Entire downward solitons to the scalar curvature flow in Minkowski space

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Abstract. We investigate existence and uniqueness in Minkowski space of entire downward translating solitons with prescribed values at infinity, for a scalar curvature flow equation. The radial case translates into an ordinary differential equation and the general case into a fully non-linear elliptic PDE on $\mathbb{R}^n$.

1. Introduction

The Minkowski space $\mathbb{R}^{n,1}$ is $\mathbb{R}^{n+1}$ with the metric

$$dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2.$$

A smooth function $u \colon \mathbb{R}^n \to \mathbb{R}$ is spacelike if $|Du(x)| < 1$ for all $x \in \mathbb{R}^n$. This alternatively means that its graph M is a spacelike hypersurface of $\mathbb{R}^{n,1}$. In the natural chart $(x_1, \ldots, x_n)$, the metric of M is

$$g_{ij} = \delta_{ij} - \partial_i u \, \partial_j u$$

and the second fundamental form is

$$h_{ij} = \frac{1}{\sqrt{1 - |Du|^2}} \, \partial_{ij}^2 u.$$

Since the inverse of the metric is

$$g^{ij} = \delta_{ij} + \frac{\partial_i u \, \partial_j u}{1 - |Du|^2}$$

the shape operator of M is given by

$$h_j^i = \frac{1}{\sqrt{1 - |Du|^2}} \sum_{k=1}^n \Big(\delta_{ik} + \frac{\partial_i u \, \partial_k u}{1 - |Du|^2} \Big) \partial_{kj}^2 u. \tag{1}$$

Let us denote by $\mathcal{H}_k[u]$ the (normalized) kth elementary symmetric function of the principal curvatures $\lambda_1, \ldots, \lambda_n$ of the graph of u,

$$\mathcal{H}_k[u] = \frac{k!(n-k)!}{n!} \sigma_k(\lambda_1, \ldots, \lambda_n).$$

We are interested in the scalar curvature $S[u]$, which is linked to $\mathcal{H}_2[u]$ by

$$S[u] = -n(n-1)\mathcal{H}_2[u],$$

and more specifically in the entire scalar curvature flow,

$$-\frac{\dot{u}}{\sqrt{1-|Du|^2}} + \mathcal{H}_2[u]^{\frac{1}{2}} = H, \tag{2}$$

where $H > 0$ is a given function. This equation expresses the evolution of a spacelike hypersurface with normal velocity given by the square root of its scalar curvature (up to a negative multiplicative constant), with a forcing term. In the paper we will suppose $n > 2$ since the case $n = 2$ is very different from the PDE point of view: it corresponds to the evolution of a surface by its Gauss curvature, which translates into an equation of Monge–Ampère type. It is well known that equation (2) is parabolic on the set of *admissible functions*

$$\{u : \mathbb{R}^n \to \mathbb{R} \text{ of class } C^2 \text{ such that } |Du| < 1,\ \mathcal{H}_1[u] > 0,\ \mathcal{H}_2[u] > 0 \text{ on } \mathbb{R}^n\},$$

and that the Maclaurin inequality

$$\mathcal{H}_1[u] \geq \mathcal{H}_2[u]^{\frac{1}{2}} \tag{3}$$

holds on that set. *Translating solitons* are solutions of (2) moving by vertical translations, i.e. such that $\dot{u} = a$ for some constant $a \in \mathbb{R}$: they are solutions of the so-called *soliton equation*

$$\mathcal{H}_2[u]^{\frac{1}{2}} = H + \frac{a}{\sqrt{1-|Du|^2}}. \tag{4}$$

Conversely, a solution $u \colon \mathbb{R}^n \to \mathbb{R}$ of (4) gives a solution of the flow (2) by setting

$$u(x, t) := u(x) + at.$$

Entire translating solitons thus readily furnish natural examples of solutions for the entire parabolic problem (2). It is moreover plausible that an entire solution of (2) necessarily converges to a translating soliton if the initial hypersurface has bounded curvature and is such that $\sup_{\mathbb{R}^n} |Du| < 1$: this was proved for the mean curvature flow in [1], if H is a constant. So the study of the soliton equation (4) is certainly important for the study of the entire flow (2). We are especially interested here in the existence and uniqueness of entire solutions of (4), and in their asymptotic properties. Since we are seeking admissible solutions (in order to apply elliptic methods), we need to suppose

$$H + a > 0.$$

Let us note that for a solution u of (4) and $\lambda > 0$, the function $u_\lambda(x) := \lambda u(x/\lambda)$ satisfies

$$\mathcal{H}_2[u_\lambda]^{\frac{1}{2}} = H_\lambda + \frac{a/\lambda}{\sqrt{1 - |Du_\lambda|^2}},$$

where $H_\lambda(x) = \frac{1}{\lambda} H(x/\lambda)$. The study of (4) thus reduces to the three cases $a = 1, -1$ or 0. The case $a = 0$ corresponds to the prescribed scalar curvature equation and was studied in [3–6, 21]. We focus here on the case $a = -1$, which is the *downward soliton equation*

$$\mathcal{H}_2[u]^{\frac{1}{2}} = H - \frac{1}{\sqrt{1 - |Du|^2}}, \tag{5}$$

with $H > 1$. We first prove that if H is a radial function, the downward soliton equation admits an entire radial solution, unique up to the addition of a constant; we will moreover specify its asymptotic behavior:

Theorem 1. *Let $C > 1$ be a constant and let $H \colon [0, +\infty) \to \mathbb{R}$ be a non-decreasing function such that $x \in \mathbb{R}^n \mapsto H(|x|) \in \mathbb{R}$ is smooth,*

$$\inf_{[0,+\infty)} H > 1 \quad \text{and} \quad H(r) \xrightarrow{r \to +\infty} C. \tag{6}$$

Then there exists a radial admissible function $u \colon \mathbb{R}^n \to \mathbb{R}$, a solution of (5) in $\mathbb{R}^n$. It is strictly convex and such that

$$|Du(x)| \xrightarrow{|x| \to +\infty} \sqrt{1 - \frac{1}{C^2}}.$$

Assuming moreover that

$$H(r) \stackrel{r \to +\infty}{=} C\left(1 - O\left(\frac{1}{r^2}\right)\right), \tag{7}$$

it satisfies

$$u(x) = \sqrt{1 - 1/C^2}|x| - \frac{1}{C^2}\sqrt{\frac{n-2}{n}} \log|x| + c_0 + o(1) \tag{8}$$

for some constant $c_0 \in \mathbb{R}$. In particular, the radial admissible solution is unique up to the addition of a constant.

We then consider the Dirichlet problem for the downward soliton equation, with constant prescribed curvature and boundary values, in uniformly convex domains:

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary, uniformly convex, and let $C > 1$ and $u_0 \in \mathbb{R}$ be constant. The Dirichlet problem*

$$\begin{cases} \mathcal{H}_2[u]^{\frac{1}{2}} = C - \dfrac{1}{\sqrt{1 - |Du|^2}} & \text{in } \Omega, \\[2mm] u = u_0 & \text{on } \partial\Omega, \end{cases} \tag{9}$$

has a unique admissible solution.

We finally prove that the downward soliton equation with constant curvature $C > 1$ admits entire admissible solutions, with prescribed asymptotic values at infinity. Let us set

$$\tilde{C} = \sqrt{1 - 1/C^2},$$

consider the sphere

$$S_{\tilde{C}} = \{x \in \mathbb{R}^n : |x| = \tilde{C}\},$$

and fix $f : S_{\tilde{C}} \to \mathbb{R}$ of class C^2.

Theorem 3. *There exists a unique admissible solution $u : \mathbb{R}^n \to \mathbb{R}$ of*

$$\mathcal{H}_2[u]^{\frac{1}{2}} = C - \frac{1}{\sqrt{1 - |Du|^2}} \tag{10}$$

such that

$$u(x) \stackrel{|x| \to +\infty}{=} \tilde{C}|x| - \frac{1}{C^2}\sqrt{\frac{n-2}{n}} \log|x| + f\left(\tilde{C}\frac{x}{|x|}\right) + o(1). \tag{11}$$

Similar results concerning the entire soliton equation for the mean curvature flow in Minkowski space were proved in [8, 16–19].

Remark 1.   (1) It may be proved that the second fundamental form of the graph of a radial solution in Theorem 1 decays as $1/r$ at infinity, and more specifically that

$$\mathcal{H}_1[u] \sim_{+\infty} \alpha_n \sqrt{C^2 - 1}\frac{1}{r} \quad \text{and} \quad \mathcal{H}_2[u] \sim_{+\infty} \beta_n (C^2 - 1)\frac{1}{r^2}$$

for some constants $\alpha_n, \beta_n > 0$ depending only on the dimension n. In particular, a radial soliton satisfies the pinching condition

$$\alpha \mathcal{H}_1[u] \leq \mathcal{H}_2[u]^{\frac{1}{2}} \leq \beta \mathcal{H}_1[u] \tag{12}$$

for some constants $\alpha, \beta > 0$. Such a pinching on the curvatures of the initial data was used in [2, 7] for the study of the euclidean scalar curvature flow. We do not know whether (12) holds for the more general solitons obtained in Theorem 3.

(2) In Theorem 2, if Ω is a ball then the solution u of the Dirichlet problem is the restriction of a radial soliton described in Theorem 1.

(3) By the standard elliptic regularity theory the solitons obtained in the three theorems above are automatically smooth (since they are classical solutions of elliptic equations with smooth coefficients).

