Møller Operators and Hadamard States for Dirac Fields with MIT Boundary Conditions

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Abstract. The aim of this paper is to prove the existence of Hadamard states for Dirac fields coupled with MIT boundary conditions on any globally hyperbolic manifold with timelike boundary once a suitable propagation of singularities theorem is assumed. To this avail, we consider particular pairs of weakly-hyperbolic symmetric systems coupled with admissible boundary conditions. We then prove the existence of an isomorphism between the solution spaces to the Cauchy problems associated with these operators — this isomorphism is in fact unitary between the spaces of L^2-initial data. In particular, we show that for Dirac fields with MIT boundary conditions, this isomorphism can be lifted to a $*$-isomorphism between the algebras of Dirac fields and that any Hadamard state can be pulled back along this $*$-isomorphism preserving the singular structure of its two-point distribution.

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1 Introduction

The initial value problem for a symmetric hyperbolic system on a Lorentzian manifold M is a classical problem that has been exhaustively studied in many contexts. If the underlying background is *globally hyperbolic*, a complete answer is known: In [3] it has been shown that the Cauchy problem is well-posed for any smooth initial data. Even if there exists a plethora of models in physics where globally hyperbolic spacetimes have been used as a background, there are many applications additionally requiring a manifold with non-empty boundary. Indeed, recent developments in quantum field theory focused their attention on manifolds with timelike boundary [13, 79], *e.g.* anti-de Sitter spacetime [30, 31] and BTZ spacetime [16]. Moreover, experimental setups for studying the Casimir effect are mathematically described by introducing timelike boundaries [34, 73]. In addition, moving walls correspond to a timelike boundary in the Lorentzian manifold. For the class of globally hyperbolic manifolds with timelike boundary, the Cauchy problem was investigated by the last two named authors. In particular, we showed in [50] that the Cauchy problem for any symmetric hyperbolic system coupled with an admissible boundary condition is well-posed and the unique solution propagates with at most the speed of light. As a byproduct the existence of a causal propagator is guaranteed. This operator plays a pivotal role in the algebraic approach to quantum field theory since it allows to construct an algebra of observables in a covariant manner, see *e.g.* [48, 14] for textbooks, [5, 44] for recent reviews and [15, 17, 40, 41, 27, 28, 32, 33, 34, 30, 29] for some applications.

In order to complete the quantization of a free field theory, it is necessary to define an (algebraic) state, *i.e.* a positive and normalized functional on the algebra of observables. Clearly not all states can be considered physically meaningful and, on globally hyperbolic spacetimes with empty boundary, only those satisfying the renowned Hadamard condition are regarded as states of physical interest. Indeed, within this setting, Hadamard states guarantee the possibility of constructing Wick polynomials following a local and covariant scheme [53, 56], moreover, they also guarantee the finiteness of the quantum fluctuations of such Wick polynomials, see *e.g.* [39]. Let us remark that in globally hyperbolic stationary spacetimes with empty boundary, the ground state and all KMS states satisfy the Hadamard condition, see *e.g.* [67, 71]. In close analogy, in the presence of a timelike boundary a generalization of the Hadamard condition has been proposed in [78]. Once the Hadamard condition is assumed, several natural questions arise. The most important one concerns the existence of such states, a problem that was answered positively for free field theories on globally hyperbolic spacetimes with empty boundary in [46] (except linearised gravity for which the method cannot be applied – see for example [12]) by means of a spacetime deformation technique. In more details, once a globally hyperbolic manifold (M, g) is assigned, the key point is to find a (ultra-)static globally hyperbolic metric g_0 on M as well as a globally hyperbolic metric g_χ interpolating between g and g_0. This is not an easy task, because the convex combination of two given globally hyperbolic metrics is not globally hyperbolic in general. If the boundary is not empty the situation gets worse. This is due to the need of a boundary condition for the Cauchy problems involved in the construction. In particular, the identification of an interpolating boundary condition is not straightforward. Hence the arguments used in [46] cannot be applied directly and a new proof has to be thought out.

The aim of this paper is to provide a geometric proof of the existence of Hadamard states for Dirac fields with a boundary condition dubbed *MIT boundary condition*. Let us recall that the MIT boundary condition is a local boundary condition which was introduced for the first time in [23] in order to reproduce the confinement of quark in a finite region of space. "Dirac waves" are indeed reflected on the boundary, see also [22, 52] for the description of hadronic states, like baryons and mesons. The MIT boundary condition has been used more recently for many other applications, like the computation of the Casimir energy in a three-dimensional rectangular box [73, 42, 43] in order to construct an integral representation for the Dirac propagator in Kerr Black Hole Geometry and finally also in [57] to prove the asymptotic completeness for linear massive Dirac fields on the Schwarzschild Anti-de Sitter spacetime. A summary of our main result is the following:

Theorem 1.1. *Let* (M, g) *be a globally hyperbolic spin spacetime with timelike boundary and let* D *be the Dirac operator coupled with the MIT boundary condition — cf. Equation* (12). *If for any* $u \in \mathsf{Sol}_{\text{MIT}}(\mathsf{D})$, *the b-wave front set* $\mathrm{WF}_b(u)$ *is the union of maximally extended generalized broken bicharacteristics, then there exists a state for the algebra of Dirac fields with MIT boundary conditions that satisfies the Hadamard condition as per Definition 3.14.*

Remark 1.2. The requirement in Theorem 1.1 is also known as "propagation of singularity theorem" and it has been used for the scalar wave equation in various settings *e.g.* [31, 47, 61, 62, 63, 74, 75, 76, 77, 9, 58]. We expect a similar result to hold since the propagation of singularity for Dirac operators reduces to the propagation of singularity for the scalar wave operator, even though it is acting on vector-valued quantities whose boundary values are coupled. In globally hyperbolic asymptotically anti-de Sitter spacetimes [1], Dappiaggi and Marta in [31] proved the propagation of singularity for the scalar wave equation for a very large class of boundary conditions, which contains all self-adjoint boundary conditions. For these reasons, we expect that the Dirac operator coupled with MIT boundary conditions (or more generally with coupled with self-adjoint boundary conditions) should also fulfill a propagation of singularity theorem in this class of spacetimes.

Our strategy to prove the existence of Hadamard states is as follows. In Section 2.5 we introduce the class of symmetric weakly-hyperbolic operators (*cf.* Definition 2.13) extending that of symmetric hyperbolic ones. We show that the Cauchy problem is well-posed for this category of operators (*cf.* Theorem 2.18). In Section 2.6 we construct a Møller operator, *i.e.* a geometric map which compares the spaces of solutions of two given symmetric weakly-hyperbolic systems coupled with admissible boundary conditions on (possibly

[1] A globally hyperbolic manifold (M, g) is called asymptotically anti-de Sitter spacetime if the following holds: (1) for any boundary function x the metric $\widehat{g} = x^2 g$ extends smoothly to a Lorentzian metric on M; (2) the pullback $\iota^*_{\partial\mathsf{M}}\widehat{g}$ is a smooth Lorentzian metric on $\partial\mathsf{M}$; (3) $\widehat{g}^\sharp(dx, dx) = 1$ on $\partial\mathsf{M}$.

different) globally hyperbolic manifolds with timelike boundary (*cf.* Theorem 2.27). In Section 2.7 we show that this geometric map can be constructed so that it preserves the natural scalar product defined on the space of solutions (*cf.* Proposition 2.33). In Section 3 we focus on the case of Dirac operators: after introducing the classical Dirac operator and MIT boundary conditions (3.1) and (3.2) respectively, we construct an isomorphism between spinor bundles defined on different Lorentzian manifolds, see Section 3.3. In Section 3.4 we construct the algebras of Dirac fields and show that the unitary map between the spaces of solutions to the Dirac equation induces a $*$-isomorphism between the corresponding algebras of Dirac fields (*cf.* Theorem 3.9). Finally in Section 3.5 we discuss and prove the existence of Hadamard states for Dirac fields with MIT boundary conditions.

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Notation and convention

- The symbol $\mathbb{K}$ denotes one of the elements of the set $\{\mathbb{R}, \mathbb{C}\}$.

- $\mathsf{M} := (\mathsf{M}, g)$ is a globally hyperbolic manifold with timelike boundary $\partial\mathsf{M}$ and we adopt the convention that g has the signature $(-, +, \ldots, +)$. If g is a Lorentzian metric such that (M, g) is globally hyperbolic, then we shall write $g \in \mathcal{GH}_{\mathsf{M}}$.

- For two Lorentzian metrics g, g', $g \leq g'$ means that any causal tangent vector for g is causal for g' or equivalently $J_g \subset J_{g'}$.

- $t : \mathsf{M} \to \mathbb{R}$ is a Cauchy temporal function and $\mathsf{M}_\mathsf{T} := t^{-1}(t_0, t_1)$ is a time strip.

- $\mathtt{n}$ is the outward unit normal vector to $\partial\mathsf{M}$.

- $\flat : \mathsf{TM} \to \mathsf{T}^*\mathsf{M}$ and $\sharp : \mathsf{T}^*\mathsf{M} \to \mathsf{TM}$ are the musical isomorphisms.

- E is a $\mathbb{K}$-vector bundle over M with N-dimensional fibers, denoted by E_p for $p \in \mathsf{M}$, and endowed with a Hermitian fiber metric $\prec \cdot \,|\, \cdot \succ_p$ that we assume to be complex linear in the second entry (and antilinear in the first one).

- $\Gamma_c(\mathsf{E}), \Gamma_{sc}(\mathsf{E})$ *resp.* $\Gamma(\mathsf{E})$ denote the spaces of compactly supported, spacelike compactly supported *resp.* smooth sections of E.

- S is a symmetric weakly-hyperbolic system of constant characteristic coupled with principal symbol denoted by σ_{S} and B is an admissible boundary space for S.
- When M is a Lorentzian spin manifold, we denote with $\mathbb{S}\mathsf{M}$ the spinor bundle over M and with D the classical Dirac operator.

2 Møller operators for symmetric weakly-hyperbolic systems

The aim of this section is to construct a geometric map, named Møller operator, to compare the solution spaces of two symmetric weakly-hyperbolic operators coupled with admissible boundary conditions on possibly different (though related) globally hyperbolic manifolds with timelike boundary. To this end, we shall first recall the theory of symmetric hyperbolic systems on globally hyperbolic manifolds with timelike boundary. Then, after showing the well-posedness of the Cauchy problem for weakly hyperbolic systems, we shall construct a family of Møller operators depending on the choice of an arbitrary smooth function f. Choosing suitably such a function, we shall prove that the resulting Møller operator is actually a unitary map between the spaces of initial data — these spaces are endowed with a naturally defined positive scalar product. Our goal is achieved with the help of [50, 65].

2.1 Globally hyperbolic manifolds

We briefly present the geometric background, referring to [50] for more details. Let (M, g) be a smooth $(n+1)$-dimensional time-oriented Lorentzian manifold with (smooth) timelike boundary $\partial\mathsf{M}$.

Definition 2.1. [1, Definition 2.14] The spacetime (M, g) is called a *globally hyperbolic manifold with timelike boundary* if

(i) M is causal, *i.e.* there are no closed causal curves;

(ii) for all points $p, q \in \mathsf{M}$, the subset $J^+(p) \cap J^-(q)$ of M is compact, where $J^+(p)$ (*resp.* $J^-(p)$) denotes the causal future (*resp.* past) of p (*resp.* q) in M.

We recall that $J^+(p)$ is the set of points in M which can be reached from p via a future-directed piecewise smooth timelike curve in (M, g), and similarly for $J^-(p)$.

If $\partial\mathsf{M} = \varnothing$, Definition 2.1 reduces to the standard one of a globally hyperbolic spacetime, see *e.g.* [10, Section 3.2] or [6, Section 1.3].

There is a characterization of globally hyperbolic manifolds with timelike boundary in terms of the existence of Cauchy hypersurfaces and which is due to Aké, Flores and Sánchez:

Theorem 2.2 ([1], Theorem 1.1). *Any globally hyperbolic manifold with timelike boundary admits a Cauchy temporal function* $t\colon \mathsf{M} \to \mathbb{R}$ *with gradient tangent to* $\partial\mathsf{M}$. *This implies that* M *splits into* $\mathbb{R} \times \Sigma$ *with metric*

$$g = -\beta^2 dt^2 \oplus h(t)\,,$$

where $\beta\colon \mathbb{R} \times \Sigma \to \mathbb{R}$ *is a smooth positive function,* $(h(t))_{t\in\mathbb{R}}$ *is a smooth one-parameter-family of Riemannian metrics on* Σ*, and each* $\Sigma_t := \{t\} \times \Sigma$ *is a smooth spacelike Cauchy hypersurface with boundary* $\partial\Sigma_t := \{t\} \times \partial\Sigma$.

We recall that a Cauchy hypersurface is an achronal set which is intersected exactly once by every inextensible piecewise smooth timelike curve in (M, g).

2.2 Symmetric hyperbolic systems of constant characteristic

Let $\mathsf{E} \to \mathsf{M}$ be a Hermitian vector bundle over a globally hyperbolic manifold with timelike boundary M, namely a $\mathbb{K}$-vector bundle with finite rank N endowed with a positive definite Riemannian or Hermitian fiber metric $\prec \cdot \,|\, \cdot \succ_p\colon \mathsf{E}_p \times \mathsf{E}_p \to \mathbb{K}$.

Definition 2.3. A linear differential operator $\mathsf{S}\colon \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ of first order is called a *symmetric hyperbolic system* over M if

(S) The principal symbol $\sigma_{\mathsf{S}}(\xi)\colon \mathsf{E}_p \to \mathsf{E}_p$ is Hermitian with respect to $\prec \cdot \,|\, \cdot \succ_p$ for every $\xi \in \mathsf{T}^*_p\mathsf{M}$ and for every $p \in \mathsf{M}$;

(H) For every future-directed timelike covector $\tau \in \mathsf{T}^*_p\mathsf{M}$, the bilinear form $\prec \sigma_{\mathsf{S}}(\tau) \cdot \,|\, \cdot \succ_p$ is positive definite on E_p for every $p \in \mathsf{M}$.

Furthermore, we say that S is *of constant characteristic* if $\dim\ker \sigma_{\mathsf{S}}(\mathtt{n}^\flat)$ is constant along the boundary. In particular, if $\sigma_{\mathsf{S}}(\mathtt{n}^\flat)$ has maximal rank equal to $\mathrm{rk}(\mathsf{E}) = N$ everywhere on $\partial\mathsf{M}$ we say that S is *nowhere characteristic*.

Remark 2.4. Notice that, if a system S is hyperbolic with respect to a metric g then it is also hyperbolic with respect to any metric in the conformal class of g. Indeed, conformal changes preserve each type of covector. Furthermore, Condition (H) implies that for any spacelike covector $\xi \in \mathsf{T}^*_p\mathsf{M}$ such that $\tau := dt + \xi$ is timelike future-directed,

$$\prec \sigma_{\mathsf{S}}(dt) \cdot \,|\, \cdot \succ_p + \prec \sigma_{\mathsf{S}}(\xi) \cdot \,|\, \cdot \succ_p = \prec \sigma_{\mathsf{S}}(dt+\xi) \cdot \,|\, \cdot \succ_p > 0$$

Therefore, a symmetric system is hyperbolic if and only if it fulfills (S) and satisfies condition (H') instead of (H):

(H') For all spacelike covector $\xi \in \mathsf{T}^*_p\mathsf{M}$ such that $dt + \xi$ is a future-directed timelike covector, the bilinear form fulfills

$$\prec \sigma_{\mathsf{S}}(\xi) \cdot \,|\, \cdot \succ_p > \; - \prec \sigma_{\mathsf{S}}(dt) \cdot \,|\, \cdot \succ_p\,.$$

2.3 Admissible boundary conditions

In order to discuss the Cauchy problem for a symmetric hyperbolic system, we have to impose suitable boundary conditions, depending of course whether we want to solve the forward or the backward Cauchy problem. We begin by fixing a Cauchy surface $\Sigma_0 := t^{-1}(\{0\})$ where we shall assign the initial data. In this paper we shall focus on a class of boundary conditions introduced by Friedrichs and Lax-Phillips respectively in [45, 60] and dubbed admissible.

Definition 2.5. A smooth linear bundle map $\pi_{\mathsf{B}_+} \colon \mathsf{E}_{|\partial\mathsf{M}} \to \mathsf{E}_{|\partial\mathsf{M}}$ is called a *future admissible boundary condition* for a first-order Friedrichs system S if

(i-f) the pointwise kernel B_+ of π_{B_+} is a smooth subbundle of $\mathsf{E}_{|\partial\mathsf{M}}$;

(ii-f) the quadratic form $\Psi \mapsto \prec \sigma_{\mathsf{S}}(\mathtt{n}^\flat)\Psi \,|\, \Psi \succ_p$ is positive semi-definite on B_+;

(iii-f) the rank of B_+ is equal to the number of pointwise non-negative eigenvalues of $\sigma_{\mathsf{S}}(\mathtt{n}^\flat)$ counting multiplicity.

Similarly we say that $\pi_{\mathsf{B}_-} : \mathsf{E}_{|\partial\mathsf{M}} \to \mathsf{E}_{|\partial\mathsf{M}}$ is *past admissible* if

(i-p) the pointwise kernel B_- of π_{B_-} is a smooth subbundle of $\mathsf{E}_{|\partial\mathsf{M}}$;

(ii-p) the quadratic form $\Psi \mapsto \prec \sigma_{\mathsf{S}}(\mathtt{n}^\flat)\Psi \,|\, \Psi \succ_p$ is negative semi-definite on B_-;

(iii-p) the rank of B_- is equal to the number of pointwise non-positive eigenvalues of $\sigma_{\mathsf{S}}(\mathtt{n}^\flat)$ counting multiplicity.

The pair $\mathsf{B} = (\mathsf{B}_+, \mathsf{B}_-)$ is called the *admissible boundary space* or *admissible boundary condition* for S.

Remark 2.6. The different assumptions on B_+ and B_- will play a relevant role in the proof of energy estimates for solutions to the symmetric hyperbolic systems S. Indeed, B_+ (*resp.* B_-) is only needed in the future (*resp.* past) of the chosen Cauchy hypersurface Σ_0.

Conditions (ii-f) and (ii-p) imply that the boundary conditions are *maximal* with respect to properties (iii-f) and (iii-p) respectively, namely no smooth vector subbundles $(\mathsf{B}')_\pm$ of E exist with the property that $\mathsf{B}_\pm \subsetneq \mathsf{B}'_\pm$ and such that, for all $\Phi' \in (\mathsf{B}')_+$ and $\Phi'' \in (\mathsf{B}')_-$,

$$\prec \sigma_{\mathsf{S}}(\mathtt{n}^\flat)\Phi' \,|\, \Phi' \succ \geq 0 \qquad \prec \sigma_{\mathsf{S}}(\mathtt{n}^\flat)\Phi'' \,|\, \Phi'' \succ \leq 0\,.$$

For further details we refer to [50, Section 2.2].
With the next lemma, we shall see that admissible boundary conditions are "stable" under conformal transformations, namely if B is a future/past admissible boundary space for a system on a globally hyperbolic manifold (M, g), then it is also future/past admissible for the same system on $(\mathsf{M}, \Omega^2 g)$, where Ω is a positive smooth function on M.

Lemma 2.7. *Let* (M, g) *be a globally hyperbolic spacetime with timelike boundary and let* $\mathsf{B}_\pm$ *be a future/past admissible boundary space for a hyperbolic Friedrichs system of constant characteristic* S. *Then* $\mathsf{B}_\pm$ *is future/past admissible w.r.t* g *if and only if it is future/past admissible w.r.t.* $\Omega^2 g$, *for any positive* $\Omega \in C^\infty(\mathsf{M})$.

Proof. We only prove the case of a future admissible boundary condition, the proof for the other case being similar. Let denote with $\mathtt{n}$ and $\widetilde{\mathtt{n}}$ the normal vector w.r.t. g and $\Omega^2 g$. Since $\widetilde{\mathtt{n}} = \Omega^{-1}\mathtt{n}$, we get $\sigma_{\mathsf{S}}(\mathtt{n}) = \Omega\sigma_{\mathsf{S}}(\widetilde{\mathtt{n}})$. This guarantees that conditions (i-f)–(ii-f) in Definition 2.5 are satisfied. □

Once a future/past admissible boundary condition π_{B} is fixed, the *adjoint boundary condition* $\pi_{\mathsf{B}}^\dagger$ is defined as the pointwise orthogonal projection (with respect to $\prec \cdot | \cdot \succ$) onto $\sigma_{\mathsf{S}}(\mathtt{n}^\flat)(\mathsf{B})$, namely

$$\mathsf{B}_+^\dagger := \Big(\sigma_{\mathsf{S}}(\mathtt{n}^\flat)(\mathsf{B}_+)\Big)^\perp \qquad \mathsf{B}_-^\dagger := \Big(\sigma_{\mathsf{S}}(\mathtt{n}^\flat)(\mathsf{B}_-)\Big)^\perp . \tag{1}$$

Definition 2.8. We say that an admissible boundary condition $\mathsf{B} = (\mathsf{B}_+, \mathsf{B}_-)$ is *self-adjoint* if and only if $\mathsf{B}_+ = \mathsf{B}_-$.

