Global rigid inner forms vs isocrystals

Tasho Kaletha and Olivier Taïbi

Abstract. We compare the cohomology of the global Galois gerbes constructed by Kottwitz in arXiv:1401.5728 and by the first author in [Invent. Math. (2018), 271–369], respectively, and give applications to the theory of endoscopy.

Contents

1. Introduction

The statement of the refined local and global Arthur–Langlands conjectures for non-quasi-split reductive groups involves the cohomology of certain Galois gerbes [9], where the notion of a Galois gerbe is that of [26]. In summary, every such group G is an inner form of its quasi-split form G^*, but it was observed by Vogan [35] that this relationship does not suffice for the normalization of various objects involved in the statement of the local Langlands conjecture. The cohomology of a local gerbe is used to provide the necessary additional data. The cohomology of a global gerbe is used to organize the local data at all places coherently, so that the local conjecture can be used in global applications.

The gerbes constructed in [20] can be used for this purpose. However, not all local inner forms can be treated when G^* does not have connected center, and not all global inner forms can be treated when G^* does not satisfy the Hasse principle. We shall refer to the formulation of the local and global conjectures involving the gerbes of [20] as the *isocrystal version*. The gerbes constructed in [10, 11] can be used without these technical hypotheses on G^*, but they are at the moment available only in characteristic zero. We shall refer to the resulting local and global conjectures as the *rigid version*.

Assume first that the ground field F is a finite extension of $\mathbb{Q}_p$ and that G^* has connected center. Then both the isocrystal and the rigid version of the refined local Langlands conjecture are available for G. It was shown in [12] that these two versions are equivalent.

Moreover, it was shown that the validity of the isocrystal version for all connected reductive groups with connected center is equivalent to the validity of the rigid version for all connected reductive groups without assumption on the center. These results were derived from a comparison theorem for the cohomology of the two local gerbes.

The current paper provides a comparison theorem for the cohomology of the two global gerbes. We give two applications to this comparison. First, when the ground field F is a finite extension of $\mathbb{Q}$ and G^* has connected center and satisfies the Hasse principle, so that both the isocrystal version and the rigid version of the global multiplicity formula are available, it is natural to ask if these two versions are equivalent. A formal argument, based on the canonicity of global transfer factors, gives an affirmative answer, but sheds no light on the relation between the normalized local pairings at each place of F. Our cohomological result allows for this finer comparison.

Second, we generalize [11, Proposition 4.4.1], which states that the product of local normalized transfer factors equal the canonical adelic transfer factor. In [11] this was proved under the assumption that there exists a pair of related F-points in the group and its endoscopic group. While this assumption was also made in [27], where transfer factors were originally defined, it was later dropped in [21] and replaced with the weaker assumption on the existence of an F-point in the endoscopic group that is related to an $\mathbb{A}$-point in the group. We use the results of the current paper to show that [11, Proposition 4.4.1] is valid under this weaker hypothesis.

We hope that the comparison of the cohomology of the two global gerbes will be useful beyond these applications, in light of Scholze's recent conjecture [30, Conjecture 9.5] on the existence of a Weil cohomology theory for varieties over $\overline{\mathbb{F}}_p$ valued in the category of representations of the global gerbe of [20].

Before we outline the comparison theorem in the global case, let us review it in the local case. Let F be a finite extension of $\mathbb{Q}_p$ and Γ the absolute Galois group of F. The local gerbe of [20], which we shall denote by $\mathcal{E}^{\mathrm{iso}}$ here, is bound by the pro-torus $\mathbb{T}^{\mathrm{iso}}$ whose character module is the trivial Γ-module $\mathbb{Q}$. The local gerbe of [10], which we shall denote by $\mathcal{E}^{\mathrm{rig}}$ here, is bound by the pro-finite algebraic group P^{rig} whose character module is the group of smooth functions $\Gamma \to \mathbb{Q}/\mathbb{Z}$, endowed with the obvious action of Γ. The map $X^*(\mathbb{T}^{\mathrm{iso}}) \to X^*(P^{\mathrm{rig}})$ sending $q \in \mathbb{Q}$ to the constant function with value q provides a homomorphism $P^{\mathrm{rig}} \to \mathbb{T}^{\mathrm{iso}}$ defined over F. One proves that the push-out of $\mathcal{E}^{\mathrm{rig}}$ along this homomorphism equals $\mathcal{E}^{\mathrm{iso}}$. The resulting map of gerbes $\mathcal{E}^{\mathrm{rig}} \to \mathcal{E}^{\mathrm{iso}}$ induces a map between their cohomology. For example, when T is an algebraic torus defined over F, we obtain a homomorphism of abelian groups $H^1(\mathcal{E}^{\mathrm{iso}}, T) \to H^1(\mathcal{E}^{\mathrm{rig}}, T)$. Both of these abelian groups have a description in terms of linear algebra. In the first case, we have the functorial isomorphism $H^1(\mathcal{E}^{\mathrm{iso}}, T) \to X_*(T)_\Gamma$, where $X_*(T)$ is the co-character module of T. In the second case, we have the functorial isomorphism

$$H^1(\mathcal{E}^{\mathrm{rig}}, T) \to \frac{X_*(T) \otimes \mathbb{Q}}{I X_*(T)}[\mathrm{tor}],$$

where $I \subset \mathbb{Z}[\Gamma]$ is the augmentation ideal, and [tor] refers to the torsion subgroup. Let

E/F be any finite Galois extension splitting T. Let $N^\natural$ denote the normalized norm map

$$[E : F]^{-1} \sum_{\sigma \in \Gamma_{E/F}} \sigma : X_*(T) \otimes \mathbb{Q} \to X_*(T) \otimes \mathbb{Q}.$$

Then the map $X_*(T)_\Gamma \to \frac{X_*(T) \otimes \mathbb{Q}}{IX_*(T)}[\mathrm{tor}]$ given by $\lambda \mapsto \lambda - N^\natural(\lambda)$ makes the following diagram commute

$$
\begin{array}{ccc}
H^1(\mathcal{E}^{\mathrm{iso}}, T) & \longrightarrow & X_*(T)_\Gamma \\
\downarrow & & \downarrow \\
H^1(\mathcal{E}^{\mathrm{rig}}, T) & \longrightarrow & \frac{X_*(T) \otimes \mathbb{Q}}{IX_*(T)}[\mathrm{tor}]
\end{array}
$$

We now come to the global case treated in this paper. Let F be a finite extension of $\mathbb{Q}$. The global gerbe of [20], which we shall denote by $\mathcal{E}^{\mathrm{iso}}$ here, is bound by a pro-torus $\mathbb{T}^{\mathrm{iso}}$, while the global gerbe of [11], which we shall denote by $\mathcal{E}^{\mathrm{rig}}$ here, is bound by a pro-finite algebraic group P^{rig}. The description of the character modules is more technical and we will not discuss it in the introduction. Unlike in the local case, we do not know of a natural map $P^{\mathrm{rig}} \to \mathbb{T}^{\mathrm{iso}}$. In fact, there is good reason to believe that one cannot expect a natural map like that to exist. The comparison of the cohomology of the two gerbes $\mathcal{E}^{\mathrm{iso}}$ and $\mathcal{E}^{\mathrm{rig}}$ proceeds via an intermediary. We define a new pro-torus $\mathbb{T}^{\mathrm{mid}}$ and natural maps $\mathbb{T}^{\mathrm{iso}} \to \mathbb{T}^{\mathrm{mid}} \leftarrow P^{\mathrm{rig}}$. We prove that the classes in $H^2(\Gamma, \mathbb{T}^{\mathrm{iso}})$ and $H^2(\Gamma, P^{\mathrm{rig}})$ of the gerbes $\mathcal{E}^{\mathrm{iso}}$ and $\mathcal{E}^{\mathrm{rig}}$ meet in $H^2(\Gamma, \mathbb{T}^{\mathrm{mid}})$. This leads to a gerbe $\mathcal{E}^{\mathrm{mid}}$ bound by $\mathbb{T}^{\mathrm{mid}}$ and equipped with homomorphisms $\mathcal{E}^{\mathrm{iso}} \to \mathcal{E}^{\mathrm{mid}} \leftarrow \mathcal{E}^{\mathrm{rig}}$. We then prove that, for every algebraic torus T defined over F, the two diagrams

$$
\begin{array}{ccc}
H^1(\mathcal{E}^{\mathrm{mid}}, T) & \longrightarrow & \mathrm{Hom}_F(\mathbb{T}^{\mathrm{mid}}, T) \\
\downarrow & & \downarrow \\
H^1(\mathcal{E}^{\mathrm{iso}}, T) & \longrightarrow & \mathrm{Hom}_F(\mathbb{T}^{\mathrm{iso}}, T)
\end{array}
\qquad
\begin{array}{ccc}
H^1(\mathcal{E}^{\mathrm{mid}}, T) & \longrightarrow & \mathrm{Hom}_F(\mathbb{T}^{\mathrm{mid}}, T) \\
\downarrow & & \downarrow \\
H^1(\mathcal{E}^{\mathrm{rig}}, T) & \longrightarrow & \mathrm{Hom}_F(P^{\mathrm{rig}}, T)
\end{array}
$$

are Cartesian and the vertical arrows in the left diagram are surjective.

An analogous discussion holds locally at each place v of F: There are maps of gerbes $\mathcal{E}_v^{\mathrm{iso}} \to \mathcal{E}_v^{\mathrm{mid}} \leftarrow \mathcal{E}_v^{\mathrm{rig}}$ over F_v that are compatible with the analogous global maps via localization maps $\mathcal{E}_v^* \to \mathcal{E}^* \times_F F_v$. The Cartesian square relating $\mathcal{E}_v^{\mathrm{iso}}$ to $\mathcal{E}_v^{\mathrm{mid}}$ shows that there is a functorial isomorphism from $H^1(\mathcal{E}_v^{\mathrm{mid}}, T)$ to the group $\{(\lambda, \mu) \mid \lambda \in X_*(T)_\Gamma,\ \mu \in X_*(T) \otimes \mathbb{Q}, N^\natural(\lambda) = N^\natural(\mu)\}$.

Recalling the comparison map $\mathcal{E}_v^{\mathrm{rig}} \to \mathcal{E}_v^{\mathrm{iso}}$ constructed in [12] and reviewed above, we now obtain the following triangle

$$
\begin{array}{ccc}
 & \mathcal{E}_v^{\mathrm{mid}} & \\
 \nearrow & & \nwarrow \\
\mathcal{E}_v^{\mathrm{rig}} & \longrightarrow & \mathcal{E}_v^{\mathrm{iso}}
\end{array}
\tag{1.0.1}
$$

This triangle does *not* commute. In order to relate the global comparison results of this paper, which concern (via the localization maps) the two diagonal arrows, to the local

comparison results of [12], which concern the bottom horizontal arrow, we need to understand the failure of commutativity.

We construct a canonical splitting $\mathcal{E}_v^{\mathrm{mid}} \to \mathcal{E}_v^{\mathrm{iso}}$ of the map $\mathcal{E}_v^{\mathrm{iso}} \to \mathcal{E}_v^{\mathrm{mid}}$ and show that composing this splitting with $\mathcal{E}_v^{\mathrm{rig}} \to \mathcal{E}_v^{\mathrm{mid}}$ equals the bottom horizontal map in (1.0.1), i.e., the comparison map $\mathcal{E}_v^{\mathrm{rig}} \to \mathcal{E}_v^{\mathrm{iso}}$ of [12]. The non-commutativity of the above triangle is then encoded in the difference between the left diagonal map $\mathcal{E}_v^{\mathrm{rig}} \to \mathcal{E}_v^{\mathrm{mid}}$ and the composition

$$\mathcal{E}_v^{\mathrm{rig}} \to \mathcal{E}_v^{\mathrm{mid}} \to \mathcal{E}_v^{\mathrm{iso}} \to \mathcal{E}_v^{\mathrm{mid}}.$$

We show that the difference between the two homomorphisms

$$H^1(\mathcal{E}_v^{\mathrm{mid}}, T) \to H^1(\mathcal{E}_v^{\mathrm{rig}}, T)$$

induced by these two maps $\mathcal{E}_v^{\mathrm{rig}} \to \mathcal{E}_v^{\mathrm{mid}}$ is given on the linear algebraic side by the map that sends (λ, μ) to $\mu - N^{\natural}(\mu) \in \frac{X_*(T) \otimes \mathbb{Q}}{I X_*(T)}[\mathrm{tor}]$.

These cohomological results allow us to compare the two isocrystal and rigid versions of the multiplicity conjecture for discrete automorphic representations. More precisely, let G^* be a quasi-split connected reductive group defined over F and let G be an inner form of G^*. Assuming the existence of the global Langlands group L_F, as well as the validity of the rigid version of the refined local Langlands correspondence, we constructed in [11, Section 4.5] a pairing between the group S_φ associated to a discrete generic global parameter $\varphi : L_F \to {}^L G$ and the adelic L-packet $\Pi_\varphi(G)$. This pairing is an ingredient in the conjectural multiplicity formula [16, (12.3)]. Its construction uses the cohomology of $\mathcal{E}^{\mathrm{rig}}$, but the result is independent of the cohomology classes used.

Assuming that G^* has connected center and satisfies the Hasse principle, another such pairing can be constructed if one assumes the isocrystal version of the refined local Langlands correspondence and uses the cohomology of $\mathcal{E}^{\mathrm{iso}}$. This construction does not yet appear in the literature. It is fairly analogous to that in [11, Section 4.5] and we give the details in Section 4.6.

As an application of our cohomological results, we show that the two constructions – using $\mathcal{E}^{\mathrm{rig}}$ and $\mathcal{E}^{\mathrm{iso}}$, respectively – produce the same pairing between S_φ and $\Pi_\varphi(G)$. More precisely, given an inner twist $\psi : G^* \to G$ we fix a cocycle $z^{\mathrm{mid}} \in Z^1(\mathcal{E}^{\mathrm{mid}}, G^*)$ that lifts the cocycle $\sigma \mapsto \psi^{-1}\sigma(\psi)$ and use it to produce cocycles $z^{\mathrm{iso}} \in Z^1(\mathcal{E}^{\mathrm{iso}}, G^*)$ and $z^{\mathrm{mid}} \in Z^1(\mathcal{E}^{\mathrm{mid}}, G^*)$. The two global pairings are constructed as products of local pairings, each normalized by the localization z_v^{iso} and z_v^{rig}, respectively. At each place v, the local pairings do depend on the choice of z^{mid}, but the resulting global pairings do not. Even though conjectural, the local pairings are related by an explicit non-conjectural factor that is a result of the normalized character identities the pairings are required to satisfy. This follows from the local comparison results of [12]. However, due to the non-commutativity of (1.0.1) the local comparison map $H^1(\mathcal{E}_v^{\mathrm{iso}}, G^*) \to H^1(\mathcal{E}_v^{\mathrm{rig}}, G^*)$ does not map $[z_v^{\mathrm{iso}}]$ to $[z_v^{\mathrm{rig}}]$. Thus the local comparison results of [12] need to be supplemented with the quantification of the non-commutativity of (1.0.1) discussed above. Combining these results, we obtain an explicit factor relating the two local pairings at a given place v.

The global comparison results of this paper imply that the product over all v of these factors equals 1 and therefore the two global pairings are equal.

Alongside this comparison result, we introduce in this paper a simplification of the construction of the global gerbe $\mathcal{E}^{\mathrm{rig}}$. In [11] this construction involved choosing a sequence $(E_i, S_i, \dot{S}_i)$, where E_i is an exhaustive tower of finite Galois extensions of F, S_i is an exhaustive tower of finite sets of places of F, and $\dot{S}_i$ is a set of lifts of S_i to places of E_i. Each triple $(E_i, S_i, \dot{S}_i)$ was required to satisfy a list of four conditions [11, Conditions 3.3.1]. In this paper we show that the resulting gerbe depends only on the set $\dot{V}$ of lifts to $\bar{F}$ of the places of F that is defined by $\dot{V} = \varprojlim \dot{S}_i$. That is, $\mathcal{E}^{\mathrm{rig}}$ is independent of the choices of E_i and S_i. Furthermore, we show that [11, Conditions 3.3.1] for each triple $(E_i, S_i, \dot{S}_i)$ are equivalent to one simple condition on $\dot{V}$, namely Condition 3.3.1 stating that $\bigcup_{v \in V} \Gamma_{\dot{v}}$ is dense in Γ, where V is the set of all places of F.

2. Definition of some local and global Galois modules

In this section we shall define some modules for the Galois group of a finite Galois extension of a ground field F that is either a number field or a local field. Taking colimits over all finite extensions of F we shall obtain modules for the absolute Galois group of a number field or a local field. These will be the character modules of $\mathbb{T}^{\mathrm{iso}}$, $\mathbb{T}^{\mathrm{mid}}$, and P^{rig}. We shall also discuss the transition maps with respect to which we take these colimits – we call these *inflation maps*. We shall also discuss *localization* maps, which relate the global Galois modules to their local counterparts.

2.1. The local modules

Let F be a local field, $\bar{F}$ an algebraic closure of F and Γ the corresponding absolute Galois group. Let E/F be a finite Galois subextension of $\bar{F}/F$, with Galois group $\Gamma_{E/F}$. Let N be natural number. We define the following $\Gamma_{E/F}$-modules:

(1) $M_E^{\mathrm{iso}} := \mathbb{Z}$.

(2) $M_{E,N}^{\mathrm{mid}}$ consists of maps $f : \Gamma_{E/F} \to \frac{1}{N}\mathbb{Z}$ satisfying $\sum_\sigma f(\sigma) \in \mathbb{Z}$.

(3) $M_{E,N}^{\mathrm{rig}}$ consists of maps $f : \Gamma_{E/F} \to \frac{1}{N}\mathbb{Z}/\mathbb{Z}$ satisfying $\sum_\sigma f(\sigma) = 0$.

The module M_E^{iso} is the module X of [20, Section 5], while $M_{E,N}^{\mathrm{rig}}$ is the module $X^*(u_{E/F,N})$ of [10, Section 3.1].

We define $\Gamma_{E/F}$-equivariant maps

$$M_E^{\mathrm{iso}} \xleftarrow{c^{\mathrm{iso}}} M_{E,N}^{\mathrm{mid}} \xrightarrow{c^{\mathrm{rig}}} M_{E,N}^{\mathrm{rig}} \tag{2.1.1}$$

via the formulas

$$f^{\mathrm{iso}} = \sum_\sigma f^{\mathrm{mid}}(\sigma), \quad f^{\mathrm{rig}}(\sigma) = -f^{\mathrm{mid}}(\sigma) \mod \mathbb{Z}.$$

Fact 2.1.1. *The maps c^{iso} and c^{rig} are surjective. The kernel of c^{rig} is induced.*

Proof. Immediate.   ∎

If N is divisible by $[E : F]$, there is a canonical splitting of c^{iso} defined by

$$s^{\mathrm{iso}} : M_E^{\mathrm{iso}} \to M_{E,N}^{\mathrm{mid}}, \quad s^{\mathrm{iso}}(f^{\mathrm{iso}}) = f^{\mathrm{mid}}, \quad f^{\mathrm{mid}}(\sigma) = [E : F]^{-1} f^{\mathrm{iso}}. \qquad (2.1.2)$$

The image of s^{iso} is precisely $(M_{E,N}^{\mathrm{mid}})^{\Gamma}$.

2.2. Local inflation maps

We continue with the notation of Section 2.1. Let K/F be another finite Galois extension with $E \subset K$, M a natural number divisible by N.

We define three maps, which we refer to as inflation maps:

(1) $M_E^{\mathrm{iso}} \to M_K^{\mathrm{iso}}$, given by multiplication by $[K : E]$.

(2) $M_{E,N}^{\mathrm{mid}} \to M_{K,M}^{\mathrm{mid}}$, given by $f^{\mathrm{mid},K}(\sigma) = f^{\mathrm{mid},E}(\sigma)$.

(3) $M_{E,N}^{\mathrm{rig}} \to M_{K,M}^{\mathrm{rig}}$, also given by $f^{\mathrm{rig},K}(\sigma) = f^{\mathrm{rig},E}(\sigma)$.

These inflation maps fit into the commutative diagram

$$
\begin{array}{ccccc}
M_E^{\mathrm{iso}} & \xleftarrow{\;c_{E,N}^{\mathrm{iso}}\;} & M_{E,N}^{\mathrm{mid}} & \xrightarrow{\;c_{E,N}^{\mathrm{rig}}\;} & M_{E,N}^{\mathrm{rig}} \\
\downarrow & & \downarrow & & \downarrow \\
M_K^{\mathrm{iso}} & \xleftarrow[c_{K,M}^{\mathrm{iso}}]{} & M_{K,M}^{\mathrm{mid}} & \xrightarrow[c_{K,M}^{\mathrm{rig}}]{} & M_{K,M}^{\mathrm{rig}}
\end{array}
$$

Using these inflation maps we can take in each case the colimit over all finite Galois extensions E/F and all natural numbers N. Denoting by $R[\Gamma]$ the set of smooth functions $\Gamma \to R$, we obtain the following:

$$M^{\mathrm{iso}} := \varinjlim M_E^{\mathrm{iso}} = \begin{cases} \mathbb{Q} & \text{if } F \text{ is non-Archimedean,} \\ \mathbb{Z} & \text{if } F \simeq \mathbb{C}, \\ \frac{1}{2}\mathbb{Z} & \text{if } F = \mathbb{R} \end{cases}$$

$$M^{\mathrm{mid}} := \varinjlim M_{E,N}^{\mathrm{mid}} = \begin{cases} \mathbb{Q}[\Gamma] & \text{if } F \text{ is non-Archimedean,} \\ \mathbb{Z}[\Gamma] = \mathbb{Z} & \text{if } F \simeq \mathbb{C}, \\ \{f : \Gamma \to \mathbb{Q} \mid \sum_{\sigma \in \Gamma} f(\sigma) \in \mathbb{Z}\} & \text{if } F = \mathbb{R} \end{cases}$$

$$M^{\mathrm{rig}} := \varinjlim M_{E,N}^{\mathrm{rig}} = \begin{cases} \mathbb{Q}/\mathbb{Z}[\Gamma] & \text{if } F \text{ is non-Archimedean,} \\ 0 & \text{if } F \simeq \mathbb{C}, \\ \{f : \Gamma \to \mathbb{Q}/\mathbb{Z} \mid \sum_{\sigma \in \Gamma} f(\sigma) = 0\} & \text{if } F = \mathbb{R} \end{cases}$$

The local comparison maps splice together to maps

$$M^{\mathrm{iso}} \xleftarrow{\;c^{\mathrm{iso}}\;} M^{\mathrm{mid}} \xrightarrow{\;c^{\mathrm{rig}}\;} M^{\mathrm{rig}},$$

the left being given by integrating over Γ with respect to the normalized Haar measure, and the right being induced by the *negative* of the natural projection $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$. The map c^{iso} has a canonical splitting s^{iso} whose image consists of constant functions $\Gamma \to \mathbb{Q}$ in the non-Archimedean case (resp. $\Gamma \to \mathbb{Z}$ in the complex case, resp. $\Gamma \to \frac{1}{2}\mathbb{Z}$ in the real case). In the non-Archimedean case the composition $c^{\mathrm{rig}} \circ s^{\mathrm{iso}}$ equals the map $X^*(\phi)$, where ϕ is the map defined in [12, (3.13)], as we see by dualizing Lemma 3.1 loc. cit. In the Archimedean case we take this equality as the definition of ϕ.

2.3. A discussion of $M_{E,N}^{\mathrm{mid},\vee}$

We now describe the $\Gamma_{E/F}$-module $M_{E,N}^{\mathrm{mid},\vee} = \mathrm{Hom}_{\mathbb{Z}}(M_{E,N}^{\mathrm{mid}}, \mathbb{Z})$ and record some of its properties.

The obvious inclusions $\mathbb{Z}[\Gamma_{E/F}] \to M_{E,N}^{\mathrm{mid}} \to N^{-1}\mathbb{Z}[\Gamma_{E/F}]$ fit into the exact sequences

$$0 \to M_{E,N}^{\mathrm{mid}} \to N^{-1}\mathbb{Z}[\Gamma_{E/F}] \xrightarrow{\sum_\sigma} N^{-1}\mathbb{Z}/\mathbb{Z} \to 0 \tag{2.3.1}$$

and

$$0 \to \mathbb{Z}[\Gamma_{E/F}] \to M_{E,N}^{\mathrm{mid}} \to N^{-1}\mathbb{Z}/\mathbb{Z}[\Gamma_{E/F}]_0 \to 0. \tag{2.3.2}$$

Here the subscript 0 denotes the subset of functions f on $\Gamma_{E/F}$ satisfying the condition $\sum_{\sigma \in \Gamma_{E/F}} f(\sigma) = 0$. In fact $N^{-1}\mathbb{Z}/\mathbb{Z}[\Gamma_{E/F}]_0$ equals $M_{E,N}^{\mathrm{rig}}$ and the map is just $-c_{E,N}^{\mathrm{rig}}$, but we do not want to emphasize this here because it distracts.

We can identify $\mathbb{Z}[\Gamma_{E/F}]$ with its own dual via the pairing $(x,y) \mapsto \sum_\sigma x(\sigma)y(\sigma)$. Then $N\mathbb{Z}[\Gamma_{E/F}]$ dualizes to $N^{-1}\mathbb{Z}[\Gamma_{E/F}]$, the inclusion $\mathbb{Z}[\Gamma_{E/F}] \to N^{-1}\mathbb{Z}[\Gamma_{E/F}]$ dualizes to the inclusion $N\mathbb{Z}[\Gamma_{E/F}] \to \mathbb{Z}[\Gamma_{E/F}]$. For a finite Galois extension K of F containing E, the inflation map $\mathbb{Z}[\Gamma_{E/F}] \to \mathbb{Z}[\Gamma_{K/F}]$ dualizes to the map $\mathbb{Z}[\Gamma_{K/F}] \to \mathbb{Z}[\Gamma_{E/F}]$ given by summing over $\Gamma_{K/E}$-cosets.

The inclusions $\mathbb{Z}[\Gamma_{E/F}] \to M_{E,N}^{\mathrm{mid}} \to N^{-1}\mathbb{Z}[\Gamma_{E/F}]$ dualize to the inclusions

$$N\mathbb{Z}[\Gamma_{E/F}] \to M_{E,N}^{\mathrm{mid},\vee} \to \mathbb{Z}[\Gamma_{E/F}]$$

and describe $M_{E,N}^{\mathrm{mid},\vee}$ as the submodule of $\mathbb{Z}[\Gamma_{E/F}]$ given by $N\mathbb{Z}[\Gamma_{E/F}] + \mathbb{Z}$, where $\mathbb{Z} = \mathbb{Z}[\Gamma_{E/F}]^\Gamma$ is the subgroup consisting of constant functions. Note that $\mathbb{Z}[\Gamma_{E/F}]^\Gamma$ coincides with $[M_{E,N}^{\mathrm{mid},\vee}]^\Gamma$.

In terms of this description of $M_{E,N}^{\mathrm{mid},\vee}$ the exact sequences dual[1] to (2.3.1) and (2.3.2) are described as follows. The dual of (2.3.1) is

$$0 \to N\mathbb{Z}[\Gamma_{E/F}] \to M_{E,N}^{\mathrm{mid},\vee} \to \mathbb{Z}/N\mathbb{Z} \to 0,$$

with the map $M_{E,N}^{\mathrm{mid},\vee} \to \mathbb{Z}/N\mathbb{Z}$ given by the natural projection on $\mathbb{Z}$ and trivial on

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[1] By dual of a short exact sequence $0 \to A \to B \to C \to 0$ where B is a finite free $\mathbb{Z}$-module we mean the short exact sequence $0 \to \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Q}/\mathbb{Z}) \to 0$.

$N\mathbb{Z}[\Gamma_{E/F}]$. The dual of (2.3.2) is

$$0 \to M_{E,N}^{\mathrm{mid},\vee} \to \mathbb{Z}[\Gamma_{E/F}] \to \frac{\mathbb{Z}/N\mathbb{Z}[\Gamma_{E/F}]}{\mathbb{Z}/N\mathbb{Z}} \to 0, \tag{2.3.3}$$

where now the last map is the natural projection.

The map $c^{\mathrm{iso}} : M_{E,N}^{\mathrm{mid}} \to \mathbb{Z}$ defined in (2.1.1) dualizes to the inclusion map

$$\mathbb{Z} = [M_{E,N}^{\mathrm{mid},\vee}]^{\Gamma} \to M_{E,N}^{\mathrm{mid},\vee}.$$

If $[E : F]$ divides N then its splitting $s^{\mathrm{iso}} : \mathbb{Z} \to M_{E,N}^{\mathrm{mid}}$ defined in (2.1.2) dualizes to $M_{E,N}^{\mathrm{mid},\vee} \to \mathbb{Z}$ given by $y \mapsto [E : F]^{-1} \sum_\sigma y(\sigma)$.

The inflation map

$$M_{E,N}^{\mathrm{mid}} \to M_{K,M}^{\mathrm{mid}}$$

dualizes to the map sending $y^K \in M_{K,M}^{\mathrm{mid},\vee} \subset \mathbb{Z}[\Gamma_{K/F}]$ to $y^E \in M_{E,N}^{\mathrm{mid},\vee} \subset \mathbb{Z}[\Gamma_{E/F}]$ given by $y^E(\sigma) = \sum_{\tau \mapsto \sigma} y^K(\tau)$.

2.4. The global modules

Let F be a number field, E/F finite Galois extension, S a (finite or infinite) set of places of F. We denote by S_E the set of places of E above those in S. Let $\dot{S}_E \subset S_E$ be a set of lifts of the places in S to places of E. We recall that of the triple $(E/F, S, \dot{S}_E)$ one can require [11, Conditions 3.3.1], which we recall here for convenience.

Conditions 2.4.1. (1) S contains all archimedean places and all places that ramify in E.

(2) Every ideal class of E contains an ideal with support in S_E.

(3) For every place w of E there exists $w' \in S_E$ such that the stabilizers in $\Gamma_{E/F}$ of w and w' coincide.

(4) For every $\sigma \in \Gamma_{E/F}$ there exists $w \in \dot{S}_E$ satisfying $\sigma w = w$.

For now, only condition (4). will be useful to us, and we will explicitly require it when it is needed. Starting with Section 3.3, we will make active use of these conditions.

We define the following $\Gamma_{E/F}$-modules:

(1) $M_{E,S}^{\mathrm{iso}} := \mathbb{Z}[S_E]_0$ consists of finitely supported functions $f : S_E \to \mathbb{Z}$ satisfying $\sum_w f(w) = 0$.

(2) $M_{E,\dot{S}_E}^{\mathrm{mid}}$ consists of finitely supported functions $f : \Gamma_{E/F} \times S_E \to \frac{1}{[E:F]}\mathbb{Z}$ satisfying $\sum_w f(\sigma, w) = 0$, $\sum_\sigma f(\sigma, w) \in \mathbb{Z}$, $\sigma^{-1}w \notin \dot{S}_E \Rightarrow f(\sigma, w) = 0$.

(3) $M_{E,\dot{S}_E}^{\mathrm{rig}}$ consists of finitely supported functions $f : \Gamma_{E/F} \times S_E \to \frac{1}{[E:F]}\mathbb{Z}/\mathbb{Z}$ satisfying $\sum_w f(\sigma, w) = 0$, $\sum_\sigma f(\sigma, w) = 0$, $\sigma^{-1}w \notin \dot{S}_E \Rightarrow f(\sigma, w) = 0$.

We shall write f^{iso} or $f^{\mathrm{iso},E}$ in the first case if we want to be more precise, and use the analogous notation in the other two cases.

The module $M_{E,S}^{\mathrm{iso}}$ was defined by Tate [34], and later by Kottwitz in [20, Section 6], where it was denoted by X_3. The module $M_{E,\dot{S}_E}^{\mathrm{rig}}$ was defined in [11], where it was denoted by $M_{E,\dot{S}_E,[E:F]}$.

We define $\Gamma_{E/F}$-equivariant maps

$$M_{E,S}^{\mathrm{iso}} \xleftarrow{c^{\mathrm{iso}}} M_{E,\dot{S}_E}^{\mathrm{mid}} \xrightarrow{c^{\mathrm{rig}}} M_{E,\dot{S}_E}^{\mathrm{rig}} \tag{2.4.1}$$

by the formulas

$$f^{\mathrm{iso}}(w) = \sum_\sigma f^{\mathrm{mid}}(\sigma, w), \quad f^{\mathrm{rig}}(\sigma, w) = -f^{\mathrm{mid}}(\sigma, w) \mod \mathbb{Z}.$$

Proposition 2.4.2. *The map c^{rig} is surjective.*

Proof. We can assume that $S \neq \emptyset$. Let $f^{\mathrm{rig}} \in M_{E,\dot{S}_E}^{\mathrm{rig}}$. For each $\sigma \in \Gamma_{E/F}$ choose $w_\sigma \in S_E$ such that $\sigma^{-1} w_\sigma \in \dot{S}_E$. Define f^{mid} as follows:

(1) For (σ, w) such that $\sigma^{-1} w \notin \dot{S}_E$, $f^{\mathrm{mid}}(\sigma, w) = 0$.

(2) For (σ, w) such that $\sigma^{-1} w \in \dot{S}_E$ but $w \neq w_\sigma$, choose an arbitrary lift $f^{\mathrm{mid}}(\sigma, w) \in \mathbb{Q}$ of $-f^{\mathrm{rig}}(\sigma, w) \in \mathbb{Q}/\mathbb{Z}$.