(4) The uniqueness properties in the three theorems above are consequences of the standard maximum principle. For instance, in Theorem 3, if u_1 and u_2 are admissible solutions of (10) such that (11) holds, and assuming by contradiction that $u_1(x_0) < u_2(x_0)$ at some point $x_0 \in \mathbb{R}^n$, we consider $u_1^\varepsilon := u_1 + \varepsilon$ with $\varepsilon > 0$ such that

$$u_1^\varepsilon(x_0) = u_1(x_0) + \varepsilon < u_2(x_0).$$

Since we have by (11),

$$(u_1^\varepsilon - u_2)(x) \xrightarrow{\;|x| \to +\infty\;} \varepsilon > 0,$$

the set $\{u_1^\varepsilon \le u_2\}$ is compact, and the maximum principle implies that $u_1^\varepsilon \equiv u_2$ on this set, a contradiction.

(5) In Theorem 3, if the function f is a constant then u is a radial function: (10)–(11) are indeed invariant by rotations of center 0 if f is constant, and we have seen above that an admissible solution is unique (up to the addition of a constant).

(6) More is known on the structure of the entire solitons to the mean curvature flow in $\mathbb{R}^{n,1}$: for example, they are convex and satisfy a splitting theorem, [19, Theorem 1.2]. The question of whether these results extend to the scalar curvature flow is open.

Let us quote some related papers. Entire mean curvature flow in Minkowski space was studied in [1, 9–11], and the corresponding soliton equation in [8, 16–19]. The scalar curvature flow in Lorentzian geometry was first studied in [12–14]. We studied in [6] the entire scalar curvature flow in Minkowski space, for an initial hypersurface at a bounded distance of a lightcone.

The paper is organized as follows. In Section 2 we study the radial soliton equation and prove Theorem 1, in Section 3 we solve the Dirichlet problem (Theorem 2) and in Section 4 we study the entire solitons with prescribed values at infinity proving Theorem 3. A short appendix ends the paper on the asymptotic behavior of the solutions of some classical ordinary differential equations.

2. Entire radial solitons

Here we prove Theorem 1 concerning radial solutions of the entire soliton equation: for radial solutions, the equation reduces to an ordinary differential equation. Let us note that a radial function $u \colon \mathbb{R}^n \to \mathbb{R}$ of class C^2 may be written in the form

$$u(x) = u(0) + \int_0^{|x|} y(r)\, dr \tag{13}$$

for some function $y \colon [0, +\infty) \to \mathbb{R}$ of class C^1 such that $y(0) = 0$. With this notation we have at $x \in \mathbb{R}^n \setminus \{0\}$,

$$\partial_i u = y \frac{x_i}{|x|} \quad \text{and} \quad \partial_{ij}^2 u = \frac{y}{|x|}\left(\delta_{ij} - \frac{x_i x_j}{|x|^2}\right) + y' \frac{x_i x_j}{|x|^2},$$

which yields, at $x = (r, 0, \ldots, 0), r > 0$,

$$Du = (y, 0, \ldots, 0), \quad D^2 u = \operatorname{diag}\left(y', \frac{y}{r}, \ldots, \frac{y}{r}\right),$$

and by (1) the following expression for the matrix of the shape operator of $M = \text{graph}(u)$,

$$S := \frac{1}{\sqrt{1-y^2}} \, \text{diag}\Big(\frac{y'}{1-y^2}, \frac{y}{r}, \dots, \frac{y}{r}\Big). \tag{14}$$

Equation (5) is thus equivalent to the ordinary differential equation

$$\frac{2}{n(1-y^2)} \frac{y}{r} \Big\{ \frac{y'}{1-y^2} + \frac{n-2}{2} \frac{y}{r} \Big\} = \Big(H(r) - \frac{1}{\sqrt{1-y^2}} \Big)^2 \tag{15}$$

on $(0, +\infty)$ with

$$\frac{1}{\sqrt{1-y^2}} < H(r).$$

The following proposition together with (13) will thus imply the first part of the theorem:

Proposition 2.1. *Under the hypotheses of Theorem* 1, *there exists* $y\colon [0, +\infty) \to \mathbb{R}$ *of class* C^1, *a solution of* (15) *on* $(0, +\infty)$ *and such that*

$$y(0) = 0 \quad \text{and} \quad 0 \le y < \sqrt{1 - \frac{1}{H(r)^2}}. \tag{16}$$

It satisfies

$$\lim_{r \to +\infty} y(r) = \sqrt{1 - \frac{1}{C^2}}. \tag{17}$$

Moreover, $y'(0) = H(0) - 1$, $y' > 0$ *on* $[0, +\infty)$ *and the matrix* S *defined in* (14) *is positive and converges to*

$$S(0) := (H(0) - 1)\,\text{diag}(1, 1, \dots, 1)$$

as r tends to 0.

Before the proof of the proposition, let us observe that, setting $v = y^2$, (15) transforms to

$$v' = -(n-2)\frac{v(1-v)}{r} + nr(1-v)\big(H(r)\sqrt{1-v} - 1\big)^2, \tag{18}$$

and, following ideas in [18], we consider, for $\varepsilon > 0$, the auxiliary equation

$$v' = -(n-2)\frac{v(1-v)}{r+\varepsilon} + n(r+\varepsilon)(1-v)\big(H(r)\sqrt{1-v} - 1\big)^2. \tag{19}$$

We first solve (19) (Lemma 2.2), then (18) (Lemma 2.3), and afterwards prove Proposition 2.1.

Lemma 2.2. *Equation* (19) *admits a smooth solution* $v_\varepsilon\colon [0, +\infty) \to \mathbb{R}$ *such that* $v_\varepsilon(0) = 0$. *It is non-decreasing on* $[0, +\infty)$, *such that*

$$0 < v_\varepsilon(r) < 1 - \frac{1}{H(r)^2} \quad \text{on } (0, +\infty), \tag{20}$$

and satisfies

$$\lim_{r \to +\infty} v_\varepsilon(r) = 1 - \frac{1}{C^2}.$$

Proof. Equation (19) is of the form $v' = f_\varepsilon(r, v)$ with the smooth function $f_\varepsilon \colon (-\varepsilon, +\infty) \times (-\infty, 1) \to \mathbb{R}$ defined by

$$f_\varepsilon(r, v) = -(n - 2)\frac{v(1 - v)}{r + \varepsilon} + n(r + \varepsilon)(1 - v)\big(H(r)\sqrt{1 - v} - 1\big)^2, \qquad (21)$$

where the function H is extended smoothly to $(-\varepsilon, +\infty)$. The Cauchy–Lipschitz theorem yields a smooth solution $v_\varepsilon \colon [0, T^*) \to (-\infty, 1)$ such that $v_\varepsilon(0) = 0$. We moreover choose T^* such that $[0, T^*)$ is maximal. Let us note that the differential equation implies that

$$v'_\varepsilon(0) = n\varepsilon(H(0) - 1)^2 > 0,$$

so that $v_\varepsilon > 0$ near $r = 0$. Let us prove by contradiction that $v_\varepsilon > 0$ on $(0, T^*)$: if $r_0 > 0$ is the first value such that $v_\varepsilon(r_0) = 0$ we have $v'_\varepsilon(r_0) \le 0$ and by the differential equation again

$$v'_\varepsilon(r_0) = n(r_0 + \varepsilon)(H(r_0) - 1)^2 > 0,$$

a contradiction. We show similarly that the second inequality in (20) holds on $[0, T^*)$: by contradiction, if $r_0 > 0$ is the first point such that

$$v_\varepsilon(r_0) = 1 - \frac{1}{H(r_0)^2},$$

the function

$$r \mapsto v_\varepsilon(r) - \left(1 - \frac{1}{H(r)^2}\right)$$

is negative on $[0, r_0)$ and vanishes at r_0; so its derivative is non-negative at r_0, which implies

$$v'_\varepsilon(r_0) \ge 2\frac{H'(r_0)}{H(r_0)^3} \ge 0.$$

But the differential equation implies

$$v'_\varepsilon(r_0) = -\frac{n - 2}{r_0 + \varepsilon}\left(1 - \frac{1}{H(r_0)^2}\right)\frac{1}{H(r_0)^2} < 0,$$

again a contradiction. So (20) holds on $(0, T^*)$, and since $H \le C$ we deduce that $v_\varepsilon \in (0, 1 - \frac{1}{C^2})$ on $(0, T^*)$, and thus $T^* = +\infty$ (since a maximal solution leaves any compact subset of $(-\varepsilon, +\infty) \times (-\infty, 1)$). We now prove that $v'_\varepsilon > 0$ on $[0, +\infty)$: by contradiction, since $v'_\varepsilon(0) > 0$ there would be a first point $r_1 > 0$ such that $v'_\varepsilon(r_1) = 0$, and differentiating (19) we would obtain at r_1,

$$v''_\varepsilon(r_1) = (n - 2)\frac{v_\varepsilon(1 - v_\varepsilon)}{(r_1 + \varepsilon)^2} + n(1 - v_\varepsilon)\big(H(r_1)\sqrt{1 - v_\varepsilon} - 1\big)^2$$

$$+ 2n(r_1 + \varepsilon)(1 - v_\varepsilon)\big(H(r_1)\sqrt{1 - v_\varepsilon} - 1\big)H'(r_1)\sqrt{1 - v_\varepsilon} > 0,$$

a contradiction since $v'_{\varepsilon|[0,r_1]}$ would also reach its minimum at r_1. Finally, if

$$l := \lim_{r \to +\infty} v_\varepsilon(r) < 1 - \frac{1}{C^2},$$

it is easy to get from (19) that $v'_\varepsilon(r) \to +\infty$ as $r \to +\infty$. This is impossible since v_ε is bounded. ∎

