map $\mathfrak{f}\colon \mathrm{Sym}(\sigma) \to F$ subject to the following conditions:

(i) σ is orthogonal if $\mathrm{char}\, F \neq 2$ and symplectic if $\mathrm{char}\, F = 2$;

(ii) $\mathfrak{f}(x + \sigma(x)) = \mathrm{Trd}_A(x)$ for $x \in A$, where Trd_A is the reduced trace.

The map $\mathfrak{f}$ is called the *semitrace* of the quadratic pair $(\sigma, \mathfrak{f})$. This terminology is motivated by the observation that when $\mathrm{char}\, F \neq 2$ every $x \in \mathrm{Sym}(\sigma)$ can be written as $x = \frac{1}{2}(x + \sigma(x))$, hence $\mathfrak{f}(x) = \frac{1}{2}\mathrm{Trd}_A(x)$. Thus, the semitrace of a quadratic pair $(\sigma, \mathfrak{f})$ is uniquely determined by the orthogonal involution σ if $\mathrm{char}\, F \neq 2$.

To simplify notation, when possible without confusion we use a single letter to denote a central simple algebra with quadratic pair, and write

$$\mathfrak{A} = (A, \sigma, \mathfrak{f}).$$

The twisted forms of orthogonal groups are defined as follows: for $\mathfrak{A}$ as above,

$$\begin{aligned}
\mathrm{O}(\mathfrak{A}) &= \{a \in A \mid \sigma(a)a = 1 \text{ and } \mathfrak{f}(asa^{-1}) = \mathfrak{f}(s) \text{ for all } s \in \mathrm{Sym}(\sigma)\},\\
\mathrm{GO}(\mathfrak{A}) &= \{a \in A \mid \sigma(a)a \in F^\times \text{ and } \mathfrak{f}(asa^{-1}) = \mathfrak{f}(s) \text{ for all } s \in \mathrm{Sym}(\sigma)\},\\
\mathrm{PGO}(\mathfrak{A}) &= \mathrm{GO}(\mathfrak{A})/F^\times.
\end{aligned}$$

The group of similitudes $\mathrm{GO}(\mathfrak{A})$ can be alternatively defined as the group of elements $a \in A^\times$ such that $\mathrm{Int}(a)$ is an automorphism of $\mathfrak{A}$. Therefore, by the Skolem–Noether theorem the group $\mathrm{PGO}(\mathfrak{A})$ can be identified with the group of automorphisms of $\mathfrak{A}$. The groups $\mathrm{O}(\mathfrak{A})$, $\mathrm{GO}(\mathfrak{A})$ and $\mathrm{PGO}(\mathfrak{A})$ are groups of rational points of algebraic groups denoted respectively by $\mathbf{O}(\mathfrak{A})$, $\mathbf{GO}(\mathfrak{A})$ and $\mathbf{PGO}(\mathfrak{A})$, see [13, §23].

For $a \in \mathrm{GO}(\mathfrak{A})$, the scalar $\sigma(a)a \in F^\times$ is called the *multiplier* of the similitude a. We write $\mu(a) = \sigma(a)a$ and thus obtain a group homomorphism

$$\mu\colon \mathrm{GO}(\mathfrak{A}) \to F^\times$$

whose kernel is $\mathrm{O}(\mathfrak{A})$. Thus, for every quadratic space (V, q) we have by definition

$$\begin{aligned}
\mathrm{O}(\mathrm{End}\, V, \sigma_b, \mathfrak{f}_q) &= \mathrm{O}(q),\\
\mathrm{GO}(\mathrm{End}\, V, \sigma_b, \mathfrak{f}_q) &= \mathrm{GO}(q),\\
\mathrm{PGO}(\mathrm{End}\, V, \sigma_b, \mathfrak{f}_q) &= \mathrm{PGO}(q).
\end{aligned}$$

The following statement is given without detailed proof in [13, (12.36)].

Proposition 2.2. *Let (V, q) and $(\widetilde{V}, \tilde{q})$ be quadratic spaces over an arbitrary field F. If $u\colon (V, q) \to (\widetilde{V}, \tilde{q})$ is a similitude, then* $\mathrm{Int}(u)$ *is an isomorphism of algebras with quadratic pair*

$$\mathrm{Int}(u)\colon (\mathrm{End}\, V, \sigma_b, \mathfrak{f}_q) \xrightarrow{\sim} (\mathrm{End}\, \widetilde{V}, \sigma_{\tilde{b}}, \mathfrak{f}_{\tilde{q}}).$$

Conversely, every isomorphism $(\operatorname{End} V, \sigma_b, \mathfrak{f}_q) \xrightarrow{\sim} (\operatorname{End} \widetilde{V}, \sigma_{\tilde{b}}, \mathfrak{f}_{\tilde{q}})$ *has the form* $\operatorname{Int}(u)$ *for some similitude* $u\colon (V,q) \to (\widetilde{V}, \tilde{q})$ *uniquely determined up to a scalar factor.*

Proof. Observe that for every linear bijection $u\colon V \to \widetilde{V}$ there exists a map $\hat{u}\colon V \to \widetilde{V}$ such that

$$\tilde{b}\big(\hat{u}(x), \tilde{y}\big) = b\big(x, u^{-1}(\tilde{y})\big) \quad \text{for all } x \in V \text{ and } \tilde{y} \in \widetilde{V},$$

since the polar forms b and $\tilde{b}$ are nonsingular. Under the standard identifications $\operatorname{End} V = V \otimes V$ and $\operatorname{End} \widetilde{V} = \widetilde{V} \otimes \widetilde{V}$ afforded by b and $\tilde{b}$, we have

$$\operatorname{Int}(u)(x \otimes y) = u(x) \otimes \hat{u}(y) \quad \text{for all } x, y \in V.$$

If u is a similitude with multiplier μ, then $\hat{u} = \mu^{-1} u$, hence $\operatorname{Int}(u) \circ \sigma_b = \sigma_{\tilde{b}} \circ \operatorname{Int}(u)$ and

$$\mathfrak{f}_{\tilde{q}}\big(\operatorname{Int}(u)(x \otimes x)\big) = \mu^{-1}\tilde{q}\big(u(x)\big) = q(x) = \mathfrak{f}_q(x \otimes x) \quad \text{for all } x \in V.$$

Since $\operatorname{Sym}(\sigma_b)$ is spanned by elements of the form $x \otimes x$, it follows that $\operatorname{Int}(u)$ is an isomorphism of algebras with quadratic pair.

For the converse, note that the Skolem–Noether theorem shows that every F-algebra isomorphism $\operatorname{End} V \xrightarrow{\sim} \operatorname{End} \widetilde{V}$ has the form $\operatorname{Int}(u)$ for some linear bijection $u\colon V \to \widetilde{V}$. If $\operatorname{Int}(u)$ is an isomorphism of algebras with quadratic pair, then $\operatorname{Int}(u)(x \otimes x) \in \operatorname{Sym}(\sigma_{\tilde{b}})$ for every $x \in V$, hence $\hat{u} = \mu^{-1}u$ for some $\mu \in F^\times$. Since $\mathfrak{f}_{\tilde{q}}(\operatorname{Int}(u)(x \otimes x)) = \mathfrak{f}_q(x \otimes x)$ for all $x \in V$, it follows that $\tilde{q}(u(x)) = \mu q(x)$ for all $x \in V$, hence u is a similitude.

To complete the proof, suppose that $u, u'\colon (V,q) \to (\widetilde{V}, \tilde{q})$ are similitudes such that $\operatorname{Int}(u) = \operatorname{Int}(u')$. Then $\operatorname{Int}(u^{-1}u') = \operatorname{Id}_V$, hence $u^{-1}u'$ lies in the center of $\operatorname{End} V$, which is F. Therefore, u and u' differ by a scalar factor. ∎

2.2. Clifford algebras

For any quadratic space (V,q) over F we let $C(V,q)$ denote the Clifford algebra of (V,q) and $C_0(V,q)$ its even Clifford algebra. We will only consider even-dimensional quadratic spaces; if $\dim V = 2m$, then the algebra $C(V,q)$ is central simple of degree 2^m and $C_0(V,q)$ is semisimple with center a quadratic étale F-algebra Z given by the discriminant or Arf invariant of q, see [17, Ch. 9]. In most cases considered through this text, the algebra Z is split, i.e., $Z \simeq F \times F$. We may then define a polarization of (V,q) as follows:

Definition 2.3. If (V,q) is an even-dimensional quadratic space with trivial discriminant or Arf invariant, a *polarization* of (V,q) is a designation of the primitive central idempotents of $C_0(V,q)$ as z_+ and z_-. Given a polarization of (V,q), we let $C_+(V,q) = C_0(V,q)z_+$ and $C_-(V,q) = C_0(V,q)z_-$, so

$$C_0(V,q) = C_+(V,q) \times C_-(V,q).$$

Each even-dimensional quadratic space of trivial discriminant or Arf invariant thus has two possible polarizations.

The algebra $C(V,q)$ carries an involution τ such that $\tau(x) = x$ for all $x \in V$. This involution preserves $C_0(V,q)$ and restricts to an involution τ_0 on $C_0(V,q)$. The type of the involutions τ and τ_0 is determined in [13, (8.4)] as follows:

- If $\dim V \equiv 2 \bmod 4$ the involution τ_0 does not leave Z fixed; we will not need to consider this case.
- If $\dim V \equiv 4 \bmod 8$, then the involutions τ and τ_0 are symplectic. When $Z \simeq F \times F$, this means that τ_0 restricts to symplectic involutions on each of the simple components of $C_0(V,q)$.
- If $\dim V \equiv 0 \bmod 8$ and $\operatorname{char} F \neq 2$, then the involutions τ and τ_0 are orthogonal.
- If $\dim V \equiv 0 \bmod 8$ and $\operatorname{char} F = 2$, then the involutions τ and τ_0 are symplectic.

Following Dolphin–Quéguiner-Mathieu [7, Prop. 6.2], a canonical quadratic pair $(\tau, \mathfrak{g})$ can be defined on $C(V,q)$ when[2] $\dim V \equiv 0 \bmod 8$ by associating to τ the following semitrace:

$$\mathfrak{g}(s) = \operatorname{Trd}_{C(V,q)}(ee's) \in F \quad \text{for } s \in \operatorname{Sym}(\tau),$$

where $e, e' \in V$ are arbitrary vectors such that $b(e,e') = 1$. If $\operatorname{char} F \neq 2$, then for any such vectors e, e' and for every $s \in \operatorname{Sym}(\tau)$ we have

$$\operatorname{Trd}_{C(V,q)}(ee's) = \operatorname{Trd}_{C(V,q)}\big(\tau(ee's)\big) = \operatorname{Trd}_{C(V,q)}(se'e) = \operatorname{Trd}_{C(V,q)}(e'es).$$

Therefore,

$$\operatorname{Trd}_{C(V,q)}(ee's) = \tfrac{1}{2}\operatorname{Trd}_{C(V,q)}\big((ee' + e'e)s\big) = \tfrac{1}{2}\operatorname{Trd}_{C(V,q)}(s),$$

as expected.

Likewise, Dolphin–Quéguiner-Mathieu show in [7, Prop. 3.6] that a canonical quadratic pair $(\tau_0, \mathfrak{g}_0)$ can be defined on $C_0(V,q)$ when $\dim V \equiv 0 \bmod 8$ by associating to τ_0 the following semitrace:

$$\mathfrak{g}_0(s) = \operatorname{Trd}_{C_0(V,q)}(ee's) \in Z \quad \text{for } s \in \operatorname{Sym}(\tau_0),$$

where $e, e' \in V$ are arbitrary vectors such that $b(e,e') = 1$. If $Z \simeq F \times F$, then

$$C_0(V,q) \simeq C_+(V,q) \times C_-(V,q)$$

for the central simple F-algebras $C_+(V,q)$, $C_-(V,q)$ defined in Definition 2.3, and the quadratic pair $(\tau_0, \mathfrak{g}_0)$ defined above is a pair of quadratic pairs $(\tau_+, \mathfrak{g}_+)$ on $C_+(V,q)$ and $(\tau_-, \mathfrak{g}_-)$ on $C_-(V,q)$.

Every similitude of quadratic spaces $u\colon (V,q) \to (\widetilde{V}, \tilde{q})$ with multiplier μ defines an F-isomorphism $C_0(u)\colon C_0(V,q) \xrightarrow{\sim} C_0(\widetilde{V}, \tilde{q})$ such that

$$C_0(u)(x \cdot y) = \mu^{-1} u(x) \cdot u(y) \quad \text{for } x, y \in V.$$

[2] Dolphin–Quéguiner-Mathieu only assume $\dim V$ even, $\dim V \geq 6$, but they restrict to $\operatorname{char} F = 2$.

It is clear from the definition that $C_0(u)$ preserves the canonical involutions τ_0 and $\widetilde{\tau}_0$ on $C_0(V,q)$ and $C_0(\widetilde{V},\tilde{q})$. If $\dim V \equiv 0 \bmod 8$, then $C_0(u)$ also preserves the semitraces $\mathfrak{g}_0$ and $\tilde{\mathfrak{g}}_0$. To see this, observe that the images $u(e)$, $u(e')$ of vectors e, $e' \in V$ such that $b(e,e') = 1$ satisfy $\tilde{b}(u(e),u(e')) = \mu$. We may therefore use $\mu^{-1}u(e)$ and $u(e')$ to compute the semitrace $\tilde{\mathfrak{g}}_0$: for $s \in \operatorname{Sym}(\tau_0)$,

$$\tilde{\mathfrak{g}}_0\big(C_0(u)(s)\big) = \operatorname{Trd}_{C_0(\widetilde{V},\tilde{q})}\big(\mu^{-1}u(e)u(e')C_0(u)(s)\big).$$

Now, $\mu^{-1}u(e)u(e') = C_0(u)(ee')$, hence by substituting in the preceding equation and using the property that algebra isomorphisms preserve reduced traces, we obtain

$$\tilde{\mathfrak{g}}_0\big(C_0(u)(s)\big) = \operatorname{Trd}_{C_0(\widetilde{V},\tilde{q})}\big(C_0(u)(ee's)\big) = C_0(u)\big(\operatorname{Trd}_{C_0(V,q)}(ee's)\big) = C_0(u)\big(\mathfrak{g}_0(s)\big).$$

Thus, $C_0(u)$ is an isomorphism of algebras with involution

$$C_0(u)\colon \big(C_0(V,q),\tau_0\big) \xrightarrow{\sim} \big(C_0(\widetilde{V},\tilde{q}),\widetilde{\tau}_0\big)$$

and an isomorphism of algebras with quadratic pair if $\dim V = \dim \widetilde{V} \equiv 0 \bmod 8$

$$C_0(u)\colon \big(C_0(V,q),\tau_0,\mathfrak{g}_0\big) \xrightarrow{\sim} \big(C_0(\widetilde{V},\tilde{q}),\widetilde{\tau}_0,\tilde{\mathfrak{g}}_0\big).$$

Among auto-similitudes $u \in \operatorname{GO}(q)$ we may distinguish proper similitudes by considering the restriction of $C_0(u)$ to the center Z of $C_0(V,q)$: the similitude u is said to be *proper* if $C_0(u)$ fixes Z and *improper* if $C_0(u)$ restricts to the nontrivial F-automorphism of Z, see [13, §13.A]. The proper similitudes form a subgroup $\operatorname{GO}^+(q)$ of index 2 in $\operatorname{GO}(q)$, and we let

$$\operatorname{O}^+(q) = \operatorname{O}(q) \cap \operatorname{GO}^+(q), \quad \operatorname{PGO}^+(q) = \operatorname{GO}^+(q)/F^\times.$$

Twisted forms. Following ideas of Jacobson and Tits, an analogue of the even Clifford algebra for a central simple algebra with quadratic pair $\mathfrak{A} = (A,\sigma,\mathfrak{f})$ of even degree is defined in [13, §8.B]. The Clifford algebra $C(\mathfrak{A})$ is obtained by a functorial construction such that for every quadratic space (V,q) of even dimension, the identification $\operatorname{End} V = V \otimes V$ set up in Section 2.1 yields an identification

$$C(\operatorname{End} V,\sigma_b,\mathfrak{f}_q) = C_0(V,q).$$

This property implies that $C(\mathfrak{A})$ is a semisimple algebra with center a quadratic étale F-algebra given by the discriminant of the quadratic pair $(\sigma,\mathfrak{f})$.

Definition 2.4. If the discriminant of $(\sigma,\mathfrak{f})$ is trivial, a *polarization* of $\mathfrak{A}$ is a designation of the primitive central idempotents of $C(\mathfrak{A})$ as z_+ and z_-. A polarization induces the labeling of the simple components of $C(\mathfrak{A})$ as $C_+(\mathfrak{A}) = C(\mathfrak{A})z_+$ and $C_-(\mathfrak{A}) = C(\mathfrak{A})z_-$, so

$$C(\mathfrak{A}) = C_+(\mathfrak{A}) \times C_-(\mathfrak{A}).$$

The algebra $C(\mathfrak{A})$ comes equipped with a canonical linear map

$$c\colon A \to C(\mathfrak{A})$$

whose image generates $C(\mathfrak{A})$ as an F-algebra. In the split case $A = \operatorname{End} V$, the map c is given by multiplication in $C(V, q)$:

$$c\colon V \otimes V \to C_0(V, q), \quad x \otimes y \mapsto x \cdot y.$$

The algebra $C(\mathfrak{A})$ carries a canonical involution $\underline{\sigma}$ characterized by the condition that $\underline{\sigma}(c(a)) = c(\sigma(a))$ for $a \in A$. If $\deg A \equiv 0 \bmod 8$, Dolphin–Quéguiner-Mathieu show that a canonical quadratic pair $(\underline{\sigma}, \underline{\mathfrak{f}})$ is defined on $C(\mathfrak{A})$ by associating to $\underline{\sigma}$ the following semitrace:

$$\underline{\mathfrak{f}}(s) = \operatorname{Trd}_{C(\mathfrak{A})}\big(c(a)s\big) \quad \text{for } s \in \operatorname{Sym}(\underline{\sigma}),$$

where $a \in A$ is any element such that $\operatorname{Trd}_A(a) = 1$, see [7, Def. 3.3]. These constructions are compatible with the corresponding definitions in the split case, in the sense that for every even-dimensional quadratic space (V, q) the standard identification $\operatorname{End} V = V \otimes V$ of Section 2.1 yields identifications of algebras with involution or quadratic pair:

$$\big(C(\operatorname{End} V, \sigma_b, \mathfrak{f}_q), \underline{\sigma}\big) = \big(C_0(V, q), \tau_0\big) \qquad \text{if } \dim V \equiv 0 \bmod 4,$$
$$\big(C(\operatorname{End} V, \sigma_b, \mathfrak{f}_q), \underline{\sigma}, \underline{\mathfrak{f}}\big) = \big(C_0(V, q), \tau_0, \mathfrak{g}_0\big) \quad \text{if } \dim V \equiv 0 \bmod 8.$$

By functoriality of the Clifford algebra construction, every isomorphism of algebras with quadratic pair $\varphi\colon \mathfrak{A} \xrightarrow{\sim} \widetilde{\mathfrak{A}}$ induces an isomorphism of algebras with involution or with quadratic pair

$$C(\varphi)\colon \big(C(\mathfrak{A}), \underline{\sigma}\big) \xrightarrow{\sim} \big(C(\widetilde{\mathfrak{A}}), \underline{\widetilde{\sigma}}\big) \quad \text{or} \quad \big(C(\mathfrak{A}), \underline{\sigma}, \underline{\mathfrak{f}}\big) \xrightarrow{\sim} \big(C(\widetilde{\mathfrak{A}}), \underline{\widetilde{\sigma}}, \underline{\widetilde{\mathfrak{f}}}\big)$$

such that

$$C(\varphi)\big(c(a)\big) = c\big(\varphi(a)\big) \quad \text{for } a \in A.$$

As in the split case, we may distinguish between proper and improper similitudes: every similitude $u \in \operatorname{GO}(\mathfrak{A})$ induces an F-automorphism $\operatorname{Int}(u)$ of $\mathfrak{A}$, hence an F-automorphism $C(\operatorname{Int}(u))$ of $C(\mathfrak{A})$. The similitude u is said to be *proper* if $C(\operatorname{Int}(u))$ leaves the center of $C(\mathfrak{A})$ elementwise fixed; otherwise it is said to be *improper*. This definition agrees with the previous definition of proper similitude in the case where $\mathfrak{A} = (\operatorname{End} V, \sigma_b, \mathfrak{f}_q)$ for a quadratic space (V, q), because $C(\operatorname{Int}(u)) = C_0(u)$ for every similitude $u \in \operatorname{GO}(q)$, see [13, (13.1)].

Proper similitudes form a subgroup $\operatorname{GO}^+(\mathfrak{A})$ of index 1 or 2 in $\operatorname{GO}(\mathfrak{A})$, and we let

$$\operatorname{O}^+(\mathfrak{A}) = \operatorname{O}(\mathfrak{A}) \cap \operatorname{GO}^+(\mathfrak{A}), \quad \operatorname{PGO}^+(\mathfrak{A}) = \operatorname{GO}^+(\mathfrak{A})/F^\times.$$

These groups are groups of rational points of linear algebraic groups $\mathbf{O}^+(\mathfrak{A})$, $\mathbf{GO}^+(\mathfrak{A})$ and $\mathbf{PGO}^+(\mathfrak{A})$, which are the connected components of the identity in $\mathbf{O}(\mathfrak{A})$, $\mathbf{GO}(\mathfrak{A})$ and $\mathbf{PGO}(\mathfrak{A})$, see [13, §23.B].

2.3. Clifford groups

Let (V, q) be a quadratic space of even dimension. The multiplicative group of $C_0(V, q)$ acts on $C(V, q)$ by conjugation. The *special Clifford group* $\mathbf{\Gamma}^+(q)$ is defined in [13, p. 349] as the normalizer of the subspace V. Thus, for every commutative F-algebra R, letting $V_R = V \otimes_F R$,

$$\mathbf{\Gamma}^+(q)(R) = \{\xi \in C_0(V, q)_R^\times \mid \xi \cdot V_R \cdot \xi^{-1} = V_R\}.$$

For $\xi \in \mathbf{\Gamma}^+(q)(R)$, the map

$$\operatorname{Int}(\xi)|_{V_R} : V_R \to V_R$$

is a proper isometry. The map carrying ξ to $\operatorname{Int}(\xi)|_{V_R}$ is a morphism of algebraic groups χ known as the *vector representation*, which fits in an exact sequence

$$1 \to \mathbf{G_m} \to \mathbf{\Gamma}^+(q) \xrightarrow{\chi} \mathbf{O}^+(q) \to 1,$$

where $\mathbf{G_m}$ is the multiplicative group, see [13, p. 349].

Mapping $\xi \in \mathbf{\Gamma}^+(q)(R)$ to $\tau_0(\xi)\xi$ defines a morphism

$$\underline{\mu}: \mathbf{\Gamma}^+(q) \to \mathbf{G_m}.$$

Its kernel is the Spin group $\mathbf{Spin}(q)$. It is an algebraic group to which we may restrict the vector representation to obtain the following exact sequence:

$$1 \to \boldsymbol{\mu}_2 \to \mathbf{Spin}(q) \xrightarrow{\chi} \mathbf{O}^+(q) \to 1,$$

where $\boldsymbol{\mu}_2$ is the algebraic group scheme defined by

$$\boldsymbol{\mu}_2(R) = \{\rho \in R \mid \rho^2 = 1\} \quad \text{for every commutative } F\text{-algebra } R.$$

Note that $\boldsymbol{\mu}_2$ is not smooth if $\operatorname{char} F = 2$.

Extended Clifford groups. Let Z be the center of $C_0(V, q)$. Henceforth, we assume $\dim V \equiv 0 \bmod 4$, so the canonical involution τ_0 acts trivially on Z.

Let $\mathbf{Sim}(\tau_0)$ be the group of similitudes of $(C_0(V, q), \tau_0)$, whose rational points over any commutative F-algebra R is

$$\mathbf{Sim}(\tau_0)(R) = \{\xi \in C_0(V, q)_R^\times \mid \tau_0(\xi)\xi \in Z_R^\times\}.$$

The multiplier map $\xi \mapsto \tau_0(\xi)\xi$ is a morphism

$$\underline{\mu}: \mathbf{Sim}(\tau_0) \to R_{Z/F}(\mathbf{G_m}),$$

where $R_{Z/F}(\mathbf{G_m})$ is the corestriction (or Weil's *restriction of scalars*) of the multiplicative group. Mapping $x \in C(V, q)_R$ and $\xi \in \mathbf{Sim}(\tau_0)(R)$ to $\tau_0(\xi)x\xi$ defines an action of $\mathbf{Sim}(\tau_0)$

on $C(V,q)$ (on the right). The *extended Clifford group* $\mathbf{\Omega}(q)$ is defined[3] as the normalizer of V. Thus, for every commutative F-algebra R,

$$\mathbf{\Omega}(q)(R) = \{\xi \in \mathbf{Sim}(\tau_0)(R) \mid \tau_0(\xi)\cdot V_R \cdot \xi = V_R\}.$$

We proceed to show that $\mathbf{\Gamma}^+(q)$ is a subgroup of $\mathbf{\Omega}(q)$ by reformulating the condition that

$$\tau_0(\xi)\cdot V_R\cdot\xi = V_R.$$

Let $\iota\colon Z\to Z$ denote the nontrivial F-automorphism of Z. Note that $xz = \iota(z)x$ for all $x\in V$ and $z\in Z$.

Lemma 2.5. *Let R be a commutative F-algebra. For $\xi\in\mathbf{Sim}(\tau_0)(R)$ and $u\in\mathbf{GL}(V)(R)$ the following are equivalent:*

(a) $\tau_0(\xi)x\xi = \sigma_b(u)(x)$ *for all* $x\in V_R$*;*

(b) $u(y) = \iota(\underline{\mu}(\xi))\xi y\xi^{-1}$ *for all* $y\in V_R$.

When these conditions hold, then $u\in\mathbf{GO}^+(q)(R)$, $C_0(u) = \operatorname{Int}(\xi)$ *and*

$$\mu(u) = N_{Z/F}\big(\underline{\mu}(\xi)\big).$$

Proof. Suppose (a) holds. Squaring each side of the equation yields

$$\tau_0(\xi)x\underline{\mu}(\xi)x\xi = q\big(\sigma_b(u)(x)\big) \quad \text{for all } x\in V_R$$

hence, since $\tau_0(\xi)x\underline{\mu}(\xi)x\xi = \tau_0(\xi)x^2\xi\iota(\underline{\mu}(\xi)) = q(x)N_{Z/F}(\underline{\mu}(\xi))$,

$$q\big(\sigma_b(u)(x)\big) = N_{Z/F}\big(\underline{\mu}(\xi)\big)q(x) \quad \text{for all } x\in V_R.$$

It follows that $\sigma_b(u)\in\mathbf{GO}(q)(R)$ and $\mu(\sigma_b(u)) = N_{Z/F}(\underline{\mu}(\xi))$, hence also $u\in\mathbf{GO}(q)(R)$ and $\mu(u) = N_{Z/F}(\underline{\mu}(\xi))$.

On the other hand, multiplying each side of (a) on the left by ξ and on the right by ξ^{-1} yields

$$\underline{\mu}(\xi)x = \xi\sigma_b(u)(x)\xi^{-1}.$$

Letting $y = \sigma_b(u)(x)$, we have $u(y) = \mu(u)x$. By substituting in the last displayed equation we obtain

$$\mu(u)^{-1}\underline{\mu}(\xi)u(y) = \xi y\xi^{-1}.$$

As $\mu(u) = N_{Z/F}(\underline{\mu}(\xi))$, condition (b) follows.

Now, suppose (b) holds. Squaring each side of the equation yields

$$q\big(u(y)\big) = \iota\big(\underline{\mu}(\xi)\big)\underline{\mu}(\xi)\xi y^2\xi^{-1} = N_{Z/F}\big(\underline{\mu}(\xi)\big)q(y) \quad \text{for all } y\in V_R,$$

[3] For a more general definition covering the case where $\dim V\equiv 2 \bmod 4$, see [13, §13.B].

hence $u \in \mathbf{GO}(q)(R)$ and $\mu(u) = N_{Z/F}(\underline{\mu}(\xi))$. On the other hand, multiplying each side of (b) by $\tau_0(\xi)$ on the left and by ξ on the right yields

$$\tau_0(\xi)u(y)\xi = N_{Z/F}\big(\underline{\mu}(\xi)\big)y = \mu(u)y \quad \text{for all } y \in V_R.$$

Letting $x = u(y)$, we have $\sigma_b(u)(x) = \mu(u)y$, hence by substituting in the last displayed equation we obtain (a).

To complete the proof, we compute $C_0(u)$ using (b). For $x, y \in V_R$, taking into account that $\mu(u) = N_{Z/F}(\underline{\mu}(\xi))$ we find

$$C_0(u)(xy) = \mu(u)^{-1}u(x)u(y) = \mu(u)^{-1}\iota\big(\underline{\mu}(\xi)\big)\xi x\xi^{-1}\iota\big(\underline{\mu}(\xi)\big)\xi y\xi^{-1} = \xi xy\xi^{-1}.$$

Since $\xi \in C_0(V, q)_R$, it follows that $C_0(u)$ restricts to the identity on Z_R, hence u is a proper similitude. ■

For $\xi \in \mathbf{\Omega}(q)(R)$, the map $x \mapsto \tau_0(\xi)x\xi$ is an invertible linear operator on V_R. If $u \in \mathbf{GL}(V)(R)$ is the image of this operator under σ_b, then condition (a) of Lemma 2.5 holds for this u. We write $u = \chi_0(\xi)$, so $\chi_0(\xi) \in \mathbf{GO}^+(q)(R)$ is equivalently defined by any of the two equations

$$\tau_0(\xi)x\xi = \sigma_b\big(\chi_0(\xi)\big)(x) \quad \text{and} \quad \chi_0(\xi)(x) = \iota\big(\underline{\mu}(\xi)\big)\xi x\xi^{-1} \quad \text{for all } x \in V_R. \tag{2.1}$$

The map χ_0 is a morphism

$$\chi_0\colon \mathbf{\Omega}(q) \to \mathbf{GO}^+(q).$$

Lemma 2.5 yields

$$\mathrm{Int}|_{\mathbf{\Omega}(q)} = C_0 \circ \chi_0 \in \mathbf{Aut}\big(C_0(V, q)\big) \quad \text{and} \quad N_{Z/F} \circ \underline{\mu} = \mu \circ \chi_0\colon \mathbf{\Omega}(q) \to \mathbf{G_m}. \tag{2.2}$$

Proposition 2.6. *The special Clifford group* $\mathbf{\Gamma}^+(q)$ *is a subgroup of* $\mathbf{\Omega}(q)$*. More precisely,*

$$\mathbf{\Gamma}^+(q) = \underline{\mu}^{-1}(\mathbf{G_m}) \subset \mathbf{\Omega}(q).$$

Moreover, $\chi_0|_{\mathbf{\Gamma}^+(q)} = \underline{\mu} \cdot \chi\colon \mathbf{\Gamma}^+(q) \to \mathbf{O}^+(q)$*, hence* χ_0 *and* χ *coincide on* $\mathbf{Spin}(q)$*.*

Proof. As pointed out in the definition of $\mathbf{Spin}(q)$ above, for every commutative F-algebra R the multiplier $\underline{\mu}(\xi)$ of any $\xi \in \mathbf{\Gamma}^+(q)(R)$ lies in $R^\times$. Therefore, Lemma 2.5 shows that $\xi V_R\xi^{-1} = V_R$ implies $\tau_0(\xi)V_R\xi = V_R$, hence $\mathbf{\Gamma}^+(q)(R) \subset \mathbf{\Omega}(q)(R)$. Conversely, if $\xi \in \mathbf{\Omega}(q)(R)$ and $\underline{\mu}(\xi) \in R^\times$, Lemma 2.5 shows that $\tau_0(\xi)V_R\xi = V_R$ implies $\xi V_R\xi^{-1} = V_R$, hence $\xi \in \mathbf{\Gamma}^+(q)(R)$. Therefore $\mathbf{\Gamma}^+(q)(R)$ is the subgroup of elements in $\mathbf{\Omega}(q)(R)$ whose multiplier lies in $R^\times$.

Moreover, for $\xi \in \mathbf{\Gamma}^+(q)(R)$ we have $\chi(\xi)(x) = \xi x\xi^{-1}$ and $\chi_0(\xi)(x) = \underline{\mu}(\xi)\xi x\xi^{-1}$ for all $x \in V_R$, hence

$$\chi_0(\xi) = \underline{\mu}(\xi)\chi(\xi).$$

■

Twisted forms. Twisted forms of $\mathbf{\Gamma}^+(q)$ and $\mathbf{\Omega}(q)$ are defined in [13, §13.B and §23.B] by using a Clifford bimodule $B(\mathfrak{A})$ associated to any central simple algebra of even degree with quadratic pair $\mathfrak{A} = (A, \sigma, \mathfrak{f})$. This bimodule is defined in [13, (9.5)] in such a way that for every even-dimensional quadratic space (V, q) the standard identification $\operatorname{End} V = V \otimes V$ yields

$$B(\operatorname{End} V, \sigma_b, \mathfrak{f}_q) = V \otimes C_1(V, q),$$

where $C_1(V, q)$ is the odd part of the Clifford algebra $C(V, q)$. The left action $*$ and the right action $\cdot$ of $C(\mathfrak{A})$ on $B(\mathfrak{A})$ are given in the split case by

$$\xi * (x \otimes \eta) = x \otimes (\xi\eta) \quad \text{and} \quad (x \otimes \eta) \cdot \xi = x \otimes (\eta\xi)$$

for $\xi \in C_0(V, q)$, $\eta \in C_1(V, q)$ and $x \in V$. The bimodule $B(\mathfrak{A})$ also carries a left A-module structure and a canonical left A-module homomorphism $b\colon A \to B(\mathfrak{A})$ (for which we use the exponential notation) given in the split case by

$$a(x \otimes \eta) = a(x) \otimes \eta \quad \text{and} \quad (x \otimes y)^b = x \otimes y \in V \otimes C_1(V, q)$$

for $a \in \operatorname{End} V$, x, $y \in V$ and $\eta \in C_1(V, q)$.

The multiplicative group of $C(\mathfrak{A})$ acts on $B(\mathfrak{A})$ on the right as follows: $\eta \mapsto \xi^{-1} * \eta \cdot \xi$ for $\xi \in C(\mathfrak{A})^\times$ and $\eta \in B(\mathfrak{A})$. The *Clifford group* $\mathbf{\Gamma}(\mathfrak{A})$ is the normalizer of the subspace $A^b \subset B(\mathfrak{A})$, hence for every commutative F-algebra R

$$\mathbf{\Gamma}(\mathfrak{A})(R) = \bigl\{\xi \in C(\mathfrak{A})_R^\times \mid \xi^{-1} * A_R^b \cdot \xi = A_R^b\bigr\}.$$

On the same model, when $\deg A \equiv 0 \bmod 4$, we define[4] the *extended Clifford group* $\mathbf{\Omega}(\mathfrak{A})$ as the normalizer of A^b under the action on $B(\mathfrak{A})$ of the group of similitudes of the canonical involution $\underline{\sigma}$ by

$$\xi \mapsto (\eta \mapsto \underline{\sigma}(\xi) * \eta \cdot \xi).$$

Thus, letting Z denote the center of $C(\mathfrak{A})$,

$$\mathbf{\Omega}(\mathfrak{A})(R) = \bigl\{\xi \in C(\mathfrak{A})_R^\times \mid \underline{\sigma}(\xi)\xi \in Z_R^\times \text{ and } \underline{\sigma}(\xi) * A^b \cdot \xi = A^b\bigr\}$$

for every commutative F-algebra R. Let $\underline{\mu}$ denote the multiplier map

$$\underline{\mu}\colon \mathbf{\Omega}(\mathfrak{A}) \to R_{Z/F}(\mathbf{G_m}), \quad \xi \mapsto \underline{\sigma}(\xi)\xi$$

and define morphisms

$$\chi\colon \mathbf{\Gamma}(\mathfrak{A}) \to \mathbf{O}^+(\mathfrak{A}) \quad \text{and} \quad \chi_0\colon \mathbf{\Omega}(\mathfrak{A}) \to \mathbf{GO}^+(\mathfrak{A})$$

by

$$\xi^{-1} * 1^b \cdot \xi = \chi(\xi)^b \quad \text{and} \quad \underline{\sigma}(\xi) * 1^b \cdot \xi = \chi_0(\xi)^b,$$

[4] An alternative definition, which also covers the case where $\deg A \equiv 2 \bmod 4$, is given in [13, §23.B].

see [13, (13.11) and (13.29)]. In the split case where $(\sigma, \mathfrak{f}) = (\sigma_b, \mathfrak{f}_q)$ for some quadratic space (V, q), the standard identification yields

$$\mathbf{\Gamma}(\operatorname{End} V, \sigma_b, \mathfrak{f}_q) = \mathbf{\Gamma}^+(q) \quad \text{and} \quad \mathbf{\Omega}(\operatorname{End} V, \sigma_b, \mathfrak{f}_q) = \mathbf{\Omega}(q),$$

and the maps χ and χ_0 are identical respectively to the vector representation and to the map χ_0 defined in (2.1). We next show that they satisfy analogues of (2.2) and Proposition 2.6.

Proposition 2.7. *Let $\mathfrak{A} = (A, \sigma, \mathfrak{f})$ be an F-algebra with quadratic pair of degree divisible by 4. The Clifford group $\mathbf{\Gamma}(\mathfrak{A})$ is a subgroup of $\mathbf{\Omega}(\mathfrak{A})$. More precisely,*

$$\mathbf{\Gamma}(\mathfrak{A}) = \underline{\mu}^{-1}(\mathbf{G_m}) \subset \mathbf{\Omega}(\mathfrak{A}).$$

Moreover, $R_{Z/F}(\mathbf{G_m}) \subset \mathbf{\Omega}(\mathfrak{A})$ and $\chi_0|_{R_{Z/F}(\mathbf{G_m})} = N_{Z/F} \colon R_{Z/F}(\mathbf{G_m}) \to \mathbf{G_m}$,

$$\operatorname{Int}|_{\mathbf{\Omega}(\mathfrak{A})} = C \circ \operatorname{Int} \circ \chi_0 \in \mathbf{Aut}\big(C(\mathfrak{A})\big) \quad \text{and} \quad N_{Z/F} \circ \underline{\mu} = \mu \circ \chi_0 \colon \mathbf{\Omega}(\mathfrak{A}) \to \mathbf{G_m},$$

and

$$\chi_0|_{\mathbf{\Gamma}(\mathfrak{A})} = \underline{\mu} \cdot \chi \colon \mathbf{\Gamma}(\mathfrak{A}) \to \mathbf{O}^+(\mathfrak{A}).$$

Proof. The first part is proved in [13, (13.25)]. (Alternatively, it follows from Proposition 2.6 by Galois descent from a Galois splitting field of A.)

Let R be a commutative F-algebra. For $z \in Z_R^\times$ we have $\underline{\sigma}(z) = z$ and $z * 1^b = 1^b \cdot \iota(z)$, hence $z \in \mathbf{\Omega}(\mathfrak{A})(R)$ with $\chi_0(z) = N_{Z/F}(z)$. The rest follows from Proposition 2.6 by scalar extension to a splitting field of A. ∎

Define $\chi' \colon \mathbf{\Omega}(\mathfrak{A}) \to \mathbf{PGO}^+(\mathfrak{A})$ by composing χ_0 with the canonical map $\mathbf{GO}^+(\mathfrak{A}) \to \mathbf{PGO}^+(\mathfrak{A})$. Recall from [13, p. 352] the following commutative diagram with exact rows, whose vertical maps are canonical:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{G_m} & \longrightarrow & \mathbf{\Gamma}(\mathfrak{A}) & \xrightarrow{\chi} & \mathbf{O}^+(\mathfrak{A}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & R_{Z/F}(\mathbf{G_m}) & \longrightarrow & \mathbf{\Omega}(\mathfrak{A}) & \xrightarrow{\chi'} & \mathbf{PGO}^+(\mathfrak{A}) & \longrightarrow & 1 \end{array} \tag{2.3}$$

The exact rows of this diagram show that $\mathbf{\Gamma}(\mathfrak{A})$ and $\mathbf{\Omega}(\mathfrak{A})$ are connected, since $\mathbf{G_m}$, $\mathbf{O}^+(\mathfrak{A})$, $R_{Z/F}(\mathbf{G_m})$ and $\mathbf{PGO}^+(\mathfrak{A})$ are connected.

In the next proposition, we write $R^1_{Z/F}(\mathbf{G_m})$ for the kernel of the norm map

$$N_{Z/F} \colon R_{Z/F}(\mathbf{G_m}) \to \mathbf{G_m}.$$

Proposition 2.8. *Let $\mathfrak{A} = (A, \sigma, \mathfrak{f})$ be an algebra with quadratic pair of degree divisible by 4. The following sequence is exact:*

$$1 \to R^1_{Z/F}(\mathbf{G_m}) \to \mathbf{\Omega}(\mathfrak{A}) \xrightarrow{\chi_0} \mathbf{GO}^+(\mathfrak{A}) \to 1.$$

Proof. Since $\ker\chi_0 \subset \ker\chi'$, it follows from the exactness of the lower row in (2.3) that $\ker\chi_0 \subset R_{Z/F}(\mathbf{G_m})$. Moreover, the following diagram is commutative with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & R_{Z/F}(\mathbf{G_m}) & \longrightarrow & \mathbf{\Omega}(\mathfrak{A}) & \xrightarrow{\chi'} & \mathbf{PGO}^+(\mathfrak{A}) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle N_{Z/F}} & & \downarrow{\scriptstyle \chi_0} & & \| & & \\ 1 & \longrightarrow & \mathbf{G_m} & \longrightarrow & \mathbf{GO}^+(\mathfrak{A}) & \longrightarrow & \mathbf{PGO}^+(\mathfrak{A}) & \longrightarrow & 1 \end{array}$$

Since we already know that $\ker\chi_0 \subset R_{Z/F}(\mathbf{G_m})$, it follows that $\ker\chi_0 = R^1_{Z/F}(\mathbf{G_m})$.

As $\mathbf{GO}^+(\mathfrak{A})$ is smooth, to prove that χ_0 is onto it suffices by [13, (22.3)] to see that χ_0 defines a surjective map on the group of rational points over an algebraic closure. This is clear from the last commutative diagram above, because the norm $N_{Z/F}$ is surjective when F is algebraically closed. ■

As in the split case, we define the Spin group

$$\mathbf{Spin}(\mathfrak{A}) = \ker(\underline{\mu}\colon \mathbf{\Gamma}(\mathfrak{A}) \to \mathbf{G_m}) = \ker\big(\underline{\mu}\colon \mathbf{\Omega}(\mathfrak{A}) \to R_{Z/F}(\mathbf{G_m})\big)$$

and we have an exact sequence (see [13, p. 352]):

$$1 \to \boldsymbol{\mu}_2 \to \mathbf{Spin}(\mathfrak{A}) \xrightarrow{\chi} \mathbf{O}^+(\mathfrak{A}) \to 1.$$

We may also restrict the map χ' to $\mathbf{Spin}(\mathfrak{A})$ to obtain a morphism

$$\chi'\colon \mathbf{Spin}(\mathfrak{A}) \to \mathbf{PGO}^+(\mathfrak{A}).$$

This morphism is surjective since the vector representation χ is surjective and the canonical map $\mathbf{O}^+(\mathfrak{A}) \to \mathbf{PGO}^+(\mathfrak{A})$ is surjective. Its kernel is

$$R_{Z/F}(\mathbf{G_m}) \cap \mathbf{Spin}(\mathfrak{A}) = R_{Z/F}(\boldsymbol{\mu}_2),$$

hence the following sequence is exact:

$$1 \to R_{Z/F}(\boldsymbol{\mu}_2) \to \mathbf{Spin}(\mathfrak{A}) \xrightarrow{\chi'} \mathbf{PGO}^+(\mathfrak{A}) \to 1. \tag{2.4}$$

The last proposition refers to the canonical quadratic pair $(\underline{\sigma}, \underline{\mathfrak{f}})$ on $C(\mathfrak{A})$ defined by Dolphin–Quéguiner-Mathieu (see Section 2.2). Assuming $\deg A \equiv 0 \bmod 8$, we write $\mathfrak{C}(\mathfrak{A})$ for the Clifford algebra of $\mathfrak{A}$ with its canonical quadratic pair:

$$\mathfrak{C}(\mathfrak{A}) = (C(\mathfrak{A}), \underline{\sigma}, \underline{\mathfrak{f}}).$$

Proposition 2.9. *Let $\mathfrak{A} = (A, \sigma, \mathfrak{f})$ be an algebra with quadratic pair. If* $\deg A \equiv 0 \bmod 8$, *then* $\mathbf{\Omega}(\mathfrak{A}) \subset \mathbf{GO}^+(\mathfrak{C}(\mathfrak{A}))$.

