

THE INTEGRAL TRACE FORM AS A COMPLETE INVARIANT
FOR REAL S_n NUMBER FIELDS

GUILLERMO MANTILLA-SOLER AND CARLOS RIVERA

Received: April 8, 2022

Revised: August 10, 2022

Communicated by Otmar Venjakob

ABSTRACT. In this paper we show that the integral trace is a complete invariant for degree n , S_n real number fields that satisfy certain ramification bound. Among the fields that our results cover, there are those of square free different ideal; for such fields we find an explicit description of the isometry group of the integral trace.

2020 Mathematics Subject Classification: 11R80, 11R29, 11R21

Keywords and Phrases: Integral trace form, tame fields, S_n -extensions

1 INTRODUCTION

In the first part of the introduction we briefly present some of the results of the paper. In the second part of the introduction we give a detailed description as to how the ideas, statements and definitions behind the main results came to be.

1.1 MAIN RESULTS

Given a number field K we denote by $(O_K, \text{tr}_{K/\mathbb{Q}})$ the pair given by the free \mathbb{Z} -module O_K together with the symmetric bilinear pairing given by the trace

$$\begin{aligned} \text{tr}_{K/\mathbb{Q}} : O_K \times O_K &\rightarrow \mathbb{Z} \\ (x, y) &\mapsto \text{Tr}_{K/\mathbb{Q}}(xy). \end{aligned}$$

Some of the results we prove in this paper are stated next.

THEOREM (cf. Theorem 5.2). *Let K be a degree n totally real S_n number field and L be any S_n number field. Let $d_s(K, L)$ be the integer defined in 1.1.*

(i) Suppose that $n \geq d_s(K, L)^2$. Then,

$$(O_K, \text{tr}_{K/\mathbb{Q}}) \cong (O_L, \text{tr}_{L/\mathbb{Q}}) \text{ if and only if } K \cong L.$$

(ii) If K does not have wild ramification, then the condition $n \geq d_s(K, L)^2$ can be improved to $n \geq \text{rad}(d_s(K, L))^2$.

(iii) If K and L do not have wild ramification except possibly for some primes with ramification index 2 lying over 2, then the condition $n \geq d_s(K, L)^2$ can also be improved to $n \geq \text{rad}(d_s(K, L))^2$.

DEFINITION 1.1. Let K be a number field and let d_K be its discriminant. For p a rational prime we let $e_p(K)$ be the maximal ramification index of a prime in K lying over p . For a number field L we define $d_f(K, L)$ as the product $\prod p^{v_p(d_K)}$ where p runs over the set of odd primes such that $e_p(K) = 2 = e_p(L)$, and set $d_s(K, L) := d_K/d_f(K, L) \in \mathbb{Z}$.

Remark 1.2. Notice that if K and L have the same discriminant d , then $d_s(K, L)$ is bounded by the non-square free part of d , i.e., $d_s(K, L) \leq \prod_{p, v_p(d) > 1} p^{v_p(d)}$.

As a consequence of the above we have that the integral trace is a complete invariant for S_n real fields with at worst quadratic ramification.

THEOREM (cf. Theorem 5.3). Let K, L be S_n number fields. Suppose that K is totally real and that the ramification index of every prime in K and L over \mathbb{Q} is at most 2. Then,

$$(O_K, \text{tr}_{K/\mathbb{Q}}) \cong (O_L, \text{tr}_{L/\mathbb{Q}})$$

if and only if $K \cong L$.

COROLLARY (Corollary 5.4). The integral trace form is a complete invariant for totally real number fields of fundamental discriminant.

EXAMPLE 1.3. Let K and L be the sextic fields defined respectively by the polynomials $x^6 - 3x^5 - 6x^4 + 17x^3 + 2x^2 - 8x + 1$ and $x^6 - x^5 - 12x^4 - x^3 + 22x^2 - 1$, and let $d_f := d_f(K, L)$ and $d_s := d_s(K, L)$. The fields K and L are totally real of discriminant $d = 2^3 \cdot 11 \cdot 619 \cdot 11411$. In this case $d_s = 8$ and $d_f = 77697499$, and since d is fundamental $\text{rad}(d_s)^2 = 4$. The fields K and L are not isomorphic, for instance a calculation shows that the prime $p = 11$ has three primes in K lying over it, while it has two in L , hence their integral traces are not isometric.

THEOREM (cf. Theorem 5.5). Let K and L be totally real number fields with square free different ideal. Then any isometry $\phi : (K, \text{tr}_{K/\mathbb{Q}}) \rightarrow (L, \text{tr}_{L/\mathbb{Q}})$ such that $\phi(O_K) = O_L$ is equal to plus or minus an isomorphism of fields $K \cong L$.

1.1.1 SOME APPLICATIONS

As a byproduct of the methods used in the proofs of the above results we can provide, for real number fields with square free different, an explicit description of the automorphism group of the integral trace.

THEOREM (cf. Corollary 5.6). *Let K be a totally real number field with square free different ideal. Then,*

$$\text{Aut}((O_K, \text{tr}_{K/\mathbb{Q}})) \cong \mathbb{Z}/2\mathbb{Z} \times \text{Aut}(K).$$

In particular, if $n > 2$ and K is a S_n number field then the automorphism group of the integral trace is “trivial”.

EXAMPLE 1.4. Let K be the biquadratic field $\mathbb{Q}(\sqrt{5}, \sqrt{13})$. This is a totally real number field with different ideal the principal generated by $\sqrt{65}$. Since $\sqrt{5}$ and $\sqrt{13}$ generate prime ideals in O_K the different ideal is square free. Since K is a Galois number field, with Galois group the Klein group, then $\text{Aut}((O_K, \text{tr}_{K/\mathbb{Q}})) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

A consequence of the above result is that for real fields of square free discriminant the automorphism group of the integral trace is trivial.

COROLLARY (Corollary 5.7). *Let K be a real number field of degree at least 3 and with square free discriminant. Then,*

$$\text{Aut}((O_K, \text{tr}_{K/\mathbb{Q}})) \cong \mathbb{Z}/2\mathbb{Z}.$$

A further application of the results here, which is ongoing work, is a method to count real number fields with square free discriminant. Using the results of this paper, together with some control of the local behavior of the trace, we can apply Siegel’s mass formula to obtain some counting results; at least for fields of small degree.

1.2 MOTIVATION ON DEFINITIONS AND RESULTS

One of the main problems in algebraic number theory is to give a satisfactory way of deciding whether or not two number fields are isomorphic. Inspired by the style of question of Cornelissen and Marcolli in [10], this query can be formulated as follows:

QUESTION 1.5. *Can one describe an isomorphism between number fields K, L from an associated mathematical/arithmetical object? In other words, can we find a complete invariant for number fields?*

Several natural objects have been at the center of study of this question by several authors; e.g., the Dedekind zeta function, the ring of Adeles, the group of Dirichlet Characters, the absolute Galois group (see [10, 21, 24, 36, 37, 44]). From all these the only one that is a complete invariant, thanks to a famous

result of Neukirch and Uchida, see [36, 44], is the absolute Galois group.

All the above invariants have in common that they refine the discriminant; i.e., having the same invariant for K and L implies that the two fields have the same discriminant. Thanks to Hermite and Minkowski it is known that, up to isomorphism, there are only finitely many number fields of a given discriminant. This result has led to the study of number fields via their discriminants, and specifically to the industry of results about the asymptotics of the number of fields up to a given discriminant bound, sometimes with specific signature types or Galois groups. Some of the examples of such investigations can be found in works of Bhargava, Cohen, Davenport-Heilbron, Datskovsky-Wright, Ellenberg-Venkatesh, Jones, Klüners, Lemke Oliver-Thorne, Malle, P. Harron, R. Harron, Roberts, Taniguchi-Thorne, Shnidman, Wang and many others (See [4, 5, 6, 11, 13, 14, 23, 22, 27, 18, 17, 38, 42, 43, 45]). Since the discriminant of a number field K of degree n is basically the signed co-volume of the lattice O_K inside of \mathbb{R}^n it is only natural to study the isometry class of such lattices as a way to refine the discriminant. In other words the isometry class of the integral bilinear pairing

$$\begin{aligned} \mathrm{tr}_{K/\mathbb{Q}} : O_K \times O_K &\rightarrow \mathbb{Z} \\ (x, y) &\mapsto \mathrm{Tr}_{K/\mathbb{Q}}(xy). \end{aligned}$$

is the first natural object that comes to mind when dealing with the problem of dividing the classification of number fields of the same discriminant in a smaller subclass. This invariant, and the closely related *shape*, are used for instance by Ellenberg and Venkatesh in their paper about asymptotics of number fields of bounded discriminant; see [14, Remark 3.2].