We now pass to the limit $\varepsilon \to 0$ in (19) and obtain a solution of (18):

Lemma 2.3. *There exists a sequence $\varepsilon_i \to_{i \to +\infty} 0$ such that $(v_{\varepsilon_i})_{i \in \mathbb{N}}$ converges smoothly on compact subsets of $(0, +\infty)$. The limit $v = \lim_{i \to +\infty} v_{\varepsilon_i}$ is a solution of (18) on $(0, +\infty)$; $v' > 0$ on $(0, +\infty)$, and v is such that*

$$0 < v(r) < 1 - \frac{1}{H(r)^2} \quad \text{on } (0, +\infty) \tag{22}$$

and satisfies

$$\lim_{r \to 0} v(r) = 0 \quad \text{and} \quad \lim_{r \to +\infty} v(r) = 1 - \frac{1}{C^2}. \tag{23}$$

Proof. Let us prove that for all $[a, b] \subset (0, +\infty)$ and $k \in \mathbb{N}$ there exists $M_k \in [0, +\infty)$ such that

$$\sup_{r \in [a,b]} |v_\varepsilon^{(k)}(r)| \le M_k \tag{24}$$

for all $\varepsilon \in (0, 1)$. Recalling expression (21) of $f_\varepsilon : (-\varepsilon, +\infty) \times (-\infty, 1) \to \mathbb{R}$, we see that f_ε and all its partial derivatives are bounded on $[a, b] \times [0, 1 - 1/C^2]$, independently of $\varepsilon \in (0, 1)$. This gives (24) by induction: for $k = 0$, v_ε takes values in $[0, 1 - 1/C^2]$ and is bounded; assuming that $v_\varepsilon, \ldots, v_\varepsilon^{(k)}$ are bounded on $[a, b]$, independently of $\varepsilon \in (0, 1)$, so is $v_\varepsilon^{(k+1)}$: it is indeed obtained after k derivatives of $v'_\varepsilon = f_\varepsilon(r, v_\varepsilon)$, all of whose terms are bounded. Thus estimates (24) hold, and extracting subsequences and using a diagonal process, we may construct a sequence $\varepsilon_i \to_{i \to +\infty} 0$ such that $(v_{\varepsilon_i})_{i \in \mathbb{N}}$ converges smoothly on compact subsets of $(0, +\infty)$. The smooth function $v = \lim_{i \to +\infty} v_{\varepsilon_i}$ is obviously a solution of (18) on $(0, +\infty)$, is non-decreasing and satisfies

$$0 \le v(r) \le 1 - \frac{1}{H(r)^2} \quad \text{on } (0, +\infty), \tag{25}$$

since so do the functions v_{ε_i}. Thus $v' \ge 0$ on $(0, +\infty)$, and assuming that $v' = 0$ at some point $r_0 > 0$, differentiating (18) we would have at r_0,

$$v''(r_0) = (n-2)\frac{v(1-v)}{r_0^2} + n(1-v)\big(H(r_0)\sqrt{1-v} - 1\big)^2$$
$$+ 2nr_0(1-v)\big(H(r_0)\sqrt{1-v} - 1\big)H'(r_0)\sqrt{1-v} > 0,$$

a contradiction since v' would reach a minimum at r_0 which implies $v''(r_0) = 0$. Thus $v' > 0$ on $(0, +\infty)$ and (25) implies (22) (also using (18) to show that the last inequality

in (25) is in fact a strict inequality). Moreover, the limit at infinity in (23) holds since, on the contrary, the differential equation would imply that $v' \to +\infty$, a contradiction with v bounded. For the limit as $r \to 0$, let us note that (18) with the condition $v' \geq 0$ yields

$$nr^2\big(H(r)\sqrt{1-v}-1\big)^2 \geq (n-2)v \geq 0.$$

Since the left-hand side tends to 0 as r tends to 0, so does the function v. ■

Proof of Proposition 2.1. Let $v: (0, +\infty) \to \mathbb{R}$ be the solution of (18) obtained in the previous lemma. The function $y := \sqrt{v} : (0, +\infty) \to \mathbb{R}$ is C^1, solves (15) on $(0, +\infty)$ and obviously satisfies the inequalities in (16) and the limit (17). Setting $y(0) = 0$ it is moreover continuous at 0. Let us prove that y is also C^1 at 0: since the first term in the left-hand side of (15) is positive we get

$$0 \leq \Big(\frac{y}{r}\Big)^2 \leq \frac{n}{n-2}\big(H(r)\sqrt{1-y^2}-1\big)^2$$

and thus

$$0 \leq \frac{y}{r} \leq \sqrt{\frac{n}{n-2}}(H(r)-1).$$

The function $w = (\frac{y}{r})^2$ is bounded and by (15) satisfies

$$w' + n\frac{w}{r} = \frac{h(r)}{r} \tag{26}$$

with

$$h(r) = (n-2)r^2w^2 + n(1-r^2w)\big(H(r)\sqrt{1-r^2w}-1\big)^2 \xrightarrow{r\to 0} n(H(0)-1)^2.$$

The unique bounded solution of (26) is

$$w(r) = \Big(\int_0^r h(s)s^{n-1}\,ds\Big)r^{-n}.$$

This implies that

$$w \sim_0 \Big(\int_0^r n(H(0)-1)^2 s^{n-1}\,ds\Big)r^{-n} = (H(0)-1)^2.$$

We thus have $y/r \to H(0) - 1$ as r tends to 0, which shows that y is differentiable at 0 with $y'(0) = H(0) - 1$. Moreover, (15) again shows that

$$\frac{y}{r}y'\frac{1}{1-y^2} + \frac{n-2}{2}\Big(\frac{y}{r}\Big)^2 \xrightarrow{r\to 0} \frac{n}{2}(H(0)-1)^2,$$

and, since $y/r \to H(0) - 1$ and $y \to 0$, yields $y' \to H(0) - 1$. Thus y is of class C^1 on $[0, +\infty)$. We now prove that $y' > 0$ on $[0, +\infty)$: $y'(0) = H(0) - 1 > 0$ and we saw in Lemma 2.3 that $v' = 2yy' > 0$ on $(0, +\infty)$. Finally, the last claim is a direct consequence of what we proved before. ■

We now study the asymptotic behavior of the solutions of (15).

Proposition 2.4. *Let us assume that*

$$H(r) \overset{r \to \pm\infty}{=} C\left(1 - O\left(\frac{1}{r^2}\right)\right).$$

The solution y of (15) has the following asymptotic expansion at infinity:

$$y(r) = \sqrt{1 - \frac{1}{C^2}} - \frac{1}{C^2}\sqrt{\frac{n-2}{n}}\frac{1}{r} + O\left(\frac{1}{r^2}\right). \tag{27}$$

Relation (13) then gives the asymptotic behavior (8) of the solution in Theorem 1, which achieves the proof of that theorem.

Proof. Let us first prove that

$$y(r) = \sqrt{1 - \frac{1}{C^2}} - \frac{1}{C^2}\sqrt{\frac{n-2}{n}}\frac{1}{r} + o\left(\frac{1}{r}\right). \tag{28}$$

Let us consider the function z such that

$$y = \sqrt{1 - \frac{1}{C^2}} - \frac{1}{r}z. \tag{29}$$

We observe that it is a solution of an equation of the form

$$z' = Az^2 + B \tag{30}$$

with A and B such that

$$\lim_{r \to \infty} A(r) = A_0 = -\frac{nC^2}{2}\sqrt{1 - \frac{1}{C^2}} \tag{31}$$

and

$$\lim_{r \to \infty} B(r) = B_0 = \frac{n-2}{2C^2}\sqrt{1 - \frac{1}{C^2}}. \tag{32}$$

To see this, let us write (15) in the form

$$2\frac{ryy'}{1 - y^2} = nr^2\left(H(r)\sqrt{1 - y^2} - 1\right)^2 - (n-2)y^2. \tag{33}$$

By a direct computation using (29) and since $z/r \to 0$ (by (17)), the left-hand side of (33) may be written in the form

$$2\frac{ryy'}{1 - y^2} = \alpha_1 z' + \beta_1$$

with $\alpha_1 \to -2C^2\sqrt{1 - \frac{1}{C^2}}$ and $\beta_1 \to 0$. Similarly, since

$$H(r)\sqrt{1 - y^2} - 1 = \left(C^2\sqrt{1 - \frac{1}{C^2}} + o(1)\right)\frac{z}{r} + O\left(\frac{1}{r^2}\right), \tag{34}$$

the right-hand side of (33) is of the form

$$nr^2\left(H(r)\sqrt{1-y^2}-1\right)^2-(n-2)y^2=\alpha_2 z^2+\beta_2$$

with $\alpha_2\to nC^4(1-\frac{1}{C^2})$ and $\beta_2\to-(n-2)(1-\frac{1}{C^2})$. Equations (30), (31) and (32) follow. Proposition A.2 in the appendix then implies that

$$\lim_{r\to+\infty}z(r)=\sqrt{-\frac{B_0}{A_0}}=\frac{1}{C^2}\sqrt{\frac{n-2}{n}},$$

which gives (28). We now prove (27). Let us consider the function α such that

$$y=\sqrt{1-\frac{1}{C^2}}-\frac{1}{C^2}\sqrt{\frac{n-2}{n}}\frac{1}{r}-\frac{\alpha}{r^2}.$$

