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Convergence of the Hesse–Koszul flow on compact Hessian manifolds

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Abstract. We study the long time behavior of the Hesse–Koszul flow on compact Hessian manifolds. When the first affine Chern class is negative, we prove that the flow converges to the unique Hesse–Einstein metric. We also derive a convergence result for a twisted Hesse–Koszul flow on any compact Hessian manifold. These results give alternative proofs for the existence of the unique Hesse–Einstein metric by Cheng–Yau and Caffarelli–Viaclovsky as well as the real Calabi theorem by Cheng–Yau, Delanoë and Caffarelli–Viaclovsky.

1. Introduction

An affine manifold is a real manifold M which admits a flat, torsion-free connection ∇ on its tangent bundle. A Riemannian metric g on an affine manifold (M, ∇) is called a *Hessian metric* if g can be locally expressed by $g = \nabla d\varphi$. We then say (M, ∇, g) is a Hessian manifold. We consider the Monge–Ampère operator:

$$MA(\varphi) := \det(\nabla d\varphi) = \det\Big(\frac{\partial^2\varphi}{\partial x^i \partial x^j}\Big), \tag{1}$$

where $\{x^1, \dots, x^n\}$ is an affine coordinate system with respect to ∇. It was observed by Cheng–Yau [10] that this is a natural operator on affine manifolds since it is invariant under affine coordinate transformations. In particular, it is very similar to the complex Monge–Ampère operator since the (real) Monge–Ampère measure $\mu_\varphi = \sqrt{\det(\varphi_{ij})}dx^1 \wedge \cdots \wedge dx^n$ is well defined. We refer the interested readers to [8, 10, 14, 22–25, 37, 44] and references therein for more details on Hessian manifolds.

Hessian manifolds and real Monge–Ampère equations play a central role in many fields, varying from mathematical physics to statistics. They appear as large complex limits of Calabi–Yau manifolds, which is in the framework of the Strominger–Yau–Zaslow [39] and the Kontsevich–Soibelman [21] conjectures (see also [3, 16, 17, 32]).

Recently, Hessian manifolds have been interpreted as particular parameter spaces of statistical models in which the Fisher–Rao metric is a Hessian metric (cf. [1, 2, 4, 28]).

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Studying geometric structures of Hessian manifolds thus could lead to many applications in statistics.

In [10], Cheng–Yau study the real Monge–Ampère equation

$$\det(g + \nabla d\varphi) = e^{\lambda\varphi + f} \det g, \tag{2}$$

with either $\lambda = 0$ or $\lambda > 0$. When $\lambda = 0$, solving this equation provides a solution to the real Calabi problem (cf. [10]): given $\eta \in c_1^a(M)$, it shows that there is a metric $\tilde{g} = g + \nabla d\varphi$ for some $\varphi \in C^\infty(M)$ such that $\kappa(g) = \eta$, where $\kappa(g)$ is the second Koszul form (see Definition 2). When $\lambda > 0$, solving this equation allows one to construct the Hesse–Einstein metric $\tilde{g} = g + \nabla d\varphi$, i.e. $\kappa(g) = -\lambda g$ which is a canonical metric on Hessian manifolds. Hesse–Einstein metrics in Hessian geometry can be seen as the real version of the Kähler–Einstein metrics in Kähler geometry (cf. [10, 31, 33, 37]).

Assuming that M is a special manifold, Cheng–Yau [10] solved (2) with $\lambda = 0$ using the continuity method. Delanoë then removed this condition in [12] but still relied on [10] for higher derivative estimates. In [10], the authors also solved (2) with $\lambda > 0$ by lifting this equation to $M + i\mathbb{R}^n$ and using methods from complex geometry. Caffarelli–Viaclovsky [8] then generalized these previous works solving (2) assuming a minimal regularity for f. They used the continuity method for $\lambda = 0$ and the viscosity method for $\lambda > 0$. We also refer to [20] for a recent variational approach with optimal transport point of view.

In this note, we give an alternative approach using a geometric flow, namely the Hesse–Koszul flow. This flow has been defined and studied by Mirghafouri–Malek [34] on compact Hessian manifolds. Given any compact Hessian manifold $(M, \nabla, g_0 = \nabla d\psi)$ we define the evolution equation

$$\frac{\partial g_{ij}}{\partial t} = -\beta_{ij}(g), \quad g|_{t=0} = g_0, \tag{3}$$

where $\beta(g) = -2\kappa(g)$, with κ the second Koszul form for (∇, g) (see Definition 2). Along the flow, the evolved metric g remains Hessian, which is why we call it the Hesse–Koszul flow.

Similarly to the Kähler–Ricci flow, we can rewrite the flow as a scalar equation, namely the parabolic Monge–Ampère equation:

$$\frac{\partial}{\partial t}\varphi = \log \frac{\det(\hat{g}(t) + \nabla d\varphi)}{\det g_0} + f, \quad \varphi_0 = 0, \tag{4}$$

where φ is the unknown function and $\hat{g}(t)$ only depends on t, g_0 and the first affine Chern class $c_1^a(M)$ (see Definition 6).

In [34], the authors proved the short time existence and the uniqueness of the flow on compact Hessian manifolds. When $\hat{g}$ is independent of t, they showed that the flow has a long time existence. In this paper, we study the characterization of the maximal existence time and the long time behavior of the flow.

Our first goal is to prove that the maximal time for the existence of a smooth solution is a cohomological constant like that of the Kähler–Ricci flow (see for instance [9, 38, 41, 42, 46]):

Theorem 1. *Let (M, ∇, g) be a compact Hessian manifold. Then the Hesse–Koszul flow has a unique smooth solution $g(t)$ on the maximal time interval [0,T), where*

$$T = \sup\{t > 0 \mid [g_0] - tc_1^a(M) > 0\}. \tag{5}$$

For the proof we adapt some Kähler–Ricci flow techniques to our case. There is indeed a natural connection between the Hessian and Kähler geometries, as first observed by Dombrowski [13]: the tangent bundle over a Hessian manifold admits a Kähler metric induced by the Hessian metric.

Our second goal is to prove that the flow converges to a Hesse–Einstein metric, assuming that the first affine Chern class is negative. This gives an alternative proof for the result in [8, 10] on the existence of Hesse–Einstein metrics:

Theorem 2. *Let M be a compact Hessian manifold. Assume that $c_1^a(M) < 0$; then starting from any Hessian metric g_0, the normalized Hesse–Koszul flow*

$$\frac{\partial g_{ij}}{\partial t} = -\beta_{ij}(g) - g$$

exists for all time and converges in C^∞ to a Hesse–Einstein metric g_∞ satisfying

$$\beta(g_\infty) = -g_\infty. \tag{6}$$

Moreover, g_∞ is the unique solution to the Hesse–Einstein equation (6).

Finally, we give another proof for a real version of Calabi's conjecture due to [8, 10, 12]. We follow Cao's approach (cf. [9]) to the Calabi conjecture to study the Hesse–Koszul flow twisted by $\eta \in c_1^a(M)$:

Theorem 3. *Let M be a compact Hessian manifold. The flow*

$$\frac{\partial g_{ij}}{\partial t} = -\beta_{ij}(g) + \eta, \quad g|_{t=0} = \hat{g}. \tag{7}$$

exists for all time and C^∞-converges to a metric g_∞ which is the unique solution to

$$\beta(g_\infty) = \eta. \tag{8}$$

The long time existence is due to [34]. For the proof of the convergence, we derive uniform *a priori* estimates by adapting some Kähler–Ricci flow techniques to our case, and by using a new approach based on [36] to prove the C^0 estimate.

Finally, we show that our approach can be applied to prove a convergence result for a parabolic Monge–Ampère equation on compact Riemannian manifolds:

Theorem 4. *Let (M, g) be a compact Riemannian manifold and ∇ be the Levi-Civita connection of g. The normalization $\tilde{\varphi} := \varphi - \frac{1}{\mathrm{Vol}_g}\int_M \varphi \, dV_g$ of the solution for the flow*

$$\frac{\partial}{\partial t}\varphi(x, t) = \log \frac{\det(g(x) + \nabla^2\varphi(t, x))}{\det g(x)} - f(x) \tag{9}$$

C^∞-converges to a function $\tilde{\varphi}_\infty$. In particular, the limit $\tilde{\varphi}_\infty$ is a solution of the following Monge–Ampère equation:

$$\det(g + \nabla^2\phi) = ce^f \det(g), \tag{10}$$

for some constant c.

This flow was studied in [19] where long time existence was proved. We establish further the convergence of the flow. Our key ingredient in the proof is a uniform C^0 estimate for the normalization of φ as in Theorem 3. Moreover, Theorem 4 gives an alternative proof of the existence of solutions for the Monge–Ampère equation (10) on compact Riemannian manifolds due to [11].

2. Preliminaries

2.1. Affine manifolds, Hessian metric and Koszul forms

Definition 1 ([37]). An *affine manifold* (M, ∇) is a differentiable manifold equipped with a flat, torsion-free connection ∇.

A Riemannian metric g on an affine manifold (M, ∇) is called a *Hessian metric* if g can be locally expressed by $g = \nabla\, d\phi$. Then (M, ∇, g) is called a *Hessian manifold.*

It is known that a manifold M is affine if and only if M admits an affine atlas such that transition functions are in the affine group $\mathrm{Aff}(n) = \{\Phi : \mathbb{R}^n \to \mathbb{R}^n,\ \Phi(x) = Ax + b\}$.

Let (M, ∇, g) be a Hessian manifold; g can be locally expressed by

$$g_{ij} = \frac{\partial^2\phi}{\partial x^i \partial x^j},$$

where $\{x^1, \ldots, x^n\}$ is an affine coordinate system with respect to ∇.

Denote by $\widehat{\nabla}$ the Levi-Civita connection of (M, g), $\gamma = \widehat{\nabla} - \nabla$. Since ∇ and $\widehat{\nabla}$ are torsion-free, we have

$$\gamma_X Y = \gamma_Y X.$$

Moreover, the components ${\gamma^i}_{jk}$ of γ with respect to affine coordinate systems coincide with the Christoffel symbols ${\Gamma^i}_{jk}$ of the Levi-Civita connection $\widehat{\nabla}$.