(3) Finally for $\sigma \in \Gamma_{E/F}$ let $f^{\mathrm{mid}}(\sigma, w_\sigma) = -\sum_{w \in S_E \smallsetminus \{w_\sigma\}} f^{\mathrm{mid}}(\sigma, w) \in \mathbb{Q}$.

Then $f^{\mathrm{mid}} \in M_{E,\dot{S}_E}^{\mathrm{mid}}$ and $c^{\mathrm{rig}}(f^{\mathrm{mid}}) = f^{\mathrm{rig}}$. ∎

Fact 2.4.3. *The kernel of c^{rig} is an induced $\Gamma_{E/F}$-module.*

Proof. After making the change of variables $\phi(\sigma, w) = f(\sigma, \sigma w)$ we see that this kernel is given by the set of functions $\phi : \Gamma_{E/F} \times S_E \to \mathbb{Z}$ satisfying $\sum_w \phi(\sigma, w) = 0$ and $w \notin \dot{S}_E \Rightarrow \phi(\sigma, w) = 0$, with $\Gamma_{E/F}$ acting by left translation on the first factor. This $\Gamma_{E/F}$-module is isomorphic to $\mathrm{Ind}_{\{1\}}^{\Gamma_{E/F}} \mathbb{Z}[S]_0$. ∎

For a $\mathbb{Z}[\Gamma_{E/F}]$-module Y denote $I_{E/F}(Y) = \sum_{\sigma \in \Gamma_{E/F}} (\sigma - 1)(Y)$.

Lemma 2.4.4. *Assume that for any $\sigma \in \Gamma_{E/F}$ there exists $w \in \dot{S}_E$ such that $\sigma w = w$. Consider the functor F_1 (resp. F_2) from the category of $\mathbb{Z}[\Gamma_{E/F}]$-modules to the category of $\mathbb{Z}$-modules which maps Y to $Y[S_E]_0$ (resp. $Y[\dot{S}_E]_0$, considered as the subgroup of $Y[S_E]_0$ consisting of functions supported on $\dot{S}_E$). There exists a morphism of functors $\alpha : F_1 \to F_2$ such that for any $\mathbb{Z}[\Gamma_{E/F}]$-module Y and any $f \in Y[S_E]_0$ we have*

$$f - \alpha_Y(f) \in I_{E/F}(Y[S_E]_0).$$

In particular we have $Y[S_E]_0 = Y[\dot{S}_E]_0 + I_{E/F}(Y[S_E]_0)$ for any $\mathbb{Z}[\Gamma_{E/F}]$-module Y.

Proof. For each $w \in S_E \smallsetminus \dot{S}_E$ choose $\sigma_w \in \Gamma_{E/F}$ such that $\sigma_w w \in \dot{S}_E$ and $\dot{v}_w \in \dot{S}_E$ such that $\sigma_w \dot{v}_w = \dot{v}_w$. Let Y be a $\mathbb{Z}[\Gamma_{E/F}]$-module and $f \in Y[S_E]_0$. Define

$$\alpha_Y(f) = f + \sum_{w \in S_E \smallsetminus \dot{S}_E} (\sigma_w - 1)\big(f(w)\delta_w - f(w)\delta_{\dot{v}_w}\big)$$

It is supported on $\dot{S}_E$. ∎

Proposition 2.4.5. *Assume that for any $\sigma \in \Gamma_{E/F}$ there exists $w \in \dot{S}_E$ such that $\sigma w = w$. The morphism c^{iso} of $\mathbb{Z}[\Gamma_{E/F}]$-modules splits.*

Proof. The $\mathbb{Z}[\Gamma_{E/F}]$-module $M_{E,S}^{\mathrm{iso}}$ is the direct limit of its submodules of finite type, which are free over $\mathbb{Z}$. Thus it is enough to construct for any $\Gamma_{E/F}$-module X that is a finitely generated free abelian group, a section s_X of the map $\mathrm{Hom}(X, M_{E,\dot{S}_E}^{\mathrm{mid}})^{\Gamma_{E/F}} \to$ $\mathrm{Hom}(X, M_{E,S_E}^{\mathrm{iso}})^{\Gamma_{E/F}}$ induced by c^{iso}, such that s_X is functorial in X. Choose a morphism of functors α as in Lemma 2.4.4. Writing $Y = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$ we have a natural identification of $\mathrm{Hom}(X, M_{E,S}^{\mathrm{iso}})^{\Gamma_{E/F}}$ with $Y[S_E]_0^{\Gamma_{E/F}}$ and a natural identification of $\mathrm{Hom}(X, M_{E,S_E}^{\mathrm{mid}})^{\Gamma_{E/F}}$ with the set of functions $f^{\mathrm{mid}} : \Gamma_{E/F} \times S_E \to [E:F]^{-1}Y$ satisfying

- for any $\sigma \in \Gamma_{E/F}$ we have $\sum_{w \in S_E} f^{\mathrm{mid}}(\sigma, w) = 0$,

- for any $w \in S_E$ we have $\sum_{\sigma \in \Gamma_{E/F}} f^{\mathrm{mid}}(\sigma, w) \in Y$,

- f^{mid} is supported on the set of (σ, w) satisfying $\sigma^{-1}w \in \dot{S}_E$.

For $f^{\mathrm{iso}} \in Y[S_E]_0$ we define a function

$$s_X(f^{\mathrm{iso}})(\sigma, w) : \Gamma_{E/F} \times S_E \to [E:F]^{-1}Y$$

by the formula

$$s_X(f^{\mathrm{iso}})(\sigma, w) = [E:F]^{-1}\sigma\big(\alpha_Y(f^{\mathrm{iso}})(\sigma^{-1}w)\big).$$

It satisfies the three conditions above and thus defines a $\Gamma_{E/F}$-equivariant map $X \to M_{E,\dot{S}_E}^{\mathrm{mid}}$: the first and third conditions are clear, the second follows from the equality

$$N_{E/F}\big(\alpha_Y(f^{\mathrm{iso}})\big) = N_{E/F}(f^{\mathrm{iso}}) = [E:F]f^{\mathrm{iso}}. \tag{2.4.2}$$

Thus we have defined a map

$$s_X : \mathrm{Hom}\big(X, M_{E,S}^{\mathrm{iso}}\big)^{\Gamma_{E/F}} \to \mathrm{Hom}\big(X, M_{E,\dot{S}_E}^{\mathrm{mid}}\big)^{\Gamma_{E/F}}.$$

Functoriality in X is clear. The fact that s_X is a section of the map induced by c^{iso} also follows from (2.4.2). ∎

Corollary 2.4.6. *Under the assumption of the proposition the map c^{iso} is surjective.*

2.5. Global inflation maps

We keep the notation of Section 2.4. Let K/F be a finite Galois extension with $E \subset K$, S' a set of places of F containing S, $\dot{S}'_K$ a set of lifts of S' to places of K such that for each $v \in S$ with lift $\dot{v} \in \dot{S}'_K$, the image of $\dot{v}$ in S_E lies in $\dot{S}_E$.

We define three inflation maps. First assume $S = S'$.

(1) $\mathbb{Z}[S_E]_0 \to \mathbb{Z}[S'_K]_0$ by $f^{\mathrm{iso},K}(u) = [K_u : E_w]f^{\mathrm{iso},E}(w)$, where $w \in S_E$ is the unique place under $u \in S_K$.

(2) $M_{E,\dot{S}_E}^{\mathrm{mid}} \to M_{K,\dot{S}_K}^{\mathrm{mid}}$ by $f^{\mathrm{mid},K}(\sigma, u) = f^{\mathrm{mid},E}(\sigma, w)$ provided $\sigma^{-1}u \in \dot{S}_K$, and $f^{\mathrm{mid},K}(\sigma, u) = 0$ otherwise.

(3) $M_{E,\dot{S}_E}^{\mathrm{rig}} \to M_{K,\dot{S}_K}^{\mathrm{rig}}$ by $f^{\mathrm{rig},K}(\sigma, u) = f^{\mathrm{rig},E}(\sigma, w)$ provided $\sigma^{-1}u \in \dot{S}_K$, and $f^{\mathrm{rig},K}(\sigma, u) = 0$ otherwise.

We now drop the assumption $S = S'$ and extend all maps defined above from S_K to S'_K by zero outside of S_K.

These maps are well-defined and fit in the following commutative diagram

$$
\begin{array}{ccccc}
M^{\mathrm{iso}}_{E,S} & \xleftarrow{\ c^{\mathrm{iso}}_{E,S}\ } & M^{\mathrm{mid}}_{E,\dot{S}_E} & \xrightarrow{\ c^{\mathrm{rig}}_{E,S}\ } & M^{\mathrm{rig}}_{E,\dot{S}_E} \\
\downarrow & & \downarrow & & \downarrow \\
M^{\mathrm{iso}}_{K,S'} & \xleftarrow{\ c^{\mathrm{iso}}_{K,S'}\ } & M^{\mathrm{mid}}_{K,\dot{S}'_K} & \xrightarrow{\ c^{\mathrm{rig}}_{K,S'}\ } & M^{\mathrm{rig}}_{K,\dot{S}'_K}
\end{array}
$$

The commutativity of the right square is immediate. The commutativity of the left square follows from the condition that if (σ, w) is in the support of f^E then $\sigma^{-1}w \in \dot{S}_E$.

2.6. Localization maps

Continue with the notation of Section 2.4. Fix $w \in \dot{S}_E$. For each of the three global modules we define localization maps $\mathrm{loc}_w : f \mapsto f_w$ as follows:

(1) $\mathrm{loc}_w : M^{\mathrm{iso}}_{E,S} \to M^{\mathrm{iso}}_{E_w}$, defined by $f_w := f(w)$.
(2) $\mathrm{loc}_w : M^{\mathrm{mid}}_{E,\dot{S}_E} \to M^{\mathrm{mid}}_{E_w,[E:F]}$, defined by $f_w(\sigma) := f(\sigma, w)$.
(3) $\mathrm{loc}_w : M^{\mathrm{rig}}_{E,\dot{S}_E} \to M^{\mathrm{rig}}_{E_w,[E:F]}$, defined by $f_w(\sigma) := f(\sigma, w)$.

These maps fit into the following commutative diagram

$$
\begin{array}{ccccc}
M^{\mathrm{iso}}_{E,S} & \longleftarrow & M^{\mathrm{mid}}_{E,\dot{S}_E} & \longrightarrow & M^{\mathrm{rig}}_{E,\dot{S}_E} \\
\downarrow & & \downarrow & & \downarrow \\
M^{\mathrm{iso}}_{E_w} & \longleftarrow & M^{\mathrm{mid}}_{E_w,[E:F]} & \longrightarrow & M^{\mathrm{rig}}_{E_w,[E:F]}
\end{array}
$$

The commutativity of the right square is immediate, while that of the left is implied by the support condition and the assumption $w \in \dot{S}_E$.

Fact 2.6.1. *The localization maps are compatible with the local and global inflation maps, i.e., in the setting of Section 2.5, for $? \in \{\mathrm{iso}, \mathrm{mid}, \mathrm{rig}\}$ and $w \in \dot{S}'_K \cap S_K$, the following diagram commutes.*

$$
\begin{array}{ccc}
M^{?}_{E,\dot{S}_E} & \longrightarrow & M^{?}_{K,\dot{S}'_K} \\
\downarrow & & \downarrow \\
M^{?}_{E_w,[E:F]} & \longrightarrow & M^{?}_{K_w,[K:F]}
\end{array}
$$

Proof. Immediate. ∎

3. Cohomology

3.1. Preliminary discussion

Let F be a local or global field of characteristic zero. Let $\bar{F}$ be an algebraic closure of F. Let Γ be the Galois group of $\bar{F}/F$. Assume given an inverse system $(D_n)_{n \in \mathbb{N}}$ of diagonalizable groups defined over F, with surjective transition maps, and an inverse system of classes in $H^2(\Gamma, D_n)$. Let $D = \varprojlim D_n$. We endow $D_n(\bar{F})$ with the discrete topology and $D(\bar{F}) = \varprojlim D_n(\bar{F})$ with the inverse limit topology. The continuous cohomology groups $H^n(\Gamma, D)$ are then the cohomology groups of the complex of continuous cochains of Γ valued in the topological group $D(\bar{F})$. This complex is the inverse limit of the complexes of continuous cochains of Γ valued in the discrete groups $D_n(\bar{F})$. Therefore [36, Theorem 3.5.8] gives the exact sequences

$$1 \to R^1 \varprojlim H^0(\Gamma, D_n) \to H^1(\Gamma, D) \to \varprojlim H^1(\Gamma, D_n) \to 1$$

and

$$1 \to R^1 \varprojlim H^1(\Gamma, D_n) \to H^2(\Gamma, D) \to \varprojlim H^2(\Gamma, D_n) \to 1.$$

If $R^1 \varprojlim H^1(\Gamma, D_n)$ vanishes, the inverse system of classes in $H^2(\Gamma, D_n)$ gives an element of $H^2(\Gamma, D)$.

Assume now that we have a class $\xi \in H^2(\Gamma, D)$ and let

$$1 \to D(\bar{F}) \to \mathcal{E} \to \Gamma \to 1$$

be an extension belonging to the corresponding isomorphism class. To be more precise, $\mathcal{E}$ is a topological group, both maps are continuous group morphisms, the topology on $D(\bar{F})$ induced by the embedding in $\mathcal{E}$ coincides with the pro-discrete one defined above, and the map $\mathcal{E} \to \Gamma$ admits a continuous section (not required to be a group morphism)[2]. This is a slight generalization of a well-known interpretation of second cohomology groups (see [29, Theorem 1.2.4] or [4, Section XIV.4]). Explicitly, start from a continuous 2-cocycle ξ that we assume to be normalized for simplicity, i.e., the restrictions of ξ to $\Gamma \times \{1\}$ and $\{1\} \times \Gamma$ are trivial. We may construct an extension $\mathcal{E} = D(\bar{F}) \boxtimes_\xi \Gamma$ by defining the topological space $\mathcal{E}$ as $D(\bar{F}) \times \Gamma$ and defining the multiplication by the formula

$$(x, \sigma) \cdot (y, \tau) = \big(x\sigma(y)\xi(\sigma, \tau), \sigma\tau\big)$$

where $x, y \in D(\bar{F})$ and $\sigma, \tau \in \Gamma$. We will denote a pair (x, σ) as above by $x \boxtimes \sigma$ to emphasize that it should be understood as an element of $D(\bar{F}) \boxtimes_\xi \Gamma$. For simplicity we will abusively denote $D \to \mathcal{E}$ instead of $D(\bar{F}) \to \mathcal{E}$. A morphism of extensions $\mathcal{E} \to \mathcal{E}'$ is a morphism of topological groups making the obvious diagram commute.

---

[2]It would be equivalent to require that the continuous bijection $\mathcal{E}/D \to \Gamma$ is a homeomorphism, i.e., this condition implies that a continuous section exists: this follows from the same argument as in [29, Exercise I.1.4].

For a linear algebraic group G over F we let $\mathcal{E}$ act on $G(\bar{F})$ via the map $\mathcal{E} \to \Gamma$ and the usual action of Γ on $G(\bar{F})$. Following Kottwitz we define the set $Z^1_{\mathrm{alg}}(\mathcal{E}, G)$ to be the set of those continuous 1-cocycles $\mathcal{E} \to G(\bar{F})$ whose restriction to $D(\bar{F})$, which by continuity factors through some projection $D(\bar{F}) \to D_n(\bar{F})$ for some n, is given by an algebraic homomorphism $D_n \to G$. In general this homomorphism is only defined over $\bar{F}$, but its $G(\bar{F})$-conjugacy class is invariant under Γ. Further, for any central algebraic subgroup $Z \subset G$ we define $Z^1(D \to \mathcal{E}, Z \to G)$ to consist of those elements of $Z^1_{\mathrm{alg}}(\mathcal{E}, G)$ whose restriction to D takes image in Z. In that case the resulting homomorphism $D \to Z$ is defined over F. Finally, we set

$$Z^1_{\mathrm{bas}}(\mathcal{E}, G) = Z^1\big(D \to \mathcal{E}, \ Z(G) \to G\big).$$

Realizing $\mathcal{E}$ as $D(\bar{F}) \boxtimes_\xi \Gamma$ as above, an element $\tilde{c}$ of $Z^1_{\mathrm{bas}}(\mathcal{E}, G)$ is concretely described by a pair (μ, c) where $\mu : D \to Z(G)$ is the morphism which on $\bar{F}$-points gives the restriction of $\tilde{c}$ to $D(\bar{F})$ and $c \in C^1(\Gamma, G(\bar{F}))$ is the composition of the section $\Gamma \to \mathcal{E}$, $\sigma \mapsto 1 \boxtimes \sigma$ with $\tilde{c}$. The cocycle condition for $\tilde{c}$ is equivalent to the condition that μ be defined over F and the equality $\mathrm{d}(c) = \mu \circ \xi$. We also define the corresponding cohomology sets

$$H^1(D \to \mathcal{E}, Z \to G) \subset H^1_{\mathrm{bas}}(\mathcal{E}, G) \subset H^1_{\mathrm{alg}}(\mathcal{E}, G)$$

to be the quotients by the action of $G(\bar{F})$ by coboundaries, i.e., g sends $z \in Z^1_{\mathrm{alg}}(\mathcal{E}, G)$ to $e \mapsto g^{-1} z(e) \sigma_e(g)$, where $\sigma_e \in \Gamma$ is the image of $e \in \mathcal{E}$.

A priori the set $H^1_{\mathrm{alg}}(\mathcal{E}, G)$ depends on the choice of the particular extension $\mathcal{E}$ in its isomorphism class. Indeed, if $\mathcal{E}'$ is another extension in the same class, then choosing an isomorphism of extensions $i : \mathcal{E}' \to \mathcal{E}$ provides an isomorphism $Z^1_{\mathrm{alg}}(\mathcal{E}, G) \to Z^1_{\mathrm{alg}}(\mathcal{E}', G)$ by pulling back along i. The induced isomorphism $H^1_{\mathrm{alg}}(\mathcal{E}, G) \to H^1_{\mathrm{alg}}(\mathcal{E}', G)$ depends only on the D-conjugacy class of i. The D-conjugacy classes of automorphisms of the extension $\mathcal{E}$ are parameterized by $H^1(\Gamma, D)$, by associating to $\alpha \in Z^1(\Gamma, D)$ the automorphism $e \mapsto \alpha(\sigma_e)e$ of $\mathcal{E}$ (see [1, Section XIII.1] for a discussion of morphisms of extensions in the case of discrete groups, similar arguments work for topological groups). Therefore, the isomorphism $H^1_{\mathrm{alg}}(\mathcal{E}, G) \to H^1_{\mathrm{alg}}(\mathcal{E}', G)$ is well-defined up to the action of $H^1(\Gamma, D)$ on $H^1_{\mathrm{alg}}(\mathcal{E}, G)$ that is given by the formula $(z \cdot \alpha)(e) = z(\alpha(\sigma_e)e)$.

It is thus clear that when $H^1(\Gamma, D) = 1$ the set $H^1_{\mathrm{alg}}(\mathcal{E}, G)$ is independent of the choice of extension $\mathcal{E}$ in its isomorphism class. In fact, the weaker condition $\varprojlim H^1(\Gamma, D_n) = 1$ turns out to be sufficient. Indeed, by assumption for any $z \in Z^1_{\mathrm{alg}}(\mathcal{E}, G)$ the restriction $z|_D$ factors through the projection $D \to D_n$ for some n and therefore $z(\alpha(\sigma_e)e) = z_n(\alpha_n(\sigma_e))z(e)$, where $z_n : D_n \to G$ composed with $D \to D_n$ equals $z|_D$, and $\alpha_n \in Z^1(\Gamma, D_n)$ is the image of $\alpha \in Z^1(\Gamma, D)$.

Assume now that D'_n is another inverse system of diagonalizable groups defined over F with surjective transition maps and that we are given homomorphisms $D'_n \to D_n$ compatible with the transition maps. These splice to a homomorphism $D' \to D$, where $D' = \varprojlim D'_n$. Assume that we are given a class $\xi' \in H^2(\Gamma, D')$ and let $\mathcal{E}'$ be the corresponding extension. If ξ' maps to ξ under the homomorphism $D' \to D$ then there exists

a homomorphism of extensions $\mathcal{E}' \to \mathcal{E}$, as above uniquely determined up to $Z^1(\Gamma, D)$. This homomorphism induces a map $H^1_{\mathrm{alg}}(\mathcal{E}, G) \to H^1_{\mathrm{alg}}(\mathcal{E}', G)$. As above, this map is well-defined if $\varprojlim H^1(\Gamma, D_n) = 1$.

We have thus seen that the vanishing of $R^i \varprojlim H^1(\Gamma, D_n)$ for $i = 0, 1$ has desirable consequences. A sufficient condition for the vanishing of both of these is the following: For any n there exists $m > n$ such that the map $H^1(\Gamma, D_m) \to H^1(\Gamma, D_n)$ is zero.

Fact 3.1.1. *Let $D \to \mathcal{E}$ be an extension of Γ as considered above, whose isomorphism class corresponds to $\xi \in H^2(\Gamma, D)$. Let G be a linear algebraic group over F and Z a central subgroup. We have the inflation-restriction exact sequence of pointed sets (abelian groups if G is abelian)*

$$1 \to H^1(\Gamma, G) \to H^1(D \to \mathcal{E}, Z \to G) \to \mathrm{Hom}_F(D, Z) \to H^2(\Gamma, G),$$

where $H^2(\Gamma, G)$ is considered only when G is abelian and in this case the last arrow is $\phi \mapsto \phi \circ \xi$.

Proof. See [10, (3.5) and Lemma 3.3]. ∎

Fact 3.1.2. *For any torus T with co-character module Y we have isomorphisms*

$$\left(Y \otimes X^*(D)\right)^\Gamma \to \mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{Hom}_{\mathbb{Z}}(Y, \mathbb{Z}), X^*(D)\right)^\Gamma \to \mathrm{Hom}_F(D, T), \qquad (3.1.1)$$

where the second map is the usual duality and the first map is the restriction of the Γ-equivariant isomorphism

$$Y \otimes X^*(D) \to \mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{Hom}_{\mathbb{Z}}(Y, \mathbb{Z}), X^*(D)\right)$$
$$y \otimes a \mapsto \left(\varphi \mapsto \varphi(y)a\right).$$

Fact 3.1.3. *Let $\mathcal{E}' \to \mathcal{E}$ be a morphism of extensions of Γ as considered above. For any algebraic group G and any central algebraic subgroup Z the square*

$$
\begin{array}{ccc}
H^1(D \to \mathcal{E}, Z \to G) & \longrightarrow & \mathrm{Hom}_F(D, Z) \\
\downarrow & & \downarrow \\
H^1(D' \to \mathcal{E}', Z \to G) & \longrightarrow & \mathrm{Hom}_F(D', Z)
\end{array}
$$

is Cartesian.

Proof. This follows directly from the fact that $\mathcal{E}$ is generated by D and the image of $\mathcal{E}'$ which have intersection the image of D'. ∎

3.2. Definition of $\mathbb{T}^{\mathrm{iso}}$, $\mathbb{T}^{\mathrm{mid}}$, and P^{rig} in the local case

Let F be a local field of characteristic zero, E/F a finite Galois extension, N a natural number.

Let $\mathbb{T}_E^{\mathrm{iso}}$ and $\mathbb{T}_{E,N}^{\mathrm{mid}}$ be the tori with character modules M_E^{iso} and $M_{E,N}^{\mathrm{mid}}$. Let $P_{E,N}^{\mathrm{rig}}$ be the finite multiplicative group with character module $M_{E,N}^{\mathrm{rig}}$. The torus $\mathbb{T}_E^{\mathrm{iso}}$ is simply $\mathbb{G}_m$. The finite multiplicative group $P_{E,N}^{\mathrm{rig}}$ was defined in [10, Section 3.1], where it was denoted by $u_{E/F,N}$.

Let $\mathbb{T}^{\mathrm{iso}}$ and $\mathbb{T}^{\mathrm{mid}}$ be the pro-tori obtained as inverse limits of the systems $\mathbb{T}_E^{\mathrm{iso}}$ and $\mathbb{T}_{E,N}^{\mathrm{mid}}$ respectively, where the transition maps are induced by the inflation maps defined in Section 2.2. Let P^{rig} be the pro-finite multiplicative group obtained as the inverse limit of $P_{E,N}^{\mathrm{rig}}$ in the same manner. The pro-torus $\mathbb{T}^{\mathrm{iso}}$ was denoted by $\mathbb{D}$ in [17, Section 3], while the group P^{rig} was denoted by u in [10, Section 3.1].

3.3. Definition of $\mathbb{T}^{\mathrm{iso}}$, $\mathbb{T}_{\dot{V}}^{\mathrm{mid}}$, and $P_{\dot{V}}^{\mathrm{rig}}$ in the global case

Let F be a global field, E/F a finite Galois extension, S a finite set of places of S, $\dot{S}_E \subset S_E$ a set of lifts for the elements of S. Let $\mathbb{T}_{E,S}^{\mathrm{iso}}$ and $\mathbb{T}_{E,\dot{S}_E}^{\mathrm{mid}}$ be the tori over F with character modules $M_{E,S}^{\mathrm{iso}}$ and $M_{E,\dot{S}_E}^{\mathrm{mid}}$. Let $P_{E,\dot{S}_E}^{\mathrm{rig}}$ be the finite multiplicative group with character module $M_{E,\dot{S}_E}^{\mathrm{rig}}$. In [11] this was denoted by $P_{E,\dot{S}_E,[E:F]}$. Note that in [11] $P_{E,\dot{S}_E}$ was used to denote

$$\varprojlim_N P_{E,\dot{S}_E,N}$$

where $P_{E,\dot{S}_E,[E:F]}$ is the finite multiplicative group denoted by $P_{E,\dot{S}_E}^{\mathrm{rig}}$ in the present paper. Since for comparison with $?^{\mathrm{mid}}$ we usually impose that this integer N equal $[E:F]$ in the present paper, we hope that this will not cause confusion.

We now choose as in [11, Section 3.3, p. 306] an exhaustive tower $(E_i)_{i \geq 0}$ of finite Galois extensions of F, exhaustive tower of finite sets of places of F, $\dot{S}_i \subset S_{i,E_i}$ a choice of lifts of S_i to places of E_i so that $\dot{S}_{i+1} \subset (\dot{S}_i)_{E_{i+1}}$ and each $(E_i/F, S_i, \dot{S}_i)$ satisfies Conditions 2.4.1. Let $\dot{V}$ be the set of places of $\bar{F}$ defined as the inverse limit of the sets $\dot{S}_i$. Thus, $\dot{V}$ is a set of lifts to $\bar{F}$ of the set V of all places of F. It is natural to ask if it is possible to formulate a condition on $\dot{V}$ that is equivalent to the fact that it arises as an inverse limit of a sequence $(E_i, S_i, \dot{S}_i)$ all of whose terms satisfy Conditions 2.4.1. This is indeed possible. For $v \in V$ denote by $\dot{v} \in \dot{V}$ its unique lift, and by $\Gamma_{\dot{v}}$ the stabilizer of $\dot{v}$ in Γ, i.e., the decomposition subgroup at $\dot{v}$.

Condition 3.3.1. $\bigcup_{v \in V} \Gamma_{\dot{v}}$ is dense in Γ.

Lemma 3.3.2. *Let $(E_i)_{i \geq 0}$ be an exhaustive tower of finite Galois extensions of F as in* [11, Section 3.3, p. 306].

(1) *Let $(S_i)_{i \geq 0}$ be an exhaustive tower of finite sets of places of F, $\dot{S}_i \subset S_{i,E_i}$ a choice of lifts of S_i to places of E_i so that $\dot{S}_{i+1} \subset (\dot{S}_i)_{E_{i+1}}$ and each $(E_i/F, S_i, \dot{S}_i)$ satisfies Conditions 2.4.1. Let $\dot{V}$ be the set of places of $\bar{F}$ defined as the inverse limit of the sets $\dot{S}_i$. Then Condition 3.3.1 holds for $\dot{V}$.*

(2) *If $\dot{V}$ is a set of lifts to $\bar{F}$ of the set V of all places of F, and $\dot{V}$ satisfies Condition 3.3.1, we can choose a finite increasing sequence $(S_i)_{i \geq 0}$ of subsets of V such that letting $\dot{S}_i$ be the intersection of $(S_i)_{E_i}$ with the image of $\dot{V}$ in V_{E_i}, the tower $(E_i, S_i, \dot{S}_i)_{i \geq 0}$ satisfies Conditions 2.4.1.*

Proof. If $(E_i/F, S_i, \dot{S}_i)$ satisfies Conditions 2.4.1 then for any i the image of $\bigcup_{v \in V} \Gamma_{\dot{v}}$ in $\Gamma_{E_i/F}$ contains $\bigcup_{v \in S_i} \Gamma_{E_{i,\dot{v}}/F_v} = \Gamma_{E_i/F}$ by the third point of Conditions 2.4.1. Since $\Gamma_{\bar{F}/E_i}$ is a basis of neighbourhoods of 1 in Γ, this means that $\bigcup_{v \in V} \Gamma_{\dot{v}}$ is dense in Γ.

The proof of the converse is similar: since all $\Gamma_{E_i/F}$ are finite any sufficiently large S_i works. ∎

In particular, this shows that sets $\dot{V}$ that satisfy Condition 3.3.1 do exist. This condition is however not automatic. Furthermore, two sets $\dot{V}$ and $\dot{V}'$ that both satisfy Condition 3.3.1 need not be conjugate under Γ. We illustrate both of these points in the following example.

Example 3.3.3. Take $F = \mathbb{Q}$ and let E/F be the extension generated by all roots of the polynomial $P = X^3 - X^2 + 1$. Then $\Gamma_{E/F} \simeq S_3$ and E/F is ramified only at 23, in fact $A := \mathbb{Z}[1/23][X]/(P)$ is finite étale over $\mathbb{Z}[1/23]$: $P' = 3X^2 - 2X = (3X - 2)X$, X is obviously invertible in A and $(9X^2 - 3X - 2)(3X - 2) = 27P - 23$. Modulo 23 we have $P(-1/3) = 0$ and $P'(-1/3) \neq 0$ and so P has a root in $\mathbb{Q}_{23}$. In particular all decomposition subgroups of $\Gamma_{E/F}$ are Abelian. Fix an isomorphism $\Gamma_{E/F} \simeq S_3$. One can choose $\dot{V}$ such that every decomposition group is either trivial, or generated by (12), or generated by (123), and thus (23) does not belong to any decomposition group.

Using the same extension E, we can give an example of two sets $\dot{V}$ and $\dot{V}'$ both satisfying Condition 3.3.1 but which are not in the same Γ-orbit. Namely, choose two places v_1, v_2 of F such that the decomposition groups in $\Gamma_{E/F}$ both have order two. Then we can choose $\dot{V}_E$ and $\dot{V}'_E$ such that $\Gamma_{E_{\dot{v}_1}/F_{v_1}} = \Gamma_{E_{\dot{v}'_1}/F_{v_1}} = \Gamma_{E_{\dot{v}_2}/F_{v_2}}$ but $\Gamma_{E_{\dot{v}'_2}/F_{v_2}} \neq \Gamma_{E_{\dot{v}_2}/F_{v_2}}$, so that even after conjugating by $\Gamma_{E/F}$ we cannot have $\Gamma_{E_{\dot{v}_i}/F_{v_i}} = \Gamma_{E_{\dot{v}'_i}/F_{v_i}}$ for $i = 1, 2$ simultaneously.

For the rest of the paper we fix $\dot{V}$ satisfying Condition 3.3.1.

Let $\mathbb{T}^{\mathrm{iso}}$ be the pro-torus over F obtained as the inverse limits of $\mathbb{T}^{\mathrm{iso}}_{E,S}$ over all pairs (E, S) as above. In the other two cases the result depends on $\dot{V}$. For each finite Galois extension E/F and each finite set of places S of F we let $\dot{S}_E = \{\dot{v}|_E \mid v \in S\}$. Consider the pro-torus

$$\mathbb{T}^{\mathrm{mid}}_{\dot{V}} = \varprojlim_{E,S} \mathbb{T}^{\mathrm{mid}}_{E,\dot{S}_E}$$

and the pro-finite group scheme $P^{\mathrm{rig}}_{\dot{V}} = \varprojlim_{E,S} P^{\mathrm{rig}}_{E,\dot{S}_E}$. Note that

$$M^{\mathrm{mid}}_{\dot{V}} = X^*(\mathbb{T}^{\mathrm{mid}}_{\dot{V}}) = \varinjlim_{E,S} M^{\mathrm{mid}}_{E,\dot{S}_E}$$

is identified with the Γ-module of functions $\phi : \Gamma \times V \to \mathbb{Q}$ continuous in the first variable and with finite support in the second variable such that for any $\sigma \in \Gamma$, $\sum_{v \in V} \phi(\sigma, v) = 0$ and for any Archimedean place $v \in V$, $\sum_{\tau \in \Gamma_{\dot{v}}} \phi(\sigma\tau, v) \in \mathbb{Z}$. This identification is obtained by mapping $f \in M^{\mathrm{mid}}_{E,\dot{S}_E}$ to ϕ defined by $\phi(\sigma, v) = f(\sigma, \sigma\dot{v})$. This description is similar to [11, Lemma 3.4.1].