Trace forms over fields, and their applications, have been extensively studied by many authors; see for instance the works of Eva Bayer-Fluckiger, J.P. Serre and their co-authors [2, 3, 41] and the work of Robert Perlis and his co-authors [9]. However, the literature around the classification results about the integral trace over number fields is scarce. These kinds of classification questions have been studied by the first named author from four different, basically disjoint, perspectives:

- (i) The first one uses the techniques developed by M. Bhargava in his thesis, specifically the higher composition laws in cubes and their relation to cubic fields. See [28] for details and for a proof of

THEOREM 1.6. *Let K, L be cubic fields such that K has positive fundamental discriminant. Then,*

$$(O_K, \mathrm{tr}_{K/\mathbb{Q}}) \cong (O_L, \mathrm{tr}_{L/\mathbb{Q}}) \text{ if and only if } K \cong L.$$

- (ii) A local-global point of view. Using the Adélization of the orthogonal group one can show that genus and the spinor genus of the integral trace

form coincide. Together with the classification genus of the integral trace, this method permits to deal with the problem in the case of non-real fields that have no wild ramification. See [31] and [33] for details and a proof of

THEOREM 1.7. *Let K, L be two number fields with the same discriminant d , same signature and without wild ramification. Suppose that the fields are non-totally real and that they do not have wild primes. Then,*

$$(O_K, \text{tr}_{K/\mathbb{Q}}) \cong (O_L, \text{tr}_{L/\mathbb{Q}}) \iff \left(\frac{\alpha_K^p}{p}\right) = \left(\frac{\alpha_L^p}{p}\right)$$

for all odd primes p that divide d .

For extra details of the construction of the elements α_K^p see also [35].

- (iii) Using that rational quadratic forms can be seen as cohomology classes in $H^1(\mathbb{Q}, O_n)$ and that Artin representations can be seen as elements of $H^1(\mathbb{Q}, \text{GL}_n(\mathbb{C}))$ one can study the relation between Dedekind zeta functions of number fields and their integral traces. See [32] for details and for a proof of

THEOREM 1.8. *Let K, L be two number fields. Suppose that K is non-totally real and that no rational prime has wild ramification in it. Then,*

$$\zeta_K(s) = \zeta_L(s) \implies (O_K, \text{tr}_{K/\mathbb{Q}}) \cong (O_L, \text{tr}_{L/\mathbb{Q}}).$$

- (iv) Imposing the Galois theoretic conditions, mainly prime power Galois cyclic extensions, a classification of the integral trace and shape can also be obtained for such fields. See [34] and [8] for details and a proof of

THEOREM 1.9. *Let n be a positive integer. Let K, L be two Galois number fields with Galois group isomorphic to $\mathbb{Z}/n\mathbb{Z}$ and without wild primes. Then,*

$$(O_K, \text{tr}_{K/\mathbb{Q}}) \cong (O_L, \text{tr}_{L/\mathbb{Q}}) \text{ if and only if } \text{disc}(K) \cong \text{disc}(L).$$

1.2.1 MOTIVATION BEHIND THE DEFINITION OF $d_s(K, L)$

Cases (i) and (iv) have an interesting duality for cubic fields. The former says that for square free discriminant the trace is a great improvement of the discriminant, while the latter says that for perfect square discriminants the trace does not provide new information. Hence, it is interesting to wonder what happens in the intermediate cases, i.e., neither square free nor perfect square discriminants. As the following example shows the answer could go either way

EXAMPLE 1.10. Let K_1, K_2 and K_3 be the cubic fields defined respectively by the cubic polynomials $f_1 := x^3 - 36x - 78$, $f_2 := x^3 - 18x - 6$ and $f_3 := x^3 - 36x - 60$. A calculation shows that these fields are not isomorphic, all have discriminant $2^2 \cdot 3^5 \cdot 23$ and

$$(O_{K_1}, \text{tr}_{K_1/\mathbb{Q}}) \cong (O_{K_3}, \text{tr}_{K_3/\mathbb{Q}}) \not\cong (O_{K_2}, \text{tr}_{K_2/\mathbb{Q}}).$$

This lead us to think that to generalize case (i) to higher degree number fields we need to generalize the square free condition of the discriminant; this was in fact supported by computational evidence. Number fields with square free discriminant have square free different ideal. It turns out, as a simple calculation shows, that for cubic fields both conditions are equivalent.

Remark 1.11. Let K be a cubic number field. Then, K has square free different ideal if and only if it has square free discriminant.

A general equivalent condition to having square free different ideal is given by the following result. We call a number field K tame if no prime has wild ramification in K .

PROPOSITION 1.12. *Let K be a number field. The field K has square free different ideal if and only if K is tame and for every prime p the maximal ramification degree is at most 2.*

Proof. The result follows from the formula for the exponents of primes appearing in the different ideal. More explicitly, recall that for a prime \mathfrak{P} in O_K lying over a prime p with ramification degree e , the exponent of \mathfrak{P} on the different ideal is at least $e - 1$. Moreover, it is $e - 1$ if and only if $p \nmid e$. \square

A possible generalization to having square free discriminant could be having symmetric group as the Galois group and having ramification exponents bounded by 2. This is explained thanks to the above and since number fields with fundamental discriminant have Galois group the full symmetric group. In fact this is the motivation behind Theorem 5.3. The main results of this paper, Theorem 5.2, pushes this generalization even further. As long as the part of the discriminant that comes from prime with ramification degrees bigger than 2 is not so big, then the integral trace is a complete invariant.

Besides the motivation to find the right generalization to Theorem 1.6, in this paper we introduce and develop the Casimir pairings. With it we introduce a fifth approach to the integral trace invariant problem. This new method, combined with several of the points of view described above, yields to most of the results of the paper.

1.3 OUTLINE OF THE PAPER

In §2 we introduce the theory of Casimir Pairings and show many of its properties which we will need later on. In §3 we study linearly disjoint number

fields, the relation of Casimir Pairings with them and we give a general criterion to decide whether or not two number fields are conjugate. In §4 we do a local-global study of Casimir elements associated to trace pairings over number fields; this section is the connection between the general theory of Casimir pairings and number fields. Finally in §5 we provide the proofs to the number theoretic applications from all the results of the previous sections.

2 CASIMIR PAIRINGS

Let V be a finite dimensional vector space over a field k endowed with a nondegenerate bilinear form B , and let $V^* = \text{Hom}(V, k)$ be its dual. The isomorphism $\Gamma_B : V \rightarrow V^*$, $v \mapsto \Gamma_B(v) := B(v, \cdot)$ determines a bilinear form on V^* , namely the pullback of B via Γ_B^{-1} ; we will denote this form by $\langle \cdot, \cdot \rangle_B$ and we call it *Casimir pairing* associated to B .

For a field k let us denote by $\text{FVect}B_k$ the category of pairs (V, B) where V is a finite dimensional k -vector space and B a nondegenerate k -bilinear form on V with morphisms given by bijective k -linear isometries.

PROPOSITION 2.1. *The functor*

$$\begin{array}{ccc} \text{FVect}B_k & \xrightarrow{\mathcal{CS}} & \text{FVect}B_k \\ \\ \begin{array}{c} (V, B_V) \\ \downarrow \phi \\ (W, B_W) \end{array} & \mapsto & \begin{array}{c} (V^*, \langle \cdot, \cdot \rangle_{B_V}) \\ \uparrow \phi^* \\ (W^*, \langle \cdot, \cdot \rangle_{B_W}) \end{array} \end{array}$$

mapping (V, B) to $(V^, \langle \cdot, \cdot \rangle_B)$ and an isometry $\phi : V \rightarrow W$ to its dual ϕ^* is a well-defined duality of categories.*

Proof. An object (V, B) in $\text{FVect}B_k$ comes equipped with the k -linear isomorphism Γ_B , which we are forcing to be an isometry between (V, B) and $\mathcal{CS}(V, B)$. In particular, $\mathcal{CS}(V, B)$ is nondegenerate, i.e., the functor is well defined on objects. To check that it is well defined on morphisms, note that an isometry $\phi : (V, B) \rightarrow (W, B')$, by definition, makes the following diagram of k -linear maps commutative:

$$\begin{array}{ccc} V & \xrightarrow{\Gamma_B} & V^* \\ \downarrow \phi & & \uparrow \phi^* \\ W & \xrightarrow{\Gamma_{B'}} & W^* \end{array}$$

Thus if ϕ is bijective, the map $\phi^* : \mathcal{CS}(V, B) \rightarrow \mathcal{CS}(W, B')$ is also a bijective isometry. An inverse for \mathcal{CS} is given by the functor \mathcal{CS}^t mapping (V, B) into V^* together with bilinear map that makes $\Gamma_B^t : V \rightarrow V^*, \Gamma_B^t(v) := B(\cdot, v)$ into an isometry. \square

We stress here that this is a very natural way to endow V^* with bilinear pairings, in a sense made precise by Remark 2.2 below.

Remark 2.2. The functors \mathcal{CS}^t and \mathcal{CS} give a (dual) adjoint equivalence, with the unit and counit induced by the usual evaluation map $V \rightarrow V^{**}$. In particular, \mathcal{CS} becomes self-dual adjoint when we restrict to symmetric bilinear forms.