The tensor $Q = \nabla\gamma$ is called the *Hessian curvature tensor* for (g, ∇). We recall here some properties of Hessian manifolds.

Proposition 1 ([37]). *Let (M, ∇) be an affine manifold and g a Riemannian metric on M. Then the following are equivalent:*

(1) *g is a Hessian metric;*

(2) *$(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z)$;*

(3) $\dfrac{\partial g_{ij}}{\partial x^k} = \dfrac{\partial g_{kj}}{\partial x^i}$;

(4) $g(\gamma_X Y, Z) = g(Y, \gamma_X Z)$;

(5) $\gamma_{ijk} = \gamma_{jik}$.

Proposition 2 ([37]). *Let $\widehat{R}$ be the Riemannian curvature of $g = \nabla d\phi$ and $Q = \nabla\gamma$ be the Hessian curvature tensor for (g, ∇). Then*

(1) $Q_{ijkl} = \dfrac{1}{2}\dfrac{\partial^4\phi}{\partial x^i \partial x^j \partial x^k \partial x^l} - \dfrac{1}{2} g^{pq} \dfrac{\partial^3\phi}{\partial x^i \partial x^k \partial x^p} \dfrac{\partial^3\phi}{\partial x^j \partial x^l \partial x^q}$;

(2) $\widehat{R}(X, Y) = -[\gamma_X, \gamma_Y]$, $\widehat{R}^i{}_{jkl} = \gamma^i{}_{lm}\gamma^m{}_{jk} - \gamma^i{}_{km}\gamma^m{}_{jl}$;

(3) $\widehat{R}_{ijkl} = \frac{1}{2}(Q_{ijkl} - Q_{jikl}) = -\frac{1}{4}\phi^{pq}(\phi_{ikp}\phi_{jlq} - \phi_{jkp}\phi_{ilq})$, *where* $\phi_{ikp} := \frac{\partial^3\phi}{\partial x^i \partial x^k \partial x^p}$ *and* $(\phi^{pq}) = (\phi_{pq})^{-1}$.

Definition 2. We define the first Koszul form α and the second Koszul form κ for (∇, g) (cf. [22, 37]) by

$$\nabla_X \mathrm{Vol}_g = \alpha(X)\mathrm{Vol}_g \quad \text{and} \quad \kappa = \nabla\alpha.$$

It follows from the definition that

$$\alpha(X) = \mathrm{Tr}\,\gamma_X, \tag{11}$$

$$\kappa(g) = \tfrac{1}{2}\nabla d(\log\det g), \tag{12}$$

so

$$\alpha_i = \frac{1}{2}\frac{\partial \log\det[g_{pq}]}{\partial x^i} = \gamma^k{}_{ki}, \tag{13}$$

$$\kappa_{ij} = \frac{1}{2}\frac{\partial^2 \log\det[g_{pq}]}{\partial x^i \partial x^j}. \tag{14}$$

In the sequel we shall use the tensor $\beta = -2\kappa$ instead of κ to define the Hesse–Koszul flow.

We shall use the operator $L_g(f) := \mathrm{Tr}_g \nabla df$ for any Riemannian metric g. In an affine coordinate system $\{x^1, \ldots, x^n\}$ with respect to ∇, we have

$$\begin{aligned} L_g(f) &= \mathrm{Tr}_g \widehat{\nabla} df - \mathrm{Tr}\,\gamma\, df \\ &= \Delta_g f - g^{ij}\gamma^k{}_{ij}\partial_k f, \end{aligned} \tag{15}$$

where $\widehat{\nabla}$ is the Levi-Civita connection of g and $\gamma = \widehat{\nabla} - \nabla$, and L is an elliptic operator for which the maximum principle holds.

2.2. Cohomology on affine manifolds and the first affine Chern class

Let (M, ∇) be an affine manifold. Denote by $(\wedge^p T^*M) \otimes (\wedge^q T^*M)$ the tensor product of vector bundles $\wedge^p T^*M$ and $\wedge^q T^*M$. Denote by $\mathcal{A}^{p,q}$ all smooth sections of $(\wedge^p T^*M) \otimes (\wedge^q T^*M)$. In an affine coordinate system with respect to ∇, a (p, q)-form

ω in $\mathcal{A}^{p,q}$ is expressed by

$$\omega = \sum \omega_{i_1\ldots i_p;j_1\ldots j_q} (dx^{i_1} \wedge \cdots \wedge dx^{i_p}) \otimes (dx^{j_1} \wedge \cdots \wedge dx^{j_q})$$
$$= \sum \omega_{I_p;J_q} dx^{I_p} \otimes dx^{J_q} \quad \text{with } I_p = (i_1, \ldots, i_p),\ J_q = (j_1, \ldots, j_q),$$

where $\omega_{I_p;J_q} = \omega_{i_1\ldots i_p;j_1\ldots j_q}$, $dx^{I_p} = dx^{i_1} \wedge \cdots \wedge dx^{i_p}$.

Definition 3. For $\alpha \in \mathcal{A}^{p,q}$ and $\beta \in \mathcal{A}^{r,s}$, we define the exterior product $\alpha \wedge \beta \in \mathcal{A}^{p+r,q+s}$ by

$$(\alpha \wedge \beta)(X_1, \ldots, X_{p+r}; Y_1, \ldots, Y_{q+s})$$
$$= \frac{1}{p!q!r!s!} \sum_{\sigma,\tau} \varepsilon_\sigma \varepsilon_\tau \alpha(X_{\sigma(1)}, \ldots, X_{\sigma(p)}; Y_{\tau(1)}, \ldots, Y_{\tau(q)})$$
$$\times \beta(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+r)}; Y_{\tau(q+1)}, \ldots, Y_{\tau(q+s)}), \tag{16}$$

where the sum is taken over all permutations σ, τ, and ε_σ (resp. ε_τ) is the sign of σ (resp. τ).

Definition 4. We define $d' = d^\nabla \otimes I \colon \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q}$ and $d'' = I \otimes d^\nabla \colon \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}$, where I is the identity operator and d^∇ is the exterior derivative induced by ∇.

Definition 5 ([10, 37]). We define a cohomology group

$$\widetilde{H}^k(M) = \{\alpha \in \mathcal{A}^{p,q} \mid d'\alpha = 0,\ d''\alpha = 0\} / d'd''(\mathcal{A}^{k-1,k-1}).$$

Let $\Omega = \Omega(x) dx^1 \wedge \cdots \wedge dx^n$ be a volume form on M. Then the second Koszul form of Ω is defined by

$$\kappa_\Omega = \sum \frac{\partial^2 \log \Omega(x)}{\partial x^i \partial x^j} dx^i \otimes dx^j.$$

Again we shall use the tensor $\beta_\Omega := -2\kappa_\Omega$ to define the Hesse–Koszul flow. Denote by $[\kappa_\Omega] \in \widetilde{H}^1(M)$ the class represented by κ_Ω. If Ω' is another volume form, then there exists a function f on M such that $\Omega' = e^f \Omega$, so we have

$$\kappa_\Omega = \kappa_{\Omega'} + \nabla df.$$

By definition, we have $[\kappa_\Omega] = [\kappa_{\Omega'}] \in \widetilde{H}^1(M)$, so we can define the *first affine Chern class* as follows:

Definition 6. We set $c_1^a(M) := -2[\kappa_\Omega] = [\beta_\Omega] \in \widetilde{H}^1(M)$ to be the *first affine Chern class* of M, for any volume form Ω.

In particular, if (M, ∇, g) is a compact Hessian manifold then $c_1^a(M) = -2[\kappa(g)] = [\beta(g)]$, where $\kappa(g)$ is the second Koszul form of (∇, g) (see Definition 2).

Let $[\alpha] \in \widetilde{H}^1(M)$; we say that $[\alpha]$ is positive (resp. semi-positive) and denote $[\alpha] > 0$ (resp. $[\alpha] \geq 0$) if there exists $\alpha' \in [\alpha]$ such that $\alpha' > 0$ (resp. $\alpha' \geq 0$). Then we have the following theorem due to Shima [37] (see also Delanoë [12]).

Theorem 5. *Let* (M, ∇, g) *be a compact Hessian manifold and* α *and* κ *be the first and the second Koszul forms respectively. Then we have*

(i) $$\int_M \mathrm{Tr}_g\, \kappa \, dV_g = \int_M \|\alpha\|^2 \, dV_g \geq 0.$$

(ii) *If* $\int_M \mathrm{Tr}_g\, \kappa \, dV_g = 0$ *then the Levi-Civita connection* $\widehat{\nabla}$ *of* g *coincides with* ∇.

In particular, the first affine Chern class $c_1^a(M)$ *cannot be positive.*

3. Maximal existence time for the flow on compact manifolds

Let (M, ∇, g) be a compact Hessian manifold of dimension n. Consider the Hesse–Koszul flow

$$\frac{\partial g_{ij}}{\partial t} = 2\kappa(g), \quad g|_{t=0} = g_0. \tag{17}$$

For our convenience, we shall write the Hesse–Koszul flow as

$$\frac{\partial g_{ij}}{\partial t} = -\beta(g), \quad g|_{t=0} = g_0, \tag{18}$$

where $\beta = -2\kappa$ represents the first affine Chern class. In this section we prove that the maximal existence time for the Hesse–Koszul flow is a cohomological constant. It only depends on the first affine Chern class. Define

$$T = \sup\big\{t > 0 \mid [g_0] - tc_1^a(M) > 0\big\}; \tag{19}$$

then the main result of this section is the following:

Theorem 6. *Let* (M, ∇, g) *be a compact Hessian manifold. Then the Hesse–Koszul flow has a unique smooth solution* $g(t)$ *on the maximal time interval [0,T).*

We follow the approach developed in Kähler geometry to establish a similar result for the Kähler–Ricci flow (cf. [9, 38, 41, 42, 46]).

3.1. Reduction to a parabolic Monge–Ampère equation

Fix any $T' < T$. Our goal is to show that there exists a solution for the flow (18) on $[0, T')$. The key ingredient is that we can rewrite the Hesse–Koszul flow (18) as a parabolic Monge–Ampère equation.