The set of lifts $\dot{V}$ being fixed, for $v \in V$ we simply denote $\Gamma_v = \Gamma_{\dot{v}}$. Denote $\overline{F_v} = \varinjlim_E E_{\dot{v}}$ where we take the limit over all finite extensions of F in $\bar{F}$. This is an algebraic

closure of F_v, strictly smaller than the completion of $\overline{F}$ for $\dot{v}$ if v is non-Archimedean. It is easy to check that the localization maps defined in Section 2.6 induce localization maps at infinite level

$$\mathbb{T}_v^{\mathrm{iso}} \to (\mathbb{T}^{\mathrm{iso}})_{F_v}, \quad \mathbb{T}_v^{\mathrm{mid}} \to (\mathbb{T}_{\dot{V}}^{\mathrm{mid}})_{F_v}, \quad \text{and} \quad P_v^{\mathrm{rig}} \to (P_{\dot{V}}^{\mathrm{rig}})_{F_v}$$

where $\mathbb{T}_v^{\mathrm{iso}}$ (resp. $\mathbb{T}_v^{\mathrm{mid}}$, P_v^{rig}) denotes the pro-torus (resp. pro-torus, pro-finite group scheme) over F_v defined in Section 3.2 for the local field F_v together with its algebraic closure $\overline{F_v}$, and a subscript F_v denotes base change from F to F_v. We will denote these three localization maps by loc_v.

3.4. The maps $\mathbb{T}^{\mathrm{iso}} \to \mathbb{T}^{\mathrm{mid}} \leftarrow P^{\mathrm{rig}}$

The maps c^{rig} and c^{iso} defined in the local case in Section 2.1 and in the global case in Section 2.4 splice together to define Cartier dual maps $c_{\mathrm{iso}} : \mathbb{T}^{\mathrm{iso}} \to \mathbb{T}^{\mathrm{mid}}$ and $c_{\mathrm{rig}} : P_{\dot{V}}^{\mathrm{rig}} \to \mathbb{T}_{\dot{V}}^{\mathrm{mid}}$.

Let F be a local or global field.

Proposition 3.4.1. *Let T be an algebraic torus defined over F.*

(1) *The map $c_{\mathrm{iso}} : \mathbb{T}^{\mathrm{iso}} \to \mathbb{T}_{\dot{V}}^{\mathrm{mid}}$ is injective. The homomorphism $\mathrm{Hom}_F(\mathbb{T}_{\dot{V}}^{\mathrm{mid}}, T) \to \mathrm{Hom}_F(\mathbb{T}^{\mathrm{iso}}, T)$ is surjective.*

(2) *The map $c_{\mathrm{rig}} : P_{\dot{V}}^{\mathrm{rig}} \to \mathbb{T}_{\dot{V}}^{\mathrm{mid}}$ is injective. The homomorphism $\mathrm{Hom}_F(\mathbb{T}_{\dot{V}}^{\mathrm{mid}}, T) \to \mathrm{Hom}_F(P_{\dot{V}}^{\mathrm{rig}}, T)$ is surjective.*

Proof. In the local case the injectivity claims follow from Fact 2.1.1, while in the global case they follow from Corollary 2.4.6 and Proposition 2.4.2.

We prove the surjectivity of $\mathrm{Hom}_F(\mathbb{T}_{\dot{V}}^{\mathrm{mid}}, T) \to \mathrm{Hom}_F(\mathbb{T}^{\mathrm{iso}}, T)$. In the local case, it follows immediately from the existence of the splitting (2.1.2). In the global case, Proposition 2.4.5 implies that $\mathrm{Hom}_F(\mathbb{T}_{E, \dot{S}_E}^{\mathrm{mid}}, T) \to \mathrm{Hom}_F(\mathbb{T}_{E,S}^{\mathrm{iso}}, T)$ is surjective for any $E, \dot{S}_E$. The surjectivity of $\mathrm{Hom}_F(\mathbb{T}_{\dot{V}}^{\mathrm{mid}}, T) \to \mathrm{Hom}_F(\mathbb{T}^{\mathrm{iso}}, T)$ follows by taking the colimit over $E, \dot{S}_E$.

We prove the surjectivity of $\mathrm{Hom}_F(\mathbb{T}_{\dot{V}}^{\mathrm{mid}}, T) \to \mathrm{Hom}_F(P_{\dot{V}}^{\mathrm{rig}}, T)$. Consider first the global case. Let $X = X^*(T)$. We claim that every $\mathbb{Z}[\Gamma]$-homomorphism $f : X \to M_{E, \dot{S}_E}^{\mathrm{rig}}$ lifts to a homomorphism $\dot{f} : X \to M_{E, \dot{S}_E}^{\mathrm{mid}}$. Since X is $\mathbb{Z}$-free we can choose a lift $\ddot{f} : X \to M_{E, \dot{S}_E}^{\mathrm{mid}}$ that is a homomorphism of $\mathbb{Z}$-modules, but not necessarily Γ-equivariant. Then $\sigma \mapsto \ddot{f} - \sigma(\ddot{f})$ is a 1-cocycle of Γ in $\mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Ker}(c_{E, \dot{S}_E}^{\mathrm{rig}}))$. By Fact 2.4.3 and [31, Chapter IX, Section 3, Proposition] this is an induced Γ-module, so $\sigma \mapsto \ddot{f} - \sigma(\ddot{f})$ is a coboundary, implying that there exists a Γ-equivariant lift $\dot{f}$ of f. This completes the proof in the global case. The proof in the local case is the same, but now based on Fact 2.1.1 in place of Fact 2.4.3. ∎

Proposition 3.4.2. *In the global case, the map $c_{\mathrm{iso}} : \mathbb{T}^{\mathrm{iso}} \to \mathbb{T}_{\dot{V}}^{\mathrm{mid}}$ splits.*

Proof. We seek a compatible family of splittings of $c_{\mathrm{iso}} : \mathbb{T}_{E_i}^{\mathrm{iso}} \to \mathbb{T}_{E_i, \dot{V}_{E_i}}^{\mathrm{mid}}$. As we saw in the proof of Proposition 2.4.5, giving such a splitting is equivalent to giving, for any torus T defined over F and split by E_i, a splitting s_i of

$$(Y \otimes M_{E_i, \dot{V}_{E_i}}^{\mathrm{mid}})^{\Gamma} = \mathrm{Hom}(\mathbb{T}_{E_i, \dot{V}_{E_i}}^{\mathrm{mid}}, T) \to \mathrm{Hom}(\mathbb{T}^{\mathrm{iso}}, T) = (Y[V_{E_i}]_0)^{\Gamma},$$

where $Y = X_*(T)$, which is functorial in T.

It is convenient to let $E_{-1} = F$. For $k \geq 0$ and $v \in V$ choose $R_{k,v} \subset \Gamma_{\bar{F}/E_{k-1}}$ representing $\Gamma_{E_k/E_{k-1}}/\Gamma_{E_k,\dot{v}/E_{k-1},\dot{v}}$. For $w \in V_{E_k} \smallsetminus \dot{V}_{E_k}$ such that $w_{E_{k-1}} \in \dot{V}_{E_{k-1}}$, let $r(k, w) \in R_{k,w_F}$ be the element such that $r(k, w)^{-1}w \in \dot{V}_{E_k}$, and choose $\dot{v}(k, w) \in \dot{V}$ such that the image of $r(k, w)$ in $\Gamma_{E_k/E_{k-1}}$ belongs to the decomposition subgroup for $\dot{v}(k, w)_{E_k}$. For $i \geq k \geq 0$ denote $\dot{V}_{i,k} = \{w \in V_{E_i} \mid w_{E_k} \in \dot{V}_{E_k}\}$. For $f \in Y[\dot{V}_{i,k-1}]$ let

$$\pi_{i,k}(f) = f + \sum_{w \in \dot{V}_{i,k-1} \smallsetminus \dot{V}_{i,k}} \big(r(k, w) - 1\big)\big(f(w)\delta_w - f(w)\delta_{\dot{v}(k,w)_{E_i}}\big).$$

It is clear that $\pi_{i,k}(f)$ is supported on $\dot{V}_{i,k}$ and that $\pi_{i,k}(f) - f \in I(Y[V_{E_i}]_0)$. Define

$$\pi_i := \pi_{i,i} \circ \pi_{i,i-1} \circ \cdots \circ \pi_{i,0} : Y[V_{E_i}] \to Y[\dot{V}_{E_i}].$$

As in [20, Section 8.3] we denote $p : Y[S_{E_i}] \to Y[S_{E_{i+1}}]$ for the inflation map and $j : Y[S_{E_{j+1}}] \to Y[S_{E_j}]$ defined by $j(f)(w) = \sum_{u|w} f(u)$, i.e., $j(\delta_u) = \delta_{u_{E_i}}$. They are both $\Gamma_{E_{i+1}/F}$-equivariant and satisfy $j \circ p = [E_{i+1} : E_i]$. It is easy to check that for $f \in Y[\dot{V}_{i+1,k-1}]$ we have

$$j\big(\pi_{i+1,k}(f)\big) = \begin{cases} \pi_{i,k}\big(j(f)\big) & \text{if } k \leq i \\ j(f) & \text{if } k = i + 1 \end{cases}$$

and thus $j \circ \pi_{i+1} = \pi_i \circ j$. Now $\pi_{i+1}p(f)$ is supported on $\dot{V}_{E_{i+1}}$ and satisfies

$$j\pi_{i+1}p(f) = [E_{i+1} : E_i]\pi_i(f),$$

and so for $w \in V_{E_{i+1}}$ we have

$$\pi_{i+1}p(f)(w) = [E_{i+1} : E_i]\pi_i(f)(w_{E_i}). \tag{3.4.1}$$

Now we can resume the proof of Proposition 2.4.5 with $\dot{f} = \pi_i(f)$ for $f \in (Y[V_{E_i}]_0)^{\Gamma}$, defining $s_i(f) \in (Y \otimes M_{E_i, \dot{V}_{E_i}}^{\mathrm{mid}})^{\Gamma}$ by

$$s_i(f)(\sigma, w) = \begin{cases} [E_i : F]^{-1}\sigma\big(\pi_i(f)(\sigma^{-1}w)\big) & \text{if } \sigma^{-1}w \in \dot{V}_{E_i} \\ 0 & \text{otherwise} \end{cases}$$

for $\sigma \in \Gamma_{E_i/F}$ and $w \in V_{E_i}$. Now (3.4.1) implies that $s_{i+1}(p(f)) \in (Y \otimes M_{E_{i+1}, \dot{V}_{E_{i+1}}}^{\mathrm{mid}})^{\Gamma}$ is the inflation of $s_i(f) \in (Y \otimes M_{E_i, \dot{V}_{E_i}}^{\mathrm{mid}})^{\Gamma}$. ∎

Remark 3.4.3. Composing such a splitting $\mathbb{T}_{\dot{V}}^{\mathrm{mid}} \to \mathbb{T}^{\mathrm{iso}}$ with c_{rig}, we obtain a map $P_{\dot{V}}^{\mathrm{rig}} \to \mathbb{T}^{\mathrm{iso}}$. Unfortunately, this splitting is not canonical, since we had to choose sets of representatives $R_{k,v}$. Therefore we cannot use it to compare $?^{\mathrm{iso}}$ and $?^{\mathrm{rig}}$ directly, as in the local case, which is why $?^{\mathrm{mid}}$ was introduced.

3.5. Review of the Tate–Nakayama isomorphism

Let E/F be a Galois extension of local fields of characteristic zero, T an algebraic torus defined over F and split over E, $Y = X_*(T)$. We have the Tate–Nakayama isomorphism $\hat{H}^i(\Gamma_{E/F}, Y) \to \hat{H}^{i+2}(\Gamma_{E/F}, T(E))$ defined by cup product against the fundamental class in $H^2(\Gamma_{E/F}, E^\times)$. Combining with the inflation $H^i(\Gamma_{E/F}, T(E)) \to H^i(\Gamma, T)$ we obtain the isomorphism

$$\hat{H}^{-1}(\Gamma_{E/F}, Y) \to H^1(\Gamma, T)$$

and the inclusion $\hat{H}^0(\Gamma_{E/F}, Y) \to H^2(\Gamma, T)$.

Given a finite multiplicative group Z defined over F and split over E, we let $A = X^*(Z)$ and $A^\vee = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$, and then have the injective map $\hat{H}^{-1}(\Gamma_{E/F}, A^\vee) \to H^2(\Gamma, Z)$ denoted by $\Theta_{E,v}$ in [11, Section 3.2].

If $1 \to Z \to T \to \bar{T} \to 1$ is an exact sequence of diagonalizable groups defined over F and split over E, where Z is finite and T and $\bar{T}$ are tori, then these maps fit in the following commutative diagram, which is the local analog of Lemma [11, Lemma 3.2.5] whose proof is easier and shall be omitted:

$$
\begin{array}{ccccc}
\hat{H}^{-1}(\Gamma_{E/F}, Y) & \xrightarrow[\mathrm{TN}]{\cong} & H^1(\Gamma_{E/F}, T(E)) & \xrightarrow{\cong} & H^1(\Gamma, T(\bar{F})) \\
\downarrow & & \downarrow & & \downarrow \\
\hat{H}^{-1}(\Gamma_{E/F}, \bar{Y}) & \xrightarrow[\mathrm{TN}]{\cong} & H^1(\Gamma_{E/F}, \bar{T}(E)) & \xrightarrow{\cong} & H^1(\Gamma, \bar{T}(\bar{F})) \\
\downarrow & & & & \downarrow \\
\hat{H}^{-1}(\Gamma_{E/F}, A^\vee) & \xrightarrow{\hspace{3cm}\Theta_E\hspace{3cm}} & & & H^2(\Gamma, Z(\bar{F})) \\
\downarrow & & & & \downarrow \\
\hat{H}^0(\Gamma_{E/F}, Y) & \xrightarrow[-\mathrm{TN}]{\cong} & H^2(\Gamma_{E/F}, T(E)) & \hookrightarrow & H^2(\Gamma, T(\bar{F})) \\
\downarrow & & \downarrow & & \downarrow \\
\hat{H}^0(\Gamma_{E/F}, \bar{Y}) & \xrightarrow[-\mathrm{TN}]{\cong} & H^2(\Gamma_{E/F}, \bar{T}(E)) & \hookrightarrow & H^2(\Gamma, \bar{T}(\bar{F}))
\end{array}
$$

Let E/F be a Galois extension of number fields, S a finite set of places of F containing all Archimedean places and all finite places ramifying in E/F and such that every ideal class of E contains an ideal with support in S (i.e., [11, Conditions 3.1.1]). Given an algebraic torus T defined over F and split over E we let $Y = X_*(T)$ and $Y[S_E]_0 = Y \otimes \mathbb{Z}[S_E]_0$. We have the Tate–Nakayama isomorphism

$$\hat{H}^i(\Gamma_{E/F}, Y[S_E]_0) \to \hat{H}^{i+2}(\Gamma_{E/F}, T(O_{E,S}))$$

defined by cup product against the fundamental class in

$$H^2\big(\Gamma_{E/F}, \mathrm{Hom}_{\mathbb{Z}}\big(\mathbb{Z}[S_E]_0, O^{\times}_{E,S}\big)\big)$$

defined in [34]. Combining with the inflation $H^i(\Gamma_{E/F}, T(O_{E,S})) \to H^i(\Gamma_S, T(O_S))$ we obtain the isomorphism

$$\hat{H}^{-1}(\Gamma_{E/F}, Y) \to H^1\big(\Gamma_S, T(O_S)\big)$$

and the inclusion $\hat{H}^0(\Gamma_{E/F}, Y) \to H^2(\Gamma_S, T(O_S))$, see [11, Lemma 3.1.9].

Given a finite multiplicative group Z defined over F and split over E, we let $A = X^*(Z)$ and $A^{\vee} = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$, and then have the injective map

$$\hat{H}^{-1}\big(\Gamma_{E/F}, A^{\vee}[S_E]_0\big) \to H^2(\Gamma, Z)$$

denoted by $\Theta_{E,S}$ in [11, Section 3.2].

If $1 \to Z \to T \to \bar{T} \to 1$ is an exact sequence of diagonalizable groups defined over F and split over E, where Z is finite and T and $\bar{T}$ are tori, and the order of Z is an S-unit, then these maps fit in the following commutative diagram according to Lemma [11, Lemma 3.2.5]:

$$
\begin{array}{ccccc}
\hat{H}^{-1}\big(\Gamma_{E/F}, Y[S_E]_0\big) & \xrightarrow[\mathrm{TN}]{\cong} & H^1\big(\Gamma_{E/F}, T(O_{E,S})\big) & \xrightarrow{\cong} & H^1\big(\Gamma_S, T(O_S)\big) \\
\downarrow & & \downarrow & & \downarrow \\
\hat{H}^{-1}\big(\Gamma_{E/F}, \bar{Y}[S_E]_0\big) & \xrightarrow[\mathrm{TN}]{\cong} & H^1\big(\Gamma_{E/F}, \bar{T}(O_{E,S})\big) & \xrightarrow{\cong} & H^1\big(\Gamma_S, \bar{T}(O_S)\big) \\
\downarrow & & & & \downarrow \\
\hat{H}^{-1}\big(\Gamma_{E/F}, A^{\vee}[S_E]_0\big) & & \xrightarrow{\Theta_{E,S}} & & H^2\big(\Gamma_S, Z(O_S)\big) \\
\downarrow & & & & \downarrow \\
\hat{H}^0\big(\Gamma_{E/F}, Y[S_E]_0\big) & \xrightarrow[-\mathrm{TN}]{\cong} & H^2\big(\Gamma_{E/F}, T(O_{E,S})\big) & \hookrightarrow & H^2\big(\Gamma_S, T(O_S)\big) \\
\downarrow & & \downarrow & & \downarrow \\
\hat{H}^0\big(\Gamma_{E/F}, \bar{Y}[S_E]_0\big) & \xrightarrow[-\mathrm{TN}]{\cong} & H^2\big(\Gamma_{E/F}, \bar{T}(O_{E,S})\big) & \hookrightarrow & H^2\big(\Gamma_S, \bar{T}(O_S)\big)
\end{array}
$$

Consider now a morphism $Z \to T$ from a finite multiplicative group Z to a torus T, not assumed injective. It induces a homomorphism $H^2(\Gamma, Z) \to H^2(\Gamma, T)$ in the local case, and $H^2(\Gamma_S, Z(O_S)) \to H^2(\Gamma_S, T(O_S))$ in the global case. We shall now define a homomorphism

$$H_T^{-1}(\Gamma_{E/F}, A^{\vee}) \to H_T^0(\Gamma_{E/F}, Y)$$

in the local case, and $H_T^{-1}(\Gamma_{E/F}, A^{\vee}[S_E]_0) \to H_T^0(\Gamma_{E/F}, Y[S_E]_0)$ in the global case, that intertwines the respective Tate–Nakayama homomorphisms.

We first consider the local case. Let $\bar{T}$ be the cokernel of $Z \to T$, an algebraic torus that is a quotient of T. We consider the complex $X^*(\bar{T}) \to X^*(T) \to X^*(Z)$. Given $\lambda : X^*(Z) \to \mathbb{Q}/\mathbb{Z}$ we compose to obtain $\lambda_T : X^*(T) \to \mathbb{Q}/\mathbb{Z}$. Since $X^*(T)$ is a free

$\mathbb{Z}$-module there exists a lift $\dot{\lambda}_T : X^*(T) \to \mathbb{Q}$ whose restriction to $X^*(\overline{T})$ necessarily takes image in $\mathbb{Z}$ and thus is an element of $X_*(\overline{T})$, well-defined modulo $X_*(T)$. Consider

$$N\dot{\lambda}_T = \sum_{\sigma \in \Gamma_{E/F}} \sigma(\dot{\lambda}_T) \in X_*(\overline{T})^\Gamma.$$

If we assume that $N\lambda = 0$ then we see that $N\dot{\lambda}_T$ belongs to the sublattice $Y = X_*(T) \subset X_*(\overline{T})$. This defines a map $H_T^{-1}(\Gamma_{E/F}, A^\vee) \to H_T^0(\Gamma_{E/F}, Y)$.

In the global case, the definition of $H_T^{-1}(\Gamma_{E/F}, A^\vee[S_E]_0) \to H_T^0(\Gamma_{E/F}, Y[S_E]_0)$ is analogous. Now instead of $\lambda : A \to \mathbb{Q}/\mathbb{Z}$ we have $\lambda : A \times S_E \to \mathbb{Q}/\mathbb{Z}$ that is a homomorphism in the first variable, and with $\sum_w \lambda(a, w) = 0$. We can choose a lift $\dot{\lambda}_T : X^*(T) \times S_E \to \mathbb{Q}$ which is also a homomorphism in the first variable and with $\sum_w \dot{\lambda}_T(x, w) = 0$, i.e., $\dot{\lambda}_T \in \mathbb{Q}Y[S_E]_0$. Again $\dot{\lambda}_T$ is well-defined modulo $Y[S_E]_0$, and if $N\lambda = 0$ then $N\dot{\lambda}_T \in Y[S_E]_0$.

The fact that these maps are compatible with the Tate–Nakayama homomorphisms is proved as follows. Define $Z' \subset T$ to be the image of Z and write $Z \to T$ as the composition of the surjective homomorphism $Z \to Z'$ and the injective homomorphism $Z' \to T$. For $Z \to Z'$ one applies the functoriality of the Tate–Nakayama homomorphism for finite multiplicative groups, and for $Z' \to T$ one uses [11, Lemma 3.2.5] and its local analog.

3.6. The local gerbes $\mathscr{E}^{\mathrm{iso}}$, $\mathscr{E}^{\mathrm{mid}}$, $\mathscr{E}^{\mathrm{rig}}$

Let F be a local field of characteristic zero, E/F a finite Galois extension, N a natural number.

There is a canonical element $\xi_E^{\mathrm{iso}} \in H^2(\Gamma, \mathbb{T}_E^{\mathrm{iso}}(\overline{F}))$: it is the element that, under the identification $\mathbb{T}_E^{\mathrm{iso}} = \mathbb{G}_m$ and the invariant map of local class field theory $H^2(\Gamma, \overline{F}^\times) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$ corresponds to $[E : F]^{-1}$, i.e., the inflation of the canonical class in $H^2(\Gamma_{E/F}, E^\times)$.

There is also a canonical element $\xi_{E,N}^{\mathrm{rig}} \in H^2(\Gamma, P_{E,N}^{\mathrm{rig}})$. It is obtained by taking $-1 \in \mathbb{Z}$, using the identification $\widehat{\mathbb{Z}} = H^2(\Gamma, u)$ of [10, Theorem 3.1], and mapping this class under the map $u \to u_{E/F,N}$.

Proposition 3.6.1. *The images of the canonical classes ξ_E^{iso} and $\xi_{E,N}^{\mathrm{rig}}$ under*

$$H^2\big(\Gamma, \mathbb{T}_E^{\mathrm{iso}}(\overline{F})\big) \to H^2\big(\Gamma, \mathbb{T}_{E,N}^{\mathrm{mid}}(\overline{F})\big) \leftarrow H^2\big(\Gamma, P_{E,N}^{\mathrm{rig}}(\overline{F})\big)$$

are equal. We denote this common image by $\xi_{E,N}^{\mathrm{mid}}$.

Proof. We use Tate–Nakayama duality to describe the canonical elements. For any algebraic torus T defined over F and split over E we have the isomorphism

$$\widehat{H}^0\big(\Gamma_{E/F}, X_*(T)\big) \to H^2\big(\Gamma_{E/F}, T(E)\big)$$

of [34], which is cup-product with the canonical class, and the inclusion

$$H^2\big(\Gamma_{E/F}, T(E)\big) \to H^2\big(\Gamma, T(\overline{F})\big).$$

We apply this with $T = \mathbb{T}_E^{\mathrm{iso}}$ and $T = \mathbb{T}_{E,N}^{\mathrm{mid}}$. For a finite multiplicative group D we write $X_*(D) = \mathrm{Hom}_{\mathbb{Z}}(X^*(D), \mathbb{Q}/\mathbb{Z})$. Then we have the injective homomorphism

$$\hat{H}^{-1}\big(\Gamma_{E/F}, X_*(D)\big) \to H^2(\Gamma, D)$$

denoted by $\Theta_{E,v}$ in [11, Section 3.2]. We apply this with $D = P_{E,N}^{\mathrm{rig}}$.

We now have the following elements:

(1) $1 \in \mathbb{Z}$, representing an element of $\mathbb{Z}/[E : F]\mathbb{Z} = \hat{H}^0(\Gamma_{E/F}, \mathbb{Z})$.

(2) The constant function with value 1 in $M_{E,N}^{\mathrm{mid},\vee}$, representing an element of $\hat{H}^0(\Gamma_{E/F}, M_{E,N}^{\mathrm{mid},\vee})$.

(3) The function $\delta_e \in \frac{\mathbb{Z}/N\mathbb{Z}[\Gamma_{E/F}]}{\mathbb{Z}/N\mathbb{Z}}$ representing an element of $\hat{H}^{-1}(\Gamma_{E/F}, -)$.

The image of the first element is the canonical class $\xi_E^{\mathrm{iso}} \in H^2(\Gamma, \mathbb{T}_E^{\mathrm{iso}})$, while the image of the third element is the canonical class $\xi_{E,N}^{\mathrm{rig}} \in H^2(\Gamma, P_{E,N}^{\mathrm{rig}})$.

It is clear that the maps $\mathbb{Z} \to M_{E,N}^{\mathrm{mid},\vee} \to \mathbb{Z}$ identify the first two elements. As for the second and third element, we consider the exact sequence (2.3.3) that is dual to (2.3.2) and see that the (-1)-cocycle δ_e in $\frac{\mathbb{Z}/N\mathbb{Z}[\Gamma_{E/F}]}{\mathbb{Z}/N\mathbb{Z}}$ lifts to the (-1)-cochain δ_e in $\mathbb{Z}[\Gamma_{E/F}]$, whose differential, i.e. $\Gamma_{E/F}$-norm, is the constant function 1 in $M_{E,N}^{\mathrm{mid},\vee}$. Since the map

$$M_{E,N}^{\mathrm{mid},\vee} \to N^{-1}\mathbb{Z}/\mathbb{Z}[\Gamma_{E/F}]_0$$

in (2.3.2) is the negative of c^{rig}, the claim now follows from the functoriality of the Tate–Nakayama isomorphism and its anticommutativity between degrees -1 and 0, i.e., the discussion of Section 3.5. ∎

Since F is local, the cohomology group $H^1(\Gamma, \mathbb{T}_{E,N}^{\mathrm{mid}})$ is finite. Therefore, the system of these groups, as E, N varies, trivially satisfies the Mittag–Leffler condition, and its $R^1 \varprojlim$ vanishes. According to [36, Theorem 3.5.8] we then have $H^2(\Gamma, \mathbb{T}^{\mathrm{mid}}) = \varprojlim H^2(\Gamma, \mathbb{T}_{E,N}^{\mathrm{mid}})$. The same argument also holds for $\mathbb{T}^{\mathrm{iso}}$ and P^{rig}. We therefore have a unique class $\xi^{\mathrm{iso}} \in H^2(\Gamma, \mathbb{T}^{\mathrm{iso}})$ (resp. $\xi^{\mathrm{rig}} \in H^2(\Gamma, P^{\mathrm{rig}})$) lifting $(\xi_E^{\mathrm{iso}})_E$ (resp. $(\xi_{E,N}^{\mathrm{rig}})_{E,N}$).

Corollary 3.6.2. (1) *Let $K/E/F$ be a tower of finite Galois extensions and $N|M$ natural numbers. The inflation map $H^2(\Gamma, \mathbb{T}_{K,M}^{\mathrm{mid}}(\overline{F})) \to H^2(\Gamma, \mathbb{T}_{E,N}^{\mathrm{mid}}(\overline{F}))$ maps $\xi_{K,M}^{\mathrm{mid}}$ to $\xi_{E,N}^{\mathrm{mid}}$ and therefore yields a canonical class $\xi^{\mathrm{mid}} \in H^2(\Gamma, \mathbb{T}^{\mathrm{mid}})$.*

(2) *The images of the canonical classes ξ^{iso} and ξ^{rig} under*

$$H^2\big(\Gamma, \mathbb{T}^{\mathrm{iso}}(\overline{F})\big) \to H^2\big(\Gamma, \mathbb{T}^{\mathrm{mid}}(\overline{F})\big) \leftarrow H^2\big(\Gamma, P^{\mathrm{rig}}(\overline{F})\big)$$

are equal to ξ^{mid}.

Proof. Both points follow from Proposition 3.6.1 and the compatibility of the canonical classes for the tori $\mathbb{T}_E^{\mathrm{iso}}$. ∎

Lemma 3.6.3. *We have $R^i \varprojlim H^1(\Gamma, \mathbb{T}_{E,N}^{\mathrm{mid}}) = 0$ for $i = 0, 1$.*

Proof. It is enough to show that for each E, N we can find M such that the map $\mathbb{T}^{\mathrm{mid}}_{E,M} \to \mathbb{T}^{\mathrm{mid}}_{E,N}$ induces the zero map on H^1. Let $\mathbb{T}^{\circ}_{E,N}$ be the cokernel of the injective map $c_{\mathrm{iso}} : \mathbb{T}^{\mathrm{iso}}_E \to \mathbb{T}^{\mathrm{mid}}_{E,N}$, so that $M^{\circ}_{E,N} := X^*(\mathbb{T}^{\circ}_{E,N})$ is the group of maps $f : \Gamma_{E/F} \to \frac{1}{N}\mathbb{Z}$ satisfying $\sum_{\sigma} f(\sigma) = 0$. The vanishing of $H^1(\Gamma, \mathbb{T}^{\mathrm{iso}}_E)$ implies that $H^1(\Gamma, \mathbb{T}^{\mathrm{mid}}_{E,N}) \to H^1(\Gamma, \mathbb{T}^{\circ}_{E,N})$ is injective, so it is enough to prove the statement with $\mathbb{T}^{\mathrm{mid}}$ replaced by $\mathbb{T}^{\circ}$. Choose $M = [E : F]N$. Then the inflation map $M^{\circ}_{E,N} \to M^{\circ}_{E,M}$ can be factored as

$$M^{\circ}_{E,N} \to M^{\circ}_{E,N} \to M^{\circ}_{E,M}$$

where the first map is multiplication by $[E : F]$ and the second map is division by $[E : F]$. The torus $\mathbb{T}^{\circ}_{E,N}$ splits over E, so it follows from Hilbert's Theorem 90 and the inflation-restriction exact sequence that $H^1(\Gamma, \mathbb{T}^{\circ}_{E,N})$ is killed by multiplication by $[E : F]$. ∎

We define the gerbes $\mathcal{E}^{\mathrm{iso}}$, $\mathcal{E}^{\mathrm{mid}}$, and $\mathcal{E}^{\mathrm{rig}}$ to be the extensions of Γ by $\mathbb{T}^{\mathrm{iso}}$, $\mathbb{T}^{\mathrm{mid}}$, and P^{rig}, respectively, given by the canonical classes ξ^{iso}, ξ^{mid} and ξ^{rig}. According to Corollary 3.6.2 there exist maps of gerbes

$$(3.6.1)$$

These dotted maps are not unique. In both cases, the set of $\mathbb{T}^{\mathrm{mid}}$-conjugacy classes of such maps is a torsor under $H^1(\Gamma, \mathbb{T}^{\mathrm{mid}})$, which by Lemma 3.6.3 and [36, Theorem 3.5.8] equals $R^1 \varprojlim H^0(\Gamma, \mathbb{T}^{\mathrm{mid}}_{E,N})$. This group is uncountable by Lemma 3.6.4. Nonetheless, the discussion of Section 3.1 and Lemma 3.6.3 shows that both the set $H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{mid}}, G)$ and the maps $H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{iso}}, G) \leftarrow H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{mid}}, G) \to H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{rig}}, G)$ are independent of the choice of $\mathcal{E}^{\mathrm{mid}}$ within its isomorphism class and of the dotted maps c_{iso} and c_{rig}, and similarly for $H^1(\mathbb{T}^{\mathrm{mid}} \to \mathcal{E}^{\mathrm{mid}}, Z \to G)$ etc.

Thanks to the canonical splitting $s_{\mathrm{iso}} : \mathbb{T}^{\mathrm{mid}} \to \mathbb{T}^{\mathrm{iso}}$ of c_{iso}, which tautologically maps ξ^{mid} to ξ^{iso}, there is also a map of gerbes $s_{\mathrm{iso}} : \mathcal{E}^{\mathrm{mid}} \to \mathcal{E}^{\mathrm{iso}}$, well-defined up to $Z^1(\Gamma, \mathbb{T}^{\mathrm{iso}})$. As above, this ambiguity disappears when considering H^1_{alg} groups. If F is non-Archimedean then the composition $s_{\mathrm{iso}} \circ c_{\mathrm{rig}} : \mathcal{E}^{\mathrm{rig}} \to \mathcal{E}^{\mathrm{iso}}$ is the morphism (3.13) of [12].

