

FREE NONCOMMUTATIVE HEREDITARY KERNELS:
JORDAN DECOMPOSITION, ARVESON EXTENSION, KERNEL DOMINATION

JOSEPH A. BALL, GREGORY MARX, AND VICTOR VINNIKOV¹

Received: February 1, 2022

Revised: August 18, 2022

Communicated by Roland Speicher

ABSTRACT. We discuss (i) a quantized version of the Jordan decomposition theorem for a complex Borel measure on a compact Hausdorff space, namely, the more general problem of decomposing a general noncommutative kernel (a quantization of the standard notion of kernel function) as a linear combination of completely positive noncommutative kernels (a quantization of the standard notion of positive definite kernel). Other special cases of (i) include: the problem of decomposing a general operator-valued kernel function as a linear combination of positive kernels (not always possible), of decomposing a general bounded linear Hilbert-space operator as a linear combination of positive linear operators (always possible), of decomposing a completely bounded linear map from a C^* -algebra \mathcal{A} to an injective C^* -algebra $\mathcal{L}(\mathcal{Y})$ as a linear combination of completely positive maps from \mathcal{A} to $\mathcal{L}(\mathcal{Y})$ (always possible). We also discuss (ii) a noncommutative kernel generalization of the Arveson extension theorem (any completely positive map ϕ from an operator system \mathbb{S} to an injective C^* -algebra $\mathcal{L}(\mathcal{Y})$ can be extended to a completely positive map ϕ_e from a C^* -algebra containing \mathbb{S} to $\mathcal{L}(\mathcal{Y})$), and (iii) a noncommutative kernel version of a Positivstellensatz (i.e., finding a certificate to explain why one kernel is positive at points where another given kernel is strictly positive).

2020 Mathematics Subject Classification: 47B32, 47A60

Keywords and Phrases: Quantized functional analysis, noncommutative function, completely positive noncommutative kernel, completely positive map, bimodule maps

¹Research partially supported by the Israel Science Foundation (ISF) grant “Free Noncommutative Analysis and Geometry”

CONTENTS

1	INTRODUCTION	1986
2	PRELIMINARIES	1991
2.1	Taxonomy of noncommutative sets	1992
2.2	Noncommutative functions on a subset Ω	1992
2.3	Noncommutative kernels	1993
2.4	Encoding of nc-kernel property via module structure	1994
3	PROBLEM A: THE DECOMPOSABILITY PROBLEM FOR HERMITIAN NONCOMMUTATIVE KERNELS/HERMITIAN MAPS	2000
4	PROBLEM B: THE ARVESON EXTENSION PROBLEM FOR NONCOM- MUTATIVE POSITIVE KERNELS	2015
5	KERNEL DOMINANCE	2016
6	EXTENSIONS OF RESULTS FOR A FINITE SET OF POINTS TO A GEN- ERAL SET OF POINTS VIA THE KUROSH METHOD	2029
6.1	Decomposition of kernels	2029
6.2	Arveson extension theorem for kernels	2030
6.3	Kernel dominance theorem	2034

1 INTRODUCTION

The idea of *quantized functional analysis* came out of the attempt to understand intrinsically spaces of operators beyond the category of concrete C^* -algebras (a closed subalgebra of $\mathcal{L}(\mathcal{H})$ closed under taking of adjoints), e.g. subalgebras of $\mathcal{L}(\mathcal{H})$ (operator algebras), a linear subspace of $\mathcal{L}(\mathcal{H})$ (operator space), a unital subspace closed under taking adjoints (operator system). To get an intrinsic characterization of such objects, unlike the C^* -algebra case where the axioms for a C^* -algebra does the job, it was found that one needs to study not only the subspace as an abstract Banach space, but also a system of compatible norms on matrices over the subspace and one must study such objects up to completely isometric isomorphism (preserving not only all the structure on the primordial space \mathcal{X} but also on all the matricial spaces $\mathcal{X}^{n \times n}$ for all positive integers $n \in \mathbb{Z}_+$ (see [26, 12, 15] for systematic treatments). The same idea has now invaded function theory with impressive results (see [20]). The purpose of this contribution is to continue this line of research with a study of the quantized version of kernels begun in [8] and continued in [9].

By a classical kernel on a set Ω we mean a function K on the Cartesian product set $\Omega \times \Omega$ with values in some space, usually a linear space, e.g., the space of operators on some Hilbert space $\mathcal{L}(\mathcal{Y})$: $(x, y) \mapsto K(x, y)$. Let us say that the kernel is HERMITIAN if $k(x, y)^* = k(y, x)$. A POSITIVE KERNEL (in the sense

of Aronszajn [5]) is one for which

$$\sum_{i,j=1}^N \langle K(z_i, z_j)y_j, y_i \rangle \geq 0$$

for all $z_1, \dots, z_N \in \Omega$ and $y_1, \dots, y_n \in \mathcal{Y}$. An equivalent characterization of positive kernels is that they all have a KOLMOGOROV DECOMPOSITION, i.e., one can find operator-valued functions $H: \Omega \rightarrow \mathcal{L}(\mathcal{X}, \mathcal{Y})$ (where \mathcal{X} is some Hilbert state space) so that K has the factorization $K(z, w) = H(z)H(w)^*$.

A profound generalization of positive kernel is that of COMPLETELY POSITIVE KERNEL given by Barreto-Bhat-Liebscher-Skeide [10] (simplified here to the Hilbert space/Hilbert-space-operator context rather than Hilbert-module/ C^* -correspondence setting in [10]) whereby K is still a function on $\Omega \times \Omega$ but takes values in the space $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))$ of linear operators from a C^* -algebra \mathcal{A} to the space of Hilbert-space operators $\mathcal{L}(\mathcal{Y})$ and is required to satisfy the more elaborate positivity condition

$$\sum_{i,j=1}^N \langle K(z_i, z_j)(a_i^* a_j)y_j, y_i \rangle \geq 0 \tag{1.1}$$

for all choices of $z_1, \dots, z_N \in \Omega$, $a_1, \dots, a_N \in \mathcal{A}$, and $y_1, \dots, y_N \in \mathcal{Y}$. The main result in [10] concerning such kernels (CP BLS KERNELS for short) is that they are characterized by having the following more elaborate KOLMOGOROV DECOMPOSITION: *there exist a Hilbert state space \mathcal{X} , an operator-valued function $H: \Omega \rightarrow \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and a $*$ -representation $\pi: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{X})$ so that*

$$K(z, w)(a) = H(z)\pi(a)H(w)^*. \tag{1.2}$$

More generally one can consider BLS kernels without the positivity condition (1.1): we say that any function K from $\Omega \times \Omega \rightarrow \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))$ is a BLS-KERNEL. If it is the case that $K(z, w)(a)^* = K(w, z)(a^*)$, we will say that K is a HERMITIAN BLS-KERNEL.

Before discussing noncommutative kernels, we discuss noncommutative functions. A classical function f is defined on some point set Ω with values in some target space which we take to be a linear space \mathcal{V}_0 . If Ω has some additional structure (topological or analytic), then we speak about f being continuous or holomorphic. The idea of a free noncommutative function (NC FUNCTION for short) f is the quantization of a classical function: the domain Ω is partitioned up into levels $\Omega = \amalg_{n=1}^\infty \Omega_n$ where Ω_n consists of $n \times n$ matrices over the ambient vector space \mathcal{V} and similarly for the target space: $\mathcal{V}_0 = \amalg_{n=1}^\infty (\mathcal{V}_0)^{n \times n}$. The nc function f is required to be GRADED

$$Z \in \Omega_n \Rightarrow f(Z) \in (\mathcal{V}_0)^{n \times n} \tag{1.3}$$

and to RESPECT INTERTWINING CONDITIONS:

$$Z \in \Omega_n, \tilde{Z} \in \Omega_{\tilde{n}}, \alpha \in \mathbb{C}^{n \times \tilde{n}}, \alpha \tilde{Z} = Z \alpha \Rightarrow \alpha f(\tilde{Z}) = f(Z) \alpha. \tag{1.4}$$

It is shown in [20] how this algebraic condition along with some weak topological conditions implies holomorphic structure for a nc function.

A noncommutative kernel (NC KERNEL for short) K is the quantization of a BBLs kernel as we now explain. The domain Ω for a nc kernel K again is partitioned up into levels $\Omega = \coprod_{n=1}^{\infty} \Omega_n$ where Ω_n consists of those elements of Ω which are in $\mathcal{V}^{n \times n}$ (where \mathcal{V} is the ambient vector space for the domain). Then the target domain for K is partitioned up as

$$\coprod_{n,m=1}^{\infty} \mathcal{L}(\mathcal{A}^{n \times m}, \mathcal{L}(\mathcal{Y})^{n \times m}) =: \mathcal{L}(\mathcal{A}_{\text{nc}}, \mathcal{L}(\mathcal{Y})_{\text{nc}}) \text{ (for short)}$$

and K is GRADED in the following sense:

$$Z \in \Omega_n, W \in \Omega_m \Rightarrow K(Z, W) \in \mathcal{L}(\mathcal{A}^{n \times m}, \mathcal{L}(\mathcal{Y})^{n \times m}). \tag{1.5}$$

A completely positive noncommutative kernel (CP NC KERNEL for short) $K: \Omega \times \Omega \rightarrow \mathcal{L}(\mathcal{A}_{\text{nc}}, \mathcal{L}(\mathcal{Y})_{\text{nc}})$ is characterized by having a *quantized BBLs-Kolmogorov representation* as follows: *there exists a Hilbert space \mathcal{Y} , a nc function $H: \Omega \rightarrow \mathcal{L}(\mathcal{X}, \mathcal{Y})_{\text{nc}}$, a unital $*$ -representation $\pi: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{X})$ so that, for $Z \in \Omega_n$ and $W \in \Omega_m$ and $P = [P_{ij}] \in \mathcal{A}^{n \times m}$,*

$$K(Z, W)([P_{ij}]) = H(Z)[\pi(P_{ij})]H(W)^*. \tag{1.6}$$

Given such a representation (1.6), one can deduce RESPECTS INTERTWINING CONDITIONS for K from both the left and the right sides (see Section 2.3 below). These conditions make sense without the Kolmogorov representation (1.6) holding but just the gradedness condition (1.5) holding; these conditions then become the definition of a *noncommutative kernel* (NC KERNEL for short) (not necessarily completely positive). A nc kernel K is said to be a HERMITIAN NC KERNEL if in addition $K(Z, W)(P)^* = K(W, Z)(P^*)$. A cp nc kernel can also be defined via a quantization of condition (1.1) being imposed on a nc kernel: *a nc kernel K is cp if and only if*

$$\sum_{i,j=1}^N \langle K(Z_i, Z_j)(P_i P_j^*) y_j, y_i \rangle \geq 0 \tag{1.7}$$

for all $Z_i \in \Omega_{n_i}, P_i \in \mathcal{A}^{n_i \times 1}, y_i \in \mathcal{Y}^{n_i}, n_i \in \mathbb{N}$ for $i = 1, \dots, N, N \in \mathbb{N}$. Let us note various special cases of cp nc kernels: a cp nc kernel with $\Omega = \Omega_1$ amounts to a cp BBLs-kernel; if in addition $\mathcal{A} = \mathbb{C}$ and one identifies $K(Z, W)$ with $K(Z, W)(1)$, a cp BBLs-kernel becomes a Aronszajn positive kernel. For more complete details, see [8]. In another direction, a completely positive map φ between C^* -algebras \mathcal{A} and \mathcal{B} can be identified with a cp nc kernel $K: \Omega \times \Omega \rightarrow \mathcal{L}(\mathcal{A}_{\text{nc}}, \mathcal{B}_{\text{nc}})$, with Ω having the special form that Ω_n consists of a single $n \times n$ diagonal matrix $\begin{bmatrix} z_0 & & \\ & \ddots & \\ & & z_0 \end{bmatrix}$, where z_0 is the unique point at level-1 of $\Omega: \Omega_1 = \{z_0\}$. This connection generalizes to the case where the point set Ω consists of finitely many points, say N , located at various levels; for this case

a cp nc kernel corresponds to a cp map from \mathcal{A}^M to $\mathcal{L}(\mathcal{Y})^M$ which is also a $(\mathcal{S}, \mathcal{S}^*)$ -bimodule map, where \mathcal{S} is a certain subalgebra of \mathbb{C}^M determined by the point set Ω (see Section 2.4 below). In fact this connection is one of the main tools for our work to follow.

Here we discuss three issues for general nc kernels.

PROBLEM A. (Jordan decomposition for nc kernels.) *Given a nc kernel*

$$K: \Omega \times \Omega \rightarrow \mathcal{L}(\mathcal{A}_{\text{nc}}, \mathcal{L}(\mathcal{Y})_{\text{nc}})$$

on a set of nc points Ω , write K as a linear combination of four cp nc kernels K_1, K_2, K_3, K_4 :

$$K(Z, W)(P) = K_1(Z, W)(P) - K_2(Z, W)(P) + iK_3(Z, W)(P) - iK_4(Z, W)(P) \tag{1.8}$$

for all $Z \in \Omega_n, W \in \Omega_m, P \in \mathcal{A}^{n \times m}$.

We shall show that Problem A has a solution if (i) the set Ω is finite, (ii) each of the maps $\{K(Z, Z): \mathcal{A}^{nz \times nz} \rightarrow \mathcal{L}(\mathcal{Y}^{nz})\}$ is completely bounded, and (iii) the intertwining matrix subalgebra \mathcal{S} associated with Ω mentioned above is a C^* -algebra (i.e., Ω is *admissible* in the sense of Definition 3.7 to come). We also show that this result covers all the previously known particular cases where the conclusion is known to hold while also excluding various counterexamples where the result is known to fail (see the discussion around Examples 3.9 and 3.11 as well as Section 6.1). In particular, the special case $\mathcal{A} = C(X)$ (continuous functions on a compact Hausdorff space X) and $\mathcal{Y} = \mathbb{C}$ gives us the classical Jordan decomposition for a complex Borel measure on X (see Example 3.9 below).

A consequence of the RESPECTS INTERTWINING CONDITIONS property for a nc kernel is the RESPECTS DIRECT SUMS property (see Section 2.3 below). If Ω is closed under the taking of direct sums, then the positivity condition (1.7) can be reformulated more succinctly simply as

$$Z \in \Omega_N, P \succeq 0 \text{ in } \mathcal{A}^{N \times N} \Rightarrow K(Z, Z)(P) \succeq 0 \text{ in } \mathcal{L}(\mathcal{Y}^N). \tag{1.9}$$

Furthermore one can cut down on the number of points Z one needs to consider if one insists that the map

$$P \in \mathcal{A}^{N \times N} \mapsto K(Z, Z)(P) \in \mathcal{L}(\mathcal{Y}^N)$$

be cp from $\mathcal{A}^{N \times N}$ to $\mathcal{L}(\mathcal{Y}^N)$. The advantage of this formulation is that it makes sense when the C^* -algebra \mathcal{A} is replaced by a unital selfadjoint linear subspace \mathbb{S} of some C^* -algebra \mathcal{A} , or more abstractly, by an *operator system* \mathfrak{S} (see e.g. [26] for additional background). This leads to the second problem to be discussed in this paper:

PROBLEM B. (Arveson extension theorem for completely positive noncommutative kernels.) *Given a cp nc kernel \mathbb{K} on a nc point set Ω with values mapping*

\mathbb{S}_{nc} into $\mathcal{L}(\mathcal{Y})_{\text{nc}}$ where \mathbb{S} is an operator system (a $*$ -closed unital linear subspace of a C^* -algebra \mathcal{A}), find an extension of K to the C^* -algebra \mathcal{A}

$$K: \Omega \times \Omega \rightarrow \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))$$

which is also a cp nc kernel.

Let us mention that for the case where the point set Ω is a single point at level 1 so the kernel amounts to a cp map from \mathbb{S} to $\mathcal{L}(\mathcal{Y})$, a result of Arveson (the Arveson extension theorem [7, 26]) resolves Problem B in the affirmative. We show how our technique of converting kernels to cp maps which are also bimodule maps with respect to a certain matrix algebra \mathcal{S} associated with the point set Ω leads to a solution of Problem B for the case where the point set is finite. Combining this result with a general procedure of Kurosh (see [21] as well as [6]) leads to a solution of Problem B for the general case via a reduction of the infinite-point case to the finite-point case.

Our next problem has the flavor of a Positivstellensatz for free nc kernels, i.e., the problem of characterizing the form that a free nc kernel must have if it is constrained to be positive at those points where another given kernel is positive. We begin with a simplified version of the problem; to actually solve the problem there are some additional hypotheses which must be incorporated. We note that the notion of FULL NC SUBSET used in the statement of Problem C is introduced in the opening paragraph of Section 2.1 below.

PROBLEM C. (Kernel dominance problem for noncommutative kernels.) *Given Hilbert spaces \mathcal{E} and \mathcal{G} , a full nc subset Ξ of \mathcal{V}_{nc} , and a Hermitian nc kernel*

$$\mathfrak{Q}: \Xi \times \Xi \rightarrow \mathcal{L}(\mathbb{C}_{\text{nc}}, \mathcal{L}(\mathcal{S})_{\text{nc}}),$$

let $\mathbb{P}_{\mathfrak{Q}}$ be the strict positivity domain for \mathfrak{Q} as defined by

$$\mathbb{P}_{\mathfrak{Q}} = \{Z \in \Xi: \mathfrak{Q}(Z, Z)(I_n) \succ 0\}.$$

Suppose that Ω is a subset of $\mathbb{P}_{\mathfrak{Q}}$ and that \mathfrak{S} is a Hermitian nc kernel defined on Ω

$$\mathfrak{S}: \Omega \times \Omega \rightarrow \mathcal{L}(\mathbb{C}_{\text{nc}}, \mathcal{L}(\mathcal{Y})_{\text{nc}})$$

which is positive semidefinite on Ω :

$$\mathfrak{S}(Z, Z)(1_{\mathcal{A}}) \succeq 0 \text{ for all } Z \in \Omega.$$

Then we seek to find two completely positive nc kernels on Ω

$$\Gamma_1: \Omega \times \Omega \rightarrow \mathcal{L}(\mathcal{L}(\mathcal{S})_{\text{nc}}, \mathbb{C}_{\text{nc}}), \quad \Gamma_2: \Omega \times \Omega \rightarrow \mathcal{L}(\mathbb{C}_{\text{nc}}, \mathcal{L}(\mathcal{Y})_{\text{nc}})$$

so that, for all $Z \in \Omega_n$, $W \in \Omega_m$, $P \in \mathbb{C}^{n \times m}$ we have the kernel decomposition

$$\mathfrak{S}(Z, W)(P) = \Gamma_1(Z, W)(\mathfrak{Q}(Z, W)(P)) + \Gamma_2(Z, W)(P). \quad (1.10)$$

Note that the representation (1.10) can be viewed as a certificate which explains why \mathfrak{S} is dominated by the kernel \mathfrak{Q} in the sense mentioned in the statement, i.e., it is immediate from the representation (1.10) that $\mathfrak{Q}(Z, Z)(I) \succ 0$ on Ω then leads to $\mathfrak{S}(Q, Q)(I) \succeq 0$ on Ω .

This formulation does not have an affirmative solution in general without imposing some additional hypotheses. The first additional hypothesis is that the kernel \mathfrak{S} should actually be defined on a somewhat larger set Ω' containing Ω , namely $\Omega' = [\Omega]_{\text{full}} \cap \mathbb{P}_{\Omega}$ where $[\Omega]_{\text{full}}$ is the *full envelope* of Ω (see again Section 2.1 below for the precise definition). In addition we need to assume that each of the kernels \mathfrak{Q} and \mathfrak{S} has the property that its restriction to a finite subset of its domain (Ξ or Ω' respectively) is decomposable. In particular this is true if \mathfrak{S} and \mathfrak{Q} are assumed to be decomposable as kernels on their respective domains (Ξ or Ω' respectively) at the start. The precise result is given as Theorem 5.1 below.

An interesting special case is the case where we take $\Omega = \mathbb{P}_{\Omega}$. Then $\Omega' = \mathbb{P}_{\Omega}$ and we assume that \mathfrak{S} as well as \mathfrak{Q} are free nc kernels defined on all of Ξ and one asks that the decomposition (1.10) holds on all of Ξ . We are not able to get such a result in general; however in case one assumes that $\Xi = (\mathbb{C}^d)_{\text{nc}}$ and \mathfrak{Q} and \mathfrak{S} are nc polynomial kernels or (more generally) nc rational kernels, then the result does hold if one imposes an additional Archimedean hypothesis (see [17, 25]); a result of this type is called a *noncommutative Positivstellensatz*. We refer to Remark 6.6 below for additional discussion.

The paper is organized as follows. After the present Introduction, we present in Section 2 preliminaries on nc functions and nc kernels and our key tool giving the connection between a completely positive map from a C^* -algebra \mathcal{A}^M to a C^* -algebra of the form $\mathcal{L}(\mathcal{Y}^M)$ which is also a $(\mathcal{S}, \mathcal{S}^*)$ -bimodule map with respect to the action of a certain subalgebra \mathcal{S} of $\mathbb{C}^{M \times M}$ on the one hand, and cp nc kernels defined on a finite set of nc points Ω on the other. Sections 3, 4, and 5 present our results on Problems A, B, C respectively, for the special case where the point set Ω consists of only finitely many points. The final section 6 shows how to extend the finite-point results from Sections 4 and 5 to the case of a general nc point set Ω by using the abstract results of Kurosh concerning nonemptiness of inverse limits for a inverse spectrum of compacta. For the case of the Jordan-decomposition problem for nc kernels (Problem A) treated in Section 3, we also show in Section 6 where the Kurosh formalism breaks down, as well as examples showing that such decomposability results are not possible in general. A final remark (Remark 6.6) makes precise the connections of our kernel-domination result with some Positivstellensätze for the noncommutative setting which have appeared relatively recently in the literature.

2 PRELIMINARIES

In this section we review some preliminaries from [9, Section 2] concerning noncommutative (nc) functions and completely positive noncommutative (cp nc)

kernels which will be needed in the sequel.

2.1 TAXONOMY OF NONCOMMUTATIVE SETS

We let \mathcal{V} be a vector space, $\mathcal{V}_{\text{nc}} = \coprod_{n=1}^{\infty} \mathcal{V}^{n \times n}$ the set of all square matrices over \mathcal{V} of arbitrary size. Note that $\mathcal{V}^{n \times m}$ is a left module over $\mathbb{C}^{n \times n}$ and a right module over $\mathbb{C}^{m \times m}$ by making use of ordinary matrix multiplication combined with the bimodular structure of \mathcal{V} over \mathbb{C} as a vector space over \mathbb{C} . We say that a subset Ξ of \mathcal{V}_{nc} is a NC SUBSET if Ξ is closed under direct sums. Following Definition 2.4 from [9], we say that the nc subset Ξ is a FULL nc subset of \mathcal{V}_{nc} if, in addition to being closed under direct sums, Ξ is INVARIANT UNDER LEFT INJECTIVE INTERTWININGS, i.e., $Z \in \Xi_n$, $\tilde{Z} \in \mathcal{V}^{m \times m}$ such that $\mathcal{I}\tilde{Z} = Z\mathcal{I}$ for some injective $\mathcal{I} \in \mathbb{C}^{n \times m}$ (so $n \geq m$) implies that $\tilde{Z} \in \Xi_m$. An equivalent statement of this latter property is that Ξ is CLOSED UNDER RESTRICTION TO INVARIANT SUBSPACES: *whenever there is an invertible $\alpha \in \mathbb{C}^{n \times n}$ and a $Z \in \Xi$ of size $n \times n$ such that $\alpha^{-1}Z\alpha = \begin{bmatrix} \tilde{Z} & Z_{12} \\ 0 & Z_{22} \end{bmatrix}$ with \tilde{Z} of size $m \times m$, then \tilde{Z} is in Ξ .* The case $n = m$ is not excluded: this special case of the condition gives us that any full nc subset Ξ is also invariant under similarities.

Given an arbitrary subset Ω of \mathcal{V}_{nc} , we have the following distinct notions of ENVELOPES:

- The NC ENVELOPE $[\Omega]_{\text{nc}}$ is the smallest superset of Ω in \mathcal{V}_{nc} which is closed under direct sums.
- The NC SIMILARITY ENVELOPE $[\Omega]_{\text{sim}}$ is the smallest superset of Ω in \mathcal{V}_{nc} which is closed under direct sums and similarity transforms.
- The FULL NC ENVELOPE $[\Omega]_{\text{full}}$ is the smallest superset of Ω in \mathcal{V}_{nc} which is a full nc subset as defined above.

Note that each of the three properties *nc set/similarity-invariant nc set/full nc set* is closed under intersections, so the each of the notions *smallest nc superset/smallest similarity-invariant nc superset/smallest full nc superset* containing a given subset is well defined. For brevity we now focus on notions related to full nc subsets as this is all that will be needed in the sequel; we leave the parallel notions concerning nc sets and similarity-invariant nc sets to the interested reader.