By (28), $\alpha/r\to 0$ as r tends to infinity. Let us see that α is a solution of an equation of the form

$$\alpha'+a\alpha=b, \tag{35}$$

with $a\to a_0>0$ and b bounded; it will imply that α is bounded by Proposition A.3 in the appendix, which will finish the proof of (27). To see that (35) holds, we write (15) in the form

$$2\frac{r^2yy'}{1-y^2}=r\left(nr^2\left(H(r)\sqrt{1-y^2}-1\right)^2-(n-2)y^2\right). \tag{36}$$

The left-hand side of (36) may be written

$$2\frac{r^2yy'}{1-y^2}=-\frac{2y}{1-y^2}\alpha'+\frac{2y}{1-y^2}(r^2y'+\alpha');$$

since

$$\frac{2y}{1-y^2}\to 2C^2\sqrt{1-\frac{1}{C^2}}$$

and

$$r^2y'+\alpha'=\frac{1}{C^2}\sqrt{\frac{n-2}{n}}+2\frac{\alpha}{r}\to\frac{1}{C^2}\sqrt{\frac{n-2}{n}},$$

it is of the form

$$2\frac{r^2yy'}{1-y^2}=c_1\alpha'+c_2, \tag{37}$$

with $c_1\to-2C^2\sqrt{1-\frac{1}{C^2}}$ and c_2 bounded. For the right-hand side of (36), using the following improvement of the expansion (34),

$$H(r)\sqrt{1-y^2}-1=\left(C^2\sqrt{1-\frac{1}{C^2}}+O\left(\frac{1}{r}\right)\right)\frac{z}{r}+O\left(\frac{1}{r^2}\right)$$

with

$$z=\frac{1}{C^2}\sqrt{\frac{n-2}{n}}+\frac{\alpha}{r}$$

bounded, it is easy to show that

$$n r^3 \left(H(r) \sqrt{1 - y^2} - 1 \right)^2 = (n - 2)\left(1 - \frac{1}{C^2} \right) r + c_3 \alpha + O(1),$$

where $c_3 \to 2\sqrt{n(n-2)}(C^2 - 1)$, and

$$r(n - 2)y^2 = (n - 2)\left(1 - \frac{1}{C^2} \right) r + O(1),$$

so that

$$r \left(n r^2 \left(H(r) \sqrt{1 - y^2} - 1 \right)^2 - (n - 2)y^2 \right) = c_3 \alpha + c_4 \tag{38}$$

with $c_3 \to 2\sqrt{n(n-2)}(C^2 - 1)$ and c_4 bounded. Equation (36) with (37) and (38) implies (35), and the result. ∎

3. The Dirichlet problem

We consider here the Dirichlet problem with constant curvature and boundary values (9), and prove Theorem 2. We may suppose that the boundary condition is $u_0 = 0$.

3.1. Method of resolution

We use the method of continuity: we consider, for $\sigma \in [0, 1]$, the family of Dirichlet problems

$$\begin{cases} \mathcal{H}_2[u]^{\frac{1}{2}} = C - \dfrac{\sigma}{\sqrt{1 - |Du|^2}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{39}$$

and set

$$S := \left\{ \sigma \in [0, 1] : \exists\, u \in C^\infty(\overline{\Omega}) \text{ an admissible solution of (39)} \right\}.$$

(1) $S \neq \emptyset$: by [3,4,21] the Dirichlet problem (39) with $\sigma = 0$ admits a smooth admissible solution.

(2) S is open in $[0, 1]$: this is a consequence of the ellipticity of $\mathcal{H}_2$ at an admissible function and the linear theory.

(3) S is closed in $[0, 1]$ if the following a priori estimates hold: there exist $\alpha \in (0, 1)$ and controlled constants $M \in \mathbb{R}^+$, $\theta \in (0, 1)$ and $\delta > 0$ such that

$$\|u\|_{2,\alpha} \leq M, \quad \sup_{\overline{\Omega}} |Du| \leq 1 - \theta \tag{40}$$

and

$$\mathcal{H}_2[u] \geq \delta \tag{41}$$

for all the admissible solutions of (39). The constants M, θ and δ are assumed to be locally independent of σ: if $\sigma_0 \in (0, 1]$ is given, M, θ and δ are independent of σ in some neighborhood of σ_0, and only depend on Ω, C, n and σ_0. These estimates indeed imply that S is closed in $[0, 1]$: if $(\sigma_n)_{n \in \mathbb{N}} \in S^{\mathbb{N}}$ converges to $\sigma_0 \in (0, 1]$, and u_n is an admissible solution of (39) with $\sigma = \sigma_n$, the estimates (40) allow us to suppose that $(u_n)_{n \in \mathbb{N}}$ converges to a spacelike function u in $C^2(\bar{\Omega})$, which is a solution of (39) with $\sigma = \sigma_0$; moreover, the a priori estimate (41) implies that u is admissible, since, passing to the limit in the Maclaurin inequality (3) for u_n, we obtain

$$\mathcal{H}_1[u] \geq \mathcal{H}_2[u]^{\frac{1}{2}} \geq \delta^{\frac{1}{2}} > 0.$$

Finally, u is smooth, by the standard elliptic regularity theory. This implies that σ_0 belongs to S.

So $S = [0, 1]$ and the Dirichlet problem (9) ((39) with $\sigma = 1$) admits a smooth admissible solution.

Let us note that an estimate of the form $\mathcal{H}_2[u] \geq \delta$ for an admissible solution of (39) is equivalent to an estimate

$$\frac{1}{\sqrt{1 - |Du|^2}} \leq C'/\sigma \tag{42}$$

for some constant C' strictly smaller than C, since by (39),

$$\mathcal{H}_2[u] \geq \delta \quad \text{if and only if} \quad \frac{1}{\sqrt{1 - |Du|^2}} \leq \frac{C - \sqrt{\delta}}{\sigma}. \tag{43}$$

Note also that (42) yields the gradient estimate required in (40). We will thus focus below on the proof of the gradient estimate (42) and afterwards on the estimate of the second derivatives (the C^0 estimate is trivial). The $C^{2,\alpha}$ estimate will then follow from the Evans, Krylov and Safonov estimates [15].

3.2. The gradient estimate

We prove here the gradient estimate (42) in two steps: we first prove that the gradient cannot attain its maximum at a point interior to the domain, and then deal with the estimate at the boundary.

3.2.1. Maximum principle for the gradient. We first show that the maximum of

$$\nu := \frac{1}{\sqrt{1 - |Du|^2}}$$

is attained at the boundary of Ω, i.e. that

$$\sup_{\bar{\Omega}} \nu = \sup_{\partial \Omega} \nu. \tag{44}$$

We will do computations on the hypersurface $M = \mathrm{graph}(u)$, and use the usual index notation and summation convention for tensors. We will denote, for $F = \mathcal{H}_2^{\frac{1}{2}}$ considered as a function of the shape operator $(h_j^i)_{i,j}$,

$$F_i^j := \frac{\partial F}{\partial h_j^i}((h_j^i)_{i,j}) \quad \text{and} \quad F^{ij} := g^{ik} F_k^j;$$

they are tensors on M. Denoting by e_{n+1}^0 the last vector of the canonical basis of $\mathbb{R}^{n,1}$ and by N the upward unit normal to M, we have

$$\nu = -\langle e_{n+1}^0, N \rangle \tag{45}$$

and

$$e_{n+1}^0 = T + \nu N, \tag{46}$$

where $T := -\nabla u \in TM$ is the opposite of the intrinsic gradient of u on M. In particular, we have the formula

$$\nu^2 = 1 + |\nabla u|^2. \tag{47}$$

Let us denote by t^k, $k = 1, \ldots, n$ the components of the vector T in some frame. Differentiating (45) twice we get the formulas

$$\nu_i = -t^k h_{ki} \tag{48}$$

and

$$\nu_{ij} = -(t^k h_{ki;j} + t_{;j}^k h_{ki}), \tag{49}$$

where we use a semicolon to denote the covariant derivative of tensors on M. Let us also recall the Gauss formula $u_{ij} = \nu h_{ij}$, which may be written here as

$$t_{;j}^k = -\nu h_j^k. \tag{50}$$

Since the tensor $h_{ij;k}$ is symmetric in all the indices (the Codazzi equations), we thus have

$$F^{ij} \nu_{ij} = -t^k F^{ij} h_{ij;k} + \nu F^{ij} h_j^k h_{ki}.$$

Since $F = C - \sigma \nu$, we have

$$F^{ij} h_{ij;k} = -\sigma \nu_k,$$

which yields

$$F^{ij} \nu_{ij} = \sigma t^k \nu_k + \nu F^{ij} h_j^k h_{ki}. \tag{51}$$

Since the second term in the right-hand side is positive, it is not possible that simultaneously $\nu_k = 0$ and $F^{ij} \nu_{ij} \leq 0$, and ν cannot achieve its maximum at a point interior to the domain Ω; property (44) follows.

3.2.2. Boundary estimate of the gradient. We will construct upper and lower barriers at the boundary for the Dirichlet problems (39). The construction will rely on the following results concerning radial solutions:

Lemma 3.1. *For all $\sigma \in [0, 1]$, there exists a function $y_\sigma \colon [0, +\infty) \to \mathbb{R}$ of class C^1, a solution of*

$$\frac{2}{n(1 - y^2)} \frac{y}{r} \left\{ \frac{y'}{1 - y^2} + \frac{n - 2}{2} \frac{y}{r} \right\} = \left(C - \frac{\sigma}{\sqrt{1 - y^2}} \right)^2 \tag{52}$$

on $(0, +\infty)$ and such that

$$y_\sigma(0) = 0 \quad and \quad 0 \le y_\sigma < \sqrt{1 - \frac{\sigma^2}{C^2}}. \tag{53}$$

It satisfies

$$\lim_{r \to +\infty} y_\sigma(r) = \sqrt{1 - \frac{\sigma^2}{C^2}}. \tag{54}$$

Moreover, $y'_\sigma > 0$ on $[0, +\infty)$, and $y'_\sigma(0) = C - \sigma$ for $\sigma \in (0, 1]$.