Remark 2.3. Notice that \mathcal{CS} commutes with scalar extension, i.e., if k'/k is a field extension, then the following diagram commutes:

$$\begin{CD} \text{FVect}B_k @>\mathcal{CS}>> \text{FVect}B_k \\ @VV(-) \otimes_k k'V @VV(-) \otimes_k k'V \\ \text{FVect}B_{k'} @>\mathcal{CS}>> \text{FVect}B_{k'} \end{CD}$$

Let $\{v_1, \dots, v_n\}$ be an ordered k -basis of V and let $\{f_1, \dots, f_n\}$ be its dual basis. We say that $\{v_1^*, \dots, v_n^*\} \subseteq V$ is the dual basis of $\{v_1, \dots, v_n\}$ with respect to B if $f_i = B(v_i^*, \cdot)$ for each i . The next lemma gives us a concrete useful formula to compute Casimir parings.

LEMMA 2.4. Let $(V, B) \in \text{FVect}B_k$ and let $\{v_1, \dots, v_n\}$ be a k -basis of V . Then,

$$\langle \psi, \phi \rangle_B = \sum_{i=1}^n \psi(v_i^*)\phi(v_i),$$

where $\{v_1^*, \dots, v_n^*\}$ is the dual basis of $\{v_1, \dots, v_n\}$ with respect to B .

Proof. Let $\{f_1, \dots, f_n\}$ be the dual basis $\{v_1, \dots, v_n\}$. Since both sides of the equality are bilinear on ϕ and ψ , it is enough to check it when $\phi = f_r$ and $\psi = f_s$, for each $1 \leq r, s \leq n$. In this case, $\sum_{i=1}^n \phi(v_i^*)\psi(v_i) = f_r(v_s^*) \cdot 1$ and $\langle \psi, \phi \rangle_B = B(v_r^*, v_s^*) = f_r(v_s^*)$. \square

Now that we have a natural way to get pairings on $V^* = \text{Hom}_k(V, k)$, we can use it to get parings on more general hom sets $\text{Hom}_k(V, W)$. We do this by identifying $\text{Hom}_k(V, W)$ with $V^* \otimes_k W$ via the usual isomorphism taking $f \otimes w \in V^* \otimes_k W$ to the map $v \mapsto f(v)w$.

DEFINITION 2.5. Let k be a field and $(V, B) \in \text{FVect}B_k$.

(i) Given k -bilinear map of vector spaces $V_1 \times V_2 \rightarrow V_3$, the associated Casimir pairing is the map

$$\text{Hom}_k(V, V_1) \otimes_k \text{Hom}_k(V, V_2) \rightarrow V_3,$$

obtained by taking the tensor product

$$\langle \cdot, \cdot \rangle_B \otimes m : (V^* \otimes_k V_1) \otimes_k (V^* \otimes_k V_2) \rightarrow (k \otimes_k V_3)$$

of the maps $\langle \cdot, \cdot \rangle_B : V^* \otimes_k V^* \rightarrow k$ and $m : V_1 \otimes_k V_2 \rightarrow V_3$.

(ii) When $V_1 = V_2 = V_3 = R$ is a k -algebra and $R \times R \rightarrow R$ is the multiplication map, we call the corresponding pairing

$$\text{Hom}_k(V, R) \otimes_k \text{Hom}_k(V, R) \rightarrow R,$$

the Casimir pairing on $\text{Hom}_k(V, R)$ associated to B .

Remark 2.6. More explicitly, Lemma 2.4 shows that the Casimir pairing associated to $V_1 \otimes V_2 \rightarrow V_3$ is the map that takes $\psi \otimes \phi \in \text{Hom}_k(V, V_1) \otimes_k \text{Hom}_k(V, V_2)$ to the image of the element

$$\sum_{i=1}^n \psi(v_i^*) \otimes \phi(v_i) \in V_1 \otimes V_2$$

under $V_1 \otimes_k V_2 \rightarrow V_3$. Where as before $\{v_i\}$ is any k -basis of V and $\{v_i^*\}$ its dual with respect to B .

By slight abuse of notation for any k -algebra R we still denote the associated Casimir pairing on $\text{Hom}_k(V, R)$ by $\langle \cdot, \cdot \rangle_B$.

EXAMPLE 2.7. Let \mathfrak{g} be an n -dimensional Lie algebra over a field k with $\text{char}(k) = 0$. By Cartan’s criterion the Killing form B on \mathfrak{g} is nondegenerate if and only if \mathfrak{g} is semisimple. In that case, if we take $V = \mathfrak{g}$, $R = U(\mathfrak{g})$ its universal enveloping algebra and $\iota : \mathfrak{g} \hookrightarrow U(\mathfrak{g})$ the canonical inclusion, then

$$C := \langle \iota, \iota \rangle_B \in U(\mathfrak{g})$$

is the well known quadratic Casimir element of \mathfrak{g} . Despite its simplicity, the Casimir element turns out to play a central role in the modern proofs of several key theorems in the representation theory of Lie algebras such as Weyl’s theorem on complete reducibility (see [20, Section 6.3]), as well as in Weyl’s and Konstant’s character formulas as shown in the appendix to chapter VI in [20].

EXAMPLE 2.8. Let (M, g) be a Riemannian manifold. Since the metric g is everywhere nondegenerate, it gives isomorphisms between the tangent bundle TM and its dual $T^*M = \Omega^1(M)$ known as the musical isomorphisms \flat and \sharp . One precisely uses $\sharp : \Omega^1(M) \rightarrow TM$ to induce an inner product on $\Omega^1(M)$, i.e., the usual inner product of 1-forms is pointwise the Casimir pairing on $\Omega_p^1(M) = \text{Hom}(T_p M, \mathbb{R})$ associated to g_p .

PROPOSITION 2.9. For a fixed $(V, B) \in \text{FVectB}_k$, the formation of Casimir pairings is functorial on $V_1 \otimes_k V_2 \rightarrow V_3$. More precisely, for each commutative diagram

$$\begin{array}{ccc} V_1 \otimes_k V_2 & \longrightarrow & V_3 \\ f \otimes g \downarrow & & \downarrow \\ V'_1 \otimes_k V'_2 & \longrightarrow & V'_3 \end{array},$$

the diagram on Casimir pairings

$$\begin{array}{ccc} \text{Hom}_k(V, V_1) \otimes_k \text{Hom}_k(V, V_2) & \longrightarrow & V_3 \\ f_* \otimes g_* \downarrow & & \downarrow \\ \text{Hom}_k(V, V'_1) \otimes_k \text{Hom}_k(V, V'_2) & \longrightarrow & V'_3 \end{array}$$

is also commutative.

Proof. This is already encoded in the definition since the identification we are using $V^* \otimes_k W = \text{Hom}_k(V, W)$ is functorial on W . \square

The following statements contain the main features of the Casimir pairings that will be needed later on.

COROLLARY 2.10. Let $(V, B) \in \text{FVectB}_k$, R be a k -algebra and $\phi, \psi \in \text{Hom}_k(V, R)$.

(i) *Functoriality on R :* If $\theta : R \rightarrow R'$ is a homomorphism of k -algebras then

$$\theta(\langle \psi, \phi \rangle_B) = \langle \theta \circ \psi, \theta \circ \phi \rangle_B.$$

(ii) *Scalar extension:* If k'/k is a field extension, then

$$\langle \psi \otimes 1, \phi \otimes 1 \rangle_{B \otimes k'} = \langle \psi, \phi \rangle_B \otimes 1 \in R \otimes_k k'.$$

Proof. A homomorphism of k -algebras is a map θ making the diagram

$$\begin{array}{ccc} R \otimes_k R & \longrightarrow & R \\ \theta \otimes \theta \downarrow & & \downarrow \theta \\ R' \otimes_k R' & \longrightarrow & R' \end{array}$$

commutative. Thus part (i) this follows from the functoriality claimed in Proposition 2.9. Part (ii) is a consequence of the compatibility in Remark 2.3. \square

COROLLARY 2.11. *Let R be an algebra over a field k .*

(i) *The functor from FVectB_k to the category of R -modules with bilinear forms taking (V, B) to R -module $\text{Hom}_k(V, R)$ endowed with its Casimir pairing $\langle \cdot, \cdot \rangle_B$ and a map $\phi : (V, B) \rightarrow (V', B')$ to ϕ^* is a contravariant embedding.*

(ii) *Let (V, B) and (V', B') be objects in FVectB_k . A k -linear map $\phi : V \rightarrow V'$ is a bijective isometry of k -spaces if only if*

$$\phi^* : \text{Hom}_k(V', R) \rightarrow \text{Hom}_k(V, R)$$

is a bijective isometry of R -modules, where we endow the hom sets with the associated Casimir pairings.