If \mathbb{D} is another subset of \mathcal{V}_{nc} and Ω' is a subset of \mathbb{D} , we say that Ω' is a \mathbb{D} -RELATIVE FULL NC SET if the full nc envelope $[\Omega']_{\text{full}}$ intersected with \mathbb{D} is again just Ω' : $\Omega' = [\Omega']_{\text{full}} \cap \mathbb{D}$. If Ω is any subset of \mathbb{D} , then the smallest \mathbb{D} -relative full nc set containing Ω is $\Omega' = \mathbb{D} \cap [\Omega]_{\text{full}}$. In this case we say that the \mathbb{D} -relative full nc set Ω' IS GENERATED BY Ω . We shall be particularly interested in the case when Ω is a finite subset of \mathbb{D} .

2.2 NONCOMMUTATIVE FUNCTIONS ON A SUBSET Ω

Let Ω be a subset of \mathcal{V}_{nc} and let \mathcal{V}_0 be another vector space. We say that $f: \Omega \rightarrow \mathcal{V}_{0,\text{nc}}$ is a (\mathcal{V}_0 -VALUED) NC FUNCTION if

- f is GRADED: $f(Z) \in \mathcal{V}_0^{n \times n}$ for $Z \in \Omega_n$,
- f RESPECTS INTERTWININGS: If $\alpha \in \mathbb{C}^{n \times m}$, $Z \in \Omega_n$, $\tilde{Z} \in \Omega_m$ are such that $Z\alpha = \alpha\tilde{Z}$, then $f(Z)\alpha = \alpha f(\tilde{Z})$.

It is known (see [20]) that the "respects intertwining" condition can be replaced by the pair of conditions:

- f RESPECTS DIRECT SUMS: If Z , \tilde{Z} , and $\begin{bmatrix} Z & 0 \\ 0 & \tilde{Z} \end{bmatrix}$ are all in Ω , then $f\left(\begin{bmatrix} Z & 0 \\ 0 & \tilde{Z} \end{bmatrix}\right) = \begin{bmatrix} f(Z) & 0 \\ 0 & f(\tilde{Z}) \end{bmatrix}$, and
- f RESPECTS SIMILARITIES: If Z, \tilde{Z} are in Ω_n , and α is invertible in $\mathbb{C}^{n \times n}$ with $\tilde{Z} = \alpha Z \alpha^{-1}$, then $f(\tilde{Z}) = \alpha f(Z) \alpha^{-1}$.

Note that in these definitions we do not insist that Ω have additional structure as a subset of \mathcal{V}_{nc} (e.g., being a nc subset, a nc similarity-invariant subset, or a full nc subset).

2.3 NONCOMMUTATIVE KERNELS

Suppose \mathcal{A} and \mathcal{B} are C^* -algebras and that K is a function from $\mathcal{V}_{nc} \times \mathcal{V}_{nc}$ into

$$\mathcal{L}(\mathcal{A}, \mathcal{B})_{nc} := \coprod_{n,m \geq 1} \mathcal{L}(\mathcal{A}^{n \times m}, \mathcal{B}^{n \times m}).$$

We say that K is a NC KERNEL if

- K is GRADED: $Z \in \Omega_n, W \in \Omega_m \Rightarrow K(Z, W) \in \mathcal{L}(\mathcal{A}^{n \times m}, \mathcal{B}^{n \times m})$,
- K RESPECTS INTERTWININGS:

$$\begin{aligned} Z \in \Omega_n, \tilde{Z} \in \Omega_{\tilde{n}}, \alpha \in \mathbb{C}^{\tilde{n} \times n} \text{ such that } \alpha Z &= \tilde{Z} \alpha, \\ W \in \Omega_m, \tilde{W} \in \Omega_{\tilde{m}}, \beta \in \mathbb{C}^{\tilde{m} \times m} \text{ such that } \beta W &= \tilde{W} \beta, \\ P \in \mathcal{A}^{n \times m} \Rightarrow \alpha K(Z, W)(P) \beta^* &= K(\tilde{Z}, \tilde{W})(\alpha P \beta^*). \end{aligned}$$

As explained in [9] and in [20], the "respects intertwining" condition can be replaced by a pair of conditions: "respects direct sums" and "respects similarities":

- K RESPECTS DIRECT SUMS: for $Z \in \Omega_n$ and $\tilde{Z} \in \Omega_{\tilde{n}}$ such that $\begin{bmatrix} Z & 0 \\ 0 & \tilde{Z} \end{bmatrix} \in \Omega_{n+\tilde{m}}$, $W \in \Omega_m$ and $\tilde{W} \in \Omega_{\tilde{m}}$ such that $\begin{bmatrix} W & 0 \\ 0 & \tilde{W} \end{bmatrix} \in \Omega_{m+\tilde{m}}$, and $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in \mathcal{A}^{(n+m) \times (\tilde{n}+\tilde{m})}$, it then holds that

$$K\left(\begin{bmatrix} Z & 0 \\ 0 & \tilde{Z} \end{bmatrix}, \begin{bmatrix} W & 0 \\ 0 & \tilde{W} \end{bmatrix}\right)\left(\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\right) = \begin{bmatrix} K(Z, W)(P_{11}) & K(Z, \tilde{W})(P_{12}) \\ K(\tilde{Z}, W)(P_{21}) & K(\tilde{Z}, \tilde{W})(P_{22}) \end{bmatrix},$$

- K RESPECTS SIMILARITIES:

$$\begin{aligned} Z, \tilde{Z} &\in \Omega_n, \alpha \in \mathbb{C}^{n \times n} \text{ invertible with } \tilde{Z} = \alpha Z \alpha^{-1}, \\ W, \tilde{W} &\in \Omega_m, \beta \in \mathbb{C}^{m \times m} \text{ invertible with } \tilde{W} = \beta W \beta^{-1}, \\ P &\in \mathcal{A}^{n \times m} \Rightarrow K(\tilde{Z}, \tilde{W})(P) = \alpha K(Z, W)(\alpha^{-1} P \beta^{-1*}) \beta^*. \end{aligned}$$

Let us say that the nc kernel K is COMPLETELY POSITIVE (CP) if, for each $Z_1, \dots, Z_K \in \Omega$, with say $Z_i \in \Omega_{n_i}$, the map $[a_{ij}] \mapsto [K(Z_i, Z_j)(a_{ij})]$ is a completely positive map between the C^* -algebras $\mathcal{A}^{N \times N}$ and $\mathcal{B}^{N \times N}$, where $N = \sum_{i=1}^K n_i$, or equivalently and more explicitly, for all $a_i \in \mathcal{A}^{n_i}$ and $b_i \in \mathcal{B}^{n_i}$, it is the case that

$$\sum_{i,j=1}^K b_i^* K(Z_i, Z_j)(a_i^* a_j) b_j \succeq 0. \tag{2.1}$$

In case Ω is a nc set, one can use the "respects direct sums" property to express this last condition more succinctly as (see [9]): *for each $Z \in \Omega_n$, $K(Z, Z)$ is a completely positive map between the C^* -algebras $\mathcal{A}^{n \times n}$ and $\mathcal{B}^{n \times n}$.* We shall be primarily interested in the case where $K: \Omega \times \Omega \rightarrow \mathcal{L}(\mathcal{A}, \mathcal{B})_{\text{nc}}$ with the C^* -algebra \mathcal{B} assumed to have the form $\mathcal{L}(\mathcal{Y})$ for some Hilbert space \mathcal{Y} ; the theory can be pushed more more generally by getting into a Hilbert C^* -module rather than Hilbert space setting (see [10, 23]), but for our purposes here the Hilbert space setting is sufficient. With this assumption in place, we rewrite (2.1) as: for all $Z_i \in \Omega_{n_i}$, $a_i \in \mathcal{A}^{n_i}$ and $y_i \in \mathcal{Y}^{n_i}$ for $1 \leq i \leq K$ we have

$$\sum_{i,j=1}^K \langle K(Z_i, Z_j)(a_i^* a_j) y_j, y_i \rangle_{\mathcal{Y}^{n_i}} \geq 0. \tag{2.2}$$

it is then known (see [8]) that K has a KOLMOGOROV DECOMPOSITION, i.e., there is a Hilbert space \mathcal{X} , and unital $*$ -representation $\pi: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{X})$ and a nc function $H: \Omega \rightarrow \mathcal{L}(\mathcal{X}, \mathcal{Y})_{\text{nc}}$ so that

$$K(Z, W)(P) = H(Z) ((\text{id}_{\mathbb{C}^{n \times m}} \otimes \pi)(P)) H(W)^* \tag{2.3}$$

for $Z \in \Omega_n$, $W \in \Omega_m$ and $P \in \mathcal{A}^{n \times m}$.

The following is a useful fact concerning cp nc kernels.

LEMMA 2.1. *A cp nc kernel $K: \Omega \times \Omega \rightarrow \mathcal{L}(\mathcal{A}, \mathcal{B})_{\text{nc}}$ is equal to the zero kernel if and only if $K(Z, Z)(1_{\mathcal{A}^{n \times n}}) = 0$ for any $Z \in \Omega_n$, $n = \mathbb{Z}_+$.*

Proof. The result follows from the fact that for any positive map ϕ between unital C^* -algebras $\|\phi\| = \|\phi(1)\|$ (see [26]). □

2.4 ENCODING OF NC-KERNEL PROPERTY VIA MODULE STRUCTURE

Suppose that we are given a point set $\Omega = \{Z_1, \dots, Z_N\}$ contained in an ambient universal nc set \mathcal{V}_{nc} together with a function $K: \Omega \times \Omega \rightarrow \mathcal{L}(\mathcal{A}_{\text{nc}}, \mathcal{L}(\mathcal{Y}))_{\text{nc}}$.

Set $Z^{(0)}$ equal to the direct sum of all the points in Ω :

$$Z^{(0)} = \bigoplus_{i=1}^N Z_i.$$

If each point Z_i say has size $n_i \times n_i$, we then see that $Z^{(0)}$ has size $N_0 \times N_0$ where $N_0 = \sum_{i=1}^N n_i$. It is useful to observe that $N_0 \times N_0$ matrices (be they over \mathbb{C} , \mathcal{A} or $\mathcal{L}(\mathcal{Y})$) can be organized as block $N \times N$ matrices with block (i, j) having size $n_i \times n_j$. We associate with the nc kernel K the linear map ϕ_K from $\mathbb{A} := \mathcal{A}^{N_0 \times N_0}$ to $\mathbb{L} := \mathcal{L}(\mathcal{Y})^{N_0 \times N_0}$ given by

$$\phi_K : P \mapsto K(Z^{(0)}, Z^{(0)})(P) := [K(Z_i, Z_j)(P_{ij})]_{1 \leq i, j \leq N} \tag{2.4}$$

if $P = [P_{ij}]_{1 \leq i, j \leq N_0}$ with $P_{ij} \in \mathcal{A}^{n_i \times n_j}$. We would like to understand how one can encode the property of K being a nc kernel (or of being a cp nc kernel) as a property of the map ϕ_K between C^* -algebras \mathbb{A} and \mathbb{L} . Toward this end, let \mathcal{S} denote the subalgebra of $N_0 \times N_0$ matrices consisting of $\alpha \in \mathbb{C}^{N_0 \times N_0}$ which intertwine the point $Z^{(0)}$ with itself:

$$\mathcal{S} = \{\alpha \in \mathbb{C}^{N_0 \times N_0} : \alpha Z^{(0)} = Z^{(0)} \alpha\}. \tag{2.5}$$

Note that \mathcal{S} is a subalgebra but not necessarily a $*$ -subalgebra of $\mathbb{C}^{N_0 \times N_0}$. Given a mapping $\phi : \mathbb{A} \rightarrow \mathbb{L}$, we say that ϕ is a $(\mathcal{S}, \mathcal{S}^*)$ -BIMODULE MAP if

$$\phi(\alpha \cdot P \cdot \beta^*) = \alpha \cdot \phi(P) \cdot \beta^* \tag{2.6}$$

for all $\alpha, \beta \in \mathcal{S}$. Note that the left-hand side in (2.6) uses the $\mathbb{C}^{N_0 \times N_0}$ -module action on \mathbb{A} while the right-hand side uses the $\mathbb{C}^{N_0 \times N_0}$ -module action on \mathbb{L} . As we are identifying $\mathcal{L}(\mathcal{Y})^{N_0 \times N_0}$ with $\mathcal{L}(\mathcal{Y}^{N_0})$, it will be convenient to introduce the notation L_α and L_β^* for the representations $\alpha \mapsto L_\alpha$, $\beta^* \mapsto L_{\beta^*}$ of \mathcal{S} and of \mathcal{S}^* respectively on \mathcal{Y}^{N_0} given by

$$L_\alpha y = \alpha \cdot y, \quad L_{\beta^*} y = \beta^* \cdot y \text{ for } y \in \mathcal{Y}^{N_0}.$$

With these conventions in place we rewrite (2.6) as

$$\phi(\alpha \cdot P \cdot \beta^*) = L_\alpha \phi(P) L_{\beta^*}. \tag{2.7}$$

Then we have the following result.

PROPOSITION 2.2. (See [23].) *Suppose that the map K from $\Omega \times \Omega$ to $\mathcal{L}(\mathcal{A}_{\text{nc}}, \mathcal{L}(\mathcal{Y})_{\text{nc}})$ and ϕ_K from \mathbb{A} to \mathbb{L} are related as in (2.4). Then:*

- (1) *K is a nc kernel on Ω if and only if ϕ_K is a $(\mathcal{S}, \mathcal{S}^*)$ -bimodule map. Conversely, if $\phi : \mathbb{A}^{N_0 \times N_0} \rightarrow \mathbb{L}^{N_0 \times N_0}$ is a $(\mathcal{S}, \mathcal{S}^*)$ -bimodule map, there is a uniquely determined nc kernel K so that ϕ has the form $\phi = \phi_K$.*
- (2) *K is a cp nc kernel on Ω if and only if ϕ_K is a $(\mathcal{S}, \mathcal{S}^*)$ -bimodule map which is also cp. Conversely, if $\phi : \mathbb{A} \rightarrow \mathbb{L}$ is a cp $(\mathcal{S}, \mathcal{S}^*)$ -bimodule map, there is a uniquely determined cp nc kernel K on Ω so that $\phi = \phi_K$.*

Proof. Suppose first that K is a nc kernel on Ω . Define $\tilde{K}(Z^{(0)}, Z^{(0)})$ to be the map ϕ_K as in (2.4) and also define

$$\begin{aligned} \tilde{K}(Z^{(0)}, Z_j)(P) &= \begin{bmatrix} K(Z_1, Z_j)(P_1) \\ \vdots \\ K(Z_N, Z_j)(P_N) \end{bmatrix} \text{ if } P = \begin{bmatrix} P_1 \\ \vdots \\ P_N \end{bmatrix}, \\ \tilde{K}(Z_i, Z^{(0)})(P) &= [K(Z_i, Z_1)(P_1) \ \cdots \ K(Z_i, Z_N)(P_N)] \\ &\text{ if } P = [P_1 \ \cdots \ P_N], \\ \tilde{K}(Z, W)(P) &= K(Z, W)(P) \text{ if } Z \in \Omega_n, W \in \Omega_m, P \in \mathcal{A}^{n \times m}. \end{aligned}$$

Then by construction \tilde{K} as a function on $\tilde{\Omega} \times \tilde{\Omega}$ is an extension of the function K defined on $\Omega \times \Omega$. It is clear from the construction that \tilde{K} is a graded kernel. Our next goal is to show that ϕ is a left \mathcal{S} -module map if and only if K satisfies the left intertwining condition:

$$\begin{aligned} \alpha \in \mathbb{C}^{i_0 \times k_0}, \alpha Z_{k_0} &= Z_{i_0} \alpha, P \in \mathcal{A}^{n_{k_0} \times n_j} \\ \Rightarrow L_\alpha K(Z_{k_0}, Z_j)(P) &= K(Z_{i_0}, Z_j)(\alpha P) \text{ for } 1 \leq i_0, k_0, j \leq N. \end{aligned} \tag{2.8}$$

Toward this end, let us assume first that ϕ is a left \mathcal{S} -module map. Let us write a $N_0 \times N_0$ complex matrix α as a block $N \times N$ matrix $\alpha = [\alpha_{ij}]_{i,j=1,\dots,N}$ where the entry α_{ij} has size $n_i \times n_j$. Similarly we write a matrix $P \in \mathcal{A}^{N_0 \times N_0}$ as $P = [P_{ij}]_{1 \leq i,j \leq N}$ where the block entry P_{ij} is in $\mathcal{A}^{n_i \times n_j}$. Then we see that

$$\begin{aligned} \phi(\alpha \cdot P) &= \left[K(Z_i, Z_j) \left(\sum_{k=1}^N \alpha_{ik} P_{kj} \right) \right]_{1 \leq i,j \leq N}, \\ L_\alpha \phi(P) &= \left[\sum_{k=1}^N \alpha_{ik} K(Z_k, Z_j)(P_{kj}) \right]_{1 \leq i,j \leq N}. \end{aligned}$$

By definition ϕ being a left \mathcal{S} -module map means that

$$\alpha Z^{(0)} = Z^{(0)} \alpha \Rightarrow \phi(\alpha P) = L_\alpha \phi(P),$$

or, in more detail,

$$\begin{aligned} \alpha_{ik} Z_k &= Z_i \alpha_{ik} \text{ for all } k \Rightarrow \\ \left[K(Z_i, Z_j) \left(\sum_{k=1}^N \alpha_{ik} P_{kj} \right) \right]_{ij} &= \left[\sum_{k=1}^N \alpha_{ik} K(Z_k, Z_j)(P_{kj}) \right]_{ij}. \end{aligned} \tag{2.9}$$

Let us specialize this to the case where α has only one nonzero block-entry:

$$\alpha_{ij} = \delta_{i i_0} \delta_{k k_0} \alpha_{i_0 k_0} \text{ for some } n_{i_0} \times n_{k_0}\text{-block matrix } \alpha_{i_0 k_0}$$

where $\delta_{i i_0}$ and $\delta_{k k_0}$ are Kronecker deltas. Then we see that the intertwining condition $\alpha_{ik} Z_k = Z_i \alpha_{ik}$ is satisfied as long as

$$\alpha_{i_0 k_0} Z_{k_0} = Z_{i_0} \alpha_{i_0 k_0} \tag{2.10}$$

since the remaining conditions $(\alpha_{ij}Z_j = Z_i\alpha_{ij})$ are automatic in the form $0 = 0$ when $(i, j) \neq (i_0, j_0)$. In this case condition (2.9) works out to be

$$[\delta_{ii_0}K(Z_{i_0}, Z_j)(\alpha_{i_0k_0}P_{k_0j})]_{ij} = [\delta_{ii_0}\alpha_{i_0k_0}K(Z_{k_0}, Z_j)(P_{k_0j})]_{ij}.$$

In particular we have equality of the (i_0, j) -entries:

$$K(Z_{i_0}, Z_j)(\alpha_{i_0k_0}P_{k_0j}) = \alpha_{i_0k_0}K(Z_{i_0}, Z_j)(P_{k_0j})$$

As the indices i_0, k_0 are arbitrary and the matrix $\alpha_{i_0k_0}$ is arbitrary subject to the condition (2.10), we can now conclude that indeed K satisfies the left intertwining conditions (2.8). That K also satisfies the right intertwining conditions

$$\begin{aligned} \beta \in \mathbb{C}^{k_0 \times j_0}, \beta Z_{k_0} = Z_{j_0}\beta, P \in \mathcal{A}^{n_i \times n_{j_0}} \\ \Rightarrow K(Z_i, Z_{j_0})(P)\beta^* = K(Z_i, Z_{k_0})(P\beta^*) \end{aligned} \tag{2.11}$$

follows similarly by using the assumption that ϕ is also a right \mathcal{S}^* -module. Conversely suppose that K satisfies the left intertwining conditions (2.8) and we seek to verify that ϕ is a left \mathcal{S} -module map. Thus by assumption we know that the implication (2.8) holds for each triple of indices i_0, k_0, j and we seek to verify condition (2.9). The premise for (2.9) is that we are given $\alpha = [\alpha_{ik}]_{i,k}$ such that $\alpha_{ik}Z_k = Z_i\alpha_{ik}$. As by assumption K satisfies (2.8), this implies that $\alpha_{ik}K(Z_k, Z_j)(P_{kj}) = K(Z_i, Z_j)(\alpha_{ik}P_{kj})$ for each fixed i, j, k . It now suffices to sum over k from 1 to N to arrive at (2.9) as wanted. One can show that K satisfying the right intertwining conditions (2.11) implies that ϕ is a right \mathcal{S}^* -module map by a dual argument.

We now suppose that we are only given $\phi: \mathbb{A} \rightarrow \mathbb{L}$ which is a $(\mathcal{S}, \mathcal{S}^*)$ -bimodule map. Let us let Q_i be the $N_0 \times N_0$ matrix over \mathbb{C} corresponding to the projection to the i -th block, i.e., Q_i is the block diagonal matrix with only nonzero block diagonal entry equal to the identity matrix I_{n_i} in the i -th diagonal block. Then it is easily checked that $Q_i \in \mathcal{S} \cap \mathcal{S}^*$, and hence $\phi(Q_i \cdot P) = L_{Q_i}\phi(P)$ and $\phi(P \cdot Q_j) = \phi(P)L_{Q_j}$ for $1 \leq i, j \leq N$. From this property one can deduce that the map ϕ then must have the form

$$\phi([P_{ij}]_{1 \leq i, j \leq N}) = [\phi_{ij}(P_{ij})]_{1 \leq i, j \leq N}$$

for linear maps $\phi_{ij}: \mathcal{A}^{n_i \times n_j} \rightarrow \mathcal{L}(\mathcal{Y})^{n_i \times n_j}$. We then define $K: \Omega \times \Omega \rightarrow \mathcal{L}(\mathcal{A}_{\text{nc}}, \mathcal{L}(\mathcal{Y})_{\text{nc}})$ by

$$K(Z_i, Z_j)(P_{ij}) = \phi_{ij}(P_{ij}). \tag{2.12}$$

Now it is a simple bookkeeping exercise to check that the $(\mathcal{S}, \mathcal{S}^*)$ -bimodule property of ϕ is exactly what is needed for K so defined to be a nc kernel on the finite set Ω .

We next analyze the claim regarding complete positivity. Suppose that K is a nc kernel on Ω . When it is assumed that Ω is a nc set (i.e., invariant under formation of direct sums), it is often taken as the definition of K being completely positive simply that the map $K(Z, Z)$ is a positive map for all $Z \in \Omega$

(see Proposition 2.2 in [8]). In case Ω is finite set $\{Z_1, \dots, Z_N\}$ augmented by the single point $Z^{(0)} = \bigoplus_1^N Z_i$, the adjustment of these observations is that K is a c.p. kernel if and only if $K(Z^{(0)}, Z^{(0)})$ is a cp map from \mathbb{A} to \mathbb{L} , i.e., if and only if ϕ is a completely positive map.

Conversely, if ϕ is a cp $(\mathcal{S}, \mathcal{S}^*)$ -bimodule map and K is defined as in (2.12), one can check that the complete positivity of ϕ is all that is required to guarantee the complete positivity of K as a nc kernel. \square

The next result characterizes the $(\mathcal{S}, \mathcal{S}^*)$ -bimodule property for a cp map ϕ from $\mathbb{A} := \mathcal{A}^{N_0 \times N_0}$ to $\mathbb{L} := \mathcal{L}(\mathcal{Y}^{N_0})$ in terms of a Stinespring representation

$$\phi(P) = V\pi(P)V^* \text{ for } P \in \mathcal{A}^{N_0 \times N_0}. \tag{2.13}$$

Here V is an operator from \mathcal{X} to \mathcal{Y}^{N_0} and $\pi: \mathcal{A}^{N_0 \times N_0} \rightarrow \mathcal{L}(\mathcal{X})$ is a $*$ -representation of $\mathcal{A}^{N_0 \times N_0}$ on a Hilbert space \mathcal{X} .

THEOREM 2.3. *Suppose that \mathcal{S} be a subalgebra of $\mathbb{C}^{N_0 \times N_0}$; use the natural $\mathbb{C}^{N_0 \times N_0}$ -bimodule structure of $\mathcal{A}^{N_0 \times N_0}$ to also view $\mathcal{A}^{N_0 \times N_0}$ as a $(\mathcal{S}, \mathcal{S}^*)$ -bimodule, and similarly for $\mathcal{L}(\mathcal{Y})^{N_0 \times N_0} \cong \mathcal{L}(\mathcal{Y}^{N_0})$. Let ϕ be a cp map from $\mathcal{A}^{N_0 \times N_0}$ to $\mathcal{L}(\mathcal{Y}^{N_0})$ with Stinespring representation (2.13). Then ϕ is a $(\mathcal{S}, \mathcal{S}^*)$ -bimodule map if and only if*

$$\pi(\beta^* \otimes 1_{\mathcal{A}})V^* = V^*L_{\beta^*} \text{ for all } \beta \in \mathcal{S}, \tag{2.14}$$

or equivalently,

$$L_{\alpha}V = V\pi(\alpha \otimes 1_{\mathcal{A}}) \text{ for all } \alpha \in \mathcal{S}. \tag{2.15}$$

Proof. Note first that (2.14) and (2.15) follow from each other by taking adjoints.