Proof. Let R be a commutative F-algebra and let $\xi \in \mathbf{\Omega}(\mathfrak{A})(R)$. Since $\chi_0(\xi) \in \mathbf{GO}(\mathfrak{A})(R)$, it follows that $\operatorname{Int}(\chi_0(\xi))$ is an automorphism of $\mathfrak{A}_R$, hence $C(\operatorname{Int}(\chi_0(\xi)))$ is an automorphism of $\mathfrak{C}(\mathfrak{A})_R$. But Proposition 2.7 shows that $C(\operatorname{Int}(\chi_0(\xi))) = \operatorname{Int}(\xi)$, hence $\xi \in \mathbf{GO}(\mathfrak{C}(\mathfrak{A}))(R)$. We thus see that $\mathbf{\Omega}(\mathfrak{A}) \subset \mathbf{GO}(\mathfrak{C}(\mathfrak{A}))$. Since $\mathbf{\Omega}(\mathfrak{A})$ is connected, it actually lies in the connected component $\mathbf{GO}^+(\mathfrak{C}(\mathfrak{A}))$. ■

2.4. Lie algebras of orthogonal groups

Throughout this subsection, A is a central simple algebra of even degree $n = 2m$ over an arbitrary field F, and $(\sigma, \mathfrak{f})$ is a quadratic pair on A. We discuss several Lie algebras related to the algebra with quadratic pair $\mathfrak{A} = (A, \sigma, \mathfrak{f})$, and obtain different results depending on whether the characteristic is 2 or not. The discrepancies derive from the observation that the Lie algebra of the algebraic group scheme $\boldsymbol{\mu}_2$ is F when char $F = 2$, whereas it vanishes when char $F \neq 2$.

The bracket $[a, b] = ab - ba$ turns A into a Lie algebra denoted by $\mathfrak{L}(A)$. As usual, for $a \in A$ we let $\mathrm{ad}_a \colon A \to A$ denote the linear operator defined by

$$\mathrm{ad}_a(x) = [a, x] \quad \text{for } x \in A.$$

The following are subalgebras of $\mathfrak{L}(A)$ associated with the quadratic pair $(\sigma, \mathfrak{f})$; they are the Lie algebras of the algebraic group schemes $\mathbf{O}(\mathfrak{A})$ and $\mathbf{GO}(\mathfrak{A})$ respectively, see [13, §23.B]:

$$\mathfrak{o}(\mathfrak{A}) = \mathrm{Alt}(\sigma) = \{a - \sigma(a) \mid a \in A\}$$
$$\mathfrak{go}(\mathfrak{A}) = \big\{g \in A \mid \sigma(g) + g \in F \text{ and } \mathfrak{f}([g, s]) = 0 \text{ for all } s \in \mathrm{Sym}(\sigma)\big\}.$$

Note that $\mathfrak{o}(\mathfrak{A})$ depends only on σ and not on $\mathfrak{f}$. Clearly, $F \subset \mathfrak{go}(\mathfrak{A})$. We let

$$\mathfrak{pgo}(\mathfrak{A}) = \mathfrak{go}(\mathfrak{A})/F$$

and define

$$\dot{\mu} \colon \mathfrak{go}(\mathfrak{A}) \to F \quad \text{by } \dot{\mu}(g) = \sigma(g) + g.$$

This map is the differential of the multiplier morphism $\mu \colon \mathbf{GO}(\mathfrak{A}) \to \mathbf{G_m}$, hence it is a Lie algebra homomorphism.

Proposition 2.10. *Let $\ell \in A$ be such that $\mathfrak{f}(s) = \mathrm{Trd}_A(\ell s)$ for all $s \in \mathrm{Sym}(\sigma)$. Then*

$$\begin{aligned}\mathfrak{go}(\mathfrak{A}) &= \big\{g \in A \mid \mathrm{ad}_g \circ \sigma = \sigma \circ \mathrm{ad}_g \text{ and } (\mathfrak{f} \circ \mathrm{ad}_g)(s) = 0 \text{ for all } s \in \mathrm{Sym}(\sigma)\big\} \\ &= \big\{g \in A \mid \mathrm{Trd}_A(gs) = (\sigma(g) + g)\,\mathfrak{f}(s) \text{ for all } s \in \mathrm{Sym}(\sigma)\big\} \\ &= \mathfrak{o}(\mathfrak{A}) + \ell F \end{aligned} \tag{2.5}$$

and the following sequence is exact:

$$0 \to \mathfrak{o}(\mathfrak{A}) \to \mathfrak{go}(\mathfrak{A}) \xrightarrow{\dot{\mu}} F \to 0. \tag{2.6}$$

Moreover,

$$\dim \mathfrak{o}(\mathfrak{A}) = \dim \mathfrak{pgo}(\mathfrak{A}) = m(2m-1) \quad \text{and} \quad \dim \mathfrak{go}(\mathfrak{A}) = m(2m-1) + 1.$$

If char $F \neq 2$*, the inclusion $\mathfrak{o}(\mathfrak{A}) \hookrightarrow \mathfrak{go}(\mathfrak{A})$ is split by the map $\frac{1}{2}(\mathrm{Id} - \sigma) \colon \mathfrak{go}(\mathfrak{A}) \to \mathfrak{o}(\mathfrak{A})$, and it induces a canonical isomorphism*

$$\mathfrak{o}(\mathfrak{A}) \xrightarrow{\sim} \mathfrak{pgo}(\mathfrak{A}).$$

If char $F = 2$*, the map* $\dot{\mu}$ *induces a map* $\mathfrak{pgo}(\mathfrak{A}) \to F$ *for which we also use the notation* $\dot{\mu}$*, and the map* $\mathfrak{o}(\mathfrak{A}) \to \mathfrak{pgo}(\mathfrak{A})$ *induced by the inclusion* $\mathfrak{o}(\mathfrak{A}) \hookrightarrow \mathfrak{go}(\mathfrak{A})$ *fits into an exact sequence*

$$0 \to F \to \mathfrak{o}(\mathfrak{A}) \to \mathfrak{pgo}(\mathfrak{A}) \xrightarrow{\dot{\mu}} F \to 0.$$

Proof. For $g, x \in A$,

$$(\mathrm{ad}_g \circ \sigma - \sigma \circ \mathrm{ad}_g)(x) = \big[g, \sigma(x)\big] - \sigma\big([g,x]\big) = \big[g + \sigma(g), \sigma(x)\big].$$

Therefore, $\mathrm{ad}_g \circ\sigma = \sigma \circ \mathrm{ad}_g$ if and only if $g + \sigma(g) \in F$, and the definition of $\mathfrak{go}(\mathfrak{A})$ readily yields

$$\mathfrak{go}(\mathfrak{A}) = \big\{g \in A \mid \mathrm{ad}_g \circ\sigma = \sigma \circ \mathrm{ad}_g \text{ and } (\mathfrak{f} \circ \mathrm{ad}_g)(s) = 0 \text{ for all } s \in \mathrm{Sym}(\sigma)\big\}.$$

Now, suppose $g \in A$ satisfies $\sigma(g) + g \in F$, and let $\mu = \sigma(g) + g$. For $s \in \mathrm{Sym}(\sigma)$ we have

$$\begin{aligned}\mathrm{Trd}_A(gs) &= \mathfrak{f}\big(gs + \sigma(gs)\big) = \mathfrak{f}\big(gs + s\sigma(g)\big)\\ &= \mathfrak{f}\big(gs + s(\mu - g)\big) = \mathfrak{f}([g,s]) + \mu\, \mathfrak{f}(s).\end{aligned} \tag{2.7}$$

Therefore, $\mathrm{Trd}_A(gs) = \mu\, \mathfrak{f}(s)$ for $g \in \mathfrak{go}(\mathfrak{A})$ and $s \in \mathrm{Sym}(\sigma)$, hence

$$\mathfrak{go}(\mathfrak{A}) \subset \big\{g \in A \mid \mathrm{Trd}_A(gs) = (\sigma(g) + g)\, \mathfrak{f}(s) \text{ for all } s \in \mathrm{Sym}(\sigma)\big\}.$$

To prove the reverse inclusion, suppose $g \in A$ satisfies $\mathrm{Trd}_A(gs) = (\sigma(g) + g)\, \mathfrak{f}(s)$ for all $s \in \mathrm{Sym}(\sigma)$. We first show that $\sigma(g) + g \in F$. If $x \in A$ is such that $\mathrm{Trd}_A(x) = 1$, then

$$\mathfrak{f}(\sigma(x) + x) = 1,$$

hence the hypothesis on g yields $\mathrm{Trd}_A(g(\sigma(x) + x)) = \sigma(g) + g$, which shows that $\sigma(g) + g \in F$. Letting $\mu = \sigma(g) + g$, we have by (2.7) above $\mathrm{Trd}_A(gs) = \mathfrak{f}([g,s]) + \mu\, \mathfrak{f}(s)$ for all $s \in \mathrm{Sym}(\sigma)$. On the other hand, $\mathrm{Trd}_A(gs) = \mu\, \mathfrak{f}(s)$ by the hypothesis on g, hence $\mathfrak{f}([g,s]) = 0$, proving $g \in \mathfrak{go}(\mathfrak{A})$.

The first two equations in (2.5) are thus proved. The second one shows that $\ell \in \mathfrak{go}(\mathfrak{A})$ since $\mathrm{Trd}_A(\ell s) = \mathfrak{f}(s)$ for all $s \in \mathrm{Sym}(\sigma)$ and $\sigma(\ell) + \ell = 1$. This last equation also reads $\dot{\mu}(\ell) = 1$, hence the map $\dot{\mu}\colon \mathfrak{go}(\mathfrak{A}) \to F$ is onto. The second characterization of $\mathfrak{go}(\mathfrak{A})$ in (2.5) also shows that

$$\ker(\dot{\mu}\colon \mathfrak{go}(\mathfrak{A}) \to F) = \big\{g \in A \mid \mathrm{Trd}_A(gs) = 0 \text{ for all } s \in \mathrm{Sym}(\sigma)\big\},$$

which means that $\ker(\dot{\mu})$ is the orthogonal complement of $\mathrm{Sym}(\sigma)$ for the bilinear form $\mathrm{Trd}_A(XY)$. This orthogonal complement is known to be $\mathrm{Alt}(\sigma)$ by [13, (2.3)]. As $\mathfrak{o}(\mathfrak{A}) = \mathrm{Alt}(\sigma)$, it follows that $\mathfrak{o}(\mathfrak{A}) \subset \mathfrak{go}(\mathfrak{A})$ and the sequence (2.6) is exact.

From the above observations it follows that

$$\mathfrak{o}(\mathfrak{A}) + \ell F \subset \mathfrak{go}(\mathfrak{A}).$$

We use dimension count to show that this inclusion is an equality, completing the proof of (2.5). Note that $\ell \notin \mathfrak{o}(\mathfrak{A})$ since $\mathfrak{o}(\mathfrak{A}) = \ker(\dot\mu)$ whereas $\dot\mu(\ell) = 1$. Therefore,

$$\dim\big(\mathfrak{o}(\mathfrak{A}) + \ell F\big) = 1 + \dim \mathfrak{o}(\mathfrak{A}).$$

On the other hand, the exact sequence (2.6) yields $\dim \mathfrak{go}(\mathfrak{A}) = 1 + \dim \mathfrak{o}(\mathfrak{A})$, hence the proof of (2.5) is complete. Since $\dim \operatorname{Alt}(\sigma) = m(2m-1)$ by [13, (2.6)], we obtain

$$\dim \mathfrak{o}(\mathfrak{A}) = m(2m-1) \quad \text{and} \quad \dim \mathfrak{go}(\mathfrak{A}) = m(2m-1) + 1.$$

It follows that $\dim \mathfrak{pgo}(\mathfrak{A}) = m(2m-1)$ because $\mathfrak{pgo}(\mathfrak{A}) = \mathfrak{go}(\mathfrak{A})/F$.

If $\operatorname{char} F \neq 2$, then we may take $\ell = \frac{1}{2}$ in the discussion above, so $\mathfrak{go}(\mathfrak{A}) = \mathfrak{o}(\mathfrak{A}) \oplus F$ and $\mathfrak{pgo}(\mathfrak{A}) \simeq \mathfrak{o}(\mathfrak{A})$ canonically.

If $\operatorname{char} F = 2$, then $F \subset \operatorname{Alt}(\sigma)$ because the involution σ is symplectic, and the map $\dot\mu\colon \mathfrak{go}(\mathfrak{A}) \to F$ vanishes on F. Therefore, $\dot\mu$ induces a map $\mathfrak{pgo}(\mathfrak{A}) \to F$ whose kernel is the image of $\mathfrak{o}(\mathfrak{A})$. ∎

When the algebra A is split, we may represent it as $A = \operatorname{End} V$ for some F-vector space V of dimension n. The quadratic pair $(\sigma, \mathfrak{f})$ is then the quadratic pair $(\sigma_b, \mathfrak{f}_q)$ adjoint to a nonsingular quadratic form q on V (see Section 2.1), and we write simply $\mathfrak{go}(q)$ for $\mathfrak{go}(\operatorname{End} V, \sigma_b, \mathfrak{f}_q)$.

Proposition 2.11. *Let $g \in \operatorname{End} V$ and $\mu \in F$. We have $g \in \mathfrak{go}(q)$ and $\dot\mu(g) = \mu$ if and only if*

$$b(g(u), u) = \mu q(u) \quad \textit{for all } u \in V. \tag{2.8}$$

Proof. We use the standard identification $V \otimes V = \operatorname{End} V$ set up in Section 2.1. For $s = u \otimes u \in \operatorname{Sym}(\sigma_b)$ we have $gs = g(u) \otimes u$, hence $\operatorname{Trd}(gs) = b(g(u), u)$. On the other hand $\mathfrak{f}_q(s) = q(u)$, hence if $g \in \mathfrak{go}(q)$ and $\dot\mu(g) = \mu$ then the second characterization of $\mathfrak{go}(\operatorname{End} V, \sigma_b, \mathfrak{f}_q)$ in (2.5) shows that (2.8) holds.

Conversely, if (2.8) holds, then $\operatorname{Trd}(gs) = \mu\, \mathfrak{f}_q(s)$ for all $s \in \operatorname{Sym}(\sigma_b)$ of the form $s = u \otimes u$ with $u \in V$. Applying this to $s = (u+v) \otimes (u+v)$ with $u, v \in V$ yields

$$\operatorname{Trd}\big(g(u \otimes v + v \otimes u)\big) = \mu\, \mathfrak{f}_q(u \otimes v + v \otimes u) = \mu \operatorname{Trd}(u \otimes v),$$

hence $b(g(u), v) + b(u, g(v)) = \mu b(u, v)$. Since σ_b is the adjoint involution of b, it follows that $\sigma_b(g) + g = \mu$. We thus see that $\operatorname{Trd}(gs) = (\sigma_b(g) + g)\, \mathfrak{f}_q(s)$ for all $s \in \operatorname{Sym}(\sigma_b)$, which proves $g \in \mathfrak{go}(q)$ by the second characterization of $\mathfrak{go}(\operatorname{End} V, \sigma_b, \mathfrak{f}_q)$ in (2.5). ∎

Returning to the general case, where the algebra A is not necessarily split, let $C(\mathfrak{A})$ denote the Clifford algebra of $\mathfrak{A} = (A, \sigma, \mathfrak{f})$, and write $c\colon A \to C(\mathfrak{A})$ for the canonical map. Every $g \in \mathfrak{go}(\mathfrak{A})$ defines a derivation δ_g of $C(\mathfrak{A})$ such that

$$\delta_g\big(c(a)\big) = c\big([g, a]\big) \quad \text{for } a \in A; \tag{2.9}$$

this can be checked directly from the definition of $C(\mathfrak{A})$ or by viewing the map $g \mapsto \delta_g$ as the differential of the morphism $\mathbf{GO}(\mathfrak{A}) \to \mathbf{Aut}(C(\mathfrak{A}))$ defined on rational points by mapping $g \in \mathrm{GO}(\mathfrak{A})$ to $C(\mathrm{Int}(g))$. The derivation δ_g is uniquely determined by (2.9), because $c(A)$ generates A as an associative algebra.

Recall from [13, §8.C] that $c(A)$ is a Lie subalgebra of $\mathfrak{L}(C(\mathfrak{A}))$. By [13, p. 351], $c(A)$ is the Lie algebra of the algebraic group $\mathbf{\Gamma}(\mathfrak{A})$, whose group of rational points is the Clifford group $\Gamma(\mathfrak{A})$, hence we call it the *Clifford Lie algebra* of $\mathfrak{A}$ and write

$$\boldsymbol{\gamma}(\mathfrak{A}) = c(A) \subset \mathfrak{L}\big(C(\mathfrak{A})\big).$$

The kernel of the map $c\colon A \to \boldsymbol{\gamma}(\mathfrak{A})$ is $\ker(\mathfrak{f}) \subset \mathrm{Sym}(\sigma)$ by [13, (8.14)], hence

$$\dim \boldsymbol{\gamma}(\mathfrak{A}) = m(2m-1) + 1.$$

Let $\underline{\sigma}$ be the canonical involution on $C(\mathfrak{A})$, which is characterized by the condition that $\underline{\sigma}(c(a)) = c(\sigma(a))$ for $a \in A$. We have

$$\underline{\sigma}\big(c(a)\big) + c(a) = c\big(\sigma(a) + a\big) = \mathfrak{f}\,\big(\sigma(a) + a\big) = \mathrm{Trd}_A(a),$$

hence $\underline{\sigma}(\xi) + \xi \in F$ for $\xi \in \boldsymbol{\gamma}(\mathfrak{A})$, and we may define a Lie algebra homomorphism

$$\underline{\dot{\mu}}\colon \boldsymbol{\gamma}(\mathfrak{A}) \to F \quad \text{by } \underline{\dot{\mu}}(\xi) = \underline{\sigma}(\xi) + \xi,$$

so $\underline{\dot{\mu}}(c(a)) = \mathrm{Trd}_A(a)$ for $a \in A$. We let $\mathfrak{spin}(\mathfrak{A})$ denote the kernel

$$\mathfrak{spin}(\mathfrak{A}) = \ker \underline{\dot{\mu}} = \big\{c(a) \mid \mathrm{Trd}_A(a) = 0\big\} \subset \boldsymbol{\gamma}(\mathfrak{A}),$$

which is the Lie algebra of the algebraic group $\mathbf{Spin}(\mathfrak{A})$ defined in Section 2.3. By definition of $\mathfrak{spin}(\mathfrak{A})$, the following sequence is exact:

$$0 \to \mathfrak{spin}(\mathfrak{A}) \to \boldsymbol{\gamma}(\mathfrak{A}) \xrightarrow{\underline{\dot{\mu}}} F \to 0,$$

and therefore

$$\dim \mathfrak{spin}(\mathfrak{A}) = m(2m-1). \tag{2.10}$$

Recall from [13, (8.15)] the Lie homomorphism

$$\dot{\chi}\colon \boldsymbol{\gamma}(\mathfrak{A}) \to \mathfrak{o}(\mathfrak{A}), \quad c(a) \mapsto a - \sigma(a) \text{ for } a \in A,$$

which fits in the following exact sequence

$$0 \to F \to \boldsymbol{\gamma}(\mathfrak{A}) \xrightarrow{\dot{\chi}} \mathfrak{o}(\mathfrak{A}) \to 0. \tag{2.11}$$

That sequence is the Lie algebra version of the following exact sequence of algebraic groups from [13, p. 352]:

$$1 \to \mathbf{G_m} \to \mathbf{\Gamma}(\mathfrak{A}) \xrightarrow{\chi} \mathbf{O}^+(\mathfrak{A}) \to 1.$$

We let

$$\mathfrak{so}(\mathfrak{A}) = \dot{\chi}\big(\mathfrak{spin}(\mathfrak{A})\big) = \big\{a - \sigma(a) \mid \mathrm{Trd}_A(a) = 0\big\} \subset \mathfrak{o}(\mathfrak{A}).$$

If char $F \neq 2$, then $\mathfrak{o}(\mathfrak{A}) = \mathrm{Skew}(\sigma)$, hence every $a \in \mathfrak{o}(\mathfrak{A})$ satisfies $\mathrm{Trd}_A(a) = 0$ and $a = \frac{1}{2}a - \sigma(\frac{1}{2}a)$, hence

$$\mathfrak{so}(\mathfrak{A}) = \mathfrak{o}(\mathfrak{A}).$$

Moreover, in $\boldsymbol{\gamma}(\mathfrak{A})$ we have $F \cap \mathfrak{spin}(\mathfrak{A}) = 0$ because $\dot{\mu}(\lambda) = 2\lambda$ for $\lambda \in F$, hence the restriction of $\dot{\chi}$ is an isomorphism

$$\dot{\chi}\colon \mathfrak{spin}(\mathfrak{A}) \xrightarrow{\sim} \mathfrak{o}(\mathfrak{A}). \tag{2.12}$$

By contrast, if char $F = 2$ we may define a map

$$\mathrm{Trp}\colon \mathfrak{o}(\mathfrak{A}) \to F \quad \text{by } \mathrm{Trp}\big(a - \sigma(a)\big) = \mathrm{Trd}_A(a),$$

because $\mathrm{Trd}_A(\mathrm{Sym}(\sigma)) = 0$. (The map Trp is the *pfaffian trace*, see [13, (2.13)].) For a, $b \in A$ we have

$$\big[a - \sigma(a), b - \sigma(b)\big] = \big[a - \sigma(a), b\big] - \sigma\big(\big[a - \sigma(a), b\big]\big),$$

hence

$$\mathrm{Trp}\left(\big[a - \sigma(a), b - \sigma(b)\big]\right) = \mathrm{Trd}_A\left(\big[a - \sigma(a), b\big]\right) = 0.$$

Therefore, Trp is a Lie algebra homomorphism. Note also that $F \subset \mathfrak{spin}(\mathfrak{A})$ because $\dot{\mu}(\lambda) = 2\lambda = 0$ for $\lambda \in F$. Therefore, there is a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & F & = & F & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathfrak{spin}(\mathfrak{A}) & \longrightarrow & \boldsymbol{\gamma}(\mathfrak{A}) & \xrightarrow{\dot{\mu}} & F & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \dot{\chi}} & & \downarrow{\scriptstyle \dot{\chi}} & & \| & & \\
0 & \longrightarrow & \mathfrak{so}(\mathfrak{A}) & \longrightarrow & \mathfrak{o}(\mathfrak{A}) & \xrightarrow{\mathrm{Trp}} & F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

2.5. Extended Clifford Lie algebras

Throughout this subsection A is a central simple algebra of degree $n = 2m$ over an arbitrary field F, and we assume m is even. Let $(\sigma, \mathfrak{f})$ be a quadratic pair on A, and let $\mathfrak{A} = (A, \sigma, \mathfrak{f})$. Recall from Section 2.3 the Clifford bimodule $B(\mathfrak{A})$ with its canonical left

A-module homomorphism $b\colon A \to B(\mathfrak{A})$. We write Z for the center of $C(\mathfrak{A})$ and ι for the nontrivial F-automorphism of Z.

Since the left A-module action on $B(\mathfrak{A})$ commutes with the left and right $C(\mathfrak{A})$-module actions, the condition $\underline{\sigma}(\xi) * A_R^b \cdot \xi = A_R^b$ in the definition of the extended Clifford group $\mathbf{\Omega}(\mathfrak{A})$ is equivalent to $\underline{\sigma}(\xi) * 1^b \cdot \xi \in A_R^b$. The Lie algebra of $\mathbf{\Omega}(\mathfrak{A})$ is therefore as follows:

Definition 2.12. The *extended Clifford Lie algebra of* $\mathfrak{A}$ is

$$\boldsymbol{\omega}(\mathfrak{A}) = \big\{\xi \in C(\mathfrak{A}) \mid \underline{\sigma}(\xi) + \xi \in Z \text{ and } \underline{\sigma}(\xi) * 1^b + 1^b \cdot \xi \in A^b\big\}.$$

It is shown in [13, p. 352] that the algebraic group scheme $\mathbf{\Omega}(\mathfrak{A})$ is smooth, because $R_{Z/F}(\mathbf{G_m})$ and $\mathbf{PGO}^+(\mathfrak{A})$ are smooth and the lower row of the diagram (2.3) is exact. Since $\dim R_{Z/F}(\mathbf{G_m}) = 2$ and $\dim \mathbf{PGO}^+(\mathfrak{A}) = m(2m-1)$, it follows that

$$\dim \mathbf{\Omega}(\mathfrak{A}) = \dim \boldsymbol{\omega}(\mathfrak{A}) = m(2m-1) + 2.$$

For $\xi \in \boldsymbol{\omega}(\mathfrak{A})$ we write

$$\dot{\underline{\mu}}(\xi) = \underline{\sigma}(\xi) + \xi \in Z.$$

Since the map b is injective, for each $\xi \in \boldsymbol{\omega}(\mathfrak{A})$ there is a uniquely determined element $\dot{\chi}_0(\xi) \in A$ such that

$$\underline{\sigma}(\xi) * 1^b + 1^b \cdot \xi = \dot{\chi}_0(\xi)^b.$$

Thus, letting $F[\varepsilon]$ denote the algebra of dual numbers, where $\varepsilon^2 = 0$, we have

$$\underline{\sigma}(1+\varepsilon\xi) * 1^b \cdot (1+\varepsilon\xi) = \big(1 + \varepsilon\dot{\chi}_0(\xi)\big)^b \quad \text{for } \xi \in \boldsymbol{\omega}(\mathfrak{A}).$$

This shows that $\dot{\chi}_0$ is the differential of $\chi_0\colon \mathbf{\Omega}(\mathfrak{A}) \to \mathbf{GO}^+(\mathfrak{A})$.

For the next statement, recall from (2.9) that every $g \in \mathfrak{go}(\mathfrak{A})$ defines a derivation δ_g of $C(\mathfrak{A})$ such that $\delta_g(c(a)) = c([g,a])$ for all $a \in A$.

Proposition 2.13. *The Lie algebra $\boldsymbol{\omega}(\mathfrak{A})$ is a subalgebra of $\mathfrak{L}(C(\mathfrak{A}))$ containing Z and $\boldsymbol{\gamma}(\mathfrak{A})$, and $\dot{\chi}_0$, $\dot{\underline{\mu}}$ are Lie algebra homomorphisms*

$$\dot{\chi}_0\colon \boldsymbol{\omega}(\mathfrak{A}) \to \mathfrak{go}(\mathfrak{A}) \quad \textit{and} \quad \dot{\underline{\mu}}\colon \boldsymbol{\omega}(\mathfrak{A}) \to Z.$$

Moreover, $\dot{\chi}_0(z) = \operatorname{Tr}_{Z/F}(z) \in F$ *for* $z \in Z$,

$$\operatorname{ad}_\xi = \delta_{\dot{\chi}_0(\xi)} \quad \textit{and} \quad \dot{\mu}\big(\dot{\chi}_0(\xi)\big) = \operatorname{Tr}_{Z/F}\big(\dot{\underline{\mu}}(\xi)\big) \quad \textit{for } \xi \in \boldsymbol{\omega}(\mathfrak{A}),$$

and

$$\dot{\chi}_0(\xi) = \dot{\underline{\mu}}(\xi) + \dot{\chi}(\xi) \quad \textit{for } \xi \in \boldsymbol{\gamma}(\mathfrak{A}).$$

Proof. That $\boldsymbol{\omega}(\mathfrak{A})$ is a Lie subalgebra of $\mathfrak{L}(C(\mathfrak{A}))$ and $\dot{\chi}_0$, $\dot{\underline{\mu}}$ are Lie algebra homomorphisms is clear because $\boldsymbol{\omega}(\mathfrak{A})$ is the Lie algebra of $\mathbf{\Omega}(\mathfrak{A})$ and $\dot{\chi}_0$, $\dot{\underline{\mu}}$ are the differentials of χ_0 and $\underline{\mu}\colon \mathbf{\Omega}(\mathfrak{A}) \to R_{Z/F}(\mathbf{G_m})$ respectively.

Over the algebra $F[\varepsilon]$ of dual numbers, Proposition 2.7 yields

$$\operatorname{Int}(1+\varepsilon\xi) = C\big(\operatorname{Int}\big(\chi_0(1+\varepsilon\xi)\big)\big) \quad \text{for } \xi \in \boldsymbol{\omega}(\mathfrak{A}).$$

Hence for $\xi \in \boldsymbol{\omega}(\mathfrak{A})$ and $a \in A$

$$(1+\varepsilon\xi)c(a)(1-\varepsilon\xi) = c\big(\big(1+\varepsilon\dot\chi_0(\xi)\big)a\big(1-\varepsilon\dot\chi_0(\xi)\big)\big).$$

Comparing the coefficients of ε yields $[\xi, c(a)] = c([\dot\chi_0(\xi), a])$. Therefore, the derivations ad_ξ and $\delta_{\dot\chi_0(\xi)}$ coincide on $c(A)$, hence $\operatorname{ad}_\xi = \delta_{\dot\chi_0(\xi)}$ because $c(A)$ generates $C(\mathfrak{A})$ as an associative algebra.

The other equations similarly follow by taking the differentials of $\chi_0(z) = N_{Z/F}(z)$ for $z \in Z^\times$, $\mu(\chi_0(\xi)) = N_{Z/F}(\underline{\mu}(\xi))$ for $\xi \in \boldsymbol{\Omega}(\mathfrak{A})$ and $\chi_0(\xi) = \underline{\mu}(\xi)\chi(\xi)$ for $\xi \in \Gamma(\mathfrak{A})$ (see Proposition 2.7). ■

Corollary 2.14. *If* char $F \neq 2$, *then* $\boldsymbol{\omega}(\mathfrak{A}) = \boldsymbol{\gamma}(\mathfrak{A}) + Z$.

Proof. If char $F \neq 2$, then $Z \cap \boldsymbol{\gamma}(\mathfrak{A}) = F$, while Proposition 2.13 shows that $\boldsymbol{\gamma}(\mathfrak{A}) + Z \subset \boldsymbol{\omega}(\mathfrak{A})$. Dimension count then shows that $\boldsymbol{\omega}(\mathfrak{A}) = \boldsymbol{\gamma}(\mathfrak{A}) + Z$. ■

Note that $Z \subset \boldsymbol{\gamma}(\mathfrak{A})$ if char $F = 2$ (see [13, (8.27)]), hence $\boldsymbol{\gamma}(\mathfrak{A}) + Z = \boldsymbol{\gamma}(\mathfrak{A}) \subsetneq \boldsymbol{\omega}(\mathfrak{A})$ in that case.

The following Lie algebra versions of the commutative diagram (2.3) and of Proposition 2.8 can be derived from their algebraic group scheme versions. We give a direct proof instead.

Proposition 2.15. *Let* $Z^0 = \ker(\operatorname{Tr}\colon Z \to F)$ *and let* $\dot\chi'\colon \boldsymbol{\omega}(\mathfrak{A}) \to \mathfrak{pgo}(\mathfrak{A})$ *be defined by* $\dot\chi'(\xi) = \dot\chi_0(\xi) + F$ *for* $\xi \in \boldsymbol{\omega}(\mathfrak{A})$. *The following sequence is exact:*

$$0 \to Z^0 \to \boldsymbol{\omega}(\mathfrak{A}) \xrightarrow{\dot\chi_0} \mathfrak{go}(\mathfrak{A}) \to 0. \tag{2.13}$$

The following diagram is commutative with exact rows and canonical vertical maps:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & \boldsymbol{\gamma}(\mathfrak{A}) & \xrightarrow{\dot\chi} & \mathfrak{o}(\mathfrak{A}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z & \longrightarrow & \boldsymbol{\omega}(\mathfrak{A}) & \xrightarrow{\dot\chi'} & \mathfrak{pgo}(\mathfrak{A}) & \longrightarrow & 0 \end{array} \tag{2.14}$$

Moreover,

$$\boldsymbol{\gamma}(\mathfrak{A}) = \big\{\xi \in \boldsymbol{\omega}(\mathfrak{A}) \mid \underline{\dot\mu}(\xi) \in F\big\} \quad \text{and} \quad \mathfrak{spin}(\mathfrak{A}) = \ker(\underline{\dot\mu}\colon \boldsymbol{\omega}(\mathfrak{A}) \to Z).$$

Proof. We first show $Z^0 = \ker \dot\chi_0$. The inclusion $Z^0 \subset \ker \dot\chi_0$ follows from Proposition 2.13. To prove the reverse inclusion, let $\xi \in \ker \dot\chi_0$. Proposition 2.13 yields $[\xi, c(a)] = 0$ for all $a \in A$. As $c(A)$ generates $C(\mathfrak{A})$, we conclude that $\xi \in Z$. But then Proposition 2.13 shows that $\dot\chi_0(\xi) = \operatorname{Tr}_{Z/F}(\xi)$, hence $\xi \in Z^0$.

Dimension count now shows that $\dot\chi_0$ is surjective, hence (2.13) is an exact sequence.

The upper sequence of diagram (2.14) is (2.11). We have just seen that $\dot\chi_0$ is surjective, hence $\dot\chi'$ also is surjective. By Proposition 2.13, its kernel contains Z. Dimension count then yields $\ker\dot\chi' = Z$, hence the lower sequence of the diagram is exact. Commutativity of the diagram follows from Proposition 2.13, since $\underline{\dot\mu}(\xi) \in F$ for $\xi \in \boldsymbol{\gamma}(\mathfrak{A})$.

This last observation shows that $\boldsymbol{\gamma}(\mathfrak{A})$ lies in the kernel of the map

$$\dot\varkappa\colon \boldsymbol{\omega}(\mathfrak{A}) \to Z/F, \quad \xi \mapsto \underline{\dot\mu}(\xi) + F.$$

We have to prove that $\boldsymbol{\gamma}(\mathfrak{A}) = \ker\dot\varkappa$. To see this, it suffices to show that $\dot\varkappa$ is onto, because $\dim\boldsymbol{\gamma}(\mathfrak{A}) = (\dim\boldsymbol{\omega}(\mathfrak{A})) - 1$ and $\dim(Z/F) = 1$.

If $\operatorname{char} F \neq 2$, surjectivity is clear because $\underline{\dot\mu}(z) = 2z$ for all $z \in Z$. If $\operatorname{char} F = 2$, we pick an element $\ell \in \mathfrak{go}(\mathfrak{A})$ such that $\dot\mu(\ell) = 1$. Since $\dot\chi_0$ is onto, we may find $\xi \in \boldsymbol{\omega}(\mathfrak{A})$ such that $\dot\chi_0(\xi) = \ell$. Then by Proposition 2.13 we have $\operatorname{Tr}_{Z/F}(\underline{\dot\mu}(\xi)) = 1$, hence $\underline{\dot\mu}(\xi) \notin F$. This shows $\dot\varkappa$ is onto.

To complete the proof, it suffices to observe that $\mathfrak{spin}(\mathfrak{A}) = \ker(\underline{\dot\mu}\colon \boldsymbol{\gamma}(\mathfrak{A}) \to F)$ by definition. ∎

When $\operatorname{char} F = 2$ we have $\underline{\dot\mu}(Z) = 0$, hence $Z \subset \mathfrak{spin}(\mathfrak{A})$ and we may define a Lie algebra homomorphism $\dot S\colon \mathfrak{pgo}(\mathfrak{A}) \to Z$ by

$$\dot S(g + F) = \underline{\dot\mu}(\xi) \quad \text{for any } \xi \in \boldsymbol{\omega}(\mathfrak{A}) \text{ such that } \dot\chi'(\xi) = g + F.$$

Corollary 2.16. *If* $\operatorname{char} F \neq 2$, *then* $\dot\chi'$ *yields an isomorphism* $\mathfrak{spin}(\mathfrak{A}) \xrightarrow{\sim} \mathfrak{pgo}(\mathfrak{A})$.

If $\operatorname{char} F = 2$, *the restriction of* $\dot\chi'$ *fits in the exact sequence*

$$0 \to Z \to \mathfrak{spin}(\mathfrak{A}) \xrightarrow{\dot\chi'} \mathfrak{pgo}(\mathfrak{A}) \xrightarrow{\dot S} Z \to 0.$$

Proof. If $\operatorname{char} F \neq 2$ we saw in (2.12) that $\dot\chi$ yields an isomorphism $\mathfrak{spin}(\mathfrak{A}) \simeq \mathfrak{o}(\mathfrak{A})$, and in Proposition 2.10 we saw that the canonical map is an isomorphism $\mathfrak{o}(\mathfrak{A}) \xrightarrow{\sim} \mathfrak{pgo}(\mathfrak{A})$, hence $\dot\chi'$ is an isomorphism $\mathfrak{spin}(\mathfrak{A}) \simeq \mathfrak{pgo}(\mathfrak{A})$.

For the rest of the proof, assume $\operatorname{char} F = 2$. Since $\dot\chi'\colon \boldsymbol{\omega}(\mathfrak{A}) \to \mathfrak{pgo}(\mathfrak{A})$ is onto and $\mathfrak{spin}(\mathfrak{A}) = \ker\underline{\dot\mu}$ by Proposition 2.15, it is clear from the definition of $\dot S$ that $\ker\dot S = \dot\chi'(\mathfrak{spin}(\mathfrak{A}))$. As $\dot S(\mathfrak{pgo}(\mathfrak{A})) \subset Z$, it follows that

$$\dim\mathfrak{pgo}(\mathfrak{A}) - \dim\dot\chi'\big(\mathfrak{spin}(\mathfrak{A})\big) \leq 2. \tag{2.15}$$

On the other hand we have $Z \subset \mathfrak{spin}(\mathfrak{A})$ because $\underline{\dot\mu}(Z) = 0$, and $Z \subset \ker\dot\chi'$ by Proposition 2.15, hence

$$\dim\dot\chi'\big(\mathfrak{spin}(\mathfrak{A})\big) \leq \dim\mathfrak{spin}(\mathfrak{A}) - 2. \tag{2.16}$$

As $\dim\mathfrak{pgo}(\mathfrak{A}) = m(2m-1) = \dim\mathfrak{spin}(\mathfrak{A})$ by Proposition 2.10 and (2.10), the inequalities (2.15) and (2.16) cannot be strict. Therefore, $Z = \ker\dot\chi' = \dot S(\mathfrak{pgo}(\mathfrak{A}))$ and the corollary is proved. ∎

Finally, we consider the case where m is divisible by 4; then $C(\mathfrak{A})$ carries a canonical quadratic pair $(\underline{\sigma}, \underline{\mathfrak{f}})$ defined by Dolphin–Quéguiner-Mathieu: see the end of Section 2.2. As in Proposition 2.9, we let

$$\mathfrak{C}(\mathfrak{A}) = \big(C(\mathfrak{A}), \underline{\sigma}, \underline{\mathfrak{f}}\big).$$

Proposition 2.17. *If* $\deg A \equiv 0 \bmod 8$, *then* $\boldsymbol{\omega}(\mathfrak{A}) \subset \mathfrak{go}(\mathfrak{C}(\mathfrak{A}))$.

Proof. The definition of $\boldsymbol{\omega}(\mathfrak{A})$ entails that $\underline{\sigma}(\xi) + \xi \in Z$ for all $\xi \in \boldsymbol{\omega}(\mathfrak{A})$, hence it suffices to prove $\underline{\mathfrak{f}}([\xi, s]) = 0$ for $\xi \in \boldsymbol{\omega}(\mathfrak{A})$ and $s \in \operatorname{Sym}(\underline{\sigma})$. By the definition of $\underline{\mathfrak{f}}$, this amounts to showing that if $a \in A$ is such that $\operatorname{Trd}_A(a) = 1$, then

$$\operatorname{Trd}_{C(\mathfrak{A})}\big(c(a)[\xi, s]\big) = 0 \quad \text{for } \xi \in \boldsymbol{\omega}(\mathfrak{A}) \text{ and } s \in \operatorname{Sym}(\underline{\sigma}).$$

For this, observe that

$$\operatorname{Trd}_{C(\mathfrak{A})}\big(c(a)(\xi s - s\xi)\big) = \operatorname{Trd}_{C(\mathfrak{A})}\big((\xi c(a) - c(a)\xi)s\big).$$

Now, by Proposition 2.13 we have $[\xi, c(a)] = c([\dot{\chi}_0(\xi), a])$. As $\operatorname{Trd}_A([\dot{\chi}_0(\xi), a]) = 0$, it follows that $c([\dot{\chi}_0(\xi), a]) \in \mathfrak{spin}(\mathfrak{A})$. Now, $\mathfrak{spin}(\mathfrak{A}) \subset \operatorname{Alt}(\underline{\sigma})$ by [7, Lemma 3.2], hence

$$\operatorname{Trd}_{C(\mathfrak{A})}\big(c\big(\big[\dot{\chi}_0(\xi), a\big]\big)s\big) = 0 \quad \text{for all } s \in \operatorname{Sym}(\underline{\sigma}). \qquad \blacksquare$$

Remark 2.18. When $\operatorname{char} F = 2$, the Lie algebra $\mathfrak{L}(A)$ has an additional structure given by the squaring map $a \mapsto a^2$, which turns it into a *restricted Lie algebra*. It can be verified that the Lie algebras $\mathfrak{o}(\mathfrak{A})$, $\mathfrak{so}(\mathfrak{A})$, $\mathfrak{go}(\mathfrak{A})$, $\mathfrak{pgo}(\mathfrak{A})$, $\boldsymbol{\gamma}(\mathfrak{A})$, $\mathfrak{spin}(\mathfrak{A})$, $\boldsymbol{\omega}(\mathfrak{A})$ are all restricted (i.e., preserved under the squaring map), and the maps $\dot{\mu}$, $\underline{\dot{\mu}}$, $\dot{\chi}$, Trp, $\dot{\chi}_0$, $\dot{S}$ are homomorphisms of restricted Lie algebras (i.e., commute with the squaring map). The proof is omitted, as the restricted Lie algebra structure will not be used in this work.

2.6. Homomorphisms from Clifford algebras

Throughout this subsection, $\mathfrak{A} = (A, \sigma, \mathfrak{f})$ is an algebra with quadratic pair of degree $2m$ over an arbitrary field F. We assume $m \equiv 0 \bmod 4$ and the discriminant of $(\sigma, \mathfrak{f})$ is trivial, which implies that the Clifford algebra $C(\mathfrak{A})$ decomposes as an algebra with quadratic pair into a direct product of two central simple F-algebras with quadratic pair of degree 2^{m-1}. We further choose a polarization of $\mathfrak{A}$ (see Definition 2.4), which provides a designation of the primitive central idempotents of $C(\mathfrak{A})$ as z_+ and z_-. The simple components of $C(\mathfrak{A})$ are then

$$C_+(\mathfrak{A}) = C(\mathfrak{A})z_+ \quad \text{and} \quad C_-(\mathfrak{A}) = C(\mathfrak{A})z_-.$$

We write $\pi_+\colon C(\mathfrak{A}) \to C_+(\mathfrak{A})$ and $\pi_-\colon C(\mathfrak{A}) \to C_-(\mathfrak{A})$ for the projections:

$$\pi_+(\xi) = \xi z_+, \quad \pi_-(\xi) = \xi z_- \quad \text{for } \xi \in C(\mathfrak{A}),$$

and let

$$\mathfrak{C}(\mathfrak{A}) = \big(C(\mathfrak{A}), \underline{\sigma}, \underline{\mathfrak{f}}\big).$$

Given another central simple F-algebra with quadratic pair $\mathfrak{A}' = (A', \sigma', \mathfrak{f}')$ of degree 2^{m-1}, we define a *homomorphism of algebras with quadratic pair*

$$\varphi\colon \mathfrak{C}(\mathfrak{A}) \to \mathfrak{A}' \tag{2.17}$$

to be an F-algebra homomorphism $\varphi\colon C(\mathfrak{A}) \to A'$ such that

$$\varphi \circ \underline{\sigma} = \sigma' \circ \varphi \quad \text{and} \quad \varphi\big(\underline{\mathfrak{f}}(s)\big) = \mathfrak{f}'\big(\varphi(s)\big) \text{ for all } s \in \operatorname{Sym}(\underline{\sigma}).$$

Since we assume $\dim A' = \frac{1}{2} \dim C(\mathfrak{A})$, such a homomorphism factors through one of the projections π_+ or π_-, and maps the center Z of $C(\mathfrak{A})$ to F. It readily follows that φ defines a morphism $\mathbf{GO}(\mathfrak{C}(\mathfrak{A})) \to \mathbf{GO}(\mathfrak{A}')$ and maps $\mathfrak{go}(\mathfrak{C}(\mathfrak{A}))$ to $\mathfrak{go}(\mathfrak{A}')$.

Definition 2.19. We say that φ *has the* $+$ *sign* if it factors through π_+ (i.e., $\varphi(z_+) = 1$ and $\varphi(z_-) = 0$), and that φ *has the* $-$ *sign* if it factors through π_- (i.e., $\varphi(z_+) = 0$ and $\varphi(z_-) = 1$).

Since $\mathbf{\Omega}(\mathfrak{A}) \subset \mathbf{GO}^+(\mathfrak{C}(\mathfrak{A}))$ by Proposition 2.9, we may restrict φ to $\mathbf{\Omega}(\mathfrak{A})$ to obtain the following commutative diagram with exact rows, where $\pm$ is the sign of φ:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & R_{Z/F}(\mathbf{G_m}) & \longrightarrow & \mathbf{\Omega}(\mathfrak{A}) & \xrightarrow{\chi'} & \mathbf{PGO}^+(\mathfrak{A}) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \pi_\pm} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \bar{\varphi}} & & \\
1 & \longrightarrow & \mathbf{G_m} & \longrightarrow & \mathbf{GO}^+(\mathfrak{A}') & \longrightarrow & \mathbf{PGO}^+(\mathfrak{A}') & \longrightarrow & 1
\end{array} \tag{2.18}$$

We also consider the corresponding diagram with exact rows involving the differentials:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z & \longrightarrow & \boldsymbol{\omega}(\mathfrak{A}) & \xrightarrow{\dot{\chi}'} & \mathfrak{pgo}(\mathfrak{A}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_\pm} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \theta} & & \\
0 & \longrightarrow & F & \longrightarrow & \mathfrak{go}(\mathfrak{A}') & \longrightarrow & \mathfrak{pgo}(\mathfrak{A}') & \longrightarrow & 0
\end{array} \tag{2.19}$$

Since $\varphi \circ \underline{\sigma} = \sigma' \circ \varphi$, it follows that $\varphi \circ \underline{\mu} = \mu \circ \varphi$ on $\mathbf{\Omega}(\mathfrak{A})$, hence φ maps $\mathbf{Spin}(\mathfrak{A})$ to $\mathbf{O}^+(\mathfrak{A}')$. Restricting the morphism φ to $\mathbf{Spin}(\mathfrak{A})$, we obtain from (2.18) the following commutative diagram of algebraic group schemes with exact rows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & R_{Z/F}(\boldsymbol{\mu}_2) & \longrightarrow & \mathbf{Spin}(\mathfrak{A}) & \xrightarrow{\chi'} & \mathbf{PGO}^+(\mathfrak{A}) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \pi_\pm} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \bar{\varphi}} & & \\
1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathbf{O}^+(\mathfrak{A}') & \longrightarrow & \mathbf{PGO}^+(\mathfrak{A}') & \longrightarrow & 1
\end{array}$$

Our goal in the rest of this subsection is to show that the map θ in (2.19) determines the homomorphism φ in (2.17) uniquely.