Proof. Since k is a field, every k -algebra is faithfully flat over k . Hence the scalar extension $(-)\otimes_k R$ from FVectB_k to the category of R -modules with bilinear forms is an embedding. The functor in (i) is the composition of this with the duality \mathcal{CS} . Since we can tell whether or not an arbitrary k -linear map is a bijective isometry after a faithfully flat extension of scalars, this factorization also proves (ii). \square

More importantly for our purposes, when $\text{char}(k) \neq 2$ and R is a commutative k -algebra, restricting the functor in part (i) to symmetric bilinear forms yields a contravariant embedding

$$\text{QSP}_k^* \hookrightarrow \text{QMod}_R,$$

from the category nondegenerate quadratic k -spaces and (bijective) isometries to the category of quadratic R -modules.

2.1 TRACE FORMS

DEFINITION 2.12. *Let k be a field and let E be an étale algebra over k . For any k -algebra R we denote by $\langle \cdot, \cdot \rangle_{\text{tr}_{E/k}}$ the Casimir pairing on $\text{Hom}_k(E, R)$ associated to the non-degenerate symmetric bilinear form given by the trace pairing*

$$\begin{aligned} \text{tr}_{E/k} : E \times E &\rightarrow k \\ (x, y) &\mapsto \text{Tr}_{E/k}(xy). \end{aligned}$$

LEMMA 2.13. *Let E be an étale algebra over k . Suppose that Ω/k is a separable field extension that splits E , i.e., such that $E \otimes_k \Omega \cong \Omega^n$. Then $\{\sigma_1, \dots, \sigma_n\} := \text{Hom}_{k\text{-alg}}(E, \Omega)$ is an orthonormal Ω -basis of $\text{Hom}_k(E, \Omega)$.*

Proof. In the case $E = k^n$ the algebra is already split over k , the trace form is the usual dot product on k^n and the corresponding isomorphism $k^n \rightarrow \text{Hom}_k(k^n, k)$ takes the standard i -th basis vector e_i to the i -th projection $\pi_i : k^n \rightarrow k$. Since $\{e_1, \dots, e_n\}$ is an orthonormal basis of k^n , the

set $\{\pi_1, \dots, \pi_n\}$ is an orthonormal basis of $\text{Hom}_k(k^n, k)$. In the general case, applying $\text{Hom}_\Omega(-, \Omega)$ to an isomorphism $\sigma : E \otimes_k \Omega \rightarrow \Omega^n$, we get an isometry of Ω -spaces

$$\Omega^n \rightarrow \text{Hom}_\Omega(\Omega^n, \Omega) \xrightarrow{\sigma^*} \text{Hom}_\Omega(E \otimes_k \Omega, \Omega) = \text{Hom}_k(E, \Omega),$$

which takes the canonical basis $\{e_1, \dots, e_n\}$ of Ω^n to $\{\sigma^*(\pi_1), \dots, \sigma^*(\pi_n)\} = \{\sigma_1, \dots, \sigma_n\}$. \square

It follows from the above that any k -linear map from E to Ω can be written in terms of Casimir elements:

COROLLARY 2.14. *Let k, E, Ω and $\{\sigma_1, \dots, \sigma_n\}$ be as above. Then, for all $\phi \in \text{Hom}_k(E, \Omega)$*

$$\phi = \sum_{i=1}^n \langle \phi, \sigma_i \rangle_{\text{tr}_{E/k}} \sigma_i.$$

Remark 2.15. Grothendieck's formulation of Galois theory gives a duality between étale k -algebras and finite \mathfrak{G}_k -sets, where $\mathfrak{G}_k = \text{Gal}(k_s/k)$ and k_s is a separable closure of k . There, an étale algebra E corresponds precisely to the \mathfrak{G}_k -set $\text{Hom}_{k\text{-alg}}(E, k_s)$. The fact that $\text{Hom}_{k\text{-alg}}(E, k_s)$ also happens to be an orthonormal k_s -basis for $\text{Hom}_k(E, k_s)$ allows us to leverage this correspondence. For instance, a decomposition of E as a product $\prod_i L_i$ of separable extensions L_i amounts to a decomposition into transitive \mathfrak{G}_k -sets

$$\text{Hom}_{k\text{-alg}}(E, k_s) = \bigsqcup_i \text{Hom}_{k\text{-alg}}(L_i, k_s),$$

which in turn induces an orthonormal decomposition of k_s -spaces

$$\text{Hom}_k(E, k_s) = \bigoplus_i \text{Hom}_k(L_i, k_s).$$

COROLLARY 2.16. *Let k be a field and let E/k and E'/k be étale algebras same dimension n . Suppose that Ω/k is a separable field extension splitting both E and E' . Let $\phi : E \rightarrow E'$ be a k -linear map. Then the following are equivalent:*

- (i) *The map ϕ is an isometry between $(E, \text{tr}_{E/k})$ and $(E', \text{tr}_{E'/k})$.*
- (ii) *The map ϕ^* is an isometry between the Ω -spaces $\text{Hom}_k(E, \Omega)$ and $\text{Hom}_k(E', \Omega)$.*
- (iii) *The matrix $U = (c_{ij})$ is orthogonal, where $c_{ij} := \langle \sigma_i, \tau_j \phi \rangle_{\text{tr}_{E/k}}$, and $\{\sigma_i\}, \{\tau_i\}$ are the sets of homomorphisms of k -algebras of E and E' into Ω .*

Proof. The equivalence of (i) and (ii) is a particular case of Corollary 2.11(ii), and the equivalence of (ii) and (iii) follows from the fact that $\{\sigma_i\}$ and $\{\tau_i\}$ are orthonormal bases by Lemma 2.13. \square

3 LINEAR DISJOINTNESS

When studying separable extensions that are trace isometric we discovered that linear disjointness makes matters simpler; more specifically in this case the entries of the matrix of Corollary 2.16(iii) are all in the conjugacy orbit of a single Casimir value. Also, when dealing with S_n number fields linear disjointness is equivalent to not being conjugate. In this section we explain all this in detail.

COROLLARY 3.1. *Let K/k and L/k be separable field extension, and suppose Ω/k is a separable field extension containing a Galois clousure of KL/k . Let us denote by ι_K and ι_L the inclusions from K (resp L) into KL . Suppose that*

$$\phi : (K, \text{tr}_{K/k}) \rightarrow (L, \text{tr}_{L/k})$$

is an isometry and let $c \in KL$ be the Casimir element $c := \langle \iota_K, \iota_L \phi \rangle_{\text{tr}_{KL/k}}$. If K and L are linearly disjoint over k , then there is an indexing

$$\text{Hom}_{k\text{-alg}}(KL, \Omega) = \{\theta_{i,j}; 1 \leq i, j \leq n\}$$

of the set of k - embeddings of KL into Ω such that the matrix $U := \theta_{i,j}(c)$ is orthogonal.

Proof. Since K and L are k -linearly disjoint the restriction map

$$\text{Hom}_{k\text{-alg}}(KL, \Omega) \rightarrow \text{Hom}_{k\text{-alg}}(K, \Omega) \times \text{Hom}_{k\text{-alg}}(L, \Omega); \theta \mapsto (\theta \circ \iota_K, \theta \circ \iota_L)$$

is a bijection. The result follows from this together with Proposition 2.10(i) and Corollary 2.16 (iii). \square

3.1 LINEAR DISJOINTNESS AND AN ISOMORPHISM CRITERION FOR NUMBER FIELDS

When looking for arithmetic invariants of number fields it is always important to have in hand a criterion to decide whether or not two fields are conjugate. As it turns out, for S_n number fields one such a criterion is that the fields are linearly disjoint. Here we prove something a little bit more general.

PROPOSITION 3.2. *Let K, L be number fields of the same degree n such that $\text{Gal}(\widetilde{K}/\mathbb{Q}) \cong S_n$ (\widetilde{K} the Galois closure of K) and L/\mathbb{Q} has no intermediate extensions. Then, $K \not\cong L$ if and only if K/\mathbb{Q} and L/\mathbb{Q} are linearly disjoint.*

Proof. “if” : Suppose K and L are linearly disjoint, then $K \not\cong L$. Otherwise,

$$K \cong L \cong \mathbb{Q}[X]/(p(X))$$

for some $p(X) \in \mathbb{Q}[X]$ and tensoring with L we get $K \otimes_{\mathbb{Q}} L \cong L[X]/(p(X))$ which is not a field, as $p(X)$ has a root in L .

“Only if” : Suppose $K \not\cong L$. To prove that K and L are linearly disjoint it is enough to show that so are \tilde{K} and L . Since L/\mathbb{Q} has no intermediate extensions and \tilde{K}/\mathbb{Q} is Galois, then L/\mathbb{Q} and \tilde{K}/\mathbb{Q} are linearly disjoint if and only if $L \cap \tilde{K} = \mathbb{Q}$, i.e., if and only if $L \not\subset \tilde{K}$.

Now, if $n \neq 6$, then every subgroup of S_n of index n is conjugate to $H := \{\sigma \in S_n : \sigma(1) = 1\}$ (see [39, Lemma 7.8.5]). Therefore, in that case $L \not\subset \tilde{K}$, otherwise, the subgroups $H_L := \text{Gal}(\tilde{K}/L)$ and $H_K := \text{Gal}(\tilde{K}/K)$ of index n in $\text{Gal}(\tilde{K}/\mathbb{Q}) \cong S_n$ would be conjugate, contrary to our assumption.