Remark 2.9. Our definition of a self-adjoint boundary condition is actually stronger than the one used in the literature, where only $\mathsf{B}_\pm = \mathsf{B}_\pm^\dagger$ are required. It immediately follows from the definition of a self-adjoint boundary condition that for any $(\Psi, \Phi) \in \mathsf{B}_+ \oplus \mathsf{B}_- = \mathsf{B} \oplus \mathsf{B}$, it holds

$$\prec \sigma_{\mathsf{S}}(\mathtt{n}^\flat)\Psi \,|\, \Phi \succ = 0 \,.$$

Actually, the vanishing of $(\Psi, \Phi) \mapsto \prec \sigma_{\mathsf{S}}(\mathtt{n}^\flat)\Psi \,|\, \Phi \succ$ on $\mathsf{B}_+ \oplus \mathsf{B}_-$ is equivalent to $\mathsf{B}_- \subset \mathsf{B}_+^\dagger$ and hence $\mathsf{B}_- = \mathsf{B}_+^\dagger$ by identity of space dimensions. As a consequence, if $\mathsf{B}_+ = \mathsf{B}_-$, then $\mathsf{B}_+^\dagger = \mathsf{B}_+$ and $\mathsf{B}_-^\dagger = \mathsf{B}_-$. Note however that $\mathsf{B}_+ = \mathsf{B}_-$ does not follow from both $\mathsf{B}_+^\dagger = \mathsf{B}_+$ and $\mathsf{B}_-^\dagger = \mathsf{B}_-$.

2.4 Well-posedness of the Cauchy problem

Let $t \colon \mathsf{M} \to \mathbb{R}$ be a Cauchy temporal function with gradient tangent to the boundary, as in Theorem 2.2. We decompose the symmetric system as

$$\mathsf{S} = \sigma_{\mathsf{S}}(dt)\nabla_{\partial_t} - \mathsf{H}\,,$$

where H is a first-order linear differential operator which differentiates only in the directions that are tangent to Σ while ∇ is an arbitrary but fixed metric connection for $\prec \cdot | \cdot \succ$. Let $\pi_{\mathsf{B}_+}, \pi_{\mathsf{B}_-} : \mathsf{E}_{|\partial\mathsf{M}} \longrightarrow \mathsf{E}_{|\partial\mathsf{M}}$ be future and past admissible boundary conditions respectively for S.

Definition 2.10. We say that $\mathfrak{h} \in \Gamma(\mathsf{E}_{|\Sigma_{t_0}})$, $t_0 \in \mathbb{R}$, and $\mathfrak{f} \in \Gamma(\mathsf{E})$ fulfills the *compatibility condition* of order $k \geq 0$ at time $t_0 \in \mathbb{R}$ if the following condition is satisfied:

$$\sum_{j=0}^{k} \binom{k}{j} \Big(\nabla_{\partial_t}^j \pi_{\mathsf{B}}\Big)\mathfrak{h}_{k-j}\Big|_{\partial\Sigma_{t_0}} = 0 \tag{2}$$

for both $\mathsf{B} = \mathsf{B}_+$ and $\mathsf{B} = \mathsf{B}_-$. Here the sequence $(\mathfrak{h}_k)_k$ of sections of $\mathsf{E}_{|\partial\Sigma_{t_0}}$ is defined inductively by $\mathfrak{h}_0 := \mathfrak{h}$ and

$$\mathfrak{h}_k := \sum_{j=0}^{k-1} \binom{k-1}{j} \mathsf{H}_j\, \mathfrak{h}_{k-1-j}\Big|_{\partial\Sigma_{t_0}} + \nabla_{\partial_t}^{k-1}\big(\sigma_{\mathsf{S}}^{-1}(dt)\mathfrak{f}\big)\Big|_{\partial\Sigma_{t_0}} \qquad \text{for all } k \geq 1,$$

where $\mathsf{H}_j := [\nabla_{\partial_t}, \mathsf{H}_{j-1}]$ and $\mathsf{H}_0 := \sigma_{\mathsf{S}}(dt)^{-1}\mathsf{H}$.

Roughly speaking, Equation (2) provides a sufficient and necessary condition to ensure C^k-regularity for the solution to the Cauchy problem (3) once Cauchy data are given on Σ_{t_0}. We recall one of the main results of [50], see [50, Theorem 1.2]:

Theorem 2.11 (Smooth solutions for symmetric hyperbolic systems). *Let* M *be a globally hyperbolic manifold with timelike boundary and let* S *be a symmetric hyperbolic system of constant characteristic. Let* $\mathsf{B} = (\mathsf{B}_+, \mathsf{B}_-)$ *be an admissible boundary space for* S *and let* Σ_{t_0} *be any smooth spacelike Cauchy hypersurface in* M*. Then, for every* $\mathfrak{f} \in \Gamma_c(\mathsf{E})$ *and* $h \in \Gamma_c(\mathsf{E}_{|\Sigma_{t_0}})$ *satisfying the compatibility conditions* (2) *up to any order, there exists a unique* $\Psi \in \Gamma(\mathsf{E})$ *satisfying the Cauchy problem*

$$\begin{cases} \mathsf{S}\Psi = \mathfrak{f} \\ \Psi_{|\Sigma_{t_0}} = \mathfrak{h} \\ \Psi_{|\partial\mathsf{M}\cap J^+(\Sigma_{t_0})} \in \mathsf{B}_+ \\ \Psi_{|\partial\mathsf{M}\cap J^-(\Sigma_{t_0})} \in \mathsf{B}_- \end{cases} \tag{3}$$

and the map $(\mathfrak{f}, \mathfrak{h}) \mapsto \Psi$ *sending a pair* $(\mathfrak{f}, \mathfrak{h}) \in \Gamma_c(\mathsf{E}) \times \Gamma_c(\mathsf{E}_{|\Sigma_{t_0}})$ *to the solution* $\Psi \in \Gamma_{sc}(\mathsf{E})$ *of* (3) *is continuous.*

The map $U_{\mathsf{S},t}\colon D(U_{\mathsf{S},t}) \subset \Gamma_c(\mathsf{E}|_{\Sigma_t}) \ni \mathfrak{h} \to U_{\mathsf{S},t}\mathfrak{h} := \Psi \in \Gamma_{sc}(\mathsf{E})$, which assigns to any smooth data $\mathfrak{h} \in D(U_{\mathsf{S},t})$ the unique solution Ψ to problem 3 with $\mathfrak{f} = 0$, is called *Cauchy evolution operator*. The space $D(U_{\mathsf{S},t})$ consists of all sections $\mathfrak{h} \in \Gamma_c(\mathsf{E}|_{\Sigma_t})$ fulfillling the compatibility conditions (2) with $\mathfrak{f} = 0$. For later convenience we shall denote by $\rho_t\colon \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E}|_{\Sigma_t})$ the restriction map for smooth sections. Notice that ρ_t is a right-inverse for $U_{\mathsf{S},t}$. As shown in [17, 19, 20, 21], on globally hyperbolic manifolds with empty boundary and compact Cauchy hypersurfaces, the evolution operator can be realized as a Fourier integral operator. As a matter of fact, the Fourier integral representation of the propagator contains the information on how singularities propagate in the manifold. As we shall see in Section 3.5, this is of fundamental importance in proving the existence of Hadamard states for a free quantum field theory on a curved spacetime.

We conclude this section with the following result:

Corollary 2.12. *Let* M *be a globally hyperbolic manifold with timelike boundary and let* B *be an admissible boundary space for a symmetric hyperbolic system of constant characteristic* S*. Then the Cauchy problem for* S *on* (M, g) *is well-posed if and only if it is well-posed on* $(\mathsf{M}, \Omega^2 g)$ *for any positive* $\Omega \in C^\infty(\mathsf{M})$.

Proof. Our claim follows immediately by Remark 2.4 and Lemma 2.7. □

2.5 Symmetric weakly-hyperbolic systems

We conclude this section by showing that the Cauchy problem for a symmetric system S is well-posed also if we assume that the principal symbol $\sigma_{\mathsf{S}}(\xi)$ is pointwise positive definite only for a suitable subset of future-directed timelike covectors ξ. We begin with the following definition.

Definition 2.13. A symmetric system of constant characteristic S over M is *weakly-hyperbolic* if there exists a positive smooth function C on M such that

(gh) The metric $g_C := -\beta^2 dt^2 \oplus C^2 h(t)$ is globally hyperbolic on M, where t is a Cauchy temporal function for g;

(wH) For any $p \in \mathsf{M}$ and any future-directed g-timelike covector τ of the form $\tau = dt + \xi \in \mathsf{T}_p^*\mathsf{M}$ with g-spacelike ξ,

$$\prec \sigma_{\mathsf{S}}(dt + C\xi) \cdot | \cdot \succ_p > 0$$

holds.

Remarks 2.14.

1. The idea behind Definition 2.13 is to allow the light cone of the dual metric $g^\sharp$ in the cotangent bundle to shrink a little while keeping global hyperbolicity. In this way condition (H) in Definition 2.3 has to be checked for a smaller class of future-directed timelike covectors (*cf.* Figure 1). Mind that, in the cotangent bundle, the causal future/past of g_C is not allowed to shrink too much along any Σ_t because of the condition (gh). Note also that we do not assume $C \leq 1$ on M.

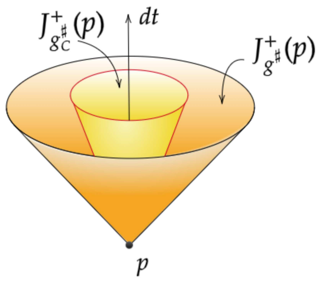


Figure 1: The future light cones of $g^\sharp$ and $g_C^\sharp$ in the cotangent bundle $\mathsf{T}^*\mathsf{M}$.

Clearly, by taking $C = 1$ in Definition 2.13 (*cf.* Remark 2.4), a symmetric hyperbolic system is also a symmetric weakly-hyperbolic system.

2. If the quadratic form $\prec \sigma_{\mathsf{S}}(dt)\cdot|\cdot\succ$ is pointwise positive definite, for all $p\in\mathsf{M}$ there exists a constant $C(p)>0$ such that $\prec \sigma_{\mathsf{S}}(\xi)\cdot|\cdot\succ$ is positive definite for every $\xi\in J^+_{g_{C(p)}}(p)$, where $J^+_{g_{C(p)}}(p)\subset T^*_p\mathsf{M}$ is the causal future in $T^*_p\mathsf{M}$ w.r.t. the metric $g_{C(p)} := -\beta^2 dt^2\oplus C(p)^2 h(t)$ — Notice that the latter is only defined at p.

The following lemma shows that a symmetric weakly-hyperbolic system can be regarded as a symmetric hyperbolic system on a suitable globally hyperbolic spacetime.

Lemma 2.15. *Let* S *be any symmetric weakly-hyperbolic system on a globally hyperbolic manifold* $(\mathsf{M},g)=(\mathbb{R}\times\Sigma,-\beta^2dt^2\oplus h(t))$ *with or without any timelike boundary. Let* $C\in C^\infty(\mathbb{R},(0,\infty))$ *be a function depending only on time which satisfies* (wH) *in Definition 2.13. Then* $(\mathsf{M},g_C):=(\mathbb{R}\times\Sigma,-\beta^2dt^2\oplus C^2h(t))$ *is globally hyperbolic and* S *is symmetric hyperbolic on* (M,g_C)*.*

Proof. Let $p\in\mathsf{M}$ and let $\xi\in\mathsf{T}^*_p\mathsf{M}$ be g_C-timelike, that is, $g^\sharp_C(\xi,\xi)<0$ where $g^\sharp_C=-\beta^{-2}\partial_t^{\otimes 2}\oplus C^{-2}h(t)^\sharp$. Then there exists unique $\lambda>0$ and $\check{\xi}\in T^*_{\pi_\Sigma(p)}\Sigma$ such that $\xi=\lambda\cdot(dt+C\check{\xi})$ — here $\pi_\Sigma\colon\mathsf{M}=\mathbb{R}\times\Sigma\to\Sigma$ is the standard projection. Condition $g^\sharp_C(\xi,\xi)<0$ is then equivalent to $g^\sharp(dt+\check{\xi},dt+\check{\xi})=g^\sharp_C(dt+C\check{\xi},dt+C\check{\xi})<0$, that is, to $dt+\check{\xi}$ being g-timelike. Then condition (wH) implies that $\sigma_{\mathsf{S}}(dt+C\check{\xi})=\lambda^{-1}\sigma_{\mathsf{S}}(\xi)$ is positive definite. This shows that S is symmetric hyperbolic on (M,g_C). We now prove that (M,g_C) is globally hyperbolic. For this, it suffices to show that $\beta^{-2}g_C=-dt^2\oplus\beta^{-2}C^2h(t)$ is globally hyperbolic when restricted to any subset of the form $(a,b)\times\Sigma$, with real $a<b$. But since for all such a,b there exists a positive constant C_0 such that $C(t)\geq C_0>0$ for all $t\in[a,b]$, we have $\beta^{-2}g_C\leq\beta^{-2}g_{C_0}$ on $[a,b]\times\Sigma$, where $g_{C_0}:=-\beta^2dt^2\oplus C_0^2h(t)$. Therefore, it suffices to show that $\beta^{-2}g_{C_0}=-dt^2\oplus\beta^{-2}C_0^2h(t)$ is globally hyperbolic on $(a,b)\times\Sigma$. To this avail, let $t_0\in(a,b)$ and let $\gamma=(\gamma_0,\widehat{\gamma})$ be an inextensible $\beta^{-2}g_{C_0}$-timelike curve (which is C^0 and piecewise C^1) in $(a,b)\times\Sigma$. Then the curve $\widetilde{\gamma}:=(C_0^{-1}\gamma_0,\widehat{\gamma})$ is $\beta^{-2}g$-timelike and still inextensible, therefore it meets $\{t_0\}\times\Sigma$ exactly once. This shows that $\beta^{-2}g_{C_0}$ is globally hyperbolic on $(a,b)\times\Sigma$. This implies that $\beta^{-2}g_{C_0}$ is globally hyperbolic and therefore so is g_C. ☐

Example 2.16. Let $\mathsf{M}=\mathbb{R}\times\Sigma$ be a product spacetime, where Σ is any complete Riemannian manifold with or without boundary. In particular, M is a globally hyperbolic spacetime, with or without timelike boundary. Then any future directed timelike vector field $X\in\Gamma(\mathsf{TM})$ defines an operator $\mathsf{S}:=\nabla_X$ acting on sections of E and which is a symmetric weakly-hyperbolic system if and only if the vector field $g(X,\partial_t)^{-1}X_\Sigma$ is bounded along Σ, where X_Σ denotes the pointwise orthogonal projection of X on Σ. This applies in particular for $X=\partial_t+v$, $v\in\Gamma(\Sigma)$ The resulting transport equation is known as Vlasov equation once applied in kinetic theory.

Definition 2.5 can be straightforwardly generalized to a symmetric weakly-hyperbolic system S. The resulting connection with standard hyperbolic systems is described by the following lemma.

Lemma 2.17. *Let B be a future/past admissible boundary space for a symmetric weakly-hyperbolic system S over a globally hyperbolic manifold (M, g). Then B is future/past admissible for S over (M, g_C). Furthermore, if S is of constant characteristic on (M, g) then it is also of constant characteristic on (M, g_C).*

Proof. To prove the claim it is enough to observe that if $\mathtt{n}_g, \mathtt{n}_{g_C}$ denote the unit vectors which are g-normal and g_C-normal respectively to $\partial\mathsf{M}$, then $\mathtt{n}_g = \lambda \mathtt{n}_{g_C}$ for a positive smooth function λ. This is due to the choice of a Cauchy temporal function whose gradient is tangent to the boundary. □

Combining Lemmas 2.15 and 2.17, we can conclude that the Cauchy problem for a symmetric weakly-hyperbolic system S is well-posed. Indeed, these two lemmas guarantee that any smooth solution propagates no faster than the speed of light (w.r.t. g_C). Therefore, the Cauchy problem can be equivalently reformulated in terms of a Cauchy problem for a symmetric positive system with $\sigma_{\mathsf{S}}(dt) > 0$ in a globally hyperbolic manifold with compact Cauchy surfaces. We summarize our results in the following theorem and we leave the details to the reader.

Theorem 2.18 (Smooth solutions for symmetric weakly-hyperbolic systems). *Let M be a globally hyperbolic spacetime with timelike boundary and let S be a symmetric weakly-hyperbolic system of constant characteristic. Assume $\pi_{\mathsf{B}_+}, \pi_{\mathsf{B}_-}$ to be future and past admissible boundary conditions for S. Let Σ_{t_0} be any smooth spacelike Cauchy hypersurface in M. Then, for every $\mathfrak{f} \in \Gamma_c(\mathsf{E})$ and $h \in \Gamma_c(\mathsf{E}_{|\Sigma_{t_0}})$ satisfying the compatibility conditions (2) up to any order, there exists a unique $\Psi \in \Gamma(\mathsf{E})$ satisfying the Cauchy problem*

$$\begin{cases} \mathsf{S}\Psi = \mathfrak{f} \\ \Psi_{|\Sigma_{t_0}} = \mathfrak{h} \\ \Psi_{|\partial\mathsf{M} \cap J^+(\Sigma_{t_0})} \in \mathsf{B}_+ \\ \Psi_{|\partial\mathsf{M} \cap J^-(\Sigma_{t_0})} \in \mathsf{B}_- \end{cases} \tag{4}$$

and the map $(\mathfrak{f}, \mathfrak{h}) \mapsto \Psi$ sending a pair $(\mathfrak{f}, \mathfrak{h}) \in \Gamma_c(\mathsf{E}) \times \Gamma_c(\mathsf{E}_{|\partial\mathsf{M}})$ to the solution $\Psi \in \Gamma_{sc}(\mathsf{E})$ of (4), is continuous.

As usual, as a byproduct of the well-posedness of the Cauchy problem, we get the existence of Green operators.