Definition 2.20. Given φ as in (2.17), of sign $\pm$, the Lie algebra homomorphism

$$\theta\colon \mathfrak{pgo}(\mathfrak{A}) \to \mathfrak{pgo}(\mathfrak{A}')$$

in diagram (2.19) is said to be *induced* by φ. Changing the perspective, a Lie algebra homomorphism $\theta\colon \mathfrak{pgo}(\mathfrak{A}) \to \mathfrak{pgo}(\mathfrak{A}')$ is said to be *liftable* if it is induced by some homomorphism of algebras with quadratic pair φ, which is then called a *lift* of θ. If θ is induced by a homomorphism φ, the *sign* of θ is defined to be the same as the sign of φ.

The following theorem shows that the latter definition is not ambiguous:

Theorem 2.21. *If a Lie algebra homomorphism $\theta\colon \mathfrak{pgo}(\mathfrak{A}) \to \mathfrak{pgo}(\mathfrak{A}')$ is liftable, then its lift is unique.*

Proof. It suffices to prove the theorem after scalar extension. We may therefore assume $\mathfrak{A} = (\operatorname{End} V, \sigma_b, \mathfrak{f}_q)$ for some hyperbolic quadratic space (V,q) of dimension $2m$. We use the standard identification $V \otimes V = \operatorname{End} V$ set up in Section 2.1.

Since q is hyperbolic, by decomposing V into an orthogonal sum of hyperbolic planes we may find a base $(e_i, e_i')_{i=1}^m$ of V such that

$$q(e_i) = q(e_i') = b(e_i, e_j) = b(e_i', e_j') = 0 \quad \text{for all } i, j = 1, \dots, m$$

and

$$b(e_i, e_j') = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The products e_ie_j, e_ie_j', $e_i'e_j$, $e_i'e_j'$ for $i, j = 1, \dots, m$ span $V \cdot V \subset C_0(V,q)$, hence they generate $C_0(V,q)$ as an F-algebra. Since $q(e_i) = q(e_i') = 0$ for all i, we do not need to count e_ie_j nor $e_i'e_j'$ among the generators if $i = j$. Moreover, $e_je_j' + e_j'e_j = b(e_j, e_j') = 1$ for all j, hence if $i \neq j$

$$e_ie_i' = e_i(e_je_j' + e_j'e_j)e_i' = (e_ie_j)(e_j'e_i') + (e_ie_j')(e_je_i')$$

and similarly

$$e_i'e_i = e_i'(e_je_j' + e_j'e_j)e_i = (e_i'e_j)(e_j'e_i) + (e_i'e_j')(e_je_i).$$

These equations show that e_ie_i' and $e_i'e_i$ lie in the subalgebra of $C_0(V,q)$ generated by e_ke_ℓ, e_ke_ℓ', $e_k'e_\ell$, $e_k'e_\ell'$ for all $k \neq \ell$ in $\{1, \dots, m\}$. Therefore, these elements generate $C_0(V,q)$.

Consequently, if $\varphi_1, \varphi_2\colon C_0(V,q) \to A'$ are two lifts of a given $\theta\colon \mathfrak{pgo}(\mathfrak{A}) \to \mathfrak{pgo}(\mathfrak{A}')$, it suffices to prove that φ_1 and φ_2 coincide on e_ke_ℓ, e_ke_ℓ', $e_k'e_\ell$, $e_k'e_\ell'$ for all $k \neq \ell$ in $\{1, \dots, m\}$ to conclude that $\varphi_1 = \varphi_2$. This is what we proceed to show.

The condition that φ_1 and φ_2 induce the same θ means that $\varphi_1(\xi) - \varphi_2(\xi) \in F$ for all $\xi \in \boldsymbol{\omega}(q)$, hence

$$\varphi_1\big([\xi_1, \xi_2]\big) = \varphi_2\big([\xi_1, \xi_2]\big) \quad \text{for all } \xi_1, \xi_2 \in \boldsymbol{\omega}(q).$$

We apply this to $\xi_1 = c(u_1 \otimes v_1) = u_1 v_1$ and $\xi_2 = c(u_2 \otimes v_2) = u_2 v_2 \in \boldsymbol{\gamma}(q) \subset \boldsymbol{\omega}(q)$ for $u_1, u_2, v_1, v_2 \in V$. If $i \neq j$, we have

$$[e_i e_j, e'_j e_j] = e_i e_j e'_j e_j - e'_j e_j e_i e_j.$$

Since e_i and e_j anticommute and $e_j^2 = 0$, the second term on the right side vanishes. In the first term, we may substitute $1 - e'_j e_j$ for $e_j e'_j$ and use $e_j^2 = 0$ to obtain

$$[e_i e_j, e'_j e_j] = e_i (1 - e'_j e_j) e_j = e_i e_j.$$

Similar computations yield for all $i \neq j$ in $\{1, \dots, m\}$

$$[e_i e'_j, e_j e'_j] = e_i e'_j, \quad [e'_i e_j, e'_j e_j] = e'_i e_j, \quad [e'_i e'_j, e_j e'_j] = e'_i e'_j.$$

Since φ_1 and φ_2 take the same value on each $[\xi_1, \xi_2]$ for $\xi_1, \xi_2 \in \boldsymbol{\omega}(q)$, it follows that φ_1 and φ_2 coincide on each $e_i e_j$, $e_i e'_j$, $e'_i e_j$ and $e'_i e'_j$ for $i \neq j$, hence $\varphi_1 = \varphi_2$. ∎

Corollary 2.22. *Let $\theta\colon \mathfrak{pgo}(\mathfrak{A}) \to \mathfrak{pgo}(\mathfrak{A}')$ be a homomorphism of Lie algebras and let K be a Galois field extension of F. If $\theta_K\colon \mathfrak{pgo}(\mathfrak{A})_K \to \mathfrak{pgo}(\mathfrak{A}')_K$ is liftable, then θ is liftable.*

Proof. Let $\varphi\colon C(\mathfrak{A})_K \to A'_K$ be the lift of θ_K, and let ρ be an element of the Galois group of K/F. Then

$$(\mathrm{Id}_{A'} \otimes \rho) \circ \varphi \circ (\mathrm{Id}_{C(\mathfrak{A})} \otimes \rho^{-1})\colon C(\mathfrak{A})_K \to A'_K$$

is a lift of $(\mathrm{Id}_{\mathfrak{pgo}(\mathfrak{A}')} \otimes \rho) \circ \theta_K \circ (\mathrm{Id}_{\mathfrak{pgo}(\mathfrak{A})} \otimes \rho^{-1}) = \theta_K$, hence, by uniqueness of the lift,

$$(\mathrm{Id}_{A'} \otimes \rho) \circ \varphi \circ (\mathrm{Id}_{C(\mathfrak{A})} \otimes \rho^{-1}) = \varphi.$$

Therefore, $\varphi|_{C(\mathfrak{A})}$ maps $C(\mathfrak{A})$ to A'; it lifts θ since φ lifts θ_K. ∎

3. Compositions of quadratic spaces

This section introduces the notion of a composition of quadratic spaces. We emphasize an important feature of compositions, which will be central to the definition of trialitarian automorphisms in the next section: each composition gives rise to two other compositions on the quadratic spaces cyclically permuted. Restricting to the case where the quadratic spaces have the same finite dimension, we show that this dimension is 1, 2, 4 or 8, the comparatively trivial case of dimension 1 arising only when the characteristic of the base field is different from 2. In order to prove this fairly classical result along the same lines as in [18, Cor. 1.12] we set up isomorphisms of algebras with involution or with quadratic pair involving Clifford algebras. In dimension 8, these isomorphisms will provide in the next section examples of trialitarian triples of split algebras. In Section 3.3 we investigate similitudes and isometries of compositions of quadratic spaces, which define algebraic groups that are close analogues of those attached to quadratic spaces.

Even though the quadratic spaces in a composition are not necessarily isometric, it is easy to see that every composition of quadratic spaces is similar to a composition of *isometric* quadratic spaces (see Proposition 3.16). The focus in the last two subsections is on this type of compositions. Using a related notion of composition of *pointed* quadratic spaces, we show in Section 3.4 that every composition of isometric quadratic spaces is isomorphic to its derivatives and also to a composition that is its own derivative, and in Section 3.5 we discuss compositions of quadratic spaces arising from the classical notion of composition algebra. To each composition algebra is associated a composition of isometric quadratic spaces, and isotopies of composition algebras are shown in Theorem 3.34 to be similitudes of the associated compositions of quadratic spaces.

Throughout this section, F is an arbitrary field. Unless explicitly specified, there is no restriction on its characteristic char F.

3.1. Composition maps and their cyclic derivatives

Let (V_1, q_1), (V_2, q_2), (V_3, q_3) be (finite-dimensional) quadratic spaces over F. Write b_1, b_2, b_3 for the associated polar bilinear forms

$$b_i\colon V_i \times V_i \to F, \quad b_i(x_i, y_i) = q_i(x_i + y_i) - q_i(x_i) - q_i(y_i) \quad \text{for } i = 1, 2, 3.$$

We assume throughout that the forms b_1, b_2, b_3 are nonsingular, hence each $\dim V_i$ is even if char $F = 2$, and we may use the polar forms to identify each V_i with its dual V_i^*. Bilinear maps $V_1 \times V_2 \to V_3$ are then identified with tensors in $V_3 \otimes V_2 \otimes V_1$, so that for $v_i \in V_i$ the tensor $v_3 \otimes v_2 \otimes v_1$ is regarded as the bilinear map

$$V_1 \times V_2 \to V_3, \quad (x_1, x_2) \mapsto v_3 b_2(v_2, x_2) b_1(v_1, x_1).$$

Let

$$\partial\colon V_3 \otimes V_2 \otimes V_1 \to V_1 \otimes V_3 \otimes V_2 \quad \text{and} \quad \partial^2\colon V_3 \otimes V_2 \otimes V_1 \to V_2 \otimes V_1 \otimes V_3$$

be the isomorphisms that permute the tensor factors cyclically. These maps allow us to derive bilinear maps $V_2 \times V_3 \to V_1$ and $V_3 \times V_1 \to V_2$ from a given bilinear map $V_1 \times V_2 \to V_3$. In our notation, bilinear maps are adorned with the same index as the target space.

Proposition 3.1. *Let $*_3\colon V_1 \times V_2 \to V_3$ be a bilinear map, and let $*_1 = \partial(*_3)$ and $*_2 = \partial^2(*_3)$ be the derived maps*

$$*_1\colon V_2 \times V_3 \to V_1, \quad *_2\colon V_3 \times V_1 \to V_2.$$

The maps $*_1$ and $*_2$ are uniquely determined by the following property: for all $x_1 \in V_1$, $x_2 \in V_2$, $x_3 \in V_3$,

$$b_1(x_1, x_2 *_1 x_3) = b_2(x_2, x_3 *_2 x_1) = b_3(x_3, x_1 *_3 x_2). \tag{3.1}$$

Proof. Uniqueness is clear because the forms b_1 and b_2 are nonsingular. By linearity, it suffices to prove (3.1) in the case where $*_3 = v_3 \otimes v_2 \otimes v_1$ for some $v_1 \in V_1$, $v_2 \in V_2$, $v_3 \in V_3$. Then $*_1 = v_1 \otimes v_3 \otimes v_2$ and $*_2 = v_2 \otimes v_1 \otimes v_3$, and each of the terms in (3.1) is equal to $b_1(v_1, x_1)b_2(v_2, x_2)b_3(v_3, x_3)$. ■

The bilinear maps of interest in this work satisfy the following multiplicativity condition:

Definition 3.2. A *composition map* $*_3\colon V_1 \times V_2 \to V_3$ is a bilinear map subject to

$$q_3(x_1 *_3 x_2) = q_1(x_1)q_2(x_2) \quad \text{for } x_1 \in V_1 \text{ and } x_2 \in V_2. \tag{3.2}$$

Even though this notion makes sense—and is studied for instance in [18, Chap. 14]—when the dimensions of V_1, V_2 and V_3 are not the same, we will always assume in the sequel that $\dim V_1 = \dim V_2 = \dim V_3$.

Proposition 3.3. *Let* $*_3\colon V_1 \times V_2 \to V_3$ *be a composition map, with* $\dim V_1 = \dim V_2 = \dim V_3$. *The derived bilinear maps* $*_1$ *and* $*_2$ *are composition maps, i.e., for all* $x_1 \in V_1$, $x_2 \in V_2$, $x_3 \in V_3$,

$$q_1(x_2 *_1 x_3) = q_2(x_2)q_3(x_3) \quad \textit{and} \quad q_2(x_3 *_2 x_1) = q_3(x_3)q_1(x_1). \tag{3.3}$$

Moreover, the following relations hold for all $x_1, y_1 \in V_1$, $x_2, y_2 \in V_2$, $x_3, y_3 \in V_3$:

$$b_3(x_1 *_3 x_2, x_1 *_3 y_2) = q_1(x_1)b_2(x_2, y_2), \tag{3.4}$$
$$b_3(x_1 *_3 x_2, y_1 *_3 x_2) = b_1(x_1, y_1)q_2(x_2), \tag{3.5}$$
$$b_1(x_2 *_1 x_3, x_2 *_1 y_3) = q_2(x_2)b_3(x_3, y_3), \tag{3.6}$$
$$b_1(x_2 *_1 x_3, y_2 *_1 x_3) = b_2(x_2, y_2)q_3(x_3), \tag{3.7}$$
$$b_2(x_3 *_2 x_1, x_3 *_2 y_1) = q_3(x_3)b_1(x_1, y_1), \tag{3.8}$$
$$b_2(x_3 *_2 x_1, y_3 *_2 x_1) = b_3(x_3, y_3)q_1(x_1), \tag{3.9}$$
$$(x_1 *_3 x_2) *_2 x_1 = x_2 q_1(x_1) \textit{ and } x_2 *_1 (x_1 *_3 x_2) = x_1 q_2(x_2), \tag{3.10}$$
$$(x_2 *_1 x_3) *_3 x_2 = x_3 q_2(x_2) \textit{ and } x_3 *_2 (x_2 *_1 x_3) = x_2 q_3(x_3), \tag{3.11}$$
$$(x_3 *_2 x_1) *_1 x_3 = x_1 q_3(x_3) \textit{ and } x_1 *_3 (x_3 *_2 x_1) = x_3 q_1(x_1), \tag{3.12}$$
$$(x_1 *_3 x_2) *_2 y_1 + (y_1 *_3 x_2) *_2 x_1 = x_2 b_1(x_1, y_1), \tag{3.13}$$
$$x_2 *_1 (x_1 *_3 y_2) + y_2 *_1 (x_1 *_3 x_2) = x_1 b_2(x_2, y_2), \tag{3.14}$$
$$(x_2 *_1 x_3) *_3 y_2 + (y_2 *_1 x_3) *_3 x_2 = x_3 b_2(x_2, y_2), \tag{3.15}$$
$$x_3 *_2 (x_2 *_1 y_3) + y_3 *_2 (x_2 *_1 x_3) = x_2 b_3(x_3, y_3), \tag{3.16}$$
$$(x_3 *_2 x_1) *_1 y_3 + (y_3 *_2 x_1) *_1 x_3 = x_1 b_3(x_3, y_3), \tag{3.17}$$
$$x_1 *_3 (x_3 *_2 y_1) + y_1 *_3 (x_3 *_2 x_1) = x_3 b_1(x_1, y_1). \tag{3.18}$$

Proof. First, (3.4) and (3.5) are obtained by linearizing (3.2). By (3.1) and (3.4) we have for $x_1 \in V_1$ and $x_2, y_2 \in V_2$

$$b_2\big((x_1 *_3 x_2) *_2 x_1, y_2\big) = b_3(x_1 *_3 x_2, x_1 *_3 y_2) = q_1(x_1)b_2(x_2, y_2).$$

Since b_2 is nonsingular, it follows that $(x_1 *_3 x_2) *_2 x_1 = x_2 q_1(x_1)$. Similarly, (3.1) and (3.5) yield

$$b_1\big(y_1, x_2 *_1 (x_1 *_3 x_2)\big) = b_3(y_1 *_3 x_2, x_1 *_3 x_2) = b_1(y_1, x_1) q_2(x_2) \quad \text{for all } y_1 \in V_1,$$

hence $x_2 *_1 (x_1 *_3 x_2) = x_1 q_2(x_2)$. We thus obtain (3.10); then (3.13), (3.14) follow by linearization.

The main part of the proof consists in proving (3.3). For this, fix an anisotropic vector $x_2 \in V_2$. The map $r_{x_2}\colon V_1 \to V_3$ defined by $r_{x_2}(x_1) = x_1 *_3 x_2$ is injective, for $x_1 *_3 x_2 = 0$ implies $x_1 = 0$ by (3.10). Since $\dim V_1 = \dim V_3$ the map r_{x_2} is also surjective, hence every $x_3 \in V_3$ can be written as $x_3 = x_1 *_3 x_2$ for some $x_1 \in V_1$. Then by (3.10)

$$x_2 *_1 x_3 = x_2 *_1 (x_1 *_3 x_2) = x_1 q_2(x_2),$$

hence

$$q_1(x_2 *_1 x_3) = q_1(x_1) q_2(x_2)^2.$$

But since $x_3 = x_1 *_3 x_2$ it follows from (3.2) that $q_3(x_3) = q_1(x_1) q_2(x_2)$, hence the right side of the last displayed equation can be rewritten as $q_2(x_2) q_3(x_3)$. We have thus proven $q_1(x_2 *_1 x_3) = q_2(x_2) q_3(x_3)$ when x_2 is anisotropic. Moreover, by (3.10) we have for all $z_2 \in V_2$

$$\begin{aligned} b_1(x_2 *_1 x_3, z_2 *_1 x_3) &= b_1\big(x_2 *_1 (x_1 *_3 x_2), z_2 *_1 (x_1 *_3 x_2)\big) \\ &= q_2(x_2) b_1\big(x_1, z_2 *_1 (x_1 *_3 x_2)\big). \end{aligned}$$

By (3.1) and (3.4),

$$b_1\big(x_1, z_2 *_1 (x_1 *_3 x_2)\big) = b_3(x_1 *_3 z_2, x_1 *_3 x_2) = q_1(x_1) b_2(z_2, x_2),$$

hence, as $q_1(x_1) q_2(x_2) = q_3(x_3)$,

$$b_1(x_2 *_1 x_3, z_2 *_1 x_3) = b_2(x_2, z_2) q_3(x_3). \tag{3.19}$$

Now, assume x_2 is isotropic. Pick anisotropic vectors $y_2, z_2 \in V_2$ such that $x_2 = y_2 + z_2$. (If $\dim V_2 > 2$, we may pick any anisotropic y_2 orthogonal to x_2 and let $z_2 = x_2 - y_2$.) By the first part of the proof we have

$$q_1(y_2 *_1 x_3) = q_2(y_2) q_3(x_3) \quad \text{and} \quad q_1(z_2 *_1 x_3) = q_2(z_2) q_3(x_3).$$

Moreover, (3.19) yields

$$b_1(y_2 *_1 x_3, z_2 *_1 x_3) = b_2(y_2, z_2) q_3(x_3).$$

Therefore,

$$\begin{aligned} q_1(x_2 *_1 x_3) &= q_1(y_2 *_1 x_3) + b_1(y_2 *_1 x_3, z_2 *_1 x_3) + q_1(z_2 *_1 x_3) \\ &= q_2(y_2) q_3(x_3) + b_2(y_2, z_2) q_3(x_3) + q_2(z_2) q_3(x_3) = q_2(x_2) q_3(x_3). \end{aligned}$$

Thus, equation $q_1(x_2 *_1 x_3) = q_2(x_2)q_3(x_3)$ is proved for all $x_2 \in V_2$ and $x_3 \in V_3$. The proof of $q_2(x_3 *_2 x_1) = q_3(x_3)q_1(x_1)$ for all $x_3 \in V_3$, $x_1 \in V_1$ is similar, using bijectivity of the map

$$\ell_{x_1} \colon V_2 \to V_3$$

carrying x_2 to $x_1 *_3 x_2$ for x_1 anisotropic. This completes the proof of (3.3), and (3.6), (3.7), (3.8), (3.9) follow by linearization.

The same arguments that gave (3.10) from (3.4) and (3.5) yield (3.11) from (3.6) and (3.7), and also (3.12) from (3.8) and (3.9). The relations (3.15) and (3.16) (resp. (3.17) and (3.18)) are derived by linearizing (3.11) (resp. (3.12)). ∎

Remark 3.4. The derived maps of a composition map between quadratic modules of constant rank 8 over a commutative ring are defined by Alsaody in [2, p. 886] using the equations (3.1). From the proof of [2, Prop. 3.7], it follows that these derived maps are composition maps.

Our main object of study in this section is defined next.

Definition 3.5. A *composition of quadratic spaces* over F is a 4-tuple

$$\mathcal{C} = \big((V_1, q_1), (V_2, q_2), (V_3, q_3), *_3\big)$$

where (V_1, q_1), (V_2, q_2), (V_3, q_3) are nonsingular quadratic spaces of the same dimension over F and $*_3 \colon V_1 \times V_2 \to V_3$ is a composition map. We write $\dim \mathcal{C} = n$ if $\dim V_1 = \dim V_2 = \dim V_3 = n$.

In view of Proposition 3.3, each composition of quadratic spaces $\mathcal{C}$ yields *derived compositions of quadratic spaces* $\partial\mathcal{C}$ and $\partial^2\mathcal{C}$ defined by

$$\partial\mathcal{C} = \big((V_2, q_2), (V_3, q_3), (V_1, q_1), *_1\big)$$

and

$$\partial^2\mathcal{C} = \big((V_3, q_3), (V_1, q_1), (V_2, q_2), *_2\big).$$

The composition maps $*_1$ and $*_2$ are called the *derived composition maps* of $*_3$. Since ∂ is a cyclic operation of period 3, we have

$$\partial(\partial\mathcal{C}) = \partial^2\mathcal{C}, \quad \partial^2(\partial\mathcal{C}) = \mathcal{C} = \partial(\partial^2\mathcal{C}), \quad \partial^2(\partial^2\mathcal{C}) = \partial\mathcal{C}.$$

Examples 3.6. (1) Let A be either F, a quadratic étale F-algebra, a quaternion F-algebra or an octonion F-algebra, and let $n_A \colon A \to F$ be (respectively) the squaring map, the norm, the quaternion (reduced) norm or the octonion norm. Assuming char $F \neq 2$ if $A = F$, we know from the properties of these algebras that multiplication in A defines a composition of quadratic spaces

$$\mathcal{C} = \big((A, n_A), (A, n_A), (A, n_A), *_3\big).$$

This particular type of composition is discussed in Section 3.5 in relation with composition algebras. Note that if $A \neq F$ the derived composition maps $*_1$ and $*_2$ are *not* simply given by the multiplication in A; see Proposition 3.29.

(2) Let U be an F-vector space of dimension 4 and let U^* be its dual space. Let also $s\colon \bigwedge^4 U \to F$ be an F-linear isomorphism. Define

- $V_1 = U^* \oplus U$ and $q_1\colon V_1 \to F$ the (hyperbolic) quadratic form defined by

$$q_1(\varphi + u) = \varphi(u)$$

for $\varphi \in U^*$ and $u \in U$;

- $V_2 = U \oplus \bigwedge^3 U$ and $q_2\colon V_2 \to F$ the (hyperbolic) quadratic form defined by

$$q_2(u + \xi) = s(u \wedge \xi)$$

for $u \in U$ and $\xi \in \bigwedge^3 U$;

- $V_3 = F \oplus \bigwedge^2 U \oplus \bigwedge^4 U$ and $q_3\colon V_3 \to F$ the (hyperbolic) quadratic form defined by

$$q_3(\alpha + \xi + \eta) = s\big(\alpha\eta - P(\xi)\big)$$

for $\alpha \in F$, $\xi \in \bigwedge^2 U$ and $\eta \in \bigwedge^4 U$, where $P\colon \bigwedge^2 U \to \bigwedge^4 U$ is the "divided square" map uniquely determined by the conditions that $P(u_1 \wedge u_2) = 0$ for $u_1, u_2 \in U$ and that its polar bilinear form satisfies $b_P(\xi_1, \xi_2) = \xi_1 \wedge \xi_2$ for $\xi_1, \xi_2 \in \bigwedge^2 U$.

For $\varphi \in U^*$, let also $d_\varphi\colon \bigwedge^3 U \to \bigwedge^2 U$ be the linear map such that

$$d_\varphi(u_1 \wedge u_2 \wedge_3) = \varphi(u_1)u_2 \wedge u_3 - \varphi(u_2)u_1 \wedge u_3 + \varphi(u_3)u_1 \wedge u_2$$

for u_1, u_2, u_3. Then the map $*_3\colon V_1 \times V_2 \to V_3$ given by

$$(\varphi + u) *_3 (u' + \xi) = \varphi(u') + d_\varphi(\xi) + u \wedge u' + u \wedge \xi$$

for u, $u' \in U$, $\varphi \in U^*$ and $\xi \in \bigwedge^3 U$ is a composition of quadratic spaces of dimension 8. This follows from straightforward computations left to the reader. (This example is inspired by the description of the Clifford algebra of a hyperbolic quadratic space in [13, (8.3)].)

The following examples are obtained in relation with a Galois F-algebra L with elementary abelian Galois group $\{1, \sigma_1, \sigma_2, \sigma_3\}$, i.e., an étale biquadratic F-algebra.

(3) Assume char $F \neq 2$, and for $i = 1, 2, 3$ let V_i denote the following 1-dimensional subspace of L:

$$V_i = \big\{x_i \in L \mid \sigma_j(x_i) = -x_i \text{ for } j \neq i\big\}.$$

Define $q_i\colon V_i \to F$ by $q_i(x_i) = x_i^2$. For $x_1 \in V_1$ and $x_2 \in V_2$ we have $x_1x_2 \in V_3$ and $(x_1x_2)^2 = x_1^2x_2^2$, hence multiplication in L defines a composition map $*_3\colon V_1 \times V_2 \to V_3$. The derived composition maps $*_1$ and $*_2$ are also given by the multiplication in L.

(4) Let A be a central simple F-algebra of degree 4 containing L. Assume char $F \neq 2$ and F contains an element ζ such that $\zeta^2 = -1$. For $i = 1, 2, 3$, define

$$V_i = \big\{x_i \in A \mid x_i\ell = \sigma_i(\ell)x_i \text{ for all } \ell \in L\big\}.$$

The F-vector space V_i has dimension 4 and carries a quadratic form q_i given by $q_i(x_i) = \mathrm{Trd}_A(x_i^2)$, where Trd_A is the reduced trace. It is shown in [16] that the following formula defines a composition map $*_3\colon V_1 \times V_2 \to V_3$:

$$x_1 *_3 x_2 = (1+\zeta)x_1x_2 + (1-\zeta)x_2x_1.$$

The derived maps are given by similar formulas:

$$\begin{aligned} x_2 *_1 x_3 &= (1+\zeta)x_2x_3 + (1-\zeta)x_3x_2 \quad \text{for } x_2 \in V_2 \text{ and } x_3 \in V_3,\\ x_3 *_2 x_1 &= (1+\zeta)x_3x_1 + (1-\zeta)x_1x_3 \quad \text{for } x_3 \in V_3 \text{ and } x_1 \in V_1. \end{aligned}$$

A characteristic 2 version of these composition maps is given in [20].

(5) Compositions of dimension 8 from central simple algebras with symplectic involution of degree 8 are given in a similar way in [4].

3.2. Canonical Clifford maps

Our goal in this subsection is to obtain structural information on the quadratic spaces for which a composition exists. This information will be derived from algebra homomorphisms defined on Clifford and even Clifford algebras.

Throughout this subsection, we fix a composition of quadratic spaces

$$\mathcal{C} = \big((V_1,q_1),(V_2,q_2),(V_3,q_3),*_3\big)$$

and we let $*_1$ and $*_2$ denote the derived composition maps of $*_3$, as per Definition 3.5. For each $x_1 \in V_1$ we may consider two linear maps

$$\ell_{x_1}\colon V_2 \to V_3, \quad x_2 \mapsto x_1 *_3 x_2 \quad \text{and} \quad r_{x_1}\colon V_3 \to V_2, \quad x_3 \mapsto x_3 *_2 x_1.$$

By (3.10) and (3.12) we have

$$\ell_{x_1} \circ r_{x_1} = q_1(x_1)\,\mathrm{Id}_{V_3} \quad \text{and} \quad r_{x_1} \circ \ell_{x_1} = q_1(x_1)\,\mathrm{Id}_{V_2}.$$

Therefore, the linear map

$$\alpha\colon V_1 \to \mathrm{End}(V_2 \oplus V_3), \quad x_1 \mapsto \begin{pmatrix} 0 & r_{x_1} \\ \ell_{x_1} & 0 \end{pmatrix}$$

extends to an F-algebra homomorphism defined on the Clifford algebra $C(V_1,q_1)$:

$$C(\alpha)\colon C(V_1,q_1) \to \mathrm{End}(V_2 \oplus V_3).$$

The image of the even Clifford algebra $C_0(V_1,q_1)$ lies in the diagonal subalgebra, hence $C(\alpha)$ restricts to an F-algebra homomorphism

$$C_0(\alpha)\colon C_0(V_1,q_1) \to (\mathrm{End}\, V_2) \times (\mathrm{End}\, V_3).$$

We write τ_1 for the involution on $C(V_1,q_1)$ that leaves every vector in V_1 fixed, and τ_{01} for the restriction of τ_1 to $C_0(V_1,q_1)$. We let $\sigma_{b_2\perp b_3}$ (resp. σ_{b_2}, resp. σ_{b_3}) denote the involution on $\mathrm{End}(V_2 \oplus V_3)$ (resp. $\mathrm{End}\, V_2$, resp. $\mathrm{End}\, V_3$) adjoint to $b_2 \perp b_3$ (resp. b_2, resp. b_3).

Theorem 3.7. *The maps $C(\alpha)$ and $C_0(\alpha)$ are homomorphisms of algebras with involution*

$$C(\alpha)\colon \big(C(V_1,q_1),\tau_1\big) \to \big(\mathrm{End}(V_2\oplus V_3),\sigma_{b_2\perp b_3}\big),$$
$$C_0(\alpha)\colon \big(C_0(V_1,q_1),\tau_{01}\big) \to (\mathrm{End}\,V_2,\sigma_{b_2})\times(\mathrm{End}\,V_3,\sigma_{b_3}).$$

Moreover, $\dim\mathcal{C} = 1$, 2, 4 *or* 8.

Proof. For the first part, it suffices to show that for $x_1\in V_1$,

$$\sigma_{b_2\perp b_3}\begin{pmatrix}0 & r_{x_1}\\ \ell_{x_1} & 0\end{pmatrix} = \begin{pmatrix}0 & r_{x_1}\\ \ell_{x_1} & 0\end{pmatrix}.$$

This amounts to proving that for $x_1\in V_1$, x_2, $y_2\in V_2$ and x_3, $y_3\in V_3$

$$b_2(x_3 *_2 x_1, y_2) + b_3(x_1 *_3 x_2, y_3) = b_2(x_2, y_3 *_2 x_1) + b_3(x_3, x_1 *_3 y_2),$$

which follows from (3.1).

To determine the various options for $\dim\mathcal{C}$, observe that the map $C(\alpha)$ endows $V_2\oplus V_3$ with a structure of left $C(V_1,q_1)$-module; similarly, V_2 and V_3 are left modules over $C_0(V_1,q_1)$ through $C_0(\alpha)$. This observation yields restrictions on the dimensions of V_2 and V_3, because the dimension of a left module over a central simple algebra A is a multiple of $(\deg A)(\mathrm{ind}\,A)$, where $\deg A$ is the degree of A and $\mathrm{ind}\,A$ is its (Schur) index.

Let $n=\dim\mathcal{C}$. If n is even, then $C(V_1,q_1)$ is a central simple F-algebra, and $V_2\oplus V_3$ is a left module over $C(V_1,q_1)$ through $C(\alpha)$, hence $(\deg C(V_1,q_1))(\mathrm{ind}\,C(V_1,q_1))$ divides $2n$. Since $\deg C(V_1,q_1) = 2^{n/2}$, it follows that $2^{n/2}$ divides $2n$, hence $n=2$, 4 or 8.

If n is odd, the even Clifford algebra $C_0(V_1,q_1)$ is central simple over F, and V_2 is a left module over $C_0(V_1,q_1)$ through $C_0(\alpha)$, hence $(\deg C_0(V_1,q_1))(\mathrm{ind}\,C_0(V_1,q_1))$ divides $\dim V_2$. As $\deg C_0(V_1,q_1) = 2^{(n-1)/2}$, this means that $2^{(n-1)/2}\,\mathrm{ind}\,C_0(V_1,q_1)$ divides n. As n is assumed to be odd, we must have $n=1$. ∎

Mimicking the construction above, we attach to the derived compositions $\partial\mathcal{C}$ and $\partial^2\mathcal{C}$ linear maps

$$\alpha'\colon V_2\to\mathrm{End}(V_3\oplus V_1),\quad x_2\mapsto\begin{pmatrix}0 & r_{x_2}\\ \ell_{x_2} & 0\end{pmatrix}$$

and

$$\alpha''\colon V_3\to\mathrm{End}(V_1\oplus V_2),\quad x_3\mapsto\begin{pmatrix}0 & r_{x_3}\\ \ell_{x_3} & 0\end{pmatrix}.$$

These maps yield homomorphisms

$$\begin{aligned} C(\alpha')&\colon \big(C(V_2,q_2),\tau_2\big)\to\big(\mathrm{End}(V_3\oplus V_1),\sigma_{b_3\perp b_1}\big),\\ C_0(\alpha')&\colon \big(C_0(V_2,q_2),\tau_{02}\big)\to(\mathrm{End}\,V_3,\sigma_{b_3})\times(\mathrm{End}\,V_1,\sigma_{b_1})\end{aligned} \tag{3.20}$$

and

$$\begin{aligned} C(\alpha'')&\colon \big(C(V_3,q_3),\tau_3\big)\to\big(\mathrm{End}(V_1\oplus V_2),\sigma_{b_1\perp b_2}\big),\\ C_0(\alpha'')&\colon \big(C_0(V_3,q_3),\tau_{03}\big)\to(\mathrm{End}\,V_1,\sigma_{b_1})\times(\mathrm{End}\,V_2,\sigma_{b_2}).\end{aligned} \tag{3.21}$$

We next take a closer look at compositions of the various degrees. If $\dim\mathcal{C}=1$, then $\operatorname{char}F\neq 2$ since odd-dimensional quadratic forms are singular in characteristic 2. If q_1 represents $\lambda_1\in F^\times$ and q_2 represents $\lambda_2\in F^\times$, then by multiplicativity q_3 represents $\lambda_1\lambda_2\in F^\times$, hence also $(\lambda_1\lambda_2)^{-1}$. Thus in this case there exist $\lambda_1,\lambda_2,\lambda_3\in F^\times$ such that $\lambda_1\lambda_2\lambda_3=1$ and

$$q_1\simeq\langle\lambda_1\rangle,\quad q_2\simeq\langle\lambda_2\rangle,\quad q_3\simeq\langle\lambda_3\rangle,$$

and $\langle 1\rangle\perp q_1\perp q_2\perp q_3$ is a 2-fold Pfister form. We will mostly ignore this easy case. (See however Example 3.6 (3).)

Proposition 3.8. *Let* $\dim\mathcal{C}=2$. *There exists a* 1*-fold Pfister form* $n_{\mathcal{C}}$, *uniquely determined up to isometry, and scalars* λ_1, λ_2, $\lambda_3\in F^\times$ *such that* $\lambda_1\lambda_2\lambda_3=1$ *and*

$$q_1\simeq\langle\lambda_1\rangle n_{\mathcal{C}},\quad q_2\simeq\langle\lambda_2\rangle n_{\mathcal{C}},\quad q_3\simeq\langle\lambda_3\rangle n_{\mathcal{C}}.$$

The form $n_{\mathcal{C}}\perp q_1\perp q_2\perp q_3$ *is a* 3*-fold Pfister form canonically associated to* $\mathcal{C}$ *up to isometry.*

Proof. Since $\dim V_1=2$, we have $q_1\simeq\langle\lambda_1\rangle n_{\mathcal{C}}$ for some $\lambda_1\in F^\times$ and some uniquely determined 1-fold Pfister form $n_{\mathcal{C}}$. For any anisotropic $y_2\in V_2$, the map $r_{y_2}\colon V_1\to V_3$ carrying x_1 to $x_1 *_3 y_2$ is a similitude with multiplier $q_2(y_2)$ by (3.2), hence $q_3\simeq\langle q_2(y_2)\rangle q_1$. Similarly, for any anisotropic $y_1\in V_1$ the map $\ell_{y_1}\colon V_2\to V_3$ is a similitude with multiplier $q_1(y_1)$, hence $q_3\simeq\langle q_1(y_1)\rangle q_2$. Therefore,

$$q_3\simeq\big\langle\lambda_1 q_2(y_2)\big\rangle n_{\mathcal{C}}\quad\text{and}\quad q_2\simeq\big\langle q_1(y_1)\big\rangle q_3\simeq\big\langle\lambda_1 q_1(y_1)q_2(y_2)\big\rangle n_{\mathcal{C}}.$$

Now, $\lambda_1 q_1(y_1)$ is represented by $n_{\mathcal{C}}$ since $q_1\simeq\langle\lambda_1\rangle n_{\mathcal{C}}$, hence $\langle\lambda_1 q_1(y_1)\rangle n_{\mathcal{C}}\simeq n_{\mathcal{C}}$. Letting $\lambda_2=q_2(y_2)$ and $\lambda_3=(\lambda_1 q_2(y_2))^{-1}$, we then have $\lambda_1\lambda_2\lambda_3=1$ and

$$q_1\simeq\langle\lambda_1\rangle n_{\mathcal{C}},\quad q_2\simeq\langle\lambda_2\rangle n_{\mathcal{C}},\quad q_3\simeq\langle\lambda_3\rangle n_{\mathcal{C}}.\qquad\blacksquare$$

Proposition 3.9. *Let* $\dim\mathcal{C}=4$. *There exists a* 2*-fold quadratic Pfister form* $n_{\mathcal{C}}$, *uniquely determined up to isometry, and scalars* λ_1, λ_2, $\lambda_3\in F^\times$ *such that* $\lambda_1\lambda_2\lambda_3=1$ *and*

$$q_1\simeq\langle\lambda_1\rangle n_{\mathcal{C}},\quad q_2\simeq\langle\lambda_2\rangle n_{\mathcal{C}},\quad q_3\simeq\langle\lambda_3\rangle n_{\mathcal{C}}.$$

The form $n_{\mathcal{C}}\perp q_1\perp q_2\perp q_3$ *is a* 4*-fold Pfister form canonically associated to* $\mathcal{C}$ *up to isometry.*

Proof. Consider the homomorphisms of algebras with involution induced by $C_0(\alpha)$:

$$\varphi_2\colon\big(C_0(V_1,q_1),\tau_{01}\big)\to(\operatorname{End}V_2,\sigma_{b_2})\quad\text{and}\quad\varphi_3\colon\big(C_0(V_1,q_1),\tau_{01}\big)\to(\operatorname{End}V_3,\sigma_{b_3}).$$

If Z is a field, then $C_0(V_1,q_1)$ is simple and its image under φ_2 is the centralizer in $\operatorname{End}V_2$ of a separable quadratic subfield fixed under σ_{b_2}. But the restriction of σ_{b_2} to such a centralizer is an orthogonal involution (see [13, (4.12)]), whereas τ_{01} is symplectic, so this case is impossible. Therefore, Z is not a field, which means that the discriminant (or Arf invariant) of q_1 is trivial. It follows that q_1 is a multiple of some uniquely determined 2-fold Pfister form $n_{\mathcal{C}}$. The same arguments as in the proof of Proposition 3.8 show that there exist $\lambda_1, \lambda_2, \lambda_3 \in F^\times$ such that $q_i \simeq \langle\lambda_i\rangle n_{\mathcal{C}}$ for $i = 1, 2, 3$. ■

Finally, we consider the case where $\dim \mathcal{C} = 8$. Recall from Section 2.2 that in this case the Clifford algebra $C(V_1, q_1)$ and the even Clifford algebra $C_0(V_1, q_1)$ carry canonical quadratic pairs. We use for these quadratic pairs the notation $(\tau_1, \mathfrak{g}_1)$ and $(\tau_{01}, \mathfrak{g}_{01})$ respectively.

Proposition 3.10. *Let* $\dim \mathcal{C} = 8$. *The canonical maps* $C(\alpha)$ *and* $C_0(\alpha)$ *are isomorphisms of algebras with quadratic pair*

$$C(\alpha)\colon \big(C(V_1, q_1), \tau_1, \mathfrak{g}_1\big) \xrightarrow{\sim} \big(\mathrm{End}(V_2 \oplus V_3), \sigma_{b_2 \perp b_3}, \mathfrak{f}_{q_2 \perp q_3}\big),$$
$$C_0(\alpha)\colon \big(C_0(V_1, q_1), \tau_{01}, \mathfrak{g}_{01}\big) \xrightarrow{\sim} (\mathrm{End}\, V_2, \sigma_{b_2}, \mathfrak{f}_{q_2}) \times (\mathrm{End}\, V_3, \sigma_{b_3}, \mathfrak{f}_{q_3}).$$

Moreover, there exists a 3*-fold quadratic Pfister form* $n_{\mathcal{C}}$*, uniquely determined up to isometry, and scalars* $\lambda_1, \lambda_2, \lambda_3 \in F^\times$ *such that* $\lambda_1\lambda_2\lambda_3 = 1$ *and*

$$q_1 \simeq \langle\lambda_1\rangle n_{\mathcal{C}}, \quad q_2 \simeq \langle\lambda_2\rangle n_{\mathcal{C}}, \quad q_3 \simeq \langle\lambda_3\rangle n_{\mathcal{C}}.$$

The form $n_{\mathcal{C}} \perp q_1 \perp q_2 \perp q_3$ *is a* 5*-fold Pfister form canonically associated to* $\mathcal{C}$ *up to isometry.*

Proof. In this case we have

$$\dim C(V_1, q_1) = \dim \mathrm{End}(V_2 \oplus V_3).$$

Since the algebra $C(V_1, q_1)$ is simple, it follows that $C(\alpha)$ is an algebra isomorphism, hence $C(V_1, q_1)$ is split. Moreover, $C_0(\alpha)$ also is an isomorphism, hence the center of $C_0(V_1, q_1)$ is isomorphic to $F \times F$, and therefore the discriminant (or Arf invariant) of q_1 is trivial. It follows that q_1 is a multiple of some uniquely determined 3-fold Pfister form $n_{\mathcal{C}}$, and the existence of $\lambda_1, \lambda_2, \lambda_3 \in F^\times$ such that $q_i \simeq \langle\lambda_i\rangle n_{\mathcal{C}}$ for $i = 1, 2, 3$ is proved as in the case where $\dim \mathcal{C} = 2$ (see Proposition 3.8).

Since we already know from Theorem 3.7 that $C(\alpha)$ and $C_0(\alpha)$ are homomorphisms of algebras with involution, it only remains to see that these maps also preserve the semi-traces.