If $n = 6$ it turns out, see [39, Lemma 7.8.6], that for the only two conjugacy classes Δ_1 and Δ_2 of index 6 in S_6 , each pair of representatives $H_1 \in \Delta_1$ and $H_2 \in \Delta_2$ satisfy

$$36 = [S_6 : H_1 \cap H_2] = [S_6 : H_1][S_6 : H_2] = 6 \cdot 6.$$

If K and L are not linearly disjoint then, arguing as above, $L \subset \tilde{K}$ and since K and L are not isomorphic the groups H_L and H_K belong to the two different conjugacy classes of index 6 in $\text{Gal}(\tilde{K}/\mathbb{Q}) \cong S_6$. By the group theoretic fact above we conclude that $[KL : \mathbb{Q}] = [L : \mathbb{Q}][K : \mathbb{Q}]$ which contradicts that assumption that K and L are not linearly disjoint. \square

COROLLARY 3.3. *Let K, L be degree n , S_n number fields. Then, $K \not\cong L$ if and only if K/\mathbb{Q} and L/\mathbb{Q} are linearly disjoint*

Proof. Since one point stabilizers are maximal subgroups of S_n , see [12, I, Corollary 1.5A], it follows from Galois correspondence that K/\mathbb{Q} and L/\mathbb{Q} have no intermediate extensions. The result follows from Proposition 3.2. \square

4 CASIMIR ELEMENTS ASSOCIATED TO THE TRACE PAIRING OVER NUMBER FIELDS

In this section we focus our attention on questions about the integrality of Casimir values associated to the trace pairing. The strategy here is a game of local vs global, where we use in our favor that we know well the local behavior of the integral trace for tame extensions (see [35, §3].)

4.1 LOCAL CONSIDERATIONS

Here we use the standard decomposition of the trace as local traces to obtain information on Casimir pairings from their local counterparts.

4.1.1 p -INTEGRALITY OF CASIMIR PAIRINGS.

LEMMA 4.1. *Let K/\mathbb{Q}_p and L/\mathbb{Q}_p be finite field extensions with ramification indexes $e := e(K)$ and $e(L) := e'$, and different ideals \mathcal{D}_K (resp. \mathcal{D}_L). Suppose*

σ and τ are \mathbb{Q}_p -embeddings of K and L into $\overline{\mathbb{Q}_p}$, and $\psi : K \rightarrow L$ is a \mathbb{Q}_p -linear map taking O_K into O_L and \mathcal{D}_K^{-1} into \mathcal{D}_L^{-1} . If either both extensions are tamely ramified or $e = e' = 2$, then the element

$$ep^{1 - (\frac{1}{e} + \frac{1}{e'})} \langle \psi^*(\tau), \sigma \rangle_{\text{tr}_{K/\mathbb{Q}_p}}$$

is p -integral. If K is unramified, then $\langle \psi^*(\tau), \sigma \rangle_{\text{tr}_{K/\mathbb{Q}_p}}$ is p -integral.

Proof. Let F be the maximal unramified subextension of K . First let us assume that L/\mathbb{Q}_p and K/\mathbb{Q}_p are tamely ramified. Then by [35, Lemma 3.1], there exists a uniformizer π of O_K and a unit $\mu_F \in O_F^*$ such that $\pi^e = \mu_F \pi$ and $O_K = O_F[\pi]$. This implies that $\text{Tr}_{K/F}(\pi^m) = 0$ whenever $e \nmid m$, hence the dual F -basis with respect to the trace pairing $\text{tr}_{K/F}$ of the integral basis $\{1, \dots, \pi^i, \dots, \pi^{e-1}\}$ for O_K over O_F is $\{\frac{1}{e}, \dots, \frac{1}{e}\pi^{-i}, \dots, \frac{1}{e}\pi^{-(e-1)}\}$.

Let $\{w_1, \dots, w_f\}$ be any integral basis for O_F over \mathbb{Z}_p with dual basis $\{w_1^*, \dots, w_f^*\}$ with respect to the trace pairing $\text{tr}_{F/\mathbb{Q}_p}$. The set $\{w_i \pi^j : 1 \leq i \leq f, 0 \leq j < e\}$ is an integral basis for O_K over \mathbb{Z}_p with dual basis $\{\frac{1}{e} w_i^* \pi^{-j} : 1 \leq i \leq f, 0 \leq j < e\}$ with respect to trace pairing $\text{tr}_{K/\mathbb{Q}_p}$. Computing the pairing in such bases we get the sum

$$\langle \psi^*(\tau), \sigma \rangle_{\text{tr}_{K/\mathbb{Q}_p}} = \sum_{i,j} \tau \psi \left(\frac{w_i^*}{e \pi^j} \right) \sigma(w_i \pi^j). \tag{*}$$

When $j = 0$, the summand $\tau \psi \left(\frac{w_i^*}{e} \right) \sigma(w_i)$ is already p -integral. This is because ψ maps O_K into O_L and F/\mathbb{Q}_p is unramified, so each $w_i^* \in \mathcal{D}_F^{-1} = O_F$. While if $j \geq 1$, since by hypothesis $x := \psi \left(\frac{w_i^*}{e \pi^j} \right) \in \mathcal{D}_L^{-1}$ and L is tame, $v_p(x) \geq -\frac{e'-1}{e'}$. Thus

$$v_p(x \sigma(w_i \pi^j)) \geq -\frac{e'}{e'-1} + \frac{j}{e} \geq -\frac{e'-1}{e'} + \frac{1}{e} = -1 + \left(\frac{1}{e} + \frac{1}{e'} \right).$$

Hence $v_p(\langle \psi^*(\tau), \sigma \rangle) \geq -1 + \left(\frac{1}{e} + \frac{1}{e'} \right)$, as claimed.

If now $e = e' = 2$, and one of the extensions is wild, then $p = 2$ and O_K still has basis of the form $\{1, \pi\}$ over O_F such that $\pi^2 \in O_F$; but now π is either a uniformizer or a unit. In the former case the same argument above applies, with the adjustment that we need to multiply by $e = 2$ and replace the tameness of L by the inequality $v_2(\mathcal{D}_L) \leq \frac{e'-1}{e'} + v_2(e') = \frac{3}{2}$; in the latter case (*) directly shows that $2 \langle \psi^* \tau, \sigma \rangle$ is 2-integral.

Finally, if $K = F$ is unramified, then (*) holds with $e = 1$ and shows that $\langle \psi^*(\tau), \sigma \rangle$ is p -integral. □

PROPOSITION 4.2. *Let K, L be number fields of degree n , and let $\phi : (K, \text{tr}_{K/\mathbb{Q}}) \rightarrow (L, \text{tr}_{L/\mathbb{Q}})$ be an isometry such that $\phi(O_K) = O_L$. Consider the matrix U representing the \mathbb{Q} -linear map*

$$\phi^* : \text{Hom}_{\mathbb{Q}}(L, \overline{\mathbb{Q}}) \rightarrow \text{Hom}_{\mathbb{Q}}(K, \overline{\mathbb{Q}})$$

in the bases that consists of the \mathbb{Q} -embeddings of K and L , and let p be a rational prime ramified in K .

- (i) *If L/\mathbb{Q} and K/\mathbb{Q} are both tamely ramified at p , then $p^{1-\frac{2}{e}}U$ has p -integral entries. Where e is the largest ramification index of a prime in L or K dividing p .*
- (ii) *If $p = 2$, and L/\mathbb{Q} and K/\mathbb{Q} are both tamely ramified at 2 except for possibly some wild primes with ramification index 2, then $2U$ has 2-integral entries.*

Proof. Since we are only concerned with p -integrality, scalar extension compatibility (Proposition 2.10(ii)) allows us to replace U with the matrix U_p representing the $\overline{\mathbb{Q}}_p$ -linear map

$$(\phi \otimes 1)^* : \text{Hom}_{\mathbb{Q}_p}(L \otimes \mathbb{Q}_p, \overline{\mathbb{Q}}_p) \rightarrow \text{Hom}_{\mathbb{Q}_p}(K \otimes \mathbb{Q}_p, \overline{\mathbb{Q}}_p)$$

in the bases that consists of \mathbb{Q}_p -algebra morphisms of $K \otimes \mathbb{Q}_p$ and $L \otimes \mathbb{Q}_p$ into $\overline{\mathbb{Q}}_p$. As noted in Remark 2.15, the decomposition of \mathbb{Q}_p -étale algebras $K \otimes \mathbb{Q}_p = \prod_{v|p} K_v$ and $L \otimes \mathbb{Q}_p = \prod_{w|p} L_w$ induces a orthogonal decomposition of $\overline{\mathbb{Q}}_p$ -spaces

$$\text{Hom}_{\mathbb{Q}_p}(K \otimes \mathbb{Q}_p, \overline{\mathbb{Q}}_p) = \prod_{v|p} \text{Hom}_{\mathbb{Q}_p}(K_v, \overline{\mathbb{Q}}_p)$$

$$\text{Hom}_{\mathbb{Q}_p}(L \otimes \mathbb{Q}_p, \overline{\mathbb{Q}}_p) = \prod_{w|p} \text{Hom}_{\mathbb{Q}_p}(L_w, \overline{\mathbb{Q}}_p).$$