Proposition 2.19. *A symmetric weakly-hyperbolic system S of constant characteristic on a globally hyperbolic manifold with timelike boundary coupled with an admissible boundary condition $\mathsf{B} = (\mathsf{B}_+, \mathsf{B}_-)$ is Green-hyperbolic, i.e., there exist two linear maps, called the advanced/retarded Green operators respectively, $\mathsf{G}^{\pm} \colon \Gamma_c(\mathsf{E}) \to \Gamma_{sc,\mathsf{B}_{\pm}}(\mathsf{E})$ satisfying* *(i)* $\mathsf{S}\circ\mathsf{G}^\pm\mathfrak{f}=\mathfrak{f}$ *for all* $\mathfrak{f}\in\Gamma_c(\mathsf{E})$ *and* $\mathsf{G}^\pm\circ\mathsf{S}\mathfrak{f}=\mathfrak{f}$ *for all* $\mathfrak{f}\in\Gamma_{c,\mathsf{B}_\pm}(\mathsf{E})$*;*

(ii) $\mathrm{supp}\,(\mathsf{G}^\pm\mathfrak{f})\subset J^\pm_{g_C}(\mathrm{supp}\,\mathfrak{f})$ *for all* $\mathfrak{f}\in\Gamma_c(\mathsf{E})$,

where $J^\pm_{g_C}$ *denote the causal future (+) and past (−) w.r.t.* g_C *and* $\Gamma_{\bullet,\mathsf{B}_\pm}(\mathsf{E})\subset\Gamma_\bullet(\mathsf{E})$*,* $\bullet\in\{sc,c\}$ *denotes the space of smooth sections on* E *(with* $\bullet$ *support property) which fulfill the* $\mathsf{B}_\pm$*-boundary condition.*
Moreover, let $\mathsf{G}:=\mathsf{G}^+-\mathsf{G}^-\colon\Gamma_c(\mathsf{E})\to\Gamma_{sc,\mathsf{B}_++\mathsf{B}_-}(\mathsf{E})$ *be the causal propagator associated with* S *and* B*. Then the following sequence is a complex*

$$0\to\Gamma_{c,\mathsf{B}_+\cap\mathsf{B}_-}(\mathsf{E})\xrightarrow{\mathsf{S}}\Gamma_c(\mathsf{E})\xrightarrow{\mathsf{G}}\Gamma_{sc,\mathsf{B}_++\mathsf{B}_-}(\mathsf{E})\xrightarrow{\mathsf{S}}\Gamma_{sc}(\mathsf{E})\to 0$$

which satisfies $\ker(\mathsf{S}_{|\Gamma_{c,\mathsf{B}_+\cap\mathsf{B}_-}(\mathsf{E})})=\{0\}$*,* $\ker(\mathsf{G})=\mathsf{S}\Gamma_{c,\mathsf{B}_+\cap\mathsf{B}_-}(\mathsf{E})$ *and* $\mathsf{S}\Gamma_{sc,\mathsf{B}_++\mathsf{B}_-}(\mathsf{E})=\Gamma_{sc}(\mathsf{E})$*. Moreover, if* B *is self-adjoint, i.e.* $\mathsf{B}_+=\mathsf{B}_-$*, then* $\ker(\mathsf{S}_{|\Gamma_{sc,\mathsf{B}_+}(\mathsf{E})})=\mathsf{G}\Gamma_c(\mathsf{E})$ *and* $\mathsf{S}\colon\Gamma_{sc,\mathsf{B}_+}(\mathsf{E})\to\Gamma_{sc}(\mathsf{E})$ *is surjective, so that the complex is exact everywhere. In that case, the solution space* $\mathsf{Sol}_{sc,\mathsf{B}}(\mathsf{S}):=\Gamma_{sc,\mathsf{B}_+}(\mathsf{E})\cap\ker(\mathsf{S})$ *fulfills*

$$\mathsf{Sol}_{sc,\mathsf{B}}(\mathsf{S})=\mathsf{G}\Gamma_c(\mathsf{E})\simeq{}^{\Gamma_c(\mathsf{E})}\!/\!_{\mathsf{S}\Gamma_{c,\mathsf{B}_+}(\mathsf{E})}\,. \tag{5}$$

Proof. Properties (i) and (ii) are satisfied by definition of $\mathsf{G}^\pm$. As a straightforward consequence, $\mathsf{GS}=0$ on $\Gamma_{c,\mathsf{B}_+\cap\mathsf{B}_-}(\mathsf{E})$ and $\mathsf{SG}=0$ on $\Gamma_c(\mathsf{E})$, therefore the sequence is a complex. Note that, since $(\mathsf{G}^\pm u)_{|\partial\mathsf{M}}\in\mathsf{B}_\pm$, we have $(\mathsf{G}u)_{|\partial\mathsf{M}}\in\Gamma_{sc,\mathsf{B}_++\mathsf{B}_-}(\mathsf{E})$, however, notice that $\mathsf{B}_++\mathsf{B}_-=\mathsf{E}|_{\partial\mathsf{M}}$ is false in general.
The injectivity of $\mathsf{S}_{|\Gamma_{c,\mathsf{B}_\pm}(\mathsf{E})}$ immediately follows from property (i) since $\mathsf{S}u=0$ for a $u\in\Gamma_{c,\mathsf{B}_\pm}(\mathsf{E})$ yields $u=\mathsf{G}^\pm\mathsf{S}u=0$. As a consequence, $\mathsf{S}_{|\Gamma_{c,\mathsf{B}_+\cap\mathsf{B}_-}(\mathsf{E})}$ is injective.
To show that $\ker(\mathsf{G})\subset\mathsf{S}\left(\Gamma_{c,\mathsf{B}_+\cap\mathsf{B}_-}(\mathsf{E})\right)$, let $u\in\Gamma_c(\mathsf{E})$ with $\mathsf{G}u=0$. Then $\mathsf{G}^+u=-\mathsf{G}^-u$, so that $\mathrm{supp}\,\mathsf{G}^+u\subset J^+_{g_C}(\mathrm{supp}\,u)\cap J^-_{g_C}(\mathrm{supp}\,u)$ must be compact by property (ii). Moreover, because $(\mathsf{G}^\pm u)_{|\partial\mathsf{M}}\in\mathsf{B}_\pm$, we have $\mathsf{G}^+u\in\Gamma_{c,\mathsf{B}_+\cap\mathsf{B}_-}(\mathsf{E})$. Therefore $\mathsf{G}^+u\in\Gamma_{c,\mathsf{B}_+\cap\mathsf{B}_-}(\mathsf{E})$ and satisfies $\mathsf{SG}^+u=u$ by property (i), from which $u\in\mathsf{S}\left(\Gamma_{c,\mathsf{B}_+\cap\mathsf{B}_-}(\mathsf{E})\right)$ follows.

From now on let us assume $\mathsf{B}_+=\mathsf{B}_-$. We prove that $\ker\left(\mathsf{S}_{|\Gamma_{sc,\mathsf{B}_+}(\mathsf{E})}\right)\subset\mathsf{G}\left(\Gamma_c(\mathsf{E})\right)$. Let $u\in\Gamma_{sc,\mathsf{B}_+}(\mathsf{E})$ be such that $\mathsf{S}u=0$. By definition, there exists a compact subset K of M such that $\mathrm{supp}\,u\subset J^+_{g_C}(K)\cup J^-_{g_C}(K)$. Up to possibly enlarging K, we may assume that $\mathrm{supp}\,u\subset I^+_{g_C}(K)\cup I^-_{g_C}(K)$, where $I^+_{g_C}$ and $I^-_{g_C}$ denote the chronological future and past w.r.t. g_C respectively. Let $\{\chi_+,\chi_-\}$ be a partition of unity subordinated to the open covering $\{I^+_{g_C}(K),I^-_{g_C}(K)\}$ of $I^+_{g_C}(K)\cup I^-_{g_C}(K)$. Let $u_\pm:=\chi_\pm u$. Then $u=u_++u_-$, where each $u_\pm$ is smooth with $\mathrm{supp}\,u_\pm\subset I^\pm_{g_C}(K)$. Furthermore, since $u_\pm$ is obtained by pointwise multiplication of u by a real number, we have $u_{\pm|\partial\mathsf{M}}\in\mathsf{B}_+$. Let $v:=\mathsf{S}u_+(=-\mathsf{S}u_-)$. Then v is smooth with support contained in $J^+_{g_C}(K)\cap J^-_{g_C}(K)$, therefore $\mathrm{supp}\,v$ is compact. We would like to check that $\mathsf{G}v=u$ in the weak – and therefore also in the strong – sense. For that, we need the following fact: if $(\mathsf{G}^{\pm})^*$ denotes the formal adjoint of $\mathsf{G}^{\pm}$, then

$$(\mathsf{G}^{\pm})^* = \mathsf{G}^{\mp}_{\dagger}$$

holds, where $\mathsf{G}^{+}_{\dagger}$ and $\mathsf{G}^{-}_{\dagger}$ are the Green operators for $\mathsf{S}^{\dagger}$ with boundary condition $\mathsf{B}^{\dagger} := (\mathsf{B}^{\dagger}_{+}, \mathsf{B}^{\dagger}_{-})$. We recall that, if B is a future/past admissible boundary condition for S, then $\mathsf{B}^{\dagger}$ is a future/past admissible boundary condition for $\mathsf{S}^{\dagger}$. Moreover, $\mathsf{S}^{\dagger}$ becomes a symmetric weakly hyperbolic system on M with reversed time orientation, in particular $\mathsf{S}^{\dagger}$ has unique advanced and retarded Green operators as well.
To check that $(\mathsf{G}^{\pm})^* = \mathsf{G}^{\mp}_{\dagger}$, let φ, ψ be arbitrary in $\Gamma_c(\mathsf{E})$. Since $\operatorname{supp} \mathsf{G}^{\pm}\varphi \cap \operatorname{supp} \mathsf{G}^{\mp}_{\dagger}\psi$ is compact, integration by parts leads to:

$$\begin{aligned}
\int_{\mathsf{M}} \prec \mathsf{G}^{\pm}\varphi \,|\, \psi \succ \mathrm{vol}_{\mathsf{M}} &= \int_{\mathsf{M}} \prec \mathsf{G}^{\pm}\varphi \,|\, \mathsf{S}^{\dagger}\mathsf{G}^{\mp}_{\dagger}\psi \succ \mathrm{vol}_{\mathsf{M}} \\
&= \int_{\mathsf{M}} \prec \mathsf{S}\mathsf{G}^{\pm}\varphi \,|\, \mathsf{G}^{\mp}_{\dagger}\psi \succ \mathrm{vol}_{\mathsf{M}} \\
&\quad - \int_{\partial\mathsf{M}} \prec \sigma_{\mathsf{S}}(\mathtt{n}^{\flat})\mathsf{G}^{\pm}\varphi \,|\, \mathsf{G}^{\mp}_{\dagger}\psi \succ \mathrm{vol}_{\partial\mathsf{M}} \\
&= \int_{\mathsf{M}} \prec \varphi \,|\, \mathsf{G}^{\mp}_{\dagger}\psi \succ \mathrm{vol}_{\mathsf{M}},
\end{aligned}$$

where the boundary term vanishes on accounto of $\mathsf{G}^{\pm}\varphi_{|\partial\mathsf{M}} \in \mathsf{B}_{\pm}$ and $\mathsf{G}^{\mp}_{\dagger}\psi_{|\partial\mathsf{M}} \in \mathsf{B}^{\dagger}_{\pm}$. This shows $(\mathsf{G}^{\pm})^* = \mathsf{G}^{\mp}_{\dagger}$. Now for all $\psi \in \Gamma_c(\mathsf{E})$, we have

$$\begin{aligned}
\int_{\mathsf{M}} \prec \mathsf{G}^{\pm}v \,|\, \psi \succ \mathrm{vol}_{\mathsf{M}} &= \int_{\mathsf{M}} \prec v \,|\, (\mathsf{G}^{\pm})^*\psi \succ \mathrm{vol}_{\mathsf{M}} \\
&= \int_{\mathsf{M}} \prec v \,|\, \mathsf{G}^{\mp}_{\dagger}\psi \succ \mathrm{vol}_{\mathsf{M}} \\
&= \pm \int_{\mathsf{M}} \prec \mathsf{S}u_{\pm} \,|\, \mathsf{G}^{\mp}_{\dagger}\psi \succ \mathrm{vol}_{\mathsf{M}} \\
&= \pm \int_{\mathsf{M}} \prec u_{\pm} \,|\, \mathsf{S}^{\dagger}\mathsf{G}^{\mp}_{\dagger}\psi \succ \mathrm{vol}_{\mathsf{M}} \\
&= \pm \int_{\mathsf{M}} \prec u_{\pm} \,|\, \psi \succ \mathrm{vol}_{\mathsf{M}},
\end{aligned}$$

where we have used in a crucial way that $\mathsf{G}^{\mp}_{\dagger}\psi_{|\partial\mathsf{M}} \in \mathsf{B}^{\dagger}_{\mp}$ and that $u_{\pm|\partial\mathsf{M}} \in \mathsf{B}_{+}$ as well as $\mathsf{B}^{\dagger}_{+} = \mathsf{B}_{+}$. Therefore, $\mathsf{G}^{\pm}v = \pm u_{\pm}$ and $\mathsf{G}v = u_{+} + u_{-} = u$, as we claimed.
It remains to prove the surjectivity of $\mathsf{S}\colon \Gamma_{sc,\mathsf{B}_+}(\mathsf{E}) \to \Gamma_{sc}(\mathsf{E})$. (We recall that we are assuming $\mathsf{B}_{+} = \mathsf{B}_{-}$.) Let $\mathfrak{f} \in \Gamma_{sc}(\mathsf{E})$ and let $K \subset \mathsf{M}$ be compact such that $\operatorname{supp} \mathfrak{f} \subset J^{+}_{g_C}(K) \cup J^{-}_{g_C}(K)$. As above, up to enlarging K we may assume

that $\mathfrak{f} = \mathfrak{f}_+ + \mathfrak{f}_-$, where $\mathfrak{f}_\pm \in \Gamma_{sc}(\mathsf{E})$ with $\operatorname{supp} \mathfrak{f}_\pm \subset J^\pm_{g_C}(K)$. By Theorem 2.2 the spacetime M is diffeomorphic to – and therefore can be identified with – $\mathbb{R} \times \Sigma$, where Σ a smooth spacelike Cauchy hypersurface of M. For each $n \in \mathbb{N}$ we let $\mathsf{M}_{(-n,n)} := (-n, n) \times \Sigma$, where $\Sigma \simeq \{0\} \times \Sigma$. Note that $\mathsf{M}_{(-n,n)}$ is still a globally hyperbolic spacetime with timelike boundary. Let χ_n be a smooth function with timelike compact support such that $\chi_{n|_{\mathsf{M}_{(-n,n)}}} = 1$. Then $\chi_n \mathfrak{f}_+$ lies in $\Gamma_c(\mathsf{E})$ and we may consider $u_n := \mathsf{G}^+ \chi_n \mathfrak{f}_+ \in \Gamma_{sc,\mathsf{B}_+}(\mathsf{E})$. Now $u^+(x) := u^+_n(x)$ for every $x \in \mathsf{M}_{(-n,n)}$ defines a smooth section of E on M with $\mathsf{S}u^+ = \mathfrak{f}_+$. Indeed if *e.g.* $m > n$ then $v := u^+_m - u^+_n$ is a smooth spacelike compactly supported section of E satisfying $\mathsf{S}v = 0$ on $\mathsf{M}_{(-n,n)}$ as well as $v_{|_{\mathsf{M}\setminus J^+(\operatorname{supp}\mathfrak{f}_+)}} = 0$ and $v_{|_{\partial \mathsf{M}_{(-n,n)}}} \in \mathsf{B}_+$, so that $v = 0$ on $\mathsf{M}_{(-n,n)}$ by uniqueness of the solution to the forward Cauchy problem. The support of u^+ must be contained in $J^+_{g_C}(K)$ since this is the case for the support of each u^+_n. Analogously, there exists a $u^- \in \Gamma_{sc,\mathsf{B}_-}(\mathsf{E}) = \Gamma_{sc,\mathsf{B}_+}(\mathsf{E})$ with $\mathsf{S}u^- = \mathfrak{f}^-$ and therefore $\mathsf{S}(u^+ + u^-) = \mathfrak{f}$. This proves the surjectivity of $\mathsf{S}\colon \Gamma_{sc,\mathsf{B}_+}(\mathsf{E}) \to \Gamma_{sc}(\mathsf{E})$ and concludes the proof of Proposition 2.19. □

For further details we refer to [50, Proposition 5.1], [26, Proposition 20] and [25, Proposition 36].

As an immediate consequence, any symmetric weakly-hyperbolic system S coupled with a self-adjoint admissible condition B satisfies the time-slice axiom.

Corollary 2.20. *Under the assumptions of Proposition 2.19, if the boundary condition B is self-adjoint, then the inclusion map $\iota : \Gamma_c(\mathsf{E}|_{t^{-1}(t_2,t_1)}) \to \Gamma_c(\mathsf{E})$ induces an isomorphism*

$$[\iota]\colon \frac{\Gamma_c(\mathsf{E}|_{t^{-1}(t_2,t_1)})}{\mathsf{S}\Gamma_{c,\mathsf{B}}(\mathsf{E}|_{t^{-1}(t_2,t_1)})} \to \frac{\Gamma_c(\mathsf{E})}{\mathsf{S}\Gamma_{c,\mathsf{B}}(\mathsf{E})}\,. \tag{6}$$

2.6 Møller operators on manifolds with timelike boundary

In [65] a geometric process to compare solutions to symmetric hyperbolic systems on different globally hyperbolic manifolds with empty boundary was realized. This works within the assumptions that: (a) the involved manifolds $\mathsf{M}_0 := (\mathsf{M}, g_0)$ and $\mathsf{M} := (\mathsf{M}, g_1)$ admit the same Cauchy temporal function; (b) $g_1 \leq g_0$, namely the set of timelike vectors for g_1 is contained in the one for g_0. The comparison of the solution spaces was achieved via the construction of a family of so-called Møller operators [24, 36, 54]. The aim of this Section is to generalize that construction to manifolds with timelike boundary.

Let us introduce the following setup:

Setup 2.21.

(i) $\mathsf{M}_0 = (\mathsf{M}, g_0)$ and $\mathsf{M}_1 = (\mathsf{M}, g_1)$ are globally hyperbolic manifolds with timelike boundary and with the same Cauchy temporal function $t\colon \mathsf{M} \to \mathbb{R}$. Moreover, by realizing $(\mathsf{M}, g_i) = (\mathbb{R} \times \Sigma, -\beta_i^2 dt^2 \oplus h_i(t))$ for $i = 0, 1$

—*cf.* Theorem 2.2— we shall assume that there exists a smooth positive function $C > 0$ such that

$$C^2\beta_1^{-2}h_1(t) \leq \beta_0^{-2}h_0(t)$$

holds for every $p \in \mathsf{M}$ and that $g_C := -\beta_1^2 dt^2 \oplus C^2 h_1(t)$ is globally hyperbolic;

(ii) E_1 (*resp.* E_0) is a $\mathbb{K}$-vector bundle over M_1 (*resp.* M_0) with finite rank and endowed with a nondegenerate bilinear or sesquilinear fiber metric $\prec \cdot | \cdot \succ_1$ (*resp.* $\prec \cdot | \cdot \succ_0$);

(iii) $\kappa_{1,0} \colon \mathsf{E}_0 \to \mathsf{E}_1$ is a fiberwise linear isometry of vector bundles with inverse $\kappa_{0,1} \colon \mathsf{E}_1 \to \mathsf{E}_0$. With a slight abuse of notation we will denote by $\kappa_{1,0}$ also the linear map $\kappa_{1,0} \colon \Gamma(\mathsf{E}_0) \to \Gamma(\mathsf{E}_1)$ defined by $[\kappa_{1,0}u](x) = \kappa_{1,0}u(x)$ for all $u \in \Gamma(\mathsf{E}_0)$ and $x \in \mathsf{M}$. Finally for any positive $f \in C^\infty(\mathsf{M})$, we set $\kappa^f_{1,0} := f\, \kappa_{1,0} \colon \Gamma(\mathsf{E}_0) \to \Gamma(\mathsf{E}_1)$ with inverse $\kappa^f_{0,1} := f^{-1}\, \kappa_{0,1}$;

(iv) S_1 (*resp.* S_0) is a symmetric weakly-hyperbolic system with self-adjoint admissible boundary space denoted by B_1 (*resp.* B_0). Moreover we shall assume that $\dim\ker \sigma_{\mathsf{S}_1}(\xi)$ is constant for all non-vanishing spacelike covectors $\xi \in T^*\mathsf{M}_1$;

(v) Let $\mathsf{S}^f_{0,1} \colon \Gamma(\mathsf{E}_1) \to \Gamma(\mathsf{E}_1)$ be the operator defined by $\mathsf{S}^f_{0,1} := \kappa^f_{1,0}\mathsf{S}_0\kappa^f_{0,1}$. We assume that there exists a linear isometry $\wp_{1,0} \colon \mathsf{T}^*\mathsf{M}_0 \to \mathsf{T}^*\mathsf{M}_1$ which preserves time orientation and such that $\sigma_{\mathsf{S}^f_{0,1}}(\xi) = \sigma_{\mathsf{S}_1}(\wp_{1,0}\xi)$ for every $\xi \in \mathsf{T}^*\mathsf{M}_1$.

Remarks 2.22.

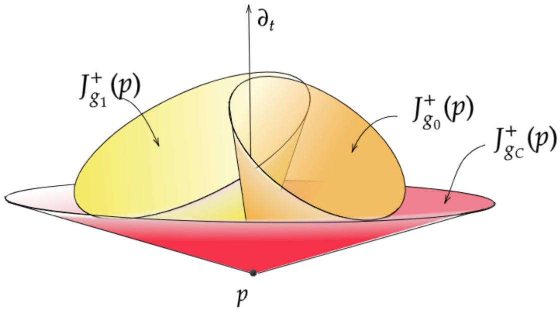
1. The assumption (i) in the Setup 2.21 implies the following. Consider the metric $g_C := -\beta_1^2 dt^2 \oplus C^2 h_1(t)$ which is globally hyperbolic on account of Definition 2.13. Then the following two situations may occur: for any vector $v \in \mathsf{TM}$

$(C \leq 1)$ $g_C(v,v) \leq \left(\frac{\beta_1}{\beta_0}\right)^2 g_0(v,v)$ and $g_C(v,v) \leq g_1(v,v)$, which implies that $J^\pm_{g_0} \cup J^\pm_{g_1} \subset J^\pm_{g_C}$.

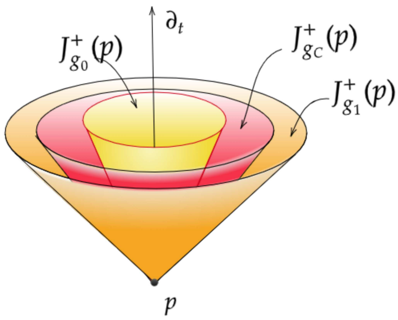
$(C \geq 1)$ $g_1(v,v) \leq g_C(v,v) \leq \left(\frac{\beta_1}{\beta_0}\right)^2 g_0(v,v)$, which implies that $J^\pm_{g_0} \subset J^\pm_{g_C} \subseteq J^\pm_{g_1}$.