The arguments in each case are similar. For $C(\alpha)$ we have to show that

$$\mathfrak{f}_{q_2 \perp q_3}\big(C(\alpha)(s)\big) = \mathfrak{g}_1(s) \quad \text{for all } s \in \mathrm{Sym}(\tau_1).$$

Fix $e_1, e_1' \in V_1$ such that $b_1(e_1, e_1') = 1$. By definition, $\mathfrak{g}_1(s) = \mathrm{Trd}_{C(V_1, q_1)}(e_1 e_1' s)$. Since isomorphisms of central simple algebras preserve reduced traces, we have for all $s \in \mathrm{Sym}(\tau_1)$

$$\mathfrak{g}_1(s) = \mathrm{Trd}_{\mathrm{End}(V_2 \oplus V_3)}\big(C(\alpha)(e_1 e_1' s)\big) = \mathrm{Trd}_{\mathrm{End}(V_2 \oplus V_3)}\big(C(\alpha)(e_1 e_1') \circ C(\alpha)(s)\big).$$

Now, $C(\alpha)(\mathrm{Sym}(\tau_1)) = \mathrm{Sym}(\sigma_{b_2\perp b_3})$ because $C(\alpha)$ is an isomorphism of algebras with involution. Therefore, we may rewrite the equation we have to prove as

$$\mathfrak{f}_{q_2\perp q_3}(s') = \mathrm{Trd}_{\mathrm{End}(V_2\oplus V_3)}\big(C(\alpha)(e_1e_1')\circ s'\big) \quad \text{for all } s' \in \mathrm{Sym}(\sigma_{b_2\perp b_3}).$$

Using the standard identification

$$\mathrm{End}(V_2\oplus V_3) = (V_2\oplus V_3)\otimes(V_2\oplus V_3)$$

set up in Section 2.1, we see that $\mathrm{Sym}(\sigma_{b_2\perp b_3})$ is spanned by elements of the form $(x_2 + x_3)\otimes(x_2+x_3)$ with $x_2\in V_2$ and $x_3\in V_3$, and that for $s' = (x_2+x_3)\otimes(x_2+x_3)$

$$\begin{aligned} C(\alpha)(e_1e_1')\circ s' &= \big(C(\alpha)(e_1e_1')(x_2+x_3)\big)\otimes(x_2+x_3)\\ &= \big(r_{e_1}\ell_{e_1'}(x_2) + \ell_{e_1}r_{e_1'}(x_3)\big)\otimes(x_2+x_3). \end{aligned}$$

Therefore, it suffices to show that for all $x_2\in V_2$ and $x_3\in V_3$

$$\begin{aligned} &\mathfrak{f}_{q_2\perp q_3}\big((x_2+x_3)\otimes(x_2+x_3)\big)\\ &\quad = \mathrm{Trd}_{\mathrm{End}(V_2\oplus V_3)}\big(\big(r_{e_1}\ell_{e_1'}(x_2)+\ell_{e_1}r_{e_1'}(x_3)\big)\otimes(x_2+x_3)\big). \end{aligned} \tag{3.22}$$

The right side is

$$\begin{aligned} &(b_2\perp b_3)\big((e_1' *_3 x_2) *_2 e_1 + e_1 *_3 (x_3 *_2 e_1'), x_2 + x_3\big)\\ &\quad = b_2\big((e_1' *_3 x_2) *_2 e_1, x_2\big) + b_3\big(e_1 *_3 (x_3 *_2 e_1'), x_3\big). \end{aligned}$$

Now, by (3.1) and (3.4) we have

$$b_2\big((e_1' *_3 x_2) *_2 e_1, x_2\big) = b_3(e_1' *_3 x_2, e_1 *_3 x_2) = b_1(e_1', e_1)q_2(x_2)$$

and, similarly,

$$b_3\big(e_1 *_3 (x_3 *_2 e_1'), x_3\big) = b_2(x_3 *_2 e_1', x_3 *_2 e_1) = q_3(x_3)b_1(e_1', e_1).$$

As $b_1(e_1, e_1') = 1$, it follows that

$$\mathrm{Trd}_{\mathrm{End}(V_2\oplus V_3)}\big(\big(r_{e_1}\ell_{e_1'}(x_2)+\ell_{e_1}r_{e_1'}(x_3)\big)\otimes(x_2+x_3)\big) = q_2(x_2)+q_3(x_3).$$

On the other hand, by definition of $\mathfrak{f}_{q_2\perp q_3}$ we have

$$\mathfrak{f}_{q_2\perp q_3}\big((x_2+x_3)\otimes(x_2+x_3)\big) = (q_2\perp q_3)(x_2+x_3) = q_2(x_2)+q_3(x_3).$$

We have thus checked (3.22).

The proof that $C_0(\alpha)$ also preserves the semitraces is obtained by a slight variation of the preceding arguments. We have to show that

$$(\mathfrak{f}_{q_2},\mathfrak{f}_{q_3})\big(C_0(\alpha)(s)\big) = C_0(\alpha)\big(\mathfrak{g}_{01}(s)\big) \quad \text{for all } s\in\mathrm{Sym}(\tau_{01}).$$

Since $C_0(\alpha)$ is an isomorphism of algebras with involution, this amounts to showing

$$\big(\mathfrak{f}_{q_2}(s_2'), \mathfrak{f}_{q_3}(s_3')\big) = \big(\mathrm{Trd}_{\mathrm{End}\,V_2}(r_{e_1}\ell_{e_1'}s_2'), \mathrm{Trd}_{\mathrm{End}\,V_3}(\ell_{e_1}r_{e_1'}s_3')\big) \tag{3.23}$$

for all $s_2' \in \mathrm{Sym}(\sigma_{b_2})$, $s_3' \in \mathrm{Sym}(\sigma_{b_3})$. It suffices to consider s_2', s_3' of the form $x_2 \otimes x_2$, $x_3 \otimes x_3$ for $x_2 \in V_2$, $x_3 \in V_3$ under the standard identifications $\mathrm{End}\,V_2 = V_2 \otimes V_2$, $\mathrm{End}\,V_3 = V_3 \otimes V_3$. For $s_2' = x_2 \otimes x_2$ we have

$$r_{e_1}\ell_{e_1'}s_2' = \big((e_1' *_3 x_2) *_2 e_1\big) \otimes x_2,$$

hence

$$\mathrm{Trd}_{\mathrm{End}\,V_2}(r_{e_1}\ell_{e_1'}s_2') = b_2\big((e_1' *_3 x_2) *_2 e_1, x_2\big) = b_3(e_1' *_3 x_2, e_1 *_3 x_2) = q_2(x_2).$$

On the other hand $\mathfrak{f}_2(x_2 \otimes x_2) = q_2(x_2)$ by definition. Likewise, for $s_3' = x_3 \otimes x_3$

$$\mathrm{Trd}_{\mathrm{End}\,V_3}(\ell_{e_1}r_{e_1'}s_3') = q_3(x_3) = \mathfrak{f}_{q_3}(s_3'),$$

hence (3.23) is proved. ∎

Remark 3.11. The map $C_0(\alpha)$ in Proposition 3.10 yields an isomorphism between the center of $C_0(V_1, q_1)$ and $F \times F$, hence also a polarization of (V_1, q_1) (see Definition 2.3): the primitive central idempotents z_+ and z_- of $C_0(V_1, q_1)$ are such that $C_0(\alpha)(z_+) = (1, 0)$ and $C_0(\alpha)(z_-) = (0, 1)$, so that $C_0(\alpha)$ induces homomorphisms

$$C_+(\alpha)\colon C_0(V_1, q_1) \to \mathrm{End}\,V_2 \quad \text{and} \quad C_-(\alpha)\colon C_0(V_1, q_1) \to \mathrm{End}\,V_3.$$

Similarly, the maps $C_0(\alpha')$ and $C_0(\alpha'')$ of (3.20) and (3.21) attached to $\partial\mathscr{C}$ and $\partial^2\mathscr{C}$ yield polarizations of (V_2, q_2) and (V_3, q_3), so that

$$C_+(\alpha')\colon C_0(V_2, q_2) \to \mathrm{End}\,V_3 \quad \text{and} \quad C_-(\alpha')\colon C_0(V_2, q_2) \to \mathrm{End}\,V_1,$$

and

$$C_+(\alpha'')\colon C_0(V_3, q_3) \to \mathrm{End}\,V_1 \quad \text{and} \quad C_-(\alpha'')\colon C_0(V_3, q_3) \to \mathrm{End}\,V_2.$$

Corollary 3.12. *For any composition of quadratic spaces $\mathscr{C}$, the following are equivalent:*

(i) $q_1 \simeq q_2 \simeq q_3$;

(ii) *q_1, q_2 and q_3 are Pfister forms;*

(iii) *q_1, q_2 and q_3 represent* 1.

Proof. According to Propositions 3.8, 3.9, and 3.10, there exist a quadratic Pfister form $n_{\mathscr{C}}$ and scalars λ_1, λ_2, λ_3 such that $\lambda_1\lambda_2\lambda_3 = 1$ and $q_1 \simeq \langle\lambda_1\rangle n_{\mathscr{C}}$, $q_2 \simeq \langle\lambda_2\rangle n_{\mathscr{C}}$ and $q_3 \simeq \langle\lambda_3\rangle n_{\mathscr{C}}$. This also holds when $\dim\mathscr{C} = 1$, with $n_{\mathscr{C}} = \langle 1\rangle$.

(i)⇒(ii) If $q_1 \simeq q_2$, then $\langle\lambda_1\lambda_2\rangle n_{\mathscr{C}} \simeq n_{\mathscr{C}}$, hence $\langle\lambda_3\rangle n_{\mathscr{C}} \simeq n_{\mathscr{C}}$. Therefore, $q_3 \simeq n_{\mathscr{C}}$.

(ii)⇒(iii) This is clear since Pfister quadratic forms represent 1.

(iii)⇒(i) For $i = 1, 2, 3$, if q_i represents 1, then $n_{\mathscr{C}}$ represents λ_i, hence

$$\langle\lambda_i\rangle n_{\mathscr{C}} \simeq n_{\mathscr{C}}.$$ ∎

3.3. Similitudes and isomorphisms

Consider two compositions of quadratic spaces $\mathscr{C} = ((V_1, q_1), (V_2, q_2), (V_3, q_3), *_3)$ and $\widetilde{\mathscr{C}} = ((\widetilde{V}_1, \tilde{q}_1), (\widetilde{V}_2, \tilde{q}_2), (\widetilde{V}_3, \tilde{q}_3), \tilde{*}_3)$ over an arbitrary field F. As in Definition 3.5, we write $*_1$ and $*_2$ (resp. $\tilde{*}_1$, $\tilde{*}_2$) for the derived composition maps of $*_3$ (resp. $\tilde{*}_3$).

Definition 3.13. For $i = 1, 2, 3$, let $g_i\colon (V_i, q_i) \to (\widetilde{V}_i, \tilde{q}_i)$ be a similitude with multiplier $\mu(g_i) \in F^\times$. The triple (g_1, g_2, g_3) is a *similitude of compositions* $\mathscr{C} \to \widetilde{\mathscr{C}}$ if there exists $\lambda_3 \in F^\times$ such that

$$\lambda_3 g_3(x_1 *_3 x_2) = g_1(x_1) \tilde{*}_3 g_2(x_2) \quad \text{for all } x_1 \in V_1,\ x_2 \in V_2. \tag{3.24}$$

Proposition 3.14. *If $g = (g_1, g_2, g_3)$ is a similitude $\mathscr{C} \to \widetilde{\mathscr{C}}$, then $\partial g := (g_2, g_3, g_1)$ is a similitude $\partial\mathscr{C} \to \partial\widetilde{\mathscr{C}}$ and $\partial^2 g := (g_3, g_1, g_2)$ is a similitude $\partial^2\mathscr{C} \to \partial^2\widetilde{\mathscr{C}}$. Moreover, the scalars λ_1, λ_2, $\lambda_3 \in F^\times$ such that for all $x_1 \in V_1$, $x_2 \in V_2$, $x_3 \in V_3$*

$$\begin{aligned}
\lambda_1 g_1(x_2 *_1 x_3) &= g_2(x_2) \tilde{*}_1 g_3(x_3),\\
\lambda_2 g_2(x_3 *_2 x_1) &= g_3(x_3) \tilde{*}_2 g_1(x_1),\\
\lambda_3 g_3(x_1 *_3 x_2) &= g_1(x_1) \tilde{*}_3 g_2(x_2)
\end{aligned}$$

are related to the multipliers of g_1, g_2, g_3 by

$$\mu(g_1) = \lambda_2\lambda_3, \quad \mu(g_2) = \lambda_3\lambda_1, \quad \mu(g_3) = \lambda_1\lambda_2. \tag{3.25}$$

Proof. For the first part, we have to prove the existence of $\lambda_1 \in F^\times$ such that

$$\lambda_1 g_1(x_2 *_1 x_3) = g_2(x_2) \tilde{*}_1 g_3(x_3) \quad \text{for all } x_2 \in V_2 \text{ and } x_3 \in V_3.$$

Multiplying each side of (3.24) on the left by $g_2(x_2)$, we obtain for all $x_1 \in V_1$ and $x_2 \in V_2$

$$\lambda_3 g_2(x_2) \tilde{*}_1 g_3(x_1 *_3 x_2) = \tilde{q}_2\big(g_2(x_2)\big) g_1(x_1) = \mu(g_2) q_2(x_2) g_1(x_1).$$

If x_2 is anisotropic, then $r_{x_2}\colon V_1 \to V_3$ is bijective with inverse $q_2(x_2)^{-1}\ell_{x_2}$, hence every $x_3 \in V_3$ can be written as $x_3 = x_1 *_3 x_2$ with $x_1 = q_2(x_2)^{-1} x_2 *_1 x_3$. Substituting in the last displayed equation, we obtain for $x_2 \in V_2$ anisotropic and $x_3 \in V_3$

$$g_2(x_2) \tilde{*}_1 g_3(x_3) = \mu(g_2)\lambda_3^{-1} g_1(x_2 *_1 x_3).$$

Since anisotropic vectors span V_2, this relation holds for all $x_2 \in V_2$ and $x_3 \in V_3$. Therefore, (g_2, g_3, g_1) is a similitude of compositions, with scalar $\lambda_1 = \mu(g_2)\lambda_3^{-1}$.

Applying the same arguments to ∂g instead of g, we see that $\partial(\partial g) = \partial^2 g$ is a similitude $\partial^2\mathscr{C} \to \partial^2\widetilde{\mathscr{C}}$, with scalar $\lambda_2 = \mu(g_3)\lambda_1^{-1}$. Applying the arguments one more time, we obtain that g is a similitude $\mathscr{C} \to \widetilde{\mathscr{C}}$ with scalar $\mu(g_1)\lambda_2^{-1}$, hence

$$\lambda_3 = \mu(g_1)\lambda_2^{-1}$$

and the proof is complete. ∎

Definition 3.15. In the situation of Proposition 3.14, the triple $(\lambda_1, \lambda_2, \lambda_3) \in F^\times \times F^\times \times F^\times$ is said to be the *composition multiplier* of the similitude of compositions $g\colon \mathscr{C} \to \widetilde{\mathscr{C}}$, and we write

$$\lambda(g) = (\lambda_1, \lambda_2, \lambda_3),$$

hence $\lambda(\partial g) = (\lambda_2, \lambda_3, \lambda_1)$ and $\lambda(\partial^2 g) = (\lambda_3, \lambda_1, \lambda_2)$. Writing $\rho(g) = \lambda_1\lambda_2\lambda_3$, we thus have by (3.25)

$$\lambda(g) = \big(\rho(g)\mu(g_1)^{-1}, \rho(g)\mu(g_2)^{-1}, \rho(g)\mu(g_3)^{-1}\big) \quad \text{and} \quad \mu(g_1)\mu(g_2)\mu(g_3) = \rho(g)^2.$$

Similitudes with composition multiplier $(1, 1, 1)$ are called *isomorphisms* of compositions.

Proposition 3.16. *Every composition of quadratic spaces is similar to a composition of isometric quadratic spaces.*

Proof. Let $\mathscr{C} = ((V_1, q_1), (V_2, q_2), (V_3, q_3), *_3)$ be an arbitrary composition of quadratic spaces. Let $\lambda_1 \in F^\times$ (resp. $\lambda_2 \in F^\times$) be a value represented by q_1 (resp. q_2) and let $\lambda_3 = \lambda_1^{-1}\lambda_2^{-1} \in F^\times$. Then λ_3 is represented by q_3; define quadratic forms $\tilde{q}_1$, $\tilde{q}_2$, $\tilde{q}_3$ on V_1, V_2, V_3 by

$$\tilde{q}_1(x_1) = \lambda_1^{-1} q_1(x_1), \quad \tilde{q}_2(x_2) = \lambda_2^{-1} q_2(x_2), \quad \tilde{q}_3(x_3) = \lambda_3^{-1} q_3(x_3)$$

for $x_1 \in V_1$, $x_2 \in V_2$ and $x_3 \in V_3$. Depending on the dimension of $\mathscr{C}$, Proposition 3.8, 3.9 or 3.10 shows that the forms $\tilde{q}_1$, $\tilde{q}_2$ and $\tilde{q}_3$ are isometric Pfister forms. Define a map $\tilde{*}_3\colon V_1 \times V_2 \to V_3$ by

$$x_1 \tilde{*}_3 x_2 = \lambda_3 x_1 *_3 x_2 \quad \text{for } x_1 \in V_1 \text{ and } x_2 \in V_2.$$

Straightforward computations show that $\widetilde{\mathscr{C}} = ((\widetilde{V}_1, \tilde{q}_1), (\widetilde{V}_2, \tilde{q}_2), (\widetilde{V}_3, \tilde{q}_3), \tilde{*}_3)$ is a composition, and that $(\mathrm{Id}_{V_1}, \mathrm{Id}_{V_2}, \mathrm{Id}_{V_3})\colon \mathscr{C} \to \widetilde{\mathscr{C}}$ is a similitude of compositions, with composition multiplier $(\lambda_1, \lambda_2, \lambda_3)$. ■

Auto-similitudes of compositions of quadratic spaces define algebraic groups which we discuss next.

For every composition $\mathscr{C} = ((V_1, q_1), (V_2, q_2), (V_3, q_3), *_3)$, we associate to each similitude $(g_1, g_2, g_3)\colon \mathscr{C} \to \mathscr{C}$ with multiplier $(\lambda_1, \lambda_2, \lambda_3)$ the 4-tuple $(g_1, g_2, g_3, \lambda_3)$, from which λ_1 and λ_2 can be determined by the relations (3.25). We may thus consider the group of similitudes of $\mathscr{C}$ as the subgroup of $\mathrm{GO}(q_1) \times \mathrm{GO}(q_2) \times \mathrm{GO}(q_3) \times F^\times$ defined by the equations

$$\lambda_3 g_3(x_1 *_3 x_2) = g_1(x_1) *_3 g_2(x_2) \quad \text{for all } x_1 \in V_1, x_2 \in V_2.$$

These equations define a closed subgroup of $\mathbf{GO}(q_1) \times \mathbf{GO}(q_2) \times \mathbf{GO}(q_3) \times \mathbf{G_m}$, hence an algebraic group scheme, for which we use the notation $\mathbf{GO}(\mathscr{C})$. From Proposition 3.14 it follows that ∂ and ∂^2 yield isomorphisms

$$\partial\colon \mathbf{GO}(\mathscr{C}) \to \mathbf{GO}(\partial\mathscr{C}) \quad \text{and} \quad \partial^2\colon \mathbf{GO}(\mathscr{C}) \to \mathbf{GO}(\partial^2\mathscr{C})$$

defined as follows: for every commutative F-algebra R and $(g_1, g_2, g_3, \lambda_3) \in \mathbf{GO}(\mathscr{C})(R)$,

$$\partial(g_1, g_2, g_3, \lambda_3) = (g_2, g_3, g_1, \lambda_1) \quad \text{and} \quad \partial(g_1, g_2, g_3, \lambda_3) = (g_3, g_1, g_2, \lambda_2),$$

with

$$\lambda_1 = \mu(g_2)\lambda_3^{-1} \quad \text{and} \quad \lambda_2 = \mu(g_1)\lambda_3^{-1}.$$

The Lie algebra $\mathfrak{go}(\mathscr{C})$ of $\mathbf{GO}(\mathscr{C})$ consists of 4-tuples

$$(g_1, g_2, g_3, \lambda_3) \in \mathfrak{go}(q_1) \times \mathfrak{go}(q_2) \times \mathfrak{go}(q_3) \times F$$

satisfying the following condition:

$$\begin{aligned} &g_3(x_1 *_3 x_2) \\ &\quad = g_1(x_1) *_3 x_2 + x_1 *_3 g_2(x_2) - \lambda_3 x_1 *_3 x_2 \quad \text{for all } x_1 \in V_1, x_2 \in V_2. \end{aligned} \tag{3.26}$$

The following is the Lie algebra version of Proposition 3.14:

Proposition 3.17. *For $g = (g_1, g_2, g_3, \lambda_3) \in \mathfrak{go}(\mathscr{C})$, there are scalars λ_1, $\lambda_2 \in F$ such that*

$$\dot{\mu}(g_1) = \lambda_2 + \lambda_3, \quad \dot{\mu}(g_2) = \lambda_3 + \lambda_1, \quad \dot{\mu}(g_3) = \lambda_1 + \lambda_2$$

and for all $x_1 \in V_1$, $x_2 \in V_2$, $x_3 \in V_3$

$$\begin{aligned} g_1(x_2 *_1 x_3) &= g_2(x_2) *_1 x_3 + x_2 *_1 g_3(x_3) - \lambda_1 x_2 *_1 x_3, \\ g_2(x_3 *_2 x_1) &= g_3(x_3) *_2 x_1 + x_3 *_2 g_1(x_1) - \lambda_2 x_3 *_2 x_1, \\ g_3(x_1 *_3 x_2) &= g_1(x_1) *_3 x_2 + x_1 *_3 g_2(x_2) - \lambda_3 x_1 *_3 x_2. \end{aligned}$$

Thus, $\partial g := (g_2, g_3, g_1, \lambda_1)$ lies in $\mathfrak{go}(\partial\mathscr{C})$ and $\partial^2 g := (g_3, g_1, g_2, \lambda_2)$ in $\mathfrak{go}(\partial^2\mathscr{C})$.

The composition multiplier map $\lambda_{\mathscr{C}}$ yields a morphism of algebraic group schemes

$$\lambda_{\mathscr{C}} : \mathbf{GO}(\mathscr{C}) \to \mathbf{G}_{\mathbf{m}}^3$$

defined as follows: for every commutative F-algebra R and $(g_1, g_2, g_3, \lambda_3) \in \mathbf{GO}(\mathscr{C})(R)$,

$$\lambda_{\mathscr{C}}(g_1, g_2, g_3, \lambda_3) = \big(\mu(g_2)\lambda_3^{-1}, \mu(g_1)\lambda_3^{-1}, \lambda_3\big) \in R^\times \times R^\times \times R^\times. \tag{3.27}$$

Its differential $\dot{\lambda}_{\mathscr{C}} : \mathfrak{go}(\mathscr{C}) \to F \times F \times F$ is given by

$$\dot{\lambda}_{\mathscr{C}}(g_1, g_2, g_3, \lambda_3) = \big(\dot{\mu}(g_2) - \lambda_3, \dot{\mu}(g_1) - \lambda_3, \lambda_3\big).$$

We let $\mathbf{O}(\mathscr{C}) = \ker \lambda_{\mathscr{C}}$ and $\mathfrak{o}(\mathscr{C}) = \ker \dot{\lambda}_{\mathscr{C}}$, so $\mathbf{O}(\mathscr{C})$ is the algebraic group scheme of automorphisms of $\mathscr{C}$ and $\mathfrak{o}(\mathscr{C})$ is its Lie algebra.

Remark 3.18. For every commutative F-algebra R and $(g_1, g_2, g_3, \lambda_3) \in \mathbf{O}(\mathscr{C})(R)$ we have $\lambda_3 = 1$ and $\mu(g_1) = \mu(g_2) = 1$, hence also $\mu(g_1) = 1$ by (3.25). Thus, (g_1, g_2, g_3) is a *related triple* of isometries according to the definition given by Springer–Veldkamp [19, §3.6], Elduque [8, §1] or Alsaody–Gille [3, §3.1] for some specific compositions of quadratic spaces arising from composition algebras. In Section 4.4 below we establish isomorphisms $\mathbf{O}(\mathcal{C}) \simeq \mathbf{Spin}(q_1) \simeq \mathbf{Spin}(q_2) \simeq \mathbf{Spin}(q_3)$, which are the analogues of the isomorphisms given in [19, Prop. 3.6.3], [8, Th. 1.1] and [3, Th. 3.12] in terms of related triples.

Proposition 3.19. *The algebraic group schemes* $\mathbf{O}(\mathcal{C})$ *and* $\mathbf{GO}(\mathcal{C})$ *are smooth, and the following sequences are exact:*

$$1 \to \mathbf{O}(\mathcal{C}) \to \mathbf{GO}(\mathcal{C}) \xrightarrow{\lambda_{\mathcal{C}}} \mathbf{G}_{\mathbf{m}}^3 \to 1 \tag{3.28}$$

and

$$0 \to \mathfrak{o}(\mathcal{C}) \to \mathfrak{go}(\mathcal{C}) \xrightarrow{\dot{\lambda}_{\mathcal{C}}} F^3 \to 0. \tag{3.29}$$

Proof. Step 1: We show that $\lambda_{\mathcal{C}}$ is surjective. Since $\mathbf{G}_{\mathbf{m}}^3$ is smooth, it suffices by [13, (22.3)] to show that $\lambda_{\mathcal{C}}$ is surjective on points over an algebraic closure F_{alg} of F. For this, we consider the homotheties: if $\nu_1, \nu_2, \nu_3 \in F_{\mathrm{alg}}^\times$, then

$$\nu_i \operatorname{Id}_{(V_i)_{F_{\mathrm{alg}}}} : (V_i, q_i)_{F_{\mathrm{alg}}} \to (V_i, q_i)_{F_{\mathrm{alg}}}$$

is a similitude with multiplier ν_i^2, and

$$\big(\nu_1 \operatorname{Id}_{(V_1)_{F_{\mathrm{alg}}}}, \nu_2 \operatorname{Id}_{(V_2)_{F_{\mathrm{alg}}}}, \nu_3 \operatorname{Id}_{(V_3)_{F_{\mathrm{alg}}}}\big) : \mathcal{C}_{F_{\mathrm{alg}}} \to \mathcal{C}_{F_{\mathrm{alg}}}$$

is a similitude with multiplier $(\nu_2\nu_3\nu_1^{-1}, \nu_3\nu_1\nu_2^{-1}, \nu_1\nu_2\nu_3^{-1})$. Therefore, the image of the map $\lambda_{\mathcal{C}}$ in $(F_{\mathrm{alg}}^\times)^3$ contains $(\nu_2\nu_3\nu_1^{-1}, \nu_3\nu_1\nu_2^{-1}, \nu_1\nu_2\nu_3^{-1})$ for all $\nu_1, \nu_2, \nu_3 \in F_{\mathrm{alg}}^\times$. Given $\lambda_1, \lambda_2, \lambda_3 \in F_{\mathrm{alg}}^\times$, we may find $\nu_1, \nu_2, \nu_3 \in F_{\mathrm{alg}}^\times$ such that $\nu_2^2 = \lambda_1\lambda_3$, $\nu_3^2 = \lambda_1\lambda_2$ and $\nu_1 = \lambda_1^{-1}\nu_2\nu_3$. Then

$$(\nu_2\nu_3\nu_1^{-1}, \nu_3\nu_1\nu_2^{-1}, \nu_1\nu_2\nu_3^{-1}) = (\lambda_1, \lambda_2, \lambda_3),$$

proving surjectivity of $\lambda_{\mathcal{C}}$.

Step 2: We show that $\dot{\lambda}_{\mathcal{C}}$ is surjective. For $u_1, v_1 \in V_1$, consider the maps

$$\begin{aligned} g_1 &: V_1 \to V_1, \quad x_1 \mapsto u_1 b_1(v_1, x_1) - v_1 b_1(u_1, x_1), \\ g_2 &: V_2 \to V_2, \quad x_2 \mapsto (v_1 *_3 x_2) *_2 u_1, \\ g_3 &: V_3 \to V_3, \quad x_3 \mapsto u_1 *_3 (x_3 *_2 v_1). \end{aligned}$$

For $x_1 \in V_1$,

$$b_1\big(g_1(x_1), x_1\big) = b_1(u_1, x_1) b_1(v_1, x_1) - b_1(v_1, x_1) b_1(u_1, x_1) = 0,$$

hence $g_1 \in \mathfrak{go}(q_1)$ with $\dot{\mu}(g_1) = 0$ by Proposition 2.11. Moreover, (3.1), (3.5) and (3.8) yield for $x_2 \in V_2$ and $x_3 \in V_3$

$$\begin{aligned} b_2\big(g_2(x_2), x_2\big) &= b_3(v_1 * x_2, u_1 *_2 x_2) = b_1(v_1, u_1) q_2(x_2), \\ b_3\big(g_3(x_3), x_3\big) &= b_2(x_3 *_2 v_1, x_3 *_2 u_1) = q_3(x_3) b_1(v_1, u_1). \end{aligned}$$

Therefore, $g_2 \in \mathfrak{go}(q_2)$ and $g_3 \in \mathfrak{go}(q_3)$ with $\dot{\mu}(g_2) = \dot{\mu}(g_3) = b_1(v_1, u_1)$.

Now, for $x_1 \in V_1$ and $x_2 \in V_2$ we compute

$$g_3(x_1 *_3 x_2) = u_1 *_3 \big((x_1 *_3 x_2) *_2 v_1\big)$$

by using (3.18) twice in succession to interchange first x_1 and v_1, and then x_1 and u_1:

$$\begin{aligned} g_3(x_1 *_3 x_2) &= (u_1 *_3 x_2)b_1(v_1, x_1) - u_1 *_3 \big((v_1 *_3 x_2) *_2 x_1\big) \\ &= (u_1 *_3 x_2)b_1(v_1, x_1) - (v_1 *_3 x_2)b_1(u_1, x_1) + x_1 *_3 \big((v_1 *_3 x_2) *_2 u_1\big) \\ &= g_1(x_1) *_3 x_2 + x_1 *_3 g_2(x_2). \end{aligned}$$

It follows that $(g_1, g_2, g_3, 0)$ lies in $\mathfrak{go}(\mathcal{C})$, and the computation of $\dot\mu(g_2)$ and $\dot\mu(g_1)$ above yields

$$\dot\lambda_{\mathcal{C}}(g_1, g_2, g_3, 0) = (b_1(v_1, u_1), 0, 0).$$

Thus, taking u_1, v_1 such that $b_1(v_1, u_1) = 1$, we see that $(1, 0, 0)$ lies in the image of $\dot\lambda_{\mathcal{C}}$. Similarly, we may find $g' \in \mathfrak{go}(\partial\mathcal{C})$ and $g'' \in \mathfrak{go}(\partial^2\mathcal{C})$ such that $\dot\lambda_{\partial\mathcal{C}}(g') = \dot\lambda_{\partial^2\mathcal{C}}(g'') = (1, 0, 0)$. Then $\partial^2(g')$, $\partial(g'') \in \mathfrak{go}(\mathcal{C})$ satisfy $\dot\lambda_{\partial^2\mathcal{C}}(\partial^2(g')) = (0, 0, 1)$ and $\dot\lambda_{\partial\mathcal{C}}(\partial(g'')) = (0, 1, 0)$, hence $\dot\lambda_{\mathcal{C}}$ is surjective.

Steps 1 and 2 establish the exactness of the sequences (3.28) and (3.29). Step 2 shows that the surjective map $\lambda_{\mathcal{C}}$ is separable,[5] hence $\mathbf{O}(\mathcal{C})$ is smooth by [13, (22.13)]. Since $\mathbf{G}_{\mathbf{m}}^3$ is also smooth, it follows that $\mathbf{GO}(\mathcal{C})$ is smooth by [13, (22.12)]. ∎

Step 1 of the proof of Proposition 3.19 introduces the subgroup of homotheties of $\mathbf{GO}(\mathcal{C})$: this subgroup $\mathbf{H}(\mathcal{C})$ is the image of the closed embedding $\mathbf{G}_{\mathbf{m}}^3 \to \mathbf{GO}(\mathcal{C})$ given by

$$(\nu_1, \nu_2, \nu_3) \mapsto (\nu_1 \operatorname{Id}_{V_1}, \nu_2 \operatorname{Id}_{V_2}, \nu_3 \operatorname{Id}_{V_3}, \nu_1\nu_2\nu_3^{-1}).$$

The algebraic group $\mathbf{H}(\mathcal{C})$ lies in the center of $\mathbf{GO}(\mathcal{C})$, hence we may consider the quotient algebraic group

$$\mathbf{PGO}(\mathcal{C}) = \mathbf{GO}(\mathcal{C}) / \mathbf{H}(\mathcal{C}).$$

This is a smooth algebraic group since $\mathbf{GO}(\mathcal{C})$ is smooth. Let also

$$\mathbf{Z}(\mathcal{C}) = \mathbf{H}(\mathcal{C}) \cap \mathbf{O}(\mathcal{C}).$$

This group is the kernel of the canonical map $\mathbf{O}(\mathcal{C}) \to \mathbf{PGO}(\mathcal{C})$. For every commutative F-algebra R,

$$\mathbf{Z}(\mathcal{C})(R) = \big\{(\nu_1, \nu_2, \nu_3, 1) \mid \nu_1^2 = \nu_2^2 = \nu_3^2 = \nu_1\nu_2\nu_3 = 1\big\} \subset R^\times \times R^\times \times R^\times \times R^\times,$$

hence $\mathbf{Z}(\mathcal{C})$ is isomorphic to the kernel of the multiplication map $m\colon \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \to \boldsymbol{\mu}_2$ carrying (ν_1, ν_2, ν_3) to $\nu_1\nu_2\nu_3$. It is thus also isomorphic to $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$, hence it is a smooth algebraic group if and only if $\operatorname{char} F \neq 2$.

[5] Following [13, p. 341], a surjective homomorphism of group schemes is said to be separable if its differential is a surjective Lie algebra homomorphism.

Proposition 3.20. *The following diagram is commutative with exact rows and columns:*

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathbf{Z}(\mathcal{C}) & \longrightarrow & \mathbf{O}(\mathcal{C}) & \longrightarrow & \mathbf{PGO}(\mathcal{C}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathbf{H}(\mathcal{C}) & \longrightarrow & \mathbf{GO}(\mathcal{C}) & \longrightarrow & \mathbf{PGO}(\mathcal{C}) & \longrightarrow & 1 \\
 & & {\scriptstyle \lambda_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle \lambda_{\mathcal{C}}} & & & & \\
 & & \mathbf{G}_{\mathbf{m}}^3 & = & \mathbf{G}_{\mathbf{m}}^3 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
$$

Proof. Commutativity of the diagram is clear, and the lower row is exact by definition of $\mathbf{PGO}(\mathcal{C})$. Step 1 of the proof of Proposition 3.19 shows that $\lambda_{\mathcal{C}}: \mathbf{H}(\mathcal{C}) \to \mathbf{G}_{\mathbf{m}}^3$ is surjective, hence the left column is exact. Moreover, the right column is exact by Proposition 3.19; therefore it only remains to prove that the canonical map $\mathbf{O}(\mathcal{C}) \to \mathbf{PGO}(\mathcal{C})$ is surjective. Since $\mathbf{PGO}(\mathcal{C})$ is smooth, it suffices to consider the group of rational points over an algebraic closure F_{alg} of F. We know $\lambda_{\mathcal{C}}: \mathbf{H}(\mathcal{C}) \to \mathbf{G}_{\mathbf{m}}^3$ is surjective, hence for every $g \in \mathbf{GO}(\mathcal{C})(F_{\text{alg}})$ there exists $h \in \mathbf{H}(\mathcal{C})(F_{\text{alg}})$ such that $\lambda_{\mathcal{C}}(g) = \lambda_{\mathcal{C}}(h)$. Then gh^{-1} lies in $\mathbf{O}(\mathcal{C})(F_{\text{alg}})$ and has the same image in $\mathbf{PGO}(\mathcal{C})(F_{\text{alg}})$ as g, hence the canonical map $\mathbf{O}(\mathcal{C})(F_{\text{alg}}) \to \mathbf{PGO}(\mathcal{C})(F_{\text{alg}})$ is surjective. ∎

Let $\mathfrak{h}(\mathcal{C})$ and $\mathfrak{pgo}(\mathcal{C})$ be the Lie algebras of $\mathbf{H}(\mathcal{C})$ and $\mathbf{PGO}(\mathcal{C})$ respectively. By definition,

$$\mathfrak{h}(\mathcal{C}) = \big\{(\nu_1 \operatorname{Id}_{V_1}, \nu_2 \operatorname{Id}_{V_2}, \nu_3 \operatorname{Id}_{V_3}, \nu_1 + \nu_2 - \nu_3) \mid \nu_1, \nu_2, \nu_3 \in F\big\} \simeq F \times F \times F.$$

On the other hand, since $\mathbf{H}(\mathcal{C})$ is smooth, the canonical map $\mathbf{GO}(\mathcal{C}) \to \mathbf{PGO}(\mathcal{C})$ is separable by [13, (22.13)], hence its differential is surjective. Therefore,

$$\mathfrak{pgo}(\mathcal{C}) = \mathfrak{go}(\mathcal{C}) / \mathfrak{h}(\mathcal{C}).$$

The following result yields an explicit description of $\mathfrak{pgo}(\mathcal{C})$ for use in Section 4.3:

Proposition 3.21. *Mapping* $(g_1, g_2, g_3, \lambda_3) + \mathfrak{h}(\mathcal{C}) \in \mathfrak{pgo}(\mathcal{C})$ *to* $(g_1 + F, g_2 + F, g_3 + F) \in \mathfrak{pgo}(q_1) \times \mathfrak{pgo}(q_2) \times \mathfrak{pgo}(q_3)$ *identifies* $\mathfrak{pgo}(\mathcal{C})$ *with the subgroup of* $\mathfrak{pgo}(q_1) \times \mathfrak{pgo}(q_2) \times \mathfrak{pgo}(q_3)$ *consisting of triples* $(g_1 + F, g_2 + F, g_3 + F)$ *where* $g_1 \in \mathfrak{go}(q_1)$, $g_2 \in \mathfrak{go}(q_2)$ *and* $g_3 \in \mathfrak{go}(q_3)$ *satisfy* (3.26) *for some* $\lambda_3 \in F$.

Proof. It suffices to show that $\mathfrak{h}(\mathcal{C})$ is the kernel of the map

$$\mathfrak{go}(\mathcal{C}) \to \mathfrak{pgo}(q_1) \times \mathfrak{pgo}(q_2) \times \mathfrak{pgo}(q_3)$$

carrying $(g_1, g_2, g_3, \lambda_3)$ to $(g_1 + F, g_2 + F, g_3 + F)$. Clearly, $\mathfrak{h}(\mathscr{C})$ lies in the kernel of this map. Conversely, if $(g_1, g_2, g_3, \lambda_3)$ lies in the kernel, then there are scalars ν_1, ν_2, $\nu_3 \in F$ such that $g_i = \nu_i \operatorname{Id}_{V_i}$ for $i = 1, 2, 3$. Then (3.26) yields $\lambda_3 = \nu_1 + \nu_2 - \nu_3$, hence $(g_1, g_2, g_3, \lambda_3)$ lies in $\mathfrak{h}(\mathscr{C})$. ∎

Remark 3.22. If char $F \neq 2$, the upper row of the diagram in Proposition 3.20 shows that the canonical map $\mathfrak{o}(\mathscr{C}) \to \mathfrak{pgo}(\mathscr{C})$ is an isomorphism, for then $\mathbf{Z}(\mathscr{C})$ is smooth and its Lie algebra is 0. This canonical map is *not* bijective if char $F = 2$, even though $\mathfrak{o}(\mathscr{C})$ and $\mathfrak{pgo}(\mathscr{C})$ have the same dimension.

3.4. Compositions of pointed quadratic spaces

Fixing a representation of 1 in a quadratic space yields a new structure:

Definitions 3.23. A *pointed quadratic space* over an arbitrary field F is a triple (V, q, e) where (V, q) is a quadratic space with nonsingular polar form over F and $e \in V$ is a vector such that $q(e) = 1$. Each pointed quadratic space is endowed with a canonical isometry $^{-}$ of order 2, defined by

$$\bar{x} = eb(e, x) - x \quad \text{for } x \in V,$$

where b is the polar form of q. Isometries of pointed quadratic spaces are required to preserve the distinguished vector representing 1.

A *composition of pointed quadratic spaces* over F is a 4-tuple

$$\mathscr{C}^\bullet = \big((V_1, q_1, e_1), (V_2, q_2, e_2), (V_3, q_3, e_3), *_3\big) \tag{3.30}$$

where (V_1, q_1, e_1), (V_2, q_2, e_2), (V_3, q_3, e_3) are pointed quadratic spaces over F and the 4-tuple $\mathscr{C} := ((V_1, q_1)(V_2, q_2), (V_3, q_3), *_3)$ obtained by forgetting the distinguished vectors is a composition of quadratic spaces such that

$$e_1 *_3 e_2 = e_3.$$

It follows from the definition that q_1, q_2 and q_3 represent 1, hence these forms are isometric Pfister forms, by Proposition 3.8, 3.9 or 3.10 (depending on the dimension). Note that (3.10) readily yields

$$e_2 *_1 e_3 = e_1 \quad \text{and} \quad e_3 *_2 e_1 = e_2.$$

Therefore, the following are compositions of pointed quadratic spaces:

$$\partial\mathscr{C}^\bullet = \big((V_2, q_2, e_2), (V_3, q_3, e_3), (V_1, q_1, e_1), *_1\big),$$
$$\partial^2\mathscr{C}^\bullet = \big((V_3, q_3, e_3), (V_1, q_1, e_1), (V_2, q_2, e_2), *_2\big).$$

Let $\mathscr{C}^\bullet$ and $\widetilde{\mathscr{C}}^\bullet$ be compositions of pointed quadratic spaces, and let $\mathscr{C}$ and $\widetilde{\mathscr{C}}$ be the compositions of quadratic spaces obtained from $\mathscr{C}^\bullet$ and $\widetilde{\mathscr{C}}^\bullet$ by forgetting the distinguished vectors. Every similitude $f: \mathcal{C} \to \widetilde{\mathcal{C}}$ that preserves the distinguished vectors must be an isometry, because the equations

$$\begin{aligned}\lambda_1 f_1(x_2 *_1 x_3) &= f_2(x_2) \tilde{*}_1 f_3(x_3),\\ \lambda_2 f_2(x_3 *_2 x_1) &= f_3(x_3) \tilde{*}_2 f_1(x_1),\\ \lambda_3 f_3(x_1 *_3 x_2) &= f_1(x_1) \tilde{*}_3 f_2(x_2)\end{aligned}$$

for $x_1 \in V_1$, $x_2 \in V_2$, $x_3 \in V_3$ imply $\lambda_1 = \lambda_2 = \lambda_3 = 1$ if $f(e_i) = \tilde{e}_i$ for $i = 1, 2, 3$. Therefore, between compositions of pointed quadratic spaces the only type of maps we consider are isomorphisms.

Definition 3.24. An *isomorphism* $f: \mathcal{C}^\bullet \to \widetilde{\mathcal{C}}^\bullet$ of compositions of pointed quadratic spaces is an isomorphism $f: \mathcal{C} \to \widetilde{\mathcal{C}}$ of compositions of quadratic spaces that maps the distinguished vectors of $\mathcal{C}^\bullet$ to the distinguished vectors of $\widetilde{\mathcal{C}}^\bullet$. The automorphisms of $\mathcal{C}^\bullet$ define an algebraic group scheme $\mathbf{O}(\mathcal{C}^\bullet)$, which is a closed subgroup of $\mathbf{O}(\mathcal{C})$.

Our goal in this subsection is to show that every composition of pointed quadratic spaces $\mathcal{C}^\bullet$ carries a canonical isomorphism $\Delta: \mathcal{C}^\bullet \to \partial\mathcal{C}^\bullet$ and is isomorphic to a composition $S(\mathcal{C}^\bullet)$ such that $\partial S(\mathcal{C}^\bullet) = S(\mathcal{C}^\bullet)$. For this, we will use the following identities relating the canonical isometry $\overline{}$ and multiplication by the distinguished vectors:

Lemma 3.25. *Let $\mathcal{C}^\bullet$ be a composition of pointed quadratic spaces as in* (3.30).

(a) *For every $x_1 \in V_1$, $x_2 \in V_2$, $x_3 \in V_3$,*

$$\begin{aligned}\overline{e_1 *_3 x_2} &= e_1 *_3 \overline{x_2}, & \overline{x_1 *_3 e_2} &= \overline{x_1} *_3 e_2,\\ \overline{e_2 *_1 x_3} &= e_2 *_1 \overline{x_3}, & \overline{x_2 *_1 e_3} &= \overline{x_2} *_1 e_3,\\ \overline{e_3 *_2 x_1} &= e_3 *_2 \overline{x_1}, & \overline{x_3 *_2 e_1} &= \overline{x_3} *_2 e_1.\end{aligned}$$

(b) *For every $x_1 \in V_1$, $x_2 \in V_2$, $x_3 \in V_3$,*

$$\begin{aligned}e_1 *_3 (x_3 *_2 x_1) &= \overline{x_1} *_3 (x_3 *_2 e_1), & (x_1 *_3 x_2) *_2 e_1 &= (e_1 *_3 x_2) *_2 \overline{x_1},\\ e_2 *_1 (x_1 *_3 x_2) &= \overline{x_2} *_1 (x_1 *_3 e_2), & (x_2 *_1 x_3) *_3 e_2 &= (e_2 *_1 x_3) *_3 \overline{x_2},\\ e_3 *_2 (x_2 *_1 x_3) &= \overline{x_3} *_2 (x_2 *_1 e_3), & (x_3 *_2 x_1) *_1 e_3 &= (e_3 *_2 x_1) *_1 \overline{x_3}.\end{aligned}$$

(c) *For every $x_1 \in V_1$, $x_2 \in V_2$, $x_3 \in V_3$,*

$$\begin{aligned}e_2 *_1 \big(e_1 *_3 (e_3 *_2 x_1)\big) &= \overline{x_1} = \big((x_1 *_3 e_2) *_2 e_1\big) *_1 e_3,\\ e_3 *_2 \big(e_2 *_1 (e_1 *_3 x_2)\big) &= \overline{x_2} = \big((x_2 *_1 e_3) *_3 e_2\big) *_2 e_1,\\ e_1 *_3 \big(e_3 *_2 (e_2 *_1 x_3)\big) &= \overline{x_3} = \big((x_3 *_2 e_1) *_1 e_3\big) *_3 e_2.\end{aligned}$$

(d) *For every $x_1 \in V_1$, $x_2 \in V_2$, $x_3 \in V_3$,*

$$\begin{aligned}\overline{x_1 *_3 x_2} &= (x_2 *_1 e_3) *_3 (e_3 *_2 x_1) = \big((e_3 *_2 \overline{x_1}) *_1 (e_1 *_3 \overline{x_2})\big) *_3 e_2,\\ \overline{x_2 *_1 x_3} &= (x_3 *_2 e_1) *_1 (e_1 *_3 x_2) = \big((e_1 *_3 \overline{x_2}) *_2 (e_2 *_1 \overline{x_3})\big) *_1 e_3,\\ \overline{x_3 *_2 x_1} &= (x_1 *_3 e_2) *_2 (e_2 *_1 x_3) = \big((e_2 *_1 \overline{x_3}) *_3 (e_3 *_2 \overline{x_1})\big) *_2 e_1.\end{aligned}$$

Proof. To avoid repetitions, we just prove the first formulas in each case.