PROOF OF SUFFICIENCY: We assume that the cp map $\phi: \mathbb{A} \rightarrow \mathbb{L}$ is a $(\mathcal{S}, \mathcal{S}^*)$ -bimodule map and that the ingredients (V, π) in a Stinespring representation (2.13) for ϕ satisfy (2.14) (and hence also (2.15)). Using the fact that π is a $*$ -representation enables us to compute

$$\begin{aligned} \phi(\alpha \cdot P \cdot \beta^*) &= V\pi(\alpha \cdot P \cdot \beta^*)V^* = V\pi(\alpha \otimes 1_{\mathcal{A}})\pi(P)\pi(\beta^* \otimes 1_{\mathcal{A}})V^* \\ &= L_{\alpha}V\pi(P)V^*L_{\beta^*} \text{ (by (2.14) and (2.15))} \\ &= L_{\alpha}\phi(P)L_{\beta^*} \end{aligned}$$

and we conclude that ϕ is a $(\mathcal{S}, \mathcal{S}^*)$ -bimodule map.

PROOF OF NECESSITY: We suppose that the cp map ϕ with Stinespring representation (2.13) is a $(\mathcal{S}, \mathcal{S}^*)$ -bimodule map and we wish to show that (V, π) satisfies (2.14) - (2.15). Let us compute

$$\begin{aligned} \|\pi(\beta^* \otimes 1_{\mathcal{A}})V^*y - V^*L_{\beta^*}y\|^2 &= \langle y, V\pi(\beta\beta^* \otimes 1_{\mathcal{A}})V^*y \rangle \\ &\quad - \langle y, V\pi(\beta \otimes 1_{\mathcal{A}})V^*L_{\beta^*}y \rangle - \langle y, L_{\beta}V\pi(\beta^* \otimes 1_{\mathcal{A}})V^*y \rangle + \langle y, L_{\beta}VV^*L_{\beta^*}y \rangle \\ &= \langle y, \phi(\beta\beta^* \otimes 1_{\mathcal{A}})y - \phi(\beta \otimes 1_{\mathcal{A}})L_{\beta^*}y - L_{\beta}\phi(\beta^* \otimes 1_{\mathcal{A}})y \\ &\quad + L_{\beta}\phi(1_{\mathcal{A}^{N_0 \times N_0}})L_{\beta^*}y \rangle. \end{aligned} \tag{2.16}$$

At this point we use the assumption that ϕ is a $(\mathcal{S}, \mathcal{S}^*)$ -bimodule map to see that

$$\begin{aligned} \phi(\beta \otimes 1_{\mathcal{A}})L_{\beta^*} &= \phi(\beta\beta^* \otimes 1_{\mathcal{A}}), \\ L_{\beta}\phi(\beta^* \otimes 1_{\mathcal{A}}) &= \phi(\beta\beta^* \otimes 1_{\mathcal{A}}), \\ L_{\beta}\phi(1_{\mathcal{A}^{N_0 \times N_0}})L_{\beta^*} &= \phi(\beta\beta^* \otimes 1_{\mathcal{A}}). \end{aligned} \tag{2.17}$$

Making use of these identities enables us to continue the computation (2.16) to arrive at

$$\begin{aligned} \|\pi(\beta^* \otimes 1_{\mathcal{A}})V^*y - V^*L_{\beta^*}y\|^2 &= \langle y, \phi(\beta\beta^* \otimes 1_{\mathcal{A}})y - \phi(\beta\beta^* \otimes 1_{\mathcal{A}})y \\ &\quad - \phi(\beta\beta^* \otimes 1_{\mathcal{A}})y + \phi(\beta\beta^* \otimes 1_{\mathcal{A}})y \rangle = 0 \end{aligned} \tag{2.18}$$

for all $y \in \mathcal{Y}^{N_0}$, and (2.14) follows.

Alternatively, given that ϕ is a cp $(\mathcal{S}, \mathcal{S}^*)$ -bimodule map, we may make use of Proposition 2.2 to see that ϕ has the form ϕ_K for a cp nc kernel K on Ω . Let us extend K to the cp nc kernel \tilde{K} defined on $\tilde{\Omega} = \Omega \cup \{Z^{(0)}\}$ as discussed in the proof of Proposition 2.2. Then it is not hard to see that one gets a Stinespring representation (2.13) for ϕ from a Kolmogorov decomposition (1.7) for \tilde{K} : namely, if \tilde{K} has Kolmogorov representation

$$\tilde{K}(Z, W)(P) = H(Z)(\text{id}_{n,m} \otimes \bar{\phi})(P)H(W)^*$$

with $Z \in \tilde{\Omega}_n, W \in \tilde{\Omega}_m, P \in \mathcal{A}^{n \times m}$ with $\bar{\pi}: \mathcal{A} \rightarrow \mathcal{L}(\bar{\mathcal{X}})$ a $*$ -representation, then we get a Stinespring representation for $\phi_K = \phi: [P_{ij}] \mapsto \tilde{K}(Z^{(0)}, Z^{(0)})([P_{ij}])$ as $\phi([P_{ij}]) = V\pi([P_{ij}])V^*$ with

$$\begin{aligned} V &= H(Z^{(0)}): \bar{\mathcal{X}}^{N_0} = \mathcal{X} \rightarrow \mathcal{Y}^{N_0}, \\ \pi &= \text{id}_{N_0 \times N_0} \otimes \bar{\pi}: \mathcal{A}^{N_0 \times N_0} \rightarrow \mathcal{L}(\mathcal{X}) = \mathcal{L}(\bar{\mathcal{X}}^{N_0}). \end{aligned}$$

As H is a nc function and $\alpha \cdot Z^{(0)} = Z^{(0)} \cdot \alpha$ since $\alpha \in \mathcal{S}$, the RESPECTS INTERTWINING CONDITIONS for nc functions implies that we must also have

$$\alpha \cdot H(Z^{(0)}) = H(Z^{(0)}) \cdot \alpha.$$

Now it is a matter of checking to see that the action of α on $\bar{\mathcal{X}}^{N_0}$ can also be viewed as the action of $\pi(\alpha \otimes 1_{\mathcal{A}}) = (\text{id}_{N_0 \times N_0} \otimes \bar{\pi})(\alpha \otimes 1_{\mathcal{A}})$ on $\mathcal{X} := \bar{\mathcal{X}}^{N_0}$. In this way we recover (2.15) with $V = H(Z^{(0)})$ and with $\pi = \text{id}_{N_0 \times N_0} \bar{\pi}$. \square

A careful look at the first proof of sufficiency in Theorem 2.3 leads to the following corollary. We recall that by an operator system \mathbb{S} contained in a unital C^* -algebra \mathcal{A} we mean a self-adjoint linear subspace of \mathcal{A} containing the identity $1_{\mathcal{A}}$ of \mathcal{A} . It is possible to define operator systems more intrinsically (see [26]) but we need not get into the details here. Furthermore the notion of cp nc kernel $K: \Omega \rightarrow \Omega \rightarrow \mathcal{L}(\mathcal{A}_{\text{nc}}, \mathcal{L}(\mathcal{Y})_{\text{nc}})$ can be extended to the setting

where the C^* -algebra \mathcal{A} is replaced by an operator system \mathbb{S} contained in some C^* -algebra \mathcal{A} . To do this one must use the first characterization of cp nc kernel for the case where $\mathbb{S} = \mathcal{A}$ is an algebra which avoids using multiplication in \mathcal{A} : we say that *the nc kernel*

$$K: \Omega \times \Omega \rightarrow \mathcal{L}(\mathbb{S}_{\text{nc}}, \mathcal{L}(\mathcal{Y})_{\text{nc}})$$

is cp if the map

$$[s_{ij}] \mapsto [K(Z_i, Z_j)(s_{ij})]$$

is a completely positive map from the operator system $\mathbb{S}^{N_0 \times N_0}$ to the C^* -algebra $\mathcal{L}(\mathcal{Y})^{N_0 \times N_0}$ for all $Z_i \in \Omega_{n_i}$, $s_{ij} \in \mathbb{S}^{n_i \times n_j}$, $1 \leq i, j \leq N$, $N_0 = \sum_{i=1}^N n_i$. Furthermore a routine extension of part (2) of Proposition 2.2 assures us that the correspondence between cp nc kernels K and cp $(\mathbb{S}, \mathbb{S}^*)$ -bimodule maps ϕ_K continues to hold when $\mathbb{S} = \mathcal{A}$ is only an operator system. With these preliminaries out of the way, we can now proceed to the following result.

COROLLARY 2.4. *Suppose that \mathbb{S} is an operator system contained in $\mathcal{A}^{N_0 \times N_0}$ such that (i) \mathbb{S} is a $(\mathcal{S}, \mathcal{S}^*)$ -sub-bimodule of $\mathcal{A}^{N_0 \times N_0}$ and (ii) $\phi_0: \mathbb{S} \rightarrow \mathcal{L}(\mathcal{Y}^{N_0})$ is a cp map which is also a $(\mathcal{S}, \mathcal{S}^*)$ -bimodule map. Let $\phi: \mathcal{A}^{M \times M} \rightarrow \mathcal{L}(\mathcal{Y}^M)$ be any cp map which extends ϕ_0 (such exist by a theorem of Arveson (see [26, Theorem 7.5]):*

$$\phi(P) = \phi_0(P) \text{ if } P \in \mathbb{S}.$$

Then ϕ is also a $(\mathcal{S}, \mathcal{S}^)$ -bimodule map.*

Proof. We have the same assumptions as in the proof of necessity direction in Theorem 2.3, but now with the hypothesis that ϕ is a $(\mathcal{S}, \mathcal{S}^*)$ -module map weakened to the hypothesis that only $\phi_0 := \phi|_{\mathbb{S}}$ is a $(\mathcal{S}, \mathcal{S}^*)$ -bimodule map for some operator system $\mathbb{S} \subset \mathcal{A}^{N_0 \times N_0}$. As a first step we seek to prove that then the identities (2.14) - (2.15) (where the pair (V, π) arises from a Stinespring representation for ϕ) still hold. Toward this end, note first that the calculation (2.16) holds as before. The key observation is that the identities (2.17) only require that $\phi|_{\mathbb{S}}$ be a $(\mathcal{S}, \mathcal{S}^*)$ -bimodule map, as an operator system by definition contains the identity $1_{\mathcal{A}^N}$ and hence also all elements of the form $\beta \cdot 1_{\mathcal{A}^{N_0 \times N_0}} = \beta \otimes 1_{\mathcal{A}}$, $1_{\mathcal{A}^{N_0 \times N_0}} \cdot \beta^* = \beta^* \otimes 1_{\mathcal{A}}$, $\beta \cdot 1_{\mathcal{A}^{N_0 \times N_0}} \beta^* = \beta \beta^* \otimes 1_{\mathcal{A}}$ must all be in \mathbb{S} for any $\beta \in \mathcal{S}$. Furthermore the identities (2.17) can be interpreted as $(\mathcal{S}, \mathcal{S}^*)$ -bimodule properties for the $(\mathcal{S}, \mathcal{S}^*)$ -submodule \mathbb{S} . Hence we again can continue the calculation (2.16) to arrive at (2.18) and conclude that the identity (2.14) (and then also (2.15) upon taking adjoints) holds for all $\alpha, \beta \in \mathcal{S}$. To complete the proof, it suffices to quote the sufficiency result in Theorem 2.3 to conclude that then ϕ itself is a $(\mathcal{S}, \mathcal{S}^*)$ -bimodule map. \square

3 PROBLEM A: THE DECOMPOSABILITY PROBLEM FOR HERMITIAN NON-COMMUTATIVE KERNELS/HERMITIAN MAPS

We are given a nc kernel K and seek to show that it has a decomposition (1.8) with each K_j ($j = 1, 2, 3, 4$) a cp nc kernel. We shall first go through a series of reductions.

Given a nc kernel $K: \Omega \times \Omega \rightarrow \mathcal{L}(\mathcal{A}_{\text{nc}}, \mathcal{L}(\mathcal{Y})_{\text{nc}})$, we define the adjoint kernel K^* also from $\Omega \times \Omega$ to $\mathcal{L}(\mathcal{A}_{\text{nc}}, \mathcal{L}(\mathcal{Y})_{\text{nc}})$ by

$$K^*(Z, W)(P) = K(W, Z)(P^*)^*.$$

We have several observations concerning this adjoint operator on nc kernels.

PROPOSITION 3.1. (1) *If K is nc kernel, then K^* is also a nc kernel.*

(2) *If K is a cp nc kernel, then K is Hermitian.*

(3) *Any nc kernel K can be decomposed as $K = K_R + iK_I$ where K_R and K_I are Hermitian kernels. Hence, to prove Conjecture A, it suffices to show that any Hermitian nc kernel K can be written as the difference of two cp kernels:*

$$K = K^* \Rightarrow K = K_+ - K_- \text{ with } K_+, K_- \text{ equal to cp kernels.}$$

Proof. (1) Suppose that $Z \in \Omega_n, \tilde{Z} \in \Omega_{\tilde{n}}, \alpha \in \mathbb{C}^{\tilde{n} \times n}$ and $\alpha Z = \tilde{Z}\alpha$, and K is a nc kernel on Ω . Then we compute

$$\begin{aligned} \alpha K^*(Z, W)(P) &= \alpha K(W, Z)(P^*)^* = (K(W, Z)(P^*)\alpha^*)^* \\ &= K^*(\tilde{Z}, W)(\alpha P) \end{aligned}$$

where the *respects intertwining*s property of K with respect to the second argument is used in the last step. In this way we verify the *respects intertwining*s property of K^* with respect to the first argument. By a similar argument, the *respects intertwining*s property of K with respect to the first argument can be used to prove the *respects intertwining*s property of K^* with respect to the second argument.

(2) Given $Z \in \Omega_n, W \in \Omega_m, P_{12} \in \mathcal{A}^{n \times m}$, choose $P_{11} \in \mathcal{A}^{n \times n}, P_{12} \in \mathcal{A}^{n \times m}$ and $P_{22} \in \mathcal{A}^{m \times m}$ so that $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix}$ is positive in $\mathcal{A}^{(m+n) \times (m+n)}$. If K is cp, then we must have

$$0 \preceq K\left(\begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix}, \begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix}\right)\left(\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix}\right) = \begin{bmatrix} K(Z, Z)(P_{11}) & K(Z, W)(P_{12}) \\ K(W, Z)(P_{12}^*) & K(W, W)(P_{22}) \end{bmatrix}.$$

In particular it follows that the (2, 1) entry is the adjoint of the (1, 2) entry:

$$K(W, Z)(P_{12}^*) = K(Z, W)(P_{12})^*.$$

where $K(W, Z)(P_{12}^*)^* = K^*(Z, W)(P_{12})$. As P_{12} is arbitrary, we conclude that $K^* = K$ as claimed.

(3) Note that the operation of forming the adjoint kernel is an involution and is conjugate linear:

$$K^{**} = K, \quad (aK_1 + bK_2)^* = \bar{a}K_1^* + \bar{b}K_2^* \text{ for } a, b \in \mathbb{C}.$$

Hence for any nc kernel, $K_R := \frac{1}{2}(K + K^*)$ and $K_I := \frac{1}{2i}(K - K^*)$ are Hermitian kernels and we recover K from K_R and K_I as

$$K = \frac{1}{2}(K + K^*) + i \frac{1}{2i}(K - K^*) = K_R + iK_I.$$

□

The next result gives an equivalent formulation on deciding if a Hermitian kernel is decomposable. These results are modelled on the corresponding results in [11] for the case where $\Omega = \Omega_1$ (all points in Ω are 1×1 matrices over \mathcal{V}).

PROPOSITION 3.2. *Let K be a Hermitian nc kernel. Then the following are equivalent:*

- (1) K is decomposable, i.e., there exist cp nc kernels K_1 and K_2 so that $K = K_1 - K_2$.
- (2) K has a Hermitian Kolmogorov decomposition: for $Z \in \Omega_n$, $W \in \Omega_m$ and $P \in \mathcal{A}^{n \times m}$,

$$K(Z, W)(P) = H(Z) ((\text{id}_{n \times m} \otimes \pi)(P) \cdot (\text{id}_{n \times m} \otimes J)) H(W)^* \quad (3.1)$$

where $\pi: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{X})$ is a $*$ -representation and $J = J^* = J^{-1} \in \mathcal{L}(\mathcal{X})$ commutes with the range of π :

$$a \in \mathcal{A} \Rightarrow \pi(a)J = J\pi(a).$$

- (3) There exist cp nc kernels \mathbb{L}_1 and \mathbb{L}_2 on Ω with values in $\mathcal{L}(\mathcal{A}_{\text{nc}}, \mathcal{L}(\mathcal{Y})_{\text{nc}})$ so that the kernel

$$\mathbb{K}: \Omega \times \Omega \rightarrow \mathcal{L}(\mathcal{A}_{\text{nc}}, \mathcal{L}(\mathcal{Y})_{\text{nc}}^{2 \times 2})$$

given by, for $Z \in \Omega_n$, $W \in \Omega_m$, $P \in \mathcal{A}^{n \times m}$,

$$\mathbb{K}(Z, W)(P) = \begin{bmatrix} \mathbb{L}_1(Z, W)(P) & K(Z, W)(P) \\ K(Z, W)(P) & \mathbb{L}_2(Z, W)(P) \end{bmatrix} \quad (3.2)$$

is cp.

Proof. (1) \Leftrightarrow (2) Suppose that K is decomposable: $K = K_1 - K_2$ for cp nc kernels K_1 and K_2 . Then each of K_1 and K_2 has a Kolmogorov decomposition: for $Z \in \Omega_n$, $W \in \Omega_m$ and $P \in \mathcal{A}^{n \times m}$ we have

$$K_j(Z, W)(P) = H_j(Z)(\text{id}_{n \times m} \otimes \pi_j)(P)H_j(W)^*$$

for a nc function $H_j: \Omega \rightarrow \mathcal{L}(\mathcal{X}_j, \mathcal{Y})_{\text{nc}}$ and a $*$ -representation $\pi_j: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{X}_j)$ for $j = 1, 2$. Then we see that

$$\begin{aligned} K(Z, W)(P) &= K_1(Z, W)(P) - K_2(Z, W)(P) \\ &= [H_1(z) \quad H_2(Z)] (\text{id}_{n \times m} \otimes \begin{bmatrix} \pi_1 & 0 \\ 0 & \pi_2 \end{bmatrix})(P) (\text{id}_{n \times m} \otimes \begin{bmatrix} I_{\mathcal{X}_1} & 0 \\ 0 & -I_{\mathcal{X}_2} \end{bmatrix}) \begin{bmatrix} H_1(W)^* \\ H_2(W)^* \end{bmatrix} \\ &= H(Z)(\text{id}_{n \times m} \otimes \pi)(P)(\text{id}_{n \times m} \otimes J)H(W)^*, \end{aligned}$$

where we set

$$\mathcal{X} = \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix}, \quad H(Z) = [H(Z_1) \quad H(Z_2)], \quad \pi = \begin{bmatrix} \pi_1 & 0 \\ 0 & \pi_2 \end{bmatrix}, \quad J = \begin{bmatrix} I_{\mathcal{X}_1} & 0 \\ 0 & -I_{\mathcal{X}_2} \end{bmatrix}.$$

has a Hermitian Kolmogorov decomposition (3.1).

Conversely, if K has a Hermitian Kolmogorov decomposition (3.1), we can choose an orthogonal decomposition of \mathcal{X} as $\mathcal{S} = \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix}$ with respect to which J has matrix representation $J = \begin{bmatrix} I_{\mathcal{X}_1} & 0 \\ 0 & -I_{\mathcal{X}_2} \end{bmatrix}$. As by definition of Hermitian Kolmogorov decomposition J commutes with π , it also follows that π has a block diagonal form $\pi(a) = \begin{bmatrix} \pi_1(a) & 0 \\ 0 & \pi_2(a) \end{bmatrix}$ for $*$ -representations π_1 and π_2 of \mathcal{A} on \mathcal{X}_1 and \mathcal{X}_2 respectively. Furthermore with respect to this decomposition of \mathcal{X} as $\mathcal{X} = \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix}$ we have a matrix representation of $H(Z)$ as $H(Z) = \begin{bmatrix} H_1(z) & H_2(Z) \end{bmatrix}$ for noncommutative functions $H_j: \Omega \rightarrow \mathcal{L}(\mathcal{X}_j, \mathcal{Y})$ for $j = 1, 2$. Hence we see that

$$K(Z, W)(P) = H_1(Z)(\text{id}_{n \times m} \otimes \pi_1)(P)H_1(W)^* - H_2(Z)(\text{id}_{n \times m} \otimes \pi_2)(P)H_2(W)^*$$

is decomposable.

(2) \Leftrightarrow (3) Assume that K has a Hermitian Kolmogorov decomposition (3.1). Define $\mathbb{L}_1 = \mathbb{L}_2 =: \mathbb{L}$ where \mathbb{L} is given by

$$\mathbb{L}(Z, W)(P) = H(Z) ((\text{id}_{n \times m} \otimes \pi)(P)) H(W)^*.$$

Then \mathbb{L} is defined via a Kolmogorov decomposition and hence is a cp nc kernel. Define \mathbb{K} as in (3.2). Then

$$\begin{aligned} \mathbb{K}(Z, W)(P) = \\ \mathbb{H}_0(Z) \begin{bmatrix} (\text{id}_{n \times m} \otimes \pi)(P) & (\text{id}_{n \times m} \otimes \pi)(P) \cdot \text{id}_{n \times m} \otimes J \\ (\text{id}_{n \times m} \otimes \pi)(P) \cdot \text{id}_{n \times m} \otimes J & (\text{id}_{n \times m} \otimes \pi)(P) \end{bmatrix} \mathbb{H}_0(W)^* \end{aligned} \tag{3.3}$$

where we set

$$\mathbb{H}_0(Z) = \begin{bmatrix} H(Z) & 0 \\ 0 & H(Z) \end{bmatrix}.$$

Let us factor the middle factor as

$$\begin{aligned} & \begin{bmatrix} (\text{id}_{n \times m} \otimes \pi)(P) & (\text{id}_{n \times m} \otimes \pi)(P) \cdot \text{id}_{n \times m} \otimes J \\ (\text{id}_{n \times m} \otimes \pi)(P) \cdot \text{id}_{n \times m} \otimes J & (\text{id}_{n \times m} \otimes \pi)(P) \end{bmatrix} \\ &= \begin{bmatrix} \text{id}_{n \times m} \otimes \pi & 0 \\ 0 & \text{id}_{n \times m} \otimes \pi \end{bmatrix} (P) \cdot \text{id}_{n \times m} \otimes \begin{bmatrix} I & J \\ J & I \end{bmatrix}. \end{aligned}$$

If we note the factorization

$$\begin{bmatrix} I & J \\ J & I \end{bmatrix} = \begin{bmatrix} I \\ J \end{bmatrix} \begin{bmatrix} I & J^* \end{bmatrix}$$

and use the fact that

$$(\text{id}_{n \times m} \otimes \pi)(P)(\text{id}_{n \times m} \otimes J) = (\text{id}_{n \times m} \otimes J)(\text{id}_{n \times m} \otimes \pi)(P)$$

which we know as a consequence of (3.1) being a Hermitian Kolmogorov factorization for K , we can continue the computation (3.3) as

$$\mathbb{K}(Z, W)(P) = \mathbb{H}(z) (\text{id}_{n \times m} \otimes \pi)(P) \mathbb{H}(W)^*$$

where now we set

$$\mathbb{H}(z) = \begin{bmatrix} H(Z) \\ H(Z)(\text{id}_{n \times m} \otimes J) \end{bmatrix},$$

thereby exhibiting a Kolmogorov decomposition for \mathbb{K} . Thus \mathbb{K} so defined is a cp nc kernel.