Since $[g_0] - T'c_1^a(M) > 0$, there exists a Hessian metric $g' \in [g_0] - T'c_1^a(M)$; then $\frac{1}{T'}(g' - g_0) \in -c_1^a(M)$. Therefore,

$$\hat{g}(t) = g_0 + \frac{t}{T'}(g' - g_0) = \frac{1}{T'}((T' - t)g_0 + tg')$$

is also a Hessian metric for all $t \in [0, T']$. Fix Ω_0, a smooth positive volume form on M. Since $\beta_{\Omega_0} \in c_1^a(M)$, there exists a function $f \in C^\infty(M)$ such that

$$\frac{1}{T'}(g' - g_0) = -\beta_{\Omega_0} + \nabla df.$$

Then we define $\Omega = e^{f/2}\Omega_0$, which is a smooth positive volume form satisfying

$$\beta_\Omega = -\frac{1}{T'}(g' - g_0).$$

By abuse of notation, we shall write Ω as the local density as well, i.e.

$$\Omega = \Omega(x)dx^1 \wedge \cdots \wedge dx^n,$$

and $\Omega^2 := \Omega^2(x)$ is the square of the density. Then we have the following:

Lemma 1. *A smooth family of Hessian metric g_t on $[0, T')$ is the solution of the Hesse–Koszul flow (18) if and only if the parabolic equation*

$$\frac{\partial}{\partial t}\varphi = \log \frac{\det(\hat{g}_t + \nabla d\varphi)}{\Omega^2}, \quad \hat{g}_t + \nabla d\varphi > 0, \quad \varphi|_{t=0} = 0 \tag{20}$$

has a smooth solution $\varphi(t)$, $t \in [0, T')$ such that $g_t = \hat{g}_t + \nabla d\varphi(t)$.

Proof. If $\varphi(t)$ satisfies equation (20), then we set $g(t) = \hat{g}(t) + \nabla d\varphi(t)$. By a straightforward computation, we get

$$\frac{\partial}{\partial t}g(t) = -\beta_\Omega + \nabla d \log \frac{\det g(t)}{\Omega^2} = -\beta_\Omega + \beta_\Omega - \beta(g(t)) = -\beta(g(t)). \tag{21}$$

Since $g(0) = g_0$, we infer that $g(t)$ is a solution of (18).

For the "only if" assertion, given $g(t)$ a solution of (18), we define for any $t \in [0, T')$,

$$\varphi(t) = \int_0^t \log \frac{\det g(s)}{\Omega^2}\, ds.$$

Then we get

$$\frac{\partial}{\partial t}\varphi(t) = \log \frac{\det g(t)}{\Omega^2}, \quad \varphi(0) = 0. \tag{22}$$

We now prove that $g(t) = \hat{g}(t) + \nabla d\varphi(t)$ on $[0, T')$. Since

$$\nabla d\dot{\varphi}(t) = \nabla d \log \frac{\det g(t)}{\Omega^2} = \beta_\Omega - \beta(g(t)),$$

we obtain

$$\frac{\partial}{\partial t}(g(t) - \hat{g}(t) - \nabla d\varphi(t)) = -\beta(g(t)) + \beta_\Omega + \beta(g(t)) - \beta_\Omega = 0. \tag{23}$$

At $t = 0$ we have $g(0) - \hat{g}(0) - \nabla d\varphi(0) = 0$, so $g(t) - \hat{g}(t) - \nabla d\varphi(t) = 0$ on $[0, T') \times M$, as required. ∎

The uniqueness in Theorem 6 now follows from the following comparison principle for parabolic Monge–Ampère equations:

Proposition 3. *Let* $(\hat{g}(t))_{t\in[0,S]}$ *be a smooth family of Riemannian metrics on* M*. Suppose that* $\varphi, \psi \in C^\infty([0,S]\times M)$ *satisfy* $\hat{g}(t)+\nabla d\varphi(t)>0$, $\hat{g}(t)+\nabla d\psi(t)>0$ *and*

$$\frac{\partial \varphi}{\partial t} \le \log \frac{\det(\hat{g}+\nabla d\varphi)}{\Omega^2} - F(t,x,\varphi), \tag{24}$$
$$\frac{\partial \psi}{\partial t} \ge \log \frac{\det(\hat{g}+\nabla d\psi)}{\Omega^2} - F(t,x,\psi), \tag{25}$$

where $F(t,x,s)$ *is a smooth function with* $\frac{\partial F}{\partial s} \ge -\lambda$*. Then*

$$\sup_{[0,S]\times M} (\varphi-\psi) \le e^{\lambda S} \max\Big\{\sup_X(\varphi_0-\psi_0); 0\Big\}. \tag{26}$$

Proof. The proof follows from the maximum principle. Fix $\varepsilon>0$ and define $u(t,x)=e^{-\lambda t}(\varphi-\psi)-\varepsilon t$. Suppose that u achieves its maximum at $(t_0,x_0)\in[0,S]\times M$. We assume that $t_0>0$, otherwise we are done. At (t_0,x_0), we have $\dot{u}\ge 0$ and $\nabla du \ge 0$, hence

$$-\lambda e^{-\lambda t}(\varphi-\psi)+e^{\lambda t}(\dot{\varphi}-\dot{\psi}) \ge \varepsilon > 0$$

and $\nabla d\varphi \le \nabla d\psi$. Therefore, at (t_0,x_0) we have

$$\dot{\varphi}-\dot{\psi} \le -F(t,z,\varphi)+F(t,x,\psi),$$

and $\dot{\varphi}-\dot{\psi} > \lambda(\varphi-\psi)$, so

$$F(t_0,x_0,\psi(t_0,x_0))+\lambda\psi(t_0,x_0) > F(t_0,x_0,\varphi(t_0,x_0))+\lambda\varphi(t_0,x_0).$$

Since $\partial F/\partial s \ge -\lambda$, $s\mapsto F(\cdot,\cdot,s)$ is increasing, we get $\varphi(t_0,x_0)\le\psi(t_0,x_0)$. Therefore, $u(t,x)\le u(t_0,x_0)\le 0$. Letting $\varepsilon\to 0$ gives

$$\sup_{[0,S]\times M} (\varphi-\psi) \le e^{\lambda S} \max\Big\{\sup_X(\varphi_0-\psi_0); 0\Big\}, \tag{27}$$

as required. ■

3.2. C^0 and C^1 estimates

We now assume that the solution φ of the parabolic Monge–Ampère equation (20) exists on $[0,T_m)$ for $0<T_m<T'<T$. We shall establish uniform estimates for φ on $[0,T_m)$. The estimates for φ and $\dot{\varphi}$ follow from the maximum principle.

Lemma 2. *There is a uniform constant* $C>0$ *such that* $\sup_M |\varphi(t)| \le C$ *for all* $t\in[0,T_m)$.

Proof. For the upper bound of φ we apply the maximum principle to $H=\varphi-A\varphi$ for

$$A = 1 + \sup_{[0,T_m]\times M} \log \frac{\det \hat{g}}{\Omega^2},$$

where $\hat{g}(t) = g_0 + \frac{t}{T'}(g' - g_0)$. For any $s \in [0, T_m)$, suppose that

$$\varphi(t_0, x_0) = \max_{[0,s]\times M} \varphi,$$

with $(t_0, x_0) \in [0, s] \times M$. If $t_0 > 0$, then using the fact that $(\varphi_{ij}(t_0, x_0)) \leq 0$, we have

$$0 \leq \frac{\partial \varphi}{\partial t} = \log \frac{\det(\hat{g} + \nabla d\varphi)}{\Omega^2} - A \leq \log \frac{\det(\hat{g})}{\Omega^2} - A \leq -1,$$

so a contradiction. Hence $t_0 = 0$ and we get the upper bound for φ.

Similarly, we use the same argument for $K = \varphi + Bt$, where

$$B = 1 - \inf_{[0,T_m]\times M} \log \frac{\det \hat{g}}{\Omega^2},$$

to get the lower bound of φ. ■

We shall use the following evolution equation for $\bar{\beta}(g) := \mathrm{Tr}_g\, \beta(g) = g^{ji}\beta_{ij}(g)$.

Proposition 4. *The trace $\bar{\beta}$ of β evolves as*

$$\frac{\partial}{\partial t}\bar{\beta} = L_g\bar{\beta} + |\beta|_g^2, \tag{28}$$

where $|\beta|_g^2 = g^{li}g^{jk}\beta_{ij}\beta_{kl}$. Therefore, we have the lower bound

$$\bar{\beta} \geq -Ce^{-t},$$

where $C = -\inf_M \bar{\beta}(0) - n$.

Proof. It follows from the flow that

$$\begin{aligned} \frac{\partial}{\partial t} g^{ij} &= -g^{il}(\partial_t g_{lk})g^{kj} \\ &= -g^{il}(-\beta_{lk})g^{kj} \\ &= \beta^{ij}, \end{aligned} \tag{29}$$

where $(g^{kl}) = (g^{-1})_{kl}$. Taking the trace of both sides of the flow (18) we get

$$g^{lk}\partial_t g_{kl} = \bar{\beta}. \tag{30}$$

Combining (30), (29) and $\beta_{ij} = \partial_i\partial_j \log\det[g_{kl}]$ yields

$$\begin{aligned} \partial_t\bar{\beta} &= g^{ji}\partial_i\partial_j(g^{kl}\partial_t g_{kl}) - \partial_t g^{ji}\beta_{ij} \\ &= g^{ji}\partial_i\partial_j(\bar{\beta} - n) - \partial_t g^{ji}\beta_{ij} \\ &= L_g\bar{\beta} - \beta^{ji}\beta_{ij} \\ &= L_g\bar{\beta} - g^{li}g^{jk}\beta_{kl}\beta_{ij} \\ &= L_g\bar{\beta} + |\beta|_g^2, \end{aligned} \tag{31}$$

as required. ■

Lemma 3. *There is a uniform constant C such that*

$$\sup_M |\dot{\varphi}(t)| \le C$$

for all $t \in [0, T_m)$.

As a consequence, we have $e^{-C}\Omega^2 \le \det g(t) \le e^C \Omega^2$, so $C'^{-1}\det g_0 \le \det g(t) \le C' \det g_0$ for some constant C' depending only on g_0 and Ω.