This shows that it is enough to consider the pairings $\langle (\phi \otimes 1)^*(\tau)|_{K_v}, \sigma \rangle_{\text{tr}_{K_v/\mathbb{Q}_p}}$ for any $\sigma \in \text{Hom}_{\mathbb{Q}_p}(K_v, \overline{\mathbb{Q}}_p)$ and $\tau \in \text{Hom}_{\mathbb{Q}_p}(L_w, \overline{\mathbb{Q}}_p)$: Since for $v' \neq v$, $\sigma|_{K_{v'}} = 0$ the corresponding entry of U_p is

$$\begin{aligned} \langle (\phi \otimes 1)^*(\tau), \sigma \rangle_{\text{tr}_{K \otimes \mathbb{Q}_p/\mathbb{Q}_p}} &= \sum_{v'|p} \langle (\phi \otimes 1)^*(\tau)|_{K_{v'}}, \sigma|_{K_{v'}} \rangle_{\text{tr}_{K_{v'}/\mathbb{Q}_p}} = \\ &= \langle (\phi \otimes 1)^*(\tau)|_{K_v}, \sigma \rangle_{\text{tr}_{K_v/\mathbb{Q}_p}}. \end{aligned}$$

Now since $\phi \otimes 1$ is an integral isometry it must map the dual of $O_K \otimes \mathbb{Z}_p = \prod_{v|p} O_{L_v}$ in $K \otimes \mathbb{Q}_p$ into the dual of $O_L \otimes \mathbb{Z}_p = \prod_{w|p} O_{L_w}$ in $L \otimes \mathbb{Q}_p$, i.e.,

$$\phi \otimes 1 : \prod_{v|p} \mathcal{D}_{K_v}^{-1} \rightarrow \prod_{w|p} \mathcal{D}_{L_w}^{-1}.$$

It follows that the local extensions K_v/\mathbb{Q}_p , L_w/\mathbb{Q}_p and the map

$$\psi : K_v \xrightarrow{(\phi \otimes 1)|_{K_v}} \prod_{w'|p} L_{w'} \rightarrow L_w$$

satisfy the hypothesis of Lemma 4.1. Since $(\phi \otimes 1)^*$ is an isometry and the same considerations apply to ϕ^{-1} , the symmetry $\langle (\phi \otimes 1)^*(\tau), \sigma \rangle = \langle \tau, (\phi^{-1} \otimes 1)^*(\sigma) \rangle$ allows to switch K_v and L_w if necessary and thus reduce to the case when $e(K_v) \leq e(L_w)$. Applying the estimates in Lemma 4.1 to

$$\langle (\phi \otimes 1)^*\tau|_{K_v}, \sigma|_{K_v} \rangle_{\text{tr}_{K_v/\mathbb{Q}_p}} = \langle \psi^*(\tau), \sigma \rangle_{\text{tr}_{K_v/\mathbb{Q}_p}}$$

finishes the proof in each case. □

4.2 INTEGRALITY OF CASIMIR ELEMENTS

Here we give some conditions to show how far the casimir pairing associated to an isometry of the trace pairings of a couple of number fields is integral.

DEFINITION 4.3. *Let K, L be number fields with fixed embeddings ι_K and ι_L into \mathbb{C} . Given $\phi : K \rightarrow L$ a \mathbb{Q} -linear map we denote by c_ϕ the Casimir element given by*

$$c_\phi := \langle \iota_K, \iota_L \phi \rangle_{\text{tr}_{K/\mathbb{Q}}}$$

and we define M_ϕ to be the least positive integer m such that mc_ϕ is an algebraic integer.

The radical of a non zero integer m , i.e., the product of all prime divisors of m , is denoted by $\text{rad}(m)$.

THEOREM 4.4. *Let K, L be number fields of degree n , and let $\phi : (K, \text{tr}_{K/\mathbb{Q}}) \rightarrow (L, \text{tr}_{L/\mathbb{Q}})$ be an isometry such that $\phi(O_K) = O_L$. Then,*

- (i) *The integer M_ϕ divides $d_s(K, L)$.*
- (ii) *If K is tamely ramified, then $M_\phi \mid \text{rad}(d_s(K, L))$.*
- (iii) *If K and L are only wildly ramified at 2 and $e_2(K) = 2 = e_2(L)$, then $M_\phi \mid \text{rad}(d_s(K, L))$.*

Proof. Recall that $e_p(K)$ is the maximal ramification index of a prime in K lying over p , and that for L the integer $d_f(K, L)$ is the product $\prod p^{v_p(d_K)}$ where p runs over the set of odd primes such that $e_p(K) = 2 = e_p(L)$. Also, $d_s(K, L) := d_K/d_f(K, L)$.

Let $(\alpha_1, \dots, \alpha_n)$ be an integral basis of K . Recall that

$$c_\phi = \sum_{i=1}^n \alpha_i^* \phi(\alpha_i).$$

Since each α_i^* belongs to the codifferent ideal \mathcal{D}_K^{-1} and $[\mathcal{D}_K^{-1} : O_K] = d_K$, we have that $d_K c_\phi$ is integral. Let p be a rational prime. If p is not ramified in K then c_ϕ is p integral since $d_K c_\phi$ is, in particular $d_s c_\phi$ is p integral. Suppose that p is ramified.

- If $p \mid d_f(K, L)$, then $e_p(K) = e_p(L) = 2$. Hence, by Proposition 4.2(i) c_ϕ is p -integral in particular $d_s c_\phi$ is as well.
- If $p \nmid d_f(K, L)$ then $d_s c_\phi = \frac{d_K c_\phi}{d_f}$ is p -integral since $d_K c_\phi$ is integral.

To prove the (ii) suppose that the ramification of p in K is tame. Let \mathfrak{p} be a prime of K lying over p . Then

$$v_{\mathfrak{p}}(p\mathcal{D}_K^{-1}) = e(\mathfrak{p}|p) - (e(\mathfrak{p}|p) - 1) = 1 \geq 0.$$

It follows that pc_ϕ is p -integral, and we see that $\text{rad}(d_s(K, L))c_\phi$ is integral. Likewise, if $p = 2$ and $e_2(K) = e_2(L) = 2$, then Proposition 4.2(ii) shows that $2c_\phi$ is 2-integral, proving part (iii). □

PROPOSITION 4.5. *Let K be a totally real number field of degree n , and let L be a number field such that the extensions L/\mathbb{Q} and K/\mathbb{Q} are linearly disjoint. Suppose that there is an isometry*

$$\phi : (O_K, \text{tr}_{K/\mathbb{Q}}) \rightarrow (O_L, \text{tr}_{L/\mathbb{Q}}).$$

Then,

$$\sqrt{n} \leq M_\phi.$$

Moreover, in such a case, the equality holds if and only if $M_\phi c_\phi = \pm 1$.

Proof. Since the isometry class of the integral trace determines the discriminant, the signature, and the degree of a field we have that L is totally real with the same discriminant and degree as K . Let m be a positive integer such that $\alpha = mc_\phi$ is an algebraic integer. Let $U = (c_{ij})$ where $c_{ij} := \langle \sigma_i, \tau_j \phi \rangle_{\text{tr}_{K/F}}$, and $\{\sigma_i\}$, $\{\tau_i\}$ are the sets of complex embeddings of K and L . Since U is orthogonal, thanks to Corollary 2.16 (iii), it follows that if we let $N = mU$ then $NN^t = m^2I$. Taking traces on both sides we obtain

$$\sum_{i,j} (mc_{ij})^2 = \text{tr}(NN^t) = m^2n$$

Since K/\mathbb{Q} and L/\mathbb{Q} are linearly disjoint, it follows from the construction of Corollary 3.1, that the $n \times n$ matrix built by the conjugates of α in KL , up to permutation of its entries, is equal to N . In particular,

$$\text{tr}_{KL/\mathbb{Q}}(\alpha^2) = \sum_{i,j} (mc_{ij})^2.$$

Since α is integral, and non-zero since U is a non-zero matrix, $N_{KL/\mathbb{Q}}(\alpha^2) \geq 1$. Thanks to the AM-GM inequality,

$$n^2 = [KL : \mathbb{Q}] \leq \text{tr}_{KL/\mathbb{Q}}(\alpha^2) = m^2n.$$

Hence $\sqrt{n} \leq m$ and the equality holds if and only if $\alpha^2 = 1$. □

5 ARITHMETIC CONSEQUENCES

Two useful and closely related quadratic invariants have been derived from the integral trace form. They are the trace-zero form and the shape of a number field (an invariant with a more geometric flavor). To define them, let K be a number field. The *trace-zero module* of K , denoted O_K^0 is

$$\{x \in O_K : \text{tr}_{K/\mathbb{Q}}(x) = 0\}.$$

This module and the quadratic form obtained by restricting the trace pairing to it (the trace-zero form) played a role in the study of the trace form for cubic fields in [28], and in the work of Ellenberg-Venkatesh on asymptotics of number fields [14]. We will use it here to cover a crucial case of our main theorem.