Conversely, suppose that one can find two cp nc kernels \mathbb{L}_1 and \mathbb{L}_2 so that the kernel (3.2) is cp. Then \mathbb{K} has a Kolmogorov decomposition

$$\mathbb{K}(Z, W)(P) = \begin{bmatrix} H_1(Z) \\ H_2(Z) \end{bmatrix} ((\text{id}_{n \times m} \otimes \pi)(P)) \begin{bmatrix} H_1(W)^* & H_2(W)^* \end{bmatrix}.$$

Thus K is given by

$$K(Z, W)(P) = H_1(Z) ((\text{id}_{n \times m} \otimes \pi)(P)) H_2(W)^*.$$

Rewrite this as

$$K(Z, W)(P) = \begin{bmatrix} H_1(Z) & H_2(Z) \end{bmatrix} \begin{bmatrix} 0 & \text{id}_{n \times m} \otimes \pi(P) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} H_1(W)^* \\ H_2(W)^* \end{bmatrix}.$$

As we are assuming that K is a Hermitian kernel, we also have $K = K^*$ so

$$\begin{aligned} K(Z, W)(P) &= K^*(Z, W)(P) \\ &= \begin{bmatrix} H_1(Z) & H_2(Z) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ (\text{id}_{n \times m} \otimes \pi)(P) & 0 \end{bmatrix} \begin{bmatrix} H_1(W)^* \\ H_2(W)^* \end{bmatrix}. \end{aligned}$$

Taking the average of these two representations for K leads to a third representation

$$K(Z, W)(P) = \mathbb{H}'(Z) ((\text{id}_{n \times m} \otimes \begin{bmatrix} \pi & 0 \\ 0 & \pi \end{bmatrix})(P)) \cdot (\text{id}_{n \times m} \otimes \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}) \mathbb{H}'(W)^*$$

where now

$$\mathbb{H}'(Z) = \frac{1}{\sqrt{2}} \begin{bmatrix} H_1(Z) & H_2(Z) \end{bmatrix}$$

and where

$$\begin{bmatrix} \pi(P) & 0 \\ 0 & \pi(P) \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} \pi(P) & 0 \\ 0 & \pi(P) \end{bmatrix}.$$

We thus see that K has a Hermitian Kolmogorov decomposition as wanted. \square

REMARK 3.3. The study of cp nc kernels was launched only recently (see [8]). There it is shown that besides characterizations via the complete positivity condition (2.2) and via the Kolmogorov decomposition (2.3), there is a third

characterization: there is a Hilbert space \mathcal{H} consisting of nc functions $f: \Omega \rightarrow \mathcal{L}(\mathcal{A}, \mathcal{Y})_{\text{nc}}$ such that K serves as the reproducing kernel for \mathcal{H} in the following sense: given $W \in \Omega_m$, $v \in \mathcal{A}^m$, $y \in \mathcal{Y}^m$, the function $K_{W,a,y}: \Omega \rightarrow \mathcal{L}(\mathcal{A}, \mathcal{Y})_{\text{nc}}$ given by

$$K_{W,v,y}(Z): u \mapsto K(Z, W)(uv^*)y$$

belongs to \mathcal{H} and reproduces the value of $f \in \mathcal{H}$ at the point $W \in \Omega_m$ evaluated at $v \in \mathcal{A}^m$ in direction $y \in \mathcal{Y}^m$ according to the formula

$$\langle f, K_{W,v,y} \rangle_{\mathcal{H}} = \langle f(W)v, y \rangle_{\mathcal{Y}^m}$$

(see [8, Theorem 3.1]). We note that the decomposition (3.1) can be viewed as an indefinite-metric analogue of the Kolmogorov decomposition (2.3) for a cp nc kernel. One can then expect that the decomposition (3.1) is equivalent to indefinite-metric analogues of the complete positivity condition (2.2) and that such a K should be the “reproducing kernel” for a nc reproducing kernel Krein space $\mathcal{H}(K)$. Indeed, for the classical Aronszajn setting, investigation of such reproducing kernel Krein spaces, in particular for the case where the Krein space carries only finitely many negative squares in which case they are called reproducing kernel Pontryagin spaces, has been an active area of research over the past several decades (see e.g. [4]). We leave this topic as a possible direction for future research in the theory of Krein spaces and the associated operator theory.

We have seen in Proposition 2.2 that problems concerning nc kernels on a finite-point set Ω with values mapping operators from \mathcal{A}_{nc} (\mathcal{A} equal to a C^* -algebra) into $\mathcal{L}(\mathcal{Y})_{\text{nc}}$ (\mathcal{Y} equal to a Hilbert space) can be reformulated as problems concerning completely bounded maps from $\mathcal{A}^{M \times M}$ to $\mathcal{L}(\mathcal{Y}^M)$ (for some in general large M) which are $(\mathcal{S}, \mathcal{S}^*)$ -bimodule maps, for a carefully specified subalgebra \mathcal{S} of finite complex matrices $\mathbb{C}^{M \times M}$. Let us translate the reductions for the reducibility problem for cb nc kernels to reductions for the corresponding reducibility problem for such cb $(\mathcal{S}, \mathcal{S}^*)$ -bimodule maps. In general we shall say that a map $\phi: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{Y})$ is *Hermitian* if $\phi(P)^* = \phi(P^*)$ for all $P \in \mathcal{A}$. It is not hard to see that any cb map ϕ can be decomposed as a linear combination $\phi = \phi_R + i\phi_I$ of two Hermitian maps $\phi_R = \frac{1}{2}(\phi + \phi^*)$ and $\phi_I = \frac{1}{2i}(\phi - \phi^*)$, i.e., $\phi_R = \phi_R^*$ and $\phi_I = \phi_I^*$ where in general

$$\phi^*(P) = \phi(P^*)^*.$$

Thus the problem of *decomposability* for a cb map ϕ (writing ϕ as a linear combination of four cp maps) reduces to decomposing a Hermitian map as the difference of two cp maps; this is less trivial than the Hermitian decomposition just discussed but has been done in work of Wittstock, Haagerup, and Paulsen (see the discussion in Example 3.9 below). Our interest here is to understand these problems for maps $\phi: \mathcal{A}^{M \times M} \rightarrow \mathcal{L}(\mathcal{Y}^M)$ which are also $(\mathcal{S}, \mathcal{S}^*)$ -bimodule maps and where we wish to maintain the $(\mathcal{S}, \mathcal{S}^*)$ -bimodule structure in the components of the decomposition, a topic also well explored in the book of Paulsen [26]. The analogue of Proposition 3.2 is as follows.

PROPOSITION 3.4. *Let \mathcal{S} be a subalgebra of $\mathbb{C}^{M \times M}$. For \mathcal{A} a C^* -algebra and \mathcal{Y} a Hilbert space, we may then view $\mathcal{A}^{M \times M}$ and $\mathcal{L}(\mathcal{Y}^M) \cong \mathcal{L}(\mathcal{Y})^{M \times M}$ as $(\mathcal{S}, \mathcal{S}^*)$ -bimodules. Let ϕ be a cb Hermitian map from $\mathcal{A}^{M \times M}$ to $\mathcal{L}(\mathcal{Y}^M)$ which is also a $(\mathcal{S}, \mathcal{S}^*)$ -bimodule map. Then the following are equivalent.*

(1) ϕ is $(\mathcal{S}, \mathcal{S}^*)$ -decomposable, i.e., there exists cp $(\mathcal{S}, \mathcal{S}^*)$ -bimodule maps ϕ_1 and ϕ_2 so that $\phi = \phi_1 - \phi_2$.

(2) ϕ has a Hermitian $(\mathcal{S}, \mathcal{S}^*)$ -bimodule Stinespring representation, i.e., there exists an operator $V: \mathcal{X} \rightarrow \mathcal{Y}^M$, a $*$ -representation $\pi: \mathcal{A}^M \rightarrow \mathcal{L}(\mathcal{X})$ and a signature operator $J \in \mathcal{L}(\mathcal{X})$ ($J = J^* = J^{-1}$) so that

$$\pi(P)J = J\pi(P), \quad \phi(P) = V(\pi(P)J)V^*$$

for all $P \in \mathcal{A}^{M \times M}$ such that

$$\pi(\beta^*)V^* - V^*L_\beta^* = 0 \text{ for all } \beta \in \mathcal{S}. \quad (3.4)$$

(3) There exists cp $(\mathcal{S}, \mathcal{S}^*)$ -bimodule maps Φ_{11} , Φ_{22} so that the map $\widehat{\Phi}$ from $\mathcal{A}^{M \times M}$ to $\mathcal{L}(\mathcal{Y}^{2M})$ given by

$$\widehat{\Phi}: P \mapsto \begin{bmatrix} \Phi_{11}(P) & \phi(P) \\ \phi(P) & \Phi_{22}(P) \end{bmatrix} \quad (3.5)$$

is cp.

Proof. When \mathcal{S} is the intertwining algebra associated with a finite nc point-set $\Omega = \{Z_1, \dots, Z_N\}$ and the map ϕ has the form $\phi = \phi_K$ for a nc kernel on Ω , then the mutual equivalences of parts (1) of Propositions 3.2 and 3.4 and of parts (3) of Propositions 3.2 and 3.4 follow from the correspondence between nc kernels/cp nc kernels K and $(\mathcal{S}, \mathcal{S}^*)$ -bimodule maps/ $(\mathcal{S}, \mathcal{S}^*)$ -bimodule cp maps ϕ_K given in Proposition 3.4. For the mutual equivalence of parts (2), one should also note the alternative proof of (2.14) in the proof of Theorem 2.3.

Alternatively, one can avoid the assumption that \mathcal{S} is an intertwining algebra associated with some point set Ω and prove directly the equivalence of (1), (2), (3) in Proposition 3.4 by imitating the proofs of the corresponding results in Proposition 3.2. When doing this, when working with the Kolmogorov decompositions in part (2), one should bear in mind the result of Theorem 2.3 that condition (2.14) is automatic in Stinespring representations for cp $(\mathcal{S}, \mathcal{S}^*)$ -bimodule maps, while its counterpart (3.4) is part of the definition for a $(\mathcal{S}, \mathcal{S}^*)$ -Hermitian Stinespring representation. \square

Note that when we apply these criteria to the case where $M = N_0 = \sum_{j=1}^N n_j$ and \mathcal{S} is the intertwining algebra for the point $Z^{(0)} = \bigoplus_{i=1}^N Z_i$ where $\Omega = \{Z_1, \dots, Z_N\}$, then the content of Proposition 3.4 is just a direct translation of the content of Proposition 3.2 via the dictionary between nc kernels and cb maps given by Proposition 2.2.

Thus the nc kernel-decomposability problem, or equivalently, the cb $(\mathcal{S}, \mathcal{S}^*)$ -bimodule-map decomposability problem, is reduced to showing that one of

the criteria in parts (2) or (3) of Proposition 3.2 or in parts (2) or (3) of Proposition 3.4 always holds. The first of our partial results dealing with criterion (3) in Proposition 3.4 needs an extra hypothesis on the subalgebra $\mathcal{S} \subset \mathbb{C}^{M \times M}$. Let us use the notation $C^*(\mathcal{S})$ for the C^* -algebra generated by \mathcal{S} inside $\mathbb{C}^{M \times M}$.

THE COMPLETE SPECTRAL FACTORIZATION PROPERTY: For any natural number $n \in \mathbb{N}$ and $\alpha \in C^*(\mathcal{S})^{n \times n}$ with $\alpha \succ 0$, there is a $\beta \in \mathcal{S}^{n \times n}$ so that $\alpha = \beta\beta^*$.

With this additional hypothesis imposed, we have the following result.

THEOREM 3.5. *Assume the same setup as in the hypotheses of Proposition 3.4 and assume that \mathcal{S} has the Complete Spectral Factorization property. Then statement (3) in Proposition 3.4 holds, and the decomposability problem for Hermitian $(\mathcal{S}, \mathcal{S}^*)$ -bimodule maps is solvable. Hence also the decomposability problem for Hermitian nc kernels on a finite set of nc points Ω is solvable, as long as the intertwining algebra \mathcal{S} associated with Ω has the Complete Spectral Factorization property.*

For the case where \mathcal{S} is a C^* -algebra, Theorem 3.5 appears in Bhattacharyya-Dritschel-Todd [11] as Theorem 4.1 and is based on the off-diagonal method of Paulsen (see Exercise 8.6 in [26]). For completeness we include a complete proof since it also illustrates an application of Corollary 2.4.

Proof of Theorem 3.5. By rescaling we may also assume that ϕ is cc. We must construct cp maps Φ_{11} and Φ_{22} so that the map $\widehat{\Phi}$ given by (3.5) is cp. To simplify the notation let us set

$$\mathbb{A} := \mathcal{A}^{M \times M}, \quad \mathbb{L} := \mathcal{L}(\mathcal{Y}^M).$$

For any $\alpha \in C^*(\mathcal{S}) \subset \mathbb{C}^{M \times M}$, tensoring with the unit $1_{\mathcal{A}}$ of \mathcal{A} gives an element $\alpha \cdot 1_{\mathcal{A}}$ in $\mathcal{A}^{M \times M} = \mathbb{A}$ and similarly $\alpha \cdot I_{\mathcal{Y}} \in \mathbb{L}$. Let us define an operator system \mathbb{S} contained in the C^* -algebra $\mathbb{A}^{2 \times 2}$ by

$$\mathbb{S} = \begin{bmatrix} C^*(\mathcal{S}) \cdot 1_{\mathcal{A}} & \mathbb{A} \\ \mathbb{A} & C^*(\mathcal{S}) \cdot 1_{\mathcal{A}} \end{bmatrix} \subset \mathbb{A}^{2 \times 2} \tag{3.6}$$

and define a map $\widehat{\Phi}_{\text{pre}}: \mathbb{S} \rightarrow \mathcal{L}(\mathcal{Y})^{2M \times 2M}$ by

$$\widehat{\Phi}_{\text{pre}}: \begin{bmatrix} \alpha \cdot 1_{\mathcal{A}} & P_1 \\ P_2^* & \beta \cdot 1_{\mathcal{A}} \end{bmatrix} \mapsto \begin{bmatrix} \alpha \cdot I_{\mathcal{Y}} & \phi(P_1) \\ \phi(P_2^*) & \beta \cdot I_{\mathcal{Y}} \end{bmatrix}. \tag{3.7}$$

We shall prove that this $\widehat{\Phi}_{\text{pre}}$ is cp. To do this, for each $n \in \mathbb{Z}$ we must consider a positive element in the inflated space $\mathbb{S}^{n \times n}$ and show that $\widehat{\Phi}_{\text{pre}}^{(n)}$ sends such an element to a positive element in the corresponding inflated space $(\mathbb{L}^{2 \times 2})^{n \times n}$. Using a canonical shuffle procedure, we write elements of \mathbb{S}^n as

$$\begin{bmatrix} H \cdot 1_{\mathcal{A}} & \mathbb{P}_1 \\ \mathbb{P}_2^* & K \cdot 1_{\mathcal{A}} \end{bmatrix} \tag{3.8}$$

where $H, K \in \mathcal{S}^{n \times n}$ and $P_1, P_2 \in \mathbb{A}^{n \times n}$, and via the same canonical shuffle we view elements of $(\mathbb{L}^{2 \times 2})^{n \times n}$ as being elements of $(\mathbb{L}^{n \times n})^{2 \times 2}$. The goal now is to show that if (3.8) is positive in $\mathbb{S}^{n \times n}$, then its image under $(\widehat{\Phi}_{\text{pre}})^{(n)}$ is positive in $\mathbb{L}^{2 \times 2}$. By an approximation argument, it is sufficient to assume that (3.8) is strictly positive definite. It then follows that $H \succ 0$ and $K \succ 0$ in $\mathcal{S}^{n \times n}$. By the assumption that \mathcal{S} has the Complete Spectral Factorization property, we then have factorizations

$$H = AA^*, \quad K = BB^*$$

where A and B are in $\mathcal{S}^{n \times n}$. As (3.8) is positive, it follows that $\mathbb{P}_1 = \mathbb{P}_2 =: \mathbb{P}$ in $\mathbb{A}^{n \times n}$. We then see that (3.8) has a factorization as

$$\begin{bmatrix} H & \mathbb{P} \\ \mathbb{P}^* & K \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \cdot \begin{bmatrix} I_{\mathcal{Y}^{nM}} & A^{-1}\mathbb{P}B^{*-1} \\ B^{-1}\mathbb{P}^*A^{*-1} & I_{\mathcal{Y}^{nM}} \end{bmatrix} \cdot \begin{bmatrix} A^* & 0 \\ 0 & B^* \end{bmatrix}$$

As ϕ is cc, we have also that

$$\|\phi^{(n)}(A^{-1}\mathbb{P}B^{-1})\| < 1. \tag{3.9}$$

Furthermore, as ϕ is a $(\mathcal{S}, \mathcal{S}^*)$ -bimodule map, it follows that $\phi^{(n)}$ is a $(\mathcal{S}^{n \times n}, (\mathcal{S}^*)^{n \times n})$ -bimodule map and hence

$$\phi^{(n)}(A^{-1}\mathbb{P}B^{*-1}) = A^{-1} \cdot \phi^{(n)}(\mathbb{P}) \cdot B^{*-1}. \tag{3.10}$$

On the other hand let us compute

$$\begin{aligned} \widehat{\Phi}_{\text{pre}}^{(n)} \left(\begin{bmatrix} H \cdot 1_{\mathcal{A}} & \mathbb{P} \\ \mathbb{P}^* & K \cdot 1_{\mathcal{A}} \end{bmatrix} \right) &= \begin{bmatrix} H \cdot I_{\mathcal{Y}} & \phi^{(n)}(\mathbb{P}) \\ \phi^{(n)}(\mathbb{P})^* & K \cdot I_{\mathcal{Y}} \end{bmatrix} \\ &= \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \cdot \begin{bmatrix} I_{\mathcal{Y}^{nM}} & A^{-1}\phi^{(n)}(\mathbb{P})B^{*-1} \\ B^{*-1}\phi^{(n)}(\mathbb{P}^*)A^{*-1} & I_{\mathcal{Y}^{nM}} \end{bmatrix} \cdot \begin{bmatrix} A^* & 0 \\ 0 & B^* \end{bmatrix} \\ &= \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \cdot \begin{bmatrix} I_{\mathcal{Y}^{nM}} & \phi^{(n)}(A^{-1}\mathbb{P}B^{*-1}) \\ \phi^{(n)}(B^{*-1}\mathbb{P}^*A^{*-1}) & I_{\mathcal{Y}^{nM}} \end{bmatrix} \cdot \begin{bmatrix} A^* & 0 \\ 0 & B^* \end{bmatrix} \end{aligned}$$

where we make use of (3.10) for the last step. Making use of (3.9), we see that the middle factor in this last expression is positive and hence so also is $\widehat{\Phi}_{\text{pre}}^{(n)} \left(\begin{bmatrix} H & \mathbb{P} \\ \mathbb{P}^* & K \end{bmatrix} \right)$, and it follows that $\widehat{\Phi}_{\text{pre}}$ is cp as wanted.

By the Arveson extension theorem (see e.g. [26, Theorem 7.5]), it follows that there is a cp map $\widehat{\Phi}_{\text{ext}}: \mathbb{A}^{2 \times 2} \rightarrow \mathbb{L}^{2 \times 2}$ which extends $\widehat{\Phi}_{\text{pre}}$: $\widehat{\Phi}_{\text{ext}}(X) = \widehat{\Phi}_{\text{pre}}(X)$ for $X \in \mathbb{S}$. We claim next that necessarily $\widehat{\Phi}_{\text{ext}}$ has the form

$$\widehat{\Phi}_{\text{ext}}: \begin{bmatrix} P_{11} & P_{12} \\ P_{21}^* & P_{22} \end{bmatrix} \mapsto \begin{bmatrix} \Phi_{11}(P_{11}) & \phi(P_{12}) \\ \phi(P_{21}^*) & \Phi_{22}(P_{22}) \end{bmatrix} \tag{3.11}$$

for cp maps Φ_{11} and Φ_{22} from \mathbb{A} to \mathbb{L} . To see this, let $\mathcal{S}_0 \subset \mathbb{C}^{2 \times 2}$ be the subalgebra generated by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. This subalgebra is in fact a C^* -subalgebra of $\mathbb{C}^{2 \times 2}$. By conventions which we have already used several times,

we can view both $\mathbb{A}^{2 \times 2}$ as well as $\mathbb{L}^{2 \times 2}$ as \mathcal{S}_0 -bimodules. It is easily checked that the operator system \mathbb{S} is invariant under multiplication on the left as well as on the right by elements of \mathcal{S}_0 , i.e., \mathbb{S} is a \mathcal{S}_0 -sub-bimodule of $\mathbb{A}^{2 \times 2}$. Furthermore it is a direct check to see that the map $\widehat{\Phi}_{\text{pre}}$ given by (3.7) is a $(\mathcal{S}_0, \mathcal{S}_0)$ -bimodule map. As a consequence of Corollary 2.4 it follows that $\widehat{\Phi}_{\text{ext}}$ is also a \mathcal{S}_0 -bimodule map. We claim that this $\widehat{\Phi}_{\text{ext}}$ being a \mathcal{S}_0 -bimodule map forces $\widehat{\Phi}_{\text{ext}}$ to have the form (3.11). Indeed, if $P_{11} \in \mathbb{A}^{2 \times 2}$ has the form $P = \begin{bmatrix} P_{11} & 0 \\ 0 & 0 \end{bmatrix}$, then $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot P \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ where $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{2} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) \in \mathcal{S}_0$. Hence

$$\widehat{\Phi}_{\text{ext}}(P) = \widehat{\Phi}_{\text{ext}} \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \widehat{\Phi}_{\text{ext}}(P) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} =: \begin{bmatrix} \Phi_{11}(P_{11}) & 0 \\ 0 & 0 \end{bmatrix}.$$

for a cp positive map Φ_{11} . Similarly, if $P = \begin{bmatrix} 0 & 0 \\ 0 & P_{22} \end{bmatrix}$, then there is a cp map Φ_{22} so that $\widehat{\Phi}_{\text{ext}}(P) = \begin{bmatrix} 0 & 0 \\ 0 & \Phi_{22}(P_{22}) \end{bmatrix}$. Finally, if $P \in \mathbb{A}^{2 \times 2}$ has the form $P = \begin{bmatrix} 0 & P_{12} \\ 0 & 0 \end{bmatrix}$, then $P \in \mathbb{S}$ and hence

$$\widehat{\Phi}_{\text{ext}}(P) = \widehat{\Phi}_{\text{pre}}(P) = \begin{bmatrix} 0 & \phi(P_{12}) \\ 0 & 0 \end{bmatrix}.$$

By a similar argument we also have

$$\widehat{\Phi}_{\text{ext}} \left(\begin{bmatrix} 0 & 0 \\ P_{21}^* & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ \phi(P_{21}^*) & 0 \end{bmatrix}.$$

By linearity it now follows that $\widehat{\Phi}_{\text{ext}}$ has the form (3.11) as claimed. Let now $\iota: \mathbb{A} \rightarrow \mathbb{A}^{2 \times 2}$ be the cp map

$$\iota: P \mapsto \begin{bmatrix} P & P \\ P & P \end{bmatrix}$$

and finally let $\widehat{\Phi}$ be the composition

$$\widehat{\Phi} = \widehat{\Phi}_{\text{ext}} \circ \iota: \mathbb{A} \rightarrow \mathbb{A}^{2 \times 2}.$$

Then $\widehat{\Phi}$ is a composition of cp maps and hence is cp itself, and has the form exactly the form (3.5) demanded in part (3) of Proposition 3.4. \square

REMARK 3.6. As an example of a subalgebra of $\mathbb{C}^{M \times M}$ which is not already a C^* -subalgebra of $\mathbb{C}^{M \times M}$ one can consider \mathcal{S} equal to the upper (or lower) triangular $M \times M$ matrices. Then it is known that any strictly positive matrix α in $\mathbb{C}^{M \times M}$ can be factored as $\alpha = \beta\beta^*$ with β upper triangular (by the LU factorization algorithm), i.e., spectral factorization holds at the level $N = 1$. However, a simple dimension argument shows that one cannot factor a positive definite block matrix A as a product of $A = BB^*$ where B is a block matrix with each block equal to an upper triangular matrix. Hence the applicability of the assumption of the complete spectral factorization property appears to

be limited to subalgebras \mathcal{S} of $\mathbb{C}^{M \times M}$ which are in fact C^* -algebras, in which case the complete spectral factorization property follows easily from standard C^* -algebra theory.

The definition of the Complete Spectral Factorization Property for \mathcal{S} also involves a notion of minimal C^* -algebra $C^*(\mathcal{S})$ containing \mathcal{S} . To define $C^*(\mathcal{S})$, we are assuming that \mathcal{S} is viewed as a concrete operator algebra inside the concrete C^* -algebra $\mathcal{L}(\mathcal{H})$ for some Hilbert space \mathcal{H} , i.e., we are using a representation $\pi: \mathcal{S} \rightarrow \mathcal{L}(\mathcal{H})$ (in this case the identity representation), and then taking $C^*(\mathcal{S})$ to be the smallest C^* -subalgebra of $\mathcal{L}(\mathcal{H})$ containing $\pi(\mathcal{S})$. In our original definition, we assumed that \mathcal{S} was already a subalgebra of $\mathbb{C}^{N \times N} = \mathcal{L}(\mathbb{C}^N)$ and then used the identity representation. As pointed out by the referee, a striking example where the choice of representation matters is the case where we take \mathcal{S} to be the disk algebra \mathbb{A} (the algebra of continuous functions on the closed unit disk which are analytic on the open unit disk). If we view \mathbb{A} as a subalgebra of the C^* -algebra $C(\mathbb{T})$ (continuous functions on the unit circle), then indeed the complete spectral factorization property holds (see e.g. [27]). However, if instead we identify $f \in \mathbb{A}$ with the Toeplitz operator T_f acting on H^2 , then the C^* -algebra generated by $\{T_f: f \in \mathbb{A}\}$ includes all the compact operators on H^2 (see e.g. [14]) and already spectral factorization fails (e.g. a strictly positive operator of the form $I_{H^2} + K$ with K compact cannot always be factored as $I_{H^2} + K = T_f T_f^*$, as can be seen by taking K to be the projection onto the constant functions and then testing both sides of the equation against the Szegő kernels $(\langle \cdot, k_w, k_z \rangle)$).