Lemma 3.6.4. *The groups* $H^1(\Gamma, \mathbb{T}^{\mathrm{mid}}) = R^1 \varprojlim H^0(\Gamma, \mathbb{T}^{\mathrm{mid}}_{E,N})$ *and* $H^1(\Gamma, \mathbb{T}^{\mathrm{iso}}) = R^1 \varprojlim H^0(\Gamma, \mathbb{T}^{\mathrm{iso}}_E)$ *are uncountable.*

Proof. We treat the case of $\mathbb{T}^{\mathrm{mid}}$, that of $\mathbb{T}^{\mathrm{iso}}$ being analogous but simpler. Consider the Kottwitz homomorphism [19, Section 7]

$$1 \to \mathbb{T}^{\mathrm{mid}}_{E,N}(F)_0 \to \mathbb{T}^{\mathrm{mid}}_{E,N}(F) \to (M^{\mathrm{mid},\vee}_{E,N})^{\mathrm{Fr}}_I \to 0.$$

Using that $R^1 \varprojlim$ is right-exact and [28, Proposition 1.1] our claim is equivalent to the uncountability of $R^1 \varprojlim (M_{E,N}^{\mathrm{mid},\vee})_I^{\mathrm{Fr}}$. The latter is a system of countable groups, so we apply [28, Proposition 1.4] and reduce to showing that this system fails the Mittag–Leffler condition.

We fix a finite Galois extension E/F and let K traverse a co-final sequence of finite Galois extensions of F containing E. We also fix N and let M traverse a co-final sequence of multiples of N. If the image of the inflation map

$$(M_{K,M}^{\mathrm{mid},\vee})_I^{\mathrm{Fr}} \to (M_{E,N}^{\mathrm{mid},\vee})_I^{\mathrm{Fr}}$$

stabilizes, so would the image of its composition with the norm map for the action of $\Gamma_{E/F}$. Recall from Section 2.3 that the inflation map is induced by the map $\mathbb{Z}[\Gamma_{K/F}] \to \mathbb{Z}[\Gamma_{E/F}]$ defined to send y^K to $y^E(\sigma) = \sum_{\tau \mapsto \sigma} y^K(\tau)$. Composing this with the norm map $\mathbb{Z}[\Gamma_{E/F}] \to \mathbb{Z}$ we obtain the norm map $\mathbb{Z}[\Gamma_{K/F}] \to \mathbb{Z}$, i.e. the map sending y^K to $\sum_{\tau \in \Gamma_{K/F}} y^K(\tau)$. Thus we are studying whether the image of this map, restricted to $M_{K,M}^{\mathrm{mid},\vee} = M\mathbb{Z}[\Gamma_{K/F}] + \mathbb{Z} \subset \mathbb{Z}[\Gamma_{K/F}]$, stabilizes. But the norm map sends $M\mathbb{Z}[\Gamma_{K/F}]$ to $M\mathbb{Z}$ and $\mathbb{Z}$ onto $[K:F]\mathbb{Z}$. Thus, as K and M grow the image of $M_{K,M}^{\mathrm{mid},\vee}$ in $\mathbb{Z}$ shrinks to $\{0\}$. ∎

3.7. Global canonical classes at finite levels

Let F be a number field, E/F a finite Galois extension, S a finite set of places of F, $\dot{S}_E$ a set of lifts of the places in S to places of E. We assume that $(E, \dot{S}_E)$ satisfies Conditions 2.4.1.

Given a torus T over F split over E with cocharacter module Y, the Tate–Nakayama isomorphism reviewed in Section 3.5 is

$$\hat{H}^{i-2}\big(\Gamma_{E/F}, Y[S_E]_0\big) \to \hat{H}^i\big(\Gamma_{E/F}, T(O_{E,S})\big).$$

We can apply this to $i = 2$ and $Y = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[S_E]_0, \mathbb{Z})$. Then the identity element in $Y[S_E]_0 = \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}[S_E]_0)$ maps to Tate's class $H^2(\Gamma_{E/F}, \mathbb{T}_{E,S}^{\mathrm{iso}}(O_{E,S}))$, which we shall denote by $\xi_{E,S}^{\mathrm{iso}}$ (it is denoted by α_3 in [34] and [20, Section 6]).

We now consider a finite multiplicative group Z with $A = X^*(Z)$ and $|A|$ invertible away from S and have the injection (introduced as $\Theta_{E,S}$ in [11, Section 3.2])

$$\hat{H}^{-1}\big(\Gamma_{E/F}, A^\vee[S_E]_0\big) \to H^2\big(\Gamma_S, Z(O_S)\big).$$

We have $A^\vee[S_E]_0 = \mathrm{Hom}_{\mathbb{Z}}(A, \mathrm{Maps}(S_E, \frac{1}{N}\mathbb{Z}/\mathbb{Z})_0)$ for any N multiple of $|A|$. Assuming that $|A|$ divides $[E:F]$, we have

$$\mathrm{Hom}_{\mathbb{Z}}\left(A, \mathrm{Maps}\left(S_E, \frac{1}{[E:F]}\mathbb{Z}/\mathbb{Z}\right)_0\right)^{N_{E/F}} = \mathrm{Hom}_{\mathbb{Z}}(A, M_{E,\dot{S}_E}^{\mathrm{rig}})^\Gamma.$$

We can apply this to $A = M_{E,\dot{S}_E}^{\mathrm{rig}}$, in which case the image of the identity is the canonical class $\xi_{E,\dot{S}_E,N}^{\mathrm{rig}} \in H^2(\Gamma_S, P_{E,\dot{S}_E}^{\mathrm{rig}}(O_S))$, denoted by $\xi_{E\dot{S}_E,N}$ in [11, Section 3.3].

Proposition 3.7.1. *The images of $\xi_{E,S}^{\mathrm{iso}}$ and $\xi_{E,\dot{S}_E,N}^{\mathrm{rig}}$ under*

$$H^2\big(\Gamma_S, \mathbb{T}_{E,S}^{\mathrm{iso}}(O_S)\big) \to H^2\big(\Gamma_S, \mathbb{T}_{E,\dot{S}_E}^{\mathrm{mid}}(O_S)\big) \leftarrow H^2\big(\Gamma_S, P_{E,\dot{S}_E}^{\mathrm{rig}}(O_S)\big)$$

are equal.

Proof. This is the global analogue of Proposition 3.6.1, and the proof is analogous, again based on the discussion in Section 3.5. ∎

3.8. Global canonical classes at infinite level

The canonical classes are compatible under the transition maps in all three cases. That is, the transition maps

(1) $H^2(\Gamma, \mathbb{T}_{K,S'}^{\mathrm{iso}}) \to H^2(\Gamma, \mathbb{T}_{E,S}^{\mathrm{iso}})$,

(2) $H^2(\Gamma, \mathbb{T}_{K,\dot{S}_K'}^{\mathrm{mid}}) \to H^2(\Gamma, \mathbb{T}_{E,\dot{S}_E}^{\mathrm{mid}})$, and

(3) $H^2(\Gamma, P_{K,\dot{S}_K'}^{\mathrm{rig}}) \to H^2(\Gamma, P_{E,\dot{S}_E}^{\mathrm{rig}})$

identify the canonical classes. In the first case the compatibility is [20, (8.18)], in the third case it is [11, Lemma 3.3.5]. The middle case follows by Proposition 3.7.1.

We now want to define a canonical class in each of these three cases. We use the exact sequence

$$1 \to R^1 \varprojlim H^1(\Gamma, \mathbb{T}_{E,S}^{\mathrm{iso}}) \to H^2(\Gamma, \mathbb{T}^{\mathrm{iso}}) \to \varprojlim H^2(\Gamma, \mathbb{T}_{E,S}^{\mathrm{iso}}) \to 1$$

and its analogs in the "mid" and "rig" cases.

Lemma 3.8.1. *Assume that for each place $w \in V_E$ there exists a place $w' \in S_E$ s.t. $\mathrm{Stab}(w, \Gamma_{E/F}) = \mathrm{Stab}(w', \Gamma_{E/F})$. Then $H^1(\Gamma, \mathbb{T}_{E,S}^{\mathrm{iso}}) = 0$.*

Proof. We have the Tate–Nakayama isomorphism

$$\hat{H}^{-1}\big(\Gamma_{E/F}, X_*(\mathbb{T}_{E,S}^{\mathrm{iso}})[V_E]_0\big) \to H^1(\Gamma, \mathbb{T}_{E,S}^{\mathrm{iso}})$$

reviewed in Section 3.5. We compute

$$X_*(\mathbb{T}_{E,S}^{\mathrm{iso}})[V_E]_0 = \mathrm{Hom}_{\mathbb{Z}}\big(\mathbb{Z}[S_E]_0, \mathbb{Z}[V_E]_0\big).$$

In the proof of [11, Lemma 3.1.10] a $\Gamma_{E/F}$-equivariant section $V_E \to S_E$ of the natural inclusion

$$S_E \to V_E$$

was constructed under the assumptions of S that are in place here. Then [20, Lemma A.11 (3)] shows that the vanishing of $\hat{H}^{-1}(\Gamma_{E/F}, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[S_E]_0, \mathbb{Z}[V_E]_0))$ is implied by the vanishing, for all intermediate fields $E/F'/F$, of $\hat{H}^{-1}(\Gamma_{E/F'}, \mathbb{Z}[V_E]_0)$. That in turn follows from [20, Lemma 6.1], as discussed in the proof of [20, Lemma 6.5]. ∎

According to this Lemma we have a canonical class $\xi^{\mathrm{iso}} \in H^2(\Gamma, \mathbb{T}^{\mathrm{iso}})$. For $\mathbb{T}_{\dot{V}}^{\mathrm{mid}}$ we need the following global analogue of Lemma 3.6.3.

Proposition 3.8.2. *If K/F is a finite Galois extension containing E and s.t. $[E : F]$ divides $[K : E]$, then the map*

$$H^1(\Gamma, \mathbb{T}_{K,\dot{S}'_K}^{\mathrm{mid}}) \to H^1(\Gamma, \mathbb{T}_{E,\dot{S}_E}^{\mathrm{mid}})$$

is zero. For every place v of F, the map

$$H^1(\Gamma_v, \mathbb{T}_{K,\dot{S}'_K}^{\mathrm{mid}}) \to H^1(\Gamma_v, \mathbb{T}_{E,\dot{S}_E}^{\mathrm{mid}})$$

is also zero.

Proof. The proofs for Γ and Γ_v are the same, so we only treat the first case. Let $\mathbb{T}_{E,\dot{S}_E}^{\circ}$ denote the cokernel of the injective morphism $\mathbb{T}_{E,S}^{\mathrm{iso}} \to \mathbb{T}_{E,\dot{S}_E}^{\mathrm{mid}}$ dual to c^{iso}. The vanishing of $H^1(\Gamma, \mathbb{T}_{E,S}^{\mathrm{iso}})$ implies that $H^1(\Gamma, \mathbb{T}_{E,\dot{S}_E}^{\mathrm{mid}}) \to H^1(\Gamma, \mathbb{T}_{E,\dot{S}_E}^{\circ})$ is injective. It thus suffices to prove the statement with $\mathbb{T}^{\mathrm{mid}}$ replaced by $\mathbb{T}^{\circ}$.

The character module $M_{E,\dot{S}_E}^{\circ}$ of $\mathbb{T}_{E,\dot{S}_E}^{\circ}$ is equal to the kernel of $M_{E,\dot{S}_E}^{\mathrm{mid}} \to M_{E,S}^{\mathrm{iso}}$ and hence is the $\mathbb{Z}[\Gamma_{E/F}]$-module consisting of functions

$$f : \Gamma_{E/F} \times S_E \to \frac{1}{[E : F]}\mathbb{Z}$$

that satisfy the conditions

$$\sum_w f(\sigma, w) = 0, \quad \sum_\sigma f(\sigma, w) = 0, \quad \text{and} \quad \sigma^{-1}w \notin \dot{S}_E \Rightarrow f(\sigma, w) = 0.$$

The restriction of the inflation map $f^{\mathrm{mid},E} \mapsto f^{\mathrm{mid},K}$ to $M_{E,\dot{S}_E}^{\circ}$ factors as the composition

$$M_{E,\dot{S}_E}^{\circ} \to M_{E,\dot{S}_E}^{\circ} \to M_{K,\dot{S}'_K}^{\circ},$$

where the first map is just multiplication by $[K : E]$, while the second map is given by $f^{\mathrm{mid},E} \mapsto [K : E]^{-1} \cdot f^{\mathrm{mid},K}$. Note that, since $f^{\mathrm{mid},K}$ takes values in $[E : F]^{-1}\mathbb{Z}$, the function $[K : E]^{-1} \cdot f^{\mathrm{mid},K}$ takes values in $[K : F]^{-1}\mathbb{Z}$ and is thus a well-defined element of $M_{K,\dot{S}'_K}^{\circ}$. On cohomology we obtain the composition

$$H^1(\Gamma, \mathbb{T}_{K,\dot{S}'_K}^{\circ}) \to H^1(\Gamma, \mathbb{T}_{E,\dot{S}_E}^{\circ}) \to H^1(\Gamma, \mathbb{T}_{E,\dot{S}_E}^{\circ}).$$

The second map is just multiplication by $[K : E]$. Since the torus $\mathbb{T}_{E,\dot{S}_E}^{\circ}$ splits over E the inflation map $H^1(\Gamma_{E/F}, \mathbb{T}_{E,\dot{S}_E}^{\circ}) \to H^1(\Gamma, \mathbb{T}_{E,\dot{S}_E}^{\circ})$ is an isomorphism, but its source is killed by multiplication by $[E : F]$. ∎

Corollary 3.8.3. *The three abelian groups $\varprojlim H^1(\Gamma, \mathbb{T}_{E_i,\dot{S}_i}^{\mathrm{mid}})$, $\varprojlim H^1(\Gamma_v, \mathbb{T}_{E_i,\dot{S}_i}^{\mathrm{mid}})$, and $R^1\varprojlim H^1(\Gamma, \mathbb{T}_{E_i,\dot{S}_i}^{\mathrm{mid}})$ vanish.*

Proof. This follows immediately Proposition 3.8.2. ∎

The vanishing of $R^1 \varprojlim H^1(\Gamma, \mathbb{T}^{\mathrm{mid}}_{E, \dot{S}_i})$ asserted in Corollary 3.8.3 implies that the natural map

$$H^2(\Gamma, \mathbb{T}^{\mathrm{mid}}_{\dot{V}}) \to \varprojlim H^2(\Gamma, \mathbb{T}^{\mathrm{mid}}_{E_i, \dot{S}_i})$$

is an isomorphism, so we obtain a unique class $\xi^{\mathrm{mid}}_{\dot{V}} \in H^2(\Gamma, \mathbb{T}^{\mathrm{mid}}_{\dot{V}})$ mapping to $(\xi^{\mathrm{mid}}_{E_i, \dot{S}_i})_{i \geq 0}$.

The case of P^{rig} is the most delicate, since $R^1 \varprojlim H^1(\Gamma, P^{\mathrm{rig}}_{E_i, \dot{S}_i})$ is known not to vanish by [33, Section 6.3]. Nonetheless, in [11, Section 3.5] a canonical class $\xi^{\mathrm{rig}}_{\dot{V}} \in H^2(\Gamma, P_{\dot{V}})$ is constructed that maps to the inverse system $(\xi^{\mathrm{rig}}_{E_i, \dot{S}_i})_{i \geq 0}$. Its construction is briefly reviewed in the proof of the following lemma.

As in Section 3.3 we have not recorded the tower $(E_i, S_i, \dot{S}_i)_i$ in the notation $\xi^{\mathrm{rig}}_{\dot{V}}$ and $\xi^{\mathrm{mid}}_{\dot{V}}$. Again the reason is that this choice does not matter, as the following lemma shows. Note that to be precise one should also choose a co-final sequence $(N_i)_{i \geq 0}$ as introduced in the proof of [11, Corollary 3.3.8], but it is clear that increasing S_i or replacing N_i by a multiple yields the same objects in the inverse limit.

Lemma 3.8.4. *If two sequences* $(E_i, S_i, \dot{S}_i)$ *lead to the same* $\dot{V}$, *then they lead to the same class* $\xi^{\mathrm{rig}}_{\dot{V}} \in H^2(\Gamma, P^{\mathrm{rig}}_{\dot{V}})$.

Proof. In order to obtain the statement about the class $\xi^{\mathrm{rig}}_{\dot{V}}$ we need to review its construction given in [11, Section 3.5]. First, an element $x \in H^2(\Gamma, P_{\dot{V}}(\overline{\mathbb{A}}))$ is constructed from the local canonical classes $\xi^{\mathrm{rig}}_v \in H^2(\Gamma_v, P^{\mathrm{rig}}_v(\overline{F}_v))$, with the help of Shapiro maps $\dot{S}^2_v : C^2(\Gamma_v, P(\overline{F}_v)) \to C^2(\Gamma, P(\overline{\mathbb{A}}_v))$ (for a suitable choice of continuous section $\Gamma_v \backslash \Gamma \to \Gamma$ as in [11, Appendix B]). These Shapiro maps can be obtained by splicing finite-level Shapiro maps $\dot{S}^2_v : C^2(\Gamma_v, P_{E_i, \dot{S}_i}(\overline{F}_v)) \to C^2(\Gamma, P_{E_i, \dot{S}_i}(\overline{\mathbb{A}}_v))$. However, taking another continuous section yields the same map

$$H^2(\Gamma_v, P(\overline{F}_v)) \to H^2(\Gamma, P(\overline{\mathbb{A}}_v))$$

(see Lemma B.4 loc. cit.). From this and the fact that for a given pair (E, S) the projection to $C^2(\Gamma, P^{\mathrm{rig}}_v(\overline{\mathbb{A}}))$ of the 2-cocycle $\dot{x} \in C^2(\Gamma, P^{\mathrm{rig}}_{\dot{V}}(\overline{\mathbb{A}}))$ introduced in [11, Section 3.5] is trivial for any $v \notin S$ we see that the class x is independent of the chosen tower. Since x and the inverse system $(\xi^{\mathrm{rig}}_{E_i, \dot{S}_i})$ uniquely determine $\xi^{\mathrm{rig}}_{\dot{V}}$, the class $\xi^{\mathrm{rig}}_{\dot{V}}$ is itself independent of the chosen tower. ∎

The argument for $\xi^{\mathrm{mid}}_{\dot{V}}$ is analogous. However, both classes do depend on $\dot{V} = \varprojlim \dot{S}_i$.

Remark 3.8.5. Let us briefly discuss the choice of $\dot{V}$. It is formal to check that the formation of $\mathbb{T}^{\mathrm{rig}}_{\dot{V}}$, $\mathbb{T}^{\mathrm{mid}}_{\dot{V}}$ and $\xi^{\mathrm{rig}}_{\dot{V}}$ is functorial in $(F, \overline{F}, \dot{V})$, i.e., any isomorphism between two such triples induces an isomorphism between the corresponding objects. In particular for $\tau \in \Gamma$, denoting $\dot{V}' = \tau(\dot{V})$ we have canonical isomorphisms $\mathbb{T}^{\mathrm{rig}}_{\dot{V}} \simeq \mathbb{T}^{\mathrm{rig}}_{\dot{V}'}$ etc. Note that in this case τ is uniquely determined by $\dot{V}$ and $\dot{V}'$ thanks to the Čebotarev density theorem.

Consider for a moment the local case, where F is a p-adic field. As in the global case the formation of the gerbe $\mathcal{E}^{\mathrm{rig}}$ is functorial in $(F, \bar{F})$. In particular any $\sigma \in \Gamma$ induces an automorphism of $\mathcal{E}^{\mathrm{rig}}$, which stabilizes P^{rig} and acts on the quotient $\mathcal{E}^{\mathrm{rig}}/P^{\mathrm{rig}} = \Gamma$ by conjugation by σ, but the action on P^{rig} is *not* the obvious one (action of an element of Γ on $\bar{F}$-points of a scheme). This is essentially due to the fact that finite extensions of F in $\bar{F}$ occur in the definition of P^{rig}. Thus automorphisms of $\bar{F}$ induce a priori non-trivial automorphisms of cohomology groups for $\mathcal{E}^{\mathrm{rig}}$, unlike usual Galois cohomology ([31, Section VII.5, Proposition 3]). This is reflected by the fact that for a connected reductive group G over F and Z a finite central subgroup, the natural action of Γ on the finite abelian group $\bar{Y}_{+,\mathrm{tor}}(Z \to G)$ defined in [10, Section 4] (see Proposition 5.3 loc. cit.) which is the source of the Tate–Nakayama isomorphism for $\mathcal{E}^{\mathrm{rig}}$, is not trivial in general.

The existence of non-trivial automorphisms in the local case has a global consequence. Consider a global field F and two arbitrary sets of lifts $\dot{V}$ and $\dot{V}'$ satisfying Condition 3.3.1. In general there does not seem to be any natural isomorphism between the corresponding gerbes $\mathcal{E}_{\dot{V}}^{\mathrm{rig}}$ and $\mathcal{E}_{\dot{V}'}^{\mathrm{rig}}$. Here "natural" means at least compatible with localization. For example assume that there is a finite place $v_0 \in V$ such that for any $v \in V \smallsetminus \{v_0\}$ the two lifts of v in $\dot{V}$ and $\dot{V}'$ coincide but the two lifts $\dot{v}_0$ and $\dot{v}_0'$ do not coincide, say with $\dot{v}_0' = \dot{v}_0 \circ \tau$ for some $\tau \in \Gamma$. Let G be $\mathrm{Res}_{E/F}\,\mathrm{SL}_2$ for some quadratic extension E/F. Let $Z \simeq \mathrm{Res}_{E/F}\,\mu_2$ be the center of G. We have an identification of $H^1(P_{\dot{V}}^{\mathrm{rig}} \to \mathcal{E}_{\dot{V}}^{\mathrm{rig}}, Z \to G)$ with the subset of $\bigoplus_{v \in V} H^1(P_v^{\mathrm{rig}} \to \mathcal{E}_v^{\mathrm{rig}}, Z \to G)$ consisting of classes $(c_v)_v$ such that c_v is trivial for almost all v and the corresponding characters $\chi_v : Z(\hat{G}_{\mathrm{sc}}) \to \mathbb{C}^\times$ are such that $\prod_v \chi_v = 1$. Assume that v_0 is not split in E. The natural isomorphism between $H^1(P_{\dot{v}_0}^{\mathrm{rig}} \to \mathcal{E}_{\dot{v}_0}^{\mathrm{rig}}, Z \to G)$ and $H^1(P_{\dot{v}_0'}^{\mathrm{rig}} \to \mathcal{E}_{\dot{v}_0'}^{\mathrm{rig}}, Z \to G)$ maps χ_{v_0} to $\chi_{v_0} \circ \tau$. But in general this ruins the product condition $\prod_v \chi_v = 1$: there is an isomorphism $Z(\hat{G}_{\mathrm{sc}}) \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$ such that $\Gamma_{E/F}$ exchanges the two factors, so if $\chi_{v_0}(1,1) \neq 1$ then the product condition fails.

We thus see that the dependence of the global gerbe on $\dot{V}$ is necessitated by the properties of the local gerbe.

Corollary 3.8.6. *The natural maps* $\mathbb{T}^{\mathrm{iso}} \to \mathbb{T}_{\dot{V}}^{\mathrm{mid}} \leftarrow P_{\dot{V}}$ *map the canonical classes* ξ^{iso} *and* $\xi_{\dot{V}}^{\mathrm{rig}}$ *to* $\xi_{\dot{V}}^{\mathrm{mid}}$.

Proof. This can be checked on finite levels, where it is the content of Proposition 3.7.1. ∎

3.9. The global gerbes $\mathcal{E}^{\mathrm{iso}}$, $\mathcal{E}^{\mathrm{mid}}$, $\mathcal{E}^{\mathrm{rig}}$

The choice of a 2-cocycle in the canonical class ξ^{iso} (resp. $\xi_{\dot{V}}^{\mathrm{mid}}$, $\xi_{\dot{V}}^{\mathrm{rig}}$) gives an extension $\mathcal{E}^{\mathrm{iso}}$ (resp. $\mathcal{E}_{\dot{V}}^{\mathrm{mid}}$, $\mathcal{E}_{\dot{V}}^{\mathrm{rig}}$) of Γ by $\mathbb{T}^{\mathrm{iso}}(\bar{F})$ (resp. $\mathbb{T}_{\dot{V}}^{\mathrm{mid}}(\bar{F})$, $P_{\dot{V}}^{\mathrm{rig}}(\bar{F})$). Using these, we define functors

$$H_{\mathrm{alg}}^1(\mathbb{T}^{\mathrm{iso}} \to \mathcal{E}^{\mathrm{iso}}, Z \to G),\ H_{\mathrm{alg}}^1(\mathbb{T}_{\dot{V}}^{\mathrm{mid}} \to \mathcal{E}_{\dot{V}}^{\mathrm{mid}}, Z \to G),\ \text{and } H^1(P_{\dot{V}} \to \mathcal{E}_{\dot{V}}^{\mathrm{rig}}, Z \to G),$$

where G is a linear algebraic group defined over F and $Z \subset G$ is a central diagonalizable group. Note that in the first two cases replacing Z by the central torus $T = Z^\circ$ has no effect.

By the discussion in Section 3.1 these functors are well-defined, independently of the choice of 2-cocycles:

(1) For $\mathcal{E}_{\dot{V}}^{\mathrm{rig}}$ we use the vanishing of $H^1(\Gamma, P_{\dot{V}}^{\mathrm{rig}})$.

(2) For $\mathcal{E}^{\mathrm{iso}}$, we use the vanishing of $H^1(\Gamma, \mathbb{T}_{E,S}^{\mathrm{iso}})$ for all finite levels.

(3) For $\mathcal{E}_{\dot{V}}^{\mathrm{mid}}$, we use the eventual vanishing of $H^1(\Gamma, \mathbb{T}_{E,\dot{S}_E}^{\mathrm{mid}})$ at finite levels (Proposition 3.8.2).

Next we fix morphisms of extensions $\mathcal{E}^{\mathrm{iso}} \to \mathcal{E}_{\dot{V}}^{\mathrm{mid}}$ and $\mathcal{E}_{\dot{V}}^{\mathrm{rig}} \to \mathcal{E}_{\dot{V}}^{\mathrm{mid}}$ extending the morphisms $\mathbb{T}^{\mathrm{iso}} \to \mathbb{T}_{\dot{V}}^{\mathrm{mid}}$ and $P_{\dot{V}}^{\mathrm{rig}} \to \mathbb{T}_{\dot{V}}^{\mathrm{mid}}$. These exist by Proposition 3.7.1. For an affine algebraic group G defined over F and a central subgroup $Z \subset G$ we have the cohomology pointed sets defined in Section 3.1, and comparison maps between them induced by c^{iso} and c^{rig}

$$H_{\mathrm{alg}}^1(\mathbb{T}^{\mathrm{iso}} \to \mathcal{E}^{\mathrm{iso}}, Z \to G) \leftarrow H_{\mathrm{alg}}^1(\mathbb{T}_{\dot{V}}^{\mathrm{mid}} \to \mathcal{E}_{\dot{V}}^{\mathrm{mid}}, Z \to G)$$
$$\to H^1(P_{\dot{V}}^{\mathrm{rig}} \to \mathcal{E}_{\dot{V}}^{\mathrm{rig}}, Z \to G). \tag{3.9.1}$$

The morphisms of extensions $\mathcal{E}^{\mathrm{iso}} \to \mathcal{E}_{\dot{V}}^{\mathrm{mid}}$ and $\mathcal{E}_{\dot{V}}^{\mathrm{rig}} \to \mathcal{E}_{\dot{V}}^{\mathrm{mid}}$ are well defined only up to multiplication by $H^1(\Gamma, \mathbb{T}_{\dot{V}}^{\mathrm{mid}})$. According to Corollary 3.8.3 this group equals $R^1 \varprojlim H^0(\Gamma, \mathbb{T}_{\dot{V}}^{\mathrm{mid}})$ and thus all maps $H^1(\Gamma, \mathbb{T}_{\dot{V}}^{\mathrm{mid}}) \to H^1(\Gamma, \mathbb{T}_{E_i,\dot{S}_i}^{\mathrm{mid}})$ vanish. It follows that the maps on cohomology (3.9.1) are independent of the morphisms of extensions used to define them.

3.10. The relationship between the cohomology of $\mathcal{E}^{\mathrm{iso}}$, $\mathcal{E}^{\mathrm{mid}}$, and $\mathcal{E}^{\mathrm{rig}}$

In this subsection F is either local or global. We omit the subscript $\dot{V}$ when F is global in order to state the following result uniformly.

Corollary 3.10.1. *Let G be an algebraic group and $T \subset G$ a central torus. Then the squares*

$$
\begin{array}{ccc}
H^1(\mathbb{T}^{\mathrm{mid}} \to \mathcal{E}^{\mathrm{mid}}, T \to G) & \longrightarrow & \mathrm{Hom}_F(\mathbb{T}^{\mathrm{mid}}, T) \\
\downarrow & & \downarrow \\
H^1(\mathbb{T}^{\mathrm{iso}} \to \mathcal{E}^{\mathrm{iso}}, T \to G) & \longrightarrow & \mathrm{Hom}_F(\mathbb{T}^{\mathrm{iso}}, T)
\end{array}
$$

and

$$
\begin{array}{ccc}
H^1(\mathbb{T}^{\mathrm{mid}} \to \mathcal{E}^{\mathrm{mid}}, T \to G) & \longrightarrow & \mathrm{Hom}_F(\mathbb{T}^{\mathrm{mid}}, T) \\
\downarrow & & \downarrow \\
H^1(P^{\mathrm{rig}} \to \mathcal{E}^{\mathrm{rig}}, T \to G) & \longrightarrow & \mathrm{Hom}_F(P^{\mathrm{rig}}, T)
\end{array}
$$

are Cartesian and the vertical arrows are surjective.

Proof. This follows from Fact 3.1.3 and Proposition 3.4.1. ∎

Remark 3.10.2. In the first square, the same is true even if T is a diagonalizable central subgroup of G, since replacing T by $T°$ does not change any of the four corners of the square. In the second square however, the surjectivity does *not* remain true when T is disconnected. In particular, if $Z = T$ is finite, then the top left corner becomes $H^1(\Gamma, G)$ and the top right corner becomes zero, while the bottom left corner is usually strictly larger than $H^1(\Gamma, G)$, and the bottom right corner is always larger than zero.

Remark 3.10.3. In the first square, we even have a splitting

$$H^1(\mathbb{T}^{\mathrm{mid}} \to \mathcal{E}^{\mathrm{mid}}, T \to G) \simeq H^1(\mathbb{T}^{\mathrm{iso}} \to \mathcal{E}^{\mathrm{iso}}, T \to G) \times \mathrm{Hom}(\mathbb{T}^{\mathrm{mid}}/\mathbb{T}^{\mathrm{iso}}, T)$$

by the splitting (2.1.2) in the local case, and by Proposition 3.4.2 in the global case. In the global case, the splitting is not canonical.

3.11. Localization

In this section v denotes a place of F.

Fact 3.11.1. *The image of the local canonical class under the map* $H^2(\Gamma_v, \mathbb{T}_v^{\mathrm{mid}}) \to H^2(\Gamma_v, \mathbb{T}_{\dot{V}}^{\mathrm{mid}})$ *induced by* loc_v *coincides with the image of the global canonical class under the restriction map* $H^2(\Gamma, \mathbb{T}_{\dot{V}}^{\mathrm{mid}}) \to H^2(\Gamma_v, \mathbb{T}_{\dot{V}}^{\mathrm{mid}})$.

Proof. This follows from Propositions 3.7.1, 3.6.1, and [11, Corollary 3.3.8]. ∎

This implies the existence of the dotted arrow in the commutative diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{T}_v^{\mathrm{mid}}(\bar{F}_v) & \longrightarrow & \mathcal{E}_v^{\mathrm{mid}} & \longrightarrow & \Gamma_v & \longrightarrow & 1 \\
& & \downarrow & & \vdots & & \| & & \\
1 & \longrightarrow & \mathbb{T}_{\dot{V}}^{\mathrm{mid}}(\bar{F}_v) & \longrightarrow & \Box_2 & \longrightarrow & \Gamma_v & \longrightarrow & 1 \\
& & \uparrow & & \uparrow & & \| & & \\
1 & \longrightarrow & \mathbb{T}_{\dot{V}}^{\mathrm{mid}}(\bar{F}) & \longrightarrow & \Box_1 & \longrightarrow & \Gamma_v & \longrightarrow & 1 \\
& & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{T}_{\dot{V}}^{\mathrm{mid}}(\bar{F}) & \longrightarrow & \mathcal{E}_{\dot{V}}^{\mathrm{mid}} & \longrightarrow & \Gamma & \longrightarrow & 1
\end{array}
$$

Here $\Box_1$ is the pullback of $\mathcal{E}_{\dot{V}}^{\mathrm{mid}}$ via the embedding of Γ_v in Γ and $\Box_2$ is the pushout of $\Box_1$ via the map $\mathbb{T}_{\dot{V}}^{\mathrm{mid}}(\bar{F}) \to \mathbb{T}_{\dot{V}}^{\mathrm{mid}}(\bar{F}_v)$. From this diagram we obtain a localization map

$$H^1_{\mathrm{alg}}(\mathcal{E}_{\dot{V}}^{\mathrm{mid}}, G) \to H^1_{\mathrm{alg}}(\mathcal{E}_v^{\mathrm{mid}}, G_{F_v})$$

for any algebraic group G over F. The vanishing of $\varprojlim_i H^1(\Gamma_v, \mathbb{T}_{E_i,\dot{S}_i}^{\mathrm{mid}})$ shown in Corollary 3.8.3 implies that the map $H^1_{\mathrm{alg}}(\Gamma_v, \mathbb{T}_{\dot{V}}^{\mathrm{mid}}) \to H^1(\Gamma_v, \mathbb{T}_{E_i,\dot{S}_i}^{\mathrm{mid}})$ is zero for every $i \geq 0$.