2. Using assumption (iv), assumption (v) implies that $\dim\ker \sigma_{\mathsf{S}_0}(\xi)$ is constant on the space of nonzero spacelike covectors ξ and that $\wp_{1,0}$ is time-orientation preserving.

Remark 2.23. For later convenience we shall compute the principal symbol $\sigma_{\mathsf{S}_{0,1}}$ of $\mathsf{S}_{0,1} := \mathsf{S}^1_{0,1}$ in a slightly more general framework that the one depicted above. Let $\mathsf{E}_0 \to \mathsf{M}_0$ and $\mathsf{E}_1 \to \mathsf{M}_1$ be two vector bundles such that there exists



(a) $0 < C \leq 1$



(b) $C > 1$

Figure 2: Future light cones of g_0 and g_1 satisfying $\beta_1^{-2}h_1(t) \leq C^2\beta_0^{-2}h_0(t)$.

a diffeomophism $\zeta_{1,0}\colon \mathsf{M}_0 \to \mathsf{M}_1$ (with inverse $\zeta_{0,1}$) which is lifted to a vector bundle isomorphism $\kappa_{1,0}\colon \mathsf{E}_0 \to \mathsf{E}_1$. With a slight abuse of notation we shall denote with $\kappa_{1,0}\colon \Gamma(\mathsf{E}_0) \to \Gamma(\mathsf{E}_1)$ the associated map of vector bundles defined by

$$(\kappa_{1,0}u_0)(x_1) := \kappa_{1,0}(u_0(\zeta_{0,1}x_1))\,.$$

for all $u_0 \in \Gamma(\mathsf{E}_0)$ and $x_1 \in \mathsf{M}_1$. Notice that $\kappa_{1,0}(fu_0) = (\zeta_{0,1}^* f)\kappa_{1,0}u_0$ for all $u_0 \in \Gamma(\mathsf{E}_0)$ and $f \in C^\infty(\mathsf{M}_0)$ where $\zeta_{0,1}^*\colon C^\infty(\mathsf{M}_0) \to C^\infty(\mathsf{M}_1)$. Moreover, $\kappa_{1,0}\colon \Gamma(\mathsf{E}_0) \to \Gamma(\mathsf{E}_1)$ is invertible with inverse $\kappa_{0,1}$.

The principal symbol of $\mathsf{S}_{0,1} := \kappa_{1,0}\mathsf{S}_0\kappa_{0,1}$ is obtained as follows. For all $u_1 \in \mathsf{E}_1|_{x_1}$ and $\xi_1 \in \mathsf{T}^*_{x_1}\mathsf{M}_1$, let $\widetilde{u}_1 \in \Gamma(\mathsf{E}_1)$ and $f \in C^\infty(\mathsf{M}_1)$ be such that $\widetilde{u}_1(x_1) =$

u_1 and $\mathrm{d}f(x_1) = \xi_1$. Then we have

$$\begin{aligned}\sigma_{\mathsf{S}_{0,1}}(\xi_1)u_1 = [\kappa_{1,0}\mathsf{S}_0\kappa_{0,1}, f_1]\widetilde{u}_1|_{x_1} &= \kappa_{1,0}\mathsf{S}_0\kappa_{0,1}f_1\widetilde{u}_1 - f_1\kappa_{1,0}\mathsf{S}_0\kappa_{0,1}\widetilde{u}_1 \\ &= \kappa_{1,0}\mathsf{S}_0(\zeta_{1,0}^* f_1\kappa_{0,1}\widetilde{u}_1) - f_1\kappa_{1,0}\mathsf{S}_0\kappa_{0,1}\widetilde{u}_1 \\ &= \kappa_{1,0}[\mathsf{S}_0, \zeta_{1,0}^* f_1]\kappa_{0,1}\widetilde{u}_1 \\ &= \kappa_{0,1}\sigma_{\mathsf{S}_0}(\mathrm{d}(\zeta_{1,0}^* f_1))\kappa_{0,1}\widetilde{u}_1|_{x_1} \\ &= \kappa_{0,1}\sigma_{\mathsf{S}_0}((\mathrm{d}\zeta_{1,0}^*\mathrm{d}f_1)\kappa_{0,1}\widetilde{u}_1|_{x_1} \\ &= \kappa_{0,1}\sigma_{\mathsf{S}_0}((\mathrm{d}\zeta_{1,0}^*\xi_1)\kappa_{0,1}\widetilde{u}_1|_{x_1} ,\end{aligned}$$

where $(\mathrm{d}\zeta_{1,0})^* \colon \mathsf{T}^*\mathsf{M}_1 \to \mathsf{T}^*\mathsf{M}_0$. Overall we have

$$\sigma_{\mathsf{S}_{0,1}}(\xi_1) = \kappa_{1,0}\sigma_{\mathsf{S}_0}[(\mathrm{d}\zeta_{1,0})^*\xi_1]\kappa_{0,1} .$$

Similarly to the case of an empty boundary, the construction of a family of Møller operators requires to control the Cauchy problem for the operator $\mathsf{S}_{0,1}^f$. The following Proposition shows that $\mathsf{S}_{0,1}^f$ is symmetric weakly-hyperbolic over M_1.

Proposition 2.24. *Assume the Setup 2.21. Then the operator $\mathsf{S}_{0,1}^f$ is a symmetric weakly-hyperbolic system of constant characteristic on M_1 and $\kappa_{1,0}^f(\mathsf{B}_0) = \kappa_{1,0}(\mathsf{B}_0)$ is a self-adjoint admissible boundary space for $\mathsf{S}_{0,1}^f$.*

Proof. On account of Remark 2.23 — with $\zeta_{0,1} = \mathrm{id}_{\mathsf{M}}$ — we have that, for every $\xi \in T^*\mathsf{M}$, $\sigma_{\mathsf{S}_{0,1}^f}(\xi) = \kappa_{1,0}^f\sigma_{\mathsf{S}_0}(\xi)\kappa_{0,1}^f = \kappa_{1,0}\sigma_{\mathsf{S}_0}(\xi)\kappa_{0,1}$. Since S_0 is symmetric and $\kappa_{1,0}$ is a fiberwise linear isometry by assumption, $\mathsf{S}_{0,1}^f$ clearly satisfies property (S) in Definition 2.3. Moreover, because S_0 has constant characteristic and $\mathtt{n}_1^\flat$ is a pointwise positive scalar multiple of $\mathtt{n}_0^\flat$, the operator $\mathsf{S}_{0,1}^f$ has constant characteristic. Because B_0 is an admissible boundary condition for S_0, the subbundle $\kappa_{1,0}(\mathsf{B}_0) = \kappa_{1,0}^f(\mathsf{B}_0)$ must be an admissible self-adjoint boundary space for $\mathsf{S}_{0,1}^f$. We shall next prove property (wH) in Definition 2.13. To this end let $g_{0,C_0} = -\beta_0^2 dt^2 \oplus C_0^2 h_0(t)$ be the globally hyperbolic metric chosen for S_0 accordingly with Lemma 2.15. Then S_0 is a symmetric hyperbolic system and $\sigma_{\mathsf{S}_0}(\tau)$ is fiberwise positive definite for any future-directed g_{0,C_0}-covector τ. Since any conformal transformation does not change the set of future-directed covectors, the operator S_0 is hyperbolic w.r.t. $\overline{g}_0 := \beta_0^{-2} g_{0,C_0} = -dt^2 \oplus C_0^2\beta_0^{-2}h_0(t)$. We now prove that $\mathsf{S}_{0,1}^f$ is symmetric hyperbolic with respect to the metric $\overline{g}_1 := -dt^2 \oplus C_0^2 C^{-2}\beta_1^{-2}h_1(t)$, where the function C is the one from Setup 2.21.(i). For that, let $\tau = dt + \xi$ be $\overline{g}_1$-timelike future directed. On account of the assumption $\beta_1^{-2}h_1(t) \leq C^2\beta_0^{-2}h_0(t)$ we find

$$\overline{g}_0^\sharp(dt+\xi, dt+\xi) \leq \overline{g}_1^\sharp(dt+\xi, dt+\xi) < 0 ,$$

so that $dt + \xi$ is $\overline{g}_0$-timelike future directed —notice that $\overline{g}_0^\sharp = \partial_t^{\otimes 2} \oplus C_0^{-2}C^2\beta_0^2 h_0(t)^\sharp$ and similarly for $\overline{g}_1$ (*cf.* Figure 3).

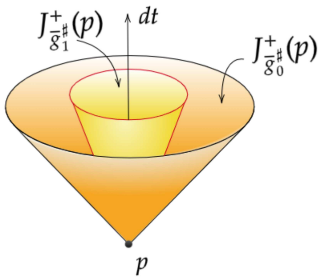


Figure 3: The future light cones of $g_1^\sharp$ and $g_0^\sharp$ in the cotangent bundle $\mathsf{T}^*\mathsf{M}$.

It follows that $\sigma_{\mathsf{S}_0}(dt+\xi) > 0$ and therefore $\sigma_{\mathsf{S}^f_{0,1}}(dt+\xi) > 0$ as well. This shows that $\mathsf{S}^f_{0,1}$ is symmetric hyperbolic with respect to $\overline{g}_1$ and therefore the same holds true for $g_{1,C_{0,1}} := \beta_1^2\overline{g}_1 = -\beta_1^2 dt^2 \oplus C^2_{0,1}h_1(t)$, where $C^2_{0,1} := C_0^2 C^{-2} > 0$ on account of the hypothesis on C and C_0. This proves that $\mathsf{S}^f_{0,1}$ is weakly-hyperbolic with respect to g_1 as per Definition 2.13. ☐

Note that the existence of a linear isometry $\wp_{1,0}$ introduced in assumption (v) is not required in the proof of Proposition 2.24.

So far, we considered a setting where the operators $\mathsf{S}_0, \mathsf{S}_1$, though being defined on different bundles, can be compared through $\kappa_{1,0}$. As a matter of fact the next step in the construction of a Møller operator intertwining S_0 and S_1 is to build an intertwining operator between $\mathsf{S}^f_{0,1}$ and S_1. To this avail, we shall first consider an interpolating operator S^f_χ defined by $\mathsf{S}^f_{\chi,1} := (1-\chi)\mathsf{S}^f_{0,1} + \chi\mathsf{S}_1$, where $\chi \in C^\infty(\mathsf{M})$ is an arbitrary smooth function with $0 \leq \chi \leq 1$.
The following proposition ensures that $\mathsf{S}^f_{\chi,1}$ is a symmetric weakly-hyperbolic system of constant characteristic as long as $\wp_{1,0}(\mathtt{n}_1^\flat)$ is not a pointwise negative scalar multiple of $\mathtt{n}_1^\flat$.

Proposition 2.25. *Assume the Setup 2.21 and that $\wp_{1,0}(\mathtt{n}_1^\flat) \neq \mu\mathtt{n}_1^\flat$ for any $\mu < 0$. Then for any $\chi \in C^\infty(\mathsf{M}, [0,1])$, the operator defined by*

$$\mathsf{S}^f_{\chi,1} := (1-\chi)\,\mathsf{S}^f_{0,1} + \chi\mathsf{S}_1 + \frac{1}{2}\left(\sigma_{\mathsf{S}_1} - \sigma_{\mathsf{S}^f_{0,1}}\right)(d\chi)\colon \Gamma(\mathsf{E}_1) \to \Gamma(\mathsf{E}_1) \tag{7}$$

is a symmetric weakly-hyperbolic system of constant characteristic over M_1.

Proof. By definition of $\mathsf{S}^f_{\chi,1}$,

$$\sigma_{\mathsf{S}^f_{\chi,1}}(\xi) = (1-\chi)\sigma_{\mathsf{S}^f_{0,1}}(\xi) + \chi\sigma_{\mathsf{S}_1}(\xi)\,.$$

Therefore, $\mathsf{S}^f_{\chi,1}$ is a symmetric system. Notice that a convex combination of weakly-hyperbolic system is still weakly-hyperbolic. As a matter of fact if

$C_{0,1}, C_1 \colon \mathsf{M} \to (0,1]$ denote the positive functions of Definition 2.13 associated with $\mathsf{S}^f_{0,1}$ and S_1 respectively, then

$$\prec \sigma_{\mathsf{S}^f_{\chi,1}}(dt + \check{C}\xi) \cdot | \cdot \succ_p > 0$$

for every future-directed g_1-timelike covector $\tau = \mathrm{d}t + \xi$, where $\check{C}$ is any smooth positive function such that $\check{C} \leq \min\{C_{0,1}, C_1\}$ and $g_{\check{C}} := -\beta_1^2 dt^2 \oplus \check{C}^2 h_1(t)$ is globally hyperbolic on M.Such a function $\check{C}$ exists on account of [11].
To conclude our proof, we shall show that $\mathsf{S}^f_{0,1}$ is of constant characteristic. To this end, we consider

$$\sigma_{\mathsf{S}^f_{\chi,1}}(\mathtt{n}_1^\flat) = (1-\chi)\sigma_{\mathsf{S}_{0,1}}(\mathtt{n}_1^\flat) + \chi\sigma_{\mathsf{S}_1}(\mathtt{n}_1^\flat) = \sigma_{\mathsf{S}_1}((1-\chi)\wp_{1,0}\mathtt{n}_1^\flat + \chi\mathtt{n}_1^\flat)\,.$$

Since by assumption $\wp_{1,0}\mathtt{n}_1^\flat \neq \mu\mathtt{n}_1^\flat$ for any $\mu < 0$, the covector $(1-\chi)\wp_{1,0}\mathtt{n}_1^\flat + \chi\mathtt{n}_1^\flat$ is non-vanishing along $\partial\mathsf{M}$, which implies that $(1-\chi)\wp_{1,0}\mathtt{n}_1^\flat + \chi\mathtt{n}_1^\flat$ is a nonzero spacelike g_1-covector. In particular, by assumption (iv) in Setup 2.21, $\mathsf{S}^f_{\chi,1}$ is of constant characteristic. □

Remarks 2.26.

1. Note that the zero-order operator $V := \frac{1}{2}\left(\sigma_{\mathsf{S}_1} - \sigma_{\mathsf{S}^f_{0,1}}\right)(d\chi)$ is a Hermitian operator which vanishes on every open subset where χ is constant – in particular on both the chronological past of Σ_- and the chronological future of Σ_+. The zero-order operator V does not play any role in the proof of Theorem 2.27. However, whenever S_1, S_0 are formally skew-adjoint, the presence of V ensures that $\mathsf{S}^f_{\chi,1}$ is formally skew-adjoint provided a suitable choice of f is made —*cf.* Proposition 2.32 for the precise statement.

2. Assumption (iv) in Setup 2.21 ensures that $\mathsf{S}^f_{\chi,1}$ is of constant characteristic. It can be dropped if $\wp_{1,0}\mathtt{n}_1^\flat = \mathtt{n}_1^\flat = \mathtt{n}_0^\flat$.

Building on Proposition 2.25, we now prove the main result of this Section.

Theorem 2.27. *Assume the Setup 2.21 and that $\wp_{1,0}(\mathtt{n}_1^\flat) \neq \mu\mathtt{n}_1^\flat$ for any $\mu < 0$. Consider two Cauchy hypersurfaces $\Sigma_\pm \subset \mathsf{M}_1$ such that $\Sigma_+ \subset J^+_{g_1}(\Sigma_-)$ — where $J^\pm_{g_1}$ denote the causal cones w.r.t. g_1 — and let $\chi \in C^\infty(\mathsf{M}_1, [0,1])$ be non-decreasing along any future-oriented timelike curve and such that*

$$\chi|_{J^+_{g_1}(\Sigma_+)} = 1\,, \qquad \text{and} \qquad \chi|_{J^-_{g_1}(\Sigma_-)} = 0\,.$$

Finally let B_χ be a self-adjoint admissible boundary space for $\mathsf{S}^f_{\chi,1}$ such that

$$\mathsf{B}_\chi = \begin{cases} \mathsf{B}_{0,1} := \kappa^f_{1,0}(\mathsf{B}_0) & \text{where } \chi = 0 \\ \mathsf{B}_1 & \text{where } \chi = 1 \end{cases}\,. \tag{8}$$

Then the Cauchy problem for $\mathsf{S}^f_{\chi,1}$ *with* B_χ*-boundary conditions is well-posed. Moreover, let* $U_{\mathsf{S}^f_{\chi,1},\pm}\colon D(U_{\mathsf{S}^f_{\chi,1},\pm})\subset\Gamma_c(\mathsf{E}_1|_{\Sigma_\pm})\to\Gamma_{sc}(\mathsf{E}_1)$ *be the Cauchy evolution operator associated with* $\mathsf{S}^f_{\chi,1}$ *and initial data on* $\Sigma_\pm$ *and let* $\rho_\pm\colon\Gamma_{sc}(\mathsf{E}_1)\to\Gamma_c(\mathsf{E}_1|_{\Sigma_\pm})$ *be the standard restriction maps.*
Then the Møller operator $\mathsf{R}_{0,1}=U_{\mathsf{S}_1,+}\circ\rho_+\circ U_{\mathsf{S}^f_{\chi,1},-}\circ\rho_-\circ\kappa^f_{1,0}$ *implements an isomorphism* $\mathsf{R}_{0,1}\colon\mathsf{Sol}_{\,\mathsf{B}_1}\to\mathsf{Sol}_{\,\mathsf{B}_0}$ *between the spaces of solutions to* S_0 *and* S_1 *defined by*

$$\mathsf{Sol}_{\,\mathsf{B}_C}(\mathsf{S}_C):=\{\Psi_C\in\Gamma(\mathsf{E}_C)\,|\,\mathsf{S}_C\Psi_C=0\ \ \textit{and}\ \ \Psi_C|_{\partial\mathsf{M}}\in\mathsf{B}_C\ \}\qquad \textit{for } C=0,1\,.$$

Proof. Since B_χ is a self-adjoint admissible boundary space, the Cauchy evolution operators and the Cauchy data map are well-defined on account of Theorem 2.18. Furthermore, for any $\Psi_0\in\mathsf{Sol}_{\,\mathsf{B}_0}(\mathsf{S}_0)$ we have $\rho_-\kappa^f_{1,0}\Psi_0\in D(U_{\mathsf{S}^f_{\chi,1},-})$ because of B_χ coincide with $\kappa^f_{1,0}(\mathsf{B}_0)$ on Σ_-. Therefore $U_{\mathsf{S}^f_{\chi,1},-}\rho_-\kappa^f_{1,0}\Psi_0$ is well defined. For a similar reason, for any $\Psi\in\mathsf{Sol}\,(\mathsf{S}^f_{\chi,1})$ we have $\rho_+\Psi\in D(U_{\mathsf{S}_1,+})$. It follows that R is well-defined.
To conclude our proof, it is enough to notice that the Møller operator is a composition of isomorphisms. As such the inverse $\mathsf{R}^{-1}_{1,0}$ of $\mathsf{R}_{1,0}$ can be computed explicitly as $\mathsf{R}^{-1}_{1,0}=\kappa^f_{0,1}\circ U_{\mathsf{S}^f_{0,1},-}\circ\rho_-\circ U_{\mathsf{S}^f_{\chi,1}+}\circ\rho_+$. ☐

Example 2.28. Let (M,g) be a globally hyperbolic spacetime with timelike boundary and let S and $\overline{\mathsf{S}}$ be symmetric weakly-hyperbolic systems of constant characteristic which differ by a zero order term, *i.e.* $\mathsf{S}-\overline{\mathsf{S}}=V$, for $V\in\Gamma(\mathrm{End}\,(\mathsf{E}))$. It follows that any self-adjoint admissible boundary condition B for S is also a self-adjoint admissible boundary condition for $\overline{\mathsf{S}}$. Therefore we can set $\mathsf{B}_\chi=\mathsf{B}$ as an interpolating boundary space for S and $\overline{\mathsf{S}}$, that is, B_χ can be chosen to be constant and independent on the function χ.

We conclude this section by showing that for any pair of admissible boundary conditions B,B' for a given symmetric weakly-hyperbolic system there exists an interpolating admissible boundary condition B_χ. In case B and B' are self-adjoint and the interpolating admissible boundary condition can be constructed to be self-adjoint, then Lemma 2.29 can be applied to Theorem 2.27 for the choices $V=\mathsf{E}_1$, $W_0=\kappa^f_{1,0}(\mathsf{B}_0)$, $W_1=\mathsf{B}_1$, see Remarks 2.30 below.