In practice by Remark 3.6 it appears to be the case that the only subalgebras of $\mathbb{C}^{M \times M}$ with the complete spectral factorization property are in fact C^* -subalgebras of $\mathbb{C}^{M \times M}$. For simplicity of terminology, let us make the following definition.

DEFINITION 3.7. Let $\Omega = \{Z_i: 1 \leq i \leq N\}$ be a finite subset of \mathcal{V}_{nc} with $Z_i \in \Omega_{n_i}$. Set $N_0 = \sum_{i=1}^N n_i$ so $Z^{(0)} = \bigoplus_{i=1}^N Z_i \in \mathcal{V}^{N_0 \times N_0}$. Associate with Ω the subalgebra \mathcal{S} of $\mathbb{C}^{N_0 \times N_0}$ given by (2.5). We say that the set of points Ω is *admissible* if \mathcal{S} is a C^* -algebra.

With this terminology in hand we can state the following immediate corollary of Theorem 3.5.

COROLLARY 3.8. *Suppose that*

$$K: \Omega \times \Omega \rightarrow \mathcal{L}(\mathcal{A}_{\text{nc}}, \mathcal{L}(\mathcal{Y})_{\text{nc}})$$

is a nc kernel on an admissible finite set of points $\Omega \subset \mathcal{V}_{\text{nc}}$. Then K is decomposable.

We now explore some examples where the admissibility hypothesis does hold and thus Problem A is guaranteed to be solvable.

EXAMPLE 3.9. HERMITIAN DECOMPOSITION FOR COMPLETELY BOUNDED MAPS. Consider the special case where $M = 1$ in Theorem 3.5. Then of course $\mathcal{S} = \mathbb{C} = C^*(\mathcal{S})$ is admissible. Then the result of Corollary 3.8 combined with the equivalence of (1) \Leftrightarrow (3) in Proposition 3.4 gives us a version of Wittstock's decomposition theorem [26, Theorem 8.5].

COROLLARY. Any cb map ϕ from a C^* -algebra \mathcal{A} to the C^* -algebra $\mathcal{L}(\mathcal{Y})$ (\mathcal{Y} equal to some Hilbert space) is decomposable.

Let us also mention that for this case the fact that ϕ being cc implies that $\widehat{\Phi}_{\text{pre}}$ (3.7) is cp appears as Lemma 8.1 in [26] with essentially the same proof as ours.

For the case where $M = 1$ and $\mathcal{S} = \mathbb{C}$, of course once item (3) in Proposition 3.4 is known, then it follows that items (1) and (2) also hold in general. Item (3) is known as the Wittstock extension theorem from [32], item (1) is known as the Wittstock decomposition theorem (see [31]) and item (2) is a Hermitian version of the generalized Stinespring representation for cb maps (see Theorem 8.4 in [26] and the Notes there at the end of the chapter). All these results are developed in Chapter 8 of Paulsen's book [26] but with a somewhat different organization.

It is of interest to specialize all this discussion to the case where $\mathcal{A} = C(X)$ (continuous functions on a compact Hausdorff space X). In case $\mathcal{Y} = \mathbb{C}$, then a map $\phi: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{Y})$ is given by a complex measure μ . In this case μ being of finite total variation is equivalent to ϕ being completely bounded, and the conclusion of Corollary 3.8 for this case amounts to the classical Jordan decomposition for a complex measure.

More generally, in case we still take $\mathcal{A} = C(X)$ but we take \mathcal{Y} to be a general Hilbert space, then a linear map ϕ from \mathcal{A} to $\mathcal{L}(\mathcal{Y})$ corresponds to an operator-valued measure μ , but μ being of finite total variation does not always match up with ϕ being completely bounded. In any case it is the complete boundedness hypothesis which guarantees a Jordan decomposition for the operator-valued measure μ (see [26, pages 104-106] for a fuller discussion).

REMARK 3.10. As pointed out by the anonymous reviewer, the Jordan decomposition theorem for a real signed measure usually also includes the assertion that the decomposition $\mu = \mu_+ - \mu_-$ can be arranged so that the positive and negative parts μ_{\pm} are mutually singular. Note that (i) if μ is a real signed-measure having a decomposition $\mu = \mu_+ - \mu_-$ as the difference between two positive measures and (ii) if μ_0 is any other fixed positive measure, then we get another such decomposition $\mu = \mu'_+ - \mu'_-$ if we set $\mu'_+ = \mu_+ + \mu_0$ and $\mu'_- = \mu_- + \mu_0$. Let us say that the decomposition $\mu = \mu_+ - \mu_-$ (with μ_{\pm} equal to positive measures) is a *minimal decomposition* for μ if this construction is the *only* way to get another such decomposition, i.e.: *whenever $\mu = \mu'_+ - \mu'_-$ is another decomposition of μ as the difference of positive measures, then there is a positive measure μ_0 such that $\mu'_+ = \mu_+ + \mu_0$ and $\mu'_- = \mu_- + \mu_0$* . It is a not difficult measure-theory exercise to show that: *the positive measures μ_+ and μ_- are mutually singular if and only if the decomposition $\mu = \mu_+ - \mu_-$ is a*

minimal decomposition for the real signed measure $\mu := \mu_+ - \mu_-$. Thus the assertion that, given that $\mu = \mu_+ - \mu_-$ is the difference of positive measures, then one can always arrange the decomposition so that μ_{\pm} are mutually singular has the same content as saying that *any decomposable real signed measure μ has a minimal decomposition $\mu = \mu_+ - \mu_-$ as the difference of positive measures.*

This alternative language for mutual singularity for a pair of positive measure μ_{\pm} carries over easily to the nc kernel context as follows. Suppose that a given Hermitian nc kernel K has a decomposition $K = K_+ - K_-$ as the difference of cp nc kernels K_{\pm} . If K_0 is any cp nc kernel, then we get another decomposition of K as $K = K'_+ - K'_-$ with $K'_+ = K_+ + K_0$, $K'_- = K_- + K_0$. If any alternative decomposition $K = K'_+ - K'_-$ arises in this way, we say that the decomposition $K = K_+ - K_-$ is a *minimal decomposition* for K . Let us formulate as a conjecture the following:

CONJECTURE. *Any decomposable Hermitian nc kernel $K = K_+ - K_-$ has a minimal decomposition $K = K_{0+} - K_{0-}$.*

As this topic is outside of the main focus of the present article, we leave the resolution of this conjecture to future work. Let us only point out here that the validity of this conjecture is the missing piece to obtain a complete generalization of the Jordan decomposition theorem for real signed measures to the cp nc kernel context. We say more about this conjecture in Remark 6.1 below.

EXAMPLE 3.11. Ω WITH BLOCK-DIAGONAL SELFADJOINT INTERTWINING SPACE Let us next consider a finite point set $\Omega = \{Z_1, \dots, Z_d\}$ (with $Z_i \in \Omega_{n_i}$) contained in the nc universal space \mathcal{V}_{nc} such that

$$\mathcal{I}(Z_i, Z_j) = \begin{cases} 0 & \text{if } i \neq j, \\ \text{a } C^* \text{-subalgebra } \mathcal{D}_i \text{ of } \mathbb{C}^{n_i \times n_j} & \text{if } i = j. \end{cases} \tag{3.12}$$

where we use the notation

$$\mathcal{I}(Z_i, Z_j) = \{\alpha \in \mathbb{C}^{n_i \times n_j} : Z_i \alpha = \alpha Z_j\}.$$

Roughly, for $i \neq j$ there is no piece of Z_i similar to a piece of Z_j , and for $i = j$, each Z_i has a nice commutant over $\mathbb{C}^{n_i \times n_i}$. For this *block-diagonal selfadjoint intertwining space* case, the intertwining algebra $\mathcal{S} = \mathcal{I}(Z^{(0)})$ is equal to $\bigoplus_{i=1}^N \mathcal{D}_i$ and hence Ω is admissible, and Corollary 3.8 leads to the following result.

COROLLARY. *Suppose that $K : \Omega \times \Omega \rightarrow \mathcal{L}(\mathcal{A}_{nc}, \mathcal{L}(\mathcal{Y})_{nc})$ is a cb nc kernel and Ω has the block-diagonal selfadjoint intertwining space property (3.12). Then K is decomposable.*

A particular instance of this scenario is the case where Ω consists exclusively of scalar points which are all distinct:

$$\Omega = \{z_1, \dots, z_N : z_i \neq z_j \text{ for } i \neq j\} = \Omega_1 \subset \mathcal{V}_{nc}.$$

In this case \mathcal{S} is the C^* -algebra consisting of the diagonal matrices

$$\mathcal{S} = \mathcal{D}_N := \{\text{diag}_{1 \leq i \leq N_0=N}[\lambda_i] \text{ with } \lambda_i \in \mathbb{C} \text{ for all } i\}.$$

A cp nc kernel on a set of distinct scalar points can be viewed as a cp kernel in the sense of Barreto-Bhat-Liebscher-Skeide [10]. By the preceding Corollary we conclude that *any BBLs-kernel from Ω to $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))$ is decomposable*. This essentially recovers the main result of [11] for the finite-point case; there the connection with \mathcal{D}_N -bimodule maps $\phi: \mathcal{A}^{n \times n} \rightarrow \mathcal{L}(\mathcal{Y})^{N \times N}$ plays a prominent role, including the characterization of \mathcal{D}_N -bimodule maps as entry-wise maps as was used here for the $N = 2$ case in the proof of (3.11), but with the connection with nc kernels not made explicit.

Let us specialize still further by taking $\mathcal{A} = \mathbb{C}$. Then we may identify ϕ with its value at $1 \in \mathbb{C}$; thus ϕ amount to an operator $\phi(1) = T \in \mathcal{L}(\mathcal{Y})$. Boundedness here is the same as complete boundedness and Corollary 3.8 gives us the decomposition $T = T_1 - T_2 + i(T_3 - T_4)$ where T_j are positive operators on \mathcal{H} .

REMARK 3.12. For this discussion let us write the set of points of Ω with superscripts

$$\Omega = \{Z^{(1)}, \dots, Z^{(N)}\} \text{ with } Z^{(i)} \in \Omega_{n_i}.$$

As Ω consists of only finitely many points, the linear span of all the coordinates of points in Ω

$$\text{span}\{z_{ij}^{(k)} : 1 \leq i, j \leq n_k, 1 \leq k \leq N\} \text{ where } Z^{(k)} = [z_{ij}^{(k)}]_{1 \leq i, j \leq n_k} \in \mathcal{V}^{n_k \times n_k}.$$

spans a finite-dimensional subspace \mathcal{M} of \mathcal{V} . As long as we are working with this finite subset of \mathcal{V}_{nc} , without loss of generality we may assume that $\mathcal{V} = \mathbb{C}^d$ for a sufficiently large d . By choosing a basis for \mathcal{M} we may identify $\mathcal{M} = \mathcal{V} = \mathbb{C}^d$ (with $d = \dim \mathcal{M}$).

We can then use this representation to determine when Ω is admissible. The result is as follows.

PROPOSITION. (1) *If $d = 1$, then Ω is admissible if and only if the matrix $Z^{(0)} = Z^{(1)} \oplus \dots \oplus Z^{(N)}$ is normal, i.e., there is an orthonormal basis which diagonalizes $Z^{(0)}$.*

(2) *For $d > 1$, the case where Ω is admissible is generic, i.e., the set of $Z^{(0)} \in (\mathbb{C}^{N_0 \times N_0})^d$ such that Ω is admissible is dense in the space of all $Z^{(0)} \in (\mathbb{C}^{N_0 \times N_0})^d$.*

PROOF. Suppose that $d = 1$ and $Z^{(0)}$ is normal. Thus $Z^{(0)} = \sum_{i=1}^M \lambda_i P_i$ where $\lambda_1, \dots, \lambda_M$ are the eigenvalues of $Z^{(0)}$ where P_1, \dots, P_M is a pairwise-orthogonal family of orthogonal projections summing to the identity. Then for p a polynomial, we have $p(Z^{(0)}) = \sum_{i=1}^M p(\lambda_i) P_i$. In particular, by solving a Lagrange interpolation problem we can always find a polynomial p_i with $p_i(\lambda_i) = 1$ and $p_i(\lambda_j) = 0$ for $j \neq i$; with this choice of polynomial we have

$p_i(Z^{(0)}) = P_i$. We conclude that if $\alpha \in \mathbb{C}^{N_0 \times N_0}$ commutes with $Z^{(0)}$ then α commutes with each P_i and hence must have the form

$$\alpha = \sum_{i=1}^M \alpha_i P_i \quad (3.13)$$

where α_i is an operator on $\text{Ran } P_i$. Conversely, any operator of this form commutes with $Z^{(0)}$, and hence the form (3.13) characterizes the commutant of $Z^{(0)}$. If α is of the form (3.13), then $\alpha^* = \sum_{i=1}^M \alpha_i^* P_i$ is also of this form, so $\mathcal{S} = \mathcal{I}(Z^{(0)})$ is selfadjoint, i.e., Ω is admissible in this case.

Suppose next that $Z^{(0)}$ is not normal. Then certainly $Z^{(0)}$ commutes with itself, but on the other hand $Z^{(0)*}$ does not commute with $Z^{(0)}$ (i.e., $Z^{(0)*} Z^{(0)} \neq Z^{(0)} Z^{(0)*}$) since $Z^{(0)}$ is not normal. Hence the commutant of $\{Z^{(0)}\}$ is not a C^* -algebra and Ω is not admissible.

Next suppose that $d > 1$ and $Z^{(0)} = (Z_1^{(0)}, \dots, Z_d^{(0)})$ where each $Z_j^{(0)} \in \mathbb{C}^{N_0 \times N_0}$. In this case it is known that generically the algebra generated by the matrices $Z_1^{(0)}, \dots, Z_d^{(0)}$ in $\mathbb{C}^{N_0 \times N_0}$ is all of $\mathbb{C}^{N_0 \times N_0}$ (see [24] for a recent treatment). If we assume that we are in this generic case, then, given any $W \in \mathbb{C}^{N_0 \times N_0}$ there is a nc polynomial p in d arguments so that $p(Z_1^{(0)}, \dots, Z_d^{(0)}) = W$. Hence, if $\alpha \in \mathbb{C}^{N_0 \times N_0}$ is in the intertwining space $\mathcal{I}(Z^{(0)})$, then

$$\alpha W = \alpha p(Z^{(0)}) = p(Z^{(0)})\alpha = W\alpha$$

i.e., α commutes with all of $\mathbb{C}^{N_0 \times N_0}$. This forces α to be a scalar multiple of the identity matrix. Hence $\mathcal{I}(Z^{(0)})$ consists of scalar matrices, and in particular is selfadjoint. We conclude that for this case a generic set of points Ω is admissible as claimed. \square

We shall discuss possible extensions of the kernel version of Theorem 3.5 to infinite point sets Ω in Section 6.1.

Our next partial result concerning the decomposability problem for nc Hermitian kernels/Hermitian maps is based on criterion (2) in Propositions 3.2 and 3.4. We shall state only for the setting of Hermitian maps.

THEOREM 3.13. *Suppose that \mathcal{S} is a subalgebra of $\mathbb{C}^{M \times M}$ and that ϕ is a Hermitian map from $\mathcal{A}^{M \times M}$ to $\mathcal{L}(\mathcal{Y})^{M \times M}$.*

(1) *If ϕ is an $(\mathcal{S}, \mathcal{S}^*)$ -bimodule decomposable map, then ϕ has a Hermitian Stinespring representation $\phi(P) = V\pi(P)JV^*$ satisfying the extra condition (3.4) as in part (2) of Proposition 3.4.*

(2) *If ϕ is a $(\mathcal{S}, \mathcal{S}^*)$ -bimodule map, then any Hermitian Stinespring representation $\phi(P) = V\pi(P)JV^*$ as in part (2) of Proposition 3.4 (taken with $\mathcal{S} = \mathbb{C}$ so that condition (3.4) can be ignored) automatically also satisfies (3.4) but only in the weaker form*

$$\text{span}_{\beta \in \mathcal{S}} \text{Ran}(\pi(\beta^*)V^* - VL_{\beta^*}) \text{ is a } J\text{-isotropic subspace of } \mathcal{X}. \quad (3.14)$$

Proof. Item (1) is just a restatement of part (2) of Proposition 3.4. As for item (2), we compute, for $y, y' \in \mathcal{Y}^M$ and $\beta, \beta' \in \mathcal{S}$,

$$\begin{aligned} & \langle J(\pi(\beta^*)V^* - V^*L_{\beta^*})y, (\pi(\beta'^*)V^* - V^*L_{\beta'^*})y' \rangle \\ &= \left\langle y, \left(V\pi(\beta\beta'^*)V^* - L_\beta V\pi(\beta'^*)V^* - V\pi(\beta)V^*L_{\beta'^*} + L_\beta V^*V L_{\beta'^*} \right) y' \right\rangle \\ &= \left\langle y, \left(\phi(\beta\beta'^*) - L_\beta\phi(\beta'^*) - \phi(\beta)L_{\beta'^*} + L_\beta\phi(1_{\mathcal{A}^{M \times M}})L_{\beta'^*} \right) y' \right\rangle \\ &= \langle y, (\phi(\beta\beta'^*) - \phi(\beta\beta'^*) - \phi(\beta\beta'^*) + \phi(\beta\beta'^*))y' \rangle = 0 \end{aligned}$$

where we used that ϕ is a $(\mathcal{S}, \mathcal{S}^*)$ -bimodule map in the penultimate step. This holding for all $y, y' \in \mathcal{Y}^M$ and $\beta, \beta' \in \mathcal{S}$ is just the statement that the subspace described in (3.14) is J -isotropic. \square

4 PROBLEM B: THE ARVESON EXTENSION PROBLEM FOR NONCOMMUTATIVE POSITIVE KERNELS

The goal of this section is to solve affirmatively the Arveson extension problem for cp nc kernels in place of a cp map on a finite set Ω . The case where Ω is allowed to be infinite is considered in Section 6.2. This proof amounts to combining Corollary 2.4 with the Arveson extension theorem for completely positive maps.

THEOREM 4.1. *Suppose that K is a nc kernel on a finite set of nc points $\Omega \subset \mathcal{V}_{\text{nc}}$ with values in $\mathcal{L}(\mathbb{S}_{\text{nc}}, \mathcal{L}(\mathcal{Y})_{\text{nc}})$, where \mathbb{S} is a operator system and $\mathcal{L}(\mathcal{Y})$ is the C^* -algebra of operators on the Hilbert space \mathcal{Y} . Let \mathcal{A} be a C^* -algebra containing \mathbb{S} . Then there exists a cp nc kernel \widehat{K}*

$$\widehat{K}: \Omega \times \Omega \rightarrow \mathcal{L}(\mathcal{A}_{\text{nc}}, \mathcal{L}(\mathcal{Y})_{\text{nc}})$$

such that $\widehat{K}(Z, W)(P) = K(Z, W)(P)$ for all $Z \in \Omega_n, W \in \Omega_m$ whenever $P \in \mathbb{S}^{n \times m}$.

Proof. Let us assume that the finite set $\Omega \subset \mathcal{V}_{\text{nc}}$ is given by

$$\Omega = \{Z_1, \dots, Z_N\} \subset \mathcal{V}_{\text{nc}}.$$

Set $Z^{(0)} = \text{diag}_{1 \leq i \leq N} [Z_i]$, set $N_0 = \sum_{i=1}^N n_i$, let \mathcal{S} be the intertwining algebra of $Z^{(0)}$ as in (2.5)

$$\mathcal{S} = \{\alpha \in \mathbb{C}^{N_0 \times N_0} : \alpha Z^{(0)} = Z^{(0)} \alpha\},$$

and let ϕ_K be the cp $(\mathcal{S}, \mathcal{S}^*)$ -bimodule map from $\mathbb{S}^{N_0 \times N_0}$ to $\mathcal{L}(\mathcal{Y}^{N_0})$ associated with K as in (2.4). By the Arveson extension theorem for completely positive maps (see [26, Theorem 7.5]), there is an extension of ϕ_K to a cp map $\widehat{\phi}: \mathcal{A}^{N_0 \times N_0} \rightarrow \mathcal{L}(\mathcal{Y}^{N_0})$. By Corollary 2.4, any such $\widehat{\phi}$ is also a $(\mathcal{S}, \mathcal{S}^*)$ -bimodule

map. Then by Proposition 2.2 applied in the reverse direction, we conclude that $\widehat{\phi} = \phi_{\widehat{K}}$ for a cp nc kernel \widehat{K} on Ω with values equal to operators from \mathcal{A}_{nc} to $\mathcal{L}(\mathcal{Y})_{\text{nc}}$. As $\phi_{\widehat{K}} = \widehat{\phi}$ is an extension of ϕ_K , we conclude that \widehat{K} is an extension of K as wanted. \square

5 PROBLEM C: KERNEL-DOMINANCE CERTIFICATES

In this section we consider the finite-point case of Problem C. Thus we assume that the point set $\Omega \subset \mathcal{V}_{\text{nc}}$ is finite. Given a point Z in some nc set Ω , we shall use the notation n_Z for the natural number n such that Z has matrix size $n \times n$ (i.e., $Z \in \Omega_n$).

THEOREM 5.1. *We let \mathcal{E} and \mathcal{N} be Hilbert spaces. We are given a full nc subset Ξ of \mathcal{V}_{nc} together with a Hermitian nc kernel*

$$\Omega: \Xi \times \Xi \rightarrow \mathcal{L}(\mathbb{C}_{\text{nc}}, \mathcal{L}(\mathcal{N})_{\text{nc}}).$$

Let \mathbb{P}_{Ω} be the strict positivity domain for Ω as defined by

$$\mathbb{P}_{\Omega} = \coprod_{n=1}^{\infty} \{Z \in \Xi_n : \Omega(Z, Z)(I_n) \succ 0\}.$$

Suppose that:

- (i) Ω is a subset of \mathbb{P}_{Ω} .
- (ii) \mathfrak{S} is a Hermitian nc kernel defined on $\Omega' := [\Omega]_{\text{full}} \cap \mathbb{P}_{\Omega}$

$$\mathfrak{S}: \Omega' \times \Omega' \rightarrow \mathcal{L}(\mathbb{C}_{\text{nc}}, \mathcal{L}(\mathcal{E})_{\text{nc}})$$

such that

$$\mathfrak{S}(Z, Z)(1_{\mathbb{C}^{n_Z \times n_Z}}) \succeq 0 \text{ for all } Z \in \Omega'. \quad (5.1)$$

- (iii) The restriction of Ω to any finite subset Ξ_{finite} of Ξ is decomposable.
- (iv) The restriction of \mathfrak{S} to any finite subset Ω'_{finite} of Ω' is decomposable.

Then there exist two completely positive nc kernels on Ω

$$\Gamma_1: \Omega \times \Omega \rightarrow \mathcal{L}(\mathcal{L}(\mathcal{N})_{\text{nc}}, \mathcal{L}(\mathcal{E})_{\text{nc}}), \quad \Gamma_2: \Omega \times \Omega \rightarrow \mathcal{L}(\mathbb{C}_{\text{nc}}, \mathcal{L}(\mathcal{E})_{\text{nc}}) \quad (5.2)$$

so that, for all $Z \in \Omega_n$, $W \in \Omega_m$, $P \in \mathbb{C}^{n \times m}$ we have

$$\mathfrak{S}(Z, W)(P) = \Gamma_1(Z, W)(\Omega(Z, W)(P)) + \Gamma_2(Z, W)(P). \quad (5.3)$$

Proof. In this section we consider only the case where two additional hypotheses are in force:

- (H1) The set $\Omega \subset \mathbb{P}_{\Omega}$ is finite.
- (H2) The coefficient Hilbert space \mathcal{E} is finite-dimensional.

In applying conditions (ii) and (iv) in the statement of the theorem, we shall write simply Ω rather than Ω_{fin} . The reduction of the general case to this special case will be discussed in Section 6.3.