Proof. We first have

$$\frac{\partial}{\partial t}\dot{\varphi} = L_g \dot{\varphi} - \operatorname{Tr}_g \beta_\Omega,$$

since $\beta_\Omega = \frac{-1}{T}(g' - g_0)$ and $\hat{g}(t) = g_0 - t\beta_\Omega$, where $L_g = \operatorname{Tr}_g \nabla d$. Therefore, we have

$$\Big(\frac{\partial}{\partial t} - L_g\Big)((T'-t)\dot{\varphi}) = -\dot{\varphi} - (T'-t)\operatorname{Tr}_g \beta_\Omega. \tag{32}$$

We also have

$$\Big(\frac{\partial}{\partial t} - L_g\Big)\varphi = \dot{\varphi} + \operatorname{Tr}_g(g - \hat{g}) = \dot{\varphi} - \operatorname{Tr}_g \hat{g} - n. \tag{33}$$

Let $H = (T'-t)\dot{\varphi} + \varphi + nt$; then combining identities above gives

$$\Big(\frac{\partial}{\partial t} - L_g\Big)H = \operatorname{Tr}_g(-(T'-t)\beta_\Omega + \hat{g}_t) = \operatorname{Tr}_g \hat{g}_{T'} > 0. \tag{34}$$

The maximum principle then implies that the minimum of H is at $t = 0$. Therefore,

$$(T'-t)\dot{\varphi}(t) + nt \ge T'\dot{\varphi}(0) \ge T' \inf_M \log \frac{\det g_0}{\Omega^2} \ge -CT',$$

so using Lemma 2 and $T' - t > T' - T_m > 0$ gives

$$\inf_M \dot{\varphi}(t) \ge -C$$

for all $t \in [0, T_m)$.

For the upper bound of $\dot{\varphi}$ we observe that

$$\partial_t \dot{\varphi} = -\operatorname{Tr}_g \beta(g) = -\bar{\beta}(g). \tag{35}$$

It follows from Proposition 4 that, along the flow, $\bar{\beta}$ satisfies

$$\frac{\partial}{\partial t}\bar{\beta} = L_g \bar{\beta} + |\beta|_g^2, \tag{36}$$

and so

$$\inf_M \bar{\beta}(t) \ge \inf_M \bar{\beta}(0) \ge -C$$

for all $t \in [0, T_m)$. Combining with (35) gives a uniform upper bound for $\dot{\varphi}$ for all $t \in [0, T_m)$, as required. ∎

3.3. C^2 estimate

Our goal in this section is to prove the following estimate:

Theorem 7. *There exists a constant $C > 0$ which depends only on g_0 such that*

$$\sup_M \operatorname{Tr}_{g_0} g \leq C$$

for all $t \in [0, T)$.

Moreover, there exists a constant C' depending only on g_0 such that

$$\frac{1}{C'} g_0 \leq g(t) \leq C' g_0 \tag{37}$$

for all $t \in [0, T_m)$.

We first prove the following estimate for the trace of the metric along the Hesse–Koszul flow, which can be seen as a real version of that of the Kähler–Ricci flow due to Cao [9] (see [38, 42] for details).

Lemma 4. *Let g_0 be a fixed Hessian metric on M. There exists a uniform constant C_0 depending only on the metric g_0 such that the solution $g(t)$ of the Hesse–Koszul flow satisfies*

$$\left(\frac{\partial}{\partial t} - L_g\right) \log \operatorname{Tr}_{g_0} g \leq C_0 \operatorname{Tr}_g g_0. \tag{38}$$

Proof. At a point x_0 we take affine coordinates $\{x^1, \ldots, x^n\}$ with respect to ∇, such that $(g_0)_{ij}(x_0) = \delta_{ij}$ and $\partial_k (g_0)_{ij}(x_0) = 0$. At x_0, for $L_g = \operatorname{Tr}_g \nabla d$, we have

$$\begin{aligned} L_g(\operatorname{Tr}_{g_0} g) &= g^{ij} \partial_i \partial_j (g_0^{kl} g_{kl}) \\ &= g^{ij} g_0^{kl} \partial_i \partial_j g_{kl} + g^{ij} \partial_i \partial_j g_0^{kl} g_{kl}. \end{aligned} \tag{39}$$

Observe that

$$g^{ij} \partial_i \partial_j g_0^{kl} g_{kl} \geq -C_0 \operatorname{Tr}_{g_0} g \operatorname{Tr}_g g_0,$$

for a constant $C_0 > 0$ depending only on g_0. We also have

$$\begin{aligned} \frac{\partial}{\partial t} \operatorname{Tr}_{g_0} g &= g_0^{lk} \partial_k \partial_l \log \det(g) \\ &= g^{ji} g_0^{kl} \partial_k \partial_l g_{ji} - g^{jp} g^{qi} g_0^{kl} \partial_k g_{ij} \partial_l g_{pq} \\ &= g^{ji} g_0^{kl} \partial_i \partial_j g_{kl} - g^{jp} g^{qi} g_0^{kl} \partial_k g_{ij} \partial_l g_{pq}. \end{aligned} \tag{40}$$

This implies that

$$(\partial_t - L_g) \operatorname{Tr}_{g_0} g \leq C_0 \operatorname{Tr}_{g_0} g \operatorname{Tr}_g g_0 - g^{jp} g^{qi} g_0^{lk} \partial_k g_{ij} \partial_l g_{pq}, \tag{41}$$

hence

$$
\begin{aligned}
&(\partial_t - L_g)\log \operatorname{Tr}_{g_0} g \\
&\quad \le C_0 \operatorname{Tr}_g g_0 + \frac{1}{\operatorname{Tr}_{g_0} g}\Big(-g^{jp} g^{qi} g_0^{lk} \partial_i g_{kj} \partial_l g_{pq} + \frac{g^{qk}\partial_k \operatorname{Tr}_{g_0} g \,\partial_q \operatorname{Tr}_{g_0} g}{\operatorname{Tr}_{g_0} g}\Big). \qquad (42)
\end{aligned}
$$

We now claim that the second term satisfies

$$
- g^{jp} g^{qi} g_0^{lk} \partial_i g_{kj} \partial_l g_{pq} + \frac{g^{lk}\partial_k \operatorname{Tr}_{g_0} g \,\partial_l \operatorname{Tr}_{g_0} g}{\operatorname{Tr}_{g_0} g} \le 0. \qquad (43)
$$

Indeed, since

$$
g_0^{li} g^{jp} g^{qk} A_{ijk} A_{lpq} \ge 0,
$$

where

$$
A_{ijk} = \partial_i g_{kj} - g_{ij}\frac{\partial_k \operatorname{Tr}_{g_0} g}{\operatorname{Tr}_{g_0} g},
$$

we get

$$
\begin{aligned}
0 &\le g^{jp} g^{qk} g_0^{li} \partial_i g_{kj} \partial_l g_{pq} + g_0^{li} g^{jp} g^{qk} g_{ij} g_{pl} \frac{\partial_k \operatorname{Tr}_{g_0} g \,\partial_q \operatorname{Tr}_{g_0} g}{\operatorname{Tr}_{g_0}^2 g} \\
&\quad - 2 g_0^{li} g^{jp} g^{qk} g_{lp} \partial_k g_{il} \frac{\partial_q \operatorname{Tr}_{g_0} g}{\operatorname{Tr}_{g_0} g} \\
&= g^{jp} g^{qk} g_0^{li} \partial_i g_{kj} \partial_l g_{pq} + g^{qk} g_0^{li} g_{il} \frac{\partial_k \operatorname{Tr}_{g_0} g \,\partial_q \operatorname{Tr}_{g_0} g}{\operatorname{Tr}_{g_0}^2 g} \\
&\quad - 2 g_0^{li} \delta_{jl} g^{qk} \partial_k g_{il} \frac{\partial_q \operatorname{Tr}_{g_0} g}{\operatorname{Tr}_{g_0} g} \\
&= g^{jp} g^{qk} g_0^{li} \partial_i g_{kj} \partial_l g_{pq} + \frac{g^{qk}\partial_k \operatorname{Tr}_{g_0} g \,\partial_q \operatorname{Tr}_{g_0} g}{\operatorname{Tr}_{g_0} g} \\
&\quad - 2 g_0^{li} g^{qk} \partial_k g_{il} \frac{\partial_q \operatorname{Tr}_{g_0} g}{\operatorname{Tr}_{g_0} g}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
0 &\le g^{jp} g^{qi} g_0^{lk} \partial_i g_{kj} \partial_l g_{pq} + \frac{g^{qk}\partial_k \operatorname{Tr}_{g_0} g \,\partial_q \operatorname{Tr}_{g_0} g}{\operatorname{Tr}_{g_0} g} \\
&\quad - 2 g_0^{li} g^{qk} \partial_k g_{il} \frac{\partial_q \operatorname{Tr}_{g_0} g}{\operatorname{Tr}_{g_0} g}. \qquad (44)
\end{aligned}
$$

Since $g_0^{li}\partial_k g_{il} = \partial_k \operatorname{Tr}_{g_0} g - g_{il}\partial_k g_0^{li} = \partial_k \operatorname{Tr}_{g_0} g$, hence

$$
0 \le g^{jp} g^{qi} g_0^{lk} \partial_i g_{kj} \partial_l g_{pq} - \frac{g^{qk}\partial_k \operatorname{Tr}_{g_0} g \,\partial_q \operatorname{Tr}_{g_0} g}{\operatorname{Tr}_{g_0} g}, \qquad (45)
$$

as required. The desired inequality now follows from (42) and (43). ■

Proof of Theorem 7. It follows from Lemma 4 that

$$\left(\frac{\partial}{\partial_t} - L_g\right) \log \mathrm{Tr}_{g_0} g \le C_0 \, \mathrm{Tr}_g \, g_0, \tag{46}$$

hence

$$\left(\frac{\partial}{\partial_t} - L_g\right)(\log \mathrm{Tr}_{g_0} g + C_0(t\dot{\varphi}(t) - \varphi(t) - nt)) \le 0. \tag{47}$$

Using the maximum principle, the function

$$H = \log \mathrm{Tr}_{g_0} g + C_0(t\dot{\varphi}(t) - \varphi(t) - nt)$$

achieves its maximum at $t = 0$. Hence

$$\log \mathrm{Tr}_{g_0} g \le C_1 - C_0 C_0(t\dot{\varphi}(t) - \varphi(t) - nt).$$

Since φ and $\dot{\varphi}$ are uniformly bounded, we infer that $\mathrm{Tr}_{g_0} g$ is uniformly bounded from above, as required. The second assertion follows from the next lemma and the fact that $C^{-1} g_0 \le \det g(t) \le C \det g_0$ (cf. Lemma 3). ■

Lemma 5. *If g_1 and g_2 are two metrics on a compact Riemannian manifold M, then*

$$\left(\frac{\det g_2}{\det g_1}\right)^{1/n} \le \frac{1}{n} \mathrm{Tr}_{g_1} g_2 \le \left(\frac{\det g_2}{\det g_1}\right)(\mathrm{Tr}_{g_2} g_1)^{n-1}. \tag{48}$$

In particular, if there exists a constant $C > 0$ such that $C^{-1} \det g_1 \le \det g_2 \le C \det g_1$, we have

$$\begin{aligned} \mathrm{Tr}_{g_1} g_2 \le C_1 &\Leftrightarrow \mathrm{Tr}_{g_2} g_1 \le C_2 \\ &\Leftrightarrow C_3^{-1} g_1 \le g_2 \le C_3 g_1, \end{aligned}$$

where, for each equivalent relation, C_i depends only on C and C_j with $j \ne i$.