(a) By definition,

$$\overline{e_1 *_3 x_2} = e_3 b_3(e_3, e_1 *_3 x_2) - e_1 *_3 x_2.$$

Substituting $e_1 *_3 e_2$ for e_3 and using $b_3(e_3, e_1 *_3 x_2) = b_2(e_3 *_2 e_1, x_2) = b_2(e_2, x_2)$ yields

$$\overline{e_1 *_3 x_2} = e_1 *_3 \big(e_2 b_2(e_2, x_2) - x_2\big) = e_1 *_3 \overline{x_2}.$$

(b) By (3.18) and (3.12),

$$e_1 *_3 (x_3 *_2 x_1) = x_3 b_1(e_1, x_1) - x_1 *_3 (x_3 *_2 e_1) \quad \text{and} \quad x_3 = e_1 *_3 (x_3 *_2 e_1),$$

hence

$$e_1 *_3 (x_3 *_2 x_1) = \big(e_1 b_1(e_1, x_1) - x_1\big) *_3 (x_3 *_2 e_1) = \overline{x_1} *_3 (x_3 *_2 e_1).$$

(c) Using (b) and (3.10), we have

$$e_2 *_1 \big(e_1 *_3 (e_3 *_2 x_1)\big) = e_2 *_1 \big(\overline{x_1} *_3 (e_3 *_2 e_1)\big) = e_2 *_1 (\overline{x_1} *_3 e_2) = \overline{x_1}.$$

(d) We compute $(x_2 *_1 e_3) *_3 (e_3 *_2 x_1)$ by using (3.18) to exchange the factors $x_2 *_1 e_3$ and x_1:

$$(x_2 *_1 e_3) *_3 (e_3 *_2 x_1) = e_3 b_1(x_2 *_1 e_3, x_1) - x_1 *_3 \big(e_3 *_2 (x_2 *_1 e_3)\big).$$

Since $e_3 *_2 (x_2 *_1 e_3) = x_2$ by (3.11) and $b_1(x_2 *_1 e_3, x_1) = b_3(e_3, x_1 *_3 x_2)$ by (3.1), it follows that

$$(x_2 *_1 e_3) *_3 (e_3 *_2 x_1) = e_3 b_3(e_3, x_1 *_3 x_2) - x_1 *_3 x_2 = \overline{x_1 *_3 x_2}.$$

On the other hand, (b) yields

$$x_2 *_1 e_3 = x_2 *_1 (e_1 *_3 e_2) = e_2 *_1 (e_1 *_3 \overline{x_2}),$$

hence, using (b) again together with (a),

$$\begin{aligned}(x_2 *_1 e_3) *_3 (e_3 *_2 x_1) &= \big(e_2 *_1 (e_1 *_3 \overline{x_2})\big) *_3 (e_3 *_2 x_1)\\ &= \big(\overline{e_3 *_2 x_1} *_1 (e_1 *_3 \overline{x_2})\big) *_3 e_2\\ &= \big((e_3 *_2 \overline{x_1}) *_1 (e_1 *_3 \overline{x_2})\big) *_3 e_2.\end{aligned}$$

∎

For a composition of pointed quadratic spaces $\mathscr{C}^\bullet$ as in (3.30), we define a composition of pointed quadratic spaces $S(\mathscr{C}^\bullet)$ as follows:

$$S(\mathscr{C}^\bullet) = \big((V_3, q_3, e_3), (V_3, q_3, e_3), (V_3, q_3, e_3), \circledast_3\big)$$

where

$$x \circledast_3 y = (e_2 *_1 \bar{x}) *_3 (\bar{y} *_2 e_1) \quad \text{for } x, y \in V_3. \tag{3.31}$$

We also define linear maps $\Delta_1\colon V_1 \to V_2$, $\Delta_2\colon V_2 \to V_3$, $\Delta_3\colon V_3 \to V_1$ as follows: for $x_1 \in V_1$, $x_2 \in V_2$ and $x_3 \in V_3$,

$$\Delta_1(x_1) = e_3 *_2 \overline{x_1}, \quad \Delta_2(x_2) = e_1 *_3 \overline{x_2}, \quad \Delta_3(x_3) = e_2 *_1 \overline{x_3}.$$

Theorem 3.26. *With the notation above,*

(a) *the triple* $\Delta = (\Delta_1, \Delta_2, \Delta_3)$ *is an isomorphism* $\Delta\colon \mathcal{C}^\bullet \to \partial\mathcal{C}^\bullet$*;*

(b) *the triple* $(\Delta_3, \Delta_2^{-1}, \mathrm{Id}_{V_3})$ *is an isomorphism* $S(\mathcal{C}^\bullet) \to \mathcal{C}^\bullet$*;*

(c) $\partial S(\mathcal{C}^\bullet) = S(\mathcal{C}^\bullet)$.

Proof. (a) It is clear that each Δ_i is an isometry of pointed quadratic spaces, so it suffices to prove $\Delta_3(x_1 *_3 x_2) = \Delta_1(x_1) *_1 \Delta_2(x_2)$ for $x_1 \in V_1$ and $x_2 \in V_2$, which amounts to

$$e_2 *_1 \overline{x_1 *_3 x_2} = (e_3 *_2 \overline{x_1}) *_1 (e_1 *_3 \overline{x_2}).$$

This readily follows from (d) of Lemma 3.25.

(b) For $y \in V_3$ we have $\Delta_2(\bar{y} *_2 e_1) = e_1 *_3 \overline{\bar{y} *_2 e_1} = y$, hence by definition

$$x \circledast_3 y = \Delta_3(x) *_3 \Delta_2^{-1}(y),$$

which proves (b).

(c) It suffices to prove

$$b_3(x \circledast_3 y, z) = b_3(x, y \circledast_3 z) \quad \text{for } x, y, z \in V_3.$$

For this, we first compute using Lemma 3.25

$$\begin{aligned}
\overline{y \circledast_3 z} &= \overline{(e_2 *_1 \bar{y}) *_3 (\bar{z} *_2 e_1)} \\
&= \big((e_3 *_2 \overline{e_2 *_1 \bar{y}}) *_1 (e_1 *_3 \overline{\bar{z} *_2 e_1})\big) *_3 e_2 \\
&= \big((e_3 *_2 (e_2 *_1 y)) *_1 \big(e_1 *_3 (z *_2 e_1)\big)\big) *_3 e_2 \\
&= \big((\bar{y} *_2 e_1) *_1 z\big) *_3 e_2.
\end{aligned}$$

Since $\bar{\ }$ and multiplication on the left by e_2 are isometries, it follows that

$$b_3(x, y \circledast_3 z) = b_1(e_2 *_1 \bar{x}, (\bar{y} *_2 e_1) *_1 z).$$

By definition of $*_1$, the right side is

$$b_3\big((e_2 *_1 \bar{x}) *_3 (\bar{y} *_2 e_1), z\big) = b_3(x \circledast_3 y, z). \qquad \blacksquare$$

For compositions of (unpointed) isometric spaces, a result similar to Theorem 3.26 easily follows:

Corollary 3.27. *For every composition of isometric quadratic spaces* $\mathcal{C}$*, there is an isomorphism* $\mathcal{C} \simeq \partial\mathcal{C}$*. Moreover,* $\mathcal{C}$ *is isomorphic to a composition* $S(\mathcal{C})$ *such that* $\partial S(\mathcal{C}) = S(\mathcal{C})$.

Proof. Let $\mathcal{C} = ((V_1, q_1), (V_2, q_2), (V_3, q_3), *_3)$ be a composition of isometric quadratic spaces. By Corollary 3.12, we know that q_1, q_2 and q_3 represent 1. We may therefore use the same constructions as in Theorem 3.26, after choosing $e_1 \in V_1$ and $e_2 \in V_2$ such that

$$q_1(e_1) = q_2(e_2) = 1$$

and letting $e_3 = e_1 *_3 e_2$, for $((V_1, q_1, e_1), (V_2, q_2, e_2), (V_3, q_3, e_3), *_3)$ is then a composition of pointed quadratic spaces. Define the maps $\Delta_1\colon V_1 \to V_2$, $\Delta_2\colon V_2 \to V_3$, $\Delta_3\colon V_3 \to V_1$ as in Theorem 3.26. The proof of that theorem shows that $(\Delta_1, \Delta_2, \Delta_3)$ is an isomorphism $\mathcal{C} \to \partial\mathcal{C}$. Moreover, letting

$$S(\mathcal{C}) = \big((V_3, q_3), (V_3, q_3), (V_3, q_3), \circledast_3\big)$$

with $\circledast_3$ as in (3.31), we see from the proof of Theorem 3.26 that $\partial S(\mathcal{C}) = S(\mathcal{C})$ and that $(\Delta_3, \Delta_2^{-1}, \mathrm{Id}_{V_3})$ is an isomorphism $S(\mathcal{C}) \xrightarrow{\sim} \mathcal{C}$. ■

Note that, in contrast with Theorem 3.26, the constructions in Corollary 3.27 are not canonical, since they depend on the choice of distinguished vectors.

3.5. Composition algebras

The purpose of this subsection is to briefly review the classical notion of composition algebras, in order to underline its connections with compositions of quadratic spaces.

Definitions 3.28. A *composition algebra* over F is a triple $\mathcal{A} = (A, q, \diamond)$ where (A, q) is a (finite-dimensional) quadratic space over F with nonsingular polar bilinear form and $\diamond\colon A \times A \to A$ is a bilinear map such that

$$q(x \diamond y) = q(x)q(y) \quad \text{for all } x, y \in A.$$

The definition can be rephrased as follows: the 4-tuple

$$\mathcal{C}(\mathcal{A}) = \big((A, q), (A, q), (A, q), \diamond\big) \tag{3.32}$$

is a composition of quadratic spaces. Theorem 3.7 shows that the dimension of a composition algebra is 1, 2, 4 or 8, with dimension 1 occurring only when char $F \neq 2$, and Corollary 3.12 shows that q is a Pfister form.

A *unital composition algebra*[6] is a 4-tuple

$$\mathcal{A}^\bullet = (A, q, e, \diamond),$$

where (A, q, e) is a pointed quadratic space and $\diamond\colon A \times A \to A$ is a bilinear map such that

$$q(x \diamond y) = q(x)q(y) \quad \text{and} \quad e \diamond x = x \diamond e = x \quad \text{for all } x, y \in A.$$

In any unital composition algebra we have $e \diamond e = e$, hence

$$\mathcal{C}^\bullet(\mathcal{A}^\bullet) = \big((A, q, e), (A, q, e), (A, q, e), \diamond\big) \tag{3.33}$$

is a composition of pointed quadratic spaces.

[6]Unital composition algebras are called *Hurwitz algebras* in [13], see [13, (33.17)].

As with more general compositions of quadratic spaces, the multiplication law $\diamond$ of a composition algebra $\mathcal{A}$ induces derived composition maps $\diamond_1$ and $\diamond_2$ of (A,q), (A,q), (A,q), defined by the conditions

$$b(x, y \diamond_1 z) = b(x \diamond y, z) \quad \text{and} \quad b(x \diamond_2 y, z) = b(x, y \diamond z) \quad \text{for } x, y, z \in A. \tag{3.34}$$

We may therefore define derived composition algebras $\partial\mathcal{A}$ and $\partial^2\mathcal{A}$ by

$$\partial\mathcal{A} = (A, q, \diamond_1) \quad \text{and} \quad \partial^2\mathcal{A} = (A, q, \diamond_2).$$

Composition algebras $\mathcal{A}$ such that $\partial\mathcal{A} = \mathcal{A}$ are called *symmetric composition algebras*. They are characterized by the condition that

$$b(x \diamond y, z) = b(x, y \diamond z) \quad \text{for all } x, y, z \in A.$$

By contrast with compositions of pointed quadratic spaces, the derivation procedure does *not* preserve unitality of composition algebras. To make this point clear, we determine below the derived composition maps of a unital composition algebra, using results from [19, Ch. 1]. Note that unital composition algebras carry a canonical involutory isometry $\overline{}$ derived from their pointed quadratic space structure as in Definitions 3.23.

Proposition 3.29. *Let $\mathcal{A}^\bullet = (A, q, e, \diamond)$ be a unital composition algebra.*

(a) *The derived composition maps $\diamond_1$ and $\diamond_2$ defined in (3.34) are given by*

$$x \diamond_1 y = y \diamond \bar{x} \quad \textit{and} \quad x \diamond_2 y = \bar{y} \diamond x \quad \textit{for } x, y \in A.$$

(b) *For the bilinear map $*\colon A \times A \to A$ defined by*

$$x * y = \bar{x} \diamond \bar{y} \quad \textit{for } x, y \in A,$$

the triple $S(\mathcal{A}^\bullet) = (A, q, *)$ is a symmetric composition algebra.

Proof. (a) Lemma 1.3.2 in [19] yields $b(x, y \diamond z) = b(z, \bar{y} \diamond x)$, hence

$$b(x \diamond_2 y, z) = b(\bar{y} \diamond x, z) \quad \text{for all } x, y, z \in A.$$

Since b is nonsingular, it follows that $x \diamond_2 y = \bar{y} \diamond x$.

Since $\overline{}$ is an isometry, [19, Lem. 1.3.2] also yields $b(\overline{x \diamond y}, \bar{z}) = b(\bar{y}, \overline{\bar{x} \diamond z})$. Now, [19, Lem. 1.3.1] shows that $\overline{x \diamond y} = \bar{y} \diamond \bar{x}$, hence the definition of $\diamond_1$ yields

$$b(\bar{y} \diamond \bar{x}, \bar{z}) = b(\bar{y}, \bar{z} \diamond x) = b(\bar{z}, x \diamond_1 \bar{y}) \quad \text{for all } x, y, z \in A.$$

Since b is nonsingular, it follows that $x \diamond_1 \bar{y} = \bar{y} \diamond \bar{x}$.

(b) Since $q(\bar{x}) = q(x)$ for all $x \in A$, it is clear that $S(\mathcal{A}^\bullet)$ is a composition algebra. To prove that the derived maps $*_1$, $*_2$ associated to $*$ are identical to $*$, it suffices to prove that $b(x * y, z) = b(x, y * z)$ for all $x, y, z \in A$, which amounts to

$$b(\bar{x} \diamond \bar{y}, z) = b(x, \bar{y} \diamond \bar{z}) \quad \text{for all } x, y, z \in A. \tag{3.35}$$

From the definition of $\diamond_1$ in (3.34) and its determination in (a), it follows that

$$b(\bar{x} \diamond \bar{y}, z) = b(\bar{x}, \bar{y} \diamond_1 z) = b(\bar{x}, z \diamond y) \quad \text{for all } x, y, z \in A.$$

Now, $z \diamond y = \overline{\bar{y} \diamond \bar{z}}$ by [19, Lem. 1.3.1], and $\bar{\ }$ is an isometry, hence the rightmost side in the last displayed equation is equal to $b(x, \bar{y} \diamond \bar{z})$, which proves (3.35). ∎

Note that in the context of Proposition 3.29, $x \diamond_1 e = \bar{x} = e \diamond_2 x$ for all $x \in A$. Hence $(A, q, e, \diamond_1)$ and $(A, q, e, \diamond_2)$ are *not* unital composition algebras, unless $\bar{\ } = \mathrm{Id}_V$, which occurs only if $\dim A = 1$.

Symmetric composition algebras $S(\mathcal{A}^\bullet)$ derived from unital composition algebras $\mathcal{A}^\bullet$ as in Proposition 3.29 are called *para-unital* composition algebras (*para-Hurwitz algebras* in the terminology of [13]). They are characterized by the property that they contain a *para-unit*, see [13, (34.8)].

Between algebras, maps that are more general than homomorphisms are considered, following Albert [1].

Definition 3.30. Let $(A, \diamond)$ and $(\widetilde{A}, \widetilde{\diamond})$ be F-algebras (i.e., F-vector spaces with a bilinear multiplication). An *isotopy* $f: (A, \diamond) \to (\widetilde{A}, \widetilde{\diamond})$ is a triple $f = (f_1, f_2, f_3)$ of linear bijections $f_i: A \to \widetilde{A}$ such that

$$f_3(x \diamond y) = f_1(x) \widetilde{\diamond} f_2(y) \quad \text{for all } x, y \in A.$$

An *autotopy* is an isotopy of an algebra to itself. Under the composition of maps, autotopies of an algebra form a group $\mathrm{Str}(A, \diamond)$ known as the *structure group* of $(A, \diamond)$. This group is the set of F-rational points of an algebraic group scheme $\mathbf{Str}(A, \diamond)$, which is a closed subgroup of $\mathbf{GL}(A) \times \mathbf{GL}(A) \times \mathbf{GL}(A)$.

For example, in the construction S of Proposition 3.29, which yields the symmetric composition algebra $S(\mathcal{A}^\bullet)$ from the unital composition algebra $\mathcal{A}^\bullet$, the algebra $(A, *)$ is isotopic to $(A, \diamond)$. The following construction, due to Kaplansky [11], shows that the algebra of every composition algebra is isotopic to the algebra of a unital composition algebra.

Proposition 3.31. *Let $\mathcal{A} = (A, q, \diamond)$ be a composition algebra. There exists a bilinear map $*: A \times A \to A$ and a vector $e \in A$ for which*

(a) *$(A, q, e, *)$ is a unital composition algebra, and*

(b) *there exists an isotopy*

$$f = (f_1, f_2, f_3): (A, \diamond) \to (A, *)$$

which is also an isomorphism $f: \mathscr{C}(\mathcal{A}) \to \mathscr{C}(A, q, *)$ of the associated compositions of quadratic spaces as in (3.32).

Proof. Since $\mathcal{A}$ is a composition algebra, Corollary 3.12 shows that there exists $u \in A$ such that $q(u) = 1$. The maps $\ell_u: A \to A$ and $r_u: A \to A$ defined by $\ell_u(x) = u \diamond x$ and

$r_u(x) = x \diamond u$ are isometries of (A, q), hence they are invertible. Define $*\colon A \times A \to A$ by

$$x * y = r_u^{-1}(x) \diamond \ell_u^{-1}(y) \quad \text{for } x, y \in A,$$

hence $x \diamond y = r_u(x) * \ell_u(y)$ for $x, y \in A$. It is clear from the definitions that $(\mathrm{Id}_A, r_u, \ell_u)$ is an isotopy $(A, \diamond) \to (A, *)$ and also a similitude $\mathcal{C}(\mathcal{A}) \to \mathcal{C}(A, q, *)$ with multiplier of the form $(\lambda_1, \lambda_2, 1)$ for some $\lambda_1, \lambda_2 \in F^\times$. Since

$$\mu(\ell_u) = \mu(r_u) = q(u) = 1,$$

it follows from (3.25) that $\lambda_1 = \lambda_2 = 1$, hence f is an isomorphism of compositions of quadratic spaces. Moreover for $e = u \diamond u$ we have $r_u^{-1}(e) = u = \ell_u^{-1}(e)$, hence

$$e * x = u \diamond \ell_u^{-1}(x) = \ell_u\big(\ell_u^{-1}(x)\big) = x = r_u\big(r_u^{-1}(x)\big) = r_u^{-1}(x) \diamond u = x * e,$$

hence $(A, q, e, *)$ is a unital composition algebra. ∎

Corollary 3.32. *Every composition of isometric quadratic spaces is isomorphic to a composition $\mathcal{C}(\mathcal{A})$ as in (3.32) for some unital composition algebra $\mathcal{A}$, and also to a composition $\mathcal{C}(\mathcal{S})$ for some symmetric composition algebra $\mathcal{S}$. Up to isomorphism, there is a unique composition of hyperbolic quadratic spaces of dimension n, for $n = 2$, 4 and 8.*

Proof. Let $\mathcal{C} = ((V_1, q_1), (V_2, q_2), (V_3, q_3), *_3)$ be a composition of isometric quadratic spaces, and let $S(\mathcal{C}) = ((V_3, q_3), (V_3, q_3), (V_3, q_3), \circledast_3)$ be the composition of quadratic spaces constructed in Corollary 3.27, which is isomorphic to $\mathcal{C}$. Clearly, $S(\mathcal{C}) = \mathcal{C}(\mathcal{S})$ for $\mathcal{S}$ the composition algebra $\mathcal{S} = (V_3, q_3, \circledast_3)$. This composition algebra is symmetric since $\partial S(\mathcal{C}) = S(\mathcal{C})$. Now, Proposition 3.31 yields a unital composition algebra $\mathcal{A} = (V_3, q_3, e, *)$ and an isomorphism $\mathcal{C}(\mathcal{S}) \simeq \mathcal{C}(\mathcal{A})$. Thus, $\mathcal{C}$ is isomorphic to $\mathcal{C}(\mathcal{S})$ and to $\mathcal{C}(\mathcal{A})$.

For $n = 2$, 4 and 8, there is up to isomorphism a unique unital composition algebra of dimension n with hyperbolic quadratic form, namely the split quadratic étale algebra, the split quaternion algebra and the split octonion algebra, see [13, (33.19)]. Therefore, there is a unique composition of hyperbolic quadratic spaces up to isomorphism. ∎

By contrast with the last argument, note that there exist more than one symmetric composition algebra of dimension 8 with hyperbolic quadratic form, see [13, (34.37)]; but their associated compositions of quadratic spaces are isomorphic.

Remark 3.33. For quadratic modules over a commutative ring, Alsaody establishes in [2, Prop. 3.7] the following result closely related to Corollary 3.32: every composition of quadratic modules of constant rank 8 is isomorphic after a faithfully flat scalar extension to the composition of quadratic spaces associated to a para-octonion algebra. The proof uses the same construction as Theorem 3.26.

We next relate isotopies of composition algebras with similitudes of the associated compositions of quadratic spaces.

Theorem 3.34. *Let $\mathcal{A} = (A, q, \diamond)$ and $\widetilde{\mathcal{A}} = (\widetilde{A}, \tilde{q}, \widetilde{\diamond})$ be composition algebras and let $\mathcal{C}(\mathcal{A})$ and $\mathcal{C}(\widetilde{\mathcal{A}})$ be the associated compositions of quadratic spaces as in (3.32). A triple $f = (f_1, f_2, f_3)$ of bijective linear maps $f_i : A \to \widetilde{A}$ is an isotopy $(A, \diamond) \to (\widetilde{A}, \widetilde{\diamond})$ if and only if it is a similitude $f : \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\widetilde{\mathcal{A}})$ with composition multiplier $(\mu(f_2), \mu(f_1), 1)$.*

Proof. It is clear by comparing Definitions 3.13 and 3.30 that similitudes $f : \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\widetilde{\mathcal{A}})$ with composition multiplier of the form $(\lambda_1, \lambda_2, 1)$ for some $\lambda_1, \lambda_2 \in F^\times$ are isotopies $(A, \diamond) \to (\widetilde{A}, \widetilde{\diamond})$. To prove the converse, it suffices to show that for every isotopy

$$f = (f_1, f_2, f_3) : (A, \diamond) \to (\widetilde{A}, \widetilde{\diamond})$$

the maps f_i are similitudes. The isotopy condition then shows that f is a similitude of compositions $\mathcal{C}(\mathcal{A}) \to \mathcal{C}(\widetilde{\mathcal{A}})$ with multiplier of the form $(\lambda_1, \lambda_2, 1)$ for some $\lambda_1, \lambda_2 \in F^\times$. Then (3.25) shows that $\lambda_1 = \mu(f_2)$ and $\lambda_2 = \mu(f_1)$.

By Proposition 3.31 we can find a bilinear map $* : A \times A \to A$ and a vector $e \in A$ such that $(A, q, e, *)$ is a unital composition algebra and there exists an isotopy $g : (A, \diamond) \to (A, *)$ whose components are isometries of (A, q). The map $f \circ g^{-1} : (A, *) \to (\widetilde{A}, \widetilde{\diamond})$ is then an isotopy, which means that for $f \circ g^{-1} = (h_1, h_2, h_3)$

$$h_3(x * y) = h_1(x) \widetilde{\diamond} h_2(y) \quad \text{for all } x, y \in A. \tag{3.36}$$

In particular, it follows that for all $x \in A$

$$h_1(e) \widetilde{\diamond} h_2(x) = h_3(e * x) = h_3(x) = h_3(x * e) = h_1(x) \widetilde{\diamond} h_2(e).$$

Since $\widetilde{\mathcal{A}}$ is a composition algebra, these equations imply

$$\tilde{q}\big(h_3(x)\big) = \tilde{q}\big(h_1(e)\big)\tilde{q}\big(h_2(x)\big) = \tilde{q}\big(h_1(x)\big)\tilde{q}\big(h_2(e)\big) \quad \text{for all } x \in A. \tag{3.37}$$

Corollary 3.12 shows that the form $\tilde{q}$ represents 1; as h_3 is bijective, there exist vectors $x \in A$ such that $\tilde{q}(h_3(x)) = 1$, hence $\tilde{q}(h_1(e))$ and $\tilde{q}(h_2(e))$ belong to $F^\times$. Equation (3.37) yields $\tilde{q}(h_3(e)) = \tilde{q}(h_1(e))\tilde{q}(h_2(e))$, hence $\tilde{q}(h_3(x)) \in F^\times$. Define $q' : A \to F$ by

$$q'(x) = \tilde{q}\big(h_3(e)\big)^{-1} \tilde{q}\big(h(x)\big) \quad \text{for } x \in A,$$

so $q'(e) = 1$. Since $\widetilde{\mathcal{A}}$ is a composition algebra, we obtain from (3.36)

$$\tilde{q}\big(h_3(x * y)\big) = \tilde{q}\big(h_1(x)\big)\tilde{q}\big(h_2(y)\big) \quad \text{for all } x, y \in A,$$

hence by (3.37)

$$\begin{aligned} \tilde{q}\big(h_3(x * y)\big) &= \tilde{q}\big(h_1(e)\big)^{-1} \tilde{q}\big(h_2(e)\big)^{-1} \tilde{q}\big(h_3(x)\big)\tilde{q}\big(h_3(y)\big) \\ &= \tilde{q}\big(h_3(e)\big)^{-1} \tilde{q}\big(h_3(x)\big)\tilde{q}\big(h_3(y)\big). \end{aligned}$$

Therefore,

$$q'(x * y) = q'(x)q'(y) \quad \text{for all } x, y \in A.$$

Thus, $(A, q', e, *)$ is a unital composition algebra, just like $(A, q, e, *)$. But the quadratic form in a unital composition algebra is uniquely determined as the "generic norm" of the algebra (see [13, (33.9)] or [19, Cor. 1.2.4]), hence $q' = q$, which means that

$$\tilde{q}\big(h_3(x)\big) = \tilde{q}\big(h_3(e)\big)q(x) \quad \text{for all } x \in A.$$

Thus, $h_3\colon (A, q) \to (\widetilde{A}, \tilde{q})$ is a similitude with multiplier $\tilde{q}(h_3(e))$. Equation (3.37) then yields

$$\tilde{q}\big(h_1(x)\big) = \tilde{q}\big(h_2(e)\big)^{-1}\tilde{q}\big(h_3(e)\big)q(x) \quad \text{and} \quad \tilde{q}\big(h_2(x)\big) = \tilde{q}\big(h_1(e)\big)^{-1}\tilde{q}\big(h_3(e)\big)q(x),$$

hence h_1 and h_2 also are similitudes. Now, $f = h \circ g$ and all the components of g are isometries, hence all the components of f are similitudes. ∎

Note that the construction in the proof of Theorem 3.34 is functorial: it also applies to isotopies $f\colon (A_R, \diamond) \to (\widetilde{A}_R, \tilde{\diamond})$ for R any commutative F-algebra. Therefore, in the case where $\widetilde{\mathcal{A}} = \mathcal{A}$ Theorem 3.34 has the following group scheme interpretation:

Corollary 3.35. *For any composition algebra $\mathcal{A} = (A, q, \diamond)$, let $\lambda'\colon \mathbf{GO}(\mathcal{C}(\mathcal{A})) \to \mathbf{G_m}$ be the third component of the composition multiplier map $\lambda_{\mathcal{C}(\mathcal{A})}\colon \mathbf{GO}(\mathcal{C}(\mathcal{A})) \to \mathbf{G_m^3}$, and let $\mu'\colon \mathbf{Str}(A, \diamond) \to \mathbf{G_m^2}$ be the map defined on rational points by mapping every autotopy (f_1, f_2, f_3) to the pair of multipliers $(\mu(f_1), \mu(f_2))$. The algebraic group scheme $\mathbf{Str}(A, \diamond)$ is smooth and fits in the following exact sequences:*

$$1 \to \mathbf{Str}(A, \diamond) \to \mathbf{GO}\big(\mathcal{C}(\mathcal{A})\big) \xrightarrow{\lambda'} \mathbf{G_m} \to 1$$

and

$$1 \to \mathbf{O}\big(\mathcal{C}(\mathcal{A})\big) \to \mathbf{Str}(A, \diamond) \xrightarrow{\mu'} \mathbf{G_m^2} \to 1.$$

Proof. Theorem 3.34 identifies $\mathbf{Str}(A, \diamond)$ as the kernel of λ'. Proposition 3.19 shows that $\lambda_{\mathcal{C}(\mathcal{A})}$ is a separable morphism, hence λ' also is separable, and it follows by [13, (22.13)] that $\mathbf{Str}(A, \diamond)$ is smooth. Theorem 3.34 also shows that the kernel of μ' is the kernel of the restriction of $\lambda_{\mathcal{C}(\mathcal{A})}$ to $\mathbf{Str}(A, \diamond)$, which is $\mathbf{O}(\mathcal{C}(\mathcal{A}))$ by definition. To complete the proof, observe that μ' is surjective because $\lambda_{\mathcal{C}(\mathcal{A})}$ is surjective. ∎

We next turn to automorphisms of an algebra $(A, \diamond)$, which are linear bijections $f\colon A \to A$ such that $f(x \diamond y) = f(x) \diamond f(y)$ for all $x, y \in A$. They form an algebraic group scheme $\mathbf{Aut}(A, \diamond)$, which can be viewed as a closed subgroup of $\mathbf{Str}(A, \diamond)$ since every automorphism f yields an autotopy (f, f, f) of $(A, \diamond)$. To relate the condition that $f_1 = f_2 = f_3$ in an autotopy (f_1, f_2, f_3) with the shift isomorphism $\partial\colon \mathbf{GO}(\mathcal{C}(\mathcal{A})) \to \mathbf{GO}(\partial\mathcal{C}(\mathcal{A}))$, we view $\mathbf{O}(\mathcal{C}(\mathcal{A}))$, $\mathbf{O}(\partial\mathcal{C}(\mathcal{A}))$ and $\mathbf{O}(\partial^2\mathcal{C}(\mathcal{A}))$ as subgroups of $\mathbf{GL}(A) \times \mathbf{GL}(A) \times \mathbf{GL}(A)$ and define

$$\overline{\mathbf{O}}\big(\mathcal{C}(\mathcal{A})\big) = \mathbf{O}\big(\mathcal{C}(\mathcal{A})\big) \cap \mathbf{O}\big(\partial\mathcal{C}(\mathcal{A})\big) \cap \mathbf{O}\big(\partial^2\mathcal{C}(\mathcal{A})\big).$$

The shift isomorphism ∂ clearly restricts to an automorphism of $\overline{\mathbf{O}}(\mathcal{C}(\mathcal{A}))$.

Proposition 3.36. *For every composition algebra $\mathcal{A} = (A, q, \diamond)$, the group* $\mathbf{Aut}(A, \diamond)$ *is the subgroup of* $\bar{\mathbf{O}}(\mathcal{C}(\mathcal{A}))$ *fixed under* ∂.

Proof. Let R be a commutative F-algebra. For every automorphism f of $(A, \diamond)_R$ the triple (f, f, f) is an autotopy of $(A, \diamond)_R$, hence by Theorem 3.34 a similitude of $\mathcal{C}(\mathcal{A})_R$ with composition multiplier $(\mu(f), \mu(f), 1)$. The relations (3.25) between composition multipliers and the multipliers of the components of similitudes yield $\mu(f) = \mu(f)^2$, hence $\mu(f) = 1$ and (f, f, f) is an automorphism of $\mathcal{C}(\mathcal{A})_R$. By Proposition 3.14 we then see that (f, f, f) is also an automorphism of $\partial\mathcal{C}(\mathcal{A})$ and of $\partial^2\mathcal{C}(\mathcal{A})$, hence

$$(f, f, f) \in \bar{\mathbf{O}}\big(\mathcal{C}(\mathcal{A})\big)(R),$$

and this triple is fixed under ∂.

On the other hand, if $(f_1, f_2, f_3) \in \bar{\mathbf{O}}(\mathcal{C}(\mathcal{A}))(R)$ is fixed under ∂, then $f_1 = f_2 = f_3$, hence f_1 is an automorphism of $(A, \diamond)_R$ because $(f_1, f_2, f_3) \in \mathbf{O}(\mathcal{C}(\mathcal{A}))(R)$. ∎

When $\mathcal{A} = (A, q, \diamond)$ is a symmetric composition algebra, then $\partial\mathcal{C}(\mathcal{A}) = \mathcal{C}(\mathcal{A})$ by definition, hence $\bar{\mathbf{O}}(\mathcal{C}(\mathcal{A})) = \mathbf{O}(\mathcal{C}(\mathcal{A}))$ and Proposition 3.36 shows that $\mathbf{Aut}(A, \diamond)$ is the subgroup of $\mathbf{O}(\mathcal{C}(\mathcal{A}))$ fixed under ∂. When $\dim A = 8$, an alternative description is given in [6, Th. 6.6]: $\mathbf{Aut}(A, \diamond)$ is shown to be isomorphic to the subgroup of $\mathbf{PGO}^+(q)$ fixed under an outer automorphism of order 3, which is the analogue of ∂.

For a unital composition algebra $\mathcal{A}^\bullet = (A, q, e, \diamond)$ with associated para-unital symmetric composition algebra $S(\mathcal{A}^\bullet) = (A, q, *)$ as in Proposition 3.29, it follows from functoriality of the S construction that $\mathbf{Aut}(A, \diamond) \subset \mathbf{Aut}(A, *)$. The reverse inclusion holds when $\dim A \geq 4$ by [13, (34.4)]. However, the group $\mathbf{Aut}(A, \diamond)$ can also be described as follows:

Proposition 3.37. *Let $\mathcal{A}^\bullet = (A, q, e, \diamond)$ be a unital composition algebra, and let $\mathcal{C}^\bullet(\mathcal{A}^\bullet)$ be the associated composition of pointed quadratic spaces as in* (3.33). *There is a canonical identification* $\mathbf{Aut}(A, \diamond) = \mathbf{O}(\mathcal{C}^\bullet(\mathcal{A}^\bullet))$.

Proof. Let R be a commutative F-algebra. Every automorphism $f \in \mathbf{Aut}(A, \diamond)(R)$ leaves e fixed, hence the triple (f, f, f) is an automorphism of $\mathcal{C}^\bullet(\mathcal{A}^\bullet)_R$. Therefore, mapping f to (f, f, f) defines an embedding $\mathbf{Aut}(A, \diamond) \subset \mathbf{O}(\mathcal{C}^\bullet(\mathcal{A}^\bullet))$.

For the reverse inclusion, let $(f_1, f_2, f_3) \in \mathbf{O}(\mathcal{C}^\bullet(\mathcal{A}^\bullet))(R)$. Substituting e for x or for y in the equation

$$f_3(x \diamond y) = f_1(x) \diamond f_2(y) \quad \text{for all } x, y \in A$$

yields $f_3(y) = f_2(y)$ and $f_3(x) = f_1(x)$ for all $x, y \in A$, hence $f_3 \in \mathbf{Aut}(A, \diamond)(R)$. ∎

4. Trialitarian triples

The focus in this section is on central simple algebras with quadratic pair of degree 8 over an arbitrary field F. Altering slightly the definition in [13, §42.A] (and extending it to characteristic 2), we define a *trialitarian triple* over F to be a 4-tuple (!)

$$\mathcal{T} = \big((A_1,\sigma_1,\mathfrak{f}_1),(A_2,\sigma_2,\mathfrak{f}_2),(A_3,\sigma_3,\mathfrak{f}_3),\varphi_0\big)$$

where $(A_i,\sigma_i,\mathfrak{f}_i)$ is a central simple F-algebra with quadratic pair of degree 8 for $i = 1$, 2, 3, and φ_0 is an isomorphism of algebras with quadratic pair

$$\varphi_0\colon \big(C(A_1,\sigma_1,\mathfrak{f}_1),\underline{\sigma}_1,\underline{\mathfrak{f}}_1\big) \xrightarrow{\sim} (A_2,\sigma_2,\mathfrak{f}_2)\times(A_3,\sigma_3,\mathfrak{f}_3).$$

To simplify notation, we denote by a single letter algebras with quadratic pair, as in Section 2, and write $\mathfrak{A}_i = (A_i,\sigma_i,\mathfrak{f}_i)$ and $\mathfrak{C}(\mathfrak{A}_1) = (C(\mathfrak{A}_1),\underline{\sigma}_1,\underline{\mathfrak{f}}_1)$.

If $\widetilde{\mathcal{T}} = (\widetilde{\mathfrak{A}}_1,\widetilde{\mathfrak{A}}_2,\widetilde{\mathfrak{A}}_3,\widetilde{\varphi}_0)$ is also a trialitarian triple, an *isomorphism of trialitarian triples* $\gamma\colon\mathcal{T}\to\widetilde{\mathcal{T}}$ is a triple $\gamma=(\gamma_1,\gamma_2,\gamma_3)$ of isomorphisms of algebras with quadratic pair

$$\gamma_i\colon\mathfrak{A}_i\to\widetilde{\mathfrak{A}}_i \quad i=1,2,3,$$

such that the following diagram commutes:

$$\begin{array}{ccc}
\mathfrak{C}(\mathfrak{A}_1) & \xrightarrow{\varphi_0} & \mathfrak{A}_2\times\mathfrak{A}_3 \\
{\scriptstyle C(\gamma_1)}\big\downarrow & & \big\downarrow{\scriptstyle \gamma_2\times\gamma_3} \\
\mathfrak{C}(\widetilde{\mathfrak{A}}_1) & \xrightarrow{\widetilde{\varphi}_0} & \widetilde{\mathfrak{A}}_2\times\widetilde{\mathfrak{A}}_3
\end{array}$$

We show in Section 4.1 that every composition $\mathcal{C}$ of quadratic spaces of dimension 8 yields a trialitarian triple $\operatorname{End}(\mathcal{C})$, and that every trialitarian triple of split algebras has the form $\operatorname{End}(\mathcal{C})$ for some composition $\mathcal{C}$ of dimension 8. In Section 4.2 we discuss the group scheme of automorphisms of a trialitarian triple $\mathcal{T}$: we show that $\mathbf{Aut}(\mathcal{T})$ is smooth (hence an algebraic group) and introduce algebraic groups $\mathbf{O}(\mathcal{T})$, $\mathbf{GO}(\mathcal{T})$, $\mathbf{PGO}(\mathcal{T})$, extending to the context of trialitarian triples the group schemes $\mathbf{O}(\mathcal{C})$, $\mathbf{GO}(\mathcal{C})$, $\mathbf{PGO}(\mathcal{C})$ defined in Section 3.3 for a composition $\mathcal{C}$ of quadratic spaces. A main result of the section is the construction of derived trialitarian triples in Section 4.3: to each trialitarian triple $\mathcal{T}$ we canonically associate trialitarian triples $\partial\mathcal{T}$ and $\partial^2\mathcal{T}$, in such a way that for split trialitarian triples $\operatorname{End}(\mathcal{C})$

$$\partial\operatorname{End}(\mathcal{C}) = \operatorname{End}(\partial\mathcal{C}) \quad\text{and}\quad \partial^2\operatorname{End}(\mathcal{C}) = \operatorname{End}(\partial^2\mathcal{C}).$$

This construction is used in Section 4.4 to define for each trialitarian triple

$$\mathcal{T} = (\mathfrak{A}_1,\mathfrak{A}_2,\mathfrak{A}_3,\varphi_0)$$

isomorphisms

$$\mathbf{O}(\mathcal{T})\simeq\mathbf{Spin}(\mathfrak{A}_1)\simeq\mathbf{Spin}(\mathfrak{A}_2)\simeq\mathbf{Spin}(\mathfrak{A}_3)$$

and

$$\mathbf{PGO}(\mathcal{T})\simeq\mathbf{PGO}^+(\mathfrak{A}_1)\simeq\mathbf{PGO}^+(\mathfrak{A}_2)\simeq\mathbf{PGO}^+(\mathfrak{A}_3),$$

which we call the *trialitarian isomorphisms* canonically attached to the trialitarian triple $\mathcal{T}$. The trialitarian isomorphisms $\mathbf{Spin}(\mathfrak{A}_1) \simeq \mathbf{Spin}(\mathfrak{A}_2) \simeq \mathbf{Spin}(\mathfrak{A}_3)$ restrict on the centers to isomorphisms $R_{Z_1/F}(\boldsymbol{\mu}_2) \simeq R_{Z_2/F}(\boldsymbol{\mu}_2) \simeq R_{Z_3/F}(\boldsymbol{\mu}_2)$ that do not preserve the subgroups $\boldsymbol{\mu}_2 \subset R_{Z_i/F}(\boldsymbol{\mu}_2)$: see Proposition 4.21. In the particular case where $\mathfrak{A}_1 = \mathfrak{A}_2 = \mathfrak{A}_3$ (where $A_1 = A_2 = A_3$ is split, since the sum of the Brauer classes of A_1, A_2 and A_3 vanishes in the Brauer group of F, see [13, (42.7)]), the trialitarian isomorphisms are therefore outer automorphisms. The close connection between trialitarian automorphisms of $\mathbf{PGO}_8^+$ and symmetric compositions is discussed in [6].

We return in Section 4.5 to the study of compositions of quadratic spaces, building on the theory of trialitarian triples developed in the previous subsections to obtain a few more results about the 8-dimensional case. Specifically, we establish criteria for the similarity or the isomorphism of compositions of quadratic spaces, which yield an analogue of the classical principle of triality, and we give an explicit description of the cohomological invariants of $\mathbf{Spin}_8$.

In the final Section 4.6 we show that the constructions of Section 4.2 readily yield a canonical isomorphism between the structure group of a composition algebra of dimension 8 and the extended Clifford group of its quadratic form.

4.1. The trialitarian triple of a composition of quadratic spaces

Let $\mathscr{C} = ((V_1,q_1),(V_2,q_2),(V_3,q_3),*_3)$ be a composition of quadratic spaces of dimension 8 over F. Recall from Proposition 3.10 the isomorphism of algebras with quadratic pair

$$C(\alpha)\colon \big(C(V_1,q_1),\tau_1,\mathfrak{g}_1\big) \xrightarrow{\sim} \big(\mathrm{End}(V_2\oplus V_3),\sigma_{b_2\perp b_3},\mathfrak{f}_{q_2\perp q_3}\big)$$

induced by the map

$$\alpha\colon x_1\in V_1 \mapsto \begin{pmatrix} 0 & r_{x_1}\\ \ell_{x_1} & 0\end{pmatrix} \in \mathrm{End}(V_2\oplus V_3)$$

where $\ell_{x_1}\colon V_2\to V_3$ carries $x_2\in V_2$ to $x_1 *_3 x_2 \in V_3$ and $r_{x_1}\colon V_3\to V_2$ carries $x_3\in V_3$ to $x_3 *_2 x_1 \in V_2$. Its restriction to the even Clifford algebra also is an isomorphism of algebras with quadratic pair

$$C_0(\alpha)\colon \big(C_0(V_1,q_1),\tau_{01},\mathfrak{g}_{01}\big) \xrightarrow{\sim} \big(\mathrm{End}(V_2),\sigma_{b_2},\mathfrak{f}_{q_2}\big)\times\big(\mathrm{End}(V_3),\sigma_{b_3},\mathfrak{f}_{q_3}\big),$$

see Proposition 3.10. Therefore, the following is a trialitarian triple:

$$\mathrm{End}(\mathscr{C}) = \Big(\big(\mathrm{End}(V_1),\sigma_{b_1},\mathfrak{f}_{q_1}\big),\big(\mathrm{End}(V_2),\sigma_{b_2},\mathfrak{f}_{q_2}\big),\big(\mathrm{End}(V_3),\sigma_{b_3},\mathfrak{f}_{q_3}\big),C_0(\alpha)\Big).$$

We next show that the construction of trialitarian triples from compositions of quadratic spaces is functorial.