Thus, even though the dotted arrow above is not unique, the localization map on cohomology that it induces is unique. To be more precise, this argument shows that for any central torus $Z \subset G$ we have a localization map

$$Z^1(\mathbb{T}_{\dot{V}}^{\mathrm{mid}} \to \mathcal{E}_{\dot{V}}^{\mathrm{mid}}, Z \to G) \to Z^1(\mathbb{T}_v^{\mathrm{mid}} \to \mathcal{E}_v^{\mathrm{mid}}, Z \to G)$$

which is uniquely determined up to coboundaries taking values in $Z(\overline{F_v})$.

Similarly we have localization maps for "iso". For "rig", see [11, Section 3.6]. It is formal to check that the localization maps for "mid", "iso", and "rig" are compatible, i.e., that the following diagram is commutative.

$$\begin{array}{ccccc}
H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{iso}}, G) & \longleftarrow & H^1_{\mathrm{alg}}(\mathcal{E}_{\dot{V}}^{\mathrm{mid}}, G) & \longrightarrow & H^1_{\mathrm{alg}}(\mathcal{E}_{\dot{V}}^{\mathrm{rig}}, G) \\
\downarrow & & \downarrow & & \downarrow \\
H^1_{\mathrm{alg}}(\mathcal{E}_v^{\mathrm{iso}}, G) & \longleftarrow & H^1_{\mathrm{alg}}(\mathcal{E}_v^{\mathrm{mid}}, G) & \longrightarrow & H^1_{\mathrm{alg}}(\mathcal{E}_v^{\mathrm{rig}}, G)
\end{array}$$

Lemma 3.11.2. *Let G be a connected reductive group over F and Z a central torus in G. Choose a model $\underline{G}$ of G over $\mathcal{O}_F[1/N]$ for some integer $N > 0$. For any*

$$z \in Z^1(\mathbb{T}^{\mathrm{iso}} \to \mathcal{E}^{\mathrm{iso}}, Z \to G),$$

there exists a finite Galois extension E/F and a finite set S of places of F containing all Archimedean places and all finite places dividing N or ramifying in E, such that for all $v \in V \smallsetminus S$ the localization $\mathrm{loc}_v(z) \in Z^1(\mathbb{T}_v^{\mathrm{iso}} \to \mathcal{E}_v^{\mathrm{iso}}, Z \to G)$ is the product of an element inflated from $Z^1(\Gamma_{E_{\dot{v}}/F_v}, \underline{G}(\mathcal{O}_{E_{\dot{v}}}))$ with an element inflated from a co-boundary $\Gamma_v \to Z(\overline{F_v})$.

Proof. The restriction of z to $\mathbb{T}^{\mathrm{iso}}$ is defined over F since Z is central, and factors through $\mathbb{T}_{E,S}^{\mathrm{iso}}$ for some pair (E, S) where E is a finite Galois extension of F and S satisfies the conditions of [34]. Up to enlarging S we can assume that $Z \to G$ comes from a closed embedding $\underline{Z} \to \underline{G}$ where $\underline{Z}$ is the canonical model over $\mathcal{O}_{F,S}$ of the torus Z, and that the restriction of $z : \mathbb{T}_{E,S}^{\mathrm{iso}} \to Z$ comes from a (uniquely determined) morphism $\underline{\mathbb{T}_{E,S}^{\mathrm{iso}}} \to \underline{Z}$. Let $\xi_{E,S}^{\mathrm{iso}} \in Z^2(\Gamma_{E/F}, \mathbb{T}_{E,S}^{\mathrm{iso}}(\mathcal{O}_{E,S}))$ be a representative of the canonical class. Consider a finite Galois extension E'/F containing E and $S' \subset V$ finite, containing S and satisfying Tate's conditions. Let $\mathcal{E}$ be the extension of $\Gamma_{E'/F}$ by $\underline{\mathbb{T}_{E,S}^{\mathrm{iso}}}(\mathcal{O}_{E',S'})$ built using $\xi_{E,S}^{\mathrm{iso}}$. We have a well-defined map

$$\frac{Z^1\left(\underline{\mathbb{T}_{E,S}^{\mathrm{iso}}}(\mathcal{O}_{E',S'}) \to \mathcal{E}, \underline{Z}(\mathcal{O}_{E',S'}) \to \underline{G}(\mathcal{O}_{E',S'})\right)}{B^1\left(\Gamma_{E'/F}, \underline{Z}(\mathcal{O}_{E',S'})\right)}$$
$$\to Z^1(\mathbb{T}^{\mathrm{iso}} \to \mathcal{E}^{\mathrm{iso}}, Z \to G)/B^1\left(\Gamma, Z(\overline{F})\right)$$

which exists because the images of $\xi_{E,S}^{\mathrm{iso}}$ (by inflation) and of ξ^{iso} (by $\mathbb{T}^{\mathrm{iso}} \to \mathbb{T}_{E,S}^{\mathrm{iso}}$) in $H^2(\Gamma, \mathbb{T}_{E,S}^{\mathrm{iso}})$ coincide, and is uniquely determined because $H^1(\Gamma, \mathbb{T}_{E,S}^{\mathrm{iso}}) = 0$. By continuity of z, up to enlarging E' and S' the class of z modulo $B^1(\Gamma, Z(\overline{F}))$ belongs to the image of this map.

Now for any $v \in V \smallsetminus S'$ we have a commutative diagram

$$
\begin{array}{ccc}
\dfrac{Z^1(\mathbb{T}^{\mathrm{iso}}_{E,S}(\mathcal{O}_{E',S'}) \to \mathcal{E}, \underline{Z}(\mathcal{O}_{E',S'}) \to \underline{G}(\mathcal{O}_{E',S'}))}{B^1(\Gamma_{E'/F}, \underline{Z}(\mathcal{O}_{E',S'}))} & \to & Z^1\big(\Gamma_{E'_{\dot{v}}/F_v}, \underline{G}(\mathcal{O}_{E'_{\dot{v}}})\big)/B^1\big(\Gamma_{E'_{\dot{v}}/F_v}, \underline{Z}(\mathcal{O}_{E'_{\dot{v}}})\big) \\
\downarrow & & \downarrow \\
Z^1(\mathbb{T}^{\mathrm{iso}} \to \mathcal{E}^{\mathrm{iso}}, Z \to G)/B^1\big(\Gamma, Z(\overline{F})\big) & \to & Z^1(\mathbb{T}^{\mathrm{iso}}_v \to \mathcal{E}^{\mathrm{iso}}_{\dot{V}}, Z \to G)/B^1\big(\Gamma_v, Z(\overline{F_v})\big)
\end{array}
$$

The top horizontal map exists because $H^2(\Gamma_{E'_{\dot{v}}/F_v}, \mathbb{T}^{\mathrm{iso}}_{E,S}(\mathcal{O}_{E'_{\dot{v}}})) = 0$ and is uniquely determined because

$$
H^1\big(\Gamma_{E'_{\dot{v}}/F_v}, \mathbb{T}^{\mathrm{iso}}_{E,S}(\mathcal{O}_{E'_{\dot{v}}})\big) = 0.
$$

Commutativity follows from $H^1(\Gamma_v, \mathbb{T}^{\mathrm{iso}}_{E,S}) = 0$. ∎

Remark 3.11.3. This is much easier than [33, Proposition 6.1.1] (see also [11, Section 3.9]) thanks to vanishing of H^1 at finite level. A similar ramification property could be proved for "mid" using Proposition 3.8.2, with an extra step.

3.12. The cohomology of $\mathcal{E}^{\mathrm{iso}}$ and $B(G)$

Let F be a local or global field. For E a finite Galois extension of F, Kottwitz introduced in [20] an extension $\mathcal{E}^{\mathrm{iso}}_E$ (simply denoted $\mathcal{E}(E/F)$ loc. cit.) of $\Gamma_{E/F}$ by $\mathbb{T}^{\mathrm{iso}}_E(E)$, where $\mathbb{T}^{\mathrm{iso}}_E$ is the protorus $\varprojlim_S \mathbb{T}^{\mathrm{iso}}_{E,S}$. Note that the transition maps are surjective morphisms having connected kernel between tori split by E, so that they induce surjective maps between groups of E-points.

For a linear algebraic group G defined over F, he defined the pointed set

$$
B(G) := \varinjlim_E H^1_{\mathrm{alg}}\big(\mathcal{E}^{\mathrm{iso}}_E, G(E)\big)
$$

in Section 10 loc. cit. The transition maps exist thanks to the compatibility of canonical classes with inflation maps $\mathbb{T}^{\mathrm{iso}}_K \to \mathbb{T}^{\mathrm{iso}}_E$, and are well defined thanks to the vanishing of $H^1(\Gamma_{K/F}, \mathbb{T}^{\mathrm{iso}}_E(K))$. Define $\overline{Z}^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{iso}}_E, G(E))$ as the quotient of $Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{iso}}_E, G(E))$ by the following equivalence relation: $z \sim z'$ if and only if there exists $t \in \mathbb{T}^{\mathrm{iso}}_E(E)$ such that $z'(w) = z(twt^{-1})$ for all $w \in \mathcal{E}^{\mathrm{iso}}_E$. Note that $z(twt^{-1}) = z(t)z(w)\sigma_w(z(t))^{-1}$ where $\sigma_w \in \Gamma_{E/F}$ is the image of w, so that we have a surjective map

$$
\overline{Z}^1_{\mathrm{alg}}\big(\mathcal{E}^{\mathrm{iso}}_E, G(E)\big) \to H^1_{\mathrm{alg}}\big(\mathcal{E}^{\mathrm{iso}}_E, G(E)\big).
$$

The inflation maps are well defined at the level of $\overline{Z}^1$ and letting

$$
\widetilde{B}(G) := \varinjlim_E \overline{Z}^1_{\mathrm{alg}}\big(\mathcal{E}^{\mathrm{iso}}_E, G(E)\big),
$$

we obtain a pointed set mapping onto $B(G)$.

Lemma 3.12.1. *The natural map $\widetilde{B}(G) \to \overline{Z}^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{iso}}, G)$, defined similarly to the inflation maps, is an isomorphism. In particular we have a natural isomorphism $B(G) \to H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{iso}}, G)$.*

Proof. Surjectivity is essentially the first part of the proof of Lemma 3.11.2. Injectivity is clear. ∎

Note that since the inflation maps $\mathbb{T}_K^{\text{iso}} \to \mathbb{T}_E^{\text{iso}}$ do not have connected kernel, there is no reason why there should exists $\xi^{\text{iso}} \in Z^2(\Gamma, \mathbb{T}^{\text{iso}}(\bar{F}))$ such that for some $E \neq F$ its image in $Z^2(\Gamma, \mathbb{T}^{\text{iso}}(\bar{F}))$ belongs to $Z^2(\Gamma_{E/F}, \mathbb{T}_E^{\text{iso}}(E))$.

It is not difficult to check that the isomorphisms in Lemma 3.12.1 are compatible with localization. In fact this is the second step of the proof of Lemma 3.11.2.

3.13. A Tate–Nakayama description of $H_{\text{alg}}^1(\mathscr{E}_{\dot{V}}^{\text{mid}}, T)$

The main goal of this subsection is to describe the failure of commutativity of (1.0.1). For this, we shall give a linear algebraic description of the group $H_{\text{alg}}^1(\mathscr{E}_{\dot{V}}^{\text{mid}}, T)$ for an algebraic torus T, both in the local and in the global case. The local description will be used to give a precise formula for the failure of commutativity of (1.0.1). The global description will be used to show that this failure of commutativity satisfies a product formula.

We first begin with the local case. Let F be local. Let T be an algebraic torus defined over F and write $Y = X_*(T)$. In all three cases we have the inflation-restriction exact sequence of Fact 3.1.1

$$1 \to H^1(\Gamma, T) \to H_{\text{alg}}^1(\mathscr{E}, T) \to \text{Hom}_F(\mathbb{D}, T) \to \cdots$$

with $\mathbb{D}$ being one of $\mathbb{T}^{\text{mid}}$, $\mathbb{T}^{\text{iso}}$, or P^{rig}. We have the Tate–Nakayama isomorphism $Y_{\Gamma,\text{tor}} \to H^1(\Gamma, T)$ and the isomorphism

$$\left(Y \otimes X^*(\mathbb{D})\right)^\Gamma \to \text{Hom}_F(\mathbb{D}, T)$$

of Fact 3.1.2. Compatible with these two isomorphisms is a third isomorphism whose target is $H_{\text{alg}}^1(\mathscr{E}, T)$. In the case of $\mathscr{E}^{\text{iso}}$ its source is Y_Γ according to [20, (13.2)]. In the case of $\mathscr{E}^{\text{rig}}$ its source is the torsion subgroup of $Y \otimes \mathbb{Q}/IY$ according to [10, Section 4]. We shall write $Y^{\text{iso}} := Y_\Gamma$ and $Y^{\text{rig}} := (Y \otimes \mathbb{Q}/IY)[\text{tor}]$ and see both of these as functors from the category of tori to the category of Γ-modules.

Define

$$Y^{\text{mid}} := Y^{\text{mid}}(T) := \{(\lambda, \mu) \mid \lambda \in Y_\Gamma, \ \mu \in Y \otimes \mathbb{Q}, \ N^\natural(\lambda) = N^\natural(\mu)\},$$

where $N^\natural$ is the normalized norm map, i.e. $N^\natural = [E : F]^{-1} \sum_{\sigma \in \Gamma_{E/F}} \sigma$ for any finite Galois extension E/F splitting T. We have a natural map $Y^{\text{mid}} \to Y^{\text{iso}}$, $(\lambda, \mu) \mapsto \lambda$.

Fact 3.13.1. *The right square in the commutative diagram*

$$
\begin{array}{ccccc}
0 \longrightarrow & Y_{\Gamma,\text{tor}} & \longrightarrow & Y^{\text{mid}} & \longrightarrow (Y \otimes M^{\text{mid}})^\Gamma \\
& \| & & \downarrow & \downarrow \\
0 \longrightarrow & Y_{\Gamma,\text{tor}} & \longrightarrow & Y^{\text{iso}} & \longrightarrow (Y \otimes M^{\text{iso}})^\Gamma
\end{array}
$$

is Cartesian. Here $Y^{\mathrm{mid}} \to (Y \otimes M^{\mathrm{mid}})^\Gamma$ maps (λ, μ) to $\sum_\sigma \sigma(\mu) \otimes \sigma \in Y \otimes \mathbb{Q}[\Gamma]$,

$$Y^{\mathrm{iso}} \to (Y \otimes M^{\mathrm{iso}})^\Gamma$$

is given by $N^\natural$ via the natural embedding $M^{\mathrm{iso}} \subset \mathbb{Q}$, and $(Y \otimes M^{\mathrm{mid}})^\Gamma \to (Y \otimes M^{\mathrm{iso}})^\Gamma$ is $\mathrm{id}_Y \otimes c^{\mathrm{iso}}$, i.e., via the isomorphisms (3.1.1) it is the "pre-composition by c_{iso}" map $\mathrm{Hom}(\mathbb{T}^{\mathrm{mid}}, T) \to \mathrm{Hom}(\mathbb{T}^{\mathrm{iso}}, T)$.

Remark 3.13.2. Note that the Γ-invariant map

$$Y \otimes M^{\mathrm{mid}} \to Y \otimes M^{\mathrm{iso}}$$

given by $\mathrm{id}_Y \otimes c^{\mathrm{iso}}$ has the Γ-equivariant splitting $\mathrm{id}_Y \otimes s^{\mathrm{iso}}$. This gives a canonical isomorphism between Y^{mid} and $Y^{\mathrm{iso}} \oplus \ker(\mathrm{id}_Y \otimes c^{\mathrm{iso}})^\Gamma$.

Proposition 3.13.3. *There exists a unique isomorphism $Y^{\mathrm{mid}} \to H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{mid}}, T)$ that is functorial in T and fits into the commutative diagrams*

$$
\begin{array}{ccc}
Y_{\Gamma,\mathrm{tor}} & \longrightarrow Y^{\mathrm{mid}} & \longrightarrow (Y \otimes \mathbb{Q}[\Gamma])^\Gamma \\
\downarrow\wr & \downarrow\wr & \downarrow\wr \\
H^1(\Gamma, T) & \longrightarrow H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{mid}}, T) \longrightarrow \mathrm{Hom}_F(\mathbb{T}^{\mathrm{mid}}, T)
\end{array}
$$

and

$$
\begin{array}{ccc}
Y^{\mathrm{mid}} & \longrightarrow & Y^{\mathrm{iso}} \\
\downarrow\wr & & \downarrow\wr \\
H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{mid}}, T) & \longrightarrow & H^1(\mathcal{E}^{\mathrm{iso}}, T)
\end{array}
$$

Proof. This follows from Fact 3.13.1 and Corollary 3.10.1 which realize both Y^{mid} and $H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{mid}}, T)$ as fiber products, and from the functoriality of the isomorphisms $Y^{\mathrm{iso}} \to H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{iso}}, T)$, $\mathrm{Hom}_F(\mathbb{T}^{\mathrm{iso}}, T) \to (Y \otimes \mathbb{Q})^\Gamma$, and $\mathrm{Hom}_F(\mathbb{T}^{\mathrm{mid}}, T) \to (Y \otimes \mathbb{Q}[\Gamma])^\Gamma$. $\blacksquare$

Corollary 3.13.4. *Let $Z \subset T$ be a subtorus defined over F. The isomorphism $Y^{\mathrm{mid}}(T) \to H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{mid}}, T)$ identifies*

$$Y^{\mathrm{mid}}(Z \to T) := \{(\lambda, \mu) \mid \lambda \in Y_\Gamma(T),\ \mu \in Y(Z) \otimes \mathbb{Q},\ N^\natural(\lambda) = N^\natural(\mu)\} \subset Y^{\mathrm{mid}}(T)$$

with $H^1(\mathbb{T}^{\mathrm{mid}} \to \mathcal{E}^{\mathrm{mid}}, Z \to T)$.

Proposition 3.13.5. *For any torus T over F, the composition*

$$Y^{\mathrm{iso}} \simeq H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{iso}}, T) \to H^1(\mathcal{E}^{\mathrm{mid}}, T) \simeq Y^{\mathrm{mid}}$$

where the middle map is pullback along $s_{\mathrm{iso}} : \mathcal{E}^{\mathrm{mid}} \to \mathcal{E}^{\mathrm{iso}}$, maps $\lambda \in Y^{\mathrm{iso}}$ to $(\lambda, N^\natural(\lambda))$.

Proof. The image of λ is of the form (λ, μ) since $s_{\text{iso}} \circ c_{\text{iso}} = \text{id}_{\mathcal{E}^{\text{iso}}}$ up to $Z^1(\Gamma, \mathbb{T}^{\text{iso}})$. We can compute $\mu \in Y \otimes \mathbb{Q}$ as the image of λ by the composition

$$Y^{\text{iso}} \to (Y \otimes \mathbb{Q})^{\Gamma} \xrightarrow{\text{id}_Y \otimes s_{\text{iso}}} \left(Y \otimes \mathbb{Q}[\Gamma] \right)^{\Gamma} \simeq Y \otimes \mathbb{Q}$$

where the last map is evaluation at $1 \in \Gamma$. $\blacksquare$

Proposition 3.13.6. *The composition*

$$Y^{\text{mid}} \xrightarrow{\sim} H^1_{\text{alg}}(\mathcal{E}^{\text{mid}}, T) \to H^1(P^{\text{rig}}, T) \xrightarrow{\sim} Y^{\text{rig}},$$

where the middle map is induced by a map of gerbes c_{rig} as in (3.6.1), is given by $(\lambda, \mu) \mapsto \lambda - \mu$.

Proof. This composition, as well as the map $(\lambda, \mu) \mapsto \lambda - \mu$, are functorial homomorphisms that fit into the commutative diagram with exact rows

$$
\begin{array}{ccccccc}
0 & \longrightarrow & Y_{\Gamma,\text{tor}} & \longrightarrow & Y^{\text{mid}} & \longrightarrow & Y \otimes \mathbb{Q} \\
& & \| & & \downarrow & & \downarrow \\
0 & \longrightarrow & Y_{\Gamma,\text{tor}} & \longrightarrow & Y^{\text{rig}} & \longrightarrow & \frac{Y \otimes \mathbb{Q}}{Y}
\end{array}
$$

where the right vertical map is $\mu \mapsto -\mu + Y$. If T is induced, that is if $T \simeq \text{Res}_{A/F} \text{GL}_1$ for some finite étale F-algebra A, then $Y_{\Gamma,\text{tor}} \simeq H^1(\Gamma, T) = 0$ by Shapiro's lemma and Hilbert's theorem 90 and so our two maps $Y^{\text{mid}} \to Y^{\text{rig}}$ are equal in this case. In general we realize T as a quotient of an induced torus $\widetilde{T}$, by realizing Y as a quotient of an induced $\mathbb{Z}[\Gamma_{E/F}]$-module $\widetilde{Y}$ for some finite Galois extension E/F. Let $K = \ker(\widetilde{Y} \to Y)$. To conclude it is enough to show that $\widetilde{Y}^{\text{mid}} \to Y^{\text{mid}}$ is surjective. Let $(\lambda, \mu) \in Y^{\text{mid}}$, choose $\widetilde{\lambda} \in \widetilde{Y}_{\Gamma}$ lifting λ and $\widetilde{\mu}_0 \in \widetilde{Y} \otimes \mathbb{Q}$ lifting μ. Then $\varepsilon := N^{\natural}(\widetilde{\lambda}) - N^{\natural}(\widetilde{\mu}_0) \in (K \otimes \mathbb{Q})^{\Gamma}$ and thus $N^{\natural}(\varepsilon) = \varepsilon$. Setting $\widetilde{\mu} = \widetilde{\mu}_0 + \varepsilon$, we obtain that $(\widetilde{\lambda}, \widetilde{\mu}) \in \widetilde{Y}^{\text{mid}}$ lifts (λ, μ). $\blacksquare$

Recall that in the non-Archimedean case, the morphism of extensions $s_{\text{iso}} \circ c_{\text{rig}} : \mathcal{E}^{\text{rig}} \to \mathcal{E}^{\text{iso}}$ equals [12, (3.13)] up to $Z^1(\Gamma, \mathbb{T}^{\text{iso}})$. Note that Proposition 3.2 loc. cit. (as well as its Archimedean analogue) follows from Propositions 3.13.5 and 3.13.6 above. This is not surprising since the proof of Proposition 3.13.6 is very similar to that of Proposition 3.2 loc. cit. For later use in Section 4 we also record the following consequence.

Corollary 3.13.7. *Consider the two homomorphisms $H^1_{\text{alg}}(\mathcal{E}^{\text{mid}}, T) \to H^1(\mathcal{E}^{\text{rig}}, T)$ obtained by pulling back along*

(1) *A homomorphism $c_{\text{rig}} : \mathcal{E}^{\text{rig}} \to \mathcal{E}^{\text{mid}}$,*

(2) *The composition of a morphism $s_{\text{iso}} \circ c_{\text{rig}} : \mathcal{E}^{\text{rig}} \to \mathcal{E}^{\text{iso}}$ with $c_{\text{iso}} : \mathcal{E}^{\text{iso}} \to \mathcal{E}^{\text{mid}}$.*

Their difference, when pre-composed with the Tate–Nakayama isomorphism

$$Y^{\text{mid}} \to H^1_{\text{alg}}(\mathcal{E}^{\text{mid}}, T)$$

and post-composed with the inverse of the Tate–Nakayama isomorphism

$$Y^{\mathrm{rig}} \to H^1(\mathcal{E}^{\mathrm{rig}}, T),$$

is given by the map

$$Y^{\mathrm{mid}} \to Y^{\mathrm{rig}}, \quad (\lambda, \mu) \mapsto \mu - N^{\natural}(\mu).$$

Proof. This follows immediately from Propositions 3.13.5 and 3.13.6 and the equality $N^{\natural}(\mu) = N^{\natural}(\lambda)$. ∎

Now turn to the global case: Let F be global. Let T be a torus defined over F and $Z \subset T$ a subtorus defined over F. As before we write $Y = X_*(T)$. Denote $Y^{\mathrm{iso}}_{E,S} := (Y[S_E]_0)_\Gamma$ and $Y^{\mathrm{iso}}_E = \varinjlim_S Y^{\mathrm{iso}}_{E,S}$. For K/F a finite Galois extension containing E define $j : Y^{\mathrm{iso}}_K \to Y^{\mathrm{iso}}_E$ by

$$j(f)(v) = \sum_{\substack{w \in V_K \\ w \mapsto v}} f(w) \quad \text{for } v \in V_E.$$

It turns out that this map is an isomorphism. Choose a section s of $V_K \to V_E$ whose image contains $\dot{V}_K$, then by [11, Lemma 3.1.7] the unique right inverse $s_! : Y[V_E]_0 \to Y[s(V_E)]_0 \subset Y[V_K]_0$ to $Y[V_K]_0 \to Y[V_E]_0$ induces a well-defined map $! : Y^{\mathrm{iso}}_E \to Y^{\mathrm{iso}}_K$ which does not depend on the choice of s (in fact $\dot{V}$ is irrelevant here) and of course is injective. It is also surjective thanks to Lemma 2.4.4, and so j is an isomorphism with inverse $!$. Denote

$$Y^{\mathrm{iso}} = \varinjlim_E Y^{\mathrm{iso}}_E = \varprojlim_E Y^{\mathrm{iso}}_E.$$

By [20, Lemma 4.1] we have a Tate–Nakayama isomorphism $Y^{\mathrm{iso}} \simeq H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{iso}}, T)$. Note that [20, Lemma 8.4] also shows a posteriori that the maps j are isomorphisms. For any (E, S) such that E splits T and S satisfies Tate's axioms the following diagram is commutative:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & (Y[S_E]_0)_{\Gamma,\mathrm{tor}} & \longrightarrow & Y^{\mathrm{iso}}_{E,S} & \xrightarrow{N_{E/F}} & (Y[S_E]_0)^\Gamma \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^1(F, T) & \longrightarrow & H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{iso}}, T) & \longrightarrow & \mathrm{Hom}(\mathbb{T}^{\mathrm{iso}}, T)
\end{array}
$$

where the left vertical arrow is the usual Tate–Nakayama map and the right vertical arrow is the obvious map. For a subtorus Z of T defined over F denote $Y_T = X_*(T)$ and $Y_Z = X_*(Z)$ and let

$$Y^{\mathrm{mid}}(Z \to T)_{E, \dot{S}_E} := \Big\{ (\lambda, \mu) \mid \lambda \in Y^{\mathrm{iso}}_{E,S}(T), \ \mu \in (M^{\mathrm{mid}}_{E, \dot{S}_E} \otimes Y_Z)^\Gamma,$$

$$\sum_{\sigma \in \Gamma_{E/F}} \sigma\big(\lambda(\sigma^{-1} w)\big) = \sum_\sigma \mu(\sigma, w) \Big\},$$

$$Y^{\mathrm{mid}}_{\dot{V}}(Z \to T) := \varinjlim_{E, S} Y^{\mathrm{mid}}(Z \to T)_{E, \dot{S}_E}$$

where the transition maps are given by $! = j^{-1}$ on λ and using the inflation maps defined in Section 2.5 on μ. It is easy to see that one could also define $Y_{\dot{V}}^{\mathrm{mid}}(Z \to T)$ as the set of pairs (λ, μ) with $\lambda \in Y^{\mathrm{iso}}(T)$ and $\mu \in (M_{\dot{V}}^{\mathrm{mid}} \otimes Y_Z)^{\Gamma}$ satisfying the above relation at any level E. More concretely, using the description of $M_{\dot{V}}^{\mathrm{mid}}$ given in Section 3.3 we may also see μ as a function $V \to \mathbb{Q} \otimes Y_Z$ with finite support such that $\sum_{v \in V} \mu(v) = 0$ and for any Archimedean place v of F, $N_{\overline{F_v}/F_v}(\mu(v)) \in Y_Z$.

Fact 3.13.8. *The right square below is Cartesian*

$$0 \longrightarrow Y_{\mathrm{tor}}^{\mathrm{iso}}(T) \longrightarrow Y_{\dot{V}}^{\mathrm{mid}}(Z \to T) \longrightarrow (M_{\dot{V}}^{\mathrm{mid}} \otimes Y_Z)^{\Gamma}$$

$$0 \longrightarrow Y_{\mathrm{tor}}^{\mathrm{iso}}(T) \longrightarrow Y^{\mathrm{iso}}(Z \to T) \longrightarrow (M^{\mathrm{iso}} \otimes Y_Z)^{\Gamma}$$

Proof. This follows directly from the definition. ∎

Remark 3.13.9. Using the same argument as in the proof of Proposition 3.4.2 one can show that the natural transformation $Y_{\dot{V}}^{\mathrm{mid}} \to Y^{\mathrm{iso}}$ admits a splitting, but as we already observed in Remark 3.4.3 this splitting is not canonical.

As in the local case we simply write $Y_{\dot{V}}^{\mathrm{mid}}(T)$ for $Y_{\dot{V}}^{\mathrm{mid}}(T \to T)$.

Proposition 3.13.10. *There is a unique functorial isomorphism*

$$Y_{\dot{V}}^{\mathrm{mid}}(T) \to H_{\mathrm{alg}}^1(\mathcal{E}_{\dot{V}}^{\mathrm{mid}}, T)$$

that fits into the commutative diagrams

$$Y_{\mathrm{tor}}^{\mathrm{iso}} \longrightarrow Y_{\dot{V}}^{\mathrm{mid}} \longrightarrow (M_{\dot{V}}^{\mathrm{mid}} \otimes Y)^{\Gamma}$$

$$H^1(\Gamma, T) \longrightarrow H^1(\mathcal{E}_{\dot{V}}^{\mathrm{mid}}, T) \longrightarrow \mathrm{Hom}_F(\mathbb{T}_{\dot{V}}^{\mathrm{mid}}, T)$$

and

$$Y_{\dot{V}}^{\mathrm{mid}} \longrightarrow Y^{\mathrm{iso}}$$

$$H^1(\mathcal{E}_{\dot{V}}^{\mathrm{mid}}, T) \longrightarrow H^1(\mathcal{E}^{\mathrm{iso}}, T).$$

Proof. Analogous to the proof of Proposition 3.13.3, but with Fact 3.13.1 now replaced by its global analog Fact 3.13.8. ∎

Corollary 3.13.11. *Let $Z \subset T$ be a subtorus defined over F. The isomorphism $Y_{\dot{V}}^{\mathrm{mid}}(T) \to H_{\mathrm{alg}}^1(\mathcal{E}_{\dot{V}}^{\mathrm{mid}}, T)$ identifies $Y_{\dot{V}}^{\mathrm{mid}}(Z \to T)$ with $H_{\mathrm{alg}}^1(\mathcal{E}_{\dot{V}}^{\mathrm{mid}}, Z \to T)$.*

Corollary 3.13.12. *For $Z \to T$ an injective map between tori over F both maps*

$$H^1(\mathbb{T}_{\dot V}^{\mathrm{mid}} \to \mathcal{E}_{\dot V}^{\mathrm{mid}}, Z \to T) \to H^1(\mathbb{T}^{\mathrm{iso}} \to \mathcal{E}^{\mathrm{iso}}, Z \to T) \to H^1(\Gamma, T/Z)$$

are surjective.

Proof. By Corollary 3.10.1 the map

$$H^1(\mathbb{T}_{\dot V}^{\mathrm{mid}} \to \mathcal{E}_{\dot V}^{\mathrm{mid}}, Z \to T) \to H^1(\mathbb{T}^{\mathrm{iso}} \to \mathcal{E}^{\mathrm{iso}}, Z \to T)$$

is surjective, and it is clear that the map

$$H^1(\mathbb{T}^{\mathrm{iso}} \to \mathcal{E}^{\mathrm{iso}}, Z \to T) \simeq Y^{\mathrm{iso}}(Z \to T) \to Y^{\mathrm{iso}}(T/Z)_{\mathrm{tor}} \simeq H^1(\Gamma, T/Z)$$

is surjective. ∎

Corollary 3.13.13. *For G a connected reductive group over F and Z a central torus in G both maps*

$$H^1(\mathbb{T}_{\dot V}^{\mathrm{mid}} \to \mathcal{E}_{\dot V}^{\mathrm{mid}}, Z \to G) \to H^1(\mathbb{T}^{\mathrm{iso}} \to \mathcal{E}^{\mathrm{iso}}, Z \to G) \to H^1(\Gamma, G/Z)$$

are surjective.