Proof. Let $0 < \lambda_1 \le \cdots \le \lambda_n$ be the eigenvalues of g_2 with respect to g_1 (at a given point in M). The assertion is now

$$(\lambda_1 \dots \lambda_n)^{1/n} \le \frac{1}{n} \sum_i \lambda_j \le \lambda_1 \dots \lambda_n \left(\sum_j \frac{1}{\lambda_j}\right)^{n-1}. \tag{49}$$

The left-hand side inequality is nothing but the arithmetic-geometric mean inequality. For the second one, we can assume that $\lambda_1 \dots \lambda_n = 1$; then

$$\left(\sum_j \frac{1}{\lambda_j}\right)^{n-1} \ge \lambda_1^{-1} \dots \lambda_{n-1}^{-1} \ge \frac{1}{n} \sum_j \lambda_j,$$

as required. The second claim is straightforward from the first one. ■

3.4. Higher estimates and proof of Theorem 6

We now can use the Evans–Krylov theorem and Schauder estimates to get C^k estimates for all $k \geq 0$

Proposition 5. *For any $k \in \mathbb{N}$, there exists a uniform constant $C_k > 0$ such that*

$$\|\varphi(t)\|_{C^k(M)} \leq C_k$$

for all $t \in [0, T_m)$.

Proof. Since the Monge–Ampère flow is a fully nonlinear parabolic equation with concave operator, the $C^{2,\alpha}$ estimate for φ follows from the Evans–Krylov theorem [26] (see also [27, 30]).

Let D be any first-order differential operator with constant coefficients. Differentiating (20), we get

$$\Big(\frac{\partial}{\partial t} - L_{g(t)}\Big) D\varphi = D \log(\Omega^2). \tag{50}$$

Since $|\varphi| + |\dot{\varphi}| + |\nabla d\varphi|$ and $[\nabla d\varphi]_\alpha$ are under control, $\|D\varphi\|_{C^0}$ is under control. Applying the parabolic Schauder estimates [27, 30] we infer that $\|D\varphi\|_{C^{2,\alpha}}$ is under control. Applying D to (50) we obtain

$$\Big(\frac{\partial}{\partial t} - L_{g(t)}\Big) D^2\varphi = D^2 \log(\Omega^2) + \sum D g^{jk} \frac{\partial^2 D\varphi}{\partial x^j \partial x^k}, \tag{51}$$

where the parabolic C^α norm of the right-hand side is under control by the argument above. Using again the parabolic Schauder estimates, we obtain a uniform bound for $D^2\varphi$. Iterating this procedure we complete the proof of Proposition 5. ∎

Proof of Theorem 6. It follows from the Arzelà–Ascoli theorem and Proposition 5 that given any sequence $t_j \to T_m$, there exists a subsequence t_{j_k} and a smooth function φ_{T_m} such that $\varphi_{t_{j_k}} \to \varphi_{T_m}$ in $C^k(M, g_0)$ for all $k \geq 0$. It follows from Lemma 3 that $\sup_M |\dot{\varphi}| \leq C$ for all $t \in [0, T_m)$ for some uniform constant C, so $\varphi(t) - Ct$ is nonincreasing in t. Since φ is uniformly bounded in $[0, T_m)$, $\varphi(t) - Ct$ converges as $t \to T_m$ to a function which is necessarily equal to φ_{T_m}. Therefore, the limit φ_{T_m} is unique and so $\varphi(t) \to \varphi_{T_m}$ in $C^k(M, g_0)$ for all $k \geq 0$.

Therefore, the metric $g(t) = \hat{g}(t) + \nabla d\varphi$ converges smoothly to the tensor $g_{T_m} = \hat{g}_{T_m} + \nabla d\varphi_{T_m}$. Moreover, it follows from Theorem 7 that $g(t) \geq C g_0$ for all $t \in [0, T_m)$, so g_{T_m} is positive definite, i.e. a Riemannian metric. Therefore, the flow can be extended to $t = T_m$, which is a contradiction of our assumption that T_m is the maximal time of existence. This implies that $T_m = T$, as required. ∎

4. Hesse–Koszul flow and Hesse–Einstein metrics

We consider the case when $c_1^a(M) < 0$. Then by Theorem 6, the Hesse–Koszul flow exists for all time. Since the class $[g(t)] = [g_0] - tc_1^a(M)$ becomes unbounded as $t \to \infty$, we cannot study the convergence of the flow. Therefore, we need to rescale the flow in time by $t = \log(s+1)$, where s is the time variable for the original flow. Then we get the *normalized Hesse–Koszul flow*:

$$\frac{\partial g}{\partial t} = -\beta(g) - g, \quad g|_{t=0} = g_0. \tag{52}$$

This flow also exists for all time and the class $[g(t)]$ satisfies

$$\frac{d}{dt}[g(t)] = -c_1^a(M) - [g(t)], \quad [g(0)] = [g_0]. \tag{53}$$

Therefore, we have

$$[g(t)] = e^{-t}[g_0] + (1 - e^{-t})[c_1^a(M)],$$

which yields the following theorem.

Theorem 8. *Assume that $c_1^a(M) < 0$; then starting from any Hessian metric g_0, the flow (52) exists for all time and converges in C^∞ to a Hesse–Einstein metric g_∞ satisfying*

$$\beta(g_\infty) = -g_\infty. \tag{54}$$

Moreover, g_∞ is the unique solution to the Hesse–Einstein equation (54).

The existence and uniqueness of a solution to (54) was proved by Cheng–Yau [10] and Caffarelli–Viaclovsky [8]. Our result gives another proof of this result using the parabolic approach.

Since $c_1^a(M) < 0$, there exists a Hessian metric $\hat{g}$ such that its affine Kähler form $\hat{g} \in -c_1^a(M)$. Fix a volume form Ω such that locally $\Omega = \Omega(x)dx^1 \wedge \cdots \wedge dx^n$ with

$$\frac{\partial^2}{\partial x^i \partial x^j} \log \Omega^2(x) = \hat{g}_{ij}(x).$$

By the same argument as Lemma 1, we can rewrite the normalized Hesse–Einstein flow as the parabolic Monge–Ampère equation

$$\frac{\partial}{\partial t}\varphi = \log \frac{\det(\tilde{g} + \nabla d\varphi)}{\Omega^2} - \varphi, \quad \tilde{g} + \nabla d\varphi > 0, \quad \varphi|_{t=0} = 0, \tag{55}$$

where $\tilde{g}(t,x) = e^{-t}g_0(x) + (1 - e^t)\hat{g}(x)$.

We now establish a priori estimates for φ which are independent of t. We follow the same strategy as for the Kähler–Ricci flow.

Lemma 6. *There exists a uniform constant $C > 0$ such that on $[0,\infty) \times M$,*

(1) $|\varphi(t)| \le C$;

(2) $|\dot{\varphi}(t)| \le C(t+1)e^{-t}$;

(3) *there exists a function $\varphi_\infty \in C^0(M)$ such that*

$$|\varphi(t) - \varphi_\infty| \le Ce^{-t/2};$$

(4) $C^{-1}\Omega \le \sqrt{\det g(t)} \le C\Omega$.

Proof. The first estimate is derived straightforwardly from the maximum principle applied to φ. For (2), we use the maximal principle for the function $H = (e^t - 1)\dot{\varphi} - \varphi - nt$. Taking the derivative in time on the two sides of the flow, we get

$$\frac{\partial}{\partial t}\dot{\varphi} = L_{g(t)}\dot{\varphi} + \mathrm{Tr}_{g(t)}(-e^{-t}g_0 + e^{-t}\hat{g}) - \dot{\varphi}, \tag{56}$$

where L_g is the elliptic operator $\mathrm{Tr}_g \nabla d$. In addition, we have

$$\Big(\frac{\partial}{\partial t} - L_{g(t)}\Big)\varphi = \dot{\varphi} - \mathrm{Tr}_{g(t)}(g(t) - \tilde{g}(t)) = \dot{\varphi} - n + \mathrm{Tr}_{g(t)}(e^{-t}g_0 + (1-e^{-t})\hat{g}). \tag{57}$$

Therefore, we have

$$\Big(\frac{\partial}{\partial t} - L_{g(t)}\Big)(e^t\dot{\varphi}) = \mathrm{Tr}_{g(t)}(-g_0 + \hat{g}) \tag{58}$$

and

$$\Big(\frac{\partial}{\partial t} - L_{g(t)}\Big)(\dot{\varphi} + \varphi + nt) = \mathrm{Tr}_{g(t)}\,\hat{g}. \tag{59}$$

Combining all the above, we get

$$\Big(\frac{\partial}{\partial t} - L_{g(t)}\Big)H = -\mathrm{Tr}_{g(t)}\,g_0 < 0.$$

Then the maximum principle implies that $H = (e^t - 1)\dot{\varphi} - \varphi - nt \leq 0$. Since φ is bounded by (1), we get $\dot{\varphi} \leq C(t+1)e^{-t}$.