Proof. The case $\sigma = 0$ corresponds to the radial prescribed scalar curvature equation and has the trivial solution (corresponding to an hyperboloid)

$$y_0(r) = \frac{r}{\sqrt{r^2 + \frac{1}{C^2}}}.$$

We thus assume that $\sigma \in (0, 1]$. Let y be the solution of (15) with $H = C/\sigma$ (Proposition 2.1 with the constant C/σ instead of C and $H = C/\sigma$), and set, for $r \in [0, +\infty)$, $y_\sigma(r) := y(\sigma r)$. It is clearly a solution of (52). The other properties of y_σ directly follow from the properties of y. ∎

Lemma 3.2. *If $\sigma, \sigma' \in [0, 1]$ are such that $\sigma < \sigma'$, the functions y_σ and $y_{\sigma'}$ defined in Lemma 3.1 are such that*

$$y_\sigma > y_{\sigma'} \quad on \ (0, +\infty). \tag{55}$$

Moreover, for all $R > 0$ there exists a constant $c_0 = c_0(n, C, R) > 0$ such that

$$\sqrt{1 - \frac{\sigma^2}{(C - c_0)^2}} \ge y_\sigma(R) \ge 0 \tag{56}$$

for all $\sigma \in (0, 1]$.

Proof. We first prove (55). Using Lemma 3.1 we see that

$$y_\sigma(0) = y_{\sigma'}(0) \quad and \quad y'_\sigma(0) > y'_{\sigma'}(0),$$

so that (55) holds in a neighborhood of 0. By contradiction, if (55) does not hold on $(0, +\infty)$ there exists a first point $r > 0$ such that

$$y_\sigma(r) = y_{\sigma'}(r). \tag{57}$$

Since $y_\sigma > y_{\sigma'}$ on $(0, r)$, we also have

$$y'_\sigma(r) \le y'_{\sigma'}(r). \tag{58}$$

Conditions (57), (58) and the differential equation (52) satisfied by y_σ and $y_{\sigma'}$ readily imply that $\sigma \ge \sigma'$, a contradiction. We now prove (56). Let us fix $R > 0$. Since $y'_\sigma > 0$ on $(0, +\infty)$, we directly have from (52) that

$$C - \frac{\sigma}{\sqrt{1 - y_\sigma(R)^2}} \ge \sqrt{\frac{n-2}{n}} \frac{y_\sigma(R)}{R}.$$

Since $y_\sigma(R) \ge y_1(R)$ (by (55)), (56) holds with $c_0 := \sqrt{\frac{n-2}{n}} \frac{y_1(R)}{R}$. ∎

We now estimate the gradient on the boundary of the domain:

Proposition 3.3. *There exists a positive controlled constant $C' < C$ such that*

$$\sup_{\partial\Omega} |Du| \le \sqrt{1 - \frac{\sigma^2}{C'^2}} \tag{59}$$

for all the admissible solutions of the Dirichlet problems (39). *The constant C' only depends on Ω, C and n.*

Proof. Since Ω is uniformly convex we may find uniform constants $R^+, R^- > 0$ and, for each boundary point $x_0 \in \partial\Omega$, two balls $B(x_0^+, R^+)$ and $B(x_0^-, R^-)$ tangent to $\partial\Omega$ at x_0 and satisfying

$$B(x_0^+, R^+) \subset \Omega \subset B(x_0^-, R^-); \tag{60}$$

$x_0^+, x_0^- \in \mathbb{R}^n$ stand for the centers and R^+, R^- for the radii of the balls. We consider, for a fixed $x_0 \in \partial\Omega$, the radial functions

$$v_\sigma^\pm(x) = \int_0^{|x - x_0^\pm|} y_\sigma(r)\, dr - \int_0^{R^\pm} y_\sigma(r)\, dr$$

of center $x_0^\pm$; by construction, $v_\sigma^\pm$ are solutions of the prescribed curvature equation (39) on $\mathbb{R}^n$ and vanish on $\partial B(x_0^\pm, R^\pm)$. In particular, they are upper and lower barriers of u at x_0: since u vanishes on $\partial\Omega$ and by (60), we have $v_\sigma^+ \ge u \ge v_\sigma^-$ on $\partial\Omega$ and $v_\sigma^+(x_0) = v_\sigma^-(x_0) = u(x_0)$, and thus $v_\sigma^+ \ge u \ge v_\sigma^-$ on Ω by the comparison principle. Denoting by $\partial_{\bar{n}}$ the derivative with respect to the inner unit normal to $\partial\Omega$ at x_0, this implies that

$$\partial_{\bar{n}} v_\sigma^+(x_0) \ge \partial_{\bar{n}} u(x_0) \ge \partial_{\bar{n}} v_\sigma^-(x_0).$$

Using (56) we deduce that

$$0 \geq \partial_{\bar{n}} u(x_0) \geq -y_\sigma(R^-) \geq -\sqrt{1 - \frac{\sigma^2}{(C - c_0)^2}}$$

and (59) follows with $C' = C - c_0$ since u is constant on the boundary. The constant C' depends on n, C and R^-. ∎

3.2.3. Global gradient estimate. We readily deduce from the last proposition and (44) the global bound (42) of the gradient:

Proposition 3.4. *There exists a controlled constant $C' < C$ such that*

$$v \leq C'/\sigma \tag{61}$$

for all the admissible solutions of the Dirichlet problems (39). *The constant C' only depends on Ω, C and n.*

Recall that this estimate also implies the required estimate (41) (see (43)).

3.3. The C^2 estimate

3.3.1. Maximum principle for the second derivatives. The estimate relies on the C^2 estimate of Urbas [21]. We assume that estimate (41) holds (it is equivalent to (61)), which implies that the largest principal curvature λ_1 of the graph of u is bounded below, since we have

$$\lambda_1 \geq \mathcal{H}_1 \geq \mathcal{H}_2^{\frac{1}{2}}$$

as a consequence of the admissibility of u. The aim is to bound λ_1 from above. Let us write the prescribed curvature equation (39) in the form

$$F = \psi \tag{62}$$

with $F = \mathcal{H}_2^{\frac{1}{2}}$ and $\psi = C - \sigma v$. The method in [21] consists in applying the maximum principle for the test function

$$\widetilde{W}(x, \xi) = \eta^\beta \sum_{i,j} h_{ij} \xi_i \xi_j \tag{63}$$

defined for $x \in M$ and $\xi \in T_x M$ such that $|\xi| = 1$, with $\eta = \exp(\Phi)$, where $\Phi : \bar{\Omega} \to \mathbb{R}$ is a spacelike strictly convex function and $\beta > 0$ is a constant to be chosen sufficiently large. In [21] the function ψ in the right-hand side of (62) does not depend on the gradient of the unknown function u; here ψ depends on the gradient, and we have to ensure that the third derivatives of u which appear on differentiating (62) twice may still be controlled; this is the aim of the following lemma:

Lemma 3.5. *Assume that $\widetilde{W}$ reaches its maximum at $(x_0, e_1) \in TM$. We have the estimate*

$$\frac{\psi_{11}}{\lambda_1} \geq -c(\beta + \lambda_1)$$

where c is a controlled constant and λ_1 is the largest principal curvature of the graph of u at x_0.

Note that the corresponding estimate used in [21] (for ψ independent of the gradient but arbitrarily given) is $\psi_{11}/\lambda_1 \geq -c(1 + \lambda_1)$ (see [21, inequality (2.8)]).

Proof of Lemma 3.5. By definition of $\widetilde{W}$ the vector e_1 is a principal direction of M corresponding to the largest principal curvature λ_1 at x_0. Let us assume that $e_1, e_2, \ldots, e_n$ is an orthonormal basis of $T_{x_0} M$ formed by principal directions. By (49) and since $h_{k1;1} = h_{11;k}$ (Codazzi equations), we have

$$\psi_{11} = -\sigma v_{11} = \sigma(t^k h_{11;k} + t^k_{;1} h_{k1}). \tag{64}$$

We estimate the first term in the right-hand side: the condition expressing that $\widetilde{W}$ reaches an extremum at (x_0, e_1) is

$$\beta \frac{\eta_i}{\eta} + \frac{h_{11;i}}{h_{11}} = 0$$

for $i = 1, \ldots, n$ (see details in [21]). Since η_i/η is bounded (the gradient is supposed to be controlled), we get

$$|h_{11;i}| \leq c\beta\lambda_1 \tag{65}$$

for a controlled constant c. We then note that by (50) the last term of (64) is

$$t^k_{;1} h_{k1} = -vh^k_1 h_{k1} = -v\lambda_1^2. \tag{66}$$

Formula (64) with the estimates (65) and (66) implies the lemma ($T = -\nabla u$ and $v = \sqrt{1 + |\nabla u|^2}$ are under control since so is the gradient). $\blacksquare$

With this lemma, we may follow the lines of the C^2 estimate in [21] and obtain the C^2 estimate in our case as well, without any relevant modification: the final key estimate [21, (2.16)] is replaced here by

$$0 \geq \frac{c_2\beta\lambda_1}{\eta} + f_1\lambda_1^2 - \frac{c(\beta + \beta^2)f_1}{\eta^2} - \frac{c\beta}{\eta} - c(\beta + \lambda_1)$$

with $f_1 = \partial\sigma_2^{1/2}/\partial\lambda_1$, and we obtain a bound for $\eta^\beta \lambda_1$ by fixing $\beta \geq 1$ large such that

$$\frac{c_2\beta}{\eta} \geq 2c.$$

Remark 2. As in [21] this estimate may also be carried out if η is a function such that $\eta = 0$ on $\partial\Omega$ and $\eta > 0$ in Ω. We will use this fact for the interior C^2 estimate in Section 4, required in the construction of entire solutions.