Proof. Lemma A.1 in [11] reduces this to the previous Corollary. ∎

Proposition 3.13.14. *Let $v \in S$. Denote by w its lift in $\dot S_E$. Denote $Y_v^{\mathrm{mid}} = Y^{\mathrm{mid}}(T_{F_v})$ for simplicity. The composition*

$$Y_{E,\dot S_E}^{\mathrm{mid}} \to H^1(\mathcal{E}_{\dot V}^{\mathrm{mid}}, T) \xrightarrow{\mathrm{loc}_v} H^1(\mathcal{E}_v^{\mathrm{mid}}, T) \simeq Y_v^{\mathrm{mid}},$$

where the middle localization map was defined in Section 3.11 and the last map is the inverse of the Tate–Nakayama isomorphism in Proposition 3.13.3, sends $(\lambda, \mu) \in Y_{E,\dot S_E}^{\mathrm{mid}}$ to $(\lambda_v, \mu_v) \in Y_v^{\mathrm{mid}}$ where

$$\lambda_v = \sum_{\Gamma_{E_w/F_v} \backslash \Gamma_{E/F}} \sigma\lambda(\sigma^{-1}w), \quad \mu_v = \mu(1, w).$$

Proof. The element $\lambda_v \in Y_{\Gamma_v}$ is the image of the pair (λ_v, μ_v) under the natural map $Y_v^{\mathrm{mid}} \to Y_{\Gamma_v}$. Therefore the formula for λ_v follows from the commutativity of

$$
\begin{array}{ccccccc}
Y_{E,\dot S_E}^{\mathrm{mid}} & \longrightarrow & H_{\mathrm{alg}}^1(\mathcal{E}_{\dot V}^{\mathrm{mid}}, T) & \xrightarrow{\mathrm{loc}_v} & H_{\mathrm{alg}}^1(\mathcal{E}_v^{\mathrm{mid}}, T) & \longrightarrow & Y_v^{\mathrm{mid}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Y_{E,S}^{\mathrm{iso}} & \longrightarrow & H_{\mathrm{alg}}^1(\mathcal{E}^{\mathrm{iso}}, T) & \xrightarrow{\mathrm{loc}_v} & H_{\mathrm{alg}}^1(\mathcal{E}_v^{\mathrm{iso}}, T) & \longrightarrow & Y_v^{\mathrm{iso}}
\end{array}
$$

and the formula for the bottom horizontal map described in [20, Section 7.7].

Analogously, the element $\mu_v \in Y \otimes \mathbb{Q}$ is the image of the pair (λ_v, μ_v) under the natural map $Y_v^{\mathrm{mid}}(T) \to Y \otimes \mathbb{Q}$. The formula for μ_v follows from the commutativity of

$$
\begin{array}{ccccccc}
Y_{E,\dot{S}_E}^{\mathrm{mid}} & \longrightarrow & H_{\mathrm{alg}}^1(\mathcal{E}_{\dot{V}}^{\mathrm{mid}}, T) & \xrightarrow{\mathrm{loc}_v} & H_{\mathrm{alg}}^1(\mathcal{E}_v^{\mathrm{mid}}, T) & \longrightarrow & Y_v^{\mathrm{mid}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(M_{E,\dot{S}_E}^{\mathrm{mid}} \otimes Y)^\Gamma & \longrightarrow & \mathrm{Hom}_F(\mathbb{T}_{\dot{V}}^{\mathrm{mid}}, T) & \xrightarrow{\mathrm{loc}_v} & \mathrm{Hom}_F(\mathbb{T}_v^{\mathrm{mid}}, T) & \longrightarrow & Y \otimes \mathbb{Q}
\end{array}
$$

and the formula for μ_v follows from the formula for the bottom horizontal map obtained by composing the localization map $M_{E,\dot{S}_E}^{\mathrm{mid}} \to M_{E_w,[E:F]}^{\mathrm{mid}}$ described in Section 2.6 with the evaluation at 1 map $M_{E_w,[E:F]}^{\mathrm{mid}} \to \frac{1}{[E:F]}\mathbb{Z} \hookrightarrow \mathbb{Q}$. ∎

Although we will not need it in the paper, there is a global analogue of Proposition 3.13.6.

Proposition 3.13.15. *For any torus T defined over F, the composition*

$$
Y_{\dot{V}}^{\mathrm{mid}} \xrightarrow{\sim} H_{\mathrm{alg}}^1(\mathcal{E}_{\dot{V}}^{\mathrm{mid}}, T) \to H^1(P_{\dot{V}}^{\mathrm{rig}}, T) \xrightarrow{\sim} Y_{\dot{V}}^{\mathrm{rig}},
$$

where the middle map is induced by a map of gerbes c_{rig} as discussed in Section 3.9, is given by $(\lambda, \mu) \mapsto \lambda - \mu$.

Proof. The proof is similar to the local case (Proposition 3.13.6), except that we cannot take $\tilde{\mu} = \tilde{\mu}_0 + \varepsilon$ since ε is not supported on $\dot{V}$, but thanks to Lemma 2.4.4 we may find $\varepsilon' \in (K \otimes \mathbb{Q})[\dot{V}]_0$ such that $N^\natural(\varepsilon') = \varepsilon$ and set $\tilde{\mu} = \tilde{\mu}_0 + \varepsilon'$. Details are left to the reader. ∎

4. The global multiplicity formula

4.1. An obstruction

Let G^* be a quasi-split connected reductive group over a global field F, $\psi : G^* \to G$ an inner twist. We consider a strongly regular semi-simple element $\delta \in G(\mathbb{A})$ with the property that there is an element of $G^*(F)$ stably conjugate to δ. In this situation, Langlands has defined a cohomological obstruction to the existence of an F-point in the $G(\mathbb{A})$-conjugacy class of δ. We shall now review its definition and properties, following material from [18, 27]. We will then reinterpret this obstruction in terms of the global gerbes.

We shall first assume that G satisfies the Hasse principle, as the obstruction takes a more transparent form in that case.

The condition on δ is that there exists $\delta^* \in G^*(F)$ and $g \in G^*(\bar{\mathbb{A}})$ so that $\delta = \psi(g\delta^* g^{-1})$. Let T^* be the centralizer of δ^*. Let $u \in C^1(\Gamma, G^*(\bar{F}))$ be a lift of the element of $Z^1(\Gamma, G_{\mathrm{ad}}^*(\bar{F}))$ corresponding to ψ. For $\sigma \in \Gamma$ the element $g^{-1}u(\sigma)\sigma(g)$ lies in $T^*(\bar{\mathbb{A}})$. Its image in $T^*(\bar{\mathbb{A}})/T^*(\bar{F})$ is independent of the choice of u and this gives

a 1-cocycle $\Gamma \to T^*(\overline{\mathbb{A}})/T^*(\overline{F})$. Its cohomology class is independent of the choice of g and will be denoted by $\mathrm{obs}(\delta) \in H^1(\Gamma, T^*(\overline{\mathbb{A}})/T^*(\overline{F}))$. Denote by T the centralizer of δ in $G_{\mathbb{A}}$, a maximal torus defined over $\mathbb{A}$ (but generally not over F). The composition $\psi \circ \mathrm{Ad}(g)$ induced an isomorphism $\varphi_{\delta^*, \delta} : T_{\mathbb{A}}^* \to T$ of tori over $\mathbb{A}$ which does not depend on the choice of g.

Let $\delta' \in G(\mathbb{A})$ be stably conjugate to δ. That is, there exists $g \in G(\overline{\mathbb{A}})$ s.t. $\delta' = g\delta g^{-1}$. We can define $\mathrm{inv}(\delta, \delta') \in H^1(\Gamma, T(\overline{\mathbb{A}}))$ just as in the local case, namely as the class of $\sigma \mapsto g^{-1}\sigma(g)$, and $\mathrm{inv}(\delta, \delta') = \mathrm{inv}(\delta, \delta'')$ if δ' and δ'' are conjugate in $G(\mathbb{A})$. Under the isomorphism $H^1(\Gamma, T(\overline{\mathbb{A}})) = \bigoplus_v H^1(\Gamma_v, T(\overline{F}_v))$ the class $\mathrm{inv}(\delta, \delta')$ corresponds to $\sum_v \mathrm{inv}(\delta_v, \delta'_v)$ where $\mathrm{inv}(\delta_v, \delta'_v) \in H^1(\Gamma_v, T(\overline{F}_v))$ is the class of $\sigma \mapsto h_v^{-1}\sigma(h_v)$ for any $h_v \in G(\overline{F}_v)$ such that $\delta'_v = h_v \delta_v h_v^{-1}$ (we may take $h_v = g_v$).

Lemma 4.1.1. (1) *For $h \in G(\mathbb{A})$ we have $\mathrm{obs}(h\delta h^{-1}) = \mathrm{obs}(\delta)$.*

(2) *The class $\mathrm{obs}(\delta)$ is independent of the choice of δ^* in the following sense. If $\delta^{**} \in G^*(F)$ is another choice then there exists $h \in G^*(\overline{F})$ for which $h\delta^{**}h^{-1} = \delta^*$, then the isomorphism $\varphi_{\delta^{**}, \delta^*} : T^{**} \to T^*$ induced by $\mathrm{Ad}(h)$ is defined over F, it does not depend on the choice of h and it identifies the two versions of $\mathrm{obs}(\delta)$ obtained from δ^{**} and δ^*, respectively.*

(3) *If $\delta' \in G(\mathbb{A})$ is stably conjugate to $\delta \in G(\mathbb{A})$, then*

$$\mathrm{obs}(\delta') = \mathrm{obs}(\delta) \cdot \varphi_{\delta^*, \delta}^{-1}\big(\mathrm{inv}(\delta, \delta')\big).$$

Proof. The first claim is immediate. For the second, the version of the obstruction obtained from δ^{**} is represented by the 1-cocycle

$$h^{-1}g^{-1}u(\sigma)\sigma(gh) = \mathrm{Ad}(h^{-1})\big(g^{-1}u(\sigma)\sigma(g)\big) \cdot h^{-1}\sigma(h),$$

and $h^{-1}\sigma(h)$ lies in $Z^1(\Gamma, T^{**}(\overline{F}))$. The third claim follows from a similar direct computation. ∎

Proposition 4.1.2. *The class $\mathrm{obs}(\delta)$ vanishes if and only if the $G(\mathbb{A})$-conjugacy class of δ contains an F-point.*

Proof. If the $G(\mathbb{A})$-conjugacy class of δ contains an F-point, Lemma 4.1.1 allows us to replace δ by that F-point without changing $\mathrm{obs}(\delta)$. Then g can be chosen in $G^*(\overline{F})$ and so $g^{-1}u(\sigma)\sigma(g) \in T^*(\overline{F})$, showing that $\mathrm{obs}(\delta)$ vanishes.

Conversely, if the class of $g^{-1}u(\sigma)\sigma(g)$ in $H^1(\Gamma, T^*(\overline{\mathbb{A}})/T^*(\overline{F}))$ is trivial there exists $t \in T^*(\overline{\mathbb{A}})$ s.t. $(gt)^{-1}u(\sigma)\sigma(gt) \in T^*(\overline{F})$ for all $\sigma \in \Gamma$. We may replace g by gt and drop t from the notation. Now $z(\sigma) := \psi(g^{-1}u(\sigma)\sigma(g)u(\sigma)^{-1})$ is an element of $Z^1(\Gamma, G(\overline{F}))$ whose image in $Z^1(\Gamma, G(\overline{\mathbb{A}}))$ is cohomologically trivial, namely the coboundary of $\psi(g)$. By the Hasse principle for G there exists $h \in G^*(\overline{F})$ s.t.

$$\psi\big((gh^{-1})^{-1}u(\sigma)\sigma(gh^{-1})u(\sigma)^{-1}\big) = 1.$$

This means $\psi(gh^{-1}) \in G(\mathbb{A})$. Therefore $\delta' = \psi(gh^{-1})^{-1}\delta\psi(gh^{-1})$ lies in the $G(\mathbb{A})$-conjugacy class of δ. At the same time, $\delta' = \psi(h\delta^*h^{-1}) \in G(F)$. ∎

We now drop the condition that G satisfies the Hasse principle. Then it turns out that $H^1(\Gamma, T^*(\bar{\mathbb{A}})/T^*(\bar{F}))$ is not a suitable home for the obstruction any more. By work of Kneser, Harder, and Chernousov, G_{sc} does satisfy the Hasse principle. This will lead to a slight modification of $H^1(\Gamma, T_{\mathrm{sc}}^*(\bar{\mathbb{A}})/T_{\mathrm{sc}}^*(\bar{F}))$ that will serve as a replacement for $H^1(\Gamma, T^*(\bar{\mathbb{A}})/T^*(\bar{F}))$.

Lemma 4.1.3. *Let $\delta^* \in G^*(F)$. There exists $g \in G^*(\bar{\mathbb{A}})$ s.t. $\delta = \psi(g\delta^*g^{-1})$ if and only if there exists $g_{\mathrm{sc}} \in G_{\mathrm{sc}}^*(\bar{\mathbb{A}})$ s.t. $\delta = \psi(g_{\mathrm{sc}}\delta^*g_{\mathrm{sc}}^{-1})$.*

Proof. This is not immediate because the map $G_{\mathrm{sc}}^*(\bar{\mathbb{A}}) \to G^*(\bar{\mathbb{A}})$ need not be surjective. However, letting T^* be the centralizer of δ^*, we have $G_{\mathrm{sc}}^*(\bar{\mathbb{A}}) \cdot T^*(\bar{\mathbb{A}}) = G^*(\bar{\mathbb{A}})$. Indeed, letting E be a sufficiently large finite Galois extension of F, for almost all places w of E we have

$$G_{\mathrm{sc}}^*(O_{E_w}) \cdot T^*(O_{E_w}) = G^*(O_{E_w})$$

by [15, (3.3.4)]. ∎

Let $g_{\mathrm{sc}} \in G_{\mathrm{sc}}^*(\bar{\mathbb{A}})$ be so that $\delta = \psi(g_{\mathrm{sc}}\delta^*g_{\mathrm{sc}}^{-1})$. Let $u_{\mathrm{sc}} \in C^1(\Gamma, G_{\mathrm{sc}}^*(\bar{F}))$ be a lift of the element of $Z^1(\Gamma, G_{\mathrm{ad}}^*(\bar{F}))$. For $\sigma \in \Gamma$ the element $g_{\mathrm{sc}}^{-1}u_{\mathrm{sc}}(\sigma)\sigma(g_{\mathrm{sc}})$ lies in $T_{\mathrm{sc}}^*(\bar{\mathbb{A}})$. Its image in $T_{\mathrm{sc}}^*(\bar{\mathbb{A}})/T_{\mathrm{sc}}^*(\bar{F})$ is independent of the choice of u_{sc} and is a 1-cocycle. Its cohomology class is independent of the choice of g_{sc} and will be denoted by

$$\mathrm{obs}_{\mathrm{sc}}(\delta) \in H^1\big(\Gamma, T_{\mathrm{sc}}^*(\bar{\mathbb{A}})/T_{\mathrm{sc}}^*(\bar{F})\big).$$

As we shall remark below, $\mathrm{obs}_{\mathrm{sc}}(\delta)$ is a refinement of the element $\mathrm{obs}(\delta)$ defined above. We can also refine the invariant $\mathrm{inv}(\delta, \delta') \in H^1(\Gamma, T(\bar{\mathbb{A}}))$ of two stably conjugate $\delta, \delta' \in G(\mathbb{A})$ to an element $\mathrm{inv}_{\mathrm{sc}}(\delta, \delta') \in H^1(\Gamma, T_{\mathrm{sc}}(\bar{\mathbb{A}}))$ whose class is represented by $\sigma \mapsto g_{\mathrm{sc}}^{-1}\sigma(g_{\mathrm{sc}})$ for any $g_{\mathrm{sc}} \in G_{\mathrm{sc}}(\bar{\mathbb{A}})$ conjugating δ to δ' (the existence of such an element g_{sc} is explained [18, Section 6.2]). Then we have

Lemma 4.1.4. (1) *For $h \in G_{\mathrm{sc}}(\mathbb{A})$ we have $\mathrm{obs}_{\mathrm{sc}}(h\delta h^{-1}) = \mathrm{obs}_{\mathrm{sc}}(\delta)$.*

(2) *The class $\mathrm{obs}_{\mathrm{sc}}(\delta)$ is independent of the choice of δ^* in the following sense. If $\delta^{**} \in G^*(\bar{F})$ is another choice then the isomorphism $\varphi_{\delta^{**},\delta^*} : T_{\mathrm{sc}}^{**} \to T_{\mathrm{sc}}^*$ defined as in Lemma 4.1.1 identifies the two versions of $\mathrm{obs}_{\mathrm{sc}}(\delta)$ obtained from δ^{**} and δ^*, respectively.*

(3) *If $\delta' \in G(\mathbb{A})$ is stably conjugate to $\delta \in G(\mathbb{A})$ then*

$$\mathrm{obs}_{\mathrm{sc}}(\delta') = \mathrm{obs}_{\mathrm{sc}}(\delta) \cdot \varphi_{\delta^*,\delta}^{-1}\big(\mathrm{inv}_{\mathrm{sc}}(\delta, \delta')\big).$$

Proof. The same as for Lemma 4.1.1. ∎

Proposition 4.1.5. *The class $\mathrm{obs}_{\mathrm{sc}}(\delta)$ vanishes if and only if the $G_{\mathrm{sc}}(\mathbb{A})$-conjugacy class of δ contains an F-point.*

Proof. The same as for Proposition 4.1.2, but now the assumption that G_{sc} satisfies the Hasse principle is automatically satisfied by the work of Kneser, Harder, and Chernousov [5–7, 14]. ∎

We now define

(1) Δ to be the image of $\ker(H^1(\Gamma, T^*_{\mathrm{sc}}(\bar{\mathbb{A}})) \to H^1(\Gamma, T^*(\bar{\mathbb{A}})))$ under the natural map

$$H^1\big(\Gamma, T^*_{\mathrm{sc}}(\bar{\mathbb{A}})\big) \to H^1\big(\Gamma, T^*_{\mathrm{sc}}(\bar{\mathbb{A}})/T^*_{\mathrm{sc}}(\bar{F})\big) \tag{4.1.1}$$

(2) $\mathscr{R}(T/F)^D$ to be the quotient of $H^1(\Gamma, T^*_{\mathrm{sc}}(\bar{\mathbb{A}})/T^*_{\mathrm{sc}}(\bar{F}))$ by Δ

(3) $\mathrm{obs}(\delta) = \mathrm{obs}(\delta)_{\mathrm{sc}} + \Delta \subset H^1(\Gamma, T^*_{\mathrm{sc}}(\bar{\mathbb{A}})/T^*_{\mathrm{sc}}(\bar{F}))$, or equivalently the image of $\mathrm{obs}(\delta)_{\mathrm{sc}}$ in $\mathscr{R}(T/F)^D$.

Remark 4.1.6. Note that Δ lies in the kernel of the map

$$H^1\big(\Gamma, T^*_{\mathrm{sc}}(\bar{\mathbb{A}})/T^*_{\mathrm{sc}}(\bar{F})\big) \to H^1\big(\Gamma, T^*(\bar{\mathbb{A}})/T^*(\bar{F})\big)$$

and $\mathrm{obs}_{\mathrm{sc}}(\delta)$ maps to the element that we denoted by $\mathrm{obs}(\delta)$ when the Hasse principle holds. Thus, the new definition of $\mathrm{obs}(\delta)$ is a refinement of the old definition. When the Hasse principle holds the following lemma shows that Δ is precisely the kernel of this map, so the two definitions agree.

Lemma 4.1.7. *Assume that G^* satisfies the Hasse principle. Then Δ is equal to the kernel of the map*

$$H^1\big(\Gamma, T^*_{\mathrm{sc}}(\bar{\mathbb{A}})/T^*_{\mathrm{sc}}(\bar{F})\big) \to H^1\big(\Gamma, T^*(\bar{\mathbb{A}})/T^*(\bar{F})\big).$$

Note that G satisfies the Hasse principle if and only if G^* satisfies the Hasse principle, by [16, (4.2.2)] (by [5] the assumption on E_8 factors may be removed).

Proof. It remains to prove that the kernel of the map

$$H^1\big(\Gamma, T^*_{\mathrm{sc}}(\bar{\mathbb{A}})/T^*_{\mathrm{sc}}(\bar{F})\big) \to H^1\big(\Gamma, T^*(\bar{\mathbb{A}})/T^*(\bar{F})\big)$$

contains Δ. First we assume that the derived subgroup G^*_{der} of G^* is simply connected. Denote $D = G^*/G^*_{\mathrm{der}} = T^*/T^*_{\mathrm{sc}}$. The following diagram is commutative and its rows are exact.

$$\begin{array}{ccccc}
D(\mathbb{A}) & \longrightarrow & H^1\big(\Gamma, T^*_{\mathrm{sc}}(\bar{\mathbb{A}})\big) & \longrightarrow & H^1\big(\Gamma, T^*(\bar{\mathbb{A}})\big) \\
\downarrow & & \downarrow & & \downarrow \\
\big(D(\bar{\mathbb{A}})/D(\bar{F})\big)^\Gamma & \longrightarrow & H^1\big(\Gamma, T^*_{\mathrm{sc}}(\bar{\mathbb{A}})/T^*_{\mathrm{sc}}(\bar{F})\big) & \longrightarrow & H^1\big(\Gamma, T^*(\bar{\mathbb{A}})/T^*(\bar{F})\big)
\end{array}$$

By [16, Lemma 4.3.1 (b)], in which the case of E_8 factors does not need to be excluded by [5], the torus D also satisfies the Hasse principle, i.e., the map

$$D(\mathbb{A}) \to \big(D(\bar{\mathbb{A}})/D(\bar{F})\big)^\Gamma$$

is surjective.

In general, without assuming that G^*_{der} is simply connected, we choose a z-extension $\widetilde{G}^* \to G^*$ with kernel an induced torus S and $\widetilde{G}^*_{\mathrm{der}}$ simply connected. By [16, Lemma 4.3.2 (b)] $\widetilde{G}^*$ also satisfies the Hasse principle. Let $\widetilde{T}^*$ be the preimage of T^* in $\widetilde{G}^*$.

Because S is induced both $H^1(\Gamma, S(\bar{F}))$ and $H^1(\Gamma, S(\bar{\mathbb{A}})/S(\bar{F}))$ vanish and so the short exact sequence

$$1 \to S \to \tilde{T}^* \to T^* \to 1$$

yields injective maps

$$H^1\big(\Gamma, \tilde{T}^*(\bar{\mathbb{A}})\big) \hookrightarrow H^1\big(\Gamma, T^*(\bar{\mathbb{A}})\big)$$
$$H^1\big(\Gamma, \tilde{T}^*(\bar{\mathbb{A}})/\tilde{T}^*(\bar{F})\big) \hookrightarrow H^1\big(\Gamma, T^*(\bar{\mathbb{A}})/T^*(\bar{F})\big)$$

and so the lemma for $\tilde{G}^*$ implies it for G^*. ∎

Proposition 4.1.8. *The element* $\mathrm{obs}(\delta)$ *depends only on the* $G(\mathbb{A})$-*conjugacy class of* δ. *It is independent of the choice of* δ^*. *It vanishes if and only if the* $G(\mathbb{A})$-*conjugacy class of* δ *contains an* F-*point.*

Proof. The independence of δ^* is immediate from the second point in Lemma 4.1.4. For the other two statements we note that all elements in the $G(\mathbb{A})$-conjugacy class of δ are $G_{\mathrm{sc}}(\bar{\mathbb{A}})$-conjugate to each other by Lemma 4.1.3. Therefore the set of $G_{\mathrm{sc}}(\mathbb{A})$-conjugacy classes inside of the $G(\mathbb{A})$-conjugacy class of δ is in bijection with the set

$$S = \ker\big(H^1\big(\Gamma, T_{\mathrm{sc}}(\bar{\mathbb{A}})\big) \to H^1\big(\Gamma, G_{\mathrm{sc}}(\bar{\mathbb{A}})\big)\big) \cap \ker\big(H^1\big(\Gamma, T_{\mathrm{sc}}(\bar{\mathbb{A}})\big) \to H^1\big(\Gamma, T(\bar{\mathbb{A}})\big)\big),$$

namely via $\delta' \leftrightarrow \mathrm{inv}_{\mathrm{sc}}(\delta, \delta')$.

Note that S may only be a subset, and not a subgroup, of $H^1(\Gamma, T_{\mathrm{sc}}(\bar{\mathbb{A}}))$. The third point in Lemma 4.1.4 and Proposition 4.1.5 reduce the outstanding two statements of this proposition to showing that the image of $\varphi_{\delta^*,\delta}^{-1}(S)$ under (4.1.1) equals Δ.

Let S^* be the subgroup

$$\ker\big(H^1\big(\Gamma, T_{\mathrm{sc}}(\bar{\mathbb{A}})\big) \to H^1\big(\Gamma, T(\bar{\mathbb{A}})\big)\big),$$

so that Δ is the image of $\varphi_{\delta^*,\delta}^{-1}(S^*)$ under (4.1.1). Evidently $S \subset S^*$ and our goal is to show that the images of $\varphi_{\delta^*,\delta}^{-1}(S)$ and $\varphi_{\delta^*,\delta}^{-1}(S^*)$ under (4.1.1) coincide. The abelian group $H^1(\Gamma, T_{\mathrm{sc}}(\bar{\mathbb{A}}))$ is isomorphic to $\bigoplus_v H^1(\Gamma_v, T_{\mathrm{sc}}(\bar{F}_v))$ and this isomorphism translates the subgroup S^* to $\bigoplus S_v^*$, where S_v^* is the analogous subgroup at the place v, and it translates the set S to $\bigoplus_v S_v$, where S_v is the analogous set at the place v. Each S_v is a pointed set and the symbol $\bigoplus_v$ has the same meaning as for groups, so that $\bigoplus_v S_v$ consists of tuples of elements indexed by v, and in each tuple all but finitely many terms are neutral. When v is finite $H^1(\Gamma_v, G_{\mathrm{sc}}(\bar{F}_v)) = \{1\}$ by Kneser's theorem, so that $S_v = S_v^*$. Let $S_\infty^* = \bigoplus_{v|\infty} S_v^*$ and $S_\infty = \bigoplus_{v|\infty} S_v$. We may have $S_\infty \subsetneq S_\infty^*$. Since the group

$$\ker\big(H^1\big(\Gamma, T_{\mathrm{sc}}^*(\bar{F})\big) \to H^1\big(\Gamma, T^*(\bar{F})\big)\big)$$

maps to $\varphi_{\delta^*,\delta}^{-1}(S^*)$ and vanishes under (4.1.1) it will be enough to show that it maps surjectively onto

$$\varphi_{\delta^*,\delta}^{-1}(S_\infty^*) = \ker\big(H^1\big(\Gamma_\infty, T_{\mathrm{sc}}^*(\bar{F}_\infty)\big) \to H^1\big(\Gamma_\infty, T^*(\bar{F}_\infty)\big)\big).$$

For this we consider the commutative diagram

$$
\begin{array}{ccc}
H^1\big(\Gamma, T^*_{\mathrm{sc}}(\bar{F}) \to T^*(\bar{F})\big) & \longrightarrow & \ker\big(H^1\big(\Gamma, T^*_{\mathrm{sc}}(\bar{F})\big) \to H^1\big(\Gamma, T^*(\bar{F})\big)\big) \\
\downarrow & & \downarrow \\
H^1\big(\Gamma_\infty, T^*_{\mathrm{sc}}(\bar{F}_\infty) \to T^*(\bar{F}_\infty)\big) & \longrightarrow & \ker\big(H^1\big(\Gamma_\infty, T^*_{\mathrm{sc}}(\bar{F}_\infty)\big) \to H^1\big(\Gamma_\infty, T^*(\bar{F}_\infty)\big)\big)
\end{array}
$$

The horizontal maps are the connecting homomorphisms in the corresponding long exact sequences for the cohomology of complexes of tori. They are surjective. Applying restriction of scalars $F/\mathbb{Q}$ to T^* and T^*_{sc} we can reduce the base field to $\mathbb{Q}$ and then use [21, Lemma C.5.A] to see that the image of the left vertical map is dense. Since the target of the right vertical map is finite the proof is complete. ∎

We now assume again that G satisfies the Hasse principle. It is clear from the definitions that if $G = G^*$ then $\mathrm{obs}(\delta)$ is simply the image of $\mathrm{inv}(\delta^*, \delta)$ under

$$
H^1\big(\Gamma, T^*(\bar{\mathbb{A}})\big) \to H^1\big(\Gamma, T^*(\bar{\mathbb{A}})/T^*(\bar{F})\big).
$$

When $G \neq G^*$ then $\mathrm{inv}(\delta^*, \delta)$ doesn't make sense. However, using the cohomology of the global gerbe $\mathcal{E}^{\mathrm{iso}}$ we can make sense of it. More precisely, we need the versions of $\mathcal{E}^{\mathrm{iso}}$ with $\mathbb{A}$-coefficients and $\mathbb{A}/F$-coefficients. These are denoted by $\mathcal{E}_2(K/F)$ and $\mathcal{E}_1(K/F)$ respectively in [20, Section 1.5]. Here K/F is any sufficiently large finite Galois extension. We shall write $H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{iso}}_2, T(\bar{\mathbb{A}}))$ for $\varinjlim_K H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{iso}}_2(K/F), T^*(\mathbb{A}_K))$ and $H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{iso}}_1, T^*(\bar{\mathbb{A}})/T^*(\bar{F}))$ for $\varinjlim_K H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{iso}}_1(K/F), T^*(\mathbb{A}_K)/T^*(K))$.

Assume that the element of $Z^1(\Gamma, G^*_{\mathrm{ad}})$ corresponding to ψ has a lift

$$
z^{\mathrm{iso}} \in Z^1_{\mathrm{bas}}(\mathcal{E}^{\mathrm{iso}}, G^*).
$$

Define $\mathrm{inv}[z^{\mathrm{iso}}](\delta^*, \delta)$ to be the class in $H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{iso}}_2, T^*(\bar{\mathbb{A}}))$ of the 1-cocycle

$$
e \mapsto g^{-1} z^{\mathrm{iso}}(e)\sigma_e(g).
$$

Fact 4.1.9. *The image in $H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{iso}}_1, T^*(\bar{\mathbb{A}})/T^*(\bar{F}))$ of $\mathrm{inv}[z^{\mathrm{iso}}](\delta^*, \delta)$ lies in the subgroup $H^1(\Gamma, T^*(\bar{\mathbb{A}})/T^*(\bar{F}))$ and equals $\mathrm{obs}(\delta)$.*

Proof. Immediate from the fact that z^{iso} takes values in $T^*(\bar{F})$. ∎

This image can be computed in terms of Tate–Nakayama isomorphisms.

Proposition 4.1.10. *Denote $Y = X_*(T^*)$. The compositions*

$$
H^1_{\mathrm{alg}}\big(\mathcal{E}^{\mathrm{iso}}_2, T^*(\bar{\mathbb{A}})\big) \to \bigoplus_{v \in V} H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{iso}}_v, T^*) \simeq \bigoplus_{v \in V} Y/I_v(Y) \to Y/IY,
$$

where the isomorphism is Kottwitz' local Tate–Nakayama isomorphism and the last map is "sum over all places", and

$$
H^1_{\mathrm{alg}}\big(\mathcal{E}^{\mathrm{iso}}_2, T^*(\bar{\mathbb{A}})\big) \to H^1\big(\Gamma, T^*(\bar{\mathbb{A}})/T^*(\bar{F})\big) \simeq (Y/IY)[\mathrm{tor}] \subset Y/IY
$$

where the isomorphism is the Tate–Nakayama isomorphism defined in [34], are equal.

Proof. This is a special case of a more general compatibility. In [20] Kottwitz defined a generalization

$$H^1\big(\mathcal{E}_1^{\mathrm{iso}}, T^*(\bar{\mathbb{A}})/T^*(\bar{F})\big) \simeq Y/IY$$

of Tate's isomorphism, and proved local-global compatibility ((6.6) loc. cit.): the compositions

$$H^1_{\mathrm{alg}}\big(\mathcal{E}_2^{\mathrm{iso}}, T^*(\bar{\mathbb{A}})\big) \to \bigoplus_{v \in V} H^1_{\mathrm{alg}}(\mathcal{E}_v^{\mathrm{iso}}, T^*) \simeq \bigoplus_{v \in V} Y/I_v(Y) \to Y/IY$$

and

$$H^1_{\mathrm{alg}}\big(\mathcal{E}_2^{\mathrm{iso}}, T^*(\bar{\mathbb{A}})\big) \to H^1_{\mathrm{alg}}\big(\mathcal{E}_1^{\mathrm{iso}}, T^*(\bar{\mathbb{A}})/T^*(\bar{F})\big) \simeq I/IY$$

are equal. ∎

4.2. Global transfer factors

We continue with $\psi : G^* \to G$ from the previous subsection. Let $(H, \mathcal{H}, s, \eta)$ be an endoscopic datum for G and (H_1, η_1) a z-pair as in [21, Section 2.2]. We assume that for every place v of F there exists a pair of related strongly G-regular elements $\gamma_{0,v}^H \in H(F_v)$ and $\delta_{0,v} \in G(F_v)$.

Lemma 4.2.1. *Under this assumption there exists a pair of related strongly G-regular elements $\gamma_0^H \in H(F)$ and $\delta_0 \in G(\mathbb{A})$.*

Proof. The assumption is equivalent to the following one: for every place v of F, there exists a maximal torus $T_{H,v}$ of H_{F_v} and an admissible embedding $T_{H,v} \hookrightarrow G_{F_v}$, i.e., an isomorphism of $T_{H,v}$ with a maximal torus of G_{F_v} as in [21, Lemma 3.3.B]. Note that this assumption is automatically satisfied at every place v such that G_{F_v} is quasi-split, by essentially the same argument as in the proof of this lemma. By [2, Exp. XIV, Theorem 6.1] the functor X parametrizing maximal tori in H is representable by a smooth affine scheme X over H and X is rational, in particular it satisfies weak approximation. Let S be the finite set of places v such that G_{F_v} is not quasi-split. For any $v \in S$ the $H(F_v)$-conjugacy class of $T_{H,v}$ is a neighborhood of $T_{H,v}$ in $X(F_v)$ (for the natural topology)[3]. Therefore there exists a maximal torus T_H of H, that is an element of $X(F)$, such that for every $v \in S$ the maximal tori $(T_H)_{F_v}$ and $T_{H,v}$ of H_{F_v} are conjugate under $H(F_v)$.