Let $\mathscr{C} = ((V_1,q_1),(V_2,q_2),(V_3,q_3),*_3)$ and $\widetilde{\mathscr{C}} = ((\widetilde{V}_1,\tilde{q}_1),(\widetilde{V}_2,\tilde{q}_2),(\widetilde{V}_3,\tilde{q}_3),\tilde{*}_3)$ be compositions of quadratic spaces of dimension 8. Recall that for every linear isomorphism $g_i\colon V_i\to\widetilde{V}_i$, we define

$$\mathrm{Int}(g_i)\colon \mathrm{End}(V_i)\to\mathrm{End}(\widetilde{V}_i)\quad \text{by } f\mapsto g_i\circ f\circ g_i^{-1}.$$

Proposition 4.1. *For every similitude* $(g_1, g_2, g_3)\colon \mathscr{C} \to \widetilde{\mathscr{C}}$*, the triple*

$$\operatorname{Int}(g_1, g_2, g_3) := \big(\operatorname{Int}(g_1), \operatorname{Int}(g_2), \operatorname{Int}(g_3)\big)\colon \operatorname{End}(\mathscr{C}) \to \operatorname{End}(\widetilde{\mathscr{C}})$$

is an isomorphism of trialitarian triples. Moreover, for every isomorphism of trialitarian triples $(\gamma_1, \gamma_2, \gamma_3)\colon \operatorname{End}(\mathscr{C}) \to \operatorname{End}(\widetilde{\mathscr{C}})$*, there exists a similitude* $(g_1, g_2, g_3)\colon \mathscr{C} \to \widetilde{\mathscr{C}}$ *such that*

$$(\gamma_1, \gamma_2, \gamma_3) = \operatorname{Int}(g_1, g_2, g_3).$$

Proof. Suppose that $(g_1, g_2, g_3)\colon \mathscr{C} \to \widetilde{\mathscr{C}}$ is a similitude. For $i = 1, 2, 3$, $\operatorname{Int}(g_i)$ is an isomorphism of algebras with quadratic pairs

$$\operatorname{Int}(g_i)\colon \big(\operatorname{End}(V_i), \sigma_{b_i}, \mathfrak{f}_{q_i}\big) \xrightarrow{\sim} \big(\operatorname{End}(\widetilde{V}_i), \sigma_{\tilde{b}_i}, \mathfrak{f}_{\tilde{q}_i}\big).$$

Note that under the identification $C(\operatorname{End}(V_1), \sigma_{b_1}, \mathfrak{f}_{q_1}) = C_0(V_1, q_1)$ the isomorphism induced by $\operatorname{Int}(g_1)$ is the isomorphism $C_0(g_1)\colon C_0(V_1, q_1) \to C_0(\widetilde{V}_1, \tilde{q}_1)$ such that

$$C_0(g_1)(x_1 y_1) = \mu_1^{-1} g_1(x_1) g_1(y_1) \quad \text{for } x_1, y_1 \in V_1,$$

where μ_1 is the multiplier of g_1. Therefore, in order to show that $(\operatorname{Int}(g_1), \operatorname{Int}(g_2), \operatorname{Int}(g_3))$ is an isomorphism of trialitarian triples $\operatorname{End}(\mathscr{C}) \to \operatorname{End}(\widetilde{\mathscr{C}})$, we have to show that the following diagram commutes:

$$\begin{array}{ccc} C_0(V_1, q_1) & \xrightarrow{C_0(\alpha)} & \operatorname{End}(V_2) \times \operatorname{End}(V_3) \\ {\scriptstyle C_0(g_1)}\big\downarrow & & \big\downarrow{\scriptstyle \operatorname{Int}(g_2)\times\operatorname{Int}(g_3)} \\ C_0(\widetilde{V}_1, \tilde{q}_1) & \xrightarrow{C_0(\tilde{\alpha})} & \operatorname{End}(\widetilde{V}_2) \times \operatorname{End}(\widetilde{V}_3) \end{array} \tag{4.1}$$

Let $(\lambda_1, \lambda_2, \lambda_3)$ be the composition multiplier of (g_1, g_2, g_3). For $x_1, y_1 \in V_1$ we have

$$C_0(\tilde{\alpha}) \circ C_0(g_1)(x_1 \cdot y_1) = \mu_1^{-1} \begin{pmatrix} r_{g_1(x_1)} \ell_{g_1(y_1)} & 0 \\ 0 & \ell_{g_1(x_1)} r_{g_1(y_1)} \end{pmatrix}$$

and

$$\big(\operatorname{Int}(g_2) \times \operatorname{Int}(g_3)\big) \circ C_0(\alpha)(x_1 \cdot y_1) = \begin{pmatrix} g_2 r_{x_1} \ell_{y_1} g_2^{-1} & 0 \\ 0 & g_3 \ell_{x_1} r_{y_1} g_3^{-1} \end{pmatrix}.$$

By Proposition 3.14, (g_3, g_1, g_2) is a similitude of $\partial^2\mathscr{C}$, hence for $x_1 \in V_1$ and $x_2 \in V_2$

$$\begin{aligned} g_2 r_{x_1} \ell_{y_1}(x_2) = g_2\big((y_1 *_3 x_2) *_2 x_1\big) &= \lambda_2^{-1} g_3(y_1 *_3 x_2) *_2 g_1(x_1) \\ &= \lambda_2^{-1}\lambda_3^{-1}\big(g_1(y_1) *_3 g_2(x_2)\big) *_2 g_1(x_1). \end{aligned}$$

Since $\lambda_2\lambda_3 = \mu_1$ by (3.25), it follows that $g_2 r_{x_1} \ell_{y_1} = \mu_1^{-1} r_{g_1(x_1)} \ell_{g_1(y_1)} g_2$. Similarly,

$$g_3 \ell_{x_1} r_{y_1} = \ell_{g_1(x_1)} r_{g_1(y_1)} g_3,$$

hence diagram (4.1) commutes. The first part of the proposition is thus proved.

Now, assume $(\gamma_1, \gamma_2, \gamma_3)\colon \operatorname{End}(\mathscr{C}) \to \operatorname{End}(\widetilde{\mathscr{C}})$ is an isomorphism of trialitarian triples. Each γ_i is an isomorphism

$$\gamma_i\colon \big(\operatorname{End}(V_i), \sigma_{b_i}, \mathfrak{f}_{q_i}\big) \xrightarrow{\sim} \big(\operatorname{End}(\widetilde{V}_i), \sigma_{\tilde{b}_i}, \mathfrak{f}_{\tilde{q}_i}\big);$$

Proposition 2.2 shows that $\gamma_i = \operatorname{Int}(g_i)$ for some similitude $g_i\colon (V_i, q_i) \to (\widetilde{V}_i, \tilde{q}_i)$. We may then also consider the isomorphism

$$\operatorname{Int}\begin{pmatrix} g_2 & 0 \\ 0 & g_3 \end{pmatrix}\colon \operatorname{End}(V_2 \oplus V_3) \to \operatorname{End}(\widetilde{V}_2 \oplus \widetilde{V}_3),$$

which makes the following diagram, where the vertical maps are the diagonal embeddings, commute:

$$\begin{array}{ccc} \operatorname{End}(V_2) \times \operatorname{End}(V_3) & \xrightarrow{\operatorname{Int}(g_2)\times\operatorname{Int}(g_3)} & \operatorname{End}(\widetilde{V}_2) \times \operatorname{End}(\widetilde{V}_3) \\ \big\downarrow & & \big\downarrow \\ \operatorname{End}(V_2 \oplus V_3) & \xrightarrow{\operatorname{Int}\left(\begin{smallmatrix} g_2 & 0 \\ 0 & g_3 \end{smallmatrix}\right)} & \operatorname{End}(\widetilde{V}_2 \oplus \widetilde{V}_3) \end{array}$$

From the hypothesis that $(\gamma_1, \gamma_2, \gamma_3)$ is an isomorphism of trialitarian triples, it follows that the diagram (4.1) commutes. Write μ_1 for the multiplier of g_1 and consider the linear map

$$\beta\colon V_1 \to \operatorname{End}(\widetilde{V}_2 \oplus \widetilde{V}_3), \quad x_1 \mapsto \begin{pmatrix} 0 & r_{g_1(x_1)} \\ \mu_1^{-1}\ell_{g_1(x_1)} & 0 \end{pmatrix}.$$

For $\tilde{x}_2 \in \widetilde{V}_2$ and $\tilde{x}_3 \in \widetilde{V}_3$, we have by (3.10) and (3.12)

$$\mu_1^{-1}\big(g_1(x_1)\tilde{*}_3\tilde{x}_2\big)\tilde{*}_2 g_1(x_1) = \mu_1^{-1}\tilde{q}_1\big(g_1(x_1)\big)\tilde{x}_2$$

and

$$\mu_1^{-1} g_1(x_1)\tilde{*}_3\big(\tilde{x}_3\tilde{*}_2 g_1(x_1)\big) = \mu_1^{-1}\tilde{q}_1\big(g_1(x_1)\big)\tilde{x}_3.$$

Since $\mu_1^{-1}(\tilde{q}_1(g_1(x_1))) = q_1(x_1)$, it follows that $\beta(x_1)^2 = q_1(x_1)\operatorname{Id}_{\widetilde{V}_2\oplus\widetilde{V}_3}$ for $x_1 \in V_1$. Therefore, β induces an F-algebra homomorphism

$$C(\beta)\colon C(V_1, q_1) \to \operatorname{End}(\widetilde{V}_2 \oplus \widetilde{V}_3).$$

Since $C(V_1, q_1)$ is a simple algebra, dimension count shows that $C(\beta)$ is an isomorphism. For $x_1, y_1 \in V_1$,

$$C(\beta)(x_1 y_1) = \begin{pmatrix} \mu_1^{-1} r_{g_1(x_1)}\ell_{g_1(y_1)} & 0 \\ 0 & \mu_1^{-1}\ell_{g_1(x_1)} r_{g_1(y_1)} \end{pmatrix} = C_0(\tilde{\alpha})\big(C_0(g_1)(x_1 y_1)\big),$$

hence $C(\beta)|_{C_0(V_1,q_1)} = C_0(\tilde{\alpha}) \circ C_0(g_1)$. Since the diagram (4.1) commutes, it follows that

$$C(\beta)|_{C_0(V_1,q_1)} = \operatorname{Int}\begin{pmatrix} g_2 & 0 \\ 0 & g_3 \end{pmatrix} \circ C(\alpha)|_{C_0(V_1,q_1)}.$$

Therefore, $\operatorname{Int}\left(\begin{smallmatrix} g_2 & 0 \\ 0 & g_3 \end{smallmatrix}\right) \circ C(\alpha) \circ C(\beta)^{-1}$ is an automorphism of $\operatorname{End}(\widetilde{V}_2 \oplus \widetilde{V}_3)$ whose restriction to $C(\beta)(C_0(V_1, q_1))$ is the identity. This automorphism is inner by the Skolem–Noether theorem. Since $C(\beta)(C_0(V_1, q_1)) = \operatorname{End}(\widetilde{V}_2) \times \operatorname{End}(\widetilde{V}_3)$ we must have

$$\operatorname{Int}\begin{pmatrix} g_2 & 0 \\ 0 & g_3 \end{pmatrix} \circ C(\alpha) \circ C(\beta)^{-1} = \operatorname{Int}\begin{pmatrix} \nu_2 & 0 \\ 0 & \nu_3 \end{pmatrix} \quad \text{for some } \nu_2, \nu_3 \in F^\times,$$

hence

$$\operatorname{Int}\begin{pmatrix} g_2 & 0 \\ 0 & g_3 \end{pmatrix} \circ C(\alpha)(x_1) = \begin{pmatrix} \nu_2 & 0 \\ 0 & \nu_3 \end{pmatrix} C(\beta)(x_1) \begin{pmatrix} \nu_2^{-1} & 0 \\ 0 & \nu_3^{-1} \end{pmatrix} \quad \text{for } x_1 \in V_1,$$

which means that

$$\begin{pmatrix} 0 & g_2 r_{x_1} g_3^{-1} \\ g_3 \ell_{x_1} g_2^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \nu_2 \nu_3^{-1} r_{g_1(x_1)} \\ \nu_2^{-1} \nu_3 \mu_1^{-1} \ell_{g_1(x_1)} & 0 \end{pmatrix}.$$

The equation $g_3 \ell_{x_1} g_2^{-1} = \nu_2^{-1} \nu_3 \mu_1^{-1} \ell_{g_1(x_1)}$ implies that for $x_2 \in V_2$

$$g_3(x_1 *_3 x_2) = \nu_2^{-1} \nu_3 \mu_1^{-1} g_1(x_1) \tilde{*}_3 g_2(x_2).$$

Therefore, (g_1, g_2, g_3) is a similitude $\mathcal{C} \to \widetilde{\mathcal{C}}$. ∎

We next show that every trialitarian triple of split algebras has the form $\operatorname{End}(\mathcal{C})$ for some composition $\mathcal{C}$ of quadratic spaces of dimension 8.

Theorem 4.2. *Let $\mathcal{T} = (\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \varphi_0)$ be a trialitarian triple over an arbitrary field F, where $\mathfrak{A}_i = (A_i, \sigma_i, \mathfrak{f}_i)$ for $i = 1, 2, 3$. If A_1, A_2 and A_3 are split, then there is a composition $\mathcal{C}$ of quadratic spaces of dimension 8 over F such that $\mathcal{T} \simeq \operatorname{End}(\mathcal{C})$. The composition $\mathcal{C}$ is uniquely determined up to similitude.*

Proof. For $i = 1, 2, 3$, let $A_i = \operatorname{End}(V_i)$ for some F-vector space V_i of dimension 8. Let also q_i be a quadratic form on V_i to which $(\sigma_i, \mathfrak{f}_i)$ is adjoint. The map φ_0 is thus an isomorphism of algebras with quadratic pairs (with the notation of Section 2.2):

$$\varphi_0 \colon (C_0(V_1, q_1), \tau_{01}, \mathfrak{g}_{01}) \xrightarrow{\sim} (\operatorname{End} V_2, \sigma_{b_2}, \mathfrak{f}_{q_2}) \times (\operatorname{End} V_3, \sigma_{b_3}, \mathfrak{f}_{q_3}).$$

The idea of the proof is to extend φ_0 (after scaling q_1 and q_2 or q_3 if necessary) into an isomorphism of algebras with involution preserving the $\mathbb{Z}/2\mathbb{Z}$-grading:

$$\varphi \colon \big(C(V_1, q_1), \tau_1\big) \xrightarrow{\sim} \big(\operatorname{End}(V_2 \oplus V_3), \sigma_{b_2 \perp b_3}\big).$$

For $x_1 \in V_1$, we then have $\varphi(x_1) = \left(\begin{smallmatrix} 0 & r_{x_1} \\ \ell_{x_1} & 0 \end{smallmatrix}\right)$ for some maps $\ell_{x_1} \colon V_2 \to V_3$ and $r_{x_1} \colon V_3 \to V_2$. The last part of the proof consists in showing that $(x_1, x_2) \mapsto \ell_{x_1}(x_2)$ is a composition map.

First, we extend φ_0 into an F-algebra isomorphism φ. Since q_1 is determined only up to a scalar factor, we may assume q_1 represents 1 and pick $e_1 \in V_1$ such that $q_1(e_1) = 1$.

The inner automorphism $\operatorname{Int}(e_1)$ of the full Clifford algebra $C(V_1,q_1)$ preserves $C_0(V_1,q_1)$ and is of order 2. It transfers under the isomorphism φ_0 to an automorphism of $\operatorname{End}(V_2)\times\operatorname{End}(V_3)$ that interchanges the two factors. Viewing $\operatorname{End}(V_2)\times\operatorname{End}(V_3)$ as a subalgebra diagonally embedded in $\operatorname{End}(V_2\oplus V_3)$, we may find an inner automorphism of $\operatorname{End}(V_2\oplus V_3)$ which restricts to $\varphi_0\circ\operatorname{Int}(e_1)\circ\varphi_0^{-1}$ by (a slight generalization of) the Skolem–Noether Theorem, see [5, Prop. 1, p. A VIII.253]. This inner automorphism is conjugation by an operator of the form $\left(\begin{smallmatrix}0&u'\\u&0\end{smallmatrix}\right)$ since it interchanges $\left(\begin{smallmatrix}\mathrm{Id}_{V_2}&0\\0&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&0\\0&\mathrm{Id}_{V_3}\end{smallmatrix}\right)$. Since $\varphi_0\circ\operatorname{Int}(e_1)\circ\varphi_0^{-1}$ has order 2, it follows that $uu'=u'u\in F^\times$, hence $\operatorname{Int}\left(\begin{smallmatrix}0&u^{-1}\\u&0\end{smallmatrix}\right)$ has the same restriction to $\operatorname{End}(V_2)\times\operatorname{End}(V_3)$ as $\operatorname{Int}\left(\begin{smallmatrix}0&u'\\u&0\end{smallmatrix}\right)$. Representing $C(V_1,q_1)$ and $\operatorname{End}(V_2\oplus V_3)$ as (generalized) crossed products

$$C(V_1,q_1)=C_0(V_1,q_1)\oplus e_1C_0(V_1,q_1),$$
$$\operatorname{End}(V_2\oplus V_3)=\big(\operatorname{End}(V_2)\times\operatorname{End}(V_3)\big)\oplus\begin{pmatrix}0&u^{-1}\\u&0\end{pmatrix}\big(\operatorname{End}(V_2)\times\operatorname{End}(V_3)\big),$$

we may extend φ_0 to an isomorphism of F-algebras

$$\varphi\colon C(V_1,q_1)\to\operatorname{End}(V_2\oplus V_3)$$

by mapping e_1 to $\left(\begin{smallmatrix}0&u^{-1}\\u&0\end{smallmatrix}\right)$. Let τ_1 be the involution on $C(V_1,q_1)$ that fixes every vector in V_1 and let $\tau'=\varphi\circ\tau_1\circ\varphi^{-1}$ be the corresponding involution on $\operatorname{End}(V_2\oplus V_3)$. The restriction of τ_1 to $C_0(V_1,q_1)$ is the canonical involution τ_{01}, and $\varphi_0\circ\tau_{01}=(\sigma_2\times\sigma_3)\circ\varphi_0$, hence τ' restricts to σ_2 and σ_3 on $\operatorname{End}(V_2)$ and $\operatorname{End}(V_3)$. This means that

$$\big(\operatorname{End}(V_2\oplus V_3),\tau'\big)\in\big(\operatorname{End}(V_2),\sigma_2\big)\boxplus\big(\operatorname{End}(V_3),\sigma_3\big),$$

i.e., that τ' is adjoint to a symmetric bilinear form that is the orthogonal sum of a multiple of b_2 and a multiple of b_3. Scaling q_2 or q_3, we may assume $\tau'=\sigma_{b_2\perp b_3}$ is the adjoint involution of $b_2\perp b_3$.

Under the isomorphism φ, the odd part $C_1(V_1,q_1)=e_1C_0(V_1,q_1)$ is mapped to the odd part of $\operatorname{End}(V_2\oplus V_3)$ for the checkerboard grading, hence for each $x_1\in V_1$ there exist $\ell_{x_1}\in\operatorname{Hom}(V_2,V_3)$ and $r_{x_1}\in\operatorname{Hom}(V_3,V_2)$ such that

$$\varphi(x_1)=\begin{pmatrix}0&r_{x_1}\\\ell_{x_1}&0\end{pmatrix}\in\operatorname{End}(V_2\oplus V_3).$$

Our next goal is to show that $\ell_{x_1}\colon V_2\to V_3$ is a similitude with multiplier $q_1(x_1)$. Since $\tau_1(x_1)=x_1$, it follows that $\varphi(x_1)$ is $\sigma_{b_2\perp b_3}$-symmetric, hence for all x_2, $y_2\in V_2$ and x_3, $y_3\in V_3$

$$(b_2\perp b_3)\left(\begin{pmatrix}0&r_{x_1}\\\ell_{x_1}&0\end{pmatrix}\begin{pmatrix}x_2\\x_3\end{pmatrix},\begin{pmatrix}y_2\\y_3\end{pmatrix}\right)=(b_2\perp b_3)\left(\begin{pmatrix}x_2\\x_3\end{pmatrix},\begin{pmatrix}0&r_{x_1}\\\ell_{x_1}&0\end{pmatrix}\begin{pmatrix}y_2\\y_3\end{pmatrix}\right).$$

This means that for all x_2, $y_2\in V_2$ and x_3, $y_3\in V_3$,

$$b_2\big(r_{x_1}(x_3),y_2\big)=b_3\big(x_3,\ell_{x_1}(y_2)\big)\quad\text{and}\quad b_3\big(\ell_{x_1}(x_2),y_2\big)=b_2\big(x_2,r_{x_1}(y_3)\big).\tag{4.2}$$

Moreover, the relations $x_1^2 = q_1(x_1)$ and $x_1y_1 + y_1x_1 = b_1(x_1, y_1)$ yield for all x_1, $y_1 \in V_1$

$$\ell_{x_1} r_{x_1} = r_{x_1}\ell_{x_1} = q_1(x_1) \text{ and } \ell_{x_1} r_{y_1} + \ell_{y_1} r_{x_1} = r_{x_1}\ell_{y_1} + r_{y_1}\ell_{x_1} = b_1(x_1, y_1). \quad (4.3)$$

Recall that the two components $\varphi_\pm$ of φ_0 are homomorphisms of algebras with quadratic pair

$$\varphi_+ : \big(C_0(V_1, q_1), \tau_{01}, \underline{\mathfrak{f}}_1\big) \to \mathfrak{A}_2, \quad \varphi_- : \big(C_0(V_1, q_1), \tau_{01}, \underline{\mathfrak{f}}_1\big) \to \mathfrak{A}_3.$$

As observed in Definition 2.20, $\varphi_+(\boldsymbol{\omega}(q_1)) \subset \mathfrak{go}(q_2)$ and $\varphi_-(\boldsymbol{\omega}(q_1)) \subset \mathfrak{go}(q_3)$, hence

$$\varphi_+(x_1y_1) \in \mathfrak{go}(q_2) \quad \text{and} \quad \varphi_-(x_1y_1) \in \mathfrak{go}(q_3) \quad \text{for all } x_1, y_1 \in V_1.$$

The definition of φ yields $\varphi_+(x_1y_1) = r_{x_1}\ell_{y_1}$ and $\varphi_-(x_1y_1) = \ell_{x_1} r_{y_1}$, hence by (4.3)

$$\dot{\mu}\big(\varphi_+(x_1y_1)\big) = \varphi_+(x_1y_1) + \varphi_+(y_1x_1) = b_1(x_1, y_1)$$

and

$$\dot{\mu}\big(\varphi_-(x_1y_1)\big) = \varphi_-(x_1y_1) + \varphi_-(y_1x_1) = b_1(x_1, y_1).$$

Since $r_{x_1}\ell_{y_1} \in \mathfrak{go}(q_2)$ and $\ell_{x_1} r_{y_1} \in \mathfrak{go}(q_3)$, it follows from Proposition 2.11 that for x_1, $y_1 \in V_1$, $x_2 \in V_2$ and $x_3 \in V_3$

$$b_2\big(r_{x_1}\ell_{y_1}(x_2), x_2\big) = b_1(x_1, y_1)q_2(x_2) \text{ and } b_3\big(\ell_{x_1} r_{y_1}(x_3), x_3\big) = b_1(x_1, y_1)q_3(x_3). \quad (4.4)$$

If $x_1 \in V_1$ is nonzero, there exists $y_1 \in V_1$ such that $b_1(x_1, y_1) = 1$. From (4.2) and (4.3) we derive for all $x_2 \in V_2$

$$b_3\big(\ell_{x_1}(x_2), \ell_{x_1} r_{y_1}\ell_{x_1}(x_2)\big) = b_2\big(r_{x_1}\ell_{x_1}(x_2), r_{y_1}\ell_{x_1}(x_2)\big) = q_1(x_1)b_2\big(x_2, r_{y_1}\ell_{x_1}(x_2)\big).$$

But (4.4) yields

$$b_3\big(\ell_{x_1}(x_2), \ell_{x_1} r_{y_1}\ell_{x_1}(x_2)\big) = q_3\big(\ell_{x_1}(x_2)\big) \quad \text{and} \quad b_2\big(x_2, r_{y_1}\ell_{x_1}(x_2)\big) = q_2(x_2),$$

hence

$$q_3\big(\ell_{x_1}(x_2)\big) = q_1(x_1)q_2(x_2) \quad \text{for all } x_1 \in V_1, x_2 \in V_2 \text{ with } x_1 \neq 0.$$

This equation obviously also holds for $x_1 = 0$. Therefore, defining

$$*_3 : V_1 \times V_2 \to V_3 \quad \text{by} \quad x_1 *_3 x_2 = \ell_{x_1}(x_2) \quad \text{for } x_1 \in V_1 \text{ and } x_2 \in V_2,$$

we see that $*_3$ is a composition of (V_1, q_1), (V_2, q_2) and (V_3, q_3). Let also

$$x_3 *_2 x_1 = r_{x_1}(x_3) \quad \text{for } x_3 \in V_3 \text{ and } x_1 \in V_1.$$

From (4.2) it follows that $b_2(x_3 *_2 x_1, x_2) = b_3(x_3, x_1 *_3 x_2)$ for $x_1 \in V_1$, $x_2 \in V_2$ and $x_3 \in V_3$, hence Proposition 3.1 shows that $*_2$ is the derived composition of (V_3, q_3), (V_1, q_1) and (V_2, q_2). Therefore, $\varphi_0 = C_0(\alpha)$ for $\alpha : V_1 \to \operatorname{End}(V_2 \oplus V_3)$ mapping $x_1 \in V_1$ to $\begin{pmatrix} 0 & r_{x_1} \\ \ell_{x_1} & 0 \end{pmatrix}$. We thus see that $\mathcal{T} = \operatorname{End}(\mathcal{C})$ for $\mathcal{C} = ((V_1, q_1), (V_2, q_2), (V_3, q_3), *_3)$. Proposition 4.1 shows that the composition $\mathcal{C}$ is uniquely determined up to similitude. ■

4.2. Similitudes of trialitarian triples

Throughout this subsection, we fix a trialitarian triple

$$\mathcal{T} = (\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \varphi_0)$$

with $\mathfrak{A}_i = (A_1, \sigma_i, \mathfrak{f}_i)$ a central simple algebra with quadratic pair of degree 8 over an arbitrary field F for $i = 1, 2, 3$. The algebraic group scheme $\mathbf{Aut}(\mathcal{T})$ of automorphisms of $\mathcal{T}$ is defined as follows: for any commutative F-algebra R, the group $\mathbf{Aut}(\mathcal{T})(R)$ consists of the triples $(\gamma_1, \gamma_2, \gamma_3) \in \mathbf{Aut}(\mathfrak{A}_1)(R) \times \mathbf{Aut}(\mathfrak{A}_2)(R) \times \mathbf{Aut}(\mathfrak{A}_3)(R)$ that make the following square commute:

$$\begin{array}{ccc} C(\mathfrak{A}_1)_R & \xrightarrow{\varphi_0} & A_{2R} \times A_{3R} \\ {\scriptstyle C(\gamma_1)}\downarrow & & \downarrow{\scriptstyle \gamma_2\times\gamma_3} \\ C(\mathfrak{A}_1)_R & \xrightarrow{\varphi_0} & A_{2R} \times A_{3R} \end{array} \tag{4.5}$$

Thus,

$$\mathbf{Aut}(\mathcal{T}) \subset \mathbf{Aut}(\mathfrak{A}_1) \times \mathbf{Aut}(\mathfrak{A}_2) \times \mathbf{Aut}(\mathfrak{A}_3).$$

Now, recall from [13, §23.B] that the map $\mathrm{Int}\colon \mathbf{GO}(\mathfrak{A}_i) \to \mathbf{Aut}(\mathfrak{A}_i)$ defines an isomorphism $\mathbf{PGO}(\mathfrak{A}_i) \xrightarrow{\sim} \mathbf{Aut}(\mathfrak{A}_i)$. Therefore, we may consider the inverse image of $\mathbf{Aut}(\mathcal{T})$ under the surjective morphism of algebraic group schemes

$$\mathrm{Int}\colon \mathbf{GO}(\mathfrak{A}_1) \times \mathbf{GO}(\mathfrak{A}_2) \times \mathbf{GO}(\mathfrak{A}_3) \to \mathbf{Aut}(\mathfrak{A}_1) \times \mathbf{Aut}(\mathfrak{A}_2) \times \mathbf{Aut}(\mathfrak{A}_3).$$

Definition 4.3. The algebraic group scheme of *similitudes* of the trialitarian triple $\mathcal{T}$ is

$$\mathbf{GO}(\mathcal{T}) = \mathrm{Int}^{-1}\big(\mathbf{Aut}(\mathcal{T})\big) \subset \mathbf{GO}(\mathfrak{A}_1) \times \mathbf{GO}(\mathfrak{A}_2) \times \mathbf{GO}(\mathfrak{A}_3).$$

From this definition, it follows that the map Int restricts to a surjective morphism of algebraic group schemes (see [13, (22.4)])

$$\mathrm{Int}\colon \mathbf{GO}(\mathcal{T}) \to \mathbf{Aut}(\mathcal{T}).$$

Its kernel is the algebraic group of *homotheties* $\mathbf{H}(\mathcal{T}) = \mathbf{G}_{\mathbf{m}}^3$, which lies in the center of $\mathbf{GO}(\mathcal{T})$. We may therefore consider the quotient

$$\mathbf{PGO}(\mathcal{T}) = \mathbf{GO}(\mathcal{T})/\,\mathbf{H}(\mathcal{T}) \subset \mathbf{PGO}(\mathfrak{A}_1) \times \mathbf{PGO}(\mathfrak{A}_2) \times \mathbf{PGO}(\mathfrak{A}_3),$$

and the map Int yields an isomorphism

$$\overline{\mathrm{Int}}\colon \mathbf{PGO}(\mathcal{T}) \xrightarrow{\sim} \mathbf{Aut}(\mathcal{T}).$$

Our goal in this subsection is to define a subgroup $\mathbf{O}(\mathcal{T}) \subset \mathbf{GO}(\mathcal{T})$ on the same model as the subgroup $\mathbf{O}(\mathcal{C})$ of the group $\mathbf{GO}(\mathcal{C})$ of similitudes of a composition of quadratic spaces, so that when $\mathcal{T} = \operatorname{End}(\mathscr{C})$ for some composition $\mathscr{C}$ of quadratic spaces of dimension 8 we may identify

$$\mathbf{O}(\mathcal{T}) = \mathbf{O}(\mathscr{C}), \quad \mathbf{GO}(\mathcal{T}) = \mathbf{GO}(\mathscr{C}) \quad \text{and} \quad \mathbf{PGO}(\mathcal{T}) = \mathbf{PGO}(\mathscr{C});$$

see Proposition 4.10. Moreover, for an arbitrary trialitarian triple $\mathcal{T}$, we relate $\mathbf{GO}(\mathcal{T})$ to the extended Clifford group $\mathbf{\Omega}(\mathfrak{A}_1)$ to obtain canonical isomorphisms

$$\mathbf{Spin}(\mathfrak{A}_1) \xrightarrow{\sim} \mathbf{O}(\mathcal{T}) \quad \text{and} \quad \mathbf{PGO}(\mathcal{T}) \xrightarrow{\sim} \mathbf{PGO}^+(\mathfrak{A}_1),$$

see Theorems 4.4 and 4.12.

A key tool is the following construction: let $\varphi_+ : \mathfrak{C}(\mathfrak{A}_1) \to \mathfrak{A}_2$ and $\varphi_- : \mathfrak{C}(\mathfrak{A}_1) \to \mathfrak{A}_3$ be the two components of the isomorphism $\varphi_0 : \mathfrak{C}(\mathfrak{A}_1) \to \mathfrak{A}_2 \times \mathfrak{A}_3$, which is part of the structure of $\mathcal{T}$. Recall from (2.18) that φ_+ and φ_- restrict to morphisms

$$\varphi_+ : \mathbf{\Omega}(\mathfrak{A}_1) \to \mathbf{GO}^+(\mathfrak{A}_2) \quad \text{and} \quad \varphi_- : \mathbf{\Omega}(\mathfrak{A}_1) \to \mathbf{GO}^+(\mathfrak{A}_3).$$

Combine φ_+ and φ_- with the morphism $\chi_0 : \mathbf{\Omega}(\mathfrak{A}_1) \to \mathbf{GO}^+(\mathfrak{A}_1)$ of Section 2.3 to obtain a morphism

$$\psi_{\mathcal{T}} : \mathbf{\Omega}(\mathfrak{A}_1) \to \mathbf{GO}(\mathcal{T}) \tag{4.6}$$

as follows: for every commutative F-algebra R and $\xi \in \mathbf{\Omega}(\mathfrak{A}_1)(R)$, let

$$\psi_{\mathcal{T}}(\xi) = \big(\chi_0(\xi), \varphi_+(\xi), \varphi_-(\xi)\big) \in \mathbf{GO}^+(\mathfrak{A}_1)(R) \times \mathbf{GO}^+(\mathfrak{A}_2)(R) \times \mathbf{GO}^+(\mathfrak{A}_3)(R).$$

Proposition 2.7 shows that $C(\operatorname{Int}(\chi_0(\xi))) = \operatorname{Int}(\xi)$, hence

$$\varphi_0 \circ C\big(\operatorname{Int}\big(\chi_0(\xi)\big)\big) \circ \varphi_0^{-1} = \operatorname{Int}\big(\varphi_0(\xi)\big) = \operatorname{Int}\big(\varphi_+(\xi)\big) \times \operatorname{Int}\big(\varphi_-(\xi)\big),$$

which means that $(\operatorname{Int}(\chi_0(\xi)), \operatorname{Int}(\varphi_+(\xi)), \operatorname{Int}(\varphi_-(\xi)))$ lies in $\mathbf{Aut}(\mathcal{T})(R)$, and therefore $\psi_{\mathcal{T}}(\xi) \in \mathbf{GO}(\mathcal{T})(R)$. Note that $\psi_{\mathcal{T}}$ is injective, since $(\varphi_+(\xi), \varphi_-(\xi)) = \varphi_0(\xi)$ and φ_0 is an isomorphism.

We first use the map $\psi_{\mathcal{T}}$ to prove:

Theorem 4.4. *Projection on the first component $\pi_{\mathcal{T}} : \mathbf{PGO}(\mathcal{T}) \to \mathbf{PGO}(\mathfrak{A}_1)$ defines an isomorphism*

$$\mathbf{PGO}(\mathcal{T}) \xrightarrow{\sim} \mathbf{PGO}^+(\mathfrak{A}_1).$$

Proof. Let R be a commutative F-algebra and $(\gamma_1, \gamma_2, \gamma_3) \in \mathbf{Aut}(\mathcal{T})(R)$. Since φ_0 is an isomorphism, γ_2 and γ_3 are uniquely determined by γ_1 and commutativity of the diagram (4.5). Therefore, $\pi_{\mathcal{T}}$ is injective. Moreover, commutativity of the diagram (4.5) shows that $C(\gamma_1)$ leaves the center of $C(\mathfrak{A}_1)$ fixed, which means that γ_1 lies in the connected component of the identity $\mathbf{Aut}^+(\mathfrak{A}_1)(R)$. It follows that the image of $\pi_{\mathcal{T}}$ lies in $\mathbf{PGO}^+(\mathfrak{A}_1)$.

To complete the proof, we show that $\pi_{\mathcal{T}}$ is surjective on $\mathbf{PGO}^+(\mathfrak{A}_1)$. Since $\mathbf{PGO}^+(\mathfrak{A}_1)$ is smooth, it suffices to consider rational points over an algebraic closure F_{alg} of F, by [13, (22.3)]. Recall from Proposition 2.8 that χ_0 is surjective. For every $g_1 \in \mathbf{GO}^+(\mathfrak{A}_1)(F_{\mathrm{alg}})$, we may therefore find $\xi \in \mathbf{\Omega}(\mathfrak{A}_1)(F_{\mathrm{alg}})$ such that $\chi_0(\xi) = g_1$. Then $\psi_{\mathcal{T}}(\xi) \in \mathbf{GO}(\mathcal{T})(F_{\mathrm{alg}})$, and its image $\overline{\psi}_{\mathcal{T}}(\xi)$ in $\mathbf{PGO}(\mathcal{T})(F_{\mathrm{alg}})$ satisfies

$$\pi_{\mathcal{T}}\big(\overline{\psi}_{\mathcal{T}}(\xi)\big) = g_1 F_{\mathrm{alg}}^{\times}.$$

We have thus found an element in $\mathbf{PGO}(\mathcal{T})(F_{\mathrm{alg}})$ that maps under $\pi_{\mathcal{T}}$ to any given $g_1 F_{\mathrm{alg}}^{\times} \in \mathbf{PGO}^+(\mathfrak{A}_1)(F_{\mathrm{alg}})$, hence $\pi_{\mathcal{T}}$ is surjective. ■

Corollary 4.5. *The algebraic group schemes* $\mathbf{GO}(\mathcal{T})$ *and* $\mathbf{PGO}(\mathcal{T})$ *are smooth and connected.*

Proof. That $\mathbf{PGO}(\mathcal{T})$ is smooth and connected readily follows from the theorem, since $\mathbf{PGO}^+(\mathfrak{A}_1)$ is smooth and connected by [13, §23.B]. Then $\mathbf{GO}(\mathcal{T})$ is also smooth and connected because $\mathbf{PGO}(\mathcal{T}) = \mathbf{GO}(\mathcal{T})/\mathbf{H}(\mathcal{T})$ with $\mathbf{H}(\mathcal{T})$ smooth and connected, see [13, (22.12)]. ■

We next use $\psi_{\mathcal{T}}$ to determine the structure of $\mathbf{GO}(\mathcal{T})$. Let $Z_1 \simeq F \times F$ denote the center of $C(\mathfrak{A}_1)$, and recall that $R_{Z_1/F}(\mathbf{G_m})$ lies in the center of $\mathbf{\Omega}(\mathfrak{A}_1)$ (see (2.3)). For every commutative F-algebra R and $z \in (Z_1)_R^{\times}$, Proposition 2.7 yields $\chi_0(z) = N_{Z_1/F}(z)$, while $\varphi_+(z), \varphi_-(z) \in R^{\times}$. Therefore,

$$\psi_{\mathcal{T}}(z) = \big(N_{Z_1/F}(z), \varphi_+(z), \varphi_-(z)\big) \in R^{\times} \times R^{\times} \times R^{\times} = \mathbf{H}(\mathcal{T})(R). \tag{4.7}$$

Proposition 4.6. *The morphism* $\psi_{\mathcal{T}}$ *and the inclusion* $i\colon \mathbf{H}(\mathcal{T}) \to \mathbf{GO}(\mathcal{T})$ *combine with the multiplication in* $\mathbf{GO}(\mathcal{T})$ *into a surjective morphism of algebraic group schemes*

$$\psi_{\mathcal{T}} \times i\colon \mathbf{\Omega}(\mathfrak{A}_1) \times \mathbf{H}(\mathcal{T}) \to \mathbf{GO}(\mathcal{T}).$$

This morphism fits in the exact sequence

$$1 \to R_{Z_1/F}(\mathbf{G_m}) \to \mathbf{\Omega}(\mathfrak{A}_1) \times \mathbf{H}(\mathcal{T}) \xrightarrow{\psi_{\mathcal{T}} \times i} \mathbf{GO}(\mathcal{T}) \to 1$$

where $R_{Z_1/F}(\mathbf{G_m})$ *is embedded into the product canonically in the first factor and by the inversion followed by* $\psi_{\mathcal{T}}$ *in the second.*

Proof. It is clear from the definition of the embedding of $R_{Z_1/F}(\mathbf{G_m})$ that $R_{Z_1/F}(\mathbf{G_m}) \subset \ker(\psi_{\mathcal{T}} \times i)$. To prove the reverse inclusion, consider an arbitrary commutative F-algebra R and pick $\xi \in \mathbf{\Omega}(\mathfrak{A}_1)(R)$ and $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbf{H}(\mathcal{T})(R)$ such that $\psi_{\mathcal{T}}(\xi) \cdot \nu = 1$ in $\mathbf{GO}(\mathcal{T})(R)$, i.e.,

$$\chi_0(\xi) = \nu_1^{-1}, \quad \varphi_+(\xi) = \nu_2^{-1} \quad \text{and} \quad \varphi_-(\xi) = \nu_3^{-1}.$$

The last two equations show that $\varphi_0(\xi) = (\nu_2^{-1}, \nu_3^{-1})$ in $(A_2)_R \times (A_3)_R$. Since φ_0 is an isomorphism, it follows that ξ lies in $(Z_1)_R^{\times}$, hence (ξ, ν) belongs to the image of $(Z_1)_R^{\times}$, for $\nu = \psi_{\mathcal{T}}(\xi)^{-1}$.