Lemma 2.29. *Let* $V\to M$ *be any smooth vector bundle of finite rank over a smooth manifold* M*. Let* q *be any smooth quadratic form on* V *and let the* n_+ *(resp.* n_-*) be the number of positive (resp. negative) pointwise eigenvalues of* q*. We assume that* $k:=\dim\ker q,n_+,n_-$ *are constant on* M*. Let* $W_0,W_1\to M$ *be any* (n_++k)*-dimensional subbundles of* V *such that* $q_{|W_i}\geq 0$ *holds pointwise for both* $i=0,1$*.*
Then there exists a smooth map $\phi\colon[0,1]\times W_0\to V$ *such that: (a) for every* $t\in[0,1]$*,* $\phi_t:=\phi(t,\cdot)$ *is a linear and injective vector-bundle-map; (b)* $q_{|\phi_t(W_0)}\geq 0$ *holds pointwise; (c)* $\phi_0=\mathrm{Id}_{W_0}$ *and* $\phi_1(W_0)=W_1$*.*

Proof. We begin with the following claim:
Lemma: *Let A be any smooth section of* $\mathrm{End}(V)$*. If $x \mapsto \dim\ker(A(x))$ is constant on M, then $\ker(A) \to M$ defines a smooth vector subbundle of V.*
Proof: Fix any Euclidean *resp.* Hermitian inner product on V and let $k := \dim\ker(A(x))$ for all $x \in U$. For any $x \in U$, we have $\ker(A(x)) = \mathrm{ran}(A(x)^*)^\perp$, where $A(x)^*$ is the adjoint of $A(x)$ w.r.t. the chosen inner product on V. Now $\mathrm{ran}(A^*) \to M$ defines a smooth subbundle of V. Namely it defines an $(n-k)$-dimensional vector subspace of V at each point of M. Moreover, for any $x_0 \in M$, there exists an open neighborhood U of x_0 in M and a family of smooth sections $v_1, \dots, v_{n-k}$ of $V_{|U}$ such that $\{A(x)^*v_1(x), \dots, A(x)^*v_{n-k}(x)\}$ is a family of linearly independent vectors and therefore a basis of $\mathrm{ran}(A(x)^*)$ for any $x \in U$. This shows $\mathrm{ran}(A^*) \to M$ to be a smooth subbundle of V. As a straightforward consequence, its pointwise orthogonal complement must be a smooth subbundle as well. This proves our claim. ✓
It can be deduced from the claim that $\ker(q) \to M$ defines a smooth subbundle of V. Therefore there exists a smooth supplementary subbundle W to $\ker(q)$. The restriction of q to W defines a smooth nondegenerate quadratic form. Its signature is also constant, in fact it is (n_+, n_-). By *e.g.* [66, Theorem C.1.4], the bundle W can therefore be split as $W = W_+ \oplus W_-$, where $W_\pm$ are smooth subbundles of W of rank $n_\pm$ and on which q restricts pointwise as a positive- *resp.* negative-definite quadratic form. Overall, we obtain the smooth splitting $V = \ker(q) \oplus W_+ \oplus W_-$. Now we set $W_0' := W_-$. Note that, since $q_{|W_-}$ is pointwise negative definite, we have $W_0 \cap W_- = W_1 \cap W_- = \{0\}$ by assumption on W_0 and W_1. Thus W_0' is a smooth subbundle of V such that $W_0 \oplus W_0' = W_1 \oplus W_0' = V$ and $q_{|W_0'} \leq 0$ hold pointwise. Therefore the map ϕ can be constructed as follows. Let π_{W_0} (*resp.* $\pi_{W_0'}$) be the pointwise linear projection onto W_0 with kernel W_0' (*resp.* onto W_0' with kernel W_0). Then the restriction $\pi_{W_0|W_1} : W_1 \to W_0$ of the map π_{W_0} to W_1 is an isomorphism because $W_1 \cap W_0' = \{0\}$ and $\dim W_0 = \dim W_1$. Let $F := \pi_{W_0'} \circ \left(\pi_{W_0|W_1}\right)^{-1} : W_0 \to W_0'$. For all $v \in W_0$ we have

$$v + F(v) = \pi_{W_0}\left((\pi_{W_0|W_1})^{-1}(v)\right) + \pi_{W_0'}\left((\pi_{W_0|W_1})^{-1}(v)\right) = \left(\pi_{W_0|W_1}\right)^{-1}(v),$$

so that $v + F(v) \in W_1$. Now define $\phi\colon [0,1] \times W_0 \to V$ by $\phi(t,v) := v + tF(v)$ for all $(t,v) \in [0,1] \times W_0$. Clearly ϕ is smooth, $\phi_t = \phi(t,\cdot)$ is a linear injective vector-bundle-map for every $t \in [0,1]$ because of $W_0 \cap W_0' = \{0\}$ and obviously $\phi_0 = \mathrm{Id}_{W_0}$ and $\phi_1(W_0) = W_1$ hold by the above observation. Moreover, for any $(t,v) \in [0,1] \times W_0$,

$$q(v + tF(v), v + tF(v)) = q(v,v) + 2q(v, F(v))t + q(F(v), F(v))t^2.$$

The r.h.s of the last identity is a degree-2-polynomial in t which is positive at $t = 0$ and $t = 1$: since $q(F(v), F(v)) \leq 0$ this implies that such a polynomial is non-negative on $[0,1]$. Therefore, because $q_{|W_0'} \leq 0$, we have $q_{|\phi_t(W_0)} \geq 0$. This concludes the proof of Lemma 2.29. □

To apply Lemma 2.29, consider $q := \prec \sigma_{\mathsf{S}_1}(\mathtt{n}^\flat)\cdot\,|\,\cdot \succ$ on $V := \mathsf{E}_{1|_{\partial\mathsf{M}}}$ as well as $W_0 := \mathsf{B}_{0,1} = \kappa^f_{1,0}(\mathsf{B}_0)$ and $W_1 := \mathsf{B}_1$. Then the map $\widehat{\phi}$ realizing the interpolation of the boundary conditions is defined by

$$\widehat{\phi}\colon \mathsf{B}_{0,1} \to \mathsf{E}_{1|_{\partial\mathsf{M}}}, \qquad v \mapsto \phi(\chi(\pi(v)), v),$$

where $\pi\colon \mathsf{E}_{1|_{\partial\mathsf{M}}} \to \partial\mathsf{M}$ is the standard projection. With these notations, $\mathsf{B}_\chi := \widehat{\phi}(\mathsf{B}_{0,1})$ at every point of $\partial\mathsf{M}$.

Remarks 2.30.

1. In case W_0 and W_1 are null spaces for q, the space $\phi_t(W_0)$ as constructed in the proof of Lemma 2.29 is not necessarily null for almost every $t \in [0,1]$ (unless q vanishes identically on V and thus $W_0 = W_1 = V$). This does not prevent the existence of a path of null subspaces connecting W_0 and W_1: namely the question is only whether the Grassmannian of $n_+ + n_0$-dimensional q-non-negative subspaces in an $n_+ + n_0 + n_-$-dimensional one is connected or not, where $n_0 = \dim\ker\sigma_{\mathsf{S}}(\mathtt{n}^\flat)$.

2. Note also that Lemma 2.29 can be applied to the situation where a stronger condition than condition (iv) on the operator S_1 is assumed, namely that the numbers n_0, n_+, n_- of vanishing, positive *resp.* negative eigenvalues of $\sigma_{\mathsf{S}_1}(\xi)$ are constant along $\partial\mathsf{M}$ whenever ξ is a nonvanishing covector in $T^*\Sigma_{|_{\partial\Sigma}}$. This applies for instance to the Dirac operators associated to two different globally hyperbolic metrics g_0, g_1 and where the boundary condition is the MIT one, see Section 3.2 below.

2.7 Conservation of positive definite Hermitian scalar products

Consider now the pre-Hilbert space given by

$$\mathsf{Sol}_{sc,\mathsf{B}}(\mathsf{S}) = \{\Psi \in \Gamma_{sc}(\mathsf{E}) \,|\, \mathsf{S}\Psi = 0\,,\ \Psi_{|_{\partial\mathsf{M}}} \in \mathsf{B}\}$$

where $(\cdot\,|\,\cdot)$ is the positive definite Hermitian form defined by

$$(\cdot\,|\,\cdot) = \int_\Sigma \prec \cdot \,|\, \sigma_{\mathsf{S}}(\mathtt{n}^\flat)\cdot \succ \mathrm{vol}_\Sigma\,, \tag{9}$$

where $\mathtt{n} = -\frac{1}{\beta}\partial_t$ is the past-directed unit normal vector to Σ while $\mathtt{n}^\flat = g(\mathtt{n},\cdot) = \beta dt$. In the next lemma, we shall prove that, if S is skew-adjoint, then the scalar product (9) does not depend on the choice of the Cauchy hypersurface $\Sigma \subset \mathsf{M}$.

Lemma 2.31. *Let $\Sigma \subset \mathsf{M}$ be a smooth spacelike Cauchy hypersurface with its past-oriented unit normal vector field $\mathtt{n}$ and its induced volume element vol_Σ.*

Furthermore, let S *be a formally skew-adjoint, symmetric weakly-hyperbolic system of constant characteristic with self-adjoint admissible boundary condition, i.e.* $\mathsf{B}_+ = \mathsf{B}_-$*, see Definition 2.8. Then*

$$(\cdot\,|\,\cdot)\colon \mathsf{Sol}_{sc,\mathsf{B}}(\mathsf{S}) \times \mathsf{Sol}_{sc,\mathsf{B}}(\mathsf{S}) \to \mathbb{C} \qquad (\Psi\,|\,\Phi) = \int_\Sigma \prec \Psi \,|\, \sigma_{\mathsf{S}}(\mathtt{n}^\flat)\Phi \succ \mathrm{vol}_\Sigma\,,$$

where $\mathtt{n}^\flat$ *denotes the future-directed unit conormal, yields a positive definite Hermitian scalar product which does not depend on the choice of* Σ*.*

Proof. The proof virtually coincides with the one of [5, Lemma 3.17]. First note that $\mathrm{supp}(\Psi)\cap\Sigma$ is compact since $\mathrm{supp}(\Psi)$ is spacelike compact, so that the integral is well-defined. Let Σ' be any other smooth spacelike Cauchy hypersurface. Without loss of generality we may assume that $\Sigma\cap\Sigma' = \emptyset$ and, up to swapping Σ and Σ', that $\Sigma' \subset J^+(\Sigma)$, otherwise a third Cauchy hypersurface lying in the common pasts of Σ and Σ' has to be chosen, see proof of [5, Lemma 3.17]. Let $\mathsf{M}_\mathsf{T} := J^+(\Sigma)\cap J^-(\Sigma')$ be the subset of M bounded by Σ and Σ'. The subset M_T is a nonempty open subset of M with boundary $\partial\mathsf{M}_\mathsf{T} = (\partial\mathsf{M}\cap\mathsf{M}_\mathsf{T})\cup\Sigma\cup\Sigma'$. By the Green identity [50, Lemma 2.11] we have

$$\int_{\mathsf{M}_\mathsf{T}} (\prec \mathsf{S}\Psi \,|\, \Phi \succ - \prec \Psi \,|\, \mathsf{S}^\dagger\Phi \succ)\mathrm{vol}_{\mathsf{M}_\mathsf{T}} = \int_{\partial\mathsf{M}_\mathsf{T}} \prec \Psi \,|\, \sigma_{\mathsf{S}}(\mathtt{n}^\flat)\Phi \succ \mathrm{vol}_{\partial\mathsf{M}_\mathsf{T}}$$

for any $\Psi,\Phi\in\mathsf{Sol}_{sc,\mathsf{B}}(\mathsf{S})$. Since S is assumed to be skew-adjoint, the left-hand side of the latter equality vanishes identically. Moreover, since $\mathsf{B} = \mathsf{B}^\dagger$ also $\prec \Psi \,|\, \sigma_{\mathsf{S}}(\mathtt{n}^\flat)\Phi \succ$ vanishes identically at $\partial\mathsf{M}\cap\mathsf{M}_\mathsf{T}$. Therefore we can conclude

$$0 = \int_{\Sigma'} \prec \Psi \,|\, \sigma_{\mathsf{S}}(\mathtt{n}^\flat)\Phi \succ \mathrm{vol}_{\Sigma'} - \int_\Sigma \prec \Psi \,|\, \sigma_{\mathsf{S}}(\mathtt{n}^\flat)\Phi \succ \mathrm{vol}_\Sigma\,.$$

This finishes our proof. □

With the next proposition, we will prove that there exists a choice of f which makes the operator $\mathsf{S}^f_{\chi,1}\colon \Gamma_{sc,\mathsf{B}_\chi}(\mathsf{E}_1)\to\Gamma_{sc}(\mathsf{E}_1)$ formally skew-adjoint on $\Gamma_{sc,\mathsf{B}_\chi}(\mathsf{E}_1)$, provided that B_χ is a self-adjoint boundary condition and S_0 (*resp.* S_1) are formally skew-adjoint with respect to the pairing $(\cdot\,|\,\cdot)_0$ (*resp.* $(\cdot\,|\,\cdot)_1$).

Proposition 2.32. *Within the setup of Theorem 2.27, let us assume that* S_0 *and* S_1 *are formally skew-adjoint with respect to the pairings* $(\cdot\,|\,\cdot)_0$ *and* $(\cdot\,|\,\cdot)_1$ *respectively. Furthermore, let assume that* B_χ *is a self-adjoint boundary condition for* $\mathsf{S}^f_{\chi,1}$*. If* $f\in C^\infty(\mathsf{M})$ *is the positive smooth function such that*

$$\mathrm{vol}_{\mathsf{M}_0} = f^2\mathrm{vol}_{\mathsf{M}_1}$$

on M*, where* $\mathrm{vol}_{\mathsf{M}_0}$ *(resp.* $\mathrm{vol}_{\mathsf{M}_1}$*) is the volume form of the metric* g_0 *(resp.* g_1*) on* M*, then* $\mathsf{S}^f_{\chi,1}$ *is formally skew-adjoint on* $\Gamma_{sc,\mathsf{B}_\chi}(\mathsf{E}_1)$*.*

Proof. First we compute the formal adjoint of $\mathsf{S}^f_{0,1}$ on (M, g_1). Let $\Psi_1, \Phi_1 \in \Gamma(\mathsf{E}_1)$ be such that $\mathrm{supp}(\Psi_1) \cap \mathrm{supp}(\Phi_1)$ is compact in the interior of M. Since by assumption $f^2 \mathrm{vol}_{\mathsf{M}_1} = \mathrm{vol}_{\mathsf{M}_0}$ and $\mathsf{S}_0^\dagger = -\mathsf{S}_0$, we have

$$\begin{aligned}
&\int_{\mathsf{M}} \prec \mathsf{S}^f_{0,1}\Psi_1 \,|\, \Phi_1 \succ_1 \mathrm{vol}_{\mathsf{M}_1} = \int_{\mathsf{M}} \prec \kappa^f_{1,0}\mathsf{S}_0\kappa^f_{0,1}\Psi_1 \,|\, \Phi_1 \succ_1 \mathrm{vol}_{\mathsf{M}_1}\\
&= \int_{\mathsf{M}} \prec \kappa^f_{1,0}\mathsf{S}_0\kappa^f_{0,1}\Psi_1 \,|\, \kappa^f_{1,0}\kappa^f_{0,1}\Phi_1 \succ_1 \mathrm{vol}_{\mathsf{M}_1}\\
&= \int_{\mathsf{M}} f^2 \prec \kappa_{1,0}\mathsf{S}_0\kappa^f_{0,1}\Psi_1 \,|\, \kappa_{1,0}\kappa^f_{0,1}\Phi_1 \succ_1 \mathrm{vol}_{\mathsf{M}_1}\\
&= \int_{\mathsf{M}} f^2 \prec \mathsf{S}_0\kappa^f_{0,1}\Psi_1 \,|\, \kappa^f_{0,1}\Phi_1 \succ_0 \mathrm{vol}_{\mathsf{M}_1} = \int_{\mathsf{M}} \prec \mathsf{S}_0\kappa^f_{0,1}\Psi_1 \,|\, \kappa^f_{0,1}\Phi_1 \succ_0 \mathrm{vol}_{\mathsf{M}_0}\\
&= \int_{\mathsf{M}} \prec \kappa^f_{0,1}\Psi_1 \,|\, \mathsf{S}_0^\dagger\kappa^f_{0,1}\Phi_1 \succ_0 \mathrm{vol}_{\mathsf{M}_0}\\
&= \int_{\mathsf{M}} \prec \kappa^f_{0,1}\Psi_1 \,|\, \kappa^f_{0,1}\kappa^f_{1,0}\mathsf{S}_0^\dagger\kappa^f_{0,1}\Phi_1 \succ_0 \mathrm{vol}_{\mathsf{M}_0}\\
&= \int_{\mathsf{M}} f^{-2} \prec \Psi_1 \,|\, \kappa^f_{1,0}\mathsf{S}_0^\dagger\kappa^f_{0,1}\Phi_1 \succ_1 \mathrm{vol}_{\mathsf{M}_0} = \int_{\mathsf{M}} \prec \Psi_1 \,|\, \kappa^f_{1,0}\mathsf{S}_0^\dagger\kappa^f_{0,1}\Phi_1 \succ_1 \mathrm{vol}_{\mathsf{M}_1}\\
&= -\int_{\mathsf{M}} \prec \Psi_1 \,|\, \mathsf{S}^f_{0,1}\Phi_1 \succ_1 \mathrm{vol}_{\mathsf{M}_1},
\end{aligned}$$

that is, $\left(\mathsf{S}^f_{0,1}\right)^\dagger = -\mathsf{S}^f_{0,1}$ on (M, g_1). As a consequence,

$$\begin{aligned}
\Big((1-\chi)\mathsf{S}^f_{0,1} + \chi\mathsf{S}_1\Big)^\dagger &= (1-\chi)\left(\mathsf{S}^f_{0,1}\right)^\dagger - \sigma_{\mathsf{S}^f_{0,1}}(d(1-\chi)) + \chi\mathsf{S}_1^\dagger - \sigma_{\mathsf{S}_1}(d\chi)\\
&= -(1-\chi)\mathsf{S}^f_{0,1} + \sigma_{\mathsf{S}^f_{0,1}}(d\chi) - \chi\mathsf{S}_1 - \sigma_{\mathsf{S}_1}(d\chi)\\
&= -(1-\chi)\mathsf{S}^f_{0,1} - \chi\mathsf{S}_1 - 2V,
\end{aligned}$$

where V is the zero-order operator defined as above by $V := \frac{1}{2}[\sigma_{\mathsf{S}_1}(d\chi) - \sigma_{\mathsf{S}^f_{0,1}}(d\chi)]$. Since V is a Hermitian operator it follows that $\mathsf{S}^f_{\chi,1} = (1-\chi)\mathsf{S}^f_{0,1} + \chi\mathsf{S}_1 + V$ is formally skew-adjoint. □

Building on Lemma 2.31 and Proposition 2.32, we can show that $\mathsf{R}_{1,0}$ is a unitary map between $\mathsf{Sol}_{sc,\mathsf{B}_0}(\mathsf{S}_0)$ and $\mathsf{Sol}_{sc,\mathsf{B}_1}(\mathsf{S}_1)$.