We are assuming that Ξ is a subset of the nc set \mathcal{V}_{nc} generated by the linear space \mathcal{V} . As Ω is finite by (H1), the span of all the matrix entries of all the finitely many points in Ω is contained in some finite-dimensional subspace \mathcal{V}_0 of \mathcal{V} . By choosing some basis for \mathcal{V}_0 , we may identify \mathcal{V}_0 with \mathbb{C}^d (d sufficiently large) and thereby view elements of Ω as points in $(\mathbb{C}^d)_{\text{nc}}$ as well. Thus we may assume without loss of generality that $\mathcal{V} = \mathbb{C}^d$. To make use of this reduction it is sometimes convenient to spell out the components of a point $Z \in \Omega$ which exhibits its membership in \mathbb{C}^d : using that $(\mathbb{C}^d)^{n \times n} \cong (\mathbb{C}^{n \times n})^d$, we shall write a typical point of Ω as

$$Z = (Z_1, \dots, Z_d) \text{ where each } Z_i \in \mathbb{C}^{n \times n} \text{ if } Z \in \Omega_n. \tag{5.4}$$

Then the action of a pair of complex matrices α, β of compatible sizes on a point $Z = (Z_1, \dots, Z_d)$ is component-wise:

$$\alpha \cdot (Z_1, \dots, Z_d) \cdot \beta = (\alpha Z_1 \beta, \dots, \alpha Z_d \beta)$$

where the multiplication on the right side of the equation is ordinary matrix multiplication. We shall use superscripts to distinguish various points of Ω ; thus we shall list the points of Ω as

$$\Omega = \{Z^{(1)}, \dots, Z^{(N)}\} \tag{5.5}$$

where $Z^{(i)}$ has the form $Z^{(i)} = (Z_1^{(i)}, \dots, Z_d^{(i)})$ with each component $Z_j^{(i)}$ in $\mathbb{C}^{n_i \times n_i}$ if $Z \in \Omega_{n_i}$.

We let \mathfrak{X} be the linear space of all graded kernels K on Ω with values in $\mathcal{L}(\mathbb{C}, \mathcal{L}(\mathcal{E}))_{\text{nc}}$; thus $K \in \mathfrak{X}$ means that K is an operator-valued function on $\Omega \times \Omega$ such that

$$K(Z, W) \in \mathcal{L}(\mathbb{C}^{n \times m}, \mathcal{L}(\mathcal{E})^{n \times m}) \text{ if } Z \in \Omega_n \text{ and } W \in \Omega_m. \tag{5.6}$$

We make \mathfrak{X} a Banach space by endowing \mathfrak{X} with the supremum norm:

$$\|K\|_{\mathfrak{X}} = \max\{\|K(Z, W)\| : Z, W \in \Omega_0\}.$$

As Ω is finite and \mathcal{E} is finite-dimensional by assumption, this Banach space is finite-dimensional. Hence bounded subsets of \mathfrak{X} are pre-compact in the norm topology. Thus bounded sequences always have convergent subsequences. Furthermore, convergence of a sequence $\{K_k\}_{k \in \mathbb{N}}$ to K in the norm topology of \mathfrak{X} concretely just means pointwise that $K_k(Z, W)(P)$ converges to $K(Z, W)(P)$ in the norm topology of $\mathcal{L}(\mathcal{E}^m, \mathcal{E}^n)$ for each $Z \in \Omega_n, W \in \Omega_m$, and $P \in \mathbb{C}^{n \times m}$ for all $n, m = 1, 2, \dots$

We define a subset \mathcal{C} of \mathfrak{X} to consist of all graded kernels K in \mathfrak{X} which have the form

$$K(Z, W)(P) = \Gamma_1(Z, W)(\Omega(Z, W)(P)) + \Gamma_2(Z, W)(P) \tag{5.7}$$

for all $Z \in \Omega_n$, $W \in \Omega_m$, $P \in \mathbb{C}^{n \times m}$ and $n, m \in \mathbb{Z}_+$ for some cp nc kernels Γ_1 and Γ_2 on Ω :

$$\Gamma_1: \Omega \times \Omega \rightarrow \mathcal{L}(\mathcal{L}(\mathcal{N})_{\text{nc}}, \mathcal{L}(\mathcal{E})_{\text{nc}}), \quad \Gamma_2: \Omega \times \Omega \rightarrow \mathcal{L}(\mathbb{C}_{\text{nc}}, \mathcal{L}(\mathcal{E})_{\text{nc}}).$$

Then we have

LEMMA 5.2. *The subset \mathcal{C} is a closed pointed cone in \mathfrak{X} .*

Proof. It is easily verified that \mathcal{C} is a cone. To see that \mathcal{C} is pointed, we must check that if $K \in \mathcal{C}$ and $-K \in \mathcal{C}$ then $K = 0$. We see that such a K must have the form

$$\begin{aligned} K(Z, W)(P) &= \Gamma_1(Z, W)(\mathfrak{Q}(Z, W)(P)) + \Gamma_2(Z, W)(P) \\ &= -\Gamma_3(Z, W)(\mathfrak{Q}(Z, W)(P)) - \Gamma_4(Z, W)(P). \end{aligned}$$

where Γ_i is a cp nc kernel for $i = 1, 2, 3, 4$. We know that $\mathfrak{Q}(Z, Z)(I) \succ 0$ for $Z \in \Omega$ which results in $0 \preceq K(Z, Z)(I) \preceq 0$. We conclude that $K(Z, Z)(I) = 0$ which results in $\Gamma_2(Z, Z)(I) = \Gamma_4(Z, Z)(I) = 0$ as well. By Lemma 2.1, the cp nc kernels Γ_2 and Γ_4 must be zero which means K has the simpler form

$$K(Z, W)(P) = \Gamma_1(Z, W)(\mathfrak{Q}(Z, W)(P)) = -\Gamma_3(Z, W)(\mathfrak{Q}(Z, W)(P)).$$

As Ω is a finite subset of \mathbb{P}_Ω , we may choose ϵ_0 with $0 < \epsilon_0 < 1$ so that

$$\mathfrak{Q}(Z, Z)(I_n) \succeq \epsilon_0 I_{G^n} \text{ for all } Z \in \Omega_n \text{ (n=1,2,\dots)}.$$

Hence, we have

$$\begin{aligned} 0 &= K(Z, Z)(I) = \Gamma_1(Z, Z)(\mathfrak{Q}(Z, Z)(I)) \succeq \epsilon_0 \Gamma_1(Z, Z)(I) \\ 0 &= K(Z, Z)(I) = -\Gamma_3(Z, Z)(\mathfrak{Q}(Z, Z)(I)) \preceq -\epsilon_0 \Gamma_3(Z, Z)(I) \end{aligned}$$

and it follows that $\Gamma_1(Z, Z)(I) = \Gamma_3(Z, Z)(I) = 0$. We apply Lemma 2.1 to conclude that $\Gamma_1 = \Gamma_3 = 0$ which gives the desired result, $K = 0$.

We now show that the cone is closed. Suppose that $\{K_k : k \in \mathbb{N}\}$ is a sequence of elements of \mathcal{C} such that $\|K - K_k\|_{\mathfrak{X}} \rightarrow 0$ as $k \rightarrow \infty$ for some $K \in \mathfrak{X}$. The problem is to show that K is again in \mathcal{C} .

By definition for each k there are cp nc kernels $\Gamma_{1,k}$ from $\Omega \times \Omega$ to $\mathcal{L}(\mathcal{L}(\mathcal{N}), \mathcal{L}(\mathcal{E}))_{\text{nc}}$ and $\Gamma_{2,k}$ from $\Omega \times \Omega$ to $\mathcal{L}(\mathbb{C}, \mathcal{L}(\mathcal{E}))_{\text{nc}}$ so that

$$K_k(Z, W)(P) = \Gamma_{1,k}(Z, W)(\mathfrak{Q}(Z, W)(P)) + \Gamma_{2,k}(Z, W)(P)$$

for all $Z \in \Omega_n$, $W \in \Omega_m$, $P \in \mathbb{C}^{n \times m}$ for all $m, n \in \mathbb{N}$. As Ω is a finite subset of \mathbb{P}_Ω , we may choose ϵ_0 with $0 < \epsilon_0 < 1$ so that

$$\mathfrak{Q}(Z, Z)(I_n) \succeq \epsilon_0 I_{\mathcal{N}^n} \text{ for all } Z \in \Omega_n \text{ (n=1,2,\dots)}.$$

Hence

$$\begin{aligned} K_k(Z, Z)(I_n) &= \Gamma_{1,k}(Z, Z)(\Omega(Z, Z)(I_{\mathcal{G}^n})) + \Gamma_{2,k}(Z, Z)(I_n) \\ &\succeq \epsilon_0 \Gamma_{1,k}(Z, Z)(I_{\mathcal{N}^n}) + \Gamma_{2,K}(Z, Z)(I_n) \end{aligned}$$

from which we get the estimates

$$\Gamma_{1,k}(Z, Z)(I_{\mathcal{G}^n}) \preceq (1/\epsilon_0)K_k(Z, Z)(I_n), \quad \Gamma_{2,k}(Z, Z)(I_n) \preceq K_k(Z, Z)(I_n) \tag{5.8}$$

for $k = 1, 2, 3, \dots$. As the sequence K_k is converging to K in \mathfrak{X} -norm, it follows that $\|K_k(Z, Z)(I_n)\|$ is uniformly bounded in norm with respect to $k = 1, 2, \dots$. We conclude that both collections $\Gamma_{1,k}(Z, Z)(I_n)$ and $\Gamma_{2,k}(Z, Z)(1_{\mathcal{A}^{n \times n}})$ are uniformly bounded in norm (with respect to $k = 1, 2, \dots$). As $\Gamma_{1,k}$ and $\Gamma_{2,k}$ are cp, it is known that

$$\begin{aligned} \|\Gamma_{1,k}(Z, Z)\|_{\mathcal{L}(\mathcal{L}(\mathcal{N})^{n \times n}, \mathcal{L}(\mathcal{E})^{n \times n})} &= \|\Gamma_{1,k}(Z, Z)(I_{\mathcal{N}^n})\|_{\mathcal{L}(\mathcal{E}^n)} \\ \|\Gamma_{2,k}(Z, Z)\|_{\mathcal{L}(\mathbb{C}^{n \times n}, \mathcal{L}(\mathcal{E})^{n \times n})} &= \|\Gamma_{2,k}(Z, Z)(I_n)\|_{\mathcal{L}(\mathcal{E}^n)} \end{aligned}$$

We conclude that the collection of numbers

$$\{\|\Gamma_{1,k}(Z, Z)\|, \|\Gamma_{2,k}(Z, Z)\| : Z \in \Omega, k = 1, 2, \dots\} \tag{5.9}$$

is bounded.

A consequence of the *respects direct sums* property for nc kernels is that

$$\Omega\left(\begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix}, \begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix}\right)\left(\begin{bmatrix} I_n & 0 \\ 0 & I_m \end{bmatrix}\right) = \begin{bmatrix} \Omega(Z, Z)(I_n) & 0 \\ 0 & \Omega(W, W)(I_m) \end{bmatrix}$$

for $Z \in \Omega_n, W \in \Omega_m$. Consequently, if $Z, W \in \mathbb{P}_\Omega$, then also $\begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix} \in \mathbb{P}_\Omega$. Then the same argument as given above leads to the conclusion that the set of numbers, a priori larger than that in (5.9),

$$\{\|\Gamma_{1,k}\left(\begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix}, \begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix}\right)\|, \|\Gamma_{2,k}\left(\begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix}, \begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix}\right)\| : Z, W \in \Omega_0, N \in \mathbb{N}\} \tag{5.10}$$

is bounded.

Another consequence of the *respects direct sums* property of nc kernels is that, for any nc kernel Γ ,

$$\Gamma\left(\begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix}, \begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix}\right)\left(\begin{bmatrix} 0 & P \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & \Gamma(Z, W)(P) \\ 0 & 0 \end{bmatrix}.$$

From this identity combined with the boundedness of the set (5.10) we can read off that the set

$$\{\|\Gamma_{1,k}(Z, W)\|, \|\Gamma_{2,k}(Z, W)\| : Z, W \in \Omega, k \in \mathbb{N}\} \tag{5.11}$$

is uniformly bounded as well.

By the aforementioned pre-compactness in the norm-topology of norm-bounded subsets in the Banach space \mathfrak{X} , we conclude that there exists a subsequence

$\{k_j\}_{j \in \mathbb{N}}$ so that Γ_{2,k_j} converges to some kernel $\Gamma_2 \in \mathfrak{X}$ in the norm topology of \mathfrak{X} .

Note that the kernels $\Gamma_{1,k}$ sit in the space \mathfrak{X}' consisting of graded kernels from $\Omega \times \Omega$ to $\mathcal{L}(\mathcal{L}(\mathcal{N})_{\text{nc}}, \mathcal{L}(\mathcal{E})_{\text{nc}})$, i.e., just the space \mathfrak{X} but with the C^* -algebra \mathbb{C} replaced by the C^* -algebra $\mathcal{L}(\mathcal{N})$. This space is infinite-dimensional whenever \mathcal{N} is infinite-dimensional. To handle this term, we need the following more sophisticated argument. It can be shown that \mathfrak{X}' is a dual Banach space and hence is equipped with a weak-* topology and furthermore, convergence of a net K_α to K in the weak-* topology of \mathfrak{X}' concretely just means pointwise convergence of $K_\alpha(Z, W)(P)$ to $K(Z, W)(P)$ in the weak-* topology of $\mathcal{L}(\mathcal{E}^m, \mathcal{E}^n)$ (in fact the norm topology for our setting here since \mathcal{E} is finite-dimensional). For additional information on this point we refer to [26, Corollary 2 page 230] and [30, Section IV.2] as well as [9, Section 4] for an application very similar to what is being done here. Therefore essentially the same analysis as above applies equally well to the sequence $\Gamma_{1,N}$ of kernels in \mathfrak{X}' but we may need to drop down to subnets rather than to subsequences. By dropping down to a subnet $\{\Gamma_{1,\alpha}\}_{\alpha \in \mathfrak{A}}$ if necessary, we may assume that the net $\Gamma_{1,\alpha}$ converges weak-* to a kernel Γ_1 in \mathfrak{X}' . By using the identification of weak-* convergence of the kernel-net Γ_α as pointwise weak-* convergence of the values-net $\Gamma_\alpha(Z, W)(P)$ in $\mathcal{L}(\mathcal{E}^{nw}, \mathcal{E}^{nz})$ for each $Z, W \in \Omega$ and $P \in \mathbb{C}_{\text{nc}}$, one can see that the weak-* limit of a net of cp nc kernels is again a cp nc kernel. In this way we see that the limit kernels Γ_1 and Γ_2 are again cp nc kernels. Furthermore, again making use of the concrete identification of weak-* convergence in \mathfrak{X} and taking the limit along the net α in the identity

$$K_\alpha(Z, W)(P) = \Gamma_{1,\alpha}(Z, W)(\mathfrak{Q}(Z, W)(P)) + \Gamma_{2,\alpha}(Z, W)(P),$$

we see that Γ_1 and Γ_2 give the desired representation (5.7) for membership of the limit kernel K in the cone \mathcal{C} , as needed to complete the proof of Lemma 5.2. \square

THE CONE SEPARATION ARGUMENT. Recall that \mathfrak{X} is the finite dimensional Banach space of graded kernels given by (5.6). We established in Lemma 5.2 that the cone of nc kernels $\mathcal{C} \subset \mathfrak{X}$ defined by (5.7) is both closed and pointed. We now show by contradiction that the kernel \mathfrak{S} lies in the cone \mathcal{C} . Suppose that \mathfrak{S} does not lie in the cone \mathcal{C} . Then there exists a separating linear functional l_1 on \mathfrak{X} such that $l_1(\mathfrak{S}) < 0$ and $l_1(\mathcal{C}) \geq 0$ [28]. Since \mathcal{C} is a closed pointed cone, there exists another linear functional l_2 on \mathfrak{X} such that $l_2(\mathcal{C} \setminus \{0\}) > 0$ [19, Theorem 3.38]. As $l_1(\mathfrak{S}) < 0$, we may choose $\epsilon > 0$ so that

$$l_1(\mathfrak{S}) + \epsilon l_2(\mathfrak{S}) < 0.$$

Let us then set $l = l_1 + l_2$. Then l is a linear function on \mathfrak{X} with the property that

$$l(\mathfrak{S}) < 0 \text{ and } l(\mathcal{C} \setminus \{0\}) > 0. \quad (5.12)$$

For $f : \Omega \rightarrow \mathcal{E}_{\text{nc}}$ a graded function on Ω , given $Z \in \Omega_n$, we see that $f(Z) \in \mathcal{E}^{n \times n} \simeq \mathcal{L}(\mathbb{C}, \mathcal{E})^{n \times n}$. We define the function $f^* : \Omega \rightarrow \mathcal{L}(\mathcal{E}, \mathbb{C})^{n \times n}$ by setting $f^*(Z) = f(Z)^*$, and we construct vector spaces \mathcal{G} and \mathcal{H} by

$$\begin{aligned} \mathcal{G} &= \{f^* \mid f: \Omega \rightarrow \mathcal{E}_{\text{nc}} \text{ a graded function}\} \\ \mathcal{H} &= \{f^* \mid f: \Omega \rightarrow \mathcal{E}_{\text{nc}} \text{ a nc function}\}. \end{aligned}$$

As any nc function is graded by definition, we have the vector-space inclusion $\mathcal{H} \subset \mathcal{G}$. Since Ω is a finite set of points and \mathcal{E} is finite-dimensional, it is easily checked that both \mathcal{G} and \mathcal{H} are finite-dimensional.

We show that the linear functional ℓ induces an inner product on \mathcal{H} . Notice that any $f^*, g^* \in \mathcal{H}$ induce a nc kernel $K_{f,g} : \Omega \times \Omega \rightarrow \mathcal{L}(\mathbb{C}_{\text{nc}}, \mathcal{L}(\mathcal{E})_{\text{nc}})$ by

$$K_{f,g}(Z, W)(P) = f(Z)Pg(W)^*.$$

With the linear functional ℓ , we define an inner product on \mathcal{H} by

$$\langle f^*, g^* \rangle_{\mathcal{H}} = \ell(K_{f,g}). \tag{5.13}$$

Note that $\langle f^*, f^* \rangle_{\mathcal{H}} = \ell(K_{f,f}) \geq 0$ by the fact that $K_{f,f}$ is in the cone \mathcal{C} (5.7) and $\langle f^*, f^* \rangle_{\mathcal{H}} = 0 \Rightarrow f^* = 0$ by (5.12). Then \mathcal{H} equipped with this inner product becomes a Hilbert space, still denoted as \mathcal{H} .

As in (5.5), let us denote the finite set of points Ω by $\Omega = \{Z^{(1)}, \dots, Z^{(N)}\}$ with $Z^{(i)} \in \Omega_{n_i}$. We let $Z^{(0)}$ denote the direct sum

$$Z^{(0)} = \bigoplus_i^N Z^{(i)} \in (\mathbb{C}^d)^{N_0 \times N_0} \text{ where } N_0 = \sum_{i=1}^N n_i. \tag{5.14}$$

We define a mapping

$$\mathcal{I} : \mathcal{G} \rightarrow \mathcal{L}(\mathcal{E}^{N_0}, \mathbb{C}^{N_0})$$

by

$$\mathcal{I} : f^* \mapsto f(Z^{(0)})^*.$$

Let $\widehat{\mathcal{G}} := \mathcal{I}(\mathcal{G}) \subset \mathcal{L}(\mathcal{E}^{N_0}, \mathbb{C}^{N_0 \times N_0})$. We make $\widehat{\mathcal{G}}$ a Hilbert space by equipping it with the inner product

$$\langle \mathcal{I}(f^*), \mathcal{I}(g^*) \rangle = \text{tr}(f(Z^{(0)})^*g(Z^{(0)})) \tag{5.15}$$

The map \mathcal{I} is a vector space isomorphism between \mathcal{G} and $\widehat{\mathcal{G}}$. We let $\widehat{\mathcal{H}} = \mathcal{I}(\mathcal{H})$, and we let \mathcal{I}_0 be the restriction of \mathcal{I} to \mathcal{H} . Then \mathcal{I}_0 is a vector-space isomorphism between the two finite-dimensional Hilbert spaces \mathcal{H} and $\widehat{\mathcal{H}}$.

We shall need higher-multiplicity versions of the spaces \mathcal{G} and \mathcal{H} defined as follows. For \mathcal{X} an auxiliary Hilbert space, we let

$$\begin{aligned} \mathcal{G}_{\mathcal{X}} &= \{f^* \mid f: \Omega \rightarrow \mathcal{L}(\mathcal{X}, \mathcal{E})_{\text{nc}} \text{ graded}\}, \\ \mathcal{H}_{\mathcal{X}} &= \{f^* \mid f: \Omega \rightarrow \mathcal{L}(\mathcal{X}, \mathcal{E})_{\text{nc}} \text{ a nc function}\}. \end{aligned}$$

The space $\mathcal{H}_{\mathcal{X}}$ can be given an inner-product again defined by (5.13) where now the associated kernel $K_{f,g}$ is equal to the nc kernel with values in $\mathcal{L}(\mathcal{L}(\mathcal{X})_{\text{nc}}, \mathcal{L}(\mathcal{E})_{\text{nc}})$. As explained on pages 78-79 of [9], we have the identifications

$$\mathcal{G}_{\mathcal{X}} \cong \mathcal{G} \otimes \mathcal{X}, \quad \mathcal{H}_{\mathcal{X}} \cong \mathcal{H} \otimes \mathcal{X}. \tag{5.16}$$

Given a nc function $Q: \Omega \rightarrow \mathcal{L}(\mathcal{R}, \mathcal{S})_{\text{nc}}$ for two auxiliary Hilbert spaces \mathcal{R} and \mathcal{S} , we let M_Q be the operator with adjoint M_Q^* equal to the multiplication operator given by

$$M_Q^*: f(W)^* \mapsto Q(W)^* f(W)^*.$$

It is routine to check from the definitions that

- if Q is graded, then M_Q^* maps $\mathcal{G}_{\mathcal{S}}$ to $\mathcal{G}_{\mathcal{R}}$, and
- if Q is a nc function, then M_Q^* maps $\mathcal{H}_{\mathcal{S}}$ to $\mathcal{H}_{\mathcal{R}}$.

The restriction of M_Q^* to $\mathcal{H}_{\mathcal{S}}$ with target space taken to be $\widehat{\mathcal{H}}_{\mathcal{R}}$ will be denoted by $M_{Q,0}^*$. Note that here $\mathcal{H}_{\mathcal{S}}$ and $\mathcal{H}_{\mathcal{R}}$ have inner products induced by (5.13) while $\widehat{\mathcal{G}}_{\mathcal{S}}$ and $\widehat{\mathcal{G}}_{\mathcal{R}}$ have their own quite different inner products induced by (5.15). In particular, the term subspace here is used loosely: $\mathcal{H}_{\mathcal{S}}$ is a vector subspace of $\mathcal{G}_{\mathcal{S}}$ but not a Hilbert subspace as the inner product on $\mathcal{H}_{\mathcal{S}}$ is not inherited from $\mathcal{G}_{\mathcal{S}}$ as a subset of $\mathcal{G}_{\mathcal{S}}$.