The shape of K , denoted $\text{Sh}(K)$, is the class of the lattice obtained by mapping

$$O_K^\perp = \{x \in \mathbb{Z} + [K : \mathbb{Q}]O_K : \text{tr}_{K/\mathbb{Q}}(x) = 0\}$$

under the Minkowski embedding up to rotation, reflections and multiplication by scalars. When K is totally real, we can equivalently describe $\text{Sh}(K)$ as the isometry class of the quadratic module obtained by restricting the trace pairing to O_K^\perp up to multiplication by \mathbb{R}^\times . This invariant, introduced for cubic fields in [43], has been recently studied for several different purposes (see [5, 17, 34]).

The following proposition shows that for totally fields number fields the integral trace form is the strongest of these three invariants.

LEMMA 5.1. *Let K, L be two totally real number fields and suppose $\phi : (O_K, \text{tr}_{K/\mathbb{Q}}) \rightarrow (O_L, \text{tr}_{L/\mathbb{Q}})$ is an isometry. Then, $\phi(1) = \pm 1$. In particular, the restriction of ϕ induces an isometry between the integral trace zero parts of K and L , and also between the shapes of K and L .*

Proof. Because the fields have isometric integral trace forms, they must have the same degree, say n . Now let $\alpha := \phi(1) \neq 0$. By hypothesis α^2 is a totally positive algebraic integer such that $\text{tr}_{L/\mathbb{Q}}(\alpha^2) = \text{tr}_{L/\mathbb{Q}}(\phi(1)^2) = \text{tr}_{K/\mathbb{Q}}(1^2) = n$. Hence, from the AM-GM inequality applied to the n conjugates of $\alpha^2 \in L$ over \mathbb{Q} , we find that the equality in

$$1 \leq N_{L/\mathbb{Q}}(\alpha^2)^{\frac{1}{n}} \leq \frac{\text{tr}_{L/\mathbb{Q}}(\alpha^2)}{n} = 1$$

implies $\alpha^2 \in \mathbb{Q}$ and thus $\alpha^2 = 1$. To prove the last claim, note that the isometry ϕ maps the elements in O_K orthogonal to 1 precisely to the elements in $\phi(O_K) = O_L$ orthogonal to $\phi(1) = \pm 1$, i.e., $\phi(O_K^0) = O_L^0$. Similarly, as $\phi(\mathbb{Z} + nO_K) = \phi(1)\mathbb{Z} + nO_L = \mathbb{Z} + nO_L$, we have $\phi(O_K^\perp) = O_L^\perp$. \square

We are now ready to prove the main results of the paper:

THEOREM 5.2. *Let K be a degree n totally real S_n number field and L be any S_n number field. Let $d_s(K, L)$ be the integer defined in 1.1.*

(i) *Suppose that $n \geq d_s(K, L)^2$. Then,*

$$(O_K, \text{tr}_{K/\mathbb{Q}}) \cong (O_L, \text{tr}_{L/\mathbb{Q}}) \text{ if and only if } K \cong L.$$

(ii) *If K does not have wild ramification, then the condition $n \geq d_s(K, L)^2$ can be improved to $n \geq \text{rad}(d_s(K, L))^2$.*

(iii) *If K and L do not have wild ramification except possibly for some primes with ramification index 2 lying over 2, then the condition $n \geq d_s(K, L)^2$ can also be improved to $n \geq \text{rad}(d_s(K, L))^2$.*

Proof. This is clear in degrees $n = 1, 2$, so let us suppose $n \geq 3$. We show the non-trivial implication. Since K and L have isometric integral traces, they share the same degree, signature and discriminant. If K and L are not isomorphic it follows from Corollary 3.3 that they are linearly disjoint. Let ϕ be an integral isometry, by Corollary 3.1, there is an orthogonal matrix U whose entries consist of the values $\theta(c_\phi)$ where θ runs over the set $\text{Hom}_{\mathbb{Q}\text{-alg}}(E, \mathbb{C})$. In particular $c_\phi \notin \mathbb{Q}$, otherwise, $\theta(c_\phi) = c_\phi$ for all θ which is a contradiction since the matrix U is invertible. Since c_ϕ is not rational it follows from Proposition 4.5 and from Theorem 4.4 that

$$\sqrt{n} < M_\phi \leq d_s(K, L).$$

The improvement on the upper bound of the above is obtained thanks to parts (ii) and (iii) of Theorem 4.4. \square

The above together with some previous results on integral traces imply that the integral trace is a complete invariant for real fields with at worst quadratic ramification.

THEOREM 5.3. *Let K, L be degree n , S_n number fields. Suppose that K is totally real and that the ramification index of every prime in K and L over \mathbb{Q} is at most 2. Then,*

$$(O_K, \text{tr}_{K/\mathbb{Q}}) \cong (O_L, \text{tr}_{L/\mathbb{Q}})$$

if and only if $K \cong L$.

Proof. This is trivial in degrees $n = 1, 2$, so let us suppose that $n \geq 3$. Thanks to Theorem 5.2, the result follows for $n \geq 4$. If $n = 3$, the restriction on the ramification indexes implies that the Galois closure \tilde{K} of K is unramified over $\mathbb{Q}(\sqrt{d_K})$, hence d_K must be fundamental (see [19]). Thus, using Lemma 5.1 we see that the case $n = 3$ follows from [28, Theorem 6.5]. \square

COROLLARY 5.4. *The integral trace form is a complete invariant for totally real number fields of fundamental discriminant.*

Proof. Since degree n number fields of fundamental discriminant are S_n number fields, see [25], and since having fundamental discriminant implies that ramification indices are at most 2, the result follows from Theorem 5.3. \square

THEOREM 5.5. *Let K and L be totally real number fields with square free different ideal. Then any isometry $\phi : (K, \text{tr}_{K/\mathbb{Q}}) \rightarrow (L, \text{tr}_{L/\mathbb{Q}})$ such that $\phi(O_K) = O_L$ is equal to plus or minus an isomorphism of fields $K \cong L$.*

Proof. From Corollary 2.16(iii) we know that $U = (c_{ij})$ is an orthogonal matrix, where $c_{ij} := \langle \sigma_i, \tau_j \circ \phi \rangle_{\text{tr}_{K/\mathbb{Q}}}$ and $\{\sigma_1, \dots, \sigma_n\}$ (resp. $\{\tau_1, \dots, \tau_n\}$) is the set of embeddings of K (resp. L). Moreover, because K and L are totally real, $U \in M_n(\mathbb{R})$ and thus $|c_{ij}| \leq 1$ for every $1 \leq i, j \leq n$. On the other hand, thanks to Proposition 2.10(ii) we have a natural action of the absolute Galois group $G_{\mathbb{Q}} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the entries of U . More specifically, if $\theta \in G_{\mathbb{Q}}$ then $\theta(c_{ij}) = c_{i'j'}$ where $\theta \circ \sigma_i = \sigma_{i'}$ and $\theta \circ \tau_j = \tau_{j'}$. In particular, every conjugate of an entry of U is again an entry of U and thus has absolute value bounded by 1, i.e., $|\theta(c_{ij})| \leq 1$ for every $\theta \in G_{\mathbb{Q}}$ and $1 \leq i, j \leq n$.

The fact that the different ideals of K and L are square free implies that $d_s(K, L) = 1$, thus each c_{ij} must be an algebraic integer by Theorem 4.4(i). From the above paragraph we deduce that each c_{ij} is either 0 or a real root of unity, or equivalently $c_{ij} \in \{0, \pm 1\}$. Furthermore, U being orthogonal implies that there is only one c_{ij} equal to ± 1 on each row and on each column of U . If i is the index such that $\langle \sigma_i, \tau_1 \circ \phi \rangle_{\text{tr}_{K/\mathbb{Q}}} \neq 0$, then by Corollary 2.14 we have

$$\tau_1 \circ \phi = \pm \sigma_i,$$

and thus $\mp \phi$ is multiplicative. \square

COROLLARY 5.6. *Let K be a totally real number field with square free different ideal. Then,*

$$\text{Aut}((O_K, \text{tr}_{K/\mathbb{Q}})) \cong \mathbb{Z}/2\mathbb{Z} \times \text{Aut}(K).$$

In particular, if $n > 2$ and K is a S_n number field then the automorphism group of the integral trace is “trivial”.