Take any G-regular semisimple $\gamma_0^H \in T_H(F)$. Since G^* is quasi-split over F there exists a strongly regular $\delta_0^* \in G^*(F)$ that is related to γ^H. We have arranged that for any place v of F, there exists a strongly regular element $\delta_{0,v}$ of $G(F_v)$ that is stably conjugate to $\delta_{0,v}^*$, and we are left to show that we may take $\delta_{0,v} \in \underline{G}(\mathcal{O}_{F_v})$ for almost all places v, where $\underline{G}$ is any model of G over $\mathcal{O}_F[1/N]$ for some integer $N > 0$.

---

[3]Let N be the normalizer of the universal maximal torus, i.e., N is the obvious subgroup scheme of $H \times_F X \to X$. Then the morphism of X-schemes $H \times_F X \to X \times_F X$ which on points maps (g, T) to $(\mathrm{Ad}(g)T, T)$ is a right N-torsor for the étale topology by [2, Exp. XIV, Theorems 3.9 and 3.18]. In particular this morphism, like $N \to X$, is smooth. So for a given maximal torus $T_{H,v}$ of H_{F_v} the morphism $H_{F_v} \to X_{F_v}$ mapping g to $\mathrm{Ad}(g)T_{H,v}$ is smooth, and the induced map of analytic manifolds (in the sense of [32, Part II, Chapter III]) is submersive.

We can find $N > 0$ and models $\underline{G^*}$ and $\underline{G}$ over $\mathcal{O}_F[1/N]$ s.t. ψ extends to a map over $\mathcal{O}[1/N]$, where $\mathcal{O}$ is the ring of integers of the maximal extension of F unramified at all places prime to N. For a place v prime to N the element of $Z^1(\Gamma_v, G^*)$ corresponding to ψ lies in $Z^1(\Gamma_v/I_v, G^*(O_v))$, so there is an element $g_v \in G^*(O_v)$ s.t. $\psi \circ \mathrm{Ad}(g_v)$ is an isomorphism defined over O_{F_v}. We let $\delta_{0,v} = \psi(g_v \delta_0^* g_v^{-1}) \in \underline{G}(O_{F_v})$. For the finitely many places v that are not prime to N we choose $\delta_{0,v} \in G(F_v)$ to be stably conjugate to $\psi(\delta_0) \in G(\overline{F})$. That is, there is $g_v \in G^*(\overline{F}_v)$ s.t. $\delta_v = \psi(g_v \delta_0 g_v^{-1})$. The resulting collection δ_0 of local elements lies in $G(\mathbb{A})$. ∎

Under that assumption an adelic transfer factor $\Delta'_{\mathbb{A}}(\gamma^{H_1}, \delta)$ is defined for all pairs of strongly G-regular elements $\gamma^{H_1} \in H_1(\mathbb{A})$ and $\delta \in G(\mathbb{A})$. It is defined in [27, Section 6.3] and [21, Section 7.3] without the prime decoration, and is discussed in [22, Section 5.4] with the prime decoration.

By the construction in [21, Section 7.3] the factor $\Delta'_{\mathbb{A}}(\gamma^{H_1}, \delta)$ is a product of local transfer factors over all places. Since at each place the local transfer factor is canonical up to a scalar multiple, their product is also. What makes the global transfer factor completely canonical is the following property: If $\gamma^H \in H(F)$ and $\delta \in G(\mathbb{A})$ are related, then

$$\Delta'_{\mathbb{A}}(\gamma^{H_1}, \delta) = \langle \mathrm{obs}(\delta), \widehat{\varphi}_{\gamma^H, \delta^*}^{-1}(s) \rangle. \tag{4.2.1}$$

To explain the notation, note first that the condition $\gamma^H \in H(F)$ implies the existence of $\delta^* \in G^*(F)$ in the $G^*(\mathbb{A})$ conjugacy class of $\psi^{-1}(\delta)$, which was assumed in the definition of $\mathrm{obs}(\delta)$ that was reviewed in Section 4.1. Let $T^* \subset G^*$ be the centralizer of δ^* and let $T^H \subset H$ be the centralizer of γ^H. There is a unique admissible isomorphism $\varphi_{\gamma^H, \delta^*} : T^H \to T^*$ sending γ^H to δ^*. It induces an isomorphism $\widehat{T}^H \to \widehat{T}$ via which we transport $s \in [Z(\widehat{H})/Z(\widehat{G})]^\Gamma$ to an element of $[\widehat{T}/Z(\widehat{G})]^\Gamma$. Then we use the Tate–Nakayama pairing $\langle -, - \rangle$ between $H^1(\Gamma, T^*_{\mathrm{sc}}(\mathbb{A})/T^*_{\mathrm{sc}}(\overline{F}))$ and $\pi_0([\widehat{T}^*/Z(\widehat{G})]^\Gamma)$. Even though the obstruction $\mathrm{obs}(\delta)$ was not just an element of $H^1(\Gamma, T^*_{\mathrm{sc}}(\mathbb{A})/T^*_{\mathrm{sc}}(\overline{F}))$, but rather a set of elements there, the pairing is well-defined, because s has the property that its image under the connecting homomorphism

$$[Z(\widehat{H})/Z(\widehat{G})]^\Gamma \to H^1(\Gamma, Z(\widehat{G}))$$

is everywhere locally trivial. The subgroup of $[Z(\widehat{H})/Z(\widehat{G})]^\Gamma$ of all elements with this property is denoted by $\bar{\mathfrak{K}}(T^*/F)$ and is dual to the quotient $\bar{\mathfrak{K}}(T^*/F)^D$ recalled in Section 4.1.

Remark 4.2.2. We need to be careful with the normalizations of the various pairings we are using. Equation (4.2.1) appears optically compatible with [21, Corollary 7.3.B]. However, the latter is stated for the transfer factor $\Delta_{\mathbb{A}}$ constructed as in [27], rather than the factor $\Delta'_{\mathbb{A}}$ that we are using here, whose construction differs from $\Delta_{\mathbb{A}}$ by inverting the element s, see [22, Section 5.1]. The reason for preferring Δ' over Δ is that Δ itself does not properly generalize to the twisted setting, see [22].

Thus now (4.2.1) would seem optically at odds with the Δ'-version of [21, Corollary 7.3.B], which is the last displayed equation in [22] before Section 5.5 there. This discrepancy stems from yet another clash of conventions between [27] and [21]. First we recall that in the twisted setting $\mathrm{obs}(\delta)$ is an element of a hypercohomology group $H^1(\mathbb{A}/F, T^*_{\mathrm{sc}} \to T^*)$, while the element κ lies in $H^1(W_F, \widehat{T}^* \to \widehat{T}^*/Z(\widehat{G}))$. The maps in the two complexes are $1 - \theta^*$ and $1 - \widehat{\theta}^*$ respectively, where θ^* is the twisting automorphism. In the untwisted setting $\theta^* = 1$ and the two hypercohomology groups each break up into direct products

$$H^1(\mathbb{A}/F, T^*_{\mathrm{sc}} \to T^*) = H^1(\mathbb{A}/F, T^*_{\mathrm{sc}}) \oplus H^0(\mathbb{A}/F, T^*)$$

and

$$H^1(W_F, \widehat{T}^* \to \widehat{T}^*/Z(\widehat{G})) = H^1(W_F, \widehat{T}^*) \oplus H^0(W_F, \widehat{T}^*/Z(\widehat{G})).$$

The pairing between the two hypercohomology groups becomes the product of the standard normalization of the Tate–Nakayama pairing between

$$H^1(\mathbb{A}/F, T^*_{\mathrm{sc}}) \quad \text{and} \quad H^0(W_F, \widehat{T}^*/Z(\widehat{G}))$$

and the *negative* of the standard normalization of the Langlands pairing between

$$H^0(\mathbb{A}/F, T^*) \quad \text{and} \quad H^1(W_F, \widehat{T}^*).$$

The occurrence of this negative is forced by the anti-commutativity of the cup product.

The reason why the Δ'-version of [21, Corollary 7.3.B] is compatible with (4.2.1) is that the projection of the element $\mathrm{obs}(\delta) \in H^1(\mathbb{A}/F, T^*_{\mathrm{sc}} \to T^*)$ constructed in [21] onto the direct factor $H^1(\mathbb{A}/F, T^*_{\mathrm{sc}})$ is the *inverse* of the element $\mathrm{obs}(\delta)$ constructed in Section 4.1 above. For this we direct the reader to [21, pp. 82–83] and point out that the element $v(\sigma)$ constructed there is the 1-cocycle that represents our element $\mathrm{obs}(\delta)$ here (despite the element g there being the inverse of our element g here), yet the class $\mathrm{obs}(\delta)$ constructed there contains the inverse of $v(\sigma)$. The conventions we have used here are those used in [27, (3.4)] in the quasi-split case, which are the opposite of those used in [21, Section 5.3] in the quasi-split case.

Finally, we remark that the same issue occurs with the local and adelic invariant $\mathrm{inv}(\delta, \delta')$ used in this paper – it follows the quasi-split convention in [27], and is the opposite of the quasi-split convention in [21], so any equation involving $\mathrm{inv}(\delta, \delta')$ in this paper will contain an inverse when compared to the Δ'-version of the corresponding equation of [22], and would thus appear optically identical to the corresponding equation in [21] for the factor Δ in place of Δ'.

4.3. Global transfer factors in terms of isocrystals

In this subsection we assume that $Z(G)$ is connected and G satisfies the Hasse principle. We will show how the canonical adelic transfer factor reviewed in Section 4.2 can be written as a product of local transfer factors that are normalized using $\widetilde{B}(G)$.

Let again $(H, \mathcal{H}, s, \eta)$ be an endoscopic datum for G and (H_1, η_1) a z-pair as in [21, Section 2.2]. By assumption $s \in [Z(\hat{H})/Z(\hat{G})]^\Gamma$ and the image of s in $Z^1(\Gamma, Z(\hat{G}))$ under the connecting homomorphism has cohomologically trivial localization at each place of F. The Hasse principle for G, reinterpreted as [16, Lemma 11.2.2], implies that the image of s in $Z^1(\Gamma, Z(\hat{G}))$ is already cohomologically trivial and thus s lifts to an element $s^\natural \in Z(\hat{H})^\Gamma$. We shall refer to $(H, \mathcal{H}, s^\natural, \eta)$ as an *isocrystal-refined* endoscopic datum. An isomorphism of such data is required to identify the two elements $s^\natural$, and not simply their images s.

By Corollary 3.13.13 we can choose $z^{\mathrm{iso}} \in Z^1_{\mathrm{bas}}(\mathcal{E}^{\mathrm{iso}}, G^*)$ such that $\psi^{-1}\sigma(\psi) = \mathrm{Ad}(\bar{z}_\sigma)$, where $\bar{z} \in Z^1(\Gamma, G^*_{\mathrm{ad}})$ is the image of z^{iso}. Let $\mathfrak{w}$ be a global Whittaker datum for G^*. At each place v of F we now have the normalized local transfer factor $\Delta[\mathfrak{w}_v, z^{\mathrm{iso}}_v] : H_1(F_v)_{\mathrm{sr}} \times G(F_v)_{\mathrm{sr}} \to \mathbb{C}$, defined as

$$\Delta[\mathfrak{w}_v, z^{\mathrm{iso}}_v](\gamma^{H_1}, \delta) = \Delta[\mathfrak{w}_v](\gamma^{H_1}, \delta^*) \cdot \langle \mathrm{inv}[z^{\mathrm{iso}}_v](\delta^*, \delta), \hat{\varphi}_{\delta^*, \gamma^H}(s^\natural) \rangle. \qquad (4.3.1)$$

We need to explain the notation. On the left we have $\gamma^{H_1} \in H_1(F_v)_{\mathrm{sr}}$ and $\delta \in G(F_v)_{\mathrm{sr}}$. We choose arbitrarily $\delta^* \in G^*(F_v)$ that is conjugate in $G^*(\bar{F}_v)$ to $\psi^{-1}(\delta)$. For $\Delta[\mathfrak{w}_v](\gamma^{H_1}, \delta^*)$ to be non-zero it is necessary that γ_1 be a norm of δ, so we make this assumption. Let $T^* \subset G^*$ be the centralizer of δ^*, a maximal torus of G^*. Given any $g \in G^*(\bar{F})$ with $\delta = \psi(g\delta^* g^{-1})$, the element $g^{-1} z^{\mathrm{iso}}(e)\sigma_e(g)$ of $G^*(\bar{F}_v)$ belongs to $T^*(\bar{F}_v)$, for all $e \in \mathcal{E}^{\mathrm{iso}}_v$, where $\sigma_e \in \Gamma_v$ denotes the image of e under $\mathcal{E}^{\mathrm{iso}}_v \to \Gamma_v$. For formal reasons the map $e \mapsto g^{-1} z^{\mathrm{iso}}(e)\sigma_e(g)$ is an element of $Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{iso}}_v, T^*)$ and its cohomology class is independent of the choice of g. We denote by $\mathrm{inv}[z^{\mathrm{iso}}_v](\delta^*, \delta)$ this cohomology class.

The element $\gamma^H \in H(F_v)_{\mathrm{sr}}$ is the image of γ^{H_1} under $H_1 \to H$. Letting $T^H \subset H$ be its centralizer, a maximal torus of H, there is a unique admissible isomorphism $\varphi_{\delta^*, \gamma^H} : T^* \to T^H$ mapping δ^* to γ^H. Its dual, when composed with the canonical embedding $Z(\hat{H}) \to \hat{T}^H$, transports $s^\natural \in Z(\hat{H})^{\Gamma_v}$ into $\hat{T}^{*, \Gamma_v}$. The pairing $\langle -, - \rangle : H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{iso}}_v, T^*) \times \hat{T}^{*, \Gamma_v}$ is given by [20, Lemma 8.1].

The transfer factor $\Delta[\mathfrak{w}_v] : H_1(F_v)_{\mathrm{sr}} \times G^*(F_v)_{\mathrm{sr}}$ is the Whittaker normalization of the factor Δ', as defined in [22, (5.5.2)].

Proposition 4.3.1. *The function* $\Delta[\mathfrak{w}_v, z^{\mathrm{iso}}_v]$ *is an absolute transfer factor.*

Proof. This is proved in [8, Proposition 2.2.1], under the assumption that the z-pair is trivial, and with the inverse of the pairing used here. Nonetheless, the proof given there carries over to this situation with trivial modifications. ∎

Proposition 4.3.2. *Let* $\gamma^{H_1} \in H_1(\mathbb{A})_{\mathrm{sr}}$ *and* $\delta \in G(\mathbb{A})_{\mathrm{sr}}$. *Let* V *denote the set of all places of* F. *For almost all* $v \in V$ *the factor* $\Delta[\mathfrak{w}_v, z^{\mathrm{iso}}_v](\gamma^{H_1}_v, \delta_v)$ *is equal to* 1 *and the product*

$$\prod_{v \in V} \Delta[\mathfrak{w}_v, z^{\mathrm{iso}}_v](\gamma^{H_1}_v, \delta_v)$$

is equal to the canonical adelic transfer factor $\Delta'_{\mathbb{A}}(\gamma^{H_1}, \delta)$.

Proof. The adelic transfer factor was reviewed in Section 4.2. By construction it is a product of local transfer factors, and so is the product in this proposition. Therefore one is a scalar multiple of the other. It is enough to show that they give the same value on one pair (γ^{H_1}, δ) of related elements. We choose the pair so that $\gamma^{H_1} \in H_1(F)$. The existence of such an element is assumed in the definition of the global transfer factor. Then we can choose $\delta^* \in G(F)$ related to γ^{H_1}.

We first claim that almost all factors in the product $\prod_v \Delta[\mathfrak{w}_v](\gamma^{H_1}, \delta^*)$ are equal to 1 and the product itself is equal to 1. For this we recall that this factor is the product of terms $\varepsilon_v \cdot \Delta_I \cdot \Delta_{II} \cdot \Delta_{III_1} \cdot \Delta_{III_2} \cdot \Delta_{IV}$. The terms ε_v are the local components of a global root number of an orthogonal Artin representation of degree 0. Therefore almost all terms are equal to 1 and their product is equal to 1. For the remaining terms we apply Theorem 6.4.A and Corollary 6.4.B of [27].

This reduces to showing that almost all factors in the product

$$\prod_v \left\langle \mathrm{inv}[z_v^{\mathrm{iso}}](\delta^*, \delta), \widehat{\varphi}_{\delta^*, \gamma^H}(s^\natural) \right\rangle$$

are equal to 1 and this product equals $\Delta'_{\mathbb{A}}(\gamma^{H_1}, \delta)$. This follows from Fact 4.1.9 and the equality

$$\prod_v \left\langle \mathrm{inv}[z_v^{\mathrm{iso}}](\delta^*, \delta), \widehat{\varphi}_{\delta^*, \gamma^H}(s^\natural) \right\rangle = \left\langle \sum_v \mathrm{inv}[z_v^{\mathrm{iso}}](\delta^*, \delta), \widehat{\varphi}_{\delta^*, \gamma^H}(s^\natural) \right\rangle$$

$$= \left\langle \mathrm{inv}[z^{\mathrm{iso}}](\delta^*, \delta), \widehat{\varphi}_{\delta^*, \gamma^H}(s^\natural) \right\rangle$$

where the pairings in the first term are the local pairings

$$H^1_{\mathrm{alg}}\big(\mathcal{E}_v^{\mathrm{iso}}, T^*(\overline{F}_v)\big) \times X^*(\widehat{T}^{*, \Gamma_v}) \to \mathbb{C},$$

the pairing in second term is the pairing

$$H^1_{\mathrm{alg}}\big(\mathcal{E}_2^{\mathrm{iso}}, T^*(\overline{\mathbb{A}})\big) \times \big(X^*(\widehat{T}^*) \otimes \mathbb{Z}[V]\big)_\Gamma \to \mathbb{C},$$

and that in the third term is

$$H^1_{\mathrm{alg}}\big(\mathcal{E}_1^{\mathrm{iso}}, T^*(\overline{\mathbb{A}})/T^*(\overline{F})\big) \times X^*(\widehat{T}^{*, \Gamma}) \to \mathbb{C}.$$

The last equality follows from Proposition 4.1.10. ∎

Remark 4.3.3. Every $f \in \mathcal{C}_c^\infty(G(\mathbb{A}))$ has a *canonical* transfer $f^{H_1} \in \mathcal{C}_c^\infty(H_1(\mathbb{A}))$, up to functions having identically vanishing stable orbital integrals. Namely, choose an arbitrary decomposition $\Delta'_{\mathbb{A}} = \prod_v \Delta'_v$ of the canonical adelic transfer factor as a product of local transfer factors, such that at almost all places Δ'_v is unramified. Assume without loss of generality that f is a decomposable function $f = \otimes_v f_v$. Let $f_v^{H_1} \in \mathcal{C}_c^\infty(H_1(F_v))$ be the transfer of f_v relative to the factor Δ'_v and let $f^{H_1} = \otimes_v f_v^{H_1}$. Each individual local component $f_v^{H_1}$ depends on the choice of Δ'_v, which is well-defined up to a complex

scalar, but the formula $\Delta'_{\mathbb{A}} = \prod_v \Delta'_v$ implies that f^{H_1} is independent of these choices. Note that, due to the Fundamental Lemma, the function $f_v^{H_1}$ is the unit in the unramified Hecke algebra for almost all v, and therefore $\otimes_v f_v^{H_1}$ is well-defined.

In particular, Proposition 4.3.2 implies that if we take $f_v^{H_1}$ to be the transfer relative to $\Delta[\mathfrak{w}_v, z_v^{\text{iso}}]$, then $\otimes_v f_v^{H_1}$ will be the canonical adelic transfer. Analogously, [11, Proposition 4.4.1] implies the same statement with $\Delta[\mathfrak{w}_v, z_v^{\text{iso}}]$ replaced by the transfer factor normalized using $\mathcal{E}^{\text{rig}}$ in place of $\mathcal{E}^{\text{iso}}$.

Remark 4.3.4. According to [23, Section 2.4] the product formula proved in Proposition 4.3.2 generalizes from the case of strongly regular pairs (δ, γ^{H_1}) to the case of (G, H)-regular pairs, i.e., the assumption of [18, Section 6.10] holds. The reader is encouraged to consult [3, Section 3.4], where a detailed argument of this claim has been provided.

4.4. Global transfer factors in the rigid setting

In this section we do not assume that the connected reductive group G over F satisfies the Hasse principle, nor that $Z(G)$ is connected. The analogue of Proposition 4.3.2 in the rigid setting was proved in [11, Section 4.4], under the assumption that there exists a pair of related elements $\gamma^H \in H(F)$ and $\delta \in G(F)$. Note that this is stronger than the assumption on the existence of pair of related elements $\gamma^H \in H(F)$ and $\delta \in G(\mathbb{A})$. The reason this stronger assumption was made in [11, Section 4.4] is that it is also made on [27, p. 268], where the authors declare that the global adelic transfer factors should vanish if this stronger assumption is not satisfied.

On the other hand, the constructions in [21] operate under the weaker assumption on the existence of related elements $\gamma^H \in H(F)$ and $\delta \in G(\mathbb{A})$. It is clear to us that if the weaker assumption is not satisfied then the endoscopic datum is not needed for the stabilization of the trace formula and its transfer factors can be declared zero, see Lemma 4.2.1. However, it is not clear to us that this is true if the stronger assumption is not satisfied. It could be that a Hasse principle for this assumption holds, but we do not know if it holds, let alone of any published proof. It may be possible to approach this problem using the classification of Dynkin diagrams (similar to [24]), after reducing to the case where G is absolutely simple and simply connected and the endoscopic datum is elliptic.

Instead, we have decided to generalize here the proof of [11, Proposition 4.4.1] by dropping the stronger assumption and only keeping the weaker assumption. To replicate the proof of Proposition 4.3.2 in the rigid setting, we want an analogue of Proposition 4.1.10. The natural strategy would be to introduce extensions of Γ bound by $\bar{\mathbb{A}}$- and "$\bar{\mathbb{A}}/\bar{F}$-points" of certain projective limits of finite multiplicative groups over F (analogous to the gerbes $\mathcal{E}^1$ and $\mathcal{E}^2$ of [20]), prove Tate–Nakayama isomorphisms and local-global compatibility (analogous to [20, (6.6)]). This is certainly possible and was in fact mostly done in the preparations to [11]. However, it takes a fair amount of pages to set up. We have thus chosen an alternative approach here, which uses the result in the "iso" setting (Proposition 4.1.10) and the comparison with "rig" via "mid" to deduce the result in the rigid setting (Proposition 4.4.3).

Lemma 4.4.1. *Let T be a torus over F, $Z \subset T$ a finite multiplicative subgroup. Then the diagram of Abelian groups*

$$H^1\big(P_{\dot{V}}^{\mathrm{rig}} \to \mathcal{E}_{\dot{V}}^{\mathrm{rig}}, Z(\bar{F}) \to T(\bar{\mathbb{A}})\big) \longrightarrow \bigoplus_{v \in V} H^1(P_v^{\mathrm{rig}} \to \mathcal{E}_v^{\mathrm{rig}}, Z \to T)$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\mathrm{Hom}(P_{\dot{V}}^{\mathrm{rig}}, Z) \longrightarrow \bigoplus_{v \in V} \mathrm{Hom}(P_v^{\mathrm{rig}}, Z)$$

is Cartesian.

Proof. We have to prove that the map from $H^1(P_{\dot{V}}^{\mathrm{rig}} \to \mathcal{E}_{\dot{V}}^{\mathrm{rig}}, Z(\bar{F}) \to T(\bar{\mathbb{A}}))$ to the fiber product is injective and surjective. Injectivity follows easily from the well-known isomorphism

$$H^1\big(\Gamma, T(\bar{\mathbb{A}})\big) \simeq \bigoplus_{v \in V} H^1\big(\Gamma_v, T(\overline{F_v})\big).$$

To prove surjectivity, let

$$\mu \in \mathrm{Hom}(P_{\dot{V}}^{\mathrm{rig}}, Z) \quad \text{and} \quad ([z_v]) \in \bigoplus_{v \in V} H^1(P_v^{\mathrm{rig}} \to \mathcal{E}_{\dot{v}}^{\mathrm{rig}}, Z \to T)$$

be such that for any $v \in V$, the restriction of z_v to P_v^{rig} equals μ composed with $\mathrm{loc}_v : P_v^{\mathrm{rig}} \to P_{\dot{V}}^{\mathrm{rig}}$. Recall from [11, Corollary 3.3.8] that for any $v \in V$ there is $\eta_v \in C^1(\Gamma_v, P_{\dot{V}}^{\mathrm{rig}})$ unique up to $Z^1(\Gamma_v, P_{\dot{V}}^{\mathrm{rig}}) = B^1(\Gamma_v, P_{\dot{V}}^{\mathrm{rig}})$ such that

$$\xi_{\dot{V}}^{\mathrm{rig}}|_{\Gamma_{\dot{v}}^2} = (\mathrm{loc}_v \circ \xi_v^{\mathrm{rig}}) \times \mathrm{d}(\eta_v).$$

Let S be a finite set of places of F containing all Archimedean places and such that $[z_v] = 1$ for $v \in V \smallsetminus S$, so that in particular $\mu \circ \mathrm{loc}_v$ is trivial. Recall from Section 3.1 that for any $v \in V$ the cocycle z_v may be seen as an element of $C^1(\Gamma_v, T(\overline{F_v}))$ satisfying $\mathrm{d}(z_v) = \mu \circ \mathrm{loc}_v \circ \xi_v^{\mathrm{rig}}$. There exists a finite Galois extension E/F satisfying

- the action of Γ on $X^*(Z)$ factors through $\Gamma_{E/F}$,
- E contains all roots of unity of order dividing the exponent of Z (in particular $Z(E) = Z(\bar{F})$), and
- for all $v \in S$ the cochain z_v is inflated from an element of $C^1(\Gamma_{E_{\dot{v}}/F_v}, T(E_{\dot{v}}))$ and $\mu \circ \eta_v$ is inflated from an element of $C^1(\Gamma_{E_{\dot{v}}/F_v}, Z(E_{\dot{v}}))$.

For $v \in S$ let $\tilde{z}_v^{(0)} = \dot{S}^1(z_v \times (\mu \circ \eta_v)) \in C^1(\Gamma_{E/F}, T(E \otimes_F F_v))$ where $\dot{S}^1$ is a Shapiro corestriction map associated to a section of $\Gamma_{E/F} \to \Gamma_{E_{\dot{v}}/F_v} \backslash \Gamma_{E/F}$ as explained in [11, Appendix B]. We have

$$\big[\mathrm{d}(\tilde{z}_v^{(0)})\big] = \mu_*[\xi_{\dot{V}}^{\mathrm{rig}}] \quad \text{in } H^2\big(\Gamma_{E/F}, Z(E \otimes_F F_v)\big) \simeq H^2\big(\Gamma_{E_{\dot{v}}/F_v}, Z(E_{\dot{v}})\big)$$

(Shapiro isomorphism), therefore there exists $a_v \in C^1(\Gamma_{E/F}, Z(E \otimes_F F_v))$ such that $\tilde{z}_v := \tilde{z}_v^{(0)} \times \mathrm{d}(a_v)$ satisfies $\mathrm{d}(\tilde{z}_v) = \mu_*(\xi_{\dot{V}}^{\mathrm{rig}})$ (via the embedding $Z(E) \subset Z(E \otimes_F F_v)$).

For $v \in V \smallsetminus S$ we have $\mu_*([\xi_{\dot{V}}^{\mathrm{rig}}]) = 1$ in $H^2(\Gamma_{E/F}, Z(E \otimes_F F_v))$, thanks to

- the Shapiro isomorphism $H^2(\Gamma_{E/F}, Z(E \otimes_F F_v)) \simeq H^2(\Gamma_{E_{\dot{v}}/F_v}, Z(E_{\dot{v}}))$,

- the compatibility between the local and global canonical classes which implies that the image of $\mu_*([\xi_{\dot{V}}^{\mathrm{rig}}])$ in $H^2(\Gamma_v, Z)$ equals $(\mu \circ \mathrm{loc}_v)_*(\xi_v^{\mathrm{rig}})$,

- the fact that $\mu \circ \mathrm{loc}_v : P_v^{\mathrm{rig}} \to Z$ is trivial because $[z_v] = 1$,

- the local analogue of [11, Lemma 3.2.7] (similar proof) which says that the inflation map $H^2(\Gamma_{E_{\dot{v}}/F_v}, Z(E_{\dot{v}})) \to H^2(\Gamma_v, Z)$ is injective.

So for $v \in V \smallsetminus S$ we can find $\tilde{z}_v \in C^1(\Gamma_{E/F}, Z(E \otimes_F F_v))$ satisfying $\mathrm{d}(\tilde{z}_v) = \mu_*(\xi_{\dot{V}}^{\mathrm{rig}})$. Now $(\tilde{z}_v)_{v \in V}$ represents an element of $H^1(P_{\dot{V}}^{\mathrm{rig}} \to \mathcal{E}_{\dot{V}}^{\mathrm{rig}}, Z(\overline{F}) \to T(\overline{\mathbb{A}}))$ whose restriction to $P_{\dot{V}}^{\mathrm{rig}}$ is μ and mapping to $([z_v])_v$. ∎

Corollary 4.4.2. *In the same setting, denote $Y = X_*(T)$ and $\overline{Y} = X_*(T/Z)$. Then there is a unique Tate–Nakayama isomorphism between*

$$Y_{\mathrm{sa}}^{\mathrm{rig}}(Z \to T) := \Big\{ (\lambda_v)_v \in \bigoplus_{v \in V} (\overline{Y}/I_{\dot{v}}(Y))[\mathrm{tor}] \mid \sum_{v \in V} (\lambda_v + IY) \in (Y/IY)[\mathrm{tor}] \Big\}$$

and $H^1(P_{\dot{V}}^{\mathrm{rig}} \to \mathcal{E}_{\dot{V}}^{\mathrm{rig}}, Z(\overline{F}) \to T(\overline{\mathbb{A}}))$ which is compatible with the local Tate–Nakayama isomorphisms and the identification

$$\mathrm{Hom}(P_{\dot{V}}^{\mathrm{rig}}, Z) \simeq \varinjlim_E \big((\overline{Y}/Y)[\dot{V}_E]_0 \big)^{N_{E/F} = 0}.$$

Proof. We have an obvious Cartesian diagram

$$
\begin{array}{ccc}
Y_{\mathrm{sa}}^{\mathrm{rig}}(Z \to T) & \longrightarrow & \bigoplus_{v \in V} (\overline{Y}/I_{\dot{v}}(Y))[\mathrm{tor}] \\
\downarrow & & \downarrow \\
\varinjlim_E \big((\overline{Y}/Y)[\dot{V}_E]_0 \big)^{N_{E/F}=0} & \longrightarrow & \bigoplus_{v \in V} (\overline{Y}/Y)^{N_{\dot{v}}=0}
\end{array}
$$

and comparing with the Cartesian diagram in the previous lemma gives the sought isomorphism. ∎

Proposition 4.4.3. *We have a commutative diagram*

$$
\begin{array}{ccc}
Y_{\mathrm{sa}}^{\mathrm{rig}}(Z \to T) & \overset{\sim}{\longrightarrow} & H^1\big(P_{\dot{V}}^{\mathrm{rig}} \to \mathcal{E}_{\dot{V}}^{\mathrm{rig}}, Z(\overline{F}) \to T(\overline{\mathbb{A}}) \big) \\
\downarrow & & \downarrow \\
(Y/IY)[\mathrm{tor}] & \overset{\sim}{\longrightarrow} & H^1\big(\Gamma, T(\overline{\mathbb{A}})/T(\overline{F}) \big)
\end{array}
$$

where the top horizontal isomorphism was defined in the previous corollary, the bottom isomorphism was defined in [34], the left vertical map is $(\lambda_v)_v \mapsto \sum_{v \in V} (\lambda_v + IY)$ and the right vertical map is the obvious one.

Proof. The kernel of the right vertical map is the image of $H^1(P_{\dot{V}}^{\text{rig}} \to \mathcal{E}_{\dot{V}}^{\text{rig}}, Z \to T)$ and by the local and global Tate–Nakayama isomorphisms for "rig" and their compatibility, we have an embedding ι^{rig} from the cokernel C^{rig} of

$$H^1(P_{\dot{V}}^{\text{rig}} \to \mathcal{E}_{\dot{V}}^{\text{rig}}, Z \to T) \to H^1\big(P_{\dot{V}}^{\text{rig}} \to \mathcal{E}_{\dot{V}}^{\text{rig}}, Z(\bar{F}) \to T(\bar{\mathbb{A}})\big)$$

into $(Y/IY)[\text{tor}]$, mapping $(\lambda_v)_v$ to $\sum_{v \in V}(\lambda_v + IY)$. To prove that the diagram in the proposition is commutative we have to show that ι^{rig} coincides with the composition

$$C^{\text{rig}} \to H^1\big(\Gamma, T(\bar{\mathbb{A}})/T(\bar{F})\big) \simeq (Y/IY)[\text{tor}].$$

For the rest of this proof we take the inductive limit over all finite multiplicative subgroups Z of T, as we may since all morphisms in sight are compatible with the transition maps induced by any inclusion $Z \subset Z'$. This has the effect of replacing $\bar{Y}$ by $\mathbb{Q}Y$.