We define higher-multiplicity versions of the spaces $\widehat{\mathcal{G}}$ and $\widehat{\mathcal{H}}$ as follows. Given an auxiliary Hilbert space \mathcal{X} together with a function $f \in \mathcal{G}_{\mathcal{X}}$ and the point $Z^{(0)} = \bigoplus_{i=1}^N Z^{(i)}$ as in (5.14), define

$$\mathcal{I}_{\mathcal{X}}: f^* \mapsto \bigoplus_1^N f(Z^{(i)})^* = f(Z^{(0)})^* \in \bigoplus_{i=1}^N \mathcal{L}(\mathcal{E}^{n_i}, \mathcal{X}^{n_i}).$$

Note that as f sweeps the space of all graded functions on Ω with values in $\mathcal{L}(\mathcal{E}, \mathcal{X})_{\text{nc}}$, the resulting space of values $\{\bigoplus_1^N f(Z^{(i)})^*: f \in \mathcal{G}_{\mathcal{X}}\}$ sweeps exactly the space

$$\widehat{\mathcal{G}}_{\mathcal{X}} := \bigoplus_{i=1}^N \mathcal{L}(\mathcal{E}^{n_i}, \mathcal{X}^{n_i}). \tag{5.17}$$

We introduce the Hilbert-Schmidt inner product on $\widehat{\mathcal{G}}_{\mathcal{X}}$ given by

$$\langle f^*, g^* \rangle = \text{tr}(f(Z^{(0)})^* g(Z^{(0)})). \tag{5.18}$$

to make $\widehat{\mathcal{G}}_{\mathcal{X}}$ a Hilbert space. Note that

$$\text{tr}(f(Z^{(0)})g(Z^{(0)})^*) = \text{tr}(g(Z^{(0)})^* f(Z^{(0)}))$$

where $g(Z^{(0)})^* f(Z^{(0)}) \in \mathcal{L}(\mathcal{E}^{N_0})$ is a finite-dimensional operator even if \mathcal{X} is infinite-dimensional, so indeed $f(Z^{(0)})g(Z^{(0)})^*$ is in the trace class and the inner product is well-defined. Note also that the direct sum decomposition

in (5.17) is orthogonal and that the map $\mathcal{I}_{\mathcal{X}}$ between $\mathcal{G}_{\mathcal{X}}$ and $\widehat{\mathcal{G}}_{\mathcal{X}}$ is unitary. Making use of the tensor-product identifications mentioned in (5.16), the map $\mathcal{I}_{\mathcal{X}}$ can be identified with

$$\mathcal{I}_{\mathcal{X}} \cong \mathcal{I} \otimes I_{\mathcal{X}}.$$

We let $\mathcal{I}_{\mathcal{X},0}$ denote the restriction of $\mathcal{I}_{\mathcal{X}}$ to $\mathcal{H}_{\mathcal{X}}$. Then $\mathcal{I}_{\mathcal{X},0}$ is a linear isomorphism from $\mathcal{H}_{\mathcal{X}}$ onto its image $\mathcal{I}_{\mathcal{X},0}(\mathcal{H}_{\mathcal{X}})$, which we denote as $\widehat{\mathcal{H}}_{\mathcal{X}}$. Again making the identification of spaces in (5.16), one can arrive at the operator identification

$$\mathcal{I}_{\mathcal{X},0} \cong \mathcal{I}_0 \otimes I_{\mathcal{X}}.$$

Given two auxiliary Hilbert spaces \mathcal{S} and \mathcal{R} and Q a function from Ω to $\mathcal{L}(\mathcal{R}, \mathcal{S})$, we write \widehat{M}_Q^* for the multiplication operator of the form

$$\widehat{M}_Q^*: f(Z^{(0)})^* \mapsto Q(Z^{(0)})^* f(Z^{(0)})^*. \tag{5.19}$$

Then it is routine to verify from the definitions that

- if $Q: \Omega \rightarrow \mathcal{L}(\mathcal{R}, \mathcal{S})_{\text{nc}}$ is graded, then \widehat{M}_Q^* maps $\widehat{\mathcal{G}}_{\mathcal{S}}$ to $\widehat{\mathcal{G}}_{\mathcal{R}}$, and
- if $Q: \Omega \rightarrow \mathcal{L}(\mathcal{R}, \mathcal{S})_{\text{nc}}$ is a nc function, then \widehat{M}_Q^* maps $\widehat{\mathcal{H}}_{\mathcal{S}}$ to $\widehat{\mathcal{H}}_{\mathcal{R}}$.

For $Q: \Omega \rightarrow \mathcal{L}(\mathcal{R}, \mathcal{S})_{\text{nc}}$ a nc function, we let $\widehat{M}_{Q,0}^*$ be the restriction of the operator \widehat{M}_Q^* to $\widehat{\mathcal{H}}_{\mathcal{S}}$. Notice that we have the intertwining relations

$$\mathcal{I}_{\mathcal{R}} M_Q^* = \widehat{M}_Q^* \mathcal{I}_{\mathcal{S}}, \quad \mathcal{I}_{\mathcal{R},0} M_{Q,0}^* = \widehat{M}_{Q,0}^* \mathcal{I}_{\mathcal{S},0}. \tag{5.20}$$

Let us identify the operator \widehat{M}_Q^* from $\widehat{\mathcal{G}}_{\mathcal{S}}$ to $\widehat{\mathcal{G}}_{\mathcal{R}}$ more explicitly as follows. First let us note that a nc function $Q: \Omega \rightarrow \mathcal{L}(\mathcal{R}, \mathcal{S})_{\text{nc}}$, can always be extended uniquely to a nc function, again denoted as Q , on the nc envelope $[\Omega]_{\text{nc}}$ by using the *respects direct sums* property as a definition (see [8, Proposition 2.17]). In particular we can extend Q to a nc function on $\Omega \cup \{Z^{(0)}\}$ by defining $Q(Z^{(0)}) = \bigoplus_{i=1}^N Q(Z_i)$. By definition we then have

$$\widehat{M}_Q^*: f(Z^{(0)})^* \mapsto Q(Z^{(0)})^* f(Z^{(0)})^*. \tag{5.21}$$

Viewing $\widehat{\mathcal{G}}_{\mathcal{S}}$ as the direct-sum space $\bigoplus_{j=1}^N \mathcal{L}(\mathcal{E}^{n_j}, \mathcal{S}^{n_j})$ and writing a generic element of $\bigoplus_{j=1}^N \mathcal{L}(\mathcal{E}^{n_j}, \mathcal{S}^{n_j})$ as $\bigoplus_{j=1}^N f_j^*$, we can rewrite (5.21) as

$$\widehat{M}_Q^*: \text{col}_{1 \leq j \leq N} [f_j^*] \mapsto \text{diag}_{1 \leq j \leq N} [Q(Z^{(j)})^*] \cdot \text{col}_{1 \leq j \leq N} [f_j^*].$$

where the generic element $f(Z^{(0)})^*$ of $\widehat{\mathcal{G}}_{\mathcal{S}}$ is parametrized by a graded function $f: \Omega \rightarrow \mathcal{L}(\mathcal{E}, \mathcal{S})_{\text{nc}}$. If we write $\widehat{\mathcal{G}}_{\mathcal{S}}$ and $\widehat{\mathcal{G}}_{\mathcal{R}}$ in column form

$$\widehat{\mathcal{G}}_{\mathcal{S}} = \text{col}_{1 \leq j \leq N} \mathcal{L}(\mathcal{E}^{n_j}, \mathcal{S}^{n_j}), \quad \widehat{\mathcal{G}}_{\mathcal{R}} = \text{col}_{1 \leq j \leq N} \mathcal{L}(\mathcal{E}^{n_j}, \mathcal{R}^{n_j}), \tag{5.22}$$

we see from the formula (5.21) that the action (5.21) from $\widehat{\mathcal{G}}_{\mathcal{S}}$ to $\widehat{\mathcal{G}}_{\mathcal{R}}$ with $\widehat{\mathcal{G}}_{\mathcal{S}}$ and $\widehat{\mathcal{G}}_{\mathcal{R}}$ given in column form (5.22) is given by

$$\text{col}_{1 \leq j \leq N} f_j^* \mapsto \text{diag}_{1 \leq j \leq N} Q(Z^{(j)})^* \cdot \text{col}_{1 \leq j \leq N} f_j^*.$$

As by assumption Q is a nc function on $\Omega \cup \{Z^{(0)}\}$, we then have

$$\text{diag}_{1 \leq j \leq N} Q(Z^{(j)})^* = Q(Z^{(0)})^*$$

and the formula above can be written as

$$\text{col}_{1 \leq j \leq N} [f_j^*] \mapsto L_{Q(Z^{(0)})^*} \cdot [\text{col}_{1 \leq j \leq N} f_j^*]$$

where here f_j^* runs through the space $\mathcal{L}(\mathcal{E}^{n_j}, \mathcal{S}^{n_j})$ and where $L_{Q(Z^{(0)})^*}$ denotes the operator of left multiplication by $Q(Z^{(0)})^*$ on the space $\text{col}_{1 \leq j \leq N} \mathcal{L}(\mathcal{E}^{n_j}, \mathcal{S}^{n_j})$ mapping into the space $\text{col}_{1 \leq j \leq N} \mathcal{L}(\mathcal{E}^{n_j}, \mathcal{R}^{n_j})$. Note the space $\mathcal{L}(\mathcal{E}^{n_j}, \mathcal{S}^{n_j}) \cong \mathcal{L}(\mathcal{E}, \mathcal{S})^{n_j \times n_j}$ can be thought of as $n_j \times n_j$ -block matrices with matrix entries equal to operators from \mathcal{S} to \mathcal{E} . To get still more granular, let us fix a basis for $\{e_1, \dots, e_{n_{\mathcal{E}}}\}$ for \mathcal{E} (where $n_{\mathcal{E}} = \dim \mathcal{E} < \infty$ by (H2)) and represent an element T of $\mathcal{L}(\mathcal{E}, \mathcal{S})^{n_j \times n_j}$ as a $n_j \times n_j \cdot n_{\mathcal{E}}$ block matrix with matrix entries in $\mathcal{L}(\mathbb{C}, \mathcal{S}^{n_j}) \cong \mathcal{S}^{n_j}$. We can decompose $\mathcal{L}(\mathcal{E}^{n_j}, \mathcal{S}^{n_j})$ into $n_j \cdot n_{\mathcal{E}}$ pairwise orthogonal (in the Hilbert-Schmidt inner product) subspaces, corresponding to the subspaces of $\mathcal{L}(\mathcal{E}, \mathcal{S})^{n_j \times n_j}$ with nonzero matrix entries supported in a single given column. These subspaces are all invariant for $L_{Q(Z^{(j)})^*}$ and each of these $n_j \cdot n_{\mathcal{E}}$ restricted operators amounts to the same operator $Q(Z^{(j)})^*$ in the standard presentation of a matrix as an operator acting on a column space \mathcal{S}^{n_j} . Therefore, if we apply the VEC operator of stacking the columns of the matrix $T \in \mathcal{L}(\mathbb{C}, \mathcal{L}(\mathcal{S}))^{n_j \times n_j \cdot n_{\mathcal{E}}}$ down from each other to form a single column

$$\text{VEC}([T_{ij}]) = \text{col}_{1 \leq j \leq N} \text{col}_{\sum_{\ell=1}^{i-1} \leq j \leq \sum_{\ell=1}^i n_{\ell} \cdot n_{\mathcal{E}}} [T_{ij}],$$

we see that the left multiplication operator $L_{Q(Z^{(0)})^*}$ acting between spaces of Hilbert-Schmidt matrices

$$L_{Q(Z^{(0)})^*} : \bigoplus_{j=1}^N \mathcal{L}(\mathcal{E}^{n_j}, \mathcal{S}^{n_j}) \mapsto \bigoplus_{j=1}^N \mathcal{L}(\mathcal{E}^{n_j}, \mathcal{R}^{n_j})$$

is unitarily equivalent to a single matrix $Q(\widehat{Z}^{(0)})^*$ (to be defined in a moment) acting on a standard complex column space. Here we set

$$\widehat{Z}^{(0)} = \bigoplus_{j=1}^N \bigoplus_1^{n_j \cdot n_{\mathcal{E}}} Z^{(j)} \in [\Omega]_{\text{nc}}, \tag{5.23}$$

extend the nc function Q on Ω uniquely to a nc function on $\Omega \cup \{\widehat{Z}^{(0)}\}$ via

$$Q(\widehat{Z}^{(0)})^* = \bigoplus_{j=1}^N \bigoplus_1^{n_j \cdot n_{\mathcal{E}}} Q(Z^{(j)})^* : \bigoplus_{j=1}^N \left(\bigoplus_1^{n_j \cdot n_{\mathcal{E}}} \mathcal{S}^{n_j} \right) \mapsto \bigoplus_{j=1}^N \left(\bigoplus_1^{n_j \cdot n_{\mathcal{E}}} \mathcal{R}^{n_j} \right).$$

The final conclusion is:

$$\widehat{M}_Q^* = Q(\widehat{Z}^{(0)})^*. \tag{5.24}$$

where $\widehat{Z}^{(0)}$ is given by (5.23).

One particular example of a multiplication operator is given by the nc function $\chi_k : \Omega \rightarrow \mathbb{C}_{\text{nc}}$ ($1 \leq k \leq d$) defined by $\chi_k(Z) = Z_k$ where $Z = (Z_1, \dots, Z_d)$ is the decomposition of $Z \in \Omega \subset \mathbb{C}^d$ as an element of \mathbb{C}^d (see (5.4)). Everything said above for a general nc function $Q : \Omega \rightarrow \mathcal{L}(\mathcal{R}, \mathcal{S})_{\text{nc}}$ applies in particular to the case $Q = \chi_k$ for each $k = 1, \dots, d$, with now $\mathcal{R} = \mathcal{S} = \mathbb{C}$. Thus, for each $k = 1, \dots, d$ we associate the operators

$$\begin{aligned} M_{\chi_k,0}^* &: f(W)^* \mapsto W_k^* f(W)^* \text{ on } \mathcal{H}, \\ M_{\chi_k}^* &: f(Z^{(0)})^* \mapsto Z_k^{(0)*} f(Z^{(0)})^* \text{ on } \mathcal{G} \end{aligned}$$

where $M_{\chi_k,0}^*$ is just the restriction of $M_{\chi_k}^*$ to \mathcal{H} , and from (5.20) we have

$$\mathcal{I}M_{\chi_k}^* = \widehat{M}_{\chi_k}^* \mathcal{I}, \quad \mathcal{I}_0 M_{\chi_k,0}^* = \widehat{M}_{\chi_k,0}^* \mathcal{I}_0. \tag{5.25}$$

and from (5.24) we have

$$\widehat{M}_{\chi_k}^* = (\widehat{Z}_k^{(0)})^*. \tag{5.26}$$

In general suppose that \mathcal{V} is a finite-dimensional vector space (say of dimension $n_{\mathcal{V}}$) and $T = (T_1, \dots, T_d)$ is a d -tuple of linear transformations on \mathcal{V} and suppose that $f : \Omega \rightarrow (\mathbb{C}^N)_d$ is a nc function, where Ω is a subset of $\mathbb{C}_{\text{nc}}^d \cong \amalg_{n=1}^{\infty} (\mathbb{C}^{n \times n})^d$ (d tuples of matrices of size $n \times n$ over all possible $n \in \mathbb{N}$). Next choose a vector space isomorphism $\mathbb{S} : \mathcal{V} \rightarrow \mathbb{C}^{n_{\mathcal{V}}}$. We choose a fixed basis of $\mathbb{C}^{n_{\mathcal{V}}}$ to identify linear transformations on $\mathbb{C}^{n_{\mathcal{V}}}$ with $n_{\mathcal{V}} \times n_{\mathcal{V}}$ matrices. Given a d -tuple of linear transformations $(\mathbb{T}_1, \dots, \mathbb{T}_d)$ acting on \mathcal{V} such that it happens that the matrix d -tuple

$$(T_1, \dots, T_d) = (\mathbb{S}\mathbb{T}_1\mathbb{S}^{-1}, \dots, \mathbb{S}\mathbb{T}_d\mathbb{S}^{-1}) \in (\mathbb{C}^{n_{\mathcal{V}} \times n_{\mathcal{V}}})^d$$

turns out to be in Ω (the domain of f), let us define $f(T_1, \dots, T_d) : \mathcal{V} \rightarrow \mathcal{V}$ by

$$f(T_1, \dots, T_d) = \mathbb{S}^{-1} f(\mathbb{S}\mathbb{T}_1\mathbb{S}^{-1}, \dots, \mathbb{S}\mathbb{T}_d\mathbb{S}^{-1}) \mathbb{S}.$$

To show that $f(T_1, \dots, T_d)$ is well-defined (i.e., independent of the choice of vector-space isomorphism $\mathbb{S} : \mathcal{V} \rightarrow \mathbb{C}^{n_{\mathcal{V}}}$), we proceed as follows. Suppose that we had instead used vector-space isomorphism $\mathbb{S}' : \mathcal{V} \rightarrow \mathbb{C}^{n_{\mathcal{V}}}$ and had defined $f(T_1, \dots, T_d)$ to be instead

$$f'(T_1, \dots, T_d) = \mathbb{S}'^{-1} f(\mathbb{S}'\mathbb{T}_1\mathbb{S}'^{-1}, \dots, \mathbb{S}'\mathbb{T}_d\mathbb{S}'^{-1}) \mathbb{S}'.$$

To show that $f'(T_1, \dots, T_d) = f(T_1, \dots, T_d)$ we must show that

$$\mathbb{S}'\mathbb{S}^{-1} f(\mathbb{S}\mathbb{T}_1\mathbb{S}^{-1}, \dots, \mathbb{S}\mathbb{T}_d\mathbb{S}^{-1}) \mathbb{S}\mathbb{S}'^{-1} = f(\mathbb{S}'\mathbb{T}_1\mathbb{S}'^{-1}, \dots, \mathbb{S}'\mathbb{T}_d\mathbb{S}'^{-1}).$$

But this is an immediate consequence of the *respects similarities* property of f as a function defined on d -tuples of matrices. Therefore, as far as the nc functional calculus is concerned, we are free to identify a point in $(\mathbb{C}^d)_{\text{nc}}$ at level n with a d -tuple of linear transformations on a finite-dimensional vector space of dimension n .

Thus we may identify $\widehat{Z}^{(0)}$ not only as the operator d -tuple on $\widehat{\mathcal{G}}$ but also as d -tuple of matrices of size $M \times M$ ($M = \dim \mathcal{G}$) and similarly for \widehat{M}_χ^* . From (5.25) we see that

$$\widehat{M}_\chi = (\widehat{M}_{\chi_1}, \dots, \widehat{M}_{\chi_d}) = \widehat{Z}^{(0)}. \quad (5.27)$$

From the definition (5.23) of $Z^{(0)}$ and the fact that $\Omega = \{Z^{(1)}, \dots, Z^{(N)}\}$, (5.27) implies that

$$\widehat{M}_\chi = (\widehat{M}_{\chi_1}, \dots, \widehat{M}_{\chi_d}) \in [\Omega]_{\text{nc}}. \quad (5.28)$$

From the first of relations (5.25) we read off that

$$M_{\chi,0} = (M_{\chi_1,0}, \dots, M_{\chi_d,0}) \text{ is similar to } \widehat{Z}^{(0)}|_{\widehat{\mathcal{H}}^*}$$

where $\widehat{\mathcal{H}}^*$ is the result of taking adjoints pointwise on $\widehat{\mathcal{H}}$ and then applying the VEC operation to get an invariant subspaces for $\widehat{Z}^{(0)}$ rather than for $\widehat{Z}^{(0)*}$. This last statement then implies that

$$M_{\chi,0} \in [\Omega]_{\text{full}}, \quad (5.29)$$

where the *full envelope* $[\Omega]_{\text{full}}$ of Ω is defined as in Subsection 2.1. We shall see below that in fact it is also the case that $M_{\chi,0} \in \mathbb{P}_\Omega$ implying that $M_{\chi,0} \in \Omega'$ (see (5.39) below), but at this point we have to work with only the knowledge that $M_{\chi,0} \in [\Omega]_{\text{full}}$.

Let us now suppose that Q is any nc function defined on the set $\Omega \cup \{M_{\chi,0}^*\} \subset \Omega_{\text{full}}$ with values in $\mathcal{L}(\mathcal{R}, \mathcal{S})_{\text{nc}}$. We seek to show that

$$Q(M_{\chi,0})^* = M_{Q,0}^* \text{ on } \mathcal{H}. \quad (5.30)$$

We already know a related version of this result

$$Q(\widehat{M}_\chi)^* = \widehat{M}_Q^* \quad (5.31)$$

as a consequence of (5.24) combined with (5.27). This then implies that

$$Q(\widehat{M}_\chi)^* \mathcal{I}_{\mathcal{S},0} = \widehat{M}_Q^* \mathcal{I}_{\mathcal{S},0} = \widehat{M}_{Q,0}^* \mathcal{I}_{\mathcal{S},0}. \quad (5.32)$$

A consequence of the *respects intertwining* property for nc functions is a *respects invariant subspaces* property (see [20]):

$$Q\left(\begin{bmatrix} T & * \\ 0 & * \end{bmatrix}\right) = \begin{bmatrix} Q(T) & * \\ 0 & * \end{bmatrix}.$$

From this general principle we can see that $Q(\widehat{M}_\chi)^* \mathcal{I}_{\mathcal{S},0} = Q(\widehat{M}_{\chi,0})^* \mathcal{I}_{\mathcal{S},0}$ and (5.32) becomes

$$Q(\widehat{M}_{\chi,0})^* \mathcal{I}_{\mathcal{S},0} = \widehat{M}_{Q,0}^* \mathcal{I}_{\mathcal{S},0}. \quad (5.33)$$

From the second of relations (5.20) we know that $\widehat{M}_{Q,0}^* \mathcal{I}_{S,0} = \mathcal{I}_{R,0} M_{Q,0}^*$. From the second set of intertwining relations (5.25) combined with the fact that nc functions respect intertwining conditions we get $Q(\widehat{M}_{\chi,0})^* \mathcal{I}_{S,0} = \mathcal{I}_{R,0} Q(M_{\chi,0})^*$. Plugging this information back into (5.33) leaves us with

$$\mathcal{I}_{R,0} Q(M_{\chi,0})^* = \mathcal{I}_{R,0} M_{Q,0}^*. \tag{5.34}$$

Cancelling off the injective factor $\mathcal{I}_{R,0}$ finally gets us to (5.30) as wanted. Since we have now established that $M_{\chi,0} \in [\Omega]_{\text{full}}$ (see (5.29)), assumption (iii) in the statement of Theorem 5.1 tells us that Ω is decomposable on $\widetilde{\Omega} := \Omega \cup \{M_{\chi,0}\}$. Hence, for $Z \in \widetilde{\Omega}_n, W \in \widetilde{\Omega}_m, P \in \mathbb{C}^{n \times m}$ we can write

$$\Omega(Z, W)(P) = Q_+(Z)(P \otimes I_{M_+})Q_+(W)^* - Q_-(Z)(P \otimes I_{M_-})Q_-(W)^* \tag{5.35}$$

for some nc functions $Q_{\pm}: \widetilde{\Omega} \rightarrow \mathcal{L}(\mathcal{M}_{\pm}, \mathcal{N})_{\text{nc}}$ for some auxiliary Hilbert spaces \mathcal{M}_{\pm} ,

Let us use this decomposition for Ω to show that

$$\Omega(M_{\chi,0}, M_{\chi,0})(I_{\mathcal{H}}) \succ 0. \tag{5.36}$$

Indeed, for any nc function $f: \Omega \rightarrow \mathcal{L}(\mathbb{C}, \mathcal{H})_{\text{nc}}$ generating a generic element f^* of \mathcal{H} , let us compute

$$\begin{aligned} \langle \Omega(M_{\chi,0}, M_{\chi,0})f^*, f^* \rangle &= \langle Q_+(M_{\chi,0})^* f^*, Q_+(M_{\chi,0})^* f^* \rangle \\ &\quad - \langle Q_-(M_{\chi,0})^* f^*, Q_-(M_{\chi,0})^* f^* \rangle \\ &= \langle M_{Q_+,0}^* f^*, M_{Q_+,0}^* f^* \rangle - \langle M_{Q_-,0}^* f^*, M_{Q_-,0}^* f^* \rangle \\ &= \ell(K_{\Omega,f,f}) \end{aligned} \tag{5.37}$$

where, now for Z, W only in Ω ,

$$\begin{aligned} K_{\Omega,f,f}(Z, W)(P) &= f(Z) \left(Q_+(Z)(P \otimes I_{M_+})Q_+(W)^* \right. \\ &\quad \left. - Q_-(Z)(P \otimes I_{M_-})Q_-(W)^* \right) f(W)^* \\ &= f(Z)\Omega(Z, W)(P)f(W)^*. \end{aligned}$$

From this representation we see that $K_{\Omega,f,f}$ has the form of the first term on the right-hand side of (5.7) and hence is in the cone \mathcal{C} . Hence either the kernel $K_{\Omega,f,f}$ is identically zero or $\ell(K_{\Omega,f,f}) > 0$. In the first case, it then follows in particular that

$$f(Z)\Omega(Z, Z)(I_n)f(Z)^* = 0 \text{ for all } Z \in \Omega_n, n \in \mathbb{N}. \tag{5.38}$$

By assumption $\Omega \subset \mathbb{P}_{\Omega}$, so $\Omega(Z, Z)(I_n)$ is strictly positive definite. Then (5.38) implies that $f(Z) = 0$ for all $Z \in \Omega$, and hence f^* is the zero element of \mathcal{H} . Thus, if f^* is not the zero element of \mathcal{H} we have that $\ell(K_{\Omega,f,f}) > 0$ which

translates to positivity of the quadratic form: $\langle \mathfrak{Q}(M_{\chi,0}, M_{\chi,0})(I_{\mathcal{H}})f^*, f^* \rangle_{\mathcal{H}} > 0$ for $0 \neq f \in \mathcal{H}$. As \mathcal{H} is finite-dimensional, this is just the statement that

$$\mathfrak{Q}(M_{\chi,0}, M_{\chi,0})(I_{\mathcal{H}}) \succ 0 \quad (5.39)$$

as an operator on \mathcal{H} , i.e., $M_{\chi,0} \in \mathbb{P}_{\Omega}$. As we have already established that $M_{\chi,0} \in [\Omega]_{\text{full}}$ (see (5.29)), we now know that $M_{\chi,0} \in \Omega' := [\Omega]_{\text{full}} \cap \mathbb{P}_{\Omega}$.