For the lower bound of $\dot{\varphi}$ we apply the maximum principle to

$$G = (e^t + B)\dot{\varphi} + B\varphi + nBt,$$

where B satisfies $B\hat{g} \geq g_0$. By a direct computation we get

$$\Big(\frac{\partial}{\partial t} - L_{g(t)}\Big)G = \mathrm{Tr}_{g(t)}(-g_0 + \hat{g} + B\hat{g}) > 0.$$

The maximum principle thus implies that $G \geq 0$, hence $\dot{\varphi} \geq -C(t+1)e^{-t}$. For (3), taking $s > t$ and $x \in M$, and using (2), we get

$$\begin{aligned}|\varphi(s,x) - \varphi(t,x)| &= \Big|\int_t^s \dot{\varphi}(\tau,x)\,d\tau\Big| \\ &\leq C\int_t^s (1+\tau)e^{-\tau} \leq C\int_t^s e^{-\tau/2}d\tau = 2C(e^{-t/2} - e^{-s/2}).\end{aligned} \tag{60}$$

Therefore, $\varphi(t)$ converges uniformly to a continuous function φ_∞. Then letting $s \to \infty$ yields (3).

Finally, we use (1) and (2) for

$$\log\frac{\det[g(t)]}{\Omega^2} = \dot{\varphi} + \varphi$$

to get (4). ∎

Proposition 6. *There exists a constant C such that*

$$C^{-1}g_0 \leq g(t) \leq Cg_0, \quad \textit{on } [0,\infty)\times M.$$

Proof. It follows from Lemma 6 (3) that there exists a uniform constant C such that

$$C^{-1}\det g_0 \leq \det g \leq C\det g_0. \tag{61}$$

Then using Lemma 5 it suffices to derive a uniform upper bound for $\operatorname{Tr}_{g_0} g$.

We now apply the maximum principle to $K = \log \operatorname{Tr}_{g_0} g - B\varphi$, where B is chosen hereafter. It follows from the proof of Lemma 4 that

$$\left(\frac{\partial}{\partial t} - L_g\right)\log \operatorname{Tr}_{g_0} g \leq C_0 \operatorname{Tr}_g g_0 - 1, \tag{62}$$

where C_0 only depends on g_0. Please note that here the constant 1 appears in the right-hand side because of the normalization of the flow. Therefore,

$$\begin{aligned}\left(\frac{\partial}{\partial t} - L_g\right)K &\leq C_0 \operatorname{Tr}_g g_0 - 1 - B\dot{\varphi} + B\operatorname{Tr}_g(g-g_0)\\ &= (C_0 - B)\operatorname{Tr}_g g_0 - B\dot{\varphi} + Bn - 1.\end{aligned} \tag{63}$$

Choosing $B = C_0 + 1$, we have

$$\left(\frac{\partial}{\partial t} - L_g\right)K \leq -\operatorname{Tr}_g g_0 - (C_0+1)\dot{\varphi} + (C_0+1)n - 1. \tag{64}$$

Suppose that K admits a maximum at (t_0, x_0) with $t_0 > 0$; then at this point

$$-\operatorname{Tr}_g g_0 - (C_0+1)\dot{\varphi} + (C_0+1)n - 1 \geq 0.$$

Since $\dot{\varphi}$ is uniformly bounded by Lemma 6, we have $\operatorname{Tr}_g g_0 \leq C_1$ at (t_0, x_0) for some uniform constant $C_1 > 0$. Using Lemma 5 and (61), we get $\operatorname{Tr}_{g_0} g(t_0, x_0) \leq C_2$. By our assumption, $K \leq K(t_0, x_0)$; hence

$$\begin{aligned}\log \operatorname{Tr}_{g_0} g(t,x) &\leq \log \operatorname{Tr}_{g_0} g(t_0,x_0) - (C_0+1)(\varphi(t,x) - \varphi(t_0,x))\\ &\leq C_2 - (C_0+1)(\varphi(t,x) - \varphi(t_0,x)).\end{aligned} \tag{65}$$

Since φ is uniformly bounded by Lemma 6, this implies that

$$\operatorname{Tr}_{g_0} g \leq C, \tag{66}$$

as required. ∎

We can now use the Evans–Krylov and Schauder estimates as in Proposition 5 to get C^k estimates for all $k \geq 0$.

Proposition 7. *For any $k \in \mathbb{N}$, there exists a uniform constant $C_k > 0$ such that on $[0, \infty)$,*

$$\|\varphi\|_{C^k(M)} \le C_k.$$

Proof of Theorem 8. Now we can complete the proof of Theorem 8. It follows from Lemma 6 that $\varphi(t)$ converges uniformly exponentially fast to φ_∞. Since we have all uniform C^k estimates by Proposition 7, we infer that $\varphi(t)$ converges to φ_∞ in C^∞, so φ_∞ is smooth.

In addition, we have $|\dot\varphi| \le C(1+t)e^{-t}$ (cf. Lemma 6), hence $\dot\varphi$ converges to 0 in C^∞. It follows that

$$\log \frac{\det(g_\infty)}{\Omega^2} - \varphi_\infty = 0.$$

Applying $\partial_i \partial_j$ to both sides gives $\beta_{ij}(g_\infty) = -(\beta_\Omega)_{ij} - (\varphi_\infty)_{ij} = -(g_\infty)_{ij}$; hence g_∞ satisfies

$$\beta(g_\infty) = -g_\infty. \tag{67}$$

Finally, the uniqueness follows from the maximum principle. Indeed, suppose g_1 and g_2 are two Hesse–Einstein metrics in $-c_1^a(M)$. Then $g_2 = g_1 + \nabla d\phi$ for some function $\phi \in C^\infty(M)$, so $\beta(g_2) = -g_2 = \beta(g_1) - \nabla d\phi$. Therefore, we have

$$\nabla d \log \frac{\det(g_1 + \nabla d\phi)}{\det g_1} = \nabla d\phi, \tag{68}$$

so

$$\log \frac{\det(g_1 + \nabla d\phi)}{\det g_1} = \phi + C. \tag{69}$$

By considering this equality at the maximum x_0 of $\phi + C$ we have

$$\phi + C = \log \frac{\det(g_1 + \nabla d\phi)}{\det g_1} \le 0, \tag{70}$$

so $\phi + C \le 0$. Similarly, we have $\phi + C \ge 0$ so $\phi + C \equiv 0$, hence $g_1 = g_2$. ∎

5. Hesse–Koszul flow and the real Calabi theorem

In this section we study the Hesse–Koszul flow twisted by $\eta \in c_1^a(M)$,

$$\frac{\partial g}{\partial t} = -\beta(g) + \eta, \quad g|_{t=0} = g_0. \tag{71}$$

Our goal is to prove the following convergence result:

Theorem 9. *The flow (71) exists for all time and converges in C^∞ to a metric g_∞ satisfying*

$$\beta(g_\infty) = \eta. \tag{72}$$

Moreover, g_∞ is the unique solution to (72).

In [34], the author proved the long time existence by proving a priori estimates possibly depending on time. In this section we derive uniform a priori estimates that allow us to prove the smooth convergence in Theorem 9.

5.1. Reduction to a parabolic Monge–Ampère equation

Since η and $\beta(g_0)$ belong to the first affine Chern class, there exists $f \in C^\infty(M)$ such that

$$\eta = \beta(g_0) + \nabla df.$$

Therefore, if we let

$$g = g_0 + \nabla d\varphi,$$

where $\varphi \in C^\infty([0,\infty) \times M)$ and $\varphi(0,\cdot) = 0$, the flow becomes

$$\begin{aligned} \nabla d\Big(\frac{\partial}{\partial t}\varphi\Big) &= -\beta(g) + \eta \\ &= -\beta(g) + \beta(g_0) + \nabla df \\ &= \nabla d \log \frac{\det(g)}{\det(g_0)} + \nabla df \\ &= \nabla d\Big(\log \frac{\det(g_0 + \nabla d\varphi)}{\det(g_0)}) + f\Big). \end{aligned} \tag{73}$$

The maximum principle on compact manifolds implies that solving (71) is equivalent to solving the parabolic Monge–Ampère equation

$$\frac{\partial}{\partial t}\varphi = \log \frac{\det(g_0 + \nabla d\varphi)}{\det(g_0)} + f, \quad g_0 + \nabla d\varphi > 0, \quad \varphi|_{t=0} = 0. \tag{74}$$

We start with the following observation:

Lemma 7. *We have the following:*

(1) *There exists a uniform constant C_1 such that*

$$\|\dot\varphi(t)\|_{C^0(M)} \le C_1 \quad \textit{for all } t \in [0,\infty).$$

(2) *There exists a uniform constant C_2 such that on $[0,\infty)$,*

$$C_2^{-1}\det(g_0) \le \det(g) \le C_2 \det(g_0). \tag{75}$$

Proof. Taking the derivative both sides of (74) with respect to t we get

$$\frac{\partial}{\partial t}\dot\varphi = L_{g(t)}\dot\varphi. \tag{76}$$

Then the first estimate follows from the maximum principle and the second estimate follows from the first one and the fact that

$$\det(g) = e^{\dot\varphi - f}\det(g_0). \qquad \blacksquare$$

Although the uniform estimate for $\dot\varphi$ is quite straightforward to obtain, it is important for the C^0 estimate and the proof of convergence of the flow.

5.1.1. C^0 estimate. We now prove a uniform C^0 estimate using the following parabolic Aleksandrov–Bakelman–Pucci (ABP)-type maximum principle due to Tso [45].

Theorem 10. *Let $u \in C^{2,1}((0,T) \times U)$ with $u \le 0$ on the parabolic boundary $\partial_P((0,T) \times U)$ and let*

$$A_u = \big\{(t,x) : u(t,x) \ge 0,\ \exists \xi, |\xi| \le M d^{-1},\ u(t,x) + \xi(y-x) \ge u(x,s)\ \forall y \in U, s \le t\big\},$$

where $M = \max u(t,x) > 0$ and d is the diameter of U. Then

$$M \le C_n d^{n/(n+1)} \left(\int_{A_u} |u_t \det(u_{ij})|\, dx\, dt \right)^{1/(n+1)}, \tag{77}$$

where C_n depends only on n.