Lemma 4.4.1 and Corollary 4.4.2 admit "iso" and "mid" analogues, with similar proofs and with

$$Y_{\text{sa}}^{\text{iso}} = \Big\{(\lambda_v)_v \in \bigoplus_{v \in V} Y/I_{\dot{v}}(Y) \mid N_{E/F}\Big(\sum_{v \in V} \lambda_v\Big) = 0\Big\}$$

and

$$Y_{\text{sa}}^{\text{mid}} = \Big\{\big((\lambda_v)_v, (\mu_v)_v\big) \mid (\lambda_v)_v \in Y_{\text{sa}}^{\text{mid}}, \ (\mu_v)_v \in \bigoplus_{v \in V} \mathbb{Q}Y, \ \sum_{v \in V} \mu_v = 0$$
$$\text{and } \forall v \in V, \ N_{E_{\dot{v}}/F_v}(\lambda_v - \mu_v) = 0\Big\}$$

where E/F is any finite Galois extension splitting T. We also have an embedding ι^{iso} (resp. ι^{mid}) from C^{iso} (resp. C^{mid}) into $(Y/IY)[\text{tor}]$, mapping the class of $(\lambda_v)_v$ (resp. $((\lambda_v)_v, \mu))$ to $\sum_{v \in V} \lambda_v + IY$. We have natural maps between cokernels

$$C^{\text{iso}} \longleftarrow C^{\text{mid}} \to C^{\text{rig}}$$

compatible with $\iota^{\text{iso}}, \iota^{\text{mid}}, \iota^{\text{rig}}$ thanks to the compatibility of local Tate–Nakayama isomorphisms in the three settings. It follows from Proposition 3.4.1 (or a direct argument) that the natural map $Y_{\text{sa}}^{\text{mid}} \to Y_{\text{sa}}^{\text{iso}}$ is surjective, thus $C^{\text{mid}} \to C^{\text{iso}}$ is bijective. It is obvious that ι^{iso} is surjective, and by Proposition 4.1.10 it coincides with the composition

$$C^{\text{iso}} \to H^1\big(\Gamma, T(\bar{\mathbb{A}})/T(\bar{F})\big) \simeq (Y/IY)[\text{tor}].$$

Therefore ι^{mid} and ι^{rig} are also bijective and coincide with the compositions

$$C^? \to H^1\big(\Gamma, T(\bar{\mathbb{A}})/T(\bar{F})\big) \simeq (Y/IY)[\text{tor}]. \qquad \blacksquare$$

Corollary 4.4.4. *Proposition 4.4.1 of* [11] *holds without the assumption that there exists a pairs of related elements in* $H_1(F)_{\text{sr}} \times G(F)_{\text{sr}}$.

Proof. Similar to the proof of Proposition 4.3.2, applying Proposition 4.4.3 to T_{sc}^* instead of Proposition 4.1.10 to T^*. Details are left to the reader. $\qquad \blacksquare$

4.5. Isocrystal local Langlands correspondence in the Archimedean case

In order to formulate the isocrystal version of the conjectural multiplicity formula for a connected reductive group over a number field we will need the isocrystal version of the refined local Langlands conjecture at all places. In the non-archimedean case this was formulated in [12]. Fortunately the Archimedean case is similar, and we shall formulate it here, as well as the comparison between the isocrystal and rigid versions.

The complex case $F \simeq \mathbb{C}$ is very simple: the group P^{rig} is trivial and so is $\mathcal{E}^{\mathrm{rig}}$. For any connected reductive group G over F and any tempered parameter $\varphi : F^\times \to \widehat{G}$ the group $S_\varphi^\natural$ defined as in [12, Section 4.1] is canonically isomorphic to $\widehat{G}/\widehat{G}_{\mathrm{der}} = \widehat{C}$ where $C = Z(G)^0$. Since $Z(\widehat{G}) \to \widehat{C}$ is an isogeny, if we fix $z^{\mathrm{iso}} \in B(G)_{\mathrm{bas}} = X_*(C)$ then the set of characters of $S_\varphi^\natural$ whose restriction to $Z(\widehat{G})$ is $[z^{\mathrm{iso}}] = z^{\mathrm{iso}}|_{Z(\widehat{G})}$ is just $\{z^{\mathrm{iso}}\}$. Thus the isocrystal version of the local Langlands correspondence for G is simply the usual correspondence with the extra datum of an element of $X_*(C)$.

We are left to consider the real case $F = \mathbb{R}$. Recall that the analogue of the morphism of extensions $\mathcal{E}^{\mathrm{rig}} \to \mathcal{E}^{\mathrm{iso}}$ of [12, (3.13)] is the composition $s^{\mathrm{iso}} \circ c^{\mathrm{rig}}$ as defined in Section 3.6. Note that the analogue of [12, Proposition 3.2] is a direct consequence of Propositions 3.13.5 and 3.13.6. Recall that Kottwitz defined a map $\kappa_G : B(G)_{\mathrm{bas}} \to X^*(Z(\widehat{G})^\Gamma)$, whose image was characterized in [20, Proposition 13.4] as

$$\{\chi|_{Z(\widehat{G})^\Gamma} \mid \chi \in X^*(Z(\widehat{G})), \ N_{\mathbb{C}/\mathbb{R}}(\chi) \in X^*(\widehat{G})\},$$

where we are using the facts that restriction induces an injective map $X^*(\widehat{G}) \to X^*(Z(\widehat{G}))$ as well as a surjective map $X^*(Z(\widehat{G})) \to X^*(Z(\widehat{G})^\Gamma)$. The analogue of [12, Proposition 3.3] is that the following diagram commutes, although the horizontal maps are not bijective in general.

$$
\begin{array}{ccc}
B(G)_{\mathrm{bas}} & \xrightarrow{\ \ \kappa_G\ \ } & X^*\big(Z(\widehat{G})^\Gamma\big) \\
\downarrow & & \downarrow \\
H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{rig}}, G) & \longrightarrow & \mathrm{Hom}\big(\pi_0\big((Z(\widehat{G}_{\mathrm{sc}}) \times Z(\widehat{G})^0_\infty)^+\big), \mathbb{C}^\times\big)
\end{array}
$$

Here the left vertical map is induced by $s^{\mathrm{iso}} \circ c^{\mathrm{rig}} : \mathcal{E}^{\mathrm{rig}} \to \mathcal{E}^{\mathrm{iso}}$, the bottom horizontal map is obtained as the composition of [10, Theorem 4.8 and Proposition 5.3] and the right vertical map is as in [12, Proposition 3.3], i.e., it is dual to the map

$$\pi_0\big((Z(\widehat{G}_{\mathrm{sc}}) \times Z(\widehat{G})^0_\infty)^+\big) \to Z(\widehat{G}^\Gamma) \tag{4.5.1}$$
$$\big(a, (b_n)_{n>0}\big) \mapsto a b_1 N_{\mathbb{C}/\mathbb{R}}(b_2)^{-1}$$

Note that this maps $N_{\mathbb{C}/\mathbb{R}}(\pi_0(Z(\widehat{G}_{\mathrm{sc}}) \times Z(\widehat{G})^0_\infty))$ to $N_{\mathbb{C}/\mathbb{R}}(Z(\widehat{G}_{\mathrm{der}}))$. The proof of [12, Proposition 3.3] applies almost verbatim, replacing "elliptic torus" by "fundamental torus" and using [20, Lemma 13.2] instead of [17, Proposition 5.3]. Note that the first argument of the proof, showing that $B(S)$ maps to $B(G)_{\mathrm{bas}}$, does not hold in the real case but this is

not necessary if one uses

$$B(S)_{G-\text{bas}} \simeq H^1(\mathbb{T}^{\text{iso}} \to \mathcal{E}, C \to S)$$

instead of $B(S)$ as in [20, Section 13.5].

For $\varphi : W_F \to {}^L G$ a Langlands parameter denote by S_φ its centralizer in $\hat{G}$ and define its quotient $S_\varphi^{\natural}$ (a complex reductive group) as in [12, Section 4.1]. We can define $\pi_0(S_\varphi^+) \to S_\varphi^{\natural}$ similarly to (4.5.1). The proof of [12, Lemma 4.1] does not use anything specific to the non-Archimedean case, so it still holds.

Now let G^* be a quasi-split connected reductive group over F. Fix a Whittaker datum $\mathfrak{w}$. Consider an inner form (G, ψ) of G^*. Let $z \in \widetilde{B}(G)_{\text{bas}} \simeq \bar{Z}^1_{\text{alg}}(\mathbb{T}^{\text{iso}} \to \mathcal{E}^{\text{iso}}, C \to G^*)$ be a lift of the cocycle $\Gamma \to G_{\text{ad}}(\bar{F})$, $\sigma \mapsto \psi^{-1}\sigma(\psi)$. Note that in general such a lift may not exist.

Theorem 4.5.1. *Fix a connected reductive quasi-split group G^* over $\mathbb{R}$ and a Whittaker datum $\mathfrak{w}$. There is a unique bijection between isomorphism classes of quadruples (G, ψ, z, π), where π is an irreducible admissible tempered representation of $G(\mathbb{R})$, and isomorphism classes of pairs (φ, ρ) where φ is a tempered Langlands parameter and ρ is an algebraic irreducible representation of $S_\varphi^{\natural}$, such that*

- *for given (G, ψ, z) and φ the L-packet Π_φ of isomorphism classes of π such that (G, ψ, z, π) corresponds to (φ, ρ) for some ρ equals the one defined by Langlands in [25], and*

- *the endoscopic character relations [12, (4.3)] hold with respect to the transfer factor (4.3.1).*

This correspondence is compatible with the rigid version proved in [10, Section 5.6], in the same sense as in [12, Section 4.2].

Proof. This is deduced from the rigid version of the local Langlands correspondence exactly as in [12, Section 4.2]. ∎

If $\mathfrak{w}$, (G, ψ, z) and φ are fixed we will denote, for $\pi \in \Pi_\varphi$, $\langle \pi, \cdot \rangle_{\psi, z, \mathfrak{w}}$ for the character ρ of $S_\varphi^{\natural}$ corresponding to π.

4.6. Multiplicity formula in the isocrystal setting

We return to considering a global field F and write V for its set of places and $\mathbb{A}$ for its adele ring. Let G be a connected reductive F-group which satisfies the Hasse principle, and whose center is connected. We shall formulate a version of the formula [16, (12.3)] for the multiplicity of an irreducible admissible representation of $G(\mathbb{A})$ in the discrete automorphic spectrum of G using Kottwitz's global set $B(G)$.

Assume the existence of a global Langlands group L_F, and consider a continuous semi-simple global tempered parameter $\varphi : L_F \to {}^L G$. For simplicity we do not consider more general Arthur parameters (which would involve adding a factor of SL_2 to L_F),

although they do not present additional difficulty for the following discussion, only requiring a slightly more complicated formulation. Recall from [16, Section 10.2] the group S_φ of self-equivalences of φ is defined as

$$\{g \in \widehat{G} \mid \forall x \in L_F : g^{-1}\varphi(x)g\varphi(x)^{-1} =: z(x) \in Z(\widehat{G})$$
$$\text{and } \forall v \in V, \ 0 = [z_v] \in H^1\big(L_{F_v}, Z(\widehat{G})\big)\}.$$

Note that z is an element of $Z^1(L_F, Z(\widehat{G}))$ for formal reasons. The Hasse principle, reinterpreted as [16, (4.2.2)], together with [16, Lemma 11.2.2] imply $S_\varphi = C_\varphi \cdot Z(\widehat{G})$, where

$$C_\varphi = \big\{g \in \widehat{G} \mid g^{-1}\varphi(x)g\varphi(x)^{-1} = 1\big\}.$$

We conclude that, if φ is discrete, then the finite group

$$\mathcal{S}_\varphi = \pi_0\big(S_\varphi/Z(\widehat{G})\big)$$

equals $C_\varphi/Z(\widehat{G})^\Gamma$.

Assume from now on that φ is discrete. For each place v we assume the isocrystal version of the refined local Langlands conjecture, as stated in [12, Section 4.1] when v is finite, and in Section 4.5 when v is infinite. In particular at each place we have the L-packet Π_{φ_v}. Choose a reductive model $\underline{G}$ of G over $\mathcal{O}_F[1/N]$ for some integer $N > 0$. For almost all finite places v of F, the L-packet Π_{φ_v} contains a unique unramified representation (with respect to $\underline{G}(\mathcal{O}(F_v))$). Given a collection of $\pi_v \in \Pi_{\varphi_v}(G)$, unramified for almost all v so that the restricted tensor product $\pi = \otimes'_v \pi_v$ is well-defined, we now define the class-function $\langle \pi, - \rangle$ on $\mathcal{S}_\varphi$ as follows. Let G^* be the quasi-split inner form of G and let $\psi : G^* \to G$ be an inner twist. By Corollary 3.13.13 we can choose $z^{\mathrm{iso}} \in Z^1_{\mathrm{bas}}(\mathcal{E}^{\mathrm{iso}}, G^*)$ such that $\psi^{-1}\sigma(\psi) = \mathrm{Ad}(\bar{z}_\sigma)$, where $\bar{z} \in Z^1(\Gamma, G^*_{\mathrm{ad}})$ is the image of z^{iso}. The choice of ψ gives an identification $^L\psi : {}^LG \to {}^LG^*$ and hence a parameter $\varphi^* = {}^L\psi \circ \varphi$ for G^*. Choose also a global Whittaker datum $\mathfrak{w}$ for G^*. For each place v we have the complex number $\langle \pi_v, s^* \rangle_{\psi_v, z_v^{\mathrm{iso}}, \mathfrak{w}_v}$ for $s^* \in C_{\varphi^*}$. For almost all finite places v this complex number equals one: this follows from the endoscopic character relations [12, (4.3)] and Lemma 3.11.2 by the same argument as in [11, Lemma 4.5.1]. Given $s \in C_\varphi$ define

$$\langle \pi, s \rangle_{\mathrm{iso}} = \prod_v \langle \pi_v, {}^L\psi(s) \rangle_{\psi_v, z_v^{\mathrm{iso}}, \mathfrak{w}_v}.$$

Lemma 4.6.1. *The complex number $\langle \pi, s \rangle_{\mathrm{iso}}$ is independent of the choice of $\mathfrak{w}$, z^{iso} and ψ and the map*

$$C_\varphi \to \mathbb{C}, \quad s \mapsto \langle \pi, s \rangle_{\mathrm{iso}}$$

is invariant under $Z(\widehat{G})^\Gamma$.

Proof. The parameter φ and s induce an elliptic endoscopic datum $(H, \mathcal{H}, s, \eta)$ with $\eta(\widehat{H}) = \mathrm{Cent}(s, \widehat{G})^0$ and $\eta(\mathcal{H}) = \varphi(L_F)\eta(\widehat{H})$. Choose a z-extension H_1 of H and an L-embedding $\eta_1 : \mathcal{H} \to {}^LH_1$ as in [21, Lemma 2.2.A]. Denote $\varphi' = \eta_1\eta^{-1}\varphi : L_F \to {}^LH_1$.

Fix a maximal compact subgroup K_∞ of $G(\mathbb{R} \otimes_{\mathbb{Q}} F)$. For $f(g)dg$ a smooth bi-K_∞-finite compactly supported distribution on $G(\mathbb{A})$, there exists a transfer $f'(h)dh$ on $H(\mathbb{A})$ in the sense of [21, (5.5)]. Note that the global transfer factors for $(H, \mathcal{H}, s, H_1, \eta_1)$ are canonical, and functorial under isomorphisms of endoscopic data. As discussed in Remark 4.3.3, the canonical adelic transfer f' of f satisfies the endoscopic character identities at each place v with respect to the normalized transfer factor $\Delta[\mathfrak{w}_v, z_v^{\text{iso}}]$. Therefore, taking the product of the local endoscopic character relations [12, (4.3)] at all places, we have

$$\sum_{\pi \in \Pi_\varphi} \langle \pi, s \rangle_{\text{iso}} \operatorname{tr} \pi \big(f(g)dg \big) = \sum_{\pi' \in \Pi_{\varphi'}} \operatorname{tr} \pi \big(f'(h)dh \big)$$

and since distributions $f(g)dg$ as above separate elements of Π_φ this shows that $\langle \pi, s \rangle_{\text{iso}}$ does not depend on any choice, and by functoriality of global transfer factors invariant under translation of s by $Z(\widehat{G})^\Gamma$.

Independence of ψ: Note first that we can replace ψ by $\psi \circ \operatorname{Ad}(g)$ and z^{iso} by $g^{-1} z^{\text{iso}} \sigma(g)$ for any $g \in G^*(\bar{F})$ without changing the numbers $\langle \pi_v, s^* \rangle_{z_v^{\text{iso}}}$. Let $\psi' : G^* \to G$ be another inner twist. Consider the automorphism $\theta = \psi^{-1} \circ \psi'$ of G^*. Changing ψ by $\psi \circ \operatorname{Ad}(g)$ if necessary we may assume that θ fixes an F-pinning of G^*. For any $\sigma \in \Gamma$ the automorphism $\theta^{-1} \sigma(\theta)$ is inner and fixes an F-pinning, hence trivial, and we conclude that θ is defined over F. Thus replacing ψ by $\psi' = \psi \circ \theta$ has the effect of replacing $^L\psi$ by $^L\psi \circ {}^L\theta$, φ^* by $^L\theta \circ \varphi^*$, and s^* by $^L\theta(s^*)$. The functoriality of the refined local Langlands conjecture [13, Appendix A] implies that $\langle \pi_v, s^* \rangle_{z_v^{\text{iso}}}$ remains unchanged. ∎

We thus obtain a function $\langle \pi, - \rangle_{\text{iso}}$ on $\mathcal{S}_\varphi = C_\varphi / Z(\widehat{G})^\Gamma$ that depends only on the Langlands parameter φ. The conjectural multiplicity formula [16, (12.3)] can now be stated using that function.

4.7. Comparison between the global pairings

In the last subsection we defined the pairing $\langle \pi, - \rangle_{\text{iso}}$ under the assumptions that G has connected center and satisfies the Hasse principle, using the cohomology of $\mathcal{E}^{\text{iso}}$. On the other hand, in [11, Section 4.5] we defined a pairing $\langle \pi, - \rangle$ without assumptions on G, using the cohomology of $\mathcal{E}^{\text{rig}}$. In [11, Section 4.5] we did not discuss the independence of the pairing defined in [11, Proposition 4.5.2] of the choice of ψ. The proof of Lemma 4.6.1 applies verbatim to the rigid version, and the following proposition follows.

Proposition 4.7.1. *Assume that the connected reductive group G has connected center and satisfies the Hasse principle. Then the pairings $\langle \pi, \cdot \rangle_{\text{iso}}$ and $\langle \pi, \cdot \rangle$ are equal.*

Using the Galois gerbe $\mathcal{E}_V^{\text{mid}}$ introduced in this paper, we can obtain a finer comparison result. Namely, we can see how the local pairings are related at each place v. This also implies Proposition 4.7.1, as follows.

Alternate proof of Proposition 4.7.1. By Corollary 3.13.13 (or Remark 3.10.3) we can choose $z^{\text{mid}} \in Z^1(\mathcal{E}_V^{\text{mid}}, Z(G^*) \to G^*)$ lifting z^{iso}. Up to pre-composing ψ with an inner automorphism of $G_{\bar{F}}^*$ we can assume that there exists a maximal torus T^* of G^* such

that $z^{\text{mid}} \in Z^1(\mathcal{E}^{\text{mid}}_{\dot{V}}, Z(G^*) \to T^*)$. Let $(\lambda, \mu) \in Y^{\text{mid}}(Z(G^*) \to T^*)$ be its linear algebraic description given by Proposition 3.13.10 and Corollary 3.13.11. Thus there is a finite level $(E, S, \dot{S}_E)$ such that $\lambda \in Y[S_E]_0 / IY[S_E]_0$ and $\mu \in (M^{\text{mid}}_{E,\dot{S}_E} \otimes X_*(Z(G^*)))^{\Gamma}$ satisfy $\sum_{\sigma \in \Gamma_{E/F}} \sigma(\lambda(\sigma^{-1}w)) = \sum_{\sigma} \mu(\sigma, w)$. Note that μ depends only on $z_{\text{mid}}|_{\mathbb{T}^{\text{mid}}} \in \text{Hom}_F(\mathbb{T}^{\text{mid}}, Z(G^*))$ and is thus independent of the choice of T^*.

Let $\bar{s} \in S_{\varphi}$. Choose a semi-simple lift $s \in C_{\varphi}$ and furthermore a lift $\dot{s} \in S_{\varphi}^+$. We use the images $z^{\text{iso}} \in Z^1_{\text{bas}}(\mathcal{E}^{\text{iso}}, G^*)$ and $z^{\text{rig}} \in Z^1(\mathcal{E}^{\text{rig}}, G^*)$ of z^{mid} and a global Whittaker datum $\mathfrak{w}$ for G^* to write as products over all places the complex numbers $\langle \pi, \bar{s} \rangle_{\text{iso}} = \prod_v \langle \pi_v, s \rangle_{\psi_v, z^{\text{iso}}}, \mathfrak{w}_v$ and $\langle \pi, \bar{s} \rangle_{\text{rig}} = \prod_v \langle \pi_v, \dot{s} \rangle_{\psi_v, z^{\text{rig}}, \mathfrak{w}_v}$.

By Lemma 4.8.1 below we have

$$\langle \pi, \bar{s} \rangle_{\text{iso}} \cdot \langle \pi, \bar{s} \rangle_{\text{rig}}^{-1} = \prod_{v \in V} \langle \mu_v, \dot{s} \rangle,$$

where for each place v the localization $\mu_v \in X_*(Z(G^*)) \otimes \mathbb{Q}$ of μ is given by $\mu(1, \dot{v})$ if $v \in S$ and equals 0 if $v \notin S$, according to Proposition 3.13.14. We are thus pairing $\dot{s}$ with $\sum_{\dot{v} \in \dot{S}_E} \mu(1, \dot{v}) = \sum_{w \in S_E} \mu(1, w) = 0$. ∎

4.8. Comparison between local pairings

In this subsection we will compare the pairings between the local compound L-packet of a tempered Langlands parameter and the centralizer of that parameter that are guaranteed to exist by the isocrystal and rigid versions of the refined local Langlands correspondence. Let G be a connected reductive group defined over a local field F, G^* its quasi-split inner form, $\psi : G^* \to G$ an inner twist, $\varphi : L_F \to {}^L G$ a tempered Langlands parameter. These local pairings are normalized by choices of elements $z^{\text{iso}} \in Z^1_{\text{bas}}(\mathcal{E}^{\text{iso}}, G^*)$ and $z^{\text{rig}} \in Z^1_{\text{bas}}(\mathcal{E}^{\text{rig}}, G^*)$ that lift the element $\psi^{-1}\sigma(\psi) \in Z^1(\Gamma, G^*_{\text{ad}})$, and of a Whittaker datum $\mathfrak{w}$ for G^*.

When $z^{\text{iso}} \mapsto z^{\text{rig}}$ under the comparison homomorphism [12, (3.14)] the two pairings were compared in [12, Section 4.2]. The global setting of Proposition 4.7.1 imposes a different relationship between z^{iso} and z^{rig} – they are the images of a an element z^{mid} under the maps c^{iso} and c^{mid}, respectively. Thus we will now combine the results of [12, Section 4.2] with the analysis of the non-commutativity of (1.0.1) that was quantified in Corollary 3.13.7.

Choose an arbitrary maximal torus $S \subset G^*$ such that the class of z^{mid} is in the image of $H^1(\mathcal{E}^{\text{mid}}, Z(G^*) \to S) \to H^1(\mathcal{E}^{\text{mid}}, Z(G^*) \to G^*)$. Let $(\lambda, \mu) \in Y^{\text{mid}}$ denote the Tate–Nakayama element corresponding to $[z^{\text{mid}}]$ under the isomorphism of Proposition 3.13.3. Thus $\lambda \in Y_{\Gamma}$, $\mu \in Y \otimes \mathbb{Q}$, and $N^{\natural}(\lambda) = N^{\natural}(\mu)$. Since z^{mid} sends $\mathbb{T}^{\text{mid}}$ into $C := Z(G^*)$, we have $\mu \in X_*(C) \otimes \mathbb{Q} \subset Y \otimes \mathbb{Q}$, and this is independent of the choice of S.

Denote by $p : \hat{G} \to \hat{C}$ the surjection dual to $C \subset G$.

Lemma 4.8.1. *For any semi-simple element $\dot{s} \in S_{\varphi}^+$ and $\pi \in \Pi_{\varphi}(G)$ we have*

$$\langle \pi, s \rangle_{\psi, z^{\text{iso}}, \mathfrak{w}} = \langle \mu, p(\dot{s}) \rangle \langle \pi, \dot{s} \rangle_{\psi, z^{\text{rig}}, \mathfrak{w}}.$$

where $p(\dot{s})$ is the image of $\dot{s}$ in $[\hat{\bar{C}}]^+$.

Proof. In the proof we suppress ψ and $\mathfrak{w}$ from the notation since they are fixed. The local pairings are conjectural. However, we know from [12, Section 4.2] that the following identity is implied by the assumed validity of the isocrystal and rigid versions of the refined local Langlands conjecture:

$$\langle \pi, t \rangle_{z^{\mathrm{iso}}} = \langle \pi, \dot{s} \rangle_{x^{\mathrm{rig}}},$$

where $t \in S_\varphi$ is equal to the image of $\dot{s}$ under the map $\pi_0(S_\varphi^+) \to S_\varphi$ of [12, (4.7)], and x^{rig} is the image of z^{iso} under the map $Z_{\mathrm{bas}}^1(\mathcal{E}^{\mathrm{iso}}, G^*) \to Z_{\mathrm{bas}}^1(\mathcal{E}^{\mathrm{rig}}, G^*)$ of [12, (3.14)]. This reduces the question to comparing $\langle \pi, t \rangle_{z^{\mathrm{iso}}}$ with $\langle \pi, s \rangle_{z^{\mathrm{iso}}}$ and $\langle \pi, \dot{s} \rangle_{x^{\mathrm{rig}}}$ with $\langle \pi, \dot{s} \rangle_{z^{\mathrm{rig}}}$.

For the first comparison, we use the notation of [12, Section 4.2] and represent $\dot{s}$ as $(a_{\mathrm{sc}}, (b_n)_n)$ with $a_{\mathrm{sc}} \in \widehat{G}_{\mathrm{sc}}$ and $b_n \in Z(\widehat{G})^\circ$, $b_m^{m/n} = b_n$. Then $s = a_{\mathrm{der}}b_1$ while $t = a_{\mathrm{der}}b_1 N_{E/F}(b_{[E:F]})^{-1}$ for E any finite Galois extension of F such that $\Gamma_{\bar{F}/E}$ acts trivially on $Z(\widehat{G})$. Thus $\langle \pi, s \rangle_{z^{\mathrm{iso}}} = \langle \pi, t \rangle_{z^{\mathrm{iso}}} \cdot \langle \pi, N_{E/F}(b_{[E:F]}) \rangle_{z^{\mathrm{iso}}}$. Now $N_{E/F}(b_{[E:F]}) \in Z(\widehat{G})^{\circ,\Gamma}$, and the restriction of $\langle \pi, - \rangle_{z^{\mathrm{iso}}}$ to $Z(\widehat{G})^\Gamma$ is the character $\langle [z^{\mathrm{iso}}], - \rangle$, so we obtain

$$\langle \pi, s \rangle_{z^{\mathrm{iso}}} = \langle \pi, t \rangle_{z^{\mathrm{iso}}} \cdot \langle [z^{\mathrm{iso}}], N_{E/F}(b_{[E:F]}) \rangle.$$

For the second comparison, we remind ourselves that z^{rig} is pulled back from z^{mid} via

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{T}^{\mathrm{rig}} & \longrightarrow & \mathcal{E}^{\mathrm{rig}} & \longrightarrow & \Gamma & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathbb{T}^{\mathrm{mid}} & \longrightarrow & \mathcal{E}^{\mathrm{mid}} & \longrightarrow & \Gamma & \longrightarrow & 1
\end{array}
$$

while x^{rig} is pulled back from z^{mid} via

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{T}^{\mathrm{rig}} & \longrightarrow & \mathcal{E}^{\mathrm{rig}} & \longrightarrow & \Gamma & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathbb{T}^{\mathrm{iso}} & \longrightarrow & \mathcal{E}^{\mathrm{iso}} & \longrightarrow & \Gamma & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathbb{T}^{\mathrm{mid}} & \longrightarrow & \mathcal{E}^{\mathrm{mid}} & \longrightarrow & \Gamma & \longrightarrow & 1
\end{array}
$$

Since the maps on Γ are all the identity, both z^{rig} and x^{rig} map to the same element of $Z^1(\Gamma, G_{\mathrm{ad}}^*)$, so their difference $x^{\mathrm{rig}}/z^{\mathrm{rig}}$ lies in $Z^1(\mathcal{E}^{\mathrm{rig}}, Z(G^*))$, and in fact in $Z^1(\mathcal{E}^{\mathrm{rig}}, Z)$ for some finite $Z \subset Z(G^*)$. Then we have

$$\langle \pi, \dot{s} \rangle_{x^{\mathrm{rig}}} = \langle x^{\mathrm{rig}}/z^{\mathrm{rig}}, (-d)\dot{s} \rangle \langle \pi, \dot{s} \rangle_{z^{\mathrm{rig}}} = \langle x^{\mathrm{rig}}/z^{\mathrm{rig}}, p(\dot{s}) \rangle \langle \pi, \dot{s} \rangle_{z^{\mathrm{rig}}}.$$

The first equality is due to [12, Lemma 6.2], where $d : S_\varphi^+ \to Z^1(\Gamma, \widehat{Z})$ is the differential, and we are using the pairing $Z^1(\Gamma, \widehat{Z}) \otimes H^1(\mathcal{E}^{\mathrm{rig}}, Z) \to \mathbb{C}^\times$ of [12, Section 6.2]. The second equality is due to the commutative diagrams (6.1) and (6.2) in [12], applied to the torus $C = Z(G^*)$, and we are using the pairing between $H^1(\mathcal{E}^{\mathrm{rig}}, C)$ and $\pi_0([\widehat{\bar{C}}]^+)$.

We now combine the comparisons and arrive at

$$\langle \pi, s \rangle_{z^{\mathrm{iso}}} = \big\langle [z^{\mathrm{iso}}], N_{E/F}(b_{[E:F]}) \big\rangle \langle x^{\mathrm{rig}}/z^{\mathrm{rig}}, p(\dot{s}) \rangle \langle \pi, \dot{s} \rangle_{z^{\mathrm{rig}}}.$$

Recall the Tate–Nakayama element (λ, μ) corresponding to z^{mid}. We can see $\lambda \in Y_\Gamma$ as an element of $X^*(\hat{S}^\Gamma)$, which via restriction to $Z(\hat{G})^\Gamma$ maps to $[z^{\mathrm{iso}}]$. For $m > 0$ a sufficiently divisible integer we have that $N_{E/F}\lambda = N_{E/F}\mu$ belongs to $m^{-1}X_*(C)$, and so

$$\big\langle [z^{\mathrm{iso}}], N_{E/F}(b_{[E:F]}) \big\rangle = \langle N_{E/F}\lambda, b_{[E:F]} \rangle = \big\langle m N_{E/F}\mu, p(b_{m[E:F]}) \big\rangle$$

where we have used the isogeny $p|_{Z(\hat{G})^0} : Z(\hat{G})^0 \to \hat{C}$.

Consider now the factor $\langle x^{\mathrm{rig}}/z^{\mathrm{rig}}, p(\dot{s}) \rangle$. According to Corollary 3.13.7 it equals $\langle \mu - N^\natural(\mu), p(\dot{s}) \rangle$. To evaluate this pairing, choose an integer $n > 0$ divisible by $m[E:F]$ so that $\mu \in n^{-1}X_*(C)$. Note that $N^\natural(\mu) \in n^{-1}X_*(C)$ also. Then $\langle x^{\mathrm{rig}}/z^{\mathrm{rig}}, p(\dot{s}) \rangle = \langle n(\mu - N^\natural(\mu)), p(b_n) \rangle$. We have

$$\big\langle n N^\natural(\mu), p(b_n) \big\rangle = \big\langle m N_{E/F}(\mu), b_{m[E:F]} \big\rangle.$$

Thus

$$\langle \pi, s \rangle_{z^{\mathrm{iso}}} = \langle \mu, \dot{s} \rangle \langle \pi, \dot{s} \rangle_{z^{\mathrm{rig}}}$$

and the lemma is proved. ∎

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Tasho Kaletha
Department of Mathematics, University of Michigan, 530 Church Street, Ann Arbor, MI 48109, USA; kaletha@umich.edu

Olivier Taïbi
Unité de Mathématiques Pures et Appliquées, ENS de Lyon Site Monod, 46 Allée d'Italie, 69364 Lyon Cedex 07, France; olivier.taibi@ens-lyon.fr