Proof. This follows from Theorem 5.5 by taking $K = L$. \square

COROLLARY 5.7. *Let K be a real number field of degree at least 3 and with square free discriminant. Then,*

$$\text{Aut}((O_K, \text{tr}_{K/\mathbb{Q}})) \cong \mathbb{Z}/2\mathbb{Z}.$$

Proof. Let \widetilde{K} be the Galois closure of K , let G be the Galois group of \widetilde{K}/\mathbb{Q} and let H be the subgroup of G corresponding to K . Since K has square free discriminant G is the full symmetric group and H is a one point stabilizer. In particular, H is self-normalizing in H and thus $\text{Aut}(K)$ is the trivial group. Since square free discriminant implies square free different ideal, the result follows from Theorem 5.6. \square

Since random lattices have “trivial” automorphism group, see for instance [7] and [1], this corollary is not completely unexpected. A derived question that comes to mind is whether or not maximal orders of real number fields with trivial automorphism group are random within the space of positive definite lattices.

Remark 5.8. Notice that Corollary 5.7 could be stated in a more general form: as long as H is self normalizing in G , and K has square free different ideal the same conclusion holds. This follows since for arbitrary K we have that $N_G(H)/H \cong \text{Aut}(K)$.

ACKNOWLEDGEMENTS

We thank Professor Schulze-Pillot for his help in providing us with references about a result on random lattices. We also thank Professor Eva Bayer-Fluckiger for very useful comments and remarks on a previous version of this paper. G. Mantilla-Soler’s work was supported in part by the Aalto Science Institute.

REFERENCES

- [1] E. Bannai. *Positive definite unimodular lattices with trivial automorphism groups*, Mem. Amer. Math. Soc. 85 (1990), no. 429.
- [2] E. Bayer-Fluckiger, H. W. Lenstra. *Forms in odd degree extensions and self-dual normal bases*, Amer. J. Math. 112 (1990), 359–373.
- [3] E. Bayer-Fluckiger, R. Parimala, J.P. Serre. *Hasse principle for G -trace forms*, Izv. Math. 77 (2013) no. 3, 437–460.
- [4] M. Bhargava. *The density of discriminants of quintic rings and fields*, Ann. of Math. (2) 172 (2004), no. 3, 1559–1591.
- [5] M. Bhargava, P. Harron. *The equidistribution of lattice shapes of ring of integers in cubic, quartic and quintic number fields*, Compos. Math. 156 (2016), no. 6, 1111–1120.
- [6] M. Bhargava, A. Shnidman. *On the number of cubic orders of bounded discriminant having automorphism group C_3 , and related problems*, Algebra Number Theory 8 (2014), no. 1, 53–88.
- [7] J. Biermann. *Gitter mit kleiner Automorphismegruppe in Geschlechtern von \mathbb{Z} -Gittern mit positiv-definiten quadratischer Form*. Ph.D. thesis, Georg-August-Universität Göttingen, 1981.
- [8] W. Bolaños. G. Mantilla-Soler. *The trace form over cyclic number fields*, Can. J. Math. 73 (2021), no. 4, 947–969.
- [9] P. E. Conner, R. Perlis. *A survey of trace forms of algebraic number fields*, Series in Pure Mathematics, 2. World Scientific, Singapore, 1984.

- [10] G. Cornelissen, B. de Smit, M. Marcolli, H. Smit. *Characterization of global fields by dirichlet L-series*, Res. Number Theory 5 (2019), 7, 15 pp.
- [11] B. Datskovsky and D.J. Wright. *Density of discriminants of cubic extensions*, J. Reine Angew. Math. 386 (1988), 116–138.
- [12] J.D. Dixon and B. Mortimer. *Permutation groups*, Graduate Texts in Mathematics, 163. Springer, New York, 1996.
- [13] J. Ellenberg, A. Venkatesh. *Statistics of number fields and function fields*, Proceedings ICM 2010, II, 383–402. Hindustan Book Agency, New Delhi, 2010.
- [14] J. Ellenberg, A. Venkatesh. *The number of extensions of a number field with fixed degree and bounded discriminant*, Ann. of Math. (2) 163 (2006), no. 2, 723–741.
- [15] B. Erez, J. Morales, R. Perlis. *Sur le genre de la form trace*, Seminaire de Théorie des Nombres de Bordeaux, 1987–1988, Talence, Exp. No. 18, 15 pp.
- [16] B. Hall. *Lie groups, Lie algebras, and representations: an elementary introduction*, 2nd ed. Graduate Texts in Mathematics, 222. Springer, Cham, 2015.
- [17] R. Harron. *The shapes of pure cubic fields*, Proc. Amer. Math. Soc. 145 (2017), no. 2, 509–524.
- [18] P.A. Harron. *The equidistribution of lattice shapes of rings of integers of cubic, quartic, and quintic number fields: an artist's rendering*, Ph.D. thesis, Princeton University, 2016. <http://arks.princeton.edu/ark:/88435/dsp01ws859j05g>.
- [19] H. Hasse. *Arithmetische Theorie der kubischen Zahlkörper auf klassenkörpertheoretischer Grundlage*, Math. Z. 31 (1930), 565–582.
- [20] J. E. Humphreys. *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, 9. Springer, New York-Berlin, 1972.
- [21] K. Iwasawa. *On the ring of valuation vectors*, Ann. of Math. (2) 57 (1953), 331–356.
- [22] J. Jones, D. Roberts. *A data base of number fields*, LMS J. Comput. Math. 17 (2014), no. 1, 595–618.
- [23] J. Klüners. *A counterexample to Malle's conjecture on the asymptotics of discriminants*, C.R. Math. Acad. Sci. Paris 340 (2005), no. 6, 411–414.
- [24] K. Komatsu. *On the Adele ring of algebraic number fields*, Kodai Math. Sem. Rep. 28 (1976), 78–84.

- [25] T. Kondo. *Algebraic number fields with the discriminant equal to that of a quadratic number field*, J. Math. Soc. Japan 47(1995), no. 1, 31–36.
- [26] D. Maurer. *The trace-form of an algebraic number field*, J. Number Theory 5 (1973), 379–384.
- [27] R. J. Lemke Oliver, F. Thorne. *Upper bounds on number fields of given degree and bounded discriminant*, Duke Math. J. 171 (2022), no. 15, 3077–3087.
- [28] G. Mantilla-Soler. *Integral trace forms associated to cubic extensions*, Algebra Number Theory 4(2010), no. 6, 681–699.
- [29] G. Mantilla-Soler. *On number fields with equivalent integral trace forms*, Int. J. Number Theory 8 (2012), no. 7, 1569–1580.
- [30] G. Mantilla-Soler. *A space of weight one modular forms attached to totally real cubic number fields*, Trends in number theory, 131–137. Contemp. Math., 649. Amer. Math. Soc., Providence, RI, 2015.
- [31] G. Mantilla-Soler. *The spinor genus of the integral trace*, Trans. Amer. Math. Soc. 369 (2017), 1547–1577.
- [32] G. Mantilla-Soler. *Weak arithmetic equivalence*, Canad. Math. Bull. 58 (2015), no. 1, 115–127.
- [33] G. Mantilla-Soler. *On the arithmetic determination of the trace*, J. Algebra 444 (2015), 272–283.
- [34] G. Mantilla-Soler, M. Monsurrò. *The shape of $\mathbb{Z}/\ell\mathbb{Z}$ number fields*, Ramanujan J. 39 (2016), no. 3, 451–463.
- [35] G. Mantilla-Soler. *The (α, β) -ramification invariants of a number field*, Math. Slovaca 71 (2021), no. 1, 251–263.
- [36] J. Neukirch. *Kennzeichnung der p -adischen und der endlichen algebraischen Zahlkörper*, Invent. Math. 6 (1969), 296–314.
- [37] R. Perlis. *On the equation $\zeta_K = \zeta_{K'}$* , J. Number Theory 9 (1977), 489–509.
- [38] D. Roberts. *Density of cubic field discriminants*, Math. Comp 70 (2001), no. 236, 1699–1705.
- [39] R. Schmidt. *Subgroup lattices of groups*, De Gruyter Expositions in Mathematics, 14. Walter de Gruyter, Berlin, 1994.
- [40] J. P. Serre. *Local fields*, Graduate Texts in Mathematics, 67. Springer, New York-Berlin, 1979.

- [41] J.-P. Serre. *L'invariant de Witt de la forme $Tr(x^2)$* , Comment. Math. Helv. 27 (1984), 651–676.
- [42] T. Taniguchi, F. Thorne. *The secondary term in the counting function for cubic fields*, Duke Math J. 162 (2013), no. 13, 2451–2508.
- [43] D. Terr. *The distribution of shapes of cubic orders*, Ph.D. thesis, University of California, Berkeley. ProQuest LLC, Ann Arbor, MI, 1997, 137 pp.
- [44] K. Ushida. *Isomorphisms of Galois groups*. J. Math. Soc. Japan 28 (1976), no. 4, 617–620.
- [45] J. Wang. *Secondary term of asymptotic distribution of $S_3 \times A$ extensions over \mathbb{Q}* . Preprint, 2017. <https://arxiv.org/abs/1710.10693>.

Guillermo Mantilla-Soler
Department of Mathematics
Universidad Nacional de Colombia
Medellín
Colombia
gmantelia@gmail.com

Carlos Rivera
Department of Mathematics
University of Washington
Seattle, WA
USA
caariv@uw.edu