Proposition 2.33. *Within the setup of Theorem 2.27, let assume that S_0 and S_1 are formally skew-adjoint and that B_χ is a self-adjoint boundary condition for $\mathsf{S}^f_{\chi,1}$. Let $\Sigma_1 \subset J^+(\Sigma_+)$ and $\Sigma_0 \subset J^-(\Sigma_-)$ be fixed spacelike Cauchy hypersurfaces of M (w.r.t. g_0 or g_1, it makes no difference). Let $f \in C^\infty(\mathsf{M})$ be the positive smooth function such that*

$$\mathrm{vol}_{\mathsf{M}_0} = f^2 \mathrm{vol}_{\mathsf{M}_1}$$

on M*, where* $\mathrm{vol}_{\mathsf{M}_0}$ *(resp.* $\mathrm{vol}_{\mathsf{M}_1}$*) is the volume form of the metric* g_0 *(resp.* g_1*) on* M*. Then the Møller operator* $\mathsf{R}_{1,0}\colon \mathsf{Sol}_{sc,\mathsf{B}_0}(\mathsf{S}_0) \to \mathsf{Sol}_{sc,\mathsf{B}_1}(\mathsf{S}_1)$ *is a unitary map once* $\mathsf{Sol}_{sc,\mathsf{B}_0}(\mathsf{S}_0)$ *(resp.* $\mathsf{Sol}_{sc,\mathsf{B}_1}(\mathsf{S}_1)$*) is equipped with the scalar product defined in Equation* (9) *associated with* S_0 *(resp. with* S_1*).*

Proof. Let $\Psi_0, \Phi_0 \in \mathsf{Sol}_{sc,\mathsf{B}_0}(\mathsf{S}_0)$ and $\Psi_1 := \mathsf{R}_{1,0}(\Psi_0), \Phi_1 := \mathsf{R}_{1,0}(\Phi_0) \in \mathsf{Sol}_{sc,\mathsf{B}_1}(\mathsf{S}_1)$, where the Møller operator $\mathsf{R}_{1,0}$ is defined using the interpolating operator $\mathsf{S}^f_{\chi,1}$. We also denote by $\Psi_{\chi,1}$ (*resp.* $\Phi_{\chi,1}$) the smooth section with spacelike compact support in $\ker\left(\mathsf{S}^f_{\chi,1}\right)$ on M with $\Psi_{\chi,1|_{\Sigma_-}} = \kappa^f_{1,0}\Psi_{0|_{\Sigma_-}}$ (*resp.* $\Phi_{\chi,1|_{\Sigma_-}} = \kappa^f_{1,0}\Phi_{0|_{\Sigma_-}}$). By Lemma 2.31 and the definition of f, we have $\mathtt{n}_0^\flat \otimes \mathrm{vol}_{\Sigma_-,g_0} = f^2 \mathtt{n}_1^\flat \otimes \mathrm{vol}_{\Sigma_-,g_1}$ along Σ_- and therefore

$$\begin{aligned}
\int_{\Sigma_0} \prec \sigma_{\mathsf{S}_0}(\mathtt{n}_0^\flat)\Psi_0 \,|\, \Phi_0 \succ_0 \mathrm{vol}_{\Sigma_0,g_0} &= \int_{\Sigma_-} \prec \sigma_{\mathsf{S}_0}(\mathtt{n}_0^\flat)\Psi_0 \,|\, \Phi_0 \succ_0 \mathrm{vol}_{\Sigma_-,g_0} \\
&= \int_{\Sigma_-} f^{-2} \prec \kappa^f_{1,0}\sigma_{\mathsf{S}_0}(\mathtt{n}_0^\flat)\kappa^f_{0,1}\kappa^f_{1,0}\Psi_0 \,|\, \kappa^f_{1,0}\Phi_0 \succ_1 \mathrm{vol}_{\Sigma_-,g_0} \\
&= \int_{\Sigma_-} f^{-2} \prec \sigma_{\mathsf{S}^f_{0,1}}(\mathtt{n}_0^\flat)\Psi_{\chi,1} \,|\, \Phi_{\chi,1} \succ_1 \mathrm{vol}_{\Sigma_-,g_0} \\
&= \int_{\Sigma_-} \prec \sigma_{\mathsf{S}^f_{0,1}}(\mathtt{n}_1^\flat)\Psi_{\chi,1} \,|\, \Phi_{\chi,1} \succ_1 \mathrm{vol}_{\Sigma_-,g_1} \\
&= \int_{\Sigma_-} \prec \sigma_{\mathsf{S}^f_{\chi,1}}(\mathtt{n}_1^\flat)\Psi_{\chi,1} \,|\, \Phi_{\chi,1} \succ_1 \mathrm{vol}_{\Sigma_-,g_1} \\
&= \int_{\Sigma_+} \prec \sigma_{\mathsf{S}^f_{\chi,1}}(\mathtt{n}_1^\flat)\Psi_{\chi,1} \,|\, \Phi_{\chi,1} \succ_1 \mathrm{vol}_{\Sigma_+,g_1} \\
&= \int_{\Sigma_+} \prec \sigma_{\mathsf{S}_1}(\mathtt{n}_1^\flat)\Psi_1 \,|\, \Phi_1 \succ_1 \mathrm{vol}_{\Sigma_+,g_1},
\end{aligned}$$

which concludes the proof of Proposition 2.33. ☐

Definition 2.34. We call *unitary Møller operator* the operator $\mathsf{R}_{1,0}$ defined in accordance with Proposition 2.33.

Remark 2.35. The unitary Møller operator $\mathsf{R}_{1,0} : \mathsf{Sol}_{sc,\mathsf{B}_0}(\mathsf{S}_0) \to \mathsf{Sol}_{sc,\mathsf{B}_1}(\mathsf{S}_1)$ can be seen as the composition of two unitary Møller operators

$$\begin{aligned}
&\mathsf{R}_{\chi,0}\colon \mathsf{Sol}_{sc,\mathsf{B}_0}(\mathsf{S}_0) \to \mathsf{Sol}_{sc,\mathsf{B}_\chi}(\mathsf{S}^f_{\chi,1}) \qquad \mathsf{R}_{\chi,0} := U_{\mathsf{S}^f_{\chi,1},-} \circ \rho_- \circ \kappa^f_{1,0}\,, \\
&\mathsf{R}_{1,\chi}\colon \mathsf{Sol}_{sc,\mathsf{B}_\chi}(\mathsf{S}^f_{\chi,1}) \to \mathsf{Sol}_{sc,\mathsf{B}_1}(\mathsf{S}_1) \qquad \mathsf{R}_{1,\chi} := U_{\mathsf{S}_1,+} \circ \rho_+\,.
\end{aligned}$$

3 The algebraic approach to quantum Dirac fields

In this section we shall compare the quantization of Dirac fields on two different (yet related) globally hyperbolic spacetimes with timelike boundary. To this

end, we follow the ideas of [24, 38, 65], where a class of Møller operator was introduced in order to construct unitary equivalent quantum field theories, together with the results of the previous Sections 2.5-2.6.
As a first step we introduce the relevant geometric objects, showing that they fit with the framework introduced in Section 2.6. In particular we shall apply Theorem 2.27 and Proposition 2.33 for the case of the Dirac operator with MIT boundary conditions — *cf.* Equation (12).

3.1 The Dirac operator

We briefly discuss the basics of spin geometry in our setting, see *e.g.* [50, Sec. 6.2] for more details. Let (M, g) be a globally hyperbolic spacetime which is assumed to admit a spin structure. Note that the definition of a spin structure depends on the metric of the underlying manifold and its existence is related to the topology of M. *e.g.* by Theorem 2.2 every 4-dimensional globally hyperbolic spacetime admits a spin structure since every 3-dimensional orientable manifold is parallelizable. Given a fixed spin structure, one can use the spinor representation to construct the spinor bundle $\mathbb{S}\mathsf{M}$, which is a complex vector bundle of complex rank $N := 2^{\lfloor \frac{n+1}{2} \rfloor}$. The spinor bundle comes together with a natural indefinite fiber metric $\prec \cdot | \cdot \succ$ and a *Clifford multiplication*, *i.e.* a fiber-preserving map $\gamma\colon \mathsf{TM} \to \mathrm{End}(\mathbb{S}\mathsf{M})$ satisfying

$$\gamma(u)\gamma(v)+\gamma(v)\gamma(u) = -2g(u,v)\mathrm{Id}_{\mathbb{S}_p\mathsf{M}} \quad \text{and} \quad \prec \gamma(u)\psi \,|\, \phi \succ_p = \prec \psi \,|\, \gamma(u)\phi \succ_p$$

at every point p of M. Alternatively, γ may be seen as a vector-bundle-homomorphism $\mathsf{T}^*\mathsf{M} \otimes \mathbb{S}\mathsf{M} \to \mathbb{S}\mathsf{M}$. Using the spin product $\prec \cdot | \cdot \succ$, the *adjunction map* is the complex anti-linear vector bundle isomorphism defined by

$$\Upsilon_p\colon \mathbb{S}_p\mathsf{M}_g \to \mathbb{S}^*_p\mathsf{M}_g \qquad \psi \mapsto \prec \psi \,|\, \cdot \succ \,, \tag{10}$$

where $\mathbb{S}^*\mathsf{M}_g$ is the so-called *cospinor bundle*, *i.e.* the dual bundle of $\mathbb{S}\mathsf{M}_g$. A natural covariant derivative $\nabla^{\mathbb{S}\mathsf{M}}$ can be defined on $\mathbb{S}\mathsf{M}$ which is induced by the Levi-Civita connection on TM. Note that, for any section ψ of $\mathbb{S}\mathsf{M}$, the section $\nabla\psi$ lies pointwise in $\mathsf{T}^*\mathsf{M} \otimes \mathbb{S}\mathsf{M}$.

Definition 3.1. The *(classical) Dirac operator* D is the operator defined as the composition of the metric connection $\nabla^{\mathbb{S}}$ and the Clifford multiplication:

$$\mathsf{D} = \gamma \circ \nabla^{\mathbb{S}\mathsf{M}} \colon \Gamma(\mathbb{S}\mathsf{M}) \to \Gamma(\mathbb{S}\mathsf{M}) \,.$$

The Dirac operator reads as

$$\mathsf{D}\psi = \sum_{\mu=0}^{n} \varepsilon_\mu \gamma(e_\mu) \nabla^{\mathbb{S}\mathsf{M}}_{e_\mu} \psi \,,$$

where $\{e_\mu\}$ is an arbitrary local Lorentzian-orthonormal frame of TM and $\varepsilon_\mu = g(e_\mu, e_\mu) = \pm 1$. On any globally hyperbolic spin spacetime M with timelike boundary, the classical Dirac operator D is a nowhere characteristic symmetric hyperbolic system, see *e.g.* [50, Proposition 6.2] and references therein.

Theorem 2.11 implies that the Cauchy problem for the Dirac operator on globally hyperbolic spacetimes with empty boundary is well-posed, therefore, it admits a Cauchy evolution operator $U_t \colon \Gamma_c(\mathbb{S}\mathsf{M}|_{\Sigma_t}) \to \Gamma_{sc}(\mathbb{S}\mathsf{M})$. Remarkably, as shown by Capoferri and Vassilliev [20], the Cauchy evolution operator for Dirac fields on Cauchy-compact ultrastatic manifolds (with empty boundary) can be realized as a Fourier integral operator. It would be desirable to extend their techniques to more general globally hyperbolic manifolds with possibly non-empty boundary.

3.2 Self-adjoint admissible boundary conditions

The aim of this section is to introduce the boundary conditions for the Dirac operator which we will use in the rest of the paper. The latter will be self-adjoint and admissible in the sense of Definition 2.5, however, Let us remark that not all physical interesting boundary conditions for Dirac fields belong to that class. Indeed there exist physically interesting non-local boundary conditions, such as the so-called APS boundary condition, which guarantees that the Cauchy problem is well-posed [37], but they are not admissible (since admissible boundary conditions in our sense are local). For further details on self-adjoint admissible boundary conditions for Dirac fields, we refer to [50, Section 6.1.1] and [51, Remark 3.19].

The first example of self-adjoint admissible boundary conditions are the so-called *chiral boundary conditions*. They are defined as follows: let $\mathcal{G}$ be a chirality operator on $\mathbb{S}\mathsf{M}$, *i.e.* a parallel involutive antiunitary (with respect to $\prec \cdot | \cdot \succ$) endomorphism-field of $\mathbb{S}\mathsf{M}$ that anti-commutes with Clifford multiplication by vectors. Notice that chirality operators exist only on even-dimensional manifolds. Then the so-called *chirality* boundary spaces $\mathsf{B}^{\pm}_{\text{CHI}}$ are the pair of boundary conditions respectively defined by the range of the maps

$$\pi^+_{\text{CHI}} := \frac{1}{2}\left(\text{Id} + \gamma(\mathtt{n})\mathcal{G}\right) , \qquad \pi^-_{\text{CHI}} := \frac{1}{2}\left(\text{Id} - \gamma(\mathtt{n})\mathcal{G}\right) , \tag{11}$$

where $\gamma(\mathtt{n})$ denotes Clifford multiplication for the outward-pointing unit normal along $\partial\mathsf{M}$. Note that here the exponent $\pm$ in $\pi^{\pm}_{\text{CHI}}$ does not refer to any future/past admissibility condition. It is not difficult to check that the range of both $\pi^{\pm}_{\text{CHI}}$ has dimension $2^{\lfloor \frac{n+1}{2} \rfloor -1}$, which is the number of non-negative eigenvalues of the endomorphism $\sigma_{\mathsf{D}}(\mathtt{n}^\flat)$, and, for all $\psi \in \mathbb{S}\mathsf{M}$,

$$\prec \sigma_{\mathsf{D}}(\mathtt{n}^\flat)\pi^{\pm}_{\text{CHI}}(\psi) \,|\, \pi^{\pm}_{\text{CHI}}\psi \succ = 0 \,.$$

Furthermore, since

$$\pi^{\pm}_{\text{CHI}}\pi^{\pm}_{\text{CHI}} = \pi^{\pm}_{\text{CHI}} \,, \qquad \pi^{\mp}_{\text{CHI}}\pi^{\pm}_{\text{CHI}} = 0 \,, \qquad \pi^{+}_{\text{CHI}} + \pi^{-}_{\text{CHI}} = \text{Id} \,,$$

it can be easily verified that both boundary conditions $\mathsf{B}^+_{\mathrm{CHI}}$ and $\mathsf{B}^-_{\mathrm{CHI}}$ are self-adjoint.

The second example of a self-adjoint boundary condition is the so-called *MIT boundary conditions.* The latter are a pair of boundary conditions whose boundary spaces $\mathsf{B}^{\pm}_{\mathrm{MIT}}$ are defined respectively as the range of

$$\pi^+_{\mathrm{MIT}} := \frac{1}{2}\left(\mathrm{Id} - \imath\gamma(\mathtt{n})\right) , \qquad \pi^-_{\mathrm{MIT}} := \frac{1}{2}\left(\mathrm{Id} + \imath\gamma(\mathtt{n})\right) , \tag{12}$$

where $\gamma(\mathtt{n})$ is again the Lorentzian Clifford multiplication by the outward-pointing unit normal vector field along $\partial\mathsf{M}$. Similarly to the chiral boundary conditions, the range of $\pi^{\pm}_{\mathrm{MIT}}$ has dimension $2^{\lfloor\frac{n+1}{2}\rfloor-1}$,

$$\prec \sigma_{\mathsf{D}}(\mathtt{n}^\flat)\pi^{\pm}_{\mathrm{MIT}}(\psi) \,|\, \pi^{\pm}_{\mathrm{MIT}}\psi \succ = 0$$

for all $\psi \in \mathbb{S}\mathsf{M}$ and we have

$$\pi^{\pm}_{\mathrm{MIT}}\pi^{\pm}_{\mathrm{MIT}} = \pi^{\pm}_{\mathrm{MIT}}\,, \qquad \pi^{\mp}_{\mathrm{MIT}}\pi^{\pm}_{\mathrm{MIT}} = 0\,, \qquad \pi^{+}_{\mathrm{MIT}} + \pi^{-}_{\mathrm{MIT}} = \mathrm{Id}\,.$$

3.3 Linear isometry between spinor bundles

We shall now apply the results obtained in Section 2.6 to compare the solution spaces associated with pairs of Dirac operators $\mathsf{D}_0, \mathsf{D}_1$ defined using different metrics $g_0, g_1 \in \mathcal{GH}_{\mathsf{M}}$ and equipped with MIT boundary conditions. In what follows $g_0, g_1 \in \mathcal{GH}_{\mathsf{M}}$ are assumed to fulfill assumption (i) of Setup 2.21.
As already underlined in Section 3.1, the space of spinors depends on the metric of the underlying manifold M_α, $\alpha = 0, 1$. Therefore, an identification between spaces of sections of spinor bundles for different metrics is needed to construct a unitary Møller operator. This can be achieved by following [4, Section 5].

Consider a family of Lorentzian spin manifolds $\mathsf{M}_\lambda := (\mathsf{M}, g_\lambda)$ with a common Cauchy temporal function, where $g_\lambda \in \mathcal{GH}_{\mathsf{M}}$ for any $\lambda \in \mathbb{R}$. For a given nonempty interval I in $\mathbb{R}$ let Z be the Lorentzian manifold

$$\mathsf{Z} = \mathsf{I} \times \mathsf{M} \qquad\qquad g_{\mathsf{Z}} = d\lambda^2 + g_\lambda\,.$$

On Z there exists a globally defined vector field which we denote by $e_\lambda := \frac{\partial}{\partial\lambda}$. For any λ, the spin structures on Z and $\mathsf{M}_\lambda \simeq \{\lambda\} \times \mathsf{M}$ are in one-to-one correspondence: Any spin structure on Z can be restricted to a spin structure on M_λ and any spin structure on M_λ induces one on Z – see [4, Section 3 and 5]. Actually, the spinor bundle $\mathbb{S}\mathsf{M}_\lambda$ on each globally hyperbolic spin manifold M_λ can be identified with the restriction of the spinor bundle $\mathbb{S}\mathsf{Z}$ on M_λ, in particular $\mathbb{S}\mathsf{M}_\lambda \simeq \mathbb{S}\mathsf{Z}|_{\mathsf{M}_\lambda}$ if n is odd, while $\mathbb{S}\mathsf{M}_\lambda \simeq \mathbb{S}^+\mathsf{Z}|_{\mathsf{M}_\lambda} \simeq \mathbb{S}^-\mathsf{Z}|_{\mathsf{M}_\lambda}$ if n is even. We recall that $\dim(\mathsf{M}) = n + 1$. Equivalently we may identify

$$\mathbb{S}\mathsf{Z}|_{\mathsf{M}_\lambda} = \begin{cases} \mathbb{S}\mathsf{M}_\lambda & \text{if } n \text{ is odd,} \\ \mathbb{S}\mathsf{Z}|_{\mathsf{M}_\lambda} \oplus \mathbb{S}\mathsf{Z}|_{\mathsf{M}_\lambda} & \text{if } n \text{ is even.} \end{cases} \tag{13}$$

By denoting with γ_{Z} (*resp.* γ_λ) the Clifford multiplication on $\mathbb{S}\mathsf{Z}$ (*resp.* on $\mathbb{S}\mathsf{M}_\lambda$), the family of Clifford multiplications γ_λ satisfies

$$\gamma_\lambda(v)\psi = \gamma_{\mathsf{Z}}(e_\lambda)\gamma_{\mathsf{Z}}(v)\psi \qquad \text{if } n \text{ is odd}\,, \tag{14}$$

$$\gamma_\lambda(v)(\psi_+ + \psi_-) = \gamma_{\mathsf{Z}}(e_\lambda)\gamma_{\mathsf{Z}}(v)(\psi_+ - \psi_-) \qquad \text{if } n \text{ is even}, \tag{15}$$

where in the second case $\psi = \psi_+ + \psi_- \in \mathbb{S}\mathsf{Z}|_{\mathsf{M}_\lambda} \oplus \mathbb{S}\mathsf{Z}|_{\mathsf{M}_\lambda}$ and each component $\psi_\pm$ is identified with an element in $\mathbb{S}^\pm\mathsf{Z}|_{\mathsf{M}_\lambda}$.

Lemma 3.2 ([65, Lemma 3.7]). *Let* Z *be the Lorentzian spin manifold given by*

$$\mathsf{Z} = \mathsf{I} \times \mathsf{M} \qquad\qquad g_{\mathsf{Z}} = d\lambda^2 + g_\lambda\,,$$

where $(\mathsf{M}, g_\lambda) := \mathsf{M}_\lambda$ *is a family of Lorentzian spin manifolds with a common Cauchy temporal function, and denote with* $\mathbb{S}\mathsf{M}_\lambda$ *the spinor bundle over* M_λ. *For any* $p \in \mathsf{M}_\lambda$, *the map*

$$\kappa_{1,0}\colon \mathbb{S}_p\mathsf{M}_0 \to \mathbb{S}_p\mathsf{M}_1\,. \tag{16}$$

defined by the parallel translation on Z *along the curve* $\lambda \mapsto (\lambda, p)$ *is a linear isometry and preserves the Clifford multiplication, i.e. for any* $v \in \Gamma(\mathsf{TM})$ *and any* $\Psi_0 \in \Gamma(\mathbb{S}\mathsf{M}_0)$,

$$\gamma_1(\wp_{1,0}v)(\kappa_{1,0}\Psi_0) = \kappa_{1,0}\left(\gamma_0(v)\Psi_0\right)$$

holds, where $\wp_{1,0}\colon \mathsf{TM}_0 \to \mathsf{TM}_1$ *is the parallel transport along the curve* $\lambda \mapsto (\lambda, p)$.

Remark 3.3. Let us remark, that for any pair of Lorentzian metrics g_0 and g_1 admitting a common Cauchy temporal function, there always exists a path of Lorentzian metrics g_λ connecting g_0 to g_1, *e.g.* $g_\lambda = \lambda g_1 + (1-\lambda)g_0$ where $\lambda \in [0,1]$. For more details we refer to [65, 64].

Lemma 3.2 provides an isomorphism $\kappa_{1,0}\colon \mathbb{S}\mathsf{M} \to \mathbb{S}\mathsf{M}$ with the same properties introduced in the Setup 2.21. We shall denote by $\mathsf{D}^f_{0,1}$ the intertwining Dirac operator as in Proposition 2.24. Similarly $\mathsf{D}^f_{\chi,1}$ will denote the operator interpolating between $\mathsf{D}^f_{0,1}$ and D_1. Here and in what follows f is chosen as per Proposition 2.33.