3.3.2. Boundary estimate of the second derivatives.

The C^2 estimate at the boundary is a consequence of the C^2 boundary estimate in [4], since the estimate in that paper is not affected if the right-hand term in the prescribed curvature equation $\mathcal{H}_m = H$ depends on the gradient of u. More precisely, the prescribed curvature equation is written in [4] in the form

$$\sum_{I,J} A^{IJ}(Du)(D^2u)_{IJ} = f(x, Du) \tag{67}$$

with

$$f(x, p) = \frac{n!}{m!(n-m)!}(1 - |p|^2)^{\frac{m}{2}} H(x), \tag{68}$$

where $A^{IJ}(Du)$ and $(D^2u)_{IJ}$ are the minors of indices the m-tuples I, J of, respectively,

$$\left(\delta_{ij} + \frac{u_i u_j}{1 - |Du|^2}\right)_{ij}$$

and the hessian D^2u. It appears that the C^2 boundary estimate in that paper is independent of the special form (68) and only requires that f is bounded below by a controlled positive constant and that f and its first derivatives are bounded functions on the set $\bar{\Omega} \times \bar{B}(0, 1 - \theta)$, where $\theta \in (0, 1)$ is a constant such that $\sup_\Omega |Du| \leq 1 - \theta$. We need here to estimate solutions of (67) with

$$f(x, p) = \frac{n!}{m!(n-m)!}(1 - |p|^2)^{\frac{m}{2}}\left(C - \frac{\sigma}{\sqrt{1 - |p|^2}}\right)^2,$$

where σ belongs to $(0, 1]$. We have by (61) that

$$f(x, Du) \geq \frac{n!}{m!(n-m)!}\frac{\sigma^m}{C'^m}(C - C')^2$$

for a solution u of the Dirichlet problem (39), and the right-hand term may be bounded below independently of σ on some neighborhood of a fixed $\sigma_0 \in (0, 1]$, e.g. on $(\sigma_0/2, 1]$. Moreover, f and its derivatives are clearly bounded on $\bar{\Omega} \times \bar{B}(0, \sqrt{1 - 1/C^2})$. So the C^2 boundary estimate in [4] applies here and gives the required estimate.

3.3.3. Global C^2 estimate.

Gathering the results obtained above we get the following C^2 estimate on $\bar{\Omega}$:

Proposition 3.6. *Let us fix $\sigma_0 \in (0, 1]$. There exists a controlled constant C'' such that*

$$\sup_{\bar{\Omega}} |D^2u| \leq C'' \tag{69}$$

for all the admissible solutions of the Dirichlet problems (39) with $\sigma \in (\sigma_0/2, 1]$. The constant C'' only depends on Ω, C, C', n and σ_0.

4. Entire solitons of prescribed values at infinity: Proof of Theorem 3

4.1. The method of resolution

We will first construct lower and upper barriers $\underline{u}, \bar{u} \colon \mathbb{R}^n \to \mathbb{R}$ with $\underline{u}$ strictly convex, $\underline{u} < \bar{u}$, and such that

$$\underline{u}(x), \bar{u}(x) = \widetilde{C}|x| - \frac{1}{C^2}\sqrt{\frac{n-2}{n}} \log|x| + f\left(\widetilde{C}\frac{x}{|x|}\right) + o(1) \tag{70}$$

as $|x|$ tends to infinity. They will be defined as the supremum and infimum of families of radial solutions of (10). Let $\Phi \colon \mathbb{R}^n \to \mathbb{R}$ be a strictly convex smooth function such that $\underline{u} \leq \Phi < \bar{u}$. We consider the uniformly convex domains $\Omega_m := \Phi^{-1}(-\infty, m)$, and a sequence $(u_m)_{m \in \mathbb{N}}$ of smooth admissible solutions of the Dirichlet problems

$$\begin{cases} \mathcal{H}_2[u]^{\frac{1}{2}} = C - \dfrac{1}{\sqrt{1 - |Du|^2}} & \text{in } \Omega_m, \\ u = m & \text{on } \partial\Omega_m, \end{cases} \tag{71}$$

whose existence (and uniqueness) is granted by Theorem 2. We have

$$\underline{u} \leq u_m \leq \bar{u} \tag{72}$$

on Ω_m. We need the following interior a priori estimates of the gradient and the higher derivatives of u_m: if $K \subset \mathbb{R}^n$ is a compact subset,

$$\sup_K \frac{1}{\sqrt{1 - |Du_m|^2}} \leq C_K < C, \tag{73}$$

and, for all $i \in \mathbb{N}$,

$$\sup_K \sum_{|\alpha|=i} |\partial^\alpha u_m| \leq C(K, i) \tag{74}$$

for constants C_K and $C(K, i)$ independent of $m \in \mathbb{N}$. With these estimates at hand, the Arzelà–Ascoli theorem and a diagonal process give an entire spacelike smooth function $u \colon \mathbb{R}^n \to \mathbb{R}$, a solution of (10) between the barriers, thus satisfying the asymptotic condition (11).

4.2. The construction of the barriers

The ideas of the construction rely on the papers [19, 20] concerning the mean curvature operator $\mathcal{H}_1$. Since the proofs are identical for the scalar curvature operator $\mathcal{H}_2$, we only recall the definitions and refer to these papers for details (such a construction was also used for $\mathcal{H}_2$ in [5, 6]): We extend f to $\mathbb{R}^n \setminus \{0\}$ by setting $f(\widetilde{C}x) = f(\widetilde{C}x/|x|)$. Let M be a constant such that

$$\left| f(\widetilde{C}x) - f(\widetilde{C}y) - df_{\widetilde{C}y}(\widetilde{C}x - \widetilde{C}y) \right| \leq M |\widetilde{C}x - \widetilde{C}y|^2$$

for all $x, y \in \mathbb{S}^{n-1}$, and set

$$p_1(\tilde{C}y) := Df(\tilde{C}y) + 2M\tilde{C}y,$$
$$p_2(\tilde{C}y) := Df(\tilde{C}y) - 2M\tilde{C}y.$$

Let us consider the radial solution $\psi : \mathbb{R}^n \to \mathbb{R}$ of (10) such that

$$\psi(x) = \tilde{C}|x| - \frac{1}{C^2}\sqrt{\frac{n-2}{n}}\log|x| + o(1)$$

as $|x|$ tends to $+\infty$ (Theorem 1). We set

$$\underline{z}(x, \tilde{C}y) := f(\tilde{C}y) - p_1(\tilde{C}y) \cdot \tilde{C}y + \psi(x + p_1(\tilde{C}y))$$

and

$$\bar{z}(x, \tilde{C}y) := f(\tilde{C}y) - p_2(\tilde{C}y) \cdot \tilde{C}y + \psi(x + p_2(\tilde{C}y)).$$

They satisfy the following properties: for all $x, y \in \mathbb{S}^{n-1}$,

$$f(\tilde{C}x) \geq \underline{z}(rx, \tilde{C}y) - \tilde{C}r + \frac{1}{C^2}\sqrt{\frac{n-2}{n}}\log r$$

and

$$f(\tilde{C}x) \leq \bar{z}(rx, \tilde{C}y) - \tilde{C}r + \frac{1}{C^2}\sqrt{\frac{n-2}{n}}\log r$$

as $r \to +\infty$ (see [19]). Setting, for $x \in \mathbb{R}^n$,

$$\underline{u}(x) := \sup_{y \in \mathbb{S}^{n-1}} \underline{z}(x, \tilde{C}y)$$

and

$$\bar{u}(x) := \inf_{y \in \mathbb{S}^{n-1}} \bar{z}(x, \tilde{C}y),$$

we obtain barriers $\underline{u}, \bar{u} : \mathbb{R}^n \to \mathbb{R}$, with $\underline{u}$ strictly convex, so that $\underline{u} \leq \bar{u}$ (by the maximum principle) and (70) holds. Note finally that we may assume that $\underline{u} < \bar{u}$ on $\mathbb{R}^n$, since otherwise there would exist $x \in \mathbb{R}^n$ and $y_1, y_2 \in \mathbb{S}^{n-1}$ so that $\underline{z}(x, \tilde{C}y_1) = \bar{z}(x, \tilde{C}y_2)$; we would then have $\underline{z}(., \tilde{C}y_1) \equiv \bar{z}(., \tilde{C}y_2)$ on $\mathbb{R}^n$ (by the strong maximum principle) and the function

$$u := \underline{z}(., \tilde{C}y_1) = \bar{z}(., \tilde{C}y_2)$$

would be a solution of (10)–(11) in Theorem 3.

4.3. The a priori estimates

The interior C^0 estimate is trivial by (72). We will focus below on the interior gradient and C^2 estimates. The higher-order estimates will then follow from the Evans–Krylov and Schauder estimates.