To complete the proof, it remains to show that $\psi_{\mathcal{T}} \times i$ is surjective. Since $\mathbf{GO}(\mathcal{T})$ is smooth by Corollary 4.5, it suffices to consider the groups of rational points over an algebraic closure F_{alg} of F. Let $g = (g_1, g_2, g_3) \in \mathbf{GO}(\mathcal{T})(F_{\mathrm{alg}})$. Note that g_1, g_2 and g_3 are proper similitudes, because $\mathbf{GO}(\mathcal{T})$ is connected by Corollary 4.5. We know from Proposition 2.8 that $\chi_0\colon \mathbf{\Omega}(\mathfrak{A}_1) \to \mathbf{GO}^+(\mathfrak{A}_1)$ is surjective, hence we may find $\xi \in \mathbf{\Omega}(\mathfrak{A}_1)(F_{\mathrm{alg}})$ such that $\chi_0(\xi) = g_1$. Then

$$\psi_{\mathcal{T}}(\xi) = (g_1, g_2', g_3')$$

for some $g_2' \in \mathbf{GO}^+(\mathfrak{A}_2)(F_{\mathrm{alg}})$ and $g_3' \in \mathbf{GO}^+(\mathfrak{A}_3)(F_{\mathrm{alg}})$. As g and $\psi_{\mathcal{T}}(\xi)$ belong to $\mathbf{GO}(\mathcal{T})(F_{\mathrm{alg}})$, the following diagrams commute:

$$\begin{array}{ccc} C(\mathfrak{A}_1)_{F_{\mathrm{alg}}} & \xrightarrow{\varphi_0} & (A_2)_{F_{\mathrm{alg}}} \times (A_3)_{F_{\mathrm{alg}}} \\ {\scriptstyle C(\mathrm{Int}(g_1))}\downarrow & & \downarrow{\scriptstyle \mathrm{Int}(g_2)\times\mathrm{Int}(g_3)} \\ C(\mathfrak{A}_1)_{F_{\mathrm{alg}}} & \xrightarrow{\varphi_0} & (A_2)_{F_{\mathrm{alg}}} \times (A_3)_{F_{\mathrm{alg}}} \end{array}$$

$$\begin{array}{ccc} C(\mathfrak{A}_1)_{F_{\mathrm{alg}}} & \xrightarrow{\varphi_0} & (A_2)_{F_{\mathrm{alg}}} \times (A_3)_{F_{\mathrm{alg}}} \\ {\scriptstyle C(\mathrm{Int}(g_1))}\downarrow & & \downarrow{\scriptstyle \mathrm{Int}(g_2')\times\mathrm{Int}(g_3')} \\ C(\mathfrak{A}_1)_{F_{\mathrm{alg}}} & \xrightarrow{\varphi_0} & (A_2)_{F_{\mathrm{alg}}} \times (A_3)_{F_{\mathrm{alg}}} \end{array}$$

Therefore, $\mathrm{Int}(g_2) = \mathrm{Int}(g_2')$ and $\mathrm{Int}(g_3) = \mathrm{Int}(g_3')$, which implies that $g_2 = g_2'\nu_2$ and $g_3 = g_3'\nu_3$ for some $\nu_2, \nu_3 \in F_{\mathrm{alg}}^\times$. With $\nu = (1, \nu_2, \nu_3) \in \mathbf{H}(\mathcal{T})(F_{\mathrm{alg}})$ we then have

$$\psi_{\mathcal{T}}(\xi) \cdot \nu = (g_1, g_2'\nu_2, g_3'\nu_3) = (g_1, g_2, g_3) = g.$$

Surjectivity of $\psi_{\mathcal{T}} \times i$ follows. ∎

Corollary 4.7. *Let $m\colon \mathbf{H}(\mathcal{T}) \to \mathbf{G_m}$ denote the multiplication map carrying (ν_1, ν_2, ν_3) to $\nu_1\nu_2\nu_3$. There is a morphism $\rho_{\mathcal{T}}\colon \mathbf{GO}(\mathcal{T}) \to \mathbf{G_m}$ uniquely determined by the condition that the following diagram commutes:*

$$\begin{array}{ccc} \mathbf{\Omega}(\mathfrak{A}_1) \times \mathbf{H}(\mathcal{T}) & \xrightarrow{\psi_{\mathcal{T}}\times i} & \mathbf{GO}(\mathcal{T}) \\ & {\scriptstyle (\mu\circ\chi_0)\times m} \searrow \quad \swarrow {\scriptstyle \rho_{\mathcal{T}}} & \\ & \mathbf{G_m} & \end{array}$$

Proof. Proposition 4.6 identifies $\mathbf{GO}(\mathcal{T})$ as a quotient of $\mathbf{\Omega}(\mathfrak{A}_1) \times \mathbf{H}(\mathcal{T})$ by $R_{Z_1/F}(\mathbf{G_m})$, hence to prove the existence and uniqueness of $\rho_{\mathcal{T}}$ it suffices to show that $(\mu \circ \chi_0)$ and m coincide on the images of $R_{Z_1/F}(\mathbf{G_m})$ in $\mathbf{\Omega}(\mathcal{T})$ by inclusion and in $\mathbf{H}(\mathcal{T})$ by $\psi_{\mathcal{T}}$. For every commutative F-algebra R and $z \in (Z_1)_R^\times$ we have $\chi_0(z) = N_{Z_1/F}(z)$ by Proposition 2.7, hence

$$(\mu \circ \chi_0)(z) = N_{Z_1/F}(z)^2.$$

On the other hand $N_{Z_1/F}(z) = \varphi_+(z)\varphi_-(z)$, hence

$$(m \circ \psi_{\mathcal{T}})(z) = m\big(N_{Z_1/F}(z), \varphi_+(z), \varphi_-(z)\big) = N_{Z_1/F}(z)^2.$$

Thus, $(\mu \circ \chi_0)$ and m coincide on the images of $R_{Z_1/F}(\mathbf{G_m})$. ■

Definition 4.8. The morphism $\lambda_{\mathcal{T}} \colon \mathbf{GO}(\mathcal{T}) \to \mathbf{G}_\mathbf{m}^3$ is defined as follows: for every commutative F-algebra R and $g = (g_1, g_2, g_3) \in \mathbf{GO}(\mathcal{T})(R)$, set

$$\lambda_{\mathcal{T}}(g) = \big(\rho_{\mathcal{T}}(g)\mu(g_1)^{-1}, \rho_{\mathcal{T}}(g)\mu(g_2)^{-1}, \rho_{\mathcal{T}}(g)\mu(g_3)^{-1}\big) \in R^\times \times R^\times \times R^\times.$$

From the definition of $\rho_{\mathcal{T}}$, it follows that for $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbf{H}(\mathcal{T})(R)$

$$\rho_{\mathcal{T}}(\nu) = m(\nu) = \nu_1\nu_2\nu_3,$$

hence

$$\lambda_{\mathcal{T}}(\nu) = (\nu_2\nu_3\nu_1^{-1}, \nu_3\nu_1\nu_2^{-1}, \nu_1\nu_2\nu_3^{-1}). \tag{4.8}$$

The definition of $\rho_{\mathcal{T}}$ also yields $\rho_{\mathcal{T}}(\psi_{\mathcal{T}}(\xi)) = \mu(\chi_0(\xi))$ for $\xi \in \mathbf{\Omega}(\mathfrak{A}_1)(R)$. Letting

$$\underline{\mu} \colon \mathbf{\Omega}(\mathfrak{A}_1) \to R_{Z_1/F}(\mathbf{G_m})$$

denote the multiplier map, we have by Proposition 2.7

$$\mu\big(\chi_0(\xi)\big) = N_{Z_1/F}\big(\underline{\mu}(\xi)\big) = \varphi_+\big(\underline{\mu}(\xi)\big) \cdot \varphi_-\big(\underline{\mu}(\xi)\big).$$

As φ_0 is an isomorphism of algebras with quadratic pair, we also have

$$\big(\varphi_+\big(\underline{\mu}(\xi)\big), \varphi_-\big(\underline{\mu}(\xi)\big)\big) = \varphi_0\big(\underline{\mu}(\xi)\big) = \big(\mu\big(\varphi_+(\xi)\big), \mu\big(\varphi_-(\xi)\big)\big). \tag{4.9}$$

Therefore, the definition of $\lambda_{\mathcal{T}}$ yields

$$\begin{aligned} \lambda_{\mathcal{T}}\big(\psi_{\mathcal{T}}(\xi)\big) &= \big(1, \mu\big(\chi_0(\xi)\big)\mu\big(\varphi_+(\xi)\big)^{-1}, \mu\big(\chi_0(\xi)\big)\mu\big(\varphi_-(\xi)\big)^{-1}\big) \\ &= \big(1, \mu\big(\varphi_-(\xi)\big), \mu\big(\varphi_+(\xi)\big)\big). \end{aligned} \tag{4.10}$$

Definition 4.9. Let

$$\mathbf{O}(\mathcal{T}) = \ker(\lambda_{\mathcal{T}} \colon \mathbf{GO}(\mathcal{T}) \to \mathbf{G}_\mathbf{m}^3).$$

As in the proof of Proposition 3.19, it follows from (4.8) that the map $\lambda_{\mathcal{T}} \colon \mathbf{H}(\mathcal{T}) \to \mathbf{G}_\mathbf{m}^3$, hence also $\lambda_{\mathcal{T}} \colon \mathbf{GO}(\mathcal{T}) \to \mathbf{G}_\mathbf{m}^3$, is surjective. Therefore, the following sequence is exact:

$$1 \to \mathbf{O}(\mathcal{T}) \to \mathbf{GO}(\mathcal{T}) \xrightarrow{\lambda_{\mathcal{T}}} \mathbf{G}_\mathbf{m}^3 \to 1$$

Now, let $\mathbf{Z}(\mathcal{T})$ be the kernel of the canonical map $\mathbf{O}(\mathcal{T}) \to \mathbf{PGO}(\mathcal{T})$, which is the composition of the inclusion $\mathbf{O}(\mathcal{T}) \subset \mathbf{GO}(\mathcal{T})$ and the canonical epimorphism $\mathbf{GO}(\mathcal{T}) \to \mathbf{PGO}(\mathcal{T})$. Thus, letting m be the multiplication map $(\nu_1, \nu_2, \nu_3) \mapsto \nu_1\nu_2\nu_3$,

$$\mathbf{Z}(\mathcal{T}) = \mathbf{H}(\mathcal{T}) \cap \mathbf{O}(\mathcal{T}) = \ker(m \colon \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \to \boldsymbol{\mu}_2) \simeq \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2.$$

The same arguments as in Proposition 3.20 yield the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathbf{Z}(\mathcal{T}) & \longrightarrow & \mathbf{O}(\mathcal{T}) & \longrightarrow & \mathbf{PGO}(\mathcal{T}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathbf{H}(\mathcal{T}) & \longrightarrow & \mathbf{GO}(\mathcal{T}) & \longrightarrow & \mathbf{PGO}(\mathcal{T}) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \lambda_{\mathcal{T}}} & & \downarrow{\scriptstyle \lambda_{\mathcal{T}}} & & & & \\
 & & \mathbf{G}_{\mathbf{m}}^3 & = & \mathbf{G}_{\mathbf{m}}^3 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array} \tag{4.11}$$

Now, we show that the definitions above are compatible with the corresponding definitions for compositions of quadratic spaces in Section 3.3.

Proposition 4.10. *For $\mathcal{C}$ any composition of quadratic spaces of dimension 8 and $\mathcal{T} = \operatorname{End}(\mathcal{C})$, canonical isomorphisms yield identifications*

$$\mathbf{H}(\mathcal{C}) = \mathbf{H}(\mathcal{T}), \quad \mathbf{O}(\mathcal{C}) = \mathbf{O}(\mathcal{T}), \quad \mathbf{GO}(\mathcal{C}) = \mathbf{GO}(\mathcal{T}), \quad \mathbf{PGO}(\mathcal{C}) = \mathbf{PGO}(\mathcal{T}).$$

Moreover, the following diagram commutes:

$$\begin{array}{ccc}
\mathbf{GO}(\mathcal{C}) & = & \mathbf{GO}(\mathcal{T}) \\
{\scriptstyle \lambda_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle \lambda_{\mathcal{T}}} \\
\mathbf{G}_{\mathbf{m}}^3 & = & \mathbf{G}_{\mathbf{m}}^3
\end{array} \tag{4.12}$$

Proof. Let R be a commutative F-algebra. For every $(g_1, g_2, g_3, \lambda_3) \in \mathbf{GO}(\mathcal{C})(R)$ the triple (g_1, g_2, g_3) lies in $\mathbf{GO}(\mathcal{T})(R)$, as seen in the first part of the proof of Proposition 4.1. Since λ_3 is uniquely determined by g_1, g_2 and g_3, mapping $(g_1, g_2, g_3, \lambda_3)$ to (g_1, g_2, g_3) defines an injective map $\mathbf{GO}(\mathcal{C}) \to \mathbf{GO}(\mathcal{T})$. Proposition 4.1 also shows that for F_{alg} an algebraic closure of F the map $\mathbf{GO}(\mathcal{C})(F_{\mathrm{alg}}) \to \mathbf{GO}(\mathcal{T})(F_{\mathrm{alg}})$ is surjective. This is sufficient to prove that the map $\mathbf{GO}(\mathcal{C}) \to \mathbf{GO}(\mathcal{T})$ is surjective, since $\mathbf{GO}(\mathcal{T})$ is smooth by Corollary 4.5. We have thus obtained a canonical isomorphism $\mathbf{GO}(\mathcal{C}) \xrightarrow{\sim} \mathbf{GO}(\mathcal{T})$. This isomorphism maps $\mathbf{H}(\mathcal{C})$ to $\mathbf{H}(\mathcal{T})$, hence it induces an isomorphism $\mathbf{PGO}(\mathcal{C}) \xrightarrow{\sim} \mathbf{PGO}(\mathcal{T})$.

In order to prove that the isomorphism $\mathbf{GO}(\mathcal{C}) = \mathbf{GO}(\mathcal{T})$ also maps $\mathbf{O}(\mathcal{C})$ to $\mathbf{O}(\mathcal{T})$, it suffices to prove that the diagram (4.12) is commutative. For this, we use the description of $\mathbf{GO}(\mathcal{T})$ in Proposition 4.6 as a quotient of the product of $\boldsymbol{\Omega}(\mathfrak{A}_1)$ and $\mathbf{H}(\mathcal{T})$. It is clear from (4.8) that $\lambda_{\mathscr{C}}$ and $\lambda_{\mathscr{T}}$ coincide on the image of $\mathbf{H}(\mathscr{C}) = \mathbf{H}(\mathscr{T})$ in $\mathbf{GO}(\mathscr{C}) = \mathbf{GO}(\mathscr{T})$. Therefore, it suffices to consider the image of $\mathbf{\Omega}(\mathfrak{A}_1)$ under $\psi_{\mathscr{T}}$.

Let $\mathscr{C} = ((V_1, q_1), (V_2, q_2), (V_3, q_3), *_3)$, so $\varphi_0 = C_0(\alpha)$, $A_1 = \operatorname{End} V_1$ and $\mathbf{\Omega}(\mathfrak{A}_1) = \mathbf{\Omega}(q_1)$. Let R be a commutative F-algebra and let $\xi \in \mathbf{\Omega}(q_1)(R)$. To simplify notation, write $g_1 = \chi_0(\xi) \in \mathbf{GO}^+(q_1)(R)$, $g_2 = C_+(\alpha)(\xi) \in \mathbf{GO}^+(q_2)(R)$ and $g_3 = C_-(\alpha)(\xi) \in \mathbf{GO}^+(q_3)(R)$, so

$$\psi_{\mathscr{T}}(\xi) = (g_1, g_2, g_3).$$

Now, (4.9) yields

$$C_0(\alpha)\big(\underline{\mu}(\xi)\big) = \big(\mu(g_2), \mu(g_3)\big),$$

and by (2.1), we have

$$g_1(x_1) = \iota\big(\underline{\mu}(\xi)\big)\xi x_1 \xi^{-1} \quad \text{for every } x_1 \in V_{1R}.$$

By taking the image of each side of the last equation under $C(\alpha)$, we obtain

$$\begin{pmatrix} 0 & r_{g_1(x_1)} \\ \ell_{g_1(x_1)} & 0 \end{pmatrix} = \begin{pmatrix} \mu(g_3) & 0 \\ 0 & \mu(g_2) \end{pmatrix} \begin{pmatrix} g_2 & 0 \\ 0 & g_3 \end{pmatrix} \begin{pmatrix} 0 & r_{x_1} \\ \ell_{x_1} & 0 \end{pmatrix} \begin{pmatrix} g_2^{-1} & 0 \\ 0 & g_3^{-1} \end{pmatrix}.$$

This equation yields

$$r_{g_1(x_1)} g_3 = \mu(g_3) g_2 r_{x_1} \quad \text{and} \quad \ell_{g_1(x_1)} g_2 = \mu(g_2) g_3 \ell_{x_1} \quad \text{for all } x_1 \in V_{1R},$$

which means that for all $x_1 \in V_{1R}$, $x_2 \in V_{2R}$ and $x_3 \in V_{3R}$

$$g_3(x_3) *_2 g_1(x_1) = \mu(g_3) g_2(x_3 *_2 x_1) \quad \text{and} \quad g_1(x_1) *_3 g_2(x_2) = \mu(g_2) g_3(x_1 *_3 x_2).$$

These equations show that $(g_1, g_2, g_3, \mu(g_2)) \in \mathbf{GO}(\mathscr{C})(R)$, hence by (3.25)

$$\lambda_{\mathscr{C}}\big(\psi_{\mathscr{T}}(\xi)\big) = \big(1, \mu(g_3), \mu(g_2)\big).$$

Therefore, $\lambda_{\mathscr{C}}(\psi_{\mathscr{T}}(\xi)) = \lambda_{\mathscr{T}}(\psi_{\mathscr{T}}(\xi))$ by (4.10), and the proof is complete. ■

Corollary 4.11. *For every trialitarian triple $\mathscr{T}$, the algebraic group scheme $\mathbf{O}(\mathscr{T})$ is smooth.*

Proof. Over an algebraic closure F_{alg} of F the trialitarian triple $\mathscr{T}$ is split, hence by Theorem 4.2 we may find a composition $\mathscr{C}$ of quadratic spaces of dimension 8 over F_{alg} such that $\mathscr{T}_{F_{\mathrm{alg}}} \simeq \operatorname{End}(\mathscr{C})$. Then $\mathbf{O}(\mathscr{T}_{F_{\mathrm{alg}}})$ is isomorphic to $\mathbf{O}(\mathscr{C})$, which is smooth by Proposition 3.19, hence $\mathbf{O}(\mathscr{T})$ is smooth by [13, (21.10)]. ■

The final result in this subsection elucidates the structure of $\mathbf{O}(\mathscr{T})$.

Theorem 4.12. *For every trialitarian triple $\mathscr{T}$, the morphism $\psi_{\mathscr{T}}$ restricts to an isomorphism*

$$\psi_{\mathscr{T}} \colon \mathbf{Spin}(\mathfrak{A}_1) \xrightarrow{\sim} \mathbf{O}(\mathscr{T}).$$

Proof. When defining $\psi_{\mathcal{T}}$, we already observed that this morphism is injective. Recall from Section 2.3 that $\mathbf{Spin}(\mathfrak{A}_1)$ is the kernel of $\underline{\mu}\colon \mathbf{\Omega}(\mathfrak{A}_1) \to R_{Z_1/F}(\mathbf{G_m})$. Therefore, (4.9) and (4.10) show that $\psi_{\mathcal{T}}$ map $\mathbf{Spin}(\mathfrak{A}_1)$ to $\mathbf{O}(\mathcal{T})$.

To prove that $\psi_{\mathcal{T}}$ maps $\mathbf{Spin}(\mathfrak{A}_1)$ onto $\mathbf{O}(\mathcal{T})$, it suffices to consider the groups of rational points over an algebraic closure F_{alg} of F, because we know by Corollary 4.11 that $\mathbf{O}(\mathcal{T})$ is smooth. Proposition 4.6 shows that

$$\psi_{\mathcal{T}} \times i\colon \mathbf{\Omega}(\mathcal{T})(F_{\text{alg}}) \times \mathbf{H}(\mathcal{T})(F_{\text{alg}}) \to \mathbf{GO}(\mathcal{T})(F_{\text{alg}})$$

is surjective, hence for any $g \in \mathbf{O}(\mathcal{T})(F_{\text{alg}})$ we may find $\xi \in \mathbf{\Omega}(\mathcal{T})(F_{\text{alg}})$ and

$$\nu = (\nu_1, \nu_2, \nu_3) \in \mathbf{H}(\mathcal{T})(F_{\text{alg}})$$

such that $\psi_{\mathcal{T}}(\xi) \cdot \nu = g$. Taking the image of each side under $\lambda_{\mathcal{T}}$ and using (4.8) and (4.10), we obtain

$$(1, \lambda_2, \lambda_3) \cdot (\nu_2\nu_3\nu_1^{-1}, \nu_3\nu_1\nu_2^{-1}, \nu_1\nu_2\nu_3^{-1}) = (1, 1, 1)$$

for some $\lambda_2, \lambda_3 \in F_{\text{alg}}^{\times}$, hence $\nu_1 = \nu_2\nu_3$. Therefore, $\nu = \psi_{\mathcal{T}}(z)$ for $z \in (Z_1)_{F_{\text{alg}}}^{\times}$ such that $\varphi_0(z) = (\nu_2, \nu_3)$, and $\psi_{\mathcal{T}}(\xi z) = g$. Since $\lambda_{\mathcal{T}}(g) = (1, 1, 1)$, (4.9) and (4.10) show that $\underline{\mu}(\xi z) = 1$, hence $\xi z \in \mathbf{Spin}(\mathfrak{A}_1)(F_{\text{alg}})$. Thus, $\psi_{\mathcal{T}}$ maps $\mathbf{Spin}(\mathfrak{A}_1)$ onto $\mathbf{O}(\mathcal{T})$. ∎

Corollary 4.13. *The following diagram, in which the vertical maps are isomorphisms, is commutative with exact rows:*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & R_{Z_1/F}(\boldsymbol{\mu}_2) & \longrightarrow & \mathbf{Spin}(\mathfrak{A}_1) & \xrightarrow{\chi'} & \mathbf{PGO}^+(\mathfrak{A}_1) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \psi_{\mathcal{T}}} & & \downarrow{\scriptstyle \psi_{\mathcal{T}}} & & \uparrow{\scriptstyle \pi_{\mathcal{T}}} & & \\
1 & \longrightarrow & \mathbf{Z}(\mathcal{T}) & \longrightarrow & \mathbf{O}(\mathcal{T}) & \longrightarrow & \mathbf{PGO}(\mathcal{T}) & \longrightarrow & 1
\end{array} \tag{4.13}$$

Proof. The upper sequence is (2.4), and the lower sequence is from (4.11). Commutativity of the right square follows from the definition of χ' as the composition of χ_0 with the canonical map $\mathbf{GO}^+(\mathfrak{A}_1) \to \mathbf{PGO}^+(\mathfrak{A}_1)$, and bijectivity of the vertical maps is proved in Theorems 4.4 and 4.12. ∎

4.3. Derived trialitarian triples

To every trialitarian triple $\mathcal{T} = (\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \varphi_0)$ we attach in this subsection two derived trialitarian triples

$$\partial\mathcal{T} = (\mathfrak{A}_2, \mathfrak{A}_3, \mathfrak{A}_1, \varphi_0') \quad \text{and} \quad \partial^2\mathcal{T} = (\mathfrak{A}_3, \mathfrak{A}_1, \mathfrak{A}_2, \varphi_0'')$$

in such a way that for every composition $\mathscr{C}$ of quadratic spaces of dimension 8

$$\partial\,\mathrm{End}(\mathscr{C}) = \mathrm{End}(\partial\mathscr{C}) \quad \text{and} \quad \partial^2\,\mathrm{End}(\mathscr{C}) = \mathrm{End}(\partial^2\mathscr{C}).$$

The two components of the isomorphisms

$$\varphi_0'\colon \mathfrak{C}(\mathfrak{A}_2) \to \mathfrak{A}_3 \times \mathfrak{A}_1 \quad \text{and} \quad \varphi_0''\colon \mathfrak{C}(\mathfrak{A}_3) \to \mathfrak{A}_1 \times \mathfrak{A}_2$$

are determined as lifts (in the sense of Definition 2.20) of Lie algebra homomorphisms

$$\begin{aligned} \theta_+'\colon \mathfrak{pgo}(\mathfrak{A}_2) \to \mathfrak{pgo}(\mathfrak{A}_3) \quad &\text{and} \quad \theta_+''\colon \mathfrak{pgo}(\mathfrak{A}_3) \to \mathfrak{pgo}(\mathfrak{A}_1), \\ \theta_-'\colon \mathfrak{pgo}(\mathfrak{A}_2) \to \mathfrak{pgo}(\mathfrak{A}_1) \quad &\text{and} \quad \theta_-''\colon \mathfrak{pgo}(\mathfrak{A}_3) \to \mathfrak{pgo}(\mathfrak{A}_2). \end{aligned}$$

Our main result is the following:

Theorem 4.14. *Let $\mathcal{T} = (\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \varphi_0)$ be a trialitarian triple over an arbitrary field F, and let*

$$\theta_+\colon \mathfrak{pgo}(\mathfrak{A}_1) \to \mathfrak{pgo}(\mathfrak{A}_2) \quad \text{and} \quad \theta_-\colon \mathfrak{pgo}(\mathfrak{A}_1) \to \mathfrak{pgo}(\mathfrak{A}_3)$$

denote the Lie algebra homomorphisms induced (as per Definition 2.20) by the two components of φ_0:

$$\varphi_+\colon \mathfrak{C}_+(\mathfrak{A}_1) \to \mathfrak{A}_2 \quad \text{and} \quad \varphi_-\colon \mathfrak{C}_-(\mathfrak{A}_1) \to \mathfrak{A}_3.$$

The homomorphisms θ_+ and θ_- are isomorphisms, and the following Lie algebra homomorphisms are liftable:

$$\theta_+' = \theta_- \circ \theta_+^{-1}, \quad \theta_-' = \theta_+^{-1}, \quad \theta_+'' = \theta_-^{-1}, \quad \theta_-'' = \theta_+ \circ \theta_-^{-1}.$$

Moreover, θ_+' and θ_-' on one hand, and θ_+'' and θ_-'' on the other hand, are of opposite signs (see Definition 2.20).

Corollary 2.22 shows that we may extend scalars to a Galois extension of F in order to show that a Lie algebra homomorphism is liftable. We may thus reduce to split trialitarian triples, i.e., triples of the form $\operatorname{End}(\mathcal{C})$. We investigate this case first. The proof of Theorem 4.14 will quickly follow after (4.17).

Let

$$\mathcal{C} = \big((V_1, q_1), (V_2, q_2), (V_3, q_3), *_3\big)$$

be a composition of quadratic spaces of dimension 8 over an arbitrary field F. Recall from Proposition 3.21 that the Lie algebra $\mathfrak{pgo}(\mathcal{C})$ can be described as a subalgebra of $\mathfrak{pgo}(q_1) \times \mathfrak{pgo}(q_2) \times \mathfrak{pgo}(q_3)$. Let

$$\pi_1\colon \mathfrak{pgo}(\mathcal{C}) \to \mathfrak{pgo}(q_1), \quad \pi_2\colon \mathfrak{pgo}(\mathcal{C}) \to \mathfrak{pgo}(q_2), \quad \pi_3\colon \mathfrak{pgo}(\mathcal{C}) \to \mathfrak{pgo}(q_3)$$

denote the projections on the three components, and let

$$\theta_+\colon \mathfrak{pgo}(q_1) \to \mathfrak{pgo}(q_2) \quad \text{and} \quad \theta_-\colon \mathfrak{pgo}(q_1) \to \mathfrak{pgo}(q_3)$$

be the Lie algebra homomorphisms induced by the two components of $C_0(\alpha)$,

$$C_+(\alpha)\colon C_+(V_1, q_1) \to \operatorname{End}(V_2) \quad \text{and} \quad C_-(\alpha)\colon C_-(V_1, q_1) \to \operatorname{End}(V_3).$$

Lemma 4.15. *The following diagram, where all the maps are isomorphisms, is commutative:*

$$\begin{array}{ccccc} \mathfrak{pgo}(q_1) & & \xrightarrow{\theta_+} & & \mathfrak{pgo}(q_2) \\ & \nwarrow^{\pi_1} & & \nearrow^{\pi_2} & \\ & & \mathfrak{pgo}(\mathscr{C}) & & \\ & \searrow^{\theta_-} & \downarrow{\scriptstyle \pi_3} & & \\ & & \mathfrak{pgo}(q_3) & & \end{array} \tag{4.14}$$

Proof. First, observe that π_1 is the differential of the morphism $\pi_{\mathrm{End}(\mathscr{C})}$ under the identification $\mathbf{PGO}(\mathscr{C}) = \mathbf{PGO}(\mathrm{End}(\mathscr{C}))$ of Proposition 4.10. The morphism $\pi_{\mathrm{End}(\mathscr{C})}$ is an isomorphism by Theorem 4.4, hence π_1 is an isomorphism. Similarly, π_2 is the differential of the isomorphism obtained by the composition

$$\mathbf{PGO}(\mathscr{C}) \xrightarrow{\partial} \mathbf{PGO}(\partial\mathscr{C}) \xrightarrow{\pi_{\mathrm{End}(\partial\mathscr{C})}} \mathbf{PGO}^+(q_2),$$

hence π_2 is an isomorphism. Likewise, π_3 is an isomorphism.

Now, recall from (2.19) that θ_+ and θ_- are defined by the following commutative diagrams, where $\overline{C_+(\alpha)}$ and $\overline{C_-(\alpha)}$ are obtained by composing $C_+(\alpha)$ and $C_-(\alpha)$ with the canonical homomorphisms $\mathfrak{go}(q_2) \to \mathfrak{pgo}(q_2)$ or $\mathfrak{go}(q_3) \to \mathfrak{pgo}(q_3)$:

$$\begin{array}{ccc} \boldsymbol{\omega}(q_1) & \xrightarrow{\dot{\chi}'} & \mathfrak{pgo}(q_1) \\ {\scriptstyle \overline{C_+(\alpha)}}\downarrow & \swarrow_{\theta_+} & \\ \mathfrak{pgo}(q_2) & & \end{array} \qquad \begin{array}{ccc} \boldsymbol{\omega}(q_1) & \xrightarrow{\dot{\chi}'} & \mathfrak{pgo}(q_1) \\ {\scriptstyle \overline{C_-(\alpha)}}\downarrow & \swarrow_{\theta_-} & \\ \mathfrak{pgo}(q_3) & & \end{array}$$

Therefore,

$$\overline{C_+(\alpha)} = \theta_+ \circ \dot{\chi}' \quad \text{and} \quad \overline{C_-(\alpha)} = \theta_- \circ \dot{\chi}'. \tag{4.15}$$

Next, define a Lie algebra homomorphism

$$\Psi_{\mathscr{C}} \colon \boldsymbol{\omega}(q_1) \to \mathfrak{pgo}(\mathscr{C})$$

by composing the differential $\dot{\psi}_{\mathrm{End}(\mathscr{C})} \colon \boldsymbol{\omega}(q_1) \to \mathfrak{go}(\mathscr{C})$ of the morphism $\psi_{\mathrm{End}(\mathscr{C})}$ of Section 4.2 with the canonical map $\mathfrak{go}(\mathscr{C}) \to \mathfrak{pgo}(\mathscr{C})$. Explicitly,

$$\Psi_{\mathscr{C}}(\xi) = \big(\dot{\chi}_0(\xi) + F,\ C_+(\alpha)(\xi) + F,\ C_-(\alpha)(\xi) + F\big) \quad \text{for } \xi \in \boldsymbol{\omega}(q_1),$$

or, since Proposition 2.15 shows that $\dot{\chi}'(\xi) = \dot{\chi}_0(\xi) + F$,

$$\Psi_{\mathscr{C}}(\xi) = \big(\dot{\chi}'(\xi),\ \overline{C_+(\alpha)}(\xi),\ \overline{C_-(\alpha)}(\xi)\big) \quad \text{for } \xi \in \boldsymbol{\omega}(q_1).$$

It follows from the definitions that the following diagrams are commutative:

$$
\begin{array}{ccc}
\boldsymbol{\omega}(q_1) & \xrightarrow{\dot{\chi}'} & \mathfrak{pgo}(q_1) \\
{\scriptstyle \overline{C_+(\alpha)}}\big\downarrow & \searrow^{\Psi_{\mathscr{C}}} & \big\uparrow{\scriptstyle \pi_1} \\
\mathfrak{pgo}(q_2) & \xleftarrow[\pi_2]{} & \mathfrak{pgo}(\mathscr{C})
\end{array}
\qquad\qquad
\begin{array}{ccc}
\boldsymbol{\omega}(q_1) & \xrightarrow{\dot{\chi}'} & \mathfrak{pgo}(q_1) \\
{\scriptstyle \overline{C_-(\alpha)}}\big\downarrow & \searrow^{\Psi_{\mathscr{C}}} & \big\uparrow{\scriptstyle \pi_1} \\
\mathfrak{pgo}(q_3) & \xleftarrow[\pi_3]{} & \mathfrak{pgo}(\mathscr{C})
\end{array}
$$

Therefore,

$$\overline{C_+(\alpha)} = \pi_2 \circ \Psi_{\mathscr{C}}, \quad \dot{\chi}' = \pi_1 \circ \Psi_{\mathscr{C}}, \quad \overline{C_-(\alpha)} = \pi_3 \circ \Psi_{\mathscr{C}}.$$

Substituting in (4.15) yields

$$\pi_2 \circ \Psi_{\mathscr{C}} = \theta_+ \circ \pi_1 \circ \Psi_{\mathscr{C}} \quad \text{and} \quad \theta_- \circ \pi_1 \circ \Psi_{\mathscr{C}} = \pi_3 \circ \Psi_{\mathscr{C}}.$$

We know from Proposition 2.15 that $\dot{\chi}'$ is surjective, hence $\Psi_{\mathscr{C}}$ also is surjective since π_1 is an isomorphism. Therefore, the last displayed equations yield $\pi_2 = \theta_+ \circ \pi_1$ and $\pi_3 = \theta_- \circ \pi_1$, proving the commutativity of diagram (4.14). Bijectivity of θ_+ and θ_- follows, since $\theta_+ = \pi_2 \circ \pi_1^{-1}$ and $\theta_- = \pi_3 \circ \pi_1^{-1}$. ■

We next apply Lemma 4.15 to the derived compositions $\partial\mathscr{C}$ and $\partial^2\mathscr{C}$. Recall the trialitarian triples obtained from (3.20) and (3.21):

$$\operatorname{End}(\partial\mathscr{C}) = \big(\big(\operatorname{End}(V_2), \sigma_{b_2}, \mathfrak{f}_{q_2}\big), \big(\operatorname{End}(V_3), \sigma_{b_3}, \mathfrak{f}_{q_3}\big), \big(\operatorname{End}(V_1), \sigma_{b_1}, \mathfrak{f}_{q_1}\big), C_0(\alpha')\big)$$

and

$$\operatorname{End}(\partial^2\mathscr{C}) = \big(\big(\operatorname{End}(V_3), \sigma_{b_3}, \mathfrak{f}_{q_3}\big), \big(\operatorname{End}(V_1), \sigma_{b_1}, \mathfrak{f}_{q_1}\big), \big(\operatorname{End}(V_2), \sigma_{b_2}, \mathfrak{f}_{q_2}\big), C_0(\alpha'')\big).$$

Let

$$\theta'_+ \colon \mathfrak{pgo}(q_2) \to \mathfrak{pgo}(q_3) \quad \text{and} \quad \theta'_- \colon \mathfrak{pgo}(q_2) \to \mathfrak{pgo}(q_1)$$

be the Lie algebra isomorphisms induced by $C_+(\alpha')$ and $C_-(\alpha')$ respectively, and

$$\theta''_+ \colon \mathfrak{pgo}(q_3) \to \mathfrak{pgo}(q_1) \quad \text{and} \quad \theta''_- \colon \mathfrak{pgo}(q_3) \to \mathfrak{pgo}(q_2)$$

those induced by $C_+(\alpha'')$ and $C_-(\alpha'')$. Let also

$$\pi'_1 \colon \mathfrak{pgo}(\partial\mathscr{C}) \to \mathfrak{pgo}(q_2), \quad \pi'_2 \colon \mathfrak{pgo}(\partial\mathscr{C}) \to \mathfrak{pgo}(q_3), \quad \pi'_3 \colon \mathfrak{pgo}(\partial\mathscr{C}) \to \mathfrak{pgo}(q_1)$$

and

$$\pi''_1 \colon \mathfrak{pgo}(\partial^2\mathscr{C}) \to \mathfrak{pgo}(q_3), \quad \pi''_2 \colon \mathfrak{pgo}(\partial^2\mathscr{C}) \to \mathfrak{pgo}(q_1), \quad \pi''_3 \colon \mathfrak{pgo}(\partial^2\mathscr{C}) \to \mathfrak{pgo}(q_2)$$

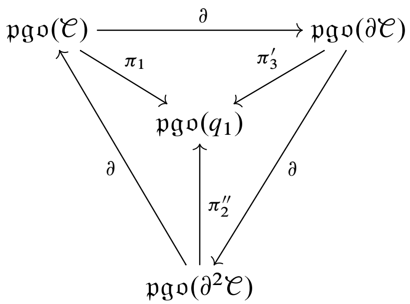
be the projections on the various components of $\mathfrak{pgo}(\partial\mathscr{C})$ and $\mathfrak{pgo}(\partial^2\mathscr{C})$. Lemma 4.15 yields

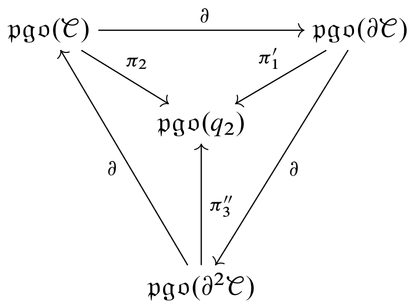
$$\theta'_+ = \pi'_2 \circ \pi'^{-1}_1, \quad \theta'_- = \pi'_3 \circ \pi'^{-1}_1, \quad \theta''_+ = \pi''_2 \circ \pi''^{-1}_1, \quad \theta''_- = \pi''_3 \circ \pi''^{-1}_1. \tag{4.16}$$

Proposition 4.16. *The following equations hold:*

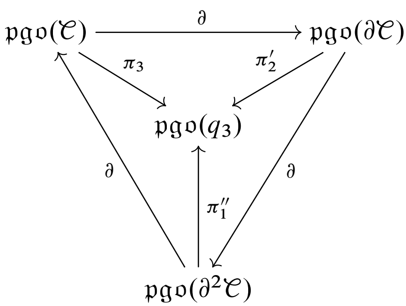
$$\theta'_+ = \theta_- \circ \theta_+^{-1}, \quad \theta'_- = \theta_+^{-1}, \quad \theta''_+ = \theta_-^{-1}, \quad \theta''_- = \theta_+ \circ \theta_-^{-1}.$$

Proof. The switch maps ∂ fit in the following commutative diagrams:





and

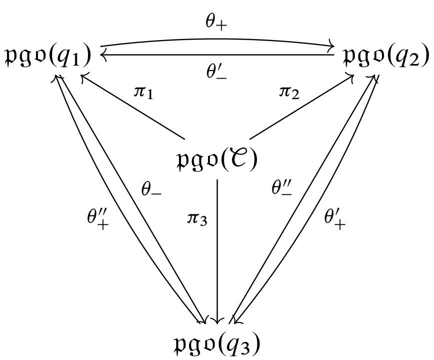


Substituting $\pi'_1 = \pi_2 \circ \partial^2$, $\pi'_2 = \pi_3 \circ \partial^2$, $\pi'_3 = \pi_1 \circ \partial^2$ and $\pi''_1 = \pi_3 \circ \partial$, $\pi''_2 = \pi_1 \circ \partial$, $\pi''_3 = \pi_2 \circ \partial$ in (4.16) yields

$$\theta'_+ = \pi_3 \circ \pi_2^{-1}, \quad \theta'_- = \pi_1 \circ \pi_2^{-1}, \quad \theta''_+ = \pi_1 \circ \pi_3^{-1}, \quad \theta''_- = \pi_2 \circ \pi_3^{-1}.$$

The proposition follows by Lemma 4.15. ∎

The maps $\theta_\pm$, $\theta'_\pm$, $\theta''_\pm$ thus fit in the following commutative diagram, in which all the maps are isomorphisms:



(4.17)

Proof of Theorem 4.14. Corollary 2.22 shows that it suffices to prove the claim after a Galois scalar extension that splits the trialitarian triple $\mathcal{T}$. We may thus assume that $\mathcal{T} = \operatorname{End}(\mathcal{C})$ for some composition $\mathcal{C}$ of quadratic spaces of dimension 8. Then Proposition 4.16 shows that $\theta_- \circ \theta_+^{-1}$ and θ_+^{-1} (resp. θ_-^{-1} and $\theta_+ \circ \theta_-^{-1}$) are the Lie algebra homomorphisms induced by the isomorphisms $C_+(\alpha')$ and $C_-(\alpha')$ (resp. $C_+(\alpha'')$ and $C_-(\alpha'')$) of the trialitarian triple $\operatorname{End}(\partial\mathcal{C})$ (resp. $\operatorname{End}(\partial^2\mathcal{C})$), hence they are liftable by definition. Moreover, $\theta_- \circ \theta_+^{-1}$ and θ_+^{-1} (resp. θ_-^{-1}, $\theta_+ \circ \theta_-^{-1}$) are of opposite signs, hence the proof is complete. ■

Definition 4.17. Given any trialitarian triple $\mathcal{T} = (\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \varphi_0)$ with Lie algebra isomorphisms

$$\theta_+ \colon \mathfrak{pgo}(\mathfrak{A}_1) \to \mathfrak{pgo}(\mathfrak{A}_2) \quad \text{and} \quad \theta_- \colon \mathfrak{pgo}(\mathfrak{A}_1) \to \mathfrak{pgo}(\mathfrak{A}_3)$$

induced by the components $\varphi_+ \colon \mathfrak{C}(\mathfrak{A}_1) \to \mathfrak{A}_2$ and $\varphi_- \colon \mathfrak{C}(\mathfrak{A}_1) \to \mathfrak{A}_3$ of φ_0, the pair of opposite Lie algebra isomorphisms

$$(\theta'_+, \theta'_-) = (\theta_- \circ \theta_+^{-1}, \theta_+^{-1}) \quad \big(\text{resp. } (\theta''_+, \theta''_-) = (\theta_-^{-1}, \theta_+ \circ \theta_-^{-1})\big)$$

lifts by Theorem 4.14 to an isomorphism

$$\varphi'_0 \colon \mathfrak{C}(\mathfrak{A}_2) \to \mathfrak{A}_3 \times \mathfrak{A}_1 \quad \big(\text{resp. } \varphi''_0 \colon \mathfrak{C}(\mathfrak{A}_3) \to \mathfrak{A}_1 \times \mathfrak{A}_2\big)$$

that defines a trialitarian triple

$$\partial\mathcal{T} = (\mathfrak{A}_2, \mathfrak{A}_3, \mathfrak{A}_1, \varphi'_0) \quad \big(\text{resp. } \partial^2\mathcal{T} = (\mathfrak{A}_3, \mathfrak{A}_1, \mathfrak{A}_2, \varphi''_0)\big).$$

The trialitarian triples $\partial\mathcal{T}$ and $\partial^2\mathcal{T}$ are called the *derived trialitarian triples* of $\mathcal{T}$.

Note that $\theta'_- \circ {\theta'_+}^{-1} = \theta''_+$ and ${\theta'_+}^{-1} = \theta''_-$, hence $\partial(\partial\mathcal{T}) = \partial^2\mathcal{T}$. Similarly, $\partial^2(\partial\mathcal{T}) = \mathcal{T} = \partial(\partial^2\mathcal{T})$ and $\partial^2(\partial^2\mathcal{T}) = \partial\mathcal{T}$.

From the proof of Theorem 4.14, it is clear that for every composition $\mathcal{C}$ of quadratic spaces of dimension 8,

$$\partial \operatorname{End}(\mathcal{C}) = \operatorname{End}(\partial\mathcal{C}) \quad \text{and} \quad \partial^2 \operatorname{End}(\mathcal{C}) = \operatorname{End}(\partial^2\mathcal{C}).$$

We next establish the functoriality of the ∂ operation.

Proposition 4.18. *If $\gamma = (\gamma_1, \gamma_2, \gamma_3) \colon \mathcal{T} \to \widetilde{\mathcal{T}}$ is an isomorphism of trialitarian triples, then $\partial\gamma := (\gamma_2, \gamma_3, \gamma_1)$ is an isomorphism of trialitarian triples $\partial\mathcal{T} \to \partial\widetilde{\mathcal{T}}$.*

Proof. Let (θ_+, θ_-) (resp. $(\widetilde{\theta}_+, \widetilde{\theta}_-)$) be the pair of liftable homomorphisms attached to $\mathcal{T}$ (resp. $\widetilde{\mathcal{T}}$). The hypothesis that γ is an isomorphism means that

$$\gamma_2 \circ \theta_+ = \widetilde{\theta}_+ \circ \gamma_1 \quad \text{and} \quad \gamma_3 \circ \theta_- = \widetilde{\theta}_- \circ \gamma_1.$$

It then follows that

$$\gamma_3 \circ (\theta_- \circ \theta_+^{-1}) = \widetilde{\theta}_- \circ \gamma_1 \circ \theta_+^{-1} = (\widetilde{\theta}_- \circ \widetilde{\theta}_+^{-1}) \circ \gamma_2 \quad \text{and} \quad \gamma_1 \circ \theta_+^{-1} = \widetilde{\theta}_+^{-1} \circ \gamma_2.$$

Since $(\theta_- \circ \theta_+^{-1}, \theta_+^{-1})$ and $(\tilde{\theta}_- \circ \tilde{\theta}_+^{-1}, \tilde{\theta}_+^{-1})$ are the pairs of liftable homomorphisms attached to $\partial\mathcal{T}$ and $\partial\tilde{\mathcal{T}}$ respectively, it follows that $(\gamma_2, \gamma_3, \gamma_1)$ is an isomorphism $\partial\mathcal{T} \to \partial\tilde{\mathcal{T}}$. ∎

For the next corollary, observe that each trialitarian triple $\mathcal{T} = (\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \varphi_0)$ yields a polarization of $\mathfrak{A}_1$ in the sense of Definition 2.4: the primitive idempotents in the center of $C(\mathfrak{A}_1)$ are designated as z_{1+} and z_{1-} according to the following convention:

$$\varphi_0(z_{1+}) = (1, 0) \quad \text{and} \quad \varphi_0(z_{1-}) = (0, 1),$$

so that the two components of φ_0 are $\varphi_+ \colon \mathfrak{C}_+(\mathfrak{A}_1) \xrightarrow{\sim} \mathfrak{A}_2$ and $\varphi_- \colon \mathfrak{C}_-(\mathfrak{A}_1) \xrightarrow{\sim} \mathfrak{A}_3$. Similarly, the maps φ_0' and φ_0'' of the derived trialitarian triples $\partial\mathcal{T}$ and $\partial^2\mathcal{T}$ yield polarizations of $\mathfrak{A}_2$ and $\mathfrak{A}_3$ so that

$$\varphi_0'(z_{2+}) = (1, 0), \quad \varphi_0'(z_{2-}) = (0, 1), \quad \varphi_0''(z_{3+}) = (1, 0), \quad \varphi_0''(z_{3-}) = (0, 1),$$

just as in the case of compositions of quadratic spaces: see Remark 3.11.

Corollary 4.19. *Let $\mathcal{T} = (\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \varphi_0)$ and $\tilde{\mathcal{T}} = (\tilde{\mathfrak{A}}_1, \tilde{\mathfrak{A}}_2, \tilde{\mathfrak{A}}_3, \tilde{\varphi}_0)$ be trialitarian triples. There are canonical one-to-one correspondences between the following sets:*

(i) *isomorphisms of trialitarian triples $\mathcal{T} \to \tilde{\mathcal{T}}$;*

(ii) *isomorphisms of algebras with quadratic pair $\mathfrak{A}_1 \to \tilde{\mathfrak{A}}_1$ preserving the polarizations induced by $\mathcal{T}$ and $\tilde{\mathcal{T}}$;*

(iii) *isomorphisms of algebras with quadratic pair $\mathfrak{A}_2 \to \tilde{\mathfrak{A}}_2$ preserving the polarizations induced by $\partial\mathcal{T}$ and $\partial\tilde{\mathcal{T}}$;*

(iv) *isomorphisms of algebras with quadratic pair $\mathfrak{A}_3 \to \tilde{\mathfrak{A}}_3$ preserving the polarizations induced by $\partial^2\mathcal{T}$ and $\partial^2\tilde{\mathcal{T}}$.*

Proof. By definition, an isomorphism $\gamma \colon \mathcal{T} \to \tilde{\mathcal{T}}$ is a triple $(\gamma_1, \gamma_2, \gamma_3)$ where each γ_i is an isomorphism $\mathfrak{A}_i \to \tilde{\mathfrak{A}}_i$ and the following square commutes:

$$\begin{array}{ccc} \mathfrak{C}(\mathfrak{A}_1) & \xrightarrow{\varphi_0} & \mathfrak{A}_2 \times \mathfrak{A}_3 \\ {\scriptstyle C(\gamma_1)}\downarrow & & \downarrow{\scriptstyle \gamma_2\times\gamma_3} \\ \mathfrak{C}(\tilde{\mathfrak{A}}_1) & \xrightarrow{\tilde{\varphi}_0} & \tilde{\mathfrak{A}}_2 \times \tilde{\mathfrak{A}}_3 \end{array} \tag{4.18}$$

Commutativity of this square implies that γ_1 preserves the polarizations of $\mathfrak{A}_1$ and $\tilde{\mathfrak{A}}_1$ induced by $\mathcal{T}$ and $\tilde{\mathcal{T}}$.