Proposition 8. *There exists a uniform constant $C > 0$ such that*

$$\|\tilde{\varphi}\|_{C^0(M)} \le C,$$

where

$$\tilde{\varphi} := \varphi - \frac{1}{\mathrm{Vol}_{g_0}(M)} \int_M \varphi\, dV_{g_0}$$

and

$$\mathrm{Vol}_{g_0}(M) := \int_M dV_{g_0}.$$

Proof. We first remark that the set $\mathcal{F}_0 = \{u \in C^\infty(M) : g_0 + \nabla du > 0, \sup_X u = 0\}$ is relatively compact in $L^1(X,\mu)$ with $\mu = \frac{1}{\int_M dV_{g_0}} dV_{g_0}$ (see for instance [18, Theorem 3.2.12]). Therefore, there exists a constant C depending only on M and g_0 such that

$$\sup_M \varphi \le \int_M \varphi d\mu + C,$$

hence $\tilde{\varphi} \le C$.

For the uniform lower bound of $\tilde{\varphi}$, we follow an idea by the second author jointly with Phong in [36] to use the parabolic ABP estimate due to Tso (Theorem 10). Now fix any $T < \infty$, and set for each t,

$$L = \min_{[0,T]\times M} \tilde{\varphi} = \tilde{\varphi}(t_0, x_0)$$

for some $(t_0, x_0) \in [0,T] \times M$. We now show that L is bounded from below by a constant independent of T. We can assume that $t_0 > 0$, otherwise we are already done. Let $(x^1, \dots, x^n)$ be affine coordinates for M (with respect to ∇) centered at x_0, $B_1 = \{x \mid |x| < 1\}$, and define the function

$$\phi = \tilde{\varphi} + \frac{\delta^2}{4}|x|^2 + |t - t_0|^2$$

on $U_\delta = B_1 \times \{t \mid -\delta \leq 2(t-t_0) < \delta\}$, where $\delta > 0$ is small. Clearly ϕ attains its minimum on U at (t_0, x_0), and $\phi \geq \min_{U_\delta} \phi + \frac{1}{4}\delta^2$ on the parabolic boundary of U_δ.

Define the set

$$S := \Big\{(x,t) \in U_\delta : \phi(x,t) \leq \phi(z_0,t_0) + \tfrac{1}{4}\delta^2,\ |D_x\phi(x,t)| < \tfrac{\delta^2}{8} \text{ and } \phi(y,s) \geq \phi(x,t) + D_x\phi(x,t)(y-x)\ \forall y \in U,\ s \leq t\Big\}. \tag{78}$$

Then applying Theorem 10 to the function $u = -\phi + \min_U \phi + \frac{\delta^2}{4}$ we obtain

$$C\delta^{2n+2} \leq \int_S (-\phi_t)\ \det(\phi_{ij})\, dx\, dt$$

for a constant $C = C(n) > 0$. It follows from Lemma 7 that $|\dot{\varphi}|$ and $\det(\phi_{ij})$ are uniform bounded from above, hence

$$C\delta^{2n+2} \leq A \int_S dx\, dt. \tag{79}$$

Next, by the definition of S, we have $\phi(x,t) \leq L + \frac{\delta^2}{4}$. Since we can assume that $|L| > \delta^2$, it follows that $\phi < 0$ and $|\phi| \geq \frac{|L|}{2}$ on S. Therefore,

$$\begin{aligned} C\delta^{2n+2} \leq A \int_S dx\, dt &\leq A\frac{|L|}{2} \int_S |\phi(x,t)|\, dx\, dt \\ &\leq A\frac{|L|}{2} \int_{U_\delta} |\phi(x,t)|\, dx\, dt. \end{aligned} \tag{80}$$

On S, we also have

$$|\phi| = -\phi = -\tilde{\varphi} - \frac{\delta^2}{4}|x|^2 - (t-t_0)^2 \leq -\tilde{\varphi} + \sup_M \tilde{\varphi}$$

since $\sup_X \tilde{\varphi} \geq 0$. Combining with (80) gives

$$C\delta^{2n+2} \leq A\frac{|L|}{2} \int_{|t|<\frac{1}{2}\delta} \Big\| \tilde{\varphi} - \sup_M \tilde{\varphi} \Big\|_{L^1(M,g_0)} dt. \tag{81}$$

Since $\mathcal{F}_0$ is relatively compact in $L^1(X,\mu)$, there is a uniform constant $C > 0$ such that $\|\tilde{\varphi} - \sup_M \tilde{\varphi}\|_{L^1(M,g_0)} \leq C$. Therefore, we get

$$C\delta^{2n+2} \leq A\frac{|L|}{2} \int_{|t|<\frac{1}{2}\delta} \Big\| \tilde{\varphi} - \sup_M \tilde{\varphi} \Big\|_{L^1(M,g_0)} dt \leq A'\delta\frac{|L|}{2}$$

from which the desired bound for L follows. ∎

Remark. Our approach here can be applied to the Kähler–Ricci flow to get a new C^0 estimate which is different from the initial approach by Cao [9] for the Kähler–Ricci flow.

5.2. C^0 estimate for elliptic Monge–Ampère equations

We now give another C^0 estimate for the flow using a uniform C^0 estimate for the elliptic Monge–Ampère equation

$$\det(g_0 + \nabla d\phi) = f \det(g_0). \tag{82}$$

The idea of the proof comes from that of the C^0 estimate for the complex Monge–Ampère equation using the ABP estimate by Blocki [6]. This approach originated in the work of Cheng–Yau (cf. [5]) and was recently revisited by Blocki [6, 7] and Székelyhidi [40].

We shall use the following ABP-type estimate (cf. [15, Lemma 9.2] and [40, Proposition 11]).

Proposition 9. *Let $u: \mathbb{B} \to \mathbb{R}$ be a smooth function such that $u(0) + \varepsilon \le \inf_{\mathbb{B}} u$, where $\mathbb{B}$ denotes the unit ball in $\mathbb{R}^n$. Define the contact set*

$$S = \big\{x \in \mathbb{B} : |Du(x)| \le \varepsilon/2 \text{ and } u(y) \ge u(x) + Du(x).(y - x)\ \forall y \in \mathbb{B}\big\}. \tag{83}$$

Then there exists a dimensional constant $C_n > 0$ such that

$$C_n \varepsilon^n \le \int_S \det(D^2 u). \tag{84}$$

Theorem 11. *Let (M, ∇, g) be a compact Hessian manifold of dimension n and $\phi \in C^2(M)$ satisfying*

$$g + \nabla d\phi > 0, \quad \det(g + \nabla d\phi) = f \det g.$$

There exists a uniform constant C depending only on $\|f\|_{L^\infty(M)}$ and g_0 such that

$$\operatorname{osc}_M \phi := \sup_M \phi - \inf_M \phi \le C.$$

Proof. By adding some constant, we assume that $\sup_M \phi = 0$. It suffices to derive a uniform bound for $I = \inf_M \phi$.

We fix affine coordinates $(U, x^1, \dots, x^n)$ with $U = \{x \in \mathbb{R}^n : |x^i| < 1\ \forall i = 1, \dots, n\}$ for which I is achieved at the origin. Let $\psi = \phi + \varepsilon|x|^2$ for some small $\varepsilon > 0$. Then on the unit ball $\mathbb{B} = \{x : |x| < 1\}$, $I = \inf_{\mathbb{B}} \psi = \psi(0)$ and $\psi(x) \ge L + \varepsilon$ for $x \in \partial\mathbb{B}$. It follows from Proposition 9 that

$$C_n \varepsilon^n \le \int_S \det(\psi_{ij}). \tag{85}$$

Since on S we have $(\psi_{ij}) = (\phi_{ij} + \varepsilon\delta_{ij}) \ge 0$ and $\det(g + \nabla d\phi) = f \det g$, we infer that $\det(\psi_{ij})$ is uniformly bounded from above on S. Therefore,

$$C_n \varepsilon^n \le \int_S \det(\psi_{ij}) \le C|S|, \tag{86}$$

where $|S|$ denotes the Lebesgue measure of S.

By the definition of S, we have $\psi(0) \geq \psi(x) - \varepsilon/2$ and $\psi(x) \leq I + \varepsilon/2 < 0$. Therefore,

$$\mathrm{Vol}(S) \leq \frac{\|\psi\|_{L^1}}{|I + \varepsilon/2|}. \tag{87}$$

Since $\mathcal{F}_0 = \{u \in C^\infty(M) : g_0 + \nabla du > 0,\ \sup_X u = 0\}$ is relatively compact in $L^1(X, \mu)$ with $\mu = \frac{1}{\int_M dV_{g_0}} dV_{g_0}$, we have that $\|\psi\|_{L^1}$ is uniformly bounded. Inequalities (86) and (87) thus give a uniform bound for I. ∎

We now can apply this theorem to the parabolic Monge–Ampère equation (74).

Another proof for Proposition 8. We rewrite the flow as

$$\det(g_0 + \nabla d\varphi) = e^{\dot{\varphi} - f} \det g_0.$$

Since $\dot{\varphi}$ is uniformly bounded (Lemma 7), we can apply Theorem 11 to obtain the uniform estimate for the oscillation of φ.

Next we have $\int_M \tilde{\varphi}\, dV_g = 0$, thus $\inf_M \tilde{\varphi} \leq 0$ and $\sup_M \tilde{\varphi} \geq 0$. Therefore, the uniform estimate for the oscillation of φ gives a uniform bound for $|\tilde{\varphi}|$. ∎

5.3. Higher-order estimates

We now prove a C^2 estimate. Note that there is a difficulty here since we only have a uniform bound for the oscillation of φ but not on φ. To overcome this, we shall use the maximum principle for a test function which contains $\tilde{\varphi}$ instead of φ.