Remark 3.4. Keeping the notation of Remark 2.23 and Lemma 3.2, the diffeomorphism $\zeta\colon \mathsf{M} \to \mathsf{M}$ is simply the identity Id. Since $\sigma_{\mathsf{D}_0}(\xi) = \gamma_0(\xi^{\sharp_0})$, where $^{\sharp_0}$ denotes the musical isomorphism with respect to g_0, we find

$$\begin{aligned}\sigma_{\mathsf{D}^f_{0,1}}(\xi_1) &= \kappa^f_{0,1}\sigma_{\mathsf{D}_0}(\xi_1)\kappa^f_{0,1} = \kappa^f_{1,0}\gamma_0(\xi_1^{\sharp_0})\kappa^f_{0,1}\\ &= \gamma_1(\wp_{1,0}\xi_1^{\sharp_0}) = \sigma_{\mathsf{D}_1}\left((\wp_{1,0}\xi_1^{\sharp_0})^{\flat_1}\right) = \sigma_{\mathsf{D}_1}(\wp_{1,0}\xi_1)\,,\end{aligned}$$

where $\sharp_1 := \flat_1^{-1}$ is the musical isomorphism associated with g_1. In the last equality we used that, for $\xi \in T_x^*\mathsf{M}$ and $X \in T_x\mathsf{M}$ we have

$$\begin{aligned}(\wp_{1,0}\xi^{\sharp_0})^{\flat_1}(X)|_x &= g_1(\wp_{1,0}\xi^{\sharp_0}, X)|_x = g_{\mathsf{Z}}(\wp_{1,0}\xi^{\sharp_0}, X)|_{(1,x)} = g_{\mathsf{Z}}(\xi^{\sharp_0}, \wp_{0,1}X)|_{(0,x)} \\ &= g_0(\xi^{\sharp_0}, \wp_{0,1}X)|_x = \xi(\wp_{0,1}X)|_x = [\wp_{1,0}\xi](X)|_x\,,\end{aligned}$$

where, with a slight abuse of notation, we denoted with $\wp_{1,0}\xi$ the parallel transport of the 1-form ξ along the curve $\lambda \to (\lambda, x)$ within Z: The latter coincides with $\wp_{0,1}^*\xi$, taking into account that $\wp_{0,1}\colon \mathsf{TM}_1 \to \mathsf{TM}_0$.

We are almost in position to apply Theorem 2.27 and Proposition 2.33. In the next lemma we shall prove that if $g_\lambda = (1-\lambda)g_0 + \lambda g_1$, $\lambda \in [0,1]$, then the parallel transport of $\mathtt{n}_1^\flat$ is not proportional to $\mu\mathtt{n}_1^\flat$ for any $\mu < 0$.

Lemma 3.5. *Let (M, g_0) and (M, g_1) be globally hyperbolic manifolds with time-like boundary split as $(\mathsf{M}, g_i) = (\mathbb{R}\times\Sigma, -\beta_i^2 dt^2 \oplus h_i(t))$ for both $i = 0, 1$. Consider the manifold $\mathsf{Z} := [0,1]\times\mathsf{M}$ endowed with the metric $g_{\mathsf{Z}} := d\lambda^2 \oplus g_\lambda$, where*

$$g_\lambda := (1-\lambda)g_0 + \lambda g_1 = -\beta_\lambda^2 dt^2 \oplus h_\lambda(t)\,,$$

where $\beta_\lambda^2 := (1-\lambda)\beta_0^2 + \lambda\beta_1^2$ and $h_\lambda(t) = (1-\lambda)h_0(t) + \lambda h_1(t)$. Then $h_1(\wp_{1,0}(\mathtt{n}_1), \mathtt{n}_1) > 0$ along $\partial\mathsf{M}$, where $\wp_{1,0}$ is the parallel transport in $(\mathsf{Z}, g_{\mathsf{Z}})$ along $[0,1]\to\mathsf{Z}$, $\lambda \mapsto (\lambda, p)$, for any $p \in \partial\mathsf{M}$.

Proof. Note that, by definition of both g_i and of g_{Z}, we have $\nabla^{\mathsf{Z}}_{\partial_\lambda}\partial_\lambda = 0 = \nabla^{\mathsf{Z}}_{\partial_\lambda}\beta_\lambda^{-1}\partial_t$, so that, for any $\lambda_0 \in [0,1]$, the parallel transport along $[0,\lambda_0]\to\mathsf{Z}$, $\lambda \mapsto (\lambda, p)$ preserves $T\Sigma$. Writing $p = (t,x)$, we fix a pointwise h_0-o.n.b. of $T_x\Sigma$ in which $h_1 = h_1(t)$ is diagonal *i.e.*, there exist $\mu_1, \ldots, \mu_n > 0$ such that $h_1(e_i, e_j) = \mu_i\delta_{ij}$ for all $1 \le i,j \le n$. This basis $(e_i)_{1\le i\le n}$ is extended constantly in λ along $\lambda \mapsto (\lambda, p)$. Splitting $\wp_{\lambda,0}\mathtt{n}_1 = \sum_{j=1}^n \alpha_j e_j$, where $\alpha_j = h_0(\wp_{\lambda,0}\mathtt{n}_1, e_j)$, we have

$$\begin{aligned} 0 &= \nabla^{\mathsf{Z}}_{\partial_\lambda}(\wp_{\lambda,0}\mathtt{n}_1) \\ &= \sum_{j=1}^n (\partial_\lambda\alpha_j)e_j + \alpha_j\nabla^{\mathsf{Z}}_{\partial_\lambda}e_j \\ &= \sum_{j=1}^n (\partial_\lambda\alpha_j)e_j + \alpha_j\left(\underbrace{[\partial_\lambda, e_j]}_{0} + \frac{1}{2}h_\lambda^{-1}\partial_\lambda h_\lambda(e_j,\cdot)\right). \end{aligned}$$

As a consequence, denoting by $Y(\lambda) := \begin{pmatrix}\alpha_1(\lambda)\\ \vdots \\ \alpha_n(\lambda)\end{pmatrix}$ and identifying h_λ (as a homomorphism $T\Sigma \to T^*\Sigma$) and $\partial_\lambda h_\lambda$ (as a symmetric 2-tensor on $T\Sigma$) with their respective matrices H_λ and $\partial_\lambda H_\lambda$ in the bases $(e_j)_{1\le j\le n}$ and $(e_j^*)_{1\le j\le n}$

respectively, the vector-valued function Y must satisfy the linear first-order ODE

$$Y'(\lambda) + \frac{1}{2} H_\lambda^{-1} \partial_\lambda H_\lambda \cdot Y(\lambda) = 0 \tag{17}$$

on $[0,1]$. In case $[H_\lambda, \partial_\lambda H_\lambda] = 0$ is fulfillled for all λ, equation (17) can be solved explicitly, namely $Y(\lambda) = H_\lambda^{-\frac{1}{2}} \cdot Y(0)$ for all $\lambda \in [0,1]$ is the solution with initial condition $Y(0) \in \mathbb{R}^n$. With $h_\lambda = (1-\lambda)h_0 + \lambda h_1$, we have $H_\lambda = (1-\lambda)\mathrm{I}_n + \lambda \mathrm{diag}(\mu_1, \ldots, \mu_n)$, so that $\partial_\lambda H_\lambda = \mathrm{diag}(\mu_1, \ldots, \mu_n) - \mathrm{I}_n$ and therefore $[H_\lambda, \partial_\lambda H_\lambda] = 0$ holds for all $\lambda \in [0,1]$. This implies that

$$Y(\lambda) = H_\lambda^{-\frac{1}{2}} \cdot Y(0) = \mathrm{diag}\left((1-\lambda+\lambda\mu_1)^{-\frac{1}{2}}, \ldots, (1-\lambda+\lambda\mu_n)^{-\frac{1}{2}}\right) \cdot Y(0)$$

holds for all $\lambda \in [0,1]$. As a consequence, $Y(1) = \mathrm{diag}\left(\mu_1^{-\frac{1}{2}}, \ldots, \mu_n^{-\frac{1}{2}}\right) \cdot Y(0)$, from which

$$h_1(\wp_{1,0}\mathtt{n}_1, \mathtt{n}_1) = H_1(Y(1), Y(0)) = \sum_{j=1}^n \mu_j^{\frac{1}{2}} \alpha_j(0)^2 > 0$$

and the claim follows. □

We conclude this section by stating Theorem 2.27 and Proposition 2.33 for the particular case of MIT boundary conditions.

Proposition 3.6. *Let assume $g_0, g_1 \in \mathcal{GH}_{\mathsf{M}}$ fulfill (i) in Setup 2.21. Let M_0 (resp. M_1) be a globally hyperbolic spin manifold with timelike boundary and let D_0 (resp. D_1) be the classical Dirac operator coupled with MIT boundary conditions $\mathsf{B}_{\mathrm{MIT}_0}$ (resp. $\mathsf{B}_{\mathrm{MIT}_1}$). Then the boundary space defined by*

$$\mathsf{B}_\chi = \ker M_\chi := \ker\left[\gamma_1(\widehat{v}) - \imath\right]$$

is a self-adjoint boundary space for the operator

$$\mathsf{D}_{\chi,1}^f := (1-\chi)\,\kappa_{1,0}^f \mathsf{D}_0 \kappa_{0,1}^f + \chi \mathsf{D}_1 + \frac{1}{2}\left(\sigma_{\mathsf{D}_1} + \sigma_{\mathsf{D}_{0,1}^f}\right)(d\chi)\,,$$

where $\widehat{v} := v/\|v\|_1$, $v = \chi\mathtt{n}_1 + (1-\chi)\wp_{1,0}\mathtt{n}_1$ and $\|v\|_1 = \sqrt{g_1(v,v)}$. Moreover, B_χ fulfills condition (8).
Therefore, letting $\mathsf{Sol}_{sc,\mathrm{MIT}}(\mathsf{D}_i) := \{\Psi \in \mathbb{S}\mathsf{M}_i \,|\, \mathsf{D}_i\Psi = 0\,,\, \Psi|_{\partial\mathsf{M}} \in \mathsf{B}_{\mathrm{MIT}}\}$, there exists a unitary isomorphism (Møller operator) $\mathsf{R}_{1,0}\colon \mathsf{Sol}_{sc,\mathrm{MIT}}(\mathsf{D}_0) \to \mathsf{Sol}_{sc,\mathrm{MIT}}(\mathsf{D}_1)$ where $\mathsf{Sol}_{sc,\mathrm{MIT}}(\mathsf{D}_0)$ (resp. $\mathsf{Sol}_{sc,\mathrm{MIT}}(\mathsf{D}_1)$) is equipped with the scalar product defined in Equation (9) associated with D_0 (resp. with D_1).

Proof. The proof is nothing but an application of Theorem 2.27 together with Proposition 2.33. To apply these results it is enough to prove that B_χ is a self-adjoint boundary space for $D_{\chi,1}^f$ which also fulfills condition (8). The second claim follows by observing that the operator $\gamma_1(\widehat{v}) - \imath$ fulfills

$$\gamma_1(\widehat{v}) - \imath = \begin{cases} \gamma_1(\mathtt{n}_1) - \imath & \text{when } \chi = 1 \\ \gamma_1(\wp_{1,0}\mathtt{n}_1) - \imath = \kappa_{1,0}[\gamma_0(\mathtt{n}_0) - \imath]\kappa_{0,1} & \text{when } \chi = 0 \end{cases}$$

where we used Remark 3.4. Moreover, B_χ is a self-adjoint admissible boundary condition for $\mathsf{D}^f_{\chi,1}$. To this end, we observe that, for all $\Psi \in \mathsf{B}_\chi$, the complex number $\prec \gamma_1(\widehat{v})\Psi \,|\, \Psi \succ_q$ satisfies simultaneously

$$\begin{aligned}\prec \gamma_1(\widehat{v})\Psi \,|\, \Psi \succ_q &= \prec \Psi \,|\, \gamma_1(\widehat{v})\Psi \succ_q = \overline{\prec \gamma_1(\widehat{v})\Psi \,|\, \Psi \succ_q} \\ \prec \gamma_1(\widehat{v})\Psi \,|\, \Psi \succ_q &= \prec \imath\Psi \,|\, \Psi \succ_q = \prec \Psi \,|\, -\imath\Psi \succ_q \\ &= - \prec \Psi \,|\, \gamma_1(\widehat{v})\Psi \succ_q = -\overline{\prec \gamma_1(\widehat{v})\Psi \,|\, \Psi \succ_q}\,,\end{aligned}$$

which implies that $\prec \gamma_1(\widehat{v})\Psi \,|\, \Psi \succ_q = \prec \sigma_{\mathsf{D}^f_{\chi,1}}(\mathtt{n}^\flat_1)\Psi \,|\, \Psi \succ_q = 0$. Furthermore B_χ coincides with the range of the projector $\pi = \frac{1}{2}(\mathrm{Id} - \imath\gamma_1(\widehat{v}))$: The latter has dimension $2^{\lfloor \frac{n+1}{2} \rfloor - 1}$, which is exactly the number of non-negative eigenvalues of $\sigma_{\mathsf{D}^f_{\chi,1}}(\mathtt{n}^\flat)$. This concludes our proof. □

Remark 3.7. Since, for any nonzero spacelike covector v on M, the operator $\sigma_{\mathsf{D}}(v)$ has vanishing kernel and $\pm|v|$ as nonvanishing eigenvalues, each with multiplicity $2^{\lfloor \frac{n+1}{2} \rfloor - 1}$, the existence of an interpolating B_χ between $\mathsf{B}_{\mathrm{MIT}_0}$ and $\mathsf{B}_{\mathrm{MIT}_1}$ for D_0 and D_1 respectively follows from Lemma 2.29, see Remark 2 above. Note however that the interpolating B_χ from Lemma 2.29 is not self-adjoint.

3.4 The algebra of Dirac fields with MIT boundary condition

In this section we shall exploit Proposition 3.6 to compare the quantization of Dirac fields with MIT boundary conditions on M_0 and M_1. To that aim we shall briefly recall the quantization procedure from the algebraic point of view. In [30, 38, 65], the quantization of a free field theory is realized as a two-step procedure. On the one hand, the physical system classically described by $\mathsf{Sol}_{sc,\mathrm{MIT}}(\mathsf{D})$ is quantized by introducing a unital $*$-algebra $\mathfrak{A}$, whose elements are interpreted as observables for the system under investigation. In a second stage, the description of possible physical states of the system is described through the choice of a suitable subclass of linear, positive and normalized functionals $\omega\colon \mathfrak{A} \to \mathbb{C}$.
By extending the analogous definition for a spacetime without boundary, we shall now introduce the $*$-algebra $\mathfrak{A}$ associated with the space $\mathsf{Sol}_{sc,\mathrm{MIT}}(\mathsf{D})$ of solutions with spacelike compact support of the Dirac operator D coupled with MIT boundary conditions and endowed with the positive definite Hermitian scalar product (9).
To this avail we shall profit from the results and definition already present in the literature, see [2]. For later convenience let $\mathsf{Sol}^{\oplus}_{sc,\mathrm{MIT}}$ be the Hilbert space obtained by completion of

$$\mathsf{Sol}_{sc,\mathrm{MIT}}(\mathsf{D}) \oplus \Upsilon \mathsf{Sol}_{sc,\mathrm{MIT}}(\mathsf{D})\,,$$

equipped with the natural scalar product $(\,,\,)_{\mathsf{Sol}^{\oplus}_{sc,\mathrm{MIT}}}$ induced by $\mathsf{Sol}_{sc,\mathrm{MIT}}(\mathsf{D})$ —*cf.* Equation (9)— in particular $(\psi_1 \,|\, \psi_2) = \int_\Sigma \prec \psi_1 \,|\, \gamma(-\beta^{-1}\partial_t)\psi_2 \succ \mathrm{vol}_\Sigma$.

Moreover, let $\Gamma\colon \mathsf{Sol}^{\oplus}_{sc,\mathrm{MIT}} \to \mathsf{Sol}^{\oplus}_{sc,\mathrm{MIT}}$ be the antilinear involution defined by $\Gamma(\psi_1 \oplus \Upsilon\psi_2) := \psi_2 \oplus \Upsilon\psi_1$ where $\Upsilon\colon \mathbb{S}\mathsf{M} \to \mathbb{S}^*\mathsf{M}$ was defined in Equation (10).

Definition 3.8. The *algebra of Dirac fields with MIT boundary conditions* is the unital, complex $*$-algebra $\mathfrak{A}$ freely generated by the abstract elements $\Xi(\psi)$, $1_{\mathfrak{A}}$, with $\psi \in \mathsf{Sol}^{\oplus}_{sc,\mathrm{MIT}}$, together with the following relations for all $\psi, \phi \in \mathsf{Sol}^{\oplus}_{sc,\mathrm{MIT}}$ and $\alpha, \beta \in \mathbb{C}$:

(i) Linearity: $\Xi(\alpha\psi + \beta\phi) = \alpha\Xi(\psi) + \beta\Xi(\phi)$

(ii) Hermiticity: $\Xi(\psi)^* = \Xi(\Gamma\psi)$

(iii) Canonical anti-commutation relations (CARs):

$$\Xi(\psi)\cdot\Xi(\phi)^* + \Xi(\phi)^*\cdot\Xi(\psi) = (\psi\,|\,\phi)\,1_{\mathfrak{A}}\,.$$

As a matter of fact $\mathfrak{A}$ can be completed in a unique way into a C^*-algebra [2] the C^*-norm being induced by the natural Hilbert structure of $\mathsf{Sol}^{\oplus}_{sc,\mathrm{MIT}}$. Occasionally we shall implicitly regard $\mathfrak{A}$ as a C^*-algebra.

Recollecting the results of the previous sections we have the following:

Theorem 3.9. *Assume that $g_0, g_1 \in \mathcal{GH}_{\mathsf{M}}$ fulfill (i) in the Setup 2.21 and let $\mathfrak{A}_\alpha$ be the algebra of Dirac fields with MIT boundary conditions on M_α. Then the unitary Møller operator $\mathsf{R}_{1,0}\colon \mathsf{Sol}\,(\mathsf{D}_0) \to \mathsf{Sol}\,(\mathsf{D}_1)$ lifts to a $*$-isomorphism $\mathfrak{R}_{1,0}\colon \mathfrak{A}_0 \to \mathfrak{A}_1$.*

Proof. Let $\Upsilon_\alpha : \mathbb{S}\mathsf{M}_\alpha \to \mathbb{S}^*\mathsf{M}_\alpha$ be the adjunction map defined in (10) between the spinor and cospinor bundle over M_α and set $\mathsf{R}^{\Upsilon}_{1,0} := \Upsilon_1\mathsf{R}_{1,0}\Upsilon_0^{-1}$. Then $\mathsf{R}^{\Upsilon}_{1,0}$ implements an isomorphism between $\Upsilon_0\mathsf{Sol}_{\,sc,\mathrm{MIT}}(\mathsf{D}_0)$ and $\Upsilon_1\mathsf{Sol}_{\,sc,\mathrm{MIT}}(\mathsf{D}_1)$. On account of Proposition 3.6 $\mathsf{R}^{\oplus}_{1,0} := \mathsf{R}_{1,0} \oplus \mathsf{R}^{\Upsilon}_{1,0}\colon \mathsf{Sol}^{\oplus}_{sc,\mathrm{MIT}} \to \mathsf{Sol}^{\oplus}_{sc,\mathrm{MIT}}$ is a unitary isomorphism. By direct inspection, the linear map $\mathfrak{R}_{1,0}\colon \mathfrak{A}_0 \to \mathfrak{A}_1$ defined by $\mathfrak{R}_{1,0}\Xi(\psi) := \Xi(\mathsf{R}^{\oplus}_{1,0}\psi)$ extends to the desired $*$-isomorphism. □

Remark 3.10. The algebra of Dirac fields with MIT boundary conditions cannot be considered as an algebra of observables, since observables are required to commute at spacelike separations and $\mathfrak{A}$ does not fulfill such a requirement. A good candidate as algebra of observables is the subalgebra $\mathfrak{A}_{\mathrm{obs}} \subset \mathfrak{A}$ consisting of elements which are even, *i.e.* invariant by replacement $\Xi(\psi) \mapsto -\Xi(\psi)$, and invariant under the action of $\mathrm{Spin}_0(1,n)$ (extended to $\mathfrak{A}$). For further details we refer to [28].

3.5 Hadamard states

In this section we study (algebraic) states and their interplay with the $*$-isomorphism $\mathfrak{R}_{1,0}$.

Definition 3.11. Given a complex $*$-algebra $\mathfrak{A}$ we call *(algebraic) state* any linear functional from $\mathfrak{A}$ into $\mathbb{C}$ that is positive, *i.e.* $\omega(\mathfrak{a}^*\mathfrak{a}) \geq 0$ for any $\mathfrak{a} \in \mathfrak{A}$, and normalized, *i.e.* $\omega(1_{\mathfrak{A}}) = 1$.