Conversely, if $\gamma_1 \colon \mathfrak{A}_1 \to \tilde{\mathfrak{A}}_1$ is an isomorphism of algebras with quadratic pair preserving polarizations, then there are isomorphisms $\gamma_2 \colon \mathfrak{A}_2 \to \tilde{\mathfrak{A}}_2$ and $\gamma_3 \colon \mathfrak{A}_3 \to \tilde{\mathfrak{A}}_3$ uniquely determined by the condition that the square (4.18) commute. The triple $(\gamma_1, \gamma_2, \gamma_3)$ is then an isomorphism $\mathcal{T} \to \tilde{\mathcal{T}}$. Thus, the sets described in (i) and (ii) are in bijection under the map carrying $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ to γ_1. Similarly, the set in (iii) is in bijection with the set of isomorphisms $\partial\mathcal{T} \to \partial\widetilde{\mathcal{T}}$, hence, by Proposition 4.18, with the set of isomorphisms $\mathcal{T} \to \widetilde{\mathcal{T}}$: to each isomorphism $\gamma\colon \mathcal{T} \to \widetilde{\mathcal{T}}$ corresponds the second component $\gamma_2\colon \mathfrak{A}_2 \to \widetilde{\mathfrak{A}}_2$. Likewise, mapping γ to γ_3 defines a one-to-one correspondence between (i) and (iv). ■

In the particular case where $\widetilde{\mathcal{T}} = \mathcal{T}$, Corollary 4.19 yields isomorphisms between the group of automorphisms of $\mathcal{T}$ and the groups of polarization-preserving automorphisms of $\mathfrak{A}_1$, $\mathfrak{A}_2$ and $\mathfrak{A}_3$, which are $\mathrm{PGO}^+(\mathfrak{A}_1)$, $\mathrm{PGO}^+(\mathfrak{A}_2)$ and $\mathrm{PGO}^+(\mathfrak{A}_3)$. We discuss this case in detail in the next subsection.

4.4. Trialitarian isomorphisms

Throughout this subsection, we fix a trialitarian triple

$$\mathcal{T} = (\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \varphi_0).$$

We show how to attach to $\mathcal{T}$ canonical isomorphisms, which we call *trialitarian isomorphisms*:

$$\mathbf{Spin}(\mathfrak{A}_1) \simeq \mathbf{Spin}(\mathfrak{A}_2) \simeq \mathbf{Spin}(\mathfrak{A}_3) \quad \text{and} \quad \mathbf{PGO}^+(\mathfrak{A}_1) \simeq \mathbf{PGO}^+(\mathfrak{A}_2) \simeq \mathbf{PGO}^+(\mathfrak{A}_3).$$

Proposition 4.18 shows that the switch map ∂ yields an isomorphism

$$\partial\colon \mathbf{GO}(\mathcal{T}) \to \mathbf{GO}(\partial\mathcal{T}).$$

This isomorphism maps $\mathbf{H}(\mathcal{T})$ to $\mathbf{H}(\partial\mathcal{T})$, hence it induces a switch isomorphism

$$\partial\colon \mathbf{PGO}(\mathcal{T}) \to \mathbf{PGO}(\partial\mathcal{T}).$$

The following proposition shows that ∂ also maps $\mathbf{O}(\mathcal{T})$ to $\mathbf{O}(\partial\mathcal{T})$:

Proposition 4.20. *The following diagram is commutative:*

$$\begin{array}{ccc} \mathbf{GO}(\mathcal{T}) & \xrightarrow{\partial} & \mathbf{GO}(\partial\mathcal{T}) \\ \downarrow{\scriptstyle \lambda_{\mathcal{T}}} & & \downarrow{\scriptstyle \lambda_{\partial\mathcal{T}}} \\ \mathbf{G}_{\mathbf{m}}^3 & \xrightarrow{\partial} & \mathbf{G}_{\mathbf{m}}^3 \end{array}$$

Proof. When $\mathcal{T} = \mathrm{End}(\mathscr{C})$ for some composition $\mathscr{C}$ of quadratic spaces of dimension 8, then $\mathbf{GO}(\mathcal{T}) = \mathbf{GO}(\mathscr{C})$ and $\mathbf{GO}(\partial\mathcal{T}) = \mathbf{GO}(\partial\mathscr{C})$ by Proposition 4.10, and commutativity of the diagram is clear from (4.12). Commutativity for an arbitrary trialitarian triple follows by scalar extension to a splitting field. ■

Recall from Corollary 4.13 the diagram (4.13) relating $\mathbf{Spin}(\mathfrak{A}_1)$ and $\mathbf{PGO}^+(\mathfrak{A}_1)$ to $\mathbf{O}(\mathcal{T})$ and $\mathbf{PGO}(\mathcal{T})$. Substituting $\partial\mathcal{T}$ for $\mathcal{T}$ in that diagram, we obtain another commutative diagram, which involves $\mathfrak{A}_2$ and $\partial\mathcal{T}$ instead of $\mathfrak{A}_1$ and $\mathcal{T}$. We may connect this new diagram to (4.13) by means of the shift map to obtain the following commutative diagram with exact rows, where all the vertical maps are isomorphisms and Z_2 denotes the center of $C(\mathfrak{A}_2)$:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & R_{Z_1/F}(\boldsymbol{\mu}_2) & \longrightarrow & \mathbf{Spin}(\mathfrak{A}_1) & \xrightarrow{\chi'} & \mathbf{PGO}^+(\mathfrak{A}_1) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \psi_{\mathcal{T}}} & & \downarrow{\scriptstyle \psi_{\mathcal{T}}} & & \uparrow{\scriptstyle \pi_{\mathcal{T}}} & & \\
1 & \longrightarrow & \mathbf{Z}(\mathcal{T}) & \longrightarrow & \mathbf{O}(\mathcal{T}) & \longrightarrow & \mathbf{PGO}(\mathcal{T}) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \\
1 & \longrightarrow & \mathbf{Z}(\partial\mathcal{T}) & \longrightarrow & \mathbf{O}(\partial\mathcal{T}) & \longrightarrow & \mathbf{PGO}(\partial\mathcal{T}) & \longrightarrow & 1 \\
 & & \uparrow{\scriptstyle \psi_{\partial\mathcal{T}}} & & \uparrow{\scriptstyle \psi_{\partial\mathcal{T}}} & & \downarrow{\scriptstyle \pi_{\partial\mathcal{T}}} & & \\
1 & \longrightarrow & R_{Z_2/F}(\boldsymbol{\mu}_2) & \longrightarrow & \mathbf{Spin}(\mathfrak{A}_2) & \xrightarrow{\chi'} & \mathbf{PGO}^+(\mathfrak{A}_2) & \longrightarrow & 1
\end{array}$$

Define

$$\Sigma_{\mathcal{T}} \colon \mathbf{Spin}(\mathfrak{A}_1) \xrightarrow{\sim} \mathbf{Spin}(\mathfrak{A}_2) \quad \text{and} \quad \Theta_{\mathcal{T}} \colon \mathbf{PGO}^+(\mathfrak{A}_1) \xrightarrow{\sim} \mathbf{PGO}^+(\mathfrak{A}_2)$$

by composing the vertical isomorphisms: $\Sigma_{\mathcal{T}} = \psi_{\partial\mathcal{T}}^{-1} \circ \partial \circ \psi_{\mathcal{T}}$ and $\Theta_{\mathcal{T}} = \pi_{\partial\mathcal{T}} \circ \partial \circ \pi_{\mathcal{T}}^{-1}$. Forgetting the two central lines of the last diagram, we obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & R_{Z_1/F}(\boldsymbol{\mu}_2) & \longrightarrow & \mathbf{Spin}(\mathfrak{A}_1) & \xrightarrow{\chi'} & \mathbf{PGO}^+(\mathfrak{A}_1) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \Sigma_{\mathcal{T}}} & & \downarrow{\scriptstyle \Sigma_{\mathcal{T}}} & & \downarrow{\scriptstyle \Theta_{\mathcal{T}}} & & \\
1 & \longrightarrow & R_{Z_2/F}(\boldsymbol{\mu}_2) & \longrightarrow & \mathbf{Spin}(\mathfrak{A}_2) & \xrightarrow{\chi'} & \mathbf{PGO}^+(\mathfrak{A}_2) & \longrightarrow & 1
\end{array}$$

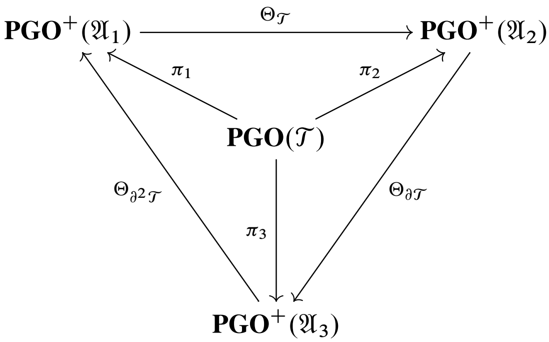
Applying the construction above to $\partial\mathcal{T}$ and $\partial^2\mathcal{T}$ instead of $\mathcal{T}$, we obtain isomorphisms $\Sigma_{\partial\mathcal{T}} \colon \mathbf{Spin}(\mathfrak{A}_2) \xrightarrow{\sim} \mathbf{Spin}(\mathfrak{A}_3)$ and $\Sigma_{\partial^2\mathcal{T}} \colon \mathbf{Spin}(\mathfrak{A}_3) \xrightarrow{\sim} \mathbf{Spin}(\mathfrak{A}_1)$ such that $\Sigma_{\partial^2\mathcal{T}} \circ \Sigma_{\partial\mathcal{T}} \circ \Sigma_{\mathcal{T}} = \mathrm{Id}$, which make the following diagram with exact rows commute:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & R_{Z_1/F}(\boldsymbol{\mu}_2) & \longrightarrow & \mathbf{Spin}(\mathfrak{A}_1) & \xrightarrow{\chi'} & \mathbf{PGO}^+(\mathfrak{A}_1) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \Sigma_{\mathcal{T}}} & & \downarrow{\scriptstyle \Sigma_{\mathcal{T}}} & & \downarrow{\scriptstyle \Theta_{\mathcal{T}}} & & \\
1 & \longrightarrow & R_{Z_2/F}(\boldsymbol{\mu}_2) & \longrightarrow & \mathbf{Spin}(\mathfrak{A}_2) & \xrightarrow{\chi'} & \mathbf{PGO}^+(\mathfrak{A}_2) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \Sigma_{\partial\mathcal{T}}} & & \downarrow{\scriptstyle \Sigma_{\partial\mathcal{T}}} & & \downarrow{\scriptstyle \Theta_{\partial\mathcal{T}}} & & \\
1 & \longrightarrow & R_{Z_3/F}(\boldsymbol{\mu}_2) & \longrightarrow & \mathbf{Spin}(\mathfrak{A}_3) & \xrightarrow{\chi'} & \mathbf{PGO}^+(\mathfrak{A}_3) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \Sigma_{\partial^2\mathcal{T}}} & & \downarrow{\scriptstyle \Sigma_{\partial^2\mathcal{T}}} & & \downarrow{\scriptstyle \Theta_{\partial^2\mathcal{C}}} & & \\
1 & \longrightarrow & R_{Z_1/F}(\boldsymbol{\mu}_2) & \longrightarrow & \mathbf{Spin}(\mathfrak{A}_1) & \xrightarrow{\chi'} & \mathbf{PGO}^+(\mathfrak{A}_1) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Letting $\pi_i : \mathbf{PGO}(\mathcal{T}) \to \mathbf{PGO}^+(\mathfrak{A}_i)$ denote the projection on the i-th component, we have

$$\pi_1 = \pi_{\mathcal{T}}, \quad \pi_2 = \pi_{\partial\mathcal{T}} \circ \partial, \quad \pi_3 = \pi_{\partial^2\mathcal{T}} \circ \partial^2,$$

hence the following diagram, in which all the maps are isomorphisms, is commutative:

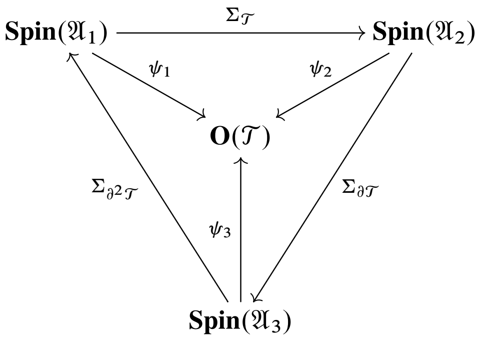


(4.19)

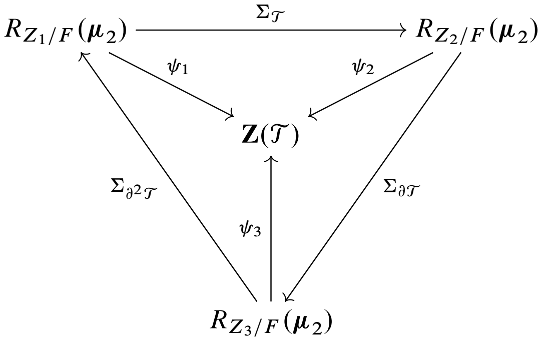
Similarly, defining $\psi_i : \mathbf{Spin}(\mathfrak{A}_i) \to \mathbf{O}(\mathcal{T})$ for $i = 1, 2, 3$ by

$$\psi_1 = \psi_{\mathcal{T}}, \quad \psi_2 = \partial^{-1} \circ \psi_{\partial\mathcal{T}}, \quad \psi_3 = \partial^{-2} \circ \psi_{\partial^2\mathcal{T}},$$

we obtain the following commutative diagram similar to (4.19), where all the maps are isomorphisms:



Restricting to the central subgroups, we also obtain a commutative diagram of isomorphisms:



The action of the trialitarian isomorphism $\Sigma_{\mathcal{T}}$ on $R_{Z_1/F}(\boldsymbol{\mu}_2)$ is easy to determine from the definition of $\psi_{\mathcal{T}}$:

Proposition 4.21. *For $i = 1, 2, 3$, let z_{i+} and z_{i-} denote the primitive idempotents of Z_i (according to the polarization). Then for every commutative F-algebra R and $a_+, a_- \in R$ such that $a_+^2 = a_-^2 = 1$,*

$$\Sigma_{\mathcal{T}}(a_+z_{1+} + a_-z_{1-}) = a_-z_{2+} + a_+a_-z_{2-}, \tag{4.20}$$

$$\Sigma_{\partial\mathcal{T}}(a_+z_{2+} + a_-z_{2-}) = a_-z_{3+} + a_+a_-z_{3-}, \tag{4.21}$$

$$\Sigma_{\partial^2\mathcal{T}}(a_+z_{3+} + a_-z_{3-}) = a_-z_{1+} + a_+a_-z_{1-}. \tag{4.22}$$

Proof. From (4.7) it follows that $\psi_{\mathcal{T}}(a_+z_{1+} + a_-z_{1-}) = (a_+a_-, a_+, a_-)$, hence

$$\partial \circ \psi_{\mathcal{T}}(a_+z_{1+} + a_-z_{1-}) = (a_+, a_-, a_+a_-) = \psi_{\partial\mathcal{T}}(a_-z_{2+} + a_+a_-z_{2-}).$$

Equation (4.20) follows, since $\Sigma_{\mathcal{T}} = \psi_{\partial\mathcal{T}}^{-1} \circ \partial \circ \psi_{\mathcal{T}}$. Equations (4.21) and (4.22) are proved similarly. ∎

Proposition 4.21 shows that $\Sigma_{\mathcal{T}}$ does *not* map the subgroup $\boldsymbol{\mu}_2$ of $R_{Z_1/F}(\boldsymbol{\mu}_2)$ to the subgroup $\boldsymbol{\mu}_2$ of $R_{Z_2/F}(\boldsymbol{\mu}_2)$; this is a characteristic feature of trialitarian isomorphisms.

4.5. Compositions of 8-dimensional quadratic spaces

Let $\mathcal{C} = ((V_1, q_1), (V_2, q_2), (V_3, q_3), *_3)$ and $\widetilde{\mathcal{C}} = ((\widetilde{V}_1, \tilde{q}_1), (\widetilde{V}_2, \tilde{q}_2), (\widetilde{V}_3, \tilde{q}_3), \tilde{*}_3)$ denote compositions of quadratic spaces of dimension 8 over F throughout this subsection. Recall from Remark 3.11 that $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ induce polarizations of (V_1, q_1) and $(\widetilde{V}_1, \tilde{q}_1)$ respectively. Our goal is to establish criteria for the existence of a similitude or an isomorphism between $\mathcal{C}$ and $\widetilde{\mathcal{C}}$.

Theorem 4.22. *For every similitude $g_1 \colon (V_1, q_1) \to (\widetilde{V}_1, \tilde{q}_1)$ preserving the polarizations induced by $\mathcal{C}$ and $\widetilde{\mathcal{C}}$, there exist similitudes $g_2 \colon (V_2, q_2) \to (\widetilde{V}_2, \tilde{q}_2)$ and $g_3 \colon (V_3, q_3) \to (\widetilde{V}_3, \tilde{q}_3)$ such that the triple (g_1, g_2, g_3) is a similitude $\mathcal{C} \to \widetilde{\mathcal{C}}$. The similitudes g_2 and g_3 are uniquely determined up to a scalar factor.*

Proof. The similitude g_1 defines an isomorphism of algebras with quadratic pair

$$\operatorname{Int}(g_1) \colon (\operatorname{End} V_1, \sigma_{b_1}, \mathfrak{f}_{q_1}) \to (\operatorname{End} \widetilde{V}_1, \sigma_{\tilde{b}_1}, \mathfrak{f}_{\tilde{q}_1}),$$

see Proposition 2.2. Since g_1 preserves the polarizations of (V_1, q_1) and $(\widetilde{V}_1, \tilde{q}_1)$, it follows that $\operatorname{Int}(g_1)$ preserves the polarizations of $(\operatorname{End} V_1, \sigma_{b_1}, \mathfrak{f}_{q_1})$ and $(\operatorname{End} \widetilde{V}_1, \sigma_{\tilde{b}_1}, \mathfrak{f}_{\tilde{q}_1})$ induced by the trialitarian triples $\operatorname{End}(\mathcal{C})$ and $\operatorname{End}(\widetilde{\mathcal{C}})$ respectively, hence Corollary 4.19 yields uniquely determined isomorphisms

$$\gamma_2 \colon (\operatorname{End} V_2, \sigma_{b_2}, \mathfrak{f}_{q_2}) \to (\operatorname{End} \widetilde{V}_2, \sigma_{\tilde{b}_2}, \mathfrak{f}_{\tilde{q}_2})$$

and

$$\gamma_3 \colon (\operatorname{End} V_3, \sigma_{b_3}, \mathfrak{f}_{q_3}) \to (\operatorname{End} \widetilde{V}_3, \sigma_{\tilde{b}_3}, \mathfrak{f}_{\tilde{q}_3})$$

such that $(\mathrm{Int}(g_1), \gamma_2, \gamma_3)$ is an isomorphism $\mathrm{End}(\mathcal{C}) \to \mathrm{End}(\widetilde{\mathcal{C}})$. Proposition 2.2 shows that there exist similitudes $g_2\colon (V_2, q_2) \to (\widetilde{V}_2, \tilde{q}_2)$ and $g_3\colon (V_3, q_3) \to (\widetilde{V}_3, \tilde{q}_3)$, uniquely determined up to a scalar factor, such that $\gamma_2 = \mathrm{Int}(g_2)$ and $\gamma_3 = \mathrm{Int}(g_3)$. It follows from Proposition 4.1 that (g_1, g_2, g_3) is a similitude $\mathcal{C} \to \widetilde{\mathcal{C}}$. ∎

Corollary 4.23. *Let $n_{\mathcal{C}}$ and $n_{\widetilde{\mathcal{C}}}$ denote the 3-fold Pfister forms associated to $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ by Proposition 3.10. The following conditions are equivalent:*

(i) *$\mathcal{C}$ is similar to $\widetilde{\mathcal{C}}$;*

(ii) $n_{\mathcal{C}} \simeq n_{\widetilde{\mathcal{C}}}$.

Proof. Recall that $q_1 \simeq \langle \lambda_1 \rangle n_{\mathcal{C}}$ and $\tilde{q}_1 \simeq \langle \tilde{\lambda}_1 \rangle n_{\widetilde{\mathcal{C}}}$ for some $\lambda_1, \tilde{\lambda}_1 \in F^\times$. If $\mathcal{C}$ is similar to $\widetilde{\mathcal{C}}$, then q_1 is similar to $\tilde{q}_1$, hence $n_{\mathcal{C}} \simeq n_{\widetilde{\mathcal{C}}}$ because similar Pfister forms are isometric. Conversely, if $n_{\mathcal{C}} \simeq n_{\widetilde{\mathcal{C}}}$, then there is a similitude $g_1\colon (V_1, q_1) \to (\widetilde{V}_1, \tilde{q}_1)$. Composing g_1 with an improper isometry if necessary, we may assume g_1 preserves the polarizations of (V_1, q_1) and $(\widetilde{V}_1, \tilde{q}_1)$. Then Theorem 4.22 yields a similitude $\mathcal{C} \to \widetilde{\mathcal{C}}$. ∎

In the particular case where $\widetilde{\mathcal{C}} = \mathcal{C}$, Theorem 4.22 is a direct generalization of the principle of triality discussed by Springer–Veldkamp [19, Th. 3.2.1], as follows:

Corollary 4.24. *For every proper similitude $g_1 \in \mathrm{GO}^+(q_1)$, there exist similitudes g_2 in $\mathrm{GO}(q_2)$ and g_3 in $\mathrm{GO}(q_3)$ such that*

$$g_1(x_2 *_1 x_3) = g_2(x_2) *_1 g_3(x_3) \quad \textit{for all } x_2 \in V_2 \textit{ and } x_3 \in V_3.$$

Proof. Theorem 4.22 yields similitudes $g_2' \in \mathrm{GO}(q_2)$ and $g_3' \in \mathrm{GO}(q_3)$ such that the triple (g_1, g_2', g_3') lies in $\mathbf{GO}(\mathcal{C})(F)$. Letting $\lambda_{\mathcal{C}}(g_1, g_2', g_3') = (\lambda_1, \lambda_2, \lambda_3)$, we have by Proposition 3.14

$$\lambda_1 g_1(x_2 *_1 x_3) = g_2'(x_2) *_1 g_3'(x_3) \quad \text{for all } x_2 \in V_2 \text{ and } x_3 \in V_3.$$

Then $g_2 = \lambda_1^{-1} g_2'$ and $g_3 = g_3'$ satisfy the requirement. ∎

In the special case where $*_1$ is the multiplication in an octonion algebra, Corollary 4.24 is (the main part of) [19, Th. 3.2.1].

Corollary 4.24 also has a "local" version:

Corollary 4.25. *For every $g_1 \in \mathfrak{go}(q_1)$, there exist $g_2 \in \mathfrak{go}(q_2)$ and $g_3 \in \mathfrak{go}(q_3)$ such that*

$$\begin{aligned} &g_1(x_2 *_1 x_3) \\ &\quad = g_2(x_2) *_1 x_3 + x_2 *_1 g_3(x_3) + \dot{\mu}(g_1) x_2 *_1 x_3 \quad \textit{for all } x_2 \in V_2 \textit{ and } x_3 \in V_3. \end{aligned}$$

Proof. Lemma 4.15 shows that projection on the first component $\pi_1\colon \mathfrak{pgo}(\mathcal{C}) \to \mathfrak{pgo}(q_1)$ is bijective, hence there exist $g_2' \in \mathfrak{go}(q_2)$ and $g_3' \in \mathfrak{go}(q_3)$ such that $(g_1 + F, g_2' + F, g_3' + F)$ lies in $\mathfrak{pgo}(\mathcal{C})$, which means that there exists $\lambda_3 \in F$ such that

$$g_3'(x_1 *_3 x_2) = g_1(x_1) *_3 x_2 + x_1 *_3 g_2'(x_2) - \lambda_3 x_1 *_3 x_2 \quad \text{for all } x_1 \in V_1 \text{ and } x_2 \in V_2.$$

By Proposition 3.17, there also exists $\lambda_1 \in F$ such that

$$g_1(x_2 *_1 x_3) = g_2'(x_2) *_1 x_3 + x_2 *_1 g_3'(x_3) - \lambda_1 x_2 *_1 x_3 \quad \text{for all } x_2 \in V_2 \text{ and } x_3 \in V_3.$$

Then $g_2 = g_2' - \lambda_1$ and $g_3 = g_3' + \dot{\mu}(g_1)$ satisfy the required condition. ∎

Specializing $*_1$ to be the multiplication in an octonion algebra (resp. the multiplication in a symmetric composition algebra of dimension 8) yields Elduque's Principle of Local Triality [8, Th. 3.2] (resp. [8, Th. 5.2]).

By contrast with similitudes in Theorem 4.22, isometries $(V_1, q_1) \to (\widetilde{V}_1, \tilde{q}_1)$ do not necessarily extend to isomorphisms $\mathscr{C} \to \widetilde{\mathscr{C}}$ since (V_i, q_i) may not be isometric to $(\widetilde{V}_i, \tilde{q}_i)$ for $i = 2, 3$. Nevertheless, we will obtain in Theorem 4.29 below an isomorphism criterion for compositions of quadratic spaces by using the following construction of similitudes.

For $\mathscr{C}$ as above, define a new composition of quadratic spaces $\mathscr{C}'$ as follows:

$$\mathscr{C}' = \big((V_2, q_2), (V_1, q_1), (V_3, q_3), *_3'\big)$$

where

$$x_2 *_3' x_1 = x_1 *_3 x_2 \quad \text{for } x_2 \in V_2 \text{ and } x_1 \in V_1.$$

To every anisotropic vector $u_3 \in V_3$, we associate the map

$$\rho_{u_3}(x_3) = u_3 q_3(u_3)^{-1} b_3(u_3, x_3) - x_3 \quad \text{for } x_3 \in V_3.$$

Computation shows that ρ_{u_3} is an isometry fixing u_3.

Proposition 4.26. *For every anisotropic vector $u_3 \in V_3$, the triples $(\ell_{u_3}, r_{u_3}, \rho_{u_3})\colon \mathscr{C} \to \mathscr{C}'$ and $(r_{u_3}, \ell_{u_3}, \rho_{u_3})\colon \mathscr{C}' \to \mathscr{C}$ are similitudes with composition multiplier* $(1, 1, q_3(u_3))$.

Proof. Since $\mu(r_{u_3}) = \mu(\ell_{u_3}) = q_3(u_3)$, to prove $(\ell_{u_3}, r_{u_3}, \rho_{u_3})$ is a similitude with composition multiplier $(1, 1, q_3(u_3))$ it suffices to show

$$q_3(u_3)\rho_{u_3}(x_1 *_3 x_2) = \ell_{u_3}(x_1) *_3' r_{u_3}(x_2) \quad \text{for all } x_1 \in V_1, x_2 \in V_2.$$

Likewise, to prove $(r_{u_3}, \ell_{u_3}, \rho_{u_3})$ is a similitude with composition multiplier $(1, 1, q_3(u_3))$ it suffices to show

$$q_3(u_3)\rho_{u_3}(x_2 *_3' x_1) = r_{u_3}(x_2) *_3 \ell_{u_3}(x_1) \quad \text{for all } x_2 \in V_2, x_1 \in V_1.$$

Each of these equations amounts to

$$u_3 b_3(u_3, x_1 *_3 x_2) - (x_1 *_3 x_2) q_3(u_3) = (x_2 *_1 u_3) *_3 (u_3 *_2 x_1).$$

By (3.18), we may rewrite the right side as

$$(x_2 *_1 u_3) *_3 (u_3 *_2 x_1) = u_3 b_1(x_2 *_1 u_3, x_1) - x_1 *_3 \big(u_3 *_2 (x_2 *_1 u_3)\big).$$

Since $b_1(x_2 *_1 u_3, x_1) = b_3(u_3, x_1 *_3 x_2)$ by (3.1), and $u_3 *_2 (x_2 *_1 u_3) = x_2 q_3(u_3)$ by (3.11), the proposition follows. ∎

Proposition 4.26 allows us to describe the group

$$G(\mathcal{C}) = \lambda_{\mathcal{C}}\big(\mathbf{GO}(\mathcal{C})(F)\big) \subset F^\times \times F^\times \times F^\times$$

of composition multipliers of auto-similitudes of $\mathcal{C}$. In the next corollary, we write $G(n_{\mathcal{C}})$ for the group of multipliers of similitudes of the Pfister form associated to $\mathcal{C}$, which is also the set of represented values of this form because Pfister forms are round (see [9, Cor. 9.9]).

Corollary 4.27. $G(\mathcal{C}) = \{(\lambda_1, \lambda_2, \lambda_3) \in F^\times \times F^\times \times F^\times \mid \lambda_1 \equiv \lambda_2 \equiv \lambda_3 \bmod G(n_{\mathcal{C}})\}$.

Proof. If $(\lambda_1, \lambda_2, \lambda_3) = \lambda_{\mathcal{C}}(g_1, g_2, g_3, \lambda_3)$ for some $(g_1, g_2, g_3, \lambda_3) \in \mathbf{GO}(\mathcal{C})(F)$, then by definition of $\lambda_{\mathcal{C}}$ (see (3.27))

$$\lambda_1 = \mu(g_2)\lambda_3^{-1} \quad \text{and} \quad \lambda_2 = \mu(g_1)\lambda_3^{-1}.$$

Since q_1 and q_2 are multiples of $n_{\mathcal{C}}$, multipliers of similitudes of q_1 and of q_2 lie in $G(n_{\mathcal{C}})$, hence $\lambda_1\lambda_3 \in G(n_{\mathcal{C}})$ and $\lambda_2\lambda_3 \in G(n_{\mathcal{C}})$. Therefore, $\lambda_1 \equiv \lambda_2 \equiv \lambda_3 \bmod G(n_{\mathcal{C}})$.

For the converse, we first establish:

Claim: $(1, 1, \nu) \in G(\mathcal{C})$ *for every* $\nu \in G(n_{\mathcal{C}})$. To see this, pick any anisotropic vector $u_3 \in V_3$, and let $v_3 \in V_3$ be the image of $u_3 q_3(u_3)^{-1}$ under any similitude of (V_3, q_3) with multiplier ν, so that $q_3(v_3) = \nu q_3(u_3)^{-1}$. By Proposition 4.26, the composition of maps $(r_{v_3}, \ell_{v_3}, \rho_{v_3}) \circ (\ell_{u_3}, r_{u_3}, \rho_{u_3})$ is an auto-similitude of $\mathcal{C}$ with multiplier

$$\big(1, 1, q_3(v_3)\big)\big(1, 1, q_3(u_3)\big) = (1, 1, \nu).$$

This proves the claim.

Since for the derived composition $\partial\mathcal{C}$ we have $n_{\partial\mathcal{C}} \simeq n_{\mathcal{C}}$, it follows that $(1, 1, \nu) \in G(\partial\mathcal{C})$ for every $\nu \in G(n_{\mathcal{C}})$, hence $(\nu, 1, 1) \in G(\mathcal{C})$ for every $\nu \in G(n_{\mathcal{C}})$.

Now, suppose $(\lambda_1, \lambda_2, \lambda_3) \in F^\times \times F^\times \times F^\times$ is such that $\lambda_1\lambda_2^{-1}$, $\lambda_2^{-1}\lambda_3 \in G(n_{\mathcal{C}})$. The previous observations show

$$(\lambda_1\lambda_2^{-1}, 1, 1), \quad (1, 1, \lambda_2^{-1}\lambda_3) \in G(\mathcal{C}).$$

Moreover, $(\lambda_2 \operatorname{Id}_{V_1}, \lambda_2 \operatorname{Id}_{V_2}, \lambda_2 \operatorname{Id}_{V_3}, \lambda_2) \in \mathbf{GO}(\mathcal{C})(F)$ is a similitude with composition multiplier $(\lambda_2, \lambda_2, \lambda_2)$. Therefore, the group $G(\mathcal{C})$ also contains the product

$$(\lambda_1\lambda_2^{-1}, 1, 1) \cdot (1, 1, \lambda_2^{-1}\lambda_3) \cdot (\lambda_2, \lambda_2, \lambda_2) = (\lambda_1, \lambda_2, \lambda_3). \qquad \blacksquare$$

Remark 4.28. Proposition 4.26 and Corollary 4.27 also hold, with the same proof, for compositions of quadratic spaces of dimension 2 or 4.

Theorem 4.29. *The compositions* $\mathcal{C}$ *and* $\widetilde{\mathcal{C}}$ *are isomorphic if and only if* (V_i, q_i) *and* $(\widetilde{V}_i, \tilde{q}_i)$ *are isometric for* $i = 1, 2$ *and* 3.

Proof. If $g = (g_1, g_2, g_3): \mathcal{C} \to \widetilde{\mathcal{C}}$ is an isomorphism, then from the relations between the multipliers of g_1, g_2, g_3 and the composition multiplier $\lambda(g)$ in (3.25) it follows that g_1, g_2 and g_3 are isometries, hence $(V_i, q_i) \simeq (\widetilde{V}_i, \tilde{q}_i)$ for all i.

For the converse, assume (V_i, q_i) is isometric to $(\widetilde{V}_i, \tilde{q}_i)$ for $i = 1, 2, 3$, and pick an isometry $g_1 : (V_1, q_1) \to (\widetilde{V}_1, \tilde{q}_1)$. Composing it with an improper isometry if needed, we may assume g_1 preserves the polarizations induced by $\mathscr{C}$ and $\widetilde{\mathscr{C}}$. Theorem 4.22 then yields a similitude $g = (g_1, g_2, g_3) : \mathscr{C} \to \widetilde{\mathscr{C}}$. Let $\lambda(g) = (\lambda_1, \lambda_2, \lambda_3)$. From the relations (3.25) between $\lambda(g)$ and the multipliers of g_1, g_2, g_3 it follows that $\mu(g_1) = \lambda_2\lambda_3$, hence $\lambda_2\lambda_3 = 1$ since g_1 is an isometry. The triple $(g_1, \lambda_2 g_2, g_3)$ also is a similitude $\mathscr{C} \to \widetilde{\mathscr{C}}$, and

$$\lambda(g_1, \lambda_2 g_2, g_3) = \lambda(g) \cdot (\lambda_2, \lambda_2^{-1}, \lambda_2) = (\lambda_1\lambda_2, 1, 1) = (\mu(g_3), 1, 1).$$

Since $(\widetilde{V}_3, \tilde{q}_3) \simeq (V_3, q_3)$, the multiplier $\mu(g_3)$ is the multiplier of a similitude of q_3, hence also of $n_{\mathscr{C}}$. Corollary 4.27 then shows that there exists an auto-similitude (g_1', g_2', g_3') of $\mathscr{C}$ such that $\lambda(g_1', g_2', g_3') = (\mu(g_3)^{-1}, 1, 1)$. Then $(g_1 \circ g_1', \lambda_2 g_2 \circ g_2', g_3 \circ g_3')$ is a similitude $\mathscr{C} \to \widetilde{\mathscr{C}}$ with composition multiplier $(1, 1, 1)$, i.e., it is an isomorphism. ■

Corollary 4.23 and Theorem 4.29 can be given a cohomological interpretation: over a separable closure of F, Corollary 3.32 (or Theorem 4.29) shows that all the compositions of quadratic spaces of dimension 8 are isomorphic. Therefore, if $\mathscr{C}_0$ is a composition of hyperbolic quadratic spaces of dimension 8 over F (such as the composition associated to the split para-octonion algebra or the composition in Examples 3.6 (2)), standard arguments of nonabelian Galois cohomology (see for instance [13, §29]) yield canonical bijections

$$H^1\big(F, \mathbf{O}(\mathscr{C}_0)\big) \leftrightarrow \boxed{\begin{array}{l}\text{isomorphism classes of compositions of}\\ \text{quadratic spaces of dimension } n \text{ over } F\end{array}}$$

and

$$H^1\big(F, \mathbf{GO}(\mathscr{C}_0)\big) \leftrightarrow \boxed{\begin{array}{l}\text{similarity classes of compositions of}\\ \text{quadratic spaces of dimension } n \text{ over } F\end{array}}$$

because $\mathbf{O}(\mathscr{C}_0)$ (resp. $\mathbf{GO}(\mathscr{C}_0)$) is the group of automorphisms (resp. auto-similitudes) of $\mathscr{C}_0$. Since by Proposition 4.10 the group $\mathbf{PGO}(\mathscr{C}_0)$ is the automorphism group of the trialitarian triple $\operatorname{End}(\mathscr{C}_0)$, there is an additional canonical bijection

$$H^1\big(F, \mathbf{PGO}(\mathscr{C}_0)\big) \leftrightarrow \boxed{\text{isomorphism classes of trialitarian triples over } F}$$

Now, Corollary 4.23 yields a bijection between $H^1(F, \mathbf{GO}(\mathscr{C}_0))$ and the set of isometry classes of 3-fold quadratic Pfister forms. Similarly, Theorem 4.29 yields a bijection between $H^1(F, \mathbf{O}(\mathscr{C}_0))$ and the set of triples of quadratic forms (q_1, q_2, q_3) up to isometry, subject to the condition that there exists a 3-fold quadratic Pfister form n such that q_1, q_2, q_3 are similar to n and the orthogonal sum $n \perp q_1 \perp q_2 \perp q_3$ is a 5-fold quadratic Pfister form. This can also be viewed as a description of $H^1(F, \mathbf{Spin}_8)$ for $\mathbf{Spin}_8$ the spin group of 8-dimensional hyperbolic quadratic forms, because Theorem 4.12 yields a canonical isomorphism $\mathbf{Spin}_8 \simeq \mathbf{O}(\mathscr{C}_0)$. We may use this description to give an interpretation of the mod 2 cohomological invariants of $\mathbf{Spin}_8$ determined by Garibaldi in [10, §18.1] under the hypothesis that char $F \neq 2$, as follows: for $n = 3, 4, 5$, let e_n denote the Elman–Lam cohomological invariant of n-fold Pfister forms, defined by

$$e_n\big(\langle 1, -a_1\rangle \cdot \ldots \cdot \langle 1, -a_n\rangle\big) = (a_1) \cup \cdots \cup (a_n) \in H^n(F, \boldsymbol{\mu}_2),$$

where $(a_i) \in H^1(F, \boldsymbol{\mu}_2)$ is the cohomology class corresponding to the square class of $a_i \in F^\times$ by Kummer theory, see [9, §16]. For every triple (q_1, q_2, q_3) as above, the cohomology classes

$$e_3(n), \quad e_4(n \perp q_1), \quad e_4(n \perp q_2), \quad e_4(n \perp q_3), \quad e_5(n \perp q_1 \perp q_2 \perp q_3)$$

define cohomological invariants, which distinguish these triples up to isometry. According to [10, §18.1], these invariants generate the $H^*(F, \mathbb{Z}/2\mathbb{Z})$-module of mod 2 invariants of $\mathbf{Spin}_8$. Note that these invariants are not independent: since

$$(n \perp q_1) \perp (n \perp q_2) \perp (n \perp q_3) = 2n \perp (n \perp q_1 \perp q_2 \perp q_3)$$

and $n \perp q_1 \perp q_2 \perp q_3$ is a 5-fold Pfister form, it follows that

$$e_4(n \perp q_1) + e_4(n \perp q_2) + e_4(n \perp q_3) = e_4(2n) = (-1) \cup e_3(n).$$

4.6. The structure group of 8-dimensional composition algebras

Let $\mathcal{A} = (A, q, \diamond)$ be a composition algebra of dimension 8. Recall from Definition 3.30 the structure group $\mathbf{Str}(A, \diamond)$, which is the group of autotopies of $(A, \diamond)$. Corollary 3.35 identifies $\mathbf{Str}(A, \diamond)$ with a subgroup of $\mathbf{GO}(\mathcal{C}(\mathcal{A}))$, for $\mathcal{C}(\mathcal{A})$ the composition of quadratic spaces associated to $\mathcal{A}$ as in (3.32).

In the trialitarian triple $\mathcal{T} = \operatorname{End}(\mathcal{C}(\mathcal{A}))$ we have $\mathfrak{A}_1 = \mathfrak{A}_2 = \mathfrak{A}_3 = (\operatorname{End} A, \sigma_b, \mathfrak{f}_q)$. Mimicking the construction in Section 4.2, we obtain a morphism

$$\psi_{\partial^2\mathcal{T}}\colon \boldsymbol{\Omega}(\mathfrak{A}_3) \to \mathbf{GO}(\partial^2\mathcal{C})$$

as in (4.6). We use it to define a morphism of algebraic groups

$$\psi_{\mathcal{A}}\colon \boldsymbol{\Omega}(q) \to \mathbf{GO}\big(\mathcal{C}(\mathcal{A})\big)$$

by specializing to the case where $\mathcal{T} = \operatorname{End}(\mathcal{C}(\mathcal{A}))$ the map $\partial \circ \psi_{\partial^2\mathcal{T}}\colon \boldsymbol{\Omega}(\mathfrak{A}_3) \to \mathbf{GO}(\mathcal{T})$, where ∂ is the shift map. Thus, for any commutative F-algebra R and $\xi \in \boldsymbol{\Omega}(q)(R)$,

$$\psi_{\mathcal{A}}(\xi) = \big(C_+(\alpha'')(\xi), C_-(\alpha'')(\xi), \chi_0(\xi)\big)$$

(viewing $\mathbf{GO}(\mathcal{C}(\mathcal{A}))$ as a subgroup of $\mathbf{GO}(q) \times \mathbf{GO}(q) \times \mathbf{GO}(q)$, as in the proof of Proposition 4.10), where $C_\pm(\alpha'')$ are the canonical Clifford maps attached to $\partial^2\mathcal{C}(\mathcal{A})$, see (3.21).

Theorem 4.30. *The map $\psi_{\mathcal{A}}$ is an isomorphism $\boldsymbol{\Omega}(q) \xrightarrow{\sim} \mathbf{Str}(A, \diamond)$.*

Proof. The map $\psi_{\mathcal{A}}$ is injective because $C(\alpha'')$ is an isomorphism

$$C(A,q) \to \mathrm{End}(A \oplus A),$$

and the computation of $\lambda_{\mathcal{T}} \circ \psi_{\mathcal{T}}$ in (4.10) together with Corollary 3.35 shows that $\psi_{\mathcal{A}}$ maps $\mathbf{\Omega}(q)$ to $\mathbf{Str}(A,\diamond)$.

To complete the proof, we show that for any commutative F-algebra R the group $\mathbf{Str}(A,\diamond)(R)$ is the image of $\mathbf{\Omega}(q)(R)$ under $\psi_{\mathcal{A}}$. Let (g_1, g_2, g_3) be an autotopy of $(A,\diamond)_R$, which means that

$$g_3(x_1 \diamond x_2) = g_1(x_1) \diamond g_2(x_2) \quad \text{for all } x_1, x_2 \in A_R.$$

By Proposition 3.14 it follows that for all $x_1, x_2, x_3 \in A_R$

$$\mu(g_2)g_1(x_2 \diamond_1 x_3) = g_2(x_2) \diamond_1 g_3(x_3) \quad \text{and} \quad \mu(g_1)g_2(x_3 \diamond_2 x_1) = g_3(x_3) \diamond_2 g_1(x_1).$$

Equivalently,

$$\mu(g_2)g_1 \circ r_{x_3} = r_{g_3(x_3)} \circ g_2 \quad \text{and} \quad \mu(g_1)g_2 \circ \ell_{x_3} = \ell_{g_3(x_3)} \circ g_1,$$

which can be reformulated as an equation in $\mathrm{End}(A \oplus A)$ as follows:

$$\begin{pmatrix} 0 & r_{g_3(x_3)} \\ \ell_{g_3(x_3)} & 0 \end{pmatrix} = \begin{pmatrix} \mu(g_2) & 0 \\ 0 & \mu(g_1) \end{pmatrix} \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \begin{pmatrix} 0 & r_{x_3} \\ \ell_{x_3} & 0 \end{pmatrix} \begin{pmatrix} g_1^{-1} & 0 \\ 0 & g_2^{-1} \end{pmatrix}. \tag{4.23}$$

Since $C(\alpha'')$ is an isomorphism, there exists $\xi \in C_0(A,q)_R$ such that $C_0(\alpha'')(\xi) = (g_1, g_2)$. Then $C_0(\alpha'')(\underline{\mu}(\xi)) = (\mu(g_1), \mu(g_2))$, and (4.23) yields

$$C(\alpha'')\big(g_3(x_3)\big) = C(\alpha'')\big(\iota\big(\underline{\mu}(\xi)\big)\xi x_3 \xi^{-1}\big) \quad \text{for all } x_3 \in A_R.$$

Since $C(\alpha'')$ is an isomorphism, it follows from Lemma 2.5 that $\tau_0(\xi)x_3\xi = \sigma_b(g_3)(x_3)$ for all $x_3 \in A_R$, hence $\xi \in \mathbf{\Omega}(q)(R)$ and $g_3 = \chi_0(\xi)$. Thus, $(g_1, g_2, g_3) = \psi_{\mathcal{A}}(\xi)$. ∎

Recall from Proposition 2.8 the exact sequence

$$1 \to R^1_{Z/F}(\mathbf{G_m}) \to \mathbf{\Omega}(q) \xrightarrow{\chi_0} \mathbf{GO}^+(q) \to 1. \tag{4.24}$$

Since the discriminant of q is trivial, we have $Z \simeq F \times F$, hence $R^1_{Z/F}(\mathbf{G_m}) \simeq \mathbf{G_m}$ and the Galois cohomology exact sequence derived from (4.24) takes the form

$$1 \to F^\times \to \mathbf{\Omega}(q)(F) \to \mathrm{GO}^+(q) \to 1.$$

Substituting $\mathbf{Str}(A,\diamond)(F)$ for $\mathbf{\Omega}(q)(F)$, we recover the exact sequence obtained by Petersson [15, (4.13)] for $\mathcal{A}$ an octonion algebra.

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