4.3.1. The interior gradient estimate. Let $K \subset \mathbb{R}^n$ be a fixed compact subset. We aim to obtain a bound

$$\sup_K v \le C_K \tag{75}$$

for a controlled constant $C_K < C$, for all the solutions of the sequence of Dirichlet problems (71) with $m \ge m_K$ sufficiently large. We construct a smooth auxiliary function $\psi \colon \bar{B}_R \to \mathbb{R}$ defined on a closed ball $\bar{B}_R$ which contains K, which is strictly convex, spacelike and such that

$$C_\psi := \frac{1}{\sqrt{1 - \sup_{\bar{B}_R} |D\psi|^2}} < C, \tag{76}$$

$$\psi \le \underline{u} - \delta_0 \text{ in } K \quad \text{and} \quad \psi \ge \bar{u} \text{ on } \partial B_R, \tag{77}$$

for some constant $\delta_0 > 0$. For the construction of ψ, let us fix $\delta_0 > 0$ and R_0 such that $K \subset \bar{B}_{R_0}$, and set, for $\tilde{C} := \sqrt{1 - 1/C^2}$,

$$\psi(x) := -A_0 + \tilde{C}\sqrt{1 + |x|^2}$$

with

$$A_0 := -\inf_{\bar{B}_{R_0}} \underline{u} + \tilde{C}\sqrt{1 + R_0^2} + \delta_0.$$

We then have

$$\inf_{\bar{B}_{R_0}} \underline{u} - \sup_{\bar{B}_{R_0}} \psi = \delta_0,$$

which in particular implies

$$\underline{u} - \psi \ge \delta_0$$

on K. We moreover have, for all $R > R_0$,

$$\sup_{\bar{B}_R} |D\psi| = \tilde{C} \frac{R}{\sqrt{1 + R^2}} < \tilde{C},$$

and, since

$$\psi(x) = \tilde{C}|x| + O(1) \quad \text{and} \quad \bar{u}(x) = \tilde{C}|x| - \frac{1}{C^2}\sqrt{\frac{n-2}{n}}\log|x| + O(1)$$

as $|x|$ tends to infinity, $\psi \ge \bar{u}$ on $\mathbb{R}^n \setminus B_R$ if R is chosen sufficiently large. For such an $R > R_0$ the spacelike function $\psi \colon \bar{B}_R \to \mathbb{R}$ is such that (76) and (77) hold. It is moreover strictly convex.

We consider a solution u of (71) with m large enough so that $\bar{B}_R \subset \Omega_m$. Let us consider the test function

$$\varphi = \frac{\eta}{C - v}$$

with

$$\eta = (u - \psi)^A, \tag{78}$$

where $A > 0$ is a constant to be chosen later. The function φ is non-negative on the compact set $\{u \geq \psi\}$, vanishes on $\{u = \psi\}$, and thus reaches a maximum at a point x_0 belonging to $\{u > \psi\}$. The aim is to prove

$$v(x_0) \leq C' \tag{79}$$

for some controlled constant $C' < C$. Such a bound will indeed imply that $\varphi(x_0) \leq c$ for some other controlled constant c, and thus that $\varphi \leq c$ on K; since $\inf_K \eta \geq \delta_0^A$, this will in turn imply that

$$\frac{1}{C - v} \leq \frac{c}{\delta_0^A} \quad \text{on } K,$$

and thus will give an estimate of the form (75). We thus focus on obtaining (79). We keep the notation introduced in Section 3.2.1. Since φ reaches its maximum at x_0, we have

$$(\log \varphi)_i = 0 \quad \text{and} \quad F^{ij}(\log \varphi)_{ij} \leq 0 \tag{80}$$

at that point. Since

$$(\log \varphi)_i = \frac{\eta_i}{\eta} + \frac{v_i}{C - v},$$

the first condition reads

$$\eta_i = -\eta \frac{v_i}{C - v}. \tag{81}$$

Using this equality, we further compute

$$(\log \varphi)_{ij} = \frac{\eta_{ij}}{\eta} - \frac{\eta_i \eta_j}{\eta^2} + \frac{v_{ij}}{C - v} + \frac{v_i v_j}{(C - v)^2} = \frac{\eta_{ij}}{\eta} + \frac{v_{ij}}{C - v},$$

and the second condition in (80) reads

$$F^{ij} v_{ij} + (C - v) \frac{1}{\eta} F^{ij} \eta_{ij} \leq 0. \tag{82}$$

By (51) (with $\sigma = 1$) and since $F^{ij} h_j^k h_{ki} \geq 0$, we have

$$F^{ij} v_{ij} \geq t^k v_k = -\frac{t^k \eta_k}{\eta}(C - v).$$

Inequality (82) thus implies

$$F^{ij} \eta_{ij} \leq t^k \eta_k, \tag{83}$$

and, since v is bounded by C,

$$F^{ij} \eta_{ij} \leq cA(u - \psi)^{A-1}. \tag{84}$$

Here and below, we use the letter c to denote a generic controlled constant (i.e. a controlled constant which may change during the computations). We now estimate the left-hand side

of (84). Let us take an orthonormal basis $(e_1, e_2, \ldots, e_n)$ formed by principal directions of the graph of u at x_0. Using (78), (81) and the fact that $v_i = \lambda_i u_i$ (by (48)), we obtain

$$A \frac{u_i - \psi_i}{u - \psi} = -\frac{\lambda_i u_i}{C - v}. \tag{85}$$

Let us consider

$$J = \{i : (1 - \alpha)u_i^2 > \psi_i u_i\} \quad \text{and} \quad J' = \{i : (1 - \alpha)u_i^2 \leq \psi_i u_i\},$$

where $\alpha > 0$ is a small constant such that

$$1 - \alpha > \sup_{\bar{B}_R} |D\psi|, \tag{86}$$

which will be chosen later. We may assume that J is not empty, since on the contrary we would have

$$(1 - \alpha)|u_i| \leq |\psi_i| \quad \text{for all } i = 1, \ldots, n,$$

which would imply (recall (47))

$$v^2 - 1 = |\nabla u|^2 \leq \frac{1}{(1-\alpha)^2} |\nabla \psi|^2 \leq \frac{1}{(1-\alpha)^2} v^2 |D\psi|^2$$

and thus

$$v^2 \leq \frac{1}{1 - \frac{1}{(1-\alpha)^2} |D\psi|^2} \leq \frac{1}{1 - \frac{1}{(1-\alpha)^2} \sup_{\bar{B}_R} |D\psi|^2},$$

i.e. recalling (76), the required estimate (79) if α is chosen sufficiently small.

Using (85), we obtain that, for all $i \in J$,

$$\lambda_i = -A \frac{(C - v)}{u - \psi} \left(1 - \frac{\psi_i}{u_i}\right) < -A \frac{(C - v)}{u - \psi} \alpha \tag{87}$$

is negative. We compute

$$F^{ij} \eta_{ij} = A(u - \psi)^{A-1} (F^{ij} u_{ij} - F^{ij} \psi_{ij})$$
$$\quad + A(A - 1)(u - \psi)^{A-2} F^{ij} (u_i - \psi_i)(u_j - \psi_j). \tag{88}$$

Since $u_{ij} = v h_{ij}$, we have

$$F^{ij} u_{ij} = v F \geq 0. \tag{89}$$

To estimate the term $F^{ij} \psi_{ij}$ we will use the formula

$$\sum_i \sigma_{1,i} \psi_{ii} = \sigma_2 \langle D\psi, Du \rangle v + \sum_i \sigma_{1,i} \sum_{\alpha, \beta} \frac{\partial^2 \psi}{\partial x_\alpha \partial x_\beta} \langle e^0_\alpha, e_i \rangle \langle e^0_\beta, e_i \rangle,$$

where $\sigma_{1,i} = \sum_{k \neq i} \lambda_k$ and e_α^0, $\alpha = 1, \dots, n$ denotes the canonical basis of $\mathbb{R}^n$. A proof may be found in [21]. Since ν is bounded by C and since $|\langle e_\alpha^0, e_i \rangle| \leq \nu$ and the derivatives of ψ are under control, it implies that

$$\sum_i \sigma_{1,i} \psi_{ii} \leq c(\sigma_2 + \sigma_1). \tag{90}$$

Now we may assume that $\sigma_2 \leq 1$ (since on the contrary, estimate (79) would be a direct consequence of the partial differential equation (71)), which implies by the Maclaurin inequality (3) that

$$\sigma_2 \leq \sqrt{\sigma_2} \leq \sqrt{\frac{n-1}{2n}} \sigma_1.$$

This inequality together with (90) implies the estimate

$$F^{ij} \psi_{ij} \leq c\sigma_2^{-\frac{1}{2}} \sigma_1 \tag{91}$$

for a controlled constant c. So (88), (89) and (91) imply that

$$F^{ij} \eta_{ij} \geq -A(u - \psi)^{A-1} c\sigma_2^{-\frac{1}{2}} \sigma_1 + \frac{A(A-1)}{2}(u - \psi)^{A-2}\sigma_2^{-\frac{1}{2}} \sum_i \sigma_{1,i}(u_i - \psi_i)^2. \tag{92}$$

Inequalities (84) and (92) yield

$$(A - 1) \sum_i \sigma_{1,i}(u_i - \psi_i)^2 \leq c(\sigma_1 + \sigma_2^{\frac{1}{2}})(u - \psi)$$

and thus, since $\sigma_2^{\frac{1}{2}} \leq c\sigma_1$ and $u - \psi$ is bounded,

$$(A - 1) \sum_i \sigma_{1,i}(u_i - \psi_i)^2 \leq c\sigma_1$$

for a further controlled constant c. For $i \in J$ we have $(u_i - \psi_i)^2 \geq u_i^2\alpha^2$ and $\sigma_{1,i} \geq \sigma_1$ (since then $\lambda_i \leq 0$ by (87)), and thus

$$(A - 1)\alpha^2 \sum_{i \in J} u_i^2 \leq c. \tag{93}$$

Note also that $(1 - \alpha)|u_i| \leq |\psi_i|$ for all $i \in J'$, which gives

$$\sum_{i \in J'} u_i^2 \leq \frac{1}{(1 - \alpha)^2} \sum_{i \in J'} \psi_i^2 \leq \frac{1}{(1 - \alpha)^2} |\nabla \psi|^2 \leq \nu^2 \frac{1}{(1 - \alpha)^2} |D\psi|^2. \tag{94}$$