Due to the natural grading on the algebra of Dirac fields with MIT boundary conditions $\mathfrak{A}$, it suffices to define ω on monomials. Among all states, the so-called quasi-free states play a distinguished role.

Definition 3.12. A state ω on $\mathfrak{A}$ is *quasifree* if it satisfies

$$\omega(\Xi(\psi_1)\cdots\Xi(\psi_n)) = \begin{cases} 0 & n \text{ odd} \\ \sum\limits_{\sigma\in S'_n}(-1)^{\mathrm{sign}(\sigma)}\prod\limits_{i=1}^{n/2}\omega\left(\Xi(\psi_{\sigma(2i-1)})\Xi(\psi_{\sigma(2i)})\right) & n \text{ even}\end{cases}$$

where S'_n denotes the set of ordered permutations of n elements.

A useful characterization of quasifree states was given by Araki in [2]: for any bounded operator $Q \in B(\mathsf{Sol}^{\oplus}_{sc,\mathrm{MIT}})$ on $\mathsf{Sol}^{\oplus}_{sc,\mathrm{MIT}}$ such that

$$0 \leq Q = Q^* \leq 1 \qquad Q + \Gamma Q \Gamma = \mathrm{Id}_{\mathsf{Sol}^{\oplus}_{sc,\mathrm{MIT}}} \tag{18}$$

there exists a quasi-free state ω on the C^*-algebra $\mathfrak{A}$ satisfying

$$\omega(\Xi(\psi_1)^*\Xi(\psi_2)) = (\psi_1, Q\psi_2)_{\mathsf{Sol}^{\oplus}_{sc,\mathrm{MIT}}}\,. \tag{19}$$

As an immediate corollary, we observe that to construct a bounded operator Q_ω as above, it is enough to construct an orthonormal projector Π on the Hilbert space $\mathsf{Sol}_{sc,\mathrm{MIT}}(\mathsf{D})$.

Corollary 3.13. *Let Υ be the adjunction map defined in Section 3.1 and* $\Pi\colon \mathsf{Sol}_{sc,\mathrm{MIT}}(\mathsf{D}) \to \mathsf{Sol}_{sc,\mathrm{MIT}}(\mathsf{D})$ *be an orthonormal projector. Then the operator* $P := \Pi \oplus (Id - \Upsilon\Pi\Upsilon^{-1})$ *satisfies*

$$0 \leq P = P^* \leq 1 \qquad P + \Gamma P \Gamma = \mathrm{Id}_{\mathsf{Sol}^{\oplus}_{sc,\mathrm{MIT}}}\,.$$

From a different perspective, we can realize $\omega(\Xi(\psi_1)^*\Xi(\psi_2))$ in terms of distributions. This turns out to be quite useful when looking for physically relevant states. To this avail we observe that, by applying Proposition 2.19, we have $\mathsf{Sol}^{\oplus}_{sc,\mathrm{MIT}} \simeq \left(\Gamma_c(\mathbb{S}\mathsf{M})/\mathsf{D}\Gamma_{c,\mathrm{MIT}}(\mathbb{S}\mathsf{M})\right)^{\oplus 2}$ —*cf.* Equation (5)— the isomorphism being given by $\left(\Gamma_c(\mathbb{S}\mathsf{M})/\mathsf{D}\Gamma_{c,\,\mathrm{MIT}}(\mathbb{S}\mathsf{M})\right)^{\oplus 2} \ni ([f_1],[f_2]) \to \mathsf{G}f_1 \oplus \Gamma\mathsf{G}f_2 \in \mathsf{Sol}^{\oplus}_{sc,\mathrm{MIT}}$. In particular we can endow $\Gamma_c(\mathbb{S}\mathsf{M})$ with the standard locally convex topology which induces a locally convex topology on the quotient $\Gamma_c(\mathbb{S}\mathsf{M})/\mathsf{D}\Gamma_{c,\,\mathrm{MIT}}(\mathbb{S}\mathsf{M})$. With those choices the map $\left(\Gamma_c(\mathbb{S}\mathsf{M})/\mathsf{D}\Gamma_{c,\,\mathrm{MIT}}(\mathbb{S}\mathsf{M})\right)^{\oplus 2} \to \mathsf{Sol}^{\oplus}_{sc,\mathrm{MIT}}$ turns out to be continuous, so that to any quasi-free state we may associate its 2-point distribution $\omega^{(2)} \in \Gamma_c(\mathbb{S}\mathsf{M}^{\oplus 2} \boxplus \mathbb{S}\mathsf{M}^{\oplus 2})'$ defined by

$$\omega^{(2)}(f_1,f_2) := \omega(\Xi(\psi_{f_1})^*\Xi(\psi_{f_2}))\,.$$

where $\psi_f \in \mathsf{Sol}^{\oplus}_{sc,\text{MIT}}$ is the element associated with $[f] \in \left[\Gamma_c(\mathbb{S}\mathsf{M})/\mathsf{D}\Gamma_{c,\text{MIT}}(\mathbb{S}\mathsf{M})\right]^{\oplus 2}$. In particular, the 2-point distribution is a solution to the Dirac equation with MIT boundary conditions, meaning that

$$\omega^{(2)}(f_1, (\mathsf{D} \oplus \mathsf{D})f_2) = 0 \qquad \forall f_1, f_2 \in \Gamma_{c,\text{MIT}}(\mathbb{S}\mathsf{M}^{\oplus 2})\,. \tag{20}$$

Notice that the restriction to compactly supported functions $f_1, f_2 \in \Gamma_{c,\text{MIT}}(\mathbb{S}\mathsf{M}^{\oplus 2})$ fulfillling MIT boundary conditions entails that $\omega^{(2)}$ is a bisolution to the Dirac equation with such boundary conditions.
A widely accepted criterion to select physically relevant states is the celebrated *Hadamard condition* [55, 68, 69, 70]. On a globally hyperbolic spacetime with empty boundary, the latter allows for the construction of Wick polynomials in a local and covariant fashion. Moreover, it guarantees the finiteness of the fluctuations of such Wick polynomials [39].
At a technical level, the Hadamard condition characterizes the wave front set $\mathrm{WF}(\omega^{(2)}) \subseteq \mathsf{T}^*\mathsf{M}^2$ of the 2-point function of a quasi-free state —generalization to non-quasi free states are possible [72]. Such a microlocal characterization is also possible for the case of a globally hyperbolic manifold with timelike boundary: therein the Hadamard condition has been formulated in [78] for the case of asymptotically anti-de Sitter spacetimes and then exploited in [31] for a wider class of boundary conditions. In these situations the proper replacement for $\mathrm{WF}(\omega^{(2)})$ is given by $\mathrm{WF}_b(\omega^{(2)}) \subset {}^b\mathsf{T}^*\mathsf{M}^2 \setminus \{0\}$, where WF_b stands for the b-wave front set (see *e.g.* [78, Appendix A]).

Definition 3.14. Let (M, g) be a globally hyperbolic spin manifold with timelike boundary. A bidistribution $\omega^{(2)} \in \Gamma_c(\mathbb{S}\mathsf{M}^{\oplus 2} \boxplus \mathbb{S}\mathsf{M}^{\oplus 2})'$ is called of *Hadamard form* if it has the following b-wave front set

$$\mathrm{WF}_b(\omega^{(2)}) = \{(x, y, k_x, -k_y) \in \mathsf{T}^*(\mathsf{M} \times \mathsf{M})\backslash\{0\}|\ (x, k_x) \sim (y, k_y),\ k_x \triangleright 0\},$$

where $\sim$ entails that (x, k_x) and (y, k_y) are connected by a generalized broken bicharacteristic, while $k_x \triangleright 0$ means that the covector k_x at $x \in \mathsf{M}$ is future pointing. Since we deal with vector-valued distributions, the standard convention for the wave front set is to take the union of the wave front set of its components in an arbitrary but fixed local frame.

For further details on Hadamard states on globally hyperbolic manifolds with empty boundary we refer to [48, 49, 56], while on globally hyperbolic manifolds with timelike boundary, we refer to [31, 78, 47].

With the next theorem, we show that the pull-back of a quasifree state along the isomorphism $\mathfrak{R}_{1,0} \colon \mathfrak{A}_0 \to \mathfrak{A}_1$ induced by the unitary Møller operator R for D preserves the singularity structure of the two-point distribution $\omega^{(2)}$.

Theorem 3.15. *Assume that $g_0, g_1 \in \mathcal{GH}_\mathsf{M}$ fulfill (i) in the Setup 2.21. Assume furthermore that a propagation of singularity theorem holds true for D with MIT boundary conditions, namely for any $u \in \mathsf{Sol}_{\text{MIT}}(\mathsf{D})$, $\mathrm{WF}_b(u)$ is the union of* *maximally extended generalized broken bicharacteristics. Denote by* $\mathfrak{A}_\alpha$*,* $\alpha = 0, 1$*, the algebras of Dirac fields with MIT boundary conditions on* M_α *and let* $\omega_\alpha : \mathfrak{A}_\alpha \to \mathbb{C}$ *be quasifree states satisfying*

$$\omega_0 = \omega_1 \circ \mathfrak{R}_{1,0} \colon \mathfrak{A}_0 \to \mathbb{C}$$

where $\mathfrak{R}_{1,0}$ *is the isomorphism induced by* $\mathsf{R}_{1,0}$ *as per Theorem 3.9. If* ω_1 *is a Hadamard state as per Definition 3.14, then so is* ω_0*.*

Proof. Since $\mathfrak{R}_{1,0}$ preserves the grading of $\mathfrak{A}_0$, $\mathfrak{A}_1$, ω_0 inherits the property of being a quasifree state from ω_1. In particular the two-point function $\omega_0^{(2)}$ satisfies

$$\omega_0^{(2)}(f_0, g_0) = \omega_0 \left(\Xi(\psi_{f_0})^* \Xi(\psi_{g_0})\right) = \omega_1^{(2)} \left(\Xi(\mathsf{R}^{\oplus}_{1,0}\psi_{f_0})^* \Xi(\mathsf{R}^{\oplus}_{1,0}\psi_{g_0})\right) .$$

We shall now prove that ω_1 fulfills the Hadamard condition. To this avail we first observe that $\mathsf{R}_{1,0}$ can in fact be decomposed as $\mathsf{R}_{1,0} = \mathsf{R}_{1,\chi} \circ \mathsf{R}_{\chi,0}$ (*cf.* Remark 2.35). With reference to Theorem 2.27, we have $\mathsf{R}_{\chi,0} := U_{\mathsf{D}^f_{\chi,1},-} \circ \rho_- \circ \kappa^f_{1,0}$ whereas $\mathsf{R}_{1,\chi} := U_{\mathsf{D}_1,+} \circ \rho_+$. Let us consider $\mathfrak{R}_{1,\chi} \colon \mathfrak{A}_\chi \to \mathfrak{A}_1$, where $\mathfrak{R}_{1,\chi}$ is the $*$-isomorphism defined in Theorem 3.9 with $\mathfrak{A}_0$ replaced with $\mathfrak{A}_\chi$. Moreover let $\omega_\chi := \omega_1 \circ \mathfrak{R}_{1,\chi}$. With reference to Theorem 2.27, let $f_1, f_2 \in \Gamma_c(\mathbb{S}\mathsf{M} \oplus \mathbb{S}\mathsf{M})$ be with support contained in a neighborhood of Σ_+. Then

$$\begin{aligned}
\omega_\chi^{(2)}(f_1, f_2) &= \omega_\chi(\Xi(\mathsf{G}_\chi f_1)^* \Xi(\mathsf{G}_\chi f_2)) \qquad \text{def. } \omega_\chi^{(2)} \\
&= \omega_1(\Xi(\mathsf{R}^{\oplus}_{1,\chi}\mathsf{G}_\chi f_1)^* \Xi(\mathsf{R}^{\oplus}_{1,\chi}\mathsf{G}_\chi f_2)) \qquad \text{def. } \mathfrak{R}_{1,\chi} \\
&= (\mathsf{R}_{1,\chi}\mathsf{G}_\chi f_1, Q_{\omega_1}\mathsf{R}_{1,\chi}\mathsf{G}_\chi f_2)_{\mathsf{Sol}_{sc,\text{MIT}}(\mathsf{D}_1)} \qquad \text{Eq. (19)} \\
&= (\rho_- \mathsf{R}^{\oplus}_{1,\chi}\mathsf{G}_\chi f_1, Q_{\omega_1}\rho_- \mathsf{R}^{\oplus}_{1,\chi}\mathsf{G}_\chi f_2)_{\Sigma_+} \qquad \text{choice of } \Sigma_+ \\
&= (\rho_- \mathsf{G}_\chi f_1, Q_{\omega_1}\rho_- \mathsf{G}_\chi f_2)_{\Sigma_+} \qquad \rho_- \mathsf{R}_{1,\chi} = \rho_- \\
&= (\rho_- \mathsf{G}_1 f_1, Q_{\omega_1}\rho_- \mathsf{G}_1 f_2)_{\Sigma_+} \\
&= \omega_1^{(2)}(f_1, f_2) ,
\end{aligned}$$

where we exploited the fact that, when computing $(\, , \,)_{\mathsf{Sol}^{\oplus}_{sc,\text{MIT}}}$, we may choose Σ arbitrarily. In the second to last equation we used that $\mathsf{G}_\chi f|_{\Sigma_+} = \mathsf{G}_1 f|_{\Sigma_+}$ for f supported in a small enough neighborhood O of Σ_+. This shows that $\omega_\chi^{(2)}$ coincides with $\omega_1^{(2)}$ in a neighborhood of Σ_+ and therefore fulfills the Hadamard condition therein. Since the 2-point function of $\omega_\chi^{(2)}$ is a bisolution to the Dirac operator, we can argue as in [78, Proposition 5.9] to show that $\omega_\chi^{(2)}$ fulfills the Hadamard condition on M.
By observing that $\omega_1 = \omega_\chi \circ \mathfrak{R}_{\chi,0}$ and proceeding with a similar argument we have that ω_1 fulfills the Hadamard condition. □

Remark 3.16. We expect the propagation of singularities to hold true because there are already positive results in this direction, see *e.g.* [31, 47, 62, 63, 77] for the scalar wave equation, [59, 74] for first order systems, and [9] for the Dirac-Coulomb system. We postpone this investigation to a forthcoming paper.

We have finally all the tools to prove the existence of Hadamard states.

Proof of Theorem 1.1. Let t be a Cauchy temporal function for g and define $g_u := -dt^2 + h$, where h is a complete Riemannian metric on $t^{-1}(s)$ for every $s \in \mathbb{R}$. On account of [64, Proposition 2.23], there exists a globally hyperbolic metric $\overline{g}$ such that $J^+_{\overline{g}} \subset J^+_{g_u} \cap J^+_g$. Denote with $\mathbb{S}\mathsf{M}_{\overline{g}}$ the spinor bundle over $(\mathsf{M}, \overline{g})$ and consider the linear isometries

$$\kappa^{f'}_{\overline{g},g} \colon \mathbb{S}\mathsf{M}_g \to \mathbb{S}\mathsf{M}_{\overline{g}} \qquad \kappa^{f''}_{\overline{g},g_u} \colon \mathbb{S}\mathsf{M}_{g_u} \to \mathbb{S}\mathsf{M}_{\overline{g}}$$

defined in Section 3.3. It is easy to see that the operators

$$\mathsf{D}^{f'}_{g,\overline{g}} := \kappa^{f'}_{g,\overline{g}} \mathsf{D}_{\overline{g}} \kappa^{f'}_{\overline{g},g} : \Gamma(\mathbb{S}\mathsf{M}_g) \to \Gamma(\mathbb{S}\mathsf{M}_g)$$

$$\mathsf{D}^{f''}_{\overline{g},g_u} := \kappa^{f''}_{g_u,\overline{g}} \mathsf{D}_{\overline{g}} \kappa^{f''}_{\overline{g},g_u} : \Gamma(\mathbb{S}\mathsf{M}_{g_u}) \to \Gamma(\mathbb{S}\mathsf{M}_{g_u})$$

are weakly-hyperbolic on (M, g) and (M, g_u) respectively, so that we can construct a unitary Møller operator $\mathsf{R}_{g_u,g} \colon \mathsf{Sol}\,(\mathsf{D}_g) \to \mathsf{Sol}\,(\mathsf{D}_{g_u})$, composing the unitary Møller operators $\mathsf{R}_{\overline{g},g_u} \colon \mathsf{Sol}\,(\mathsf{D}_{g_u}) \to \mathsf{Sol}\,(\mathsf{D}_{\overline{g}})$ and $\mathsf{R}_{g,\overline{g}} \colon \mathsf{Sol}\,(\mathsf{D}_{\overline{g}}) \to \mathsf{Sol}\,(\mathsf{D}_g)$ obtained using the same arguments as in Sections 2.6 and 2.7. In particular, we can lift the action of the unitary Møller operator to a $*$-isomorphism between the algebras of Dirac fields on (M, g) and (M, g_u) respectively. Hence for any Hadamard state ω_H on $\mathfrak{A}_u$, the state defined by

$$\omega = \omega_H \circ \mathfrak{R}_{1,0} \, : \mathfrak{A} \to \mathbb{C}\,,$$

is also a Hadamard state on account of Theorem 3.15.
It remains to show that there exists a Hadamard state ω_H for $\mathfrak{A}_u$ in a time-strip $\mathsf{M}_T := t^{-1}[0, T]$. Indeed, by the time-slice axiom (Corollary 2.20), we can extend ω_H to the whole manifold, by preserving the positivity and, arguing as in [78, Lemma 5.10], also the Hadamard form. To this end, let us write the Dirac equation as $\mathsf{D} = \sigma(dt)\partial_t + \mathsf{L}$, where L differentiates only in the tangential part of Σ. Since we coupled D with a self-adjoint boundary condition, it follows that L is skew-adjoint. As a consequence we may define the self-adjoint operator $\mathsf{H} = i\mathsf{L}$. On account of Corollary 3.13 the spectral projection $P_+(\mathsf{H})$ in the positive spectrum of H defines a quasifree state ω. Let us remark that, on a manifold with empty boundary, the state above is the so-called ground state and it is of Hadamard form. For more details we refer to [49]. It remains to verify that the two-point distribution $\omega^{(2)}(x, y)$ is of Hadamard form. To this end, we first notice that the spectral projection satisfies $[\mathsf{H}, P_+(\mathsf{H})] = 0$ modulo smoothing (see *e.g.* [20]) and, in particular, we obtain that $\mathsf{D}UP_+(\mathsf{H}) = 0$, where $U := \exp(it\mathsf{H})$ the Cauchy evolution operator. Arguing again as in [78, Proposition 5.9], we can conclude. □

Remark 3.17. The main drawback of the definition of the Møller $*$-isomorphism $\mathfrak{R}$, used in Theorem 3.15, is the lack of control on the action of the group of $*$-automorphisms induced by the isometry group of M on ω_2.

Let us remark that the study of invariant states is a well-established research topic (*cf.* [8, 7]). Indeed, the type of factor can be inferred by analyzing which and how many states are invariant. From a more physical perspective instead, invariant states can represent equilibrium states in statistical mechanics *e.g.* KMS-states or ground states.

The previous remark leads us to the following open question: Under which conditions it is possible to perform an adiabatic limit, namely when is $\lim_{\chi\to 1}\omega_1$ well-defined?

A priori we expect that there is no positive answer in general, because it is known that certain free-field theories, *e.g.*, the massless and minimally coupled (scalar or Dirac) fields on four-dimensional de Sitter spacetime do not possess a ground state, even though their massive counterparts do. Note that this is not a no-go Theorem, but at least an indication that, in these situations, the map $\omega \to \omega \circ \mathfrak{R}$ cannot be expected to preserve the ground state property. A partial investigation in this direction has been carried out in [24, 35] for the case of a scalar field theory on globally hyperbolic spacetimes with empty boundary. In this situation it has been shown that, under suitable hypotheses the adiabatic limit can be performed preserving the invariance property under time translation but spoiling in general the ground state or KMS property.

Since our results depend only on the principal symbol of the Dirac operator and on the chosen boundary condition, we conclude our paper with the following corollary.

Corollary 3.18. *Let (M, g) be a globally hyperbolic spin spacetime with timelike boundary and let $\mathsf{D}_{\mathcal{V}} = \mathsf{D} + \mathcal{V}$ be the Dirac operator coupled with a external skew-symmetric potential $\mathcal{V} \in End(\mathbb{S}\mathsf{M})$ and equipped with MIT boundary conditions. Assume furthermore that a propagation of singularities theorem holds true for D with MIT boundary conditions, namely for any $u \in \mathsf{Sol}_{\text{MIT}}(\mathsf{D})$, $\mathrm{WF}_b(u)$ is the union of maximally extended generalized broken bicharacteristics. Then there exists a state for the algebra of Dirac fields with MIT boundary conditions which satisfies the Hadamard condition.*

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