Lemma 8. *There exist uniform positive constants C such that*

$$\mathrm{Tr}_{g_0}\, g(t, x) \leq C. \tag{88}$$

In consequence, there exists a uniform C such that

$$C^{-1} g_0 \leq g \leq C g_0. \tag{89}$$

Proof. To prove (88) we apply the maximal principle to $G = \log \mathrm{Tr}_{g_0} g - B\tilde{\varphi}$. It follows from the proof of Lemma 4 that

$$\Big(\frac{\partial}{\partial t} - L_g\Big) G \leq C_0\, \mathrm{Tr}_g\, g_0 - \dot{\tilde{\varphi}} + B\, \mathrm{Tr}_g (g - g_0) + \frac{\mathrm{Tr}_{g_0} \eta}{\mathrm{Tr}_{g_0} g}, \tag{90}$$

where C_0 only depends on g_0. Suppose that $\eta \leq C g_0$ and $\mathrm{Tr}_{g_0} g \geq 1$ (otherwise we are done); then choosing $B = C_0 + 1$ and using the fact that $\dot{\varphi}$ is uniformly bounded (cf. Lemma 7) we have

$$\Big(\frac{\partial}{\partial t} - L_g\Big) G \leq -\mathrm{Tr}_g\, g_0 + C. \tag{91}$$

Suppose that G admits its maximum at (t_0, x_0) at the point (t_0, x_0) and assume without of loss of generality that $t_0 > 0$. Then the maximum principle implies that $\mathrm{Tr}_g\, g_0 \leq C$.

Again using Lemma 5 gives $\mathrm{Tr}_{g_0} g \le C$. Then for any $(t, x) \in [0, \infty) \times M$, we have $\log \mathrm{Tr}_{g_0} g(t, x) - B\tilde{\varphi}(t, x) \le \log C - B\tilde{\varphi}(x_0, t_0)$; thus (88) follows from the uniform estimate for $\tilde{\varphi}$ (cf. Proposition 8). ∎

Again we can use the Evans–Krylov and Schauder estimates to get C^k estimates.

Proposition 10. *For any $k \in \mathbb{N}$, there exists a uniform constant $C_k > 0$ such that on $[0, \infty)$,*

$$\|\tilde{\varphi}\|_{C^k(M)} \le C_k.$$

5.4. Convergence

We now finish the proof for the convergence of the flow, following the same argument as in Cao [9]. Set $\psi = \dot{\varphi} + A$ for some large constant $A > 0$ such that $\psi > 0$. Then ψ satisfies the heat equation

$$\dot{\psi} = g^{ij} \partial_i \partial_j \psi. \tag{92}$$

This is uniformly elliptic by the uniform estimates for g in the previous section. It follows from a straightforward modification of the Li–Yau Harnack inequality for heat equations (cf. [29]) that there exist positive constants C_1, C_2, C_3 depending on ellipticity bounds, such that for $0 < t_1 < t_2$ we have

$$\sup_M \psi(t_1, \cdot) \le \inf_M \psi(t_2, \cdot) \Big(\frac{t_2}{t_1}\Big)^{C_3} \exp\Big(\frac{C_2}{t_2 - t_1} + C_1(t_2 - t_1)\Big). \tag{93}$$

Using this inequality we infer that there exist positive constants C_4 and α so that

$$\mathrm{osc}_M\, \psi(t, \cdot) \le C_4 e^{-\alpha t}. \tag{94}$$

Indeed, we define

$$u_s(t, x) = \sup_{x \in M} \psi(s - 1, x) - \psi(s - 1 + t, x), \tag{95}$$

$$v_s(t, x) = \psi(s, x) - \inf_{x \in M} \psi(s - 1, x), \tag{96}$$

$$\omega(t) = \sup_{x \in M} \psi(t, x) - \inf_{x \in M} \psi(t, x) =: \mathrm{osc}_M\, \psi(t, \cdot). \tag{97}$$

Then u_s and v_s satisfy equation (92). Applying inequality (93) with $t_1 = 1/2$ and $t_2 = 1$, we obtain

$$\sup_M \psi(s - 1, x) - \inf_M \psi(s - \tfrac{1}{2}, x) \le C\Big(\sup_M \psi(s - 1, x) - \sup_M \psi(s, x)\Big), \tag{98}$$

$$\sup_M \psi(s - \tfrac{1}{2}, x) - \inf_M \psi(s - 1, x) \le C\Big(\inf_M \psi(s, x) - \sup_M \psi(s - 1, x)\Big), \tag{99}$$

where $C > 1$ is independent of s. Therefore, we infer that

$$\omega(s - 1) + \omega(s - \tfrac{1}{2}) \le C(\omega(s - 1) - \omega(s)),$$

so

$$\omega(s) \le \delta\omega(s-1),$$

where $\delta = \frac{C-1}{C} < 1$. By induction we infer that $\omega(t) \le Ce^{-\alpha t}$, where $\alpha = -\log\delta$, as required.

It follows from (94) that $|\tilde{\psi}(t,x)| \le C_4 e^{-\alpha t}$ for all $x \in M$, where

$$\tilde{\psi} := \psi - \frac{1}{\mathrm{Vol}_{g_o}(M)} \int_M \psi \, dV_{g_0} = \partial_t \tilde{\varphi};$$

hence

$$\partial_t \Big(\tilde{\varphi} + \frac{C_4}{\alpha} e^{-\alpha t} \Big) = \tilde{\psi} - C_4 e^{-\alpha t} \le 0 \tag{100}$$

and $\tilde{\varphi} + \frac{C_4}{\alpha} e^{-\alpha t}$ is decreasing in t. Since $\operatorname{osc}_M \varphi$ is uniformly bounded, so is $\tilde{\varphi} + \frac{C_4}{\alpha} e^{-\alpha t}$. Thus this function converges to a function φ_∞. By the estimates in Proposition 10, the function $\tilde{\varphi} + \frac{C_4}{\alpha} e^{-\alpha t}$ converges in C^∞ to φ_∞. Therefore, $\tilde{\varphi}(t,x)$ also converges to φ_∞. Now $\tilde{\varphi}$ satisfies

$$\frac{\partial}{\partial t}\tilde{\varphi} + \frac{1}{\mathrm{Vol}_{g_o}(M)} \int_M \dot{\varphi} \, dV_{g_0} = \log \frac{\det(g_0 + \nabla d\tilde{\varphi})}{\det g_0} + f. \tag{101}$$

Letting $t \to \infty$, we get

$$\log \frac{\det(g_0 + \nabla d\varphi_\infty)}{\det g_0} + f - c = 0, \tag{102}$$

where $c = \lim_{t\to\infty} \frac{1}{\mathrm{Vol}_{g_o}(M)} \int_M \dot{\varphi} \, dV_{g_0}$. It follows that $\beta(g_\infty) = \eta$, as required, where $g_\infty = g_0 + \nabla d\varphi_\infty$. This completes the proof of Theorem 9.

5.5. Application of the uniform estimate

We finish this section by showing that our method can be used to prove the convergence of the following parabolic Monge–Ampère equation on a smooth compact Riemannian manifold:

$$\frac{\partial}{\partial t}\varphi(x,t) = \log \frac{\det(g(x) + \nabla^2\varphi(t,x))}{\det g(x)} - \lambda\varphi - f(x), \tag{103}$$

with $\lambda = 0$, where ∇ is the Levi-Civita connection of g and $\lambda \in \mathbb{R}$. This flow is studied by Huisken in [19] where the author shows that the flow (103) has a long time existence for all $\lambda \in \mathbb{R}$. She also proved that when $\lambda > 0$, the flow converges in C^∞ to a smooth function. The convergence of the flow for $\lambda \le 0$ is still unknown. In this section, using our approach for the uniform C^0 estimate (cf. Proposition 8), we prove that for $\lambda = 0$ the normalization of φ also converges in C^∞ to a smooth function. In particular, this result gives an alternative proof of the existence of solutions for the Monge–Ampère equations on compact Riemannian manifolds due to [11].

Theorem 12. *The normalization of the solution φ of the flow* (103)*:*

$$\tilde{\varphi} := \varphi - \frac{1}{\mathrm{Vol}_g} \int_M \varphi \, dV_g$$

converges in C^∞ to a function $\tilde{\varphi}_\infty$. In particular, the limit $\tilde{\varphi}$ is a solution of the following Monge–Ampère equation:

$$\det(g + \nabla^2 \phi) = c e^f \det(g). \tag{104}$$

Proof. The key ingredient of the convergence is to prove that $\tilde{\varphi}$, $\dot{\varphi}$ and $\mathrm{Tr}_g(g + \nabla^2 \varphi)$ are uniformly bounded since all higher uniform estimates are derived from that of $\tilde{\varphi}$ and $\dot{\varphi}$.

The uniform estimate for $\dot{\varphi}$ is straightforward from the maximum principle since $\phi = \dot{\varphi}$ satisfies the heat equation

$$\frac{\partial}{\partial t} \phi = \tilde{g}^{ij} \nabla_{ij} \phi, \tag{105}$$

where $\tilde{g} = g + \nabla^2 \phi$.

For the uniform bound of $\tilde{\varphi}$ we follow the same argument as in the proof of Proposition 8, using any local coordinates instead of affine coordinates as in the previous proof.

For the uniform estimate for $\mathrm{Tr}_g(g + \nabla \phi)$ we follow the same argument as in Section 5.3. By the same computation as in Lemma 4, using normal coordinates for g, we have

$$\Big(\frac{\partial}{\partial t} - L_{\tilde{g}} \Big) \log \mathrm{Tr}_g \, \tilde{g} \le C_0 \, \mathrm{Tr}_{\tilde{g}} \, g + C_1, \tag{106}$$

where $\tilde{g} = g + \nabla^2 \varphi$, C_0 depends only on g and C_1 depends only on f and g. The constant C_1 appears when we compute $\frac{\partial}{\partial t} \mathrm{Tr}_g \, \tilde{g}$. Applying the maximum principle for $G = \log \mathrm{Tr}_g \, \tilde{g} - (C_0 + 1)\tilde{\varphi}$ and following the same argument as in the proof of Lemma 8, we get a uniform bound for $\mathrm{Tr}_g \, \tilde{g}$, as required.

Higher-order estimates now follow from the Evans–Krylov and Schauder estimates. Finally, the convergence result follows by the same lines as in Section 5.4. ∎

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