From (93) and (94) we deduce

$$|\nabla u|^2 \leq \frac{c}{(A - 1)\alpha^2} + \nu^2 \frac{1}{(1 - \alpha)^2} |D\psi|^2,$$

and since $|\nabla u|^2 = v^2 - 1$ that

$$v^2 \leq \left(1 + \frac{c}{(A-1)\alpha^2}\right) \frac{1}{1 - \frac{1}{(1-\alpha)^2}|D\psi|^2}. \tag{95}$$

Let C', C'' be constants such that

$$C_\psi := \frac{1}{\sqrt{1 - \sup_{\bar{B}_R}|D\psi|^2}} < C'' < C' < C.$$

Choosing $\alpha = \alpha(C'')$ sufficiently small so that

$$\frac{1}{\sqrt{1 - \frac{1}{(1-\alpha)^2} \sup_{\bar{B}_R} |D\psi|^2}} < C'',$$

we deduce from (95) that

$$v^2 \leq \left(1 + \frac{c}{(A-1)\alpha^2}\right) C''^2.$$

We afterwards choose $A = A(\alpha, c, C', C'')$ sufficiently large so that

$$\left(1 + \frac{c}{(A-1)\alpha^2}\right) C''^2 \leq C'^2.$$

This implies $v \leq C'$, which is the required estimate (79), and the interior C^1 estimate (75) follows.

4.3.2. The interior C^2 estimate.

Here again, the estimate relies on the C^2 estimate of Urbas [21]. We need first to construct an auxiliary smooth spacelike function $\psi_2 \colon \bar{B}_R \to \mathbb{R}$ defined on a large ball $\bar{B}_R$ containing K, strictly convex and such that

$$\psi_2 \geq \bar{u} + 1 \quad \text{on } K \tag{96}$$

and

$$\psi_2 < \underline{u} \quad \text{on } \partial B_R. \tag{97}$$

Let us fix R_0 such that $K \subset B_{R_0}$ and a constant C' smaller than C, and consider, for $\widetilde{C}' = \sqrt{1 - 1/C'^2}$, the function

$$\psi_2(x) = A + \sqrt{1 + \widetilde{C}'^2 |x|^2}$$

on $\mathbb{R}^n$, where A is a large constant such that (96) holds. Since

$$\psi_2(x) \sim \widetilde{C}'|x| \quad \text{and} \quad \underline{u}(x) \sim \widetilde{C}|x|$$

as $|x|$ tends to infinity, with $\widetilde{C}' < \widetilde{C}$, (97) also holds if R is sufficiently large. We then follow the lines of the C^2 estimate in [21]: we set

$$\eta := \psi_2 - u,$$

and apply the maximum principle to the function $\widetilde{W}$ introduced in (63), with that new definition of η, on the set $\{\psi_2 \geq u\}$ (which is compact by (97)). The estimate in Lemma 3.5 permits us to carry out the method in [21], without any modification: $\eta^\beta \lambda_1$ is controlled at a point where $\widetilde{W}$ reaches its maximum, which implies that $\widetilde{W}$ is bounded by a controlled constant; see Section 3.3.1 and especially Remark 2 in that section. Using (96) we deduce an upper bound of $D^2 u$ on K by a controlled constant.

A. Auxiliary results on classical ODEs

We gather here auxiliary results concerning the asymptotic behavior of solutions of some classical ODEs. Although these results are elementary and probably well known, since we do not find them in the literature, we include the proofs.

Proposition A.1.   (1) *Assume that* $z\colon (r_0, +\infty) \to \mathbb{R}$ *is a positive solution of*

$$z' = A_0 z^2 + B_0, \tag{98}$$

with A_0, B_0 *constants such that* $A_0 < 0$ *and* $B_0 > 0$*. Then*

$$\lim_{r \to +\infty} z(r) = \sqrt{-\frac{B_0}{A_0}}.$$

(2) *For all* $r_0 \geq 0$ *and* $z_0 \geq 0$*, equation* (98) *admits a unique solution* $z\colon [r_0, +\infty) \to \mathbb{R}$ *such that* $z(r_0) = z_0$*; it is positive on* $(r_0, +\infty)$*.*

Proof. (1) Assuming that $z \neq \sqrt{-\frac{B_0}{A_0}}$ (the contrary result is obvious), we have

$$dr = \frac{dz}{A_0 z^2 + B_0} = \frac{dz}{2\sqrt{-\frac{B_0}{A_0}} A_0} \left\{ \frac{1}{z - \sqrt{-\frac{B_0}{A_0}}} - \frac{1}{z + \sqrt{-\frac{B_0}{A_0}}} \right\}$$

and thus

$$\left| \frac{z - \sqrt{-\frac{B_0}{A_0}}}{z + \sqrt{-\frac{B_0}{A_0}}} \right| = C \exp\left(2\sqrt{-\frac{B_0}{A_0}} A_0 (r - r_0) \right) \tag{99}$$

for some constant $C > 0$. Letting r tend to $+\infty$, the right-hand term tends to 0, and thus

$$\frac{z - \sqrt{-\frac{B_0}{A_0}}}{z + \sqrt{-\frac{B_0}{A_0}}} \xrightarrow{r \to +\infty} 0,$$

which gives the result.

(2) Equation (99) yields

$$z = -\sqrt{-\frac{B_0}{A_0}} + \frac{2\sqrt{-\frac{B_0}{A_0}}}{1 - C\exp\left(2\sqrt{-\frac{B_0}{A_0}}A_0(r - r_0)\right)} \tag{100}$$

for some constant $C \in \mathbb{R}$. Conversely, (100) with

$$C = 1 - \frac{2\sqrt{-\frac{B_0}{A_0}}}{z_0 + \sqrt{-\frac{B_0}{A_0}}}$$

defines a solution $z\colon [r_0, +\infty) \to \mathbb{R}$ of (98) such that $z(r_0) = z_0$. Since z is increasing if $z_0 \in [0, \sqrt{-\frac{B_0}{A_0}})$ (this indeed corresponds to $C \in [-1, 0)$), decreasing if $z_0 > \sqrt{-\frac{B_0}{A_0}}$ (this corresponds to $C \in (0, 1)$) and constant if $z_0 = \sqrt{-\frac{B_0}{A_0}}$ (i.e. for $C = 0$), it follows from the positive sign of the initial condition z_0 or of the limit at infinity that it is positive on $(r_0, +\infty)$. ∎

Proposition A.2. *Assume that $z\colon (0, +\infty) \to \mathbb{R}$ is a positive solution of*

$$z' = Az^2 + B,$$

where $A, B\colon (0, +\infty) \to \mathbb{R}$ are continuous functions such that

$$A(r) \xrightarrow{r\to+\infty} A_0 < 0 \quad \text{and} \quad B(r) \xrightarrow{r\to+\infty} B_0 > 0.$$

Then

$$\lim_{r\to+\infty} z(r) = \sqrt{-\frac{B_0}{A_0}}.$$

Proof. Let us fix constants A_1, A_2, B_1, B_2 such that

$$A_1 < A_0 < A_2 < 0 \quad \text{and} \quad 0 < B_1 < B_0 < B_2.$$

For $r \geq r_0$ sufficiently large,

$$A_1 z^2 + B_1 < z' < A_2 z^2 + B_2.$$

Let us consider z_1 and z_2 solutions of

$$z_1' = A_1 z_1^2 + B_1 \quad \text{and} \quad z_2' = A_2 z_2^2 + B_2$$

such that

$$z_1(r_0) = z_2(r_0) = z(r_0)$$

(by Proposition A.1, 2.). By comparison we have

$$z_1 < z < z_2 \quad \text{in } (r_0, +\infty)$$

and thus, by Proposition A.1 (1),

$$\liminf z \geq \liminf z_1 = \sqrt{-\frac{B_1}{A_1}}$$

and

$$\limsup z \leq \limsup z_2 = \sqrt{-\frac{B_2}{A_2}}.$$

We finally take the limits $A_1, A_2 \to A_0$ and $B_1, B_2 \to B_0$ to obtain the result.   ■

Proposition A.3. *If* $y: [r_0, +\infty) \to \mathbb{R}$ *is a solution of*

$$y' + ay = b,$$

where $a, b: [r_0, +\infty) \to \mathbb{R}$ *are continuous functions such that*

$$0 < a_0 \leq a \quad and \quad |b| \leq \|b\|_\infty < +\infty$$

for some constant a_0, *then* y *is bounded on* $[r_0, +\infty)$:

$$\|y\|_\infty \leq C(y(r_0), a_0, \|b\|_\infty).$$

Proof. Setting, for $r \geq r_0$,

$$A(r) = \int_{r_0}^{r} a(t) \, dt$$

we have

$$y(r) = \left(y(r_0) + \int_{r_0}^{r} b(u)e^{A(u)} \, du \right) e^{-A(r)}.$$

We have $A(r) \geq 0$ and

$$A(u) - A(r) \leq -a_0(r - u)$$

if $r_0 \leq u \leq r$, which implies that

$$|y(r)| \leq |y(r_0)| + \|b\|_\infty \int_{r_0}^{r} e^{-a_0(r-u)} \, du$$

$$\leq |y(r_0)| + \frac{\|b\|_\infty}{a_0}.$$ ■

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