Consequently, we can now invoke condition (iv) in the statement of the theorem to conclude that \mathfrak{S} is also decomposable on $\Omega \cup \{M_{\chi,0}\}$. Hence we can find nc functions $S_{\pm} : \Omega \cup \{M_{\chi,0}\} \rightarrow \mathcal{L}(\mathcal{D}_{\pm}, \mathcal{E})_{\text{nc}}$ for some additional auxiliary Hilbert spaces \mathcal{D}_{\pm} so that

$$\mathfrak{S}(Z, W)(P) = S_+(Z)(P \otimes I_{\mathcal{D}_+})S_+(W)^* - S_-(Z)(P \otimes I_{\mathcal{D}_-})S_-(W)^* \quad (5.40)$$

for $Z, W \in \tilde{\Omega}$. As we have already observed that $\mathfrak{Q}(M_{\chi,0}, M_{\chi,0})(I_{\mathcal{H}}) \succ 0$, we now can apply the standing kernel-dominance hypothesis (5.1) to conclude that

$$\mathfrak{S}(M_{\chi,0}, M_{\chi,0})(I_{\mathcal{H}}) \succeq 0.$$

Using the assumed decomposition (5.40) for \mathfrak{S} , by a repeat of the computation (5.37) with \mathfrak{S} in place of \mathfrak{Q} we see that

$$0 \leq \langle \mathfrak{S}(M_{\chi,0}, M_{\chi,0})f^*, f^* \rangle = \ell(K_{\mathfrak{S},f,f})$$

where, now again for Z, W only in Ω ,

$$K_{\mathfrak{S},f,f}(Z, W)(P) = f(Z)\mathfrak{S}(Z, W)(P)f(W)^*.$$

In particular, we make take $1_{\mathcal{E}}^* \in \mathcal{H}_{\mathcal{E}}$ with $1_{\mathcal{E}}(Z) = I_{\mathcal{E}^n}$ for $Z \in \Omega_n$ to conclude that

$$\ell(\mathfrak{S}) = \langle \mathfrak{S}(M_{\chi,0}, M_{\chi,0})1_{\mathcal{E}}^*, 1_{\mathcal{E}}^* \rangle_{\mathcal{H}} \geq 0,$$

in contradiction with our earlier conclusion that $\ell(\mathfrak{S}) < 0$. The existence of an ℓ with $\ell(\mathfrak{S}) < 0$ was immediate from the supposition that \mathfrak{S} was not in the cone \mathcal{C} . We conclude that in the presence of the kernel-dominance condition (5.1), it must be the case that \mathfrak{S} is in \mathcal{C} , i.e., that \mathfrak{S} has a representation as in (5.3). This finally completes the proof of Theorem 5.1 for the case where Ω consists of only finitely many points. The proof of the general case will be completed in Section 6.3 to come. \square

We note that Theorem 5.1 handles only the case where Ω consists of finitely many points. The general case can be handled by using a theorem of Kurosh to reduce the general case to the finite-point case. This will be taken up in the next section.

6 EXTENSIONS OF RESULTS FOR A FINITE SET OF POINTS TO A GENERAL SET OF POINTS VIA THE KUROSH METHOD

The theorem of Kurosh (see [2, Theorem 2.56] as well as [6, pages 73-75]) asserts that the limit of an inverse spectrum of nonempty compact sets is a nonempty compact set. In practice one is given a family of compact sets \mathfrak{X}_μ indexed by a directed set \mathfrak{A} ($\mu \in \mathfrak{A}$). Here \mathfrak{A} being a *directed set* means that \mathfrak{A} is equipped with a partial order \preceq satisfying reflexivity and transitivity

$$\begin{aligned} \alpha \preceq \beta \text{ and } \beta \preceq \alpha &\Rightarrow \alpha = \alpha, \\ \alpha \preceq \beta \text{ and } \beta \preceq \gamma &\Rightarrow \alpha \preceq \gamma. \end{aligned}$$

as well as

$$\text{given } \alpha, \beta \in \mathfrak{A}, \exists \gamma \in \mathfrak{A} \text{ so that } \alpha \preceq \gamma \text{ and } \beta \preceq \gamma. \tag{6.1}$$

We suppose that we are given an *inverse spectrum*, i.e., a family of nonempty compact subsets $\mathbb{K}_\alpha \subset \mathfrak{X}_\alpha$ for each $\alpha \in \mathfrak{A}$ together with a collection of continuous *restriction maps* $\pi_\beta^\alpha: \mathbb{K}_\alpha \rightarrow \mathbb{K}_\beta$ for each $\alpha, \beta \in \mathfrak{A}$ with $\beta \preceq \alpha$ such that

$$\pi_\gamma^\beta \circ \pi_\beta^\alpha = \pi_\gamma^\alpha \text{ for } \gamma \preceq \beta \preceq \alpha, \quad \pi_\alpha^\alpha = \text{id}_{\mathbb{K}_\alpha} \text{ for all } \alpha \in \mathfrak{A}. \tag{6.2}$$

Then an element $\Gamma = \{\Gamma_\alpha\}_{\alpha \in \mathfrak{A}}$ of the Cartesian product set $\prod_{\alpha \in \mathfrak{A}} \mathbb{K}_\alpha$ is said to be a limit point of the inverse spectrum $\prod_{\alpha \in \mathfrak{A}} \mathbb{K}_\alpha$ if it is the case that that $\pi_\beta^\alpha \Gamma_\alpha = \Gamma_\beta$ for all $\alpha, \beta \in \mathfrak{A}$ with $\beta \preceq \alpha$. The assertion of the Kurosh theorem is that, with all the compactness assumptions and definitions as listed above, it is always the case that the set of limit points of such an inverse spectrum is nonempty.

6.1 DECOMPOSITION OF KERNELS

Let us consider the decomposability problem for classical Hermitian kernels on an infinite point set. Thus we let Ω be an abstract point set (considered as distinct 1×1 matrices over some vector space to fit into the noncommutative kernel theory) and we consider a complex-valued function $K: \Omega \times \Omega \rightarrow \mathbb{C}$ with $K(z, w)^* = K(w, z)$. This can be considered as a special case of the theory of nc kernels by taking \mathcal{A} to be \mathbb{C} , $\mathcal{Y} = \mathbb{C} = \mathcal{L}(\mathcal{Y})$, and identifying $K(z, w)$ with $K(z, w)(1)$. As a consequence of the Corollary in Example 2 (or by elementary linear algebra), we know that the restriction of K to any finite subset of Ω ($K|_F$) is decomposable (i.e., can be represented as the difference of two positive kernels). One can use the Kurosh theorem to try to find a global such decomposition for all of K as follows.

We wish to denote elements of \mathfrak{A} by lower case Greek letters, e.g., μ . We identify $\mu \in \mathfrak{A}$ with a finite subset of Ω , denoted as Ω_μ , so \mathfrak{A} consists of all finite subsets of Ω . We say that $\mu \preceq \nu$ in \mathfrak{A} if $\Omega_\mu \subset \Omega_\nu$. It is easily checked that \mathfrak{A} so defined is a directed set. Given a $\mu \in \mathfrak{A}$, we let \mathfrak{X}_μ be the set of all pairs of positive kernels (K_+, K_-) defined on $\Omega_\mu \subset \Omega$, and we let \mathbb{K}_μ consist

of all positive-kernel pairs (K_+, K_-) defined on $\Omega_\mu \times \Omega_\mu$ so that the given kernel K restricted to Ω_μ has the form

$$K(z, w) = K_+(z, w) - K_-(z, w) \text{ for } z, w \in \Omega_\mu.$$

By our remarks above, each \mathbb{K}_μ so defined is non-empty. For $\mu \preceq \nu$, we define the restriction maps π_μ^ν by $\pi_\mu^\nu: (K_+, K_-) \mapsto (K_+|_{\Omega_\mu}, K_-|_{\Omega_\mu})$ for $(K_+, K_-) \in \mathbb{K}_\nu$. Then it is easily checked that $\{\mathbb{K}_\mu: \mu \in \mathfrak{A}\}$ with the system of maps $\{\pi_\mu^\nu: \mu \subset \nu \text{ in } \mathfrak{A}\}$ is an inverse spectrum. If one can find a limit point $\{K_{+, \mu}, K_{-, \mu}\}_{\mu \in \mathfrak{A}}$ of this inverse spectrum, then one can construct a global decomposition $K = K_+ - K_-$ of K as the difference of positive kernels on all of Ω as follows. Given any two points z, w in Ω , define $K_+(z, w)$ and $K_-(z, w)$ by

$$K_+(z, w) = K_{+, \mu}(z, w), \quad K_-(z, w) = K_{-, \mu}(z, w)$$

where μ is any choice of element of \mathfrak{A} such that $z, w \in \Omega_\mu$. One can check that this definition is well-defined, that the global functions K_+ and K_- so defined are positive kernels (since the condition of positivity involves checking at only finitely many points at a time), and each K_+^F and K_-^F are positive kernels on Ω , and furthermore $K(z, w) = K_+(z, w) - K_-(z, w)$. However there is an example due to Schwartz [29] (see also [3] for additional information) that this is not always possible, so the Kurosh theorem fails in this case. Of course the reason is that there is a hypothesis missing: the set of all pairs (K_+, K_-) of cp nc kernels providing a representation of the given Hermitian kernel K as a difference of cp nc kernels can fail to be bounded. Consequently the weak-* topology (the “right” topology for this problem as we shall see in the examples to follow) restricted to this fiber fails to be compact, and the Kurosh theorem does not apply.

REMARK 6.1. We have seen that not all Hermitian kernels over an infinite point-set are decomposable. However we may still consider the class of Hermitian kernels $K = K_+ - K_-$ which are decomposable. Then Lemma 2.2 in the paper of Alpay [3] (see also the original paper of Schwartz [29]) is precisely the statement that any such K has a *minimal decomposition* (as defined in Remark 3.10), i.e., the Conjecture in Remark 3.10 holds for this setting (standard Hermitian kernels rather than nc Hermitian kernels). This suggests a starting point for attacking the Conjecture in Remark 3.10 for the general case.

6.2 ARVESON EXTENSION THEOREM FOR KERNELS

The goal of this section is to prove general Theorem 4.1 to the general case where Ω is allowed to consist of infinitely many points.

THEOREM 6.2. *Suppose that K is a nc kernel on a set of nc points $\Omega \subset \mathcal{V}_{\text{nc}}$ with values in $\mathcal{L}(\mathbb{S}_{\text{nc}}, \mathcal{L}(\mathcal{Y})_{\text{nc}})$, where \mathbb{S} is a operator system and $\mathcal{L}(\mathcal{Y})$ is the C^* -algebra of operators on the Hilbert space \mathcal{Y} . Let \mathcal{A} be a C^* -algebra containing \mathbb{S} . Then there exists a cp nc kernel \widehat{K}*

$$\widehat{K}: \Omega \times \Omega \rightarrow \mathcal{L}(\mathcal{A}_{\text{nc}}, \mathcal{L}(\mathcal{Y})_{\text{nc}})$$

such that $\widehat{K}(Z, W)(P) = K(Z, W)(P)$ for all $Z \in \Omega_n, W \in \Omega_m$ whenever $P \in \mathbb{S}^{n \times m}$.

Proof. By Theorem 4.1 we know that the result is true in case Ω is a finite set of points. We wish to reduce the general case to the finite-point case by applying the Theorem of Kurosh.

Toward this goal, we set up the directed set \mathfrak{A} exactly as in Section 6.1. We let \mathfrak{X}_μ be the linear space of all kernels $\Gamma: \Omega_\mu \times \Omega_\mu \rightarrow \mathcal{L}(\mathcal{A}_{\text{nc}}, \mathcal{L}(\mathcal{Y})_{\text{nc}})$. For $Z \in \Omega_{\mu, n_Z}$ (i.e., $Z \in \Omega_\mu$ has size $n_Z \times n_Z$), we have that $\Gamma(Z, W)$ is an element of the Banach space $\mathcal{L}(\mathcal{A}^{n_Z \times n_W}, \mathcal{L}(\mathcal{Y})^{n_Z \times n_W})$, and the fiber \mathfrak{X}_μ can be identified with the direct sum of Banach spaces

$$\mathfrak{X}_\mu = \bigoplus_{Z, W \in \Omega_\mu} \mathcal{L}(\mathcal{A}^{n_Z \times n_W}, \mathcal{L}(\mathcal{Y})^{n_Z \times n_W}).$$

If we endow \mathfrak{X}_μ with the supremum norm

$$\|\Gamma\| = \sup_{Z, W \in \Omega_\mu} \|\Gamma(Z, W)\|, \tag{6.3}$$

then \mathfrak{X}_μ is a Banach space. To finish the proof, we wish to make explicit the following additional information.

REMARK 6.3. A Banach space of the form $\mathcal{L}(\mathcal{A}^{n \times m}, \mathcal{L}(\mathcal{Y})^{n \times m})$ has a predual which we denote as $\mathcal{L}(\mathcal{A}^{n \times m}, \mathcal{L}(\mathcal{Y})^{n \times m})_*$ such that the weak-* topology on $\mathcal{L}(\mathcal{A}^{n \times m}, \mathcal{L}(\mathcal{Y})^{n \times m})$ is the same as the pointwise weak-* topology, i.e., convergence of a net $\lambda \rightarrow \phi_\lambda$ with $\phi_\lambda \in \mathcal{L}(\mathcal{A}^{n \times m}, \mathcal{L}(\mathcal{Y})^{n \times m})$ to ϕ in the weak-* topology is equivalent to pointwise weak-* convergence on $\mathcal{L}(\mathcal{Y})^{n \times m}$: $\phi_\lambda(T)$ converges to $\phi(T)$ in the weak-* topology of $\mathcal{L}(\mathcal{Y})^{n \times m}$ for each $T \in \mathcal{A}^{n \times m}$. As weak and weak-* topologies agree on bounded sets, this topology is sometimes called the bounded-weak topology or BW-topology (see [26] for more details). It is possible to identify the pre-dual spaces $\mathcal{L}(\mathcal{A}^{n \times m}, \mathcal{L}(\mathcal{Y})^{n \times m})_*$ more explicitly, but we shall not need this.

Since the direct summands of \mathfrak{X}_μ have a predual $\mathcal{L}(\mathcal{A}^{n \times m}, \mathcal{L}(\mathcal{Y})^{n \times m})_*$, so also does \mathfrak{X}_μ itself:

$$(\mathfrak{X}_\mu)_* = \bigoplus_{Z, W \in \Omega_\mu} \left(\mathcal{L}(\mathcal{A}^{N_Z \times N_W}, \mathcal{L}(\mathcal{Y})^{n_Z \times n_W}) \right)_*$$

with pairing

$$\langle \Gamma, \tau \rangle = \sum_{Z, W \in \Omega_\mu} \langle \Gamma(Z, W), \tau(Z, W) \rangle$$

for $\Gamma = \bigoplus_{Z, W \in \Omega_\mu} \Gamma \in \mathfrak{X}_\mu$ and $\tau = \bigoplus_{Z, W \in \Omega_\mu} \tau(Z, W) \in (\mathfrak{X}_\mu)_*$.

For each $\mu \in \mathfrak{A}$, we let \mathbb{K}_μ be the subset of \mathfrak{X}_μ consisting of all cp nc kernels

$$\Gamma: \Omega_\mu \times \Omega_\mu \rightarrow \mathcal{L}(\mathcal{A}_{\text{nc}}, \mathcal{L}(\mathcal{Y})_{\text{nc}})$$

so that, for $Z, W \in \Omega_\mu$,

$$K_\mu(Z, W)(P) = K(Z, W)(P) \text{ for } P \in \mathbb{S}^{n_Z \times n_W}. \tag{6.4}$$

Note that since Ω_μ is a finite set it is a consequence of Theorem 4.1 that each set \mathbb{K}_μ is nonempty.

The crucial next point is to show that each \mathbb{K}_μ is compact (the part that failed for the kernel-decomposition problem in Section 6.1). By the Banach-Alaoglu Theorem, norm-closed and bounded sets are pre-compact in the weak-* topology on the dual of a Banach space. Thus, to show that \mathbb{K}_μ is compact in the weak-* topology, it suffices to show that \mathbb{K}_μ is weak-* closed (which implies norm closed) and bounded.

For boundedness, it suffices to show that $\|\Gamma(Z, W)\|$ is uniformly bounded for $Z, W \in \Omega_\mu$ and $\Gamma \in \mathbb{K}_\mu$. Let $\Omega_\mu = \{Z_1, \dots, Z_N\}$ where $Z_i \in \Omega_{n_i}$. Consider the direct sum $Z^{(0)} = \bigoplus_{i=1}^N Z_i \in \Omega_{N_0}$ where $N_0 = \sum_{i=1}^N n_i$. We have

$$\begin{aligned} \|\Gamma(Z_i, Z_j)\| &\leq \left\| \Gamma \left(\begin{bmatrix} Z_i & 0 \\ 0 & Z_j \end{bmatrix}, \begin{bmatrix} Z_i & 0 \\ 0 & Z_j \end{bmatrix} \right) \right\| \leq \|\Gamma(Z^{(0)}, Z^{(0)})\| \\ &= \|\Gamma(Z^{(0)}, Z^{(0)})(I_{N_0})\| = \|K(Z^{(0)}, Z^{(0)})(I_{N_0})\| \end{aligned}$$

where the first two inequalities follow from the fact that Γ respects direct sums and the next equality is a consequence of $\Gamma(Z^{(0)}, Z^{(0)})$ being a completely positive map.

To analyze closedness of \mathbb{K}_μ in the weak-* topology, let $\{\Gamma_\lambda\}$ be a net in \mathbb{K}_μ which is weak-* convergent to $\Gamma \in \mathfrak{X}_\mu$. We must show that in fact Γ is back in \mathbb{K}_μ . By Remark 6.3, it suffices to work with the weak-* topology on $\mathcal{L}(\mathcal{Y})$, i.e., we can say that $\Gamma_\lambda(Z, W) \xrightarrow[\text{weak-*}]{} \Gamma(Z, W)$ if and only if

$$\langle y, \Gamma_\lambda(Z, W)(P)y' \rangle \rightarrow \langle y, \Gamma(Z, W)(P)y' \rangle$$

for every $P \in \mathcal{A}^{n \times m}$, $y \in \mathcal{Y}^n$, $y' \in \mathcal{Y}^m$. To show that $\Gamma \in \mathbb{K}_\mu$, we must show that

- (i) Γ is an extension of K from \mathbb{S}_{nc} to \mathcal{A}_{nc} , i.e. $K_\mu = \Gamma$ satisfies (6.4), and
- (ii) Γ is a cp nc kernel.

As for (i), note that $\Gamma_\lambda(Z, W)(P) = K(Z, W)(P)$ for every $Z \in \Omega_{\mu, n}$ and $W \in \Omega_{m\mu, m}$, and $P \in \mathbb{S}^{n \times m}$, so in the weak-* limit we also have $\Gamma(Z, W)(P) = K(Z, W)(P)$.

As for (ii), Γ is graded by construction, so it remains to check that Γ respects intertwining and is completely positive. Let $Z \in \Omega_{\mu, n}$, $\tilde{Z} \in \Omega_{\mu, \tilde{n}}$, $W \in \Omega_{\mu, m}$, $\tilde{W} \in \Omega_{\mu, \tilde{m}}$ and suppose that $\alpha Z = \tilde{Z}\alpha$ and $\beta W = \tilde{W}\beta$ where $\alpha \in \mathbb{C}^{\tilde{n} \times n}$ and $\beta \in \mathbb{C}^{\tilde{m} \times m}$. For $P \in \mathcal{A}^{n \times m}$, $y \in \mathcal{Y}^n$, $y' \in \mathcal{Y}^{\tilde{m}}$, we have

$$\langle y, (\alpha\Gamma_\lambda(Z, W)(P)\beta^*)y' \rangle = \langle y, \Gamma_\lambda(\tilde{Z}, \tilde{W})(\alpha P\beta^*)y' \rangle.$$

Taking the limit in λ and using weak-* convergence of Γ_λ to Γ in the equivalent sense explained in Remark 6.3, we see that the limit in λ of this last expression gives us

$$\langle y, \alpha\Gamma(Z, W)(P)\beta^*y' \rangle = \langle y, \Gamma(\tilde{Z}, \tilde{W})(\alpha P\beta^*)y' \rangle.$$

As $y \in \mathcal{Y}^{\tilde{n}}$ and $y' \in \mathcal{Y}^{\tilde{m}}$ are arbitrary, we conclude that Γ respects intertwining and hence is a noncommutative kernel. This completes the proof that \mathbb{K}_μ is compact for each $\mu \in \mathfrak{A}$.

Let us next define the system of restriction mappings $\pi_\mu^\nu: \mathbb{K}_\mu \rightarrow \mathbb{K}_\nu$ for $\nu \preceq \mu$ in \mathfrak{A} by

$$(\pi_\mu^\nu: K)(Z, W)(P) = K(Z, W)(P) \text{ for } Z, W \in \Omega_\nu, P \in \mathcal{A}^{nz \times nw}, \text{ i.e.,}$$

given an element K of \mathbb{K}_ν which is a kernel on Ω_ν , we get a kernel $\pi_\mu^\nu(K)$ on Ω_ν simply by restricting the arguments $Z, W \in \Omega_\nu$ to be in the subset $\Omega_\mu \subset \Omega_\nu$. Then one can check that the inverse-system-axiom (6.2) is satisfied. Thus we can apply the theorem of Kurosh to conclude that there exist limit points for this inverse system, i.e., an element $\Gamma = \{\Gamma_\mu\}_{\mu \in \mathfrak{A}}$ in $\mathbb{K} := \prod_{\mu \in \mathfrak{A}} \mathbb{K}_\mu$ such that $\pi_\mu^\nu \Gamma_\nu = \Gamma_\mu$ for all $\mu \preceq \nu$ in \mathfrak{A} . Given such a Γ , define a kernel K on all of $\Omega \times \Omega$ mapping into $\mathcal{L}(\mathcal{A}_{\text{nc}}, \mathcal{L}(\mathcal{Y})_{\text{nc}})$ by

$$K(Z, W)(P) = \Gamma_\mu(Z, W)(P) \tag{6.5}$$

for $Z, W \in \Omega$ and $P \in \mathcal{A}^{nz \times nw}$, where we choose μ to be any element of \mathfrak{A} such that both Z and W are in the finite set Ω_μ . Using properties of directed sets, one can show that the formula (6.5) defining $K(Z, W)(P)$ is independent of the choice of μ satisfying $Z, W \in \Omega_\mu$. As checking whether a given kernel is a cp nc kernel involves only working with finitely many points at a time, it follows that K is a cp nc kernel on Ω since each Γ_μ is a cp nc kernel on the finite set Ω_μ for each $\mu \in \mathfrak{A}$. Furthermore, for $P \in \mathbb{S}^{nz \times nw}$, it is the case that (6.4) holds with $K_\mu = K$ with $Z, W \in \Omega$ since this is the case with $K_\mu = \Gamma_\mu$ with $Z, W \in \Omega_\mu$ for each μ . The proof of Theorem 6.2 is now complete. \square

REMARK 6.4. We note that the idea of the preceding proof was to use the Kurosh argument to reduce the general case to the finite-point case which in turn is handled in Section 4 by using the results of Subsection 2.4 to reduce to the Arveson extension theorem for a cp map from an operator space \mathbb{S} to a C^* -algebra $\mathcal{L}(\mathcal{Y})$. However the proof of the Arveson extension theorem in Paulsen’s book [26, Theorem 7.5] actually proceeds by first reducing to the case where $\dim \mathcal{Y} < \infty$ and then using finite-dimensional analysis to handle this simpler case [26, Theorem 6.2]. Our comment here is that it is also possible to reduce the kernel-version of the Arveson extension theorem Theorem 6.2 directly to the finite-dimensional version of the Arveson extension theorem ([26, Theorem 6.2] by applying the Kurosh argument to an inverse-system based on a net \mathfrak{A} indexed by both finite point-sets Ω_μ and finite-dimensional subspaces $\mathcal{Y}_\mu \subset \mathcal{Y}$ ($\mu = (\Omega_\mu, \mathcal{Y}_\mu)$). This in fact is what we do in the next

Subsection 6.3 coming up, where we show how the general case of the kernel-dominance theorem (Theorem 5.1) can be reduced to the special case involving both finite-point sets $\Omega_\mu \subset \Omega$ and finite-dimensional subspaces $\mathcal{E}_\mu \subset \mathcal{E}$.

6.3 KERNEL DOMINANCE THEOREM

In this Section we use the theorem of Kurosh to show how the general case of Theorem 5.1 can be completed by reducing the general case to the special case with hypotheses (H1) and (H2) in force which was completed in Section 5.

Proof. Let us assume that we have the setup of Theorem 5.1 but without assuming hypotheses (H1) and (H2). To use the Kurosh theorem we must define the inverse spectrum of interest. We denote elements of the directed set \mathfrak{A} by lower case Greek letters (μ, ν, γ , etc.). To specify an element μ of \mathfrak{A} , we specify a finite subset Ω_μ of Ω together with a finite-dimensional subspace \mathcal{E}_μ of \mathcal{E} . We then say that $\mu \preceq \nu$ exactly when both

$$\Omega_\mu \subset \Omega_\nu \text{ and } \mathcal{E}_\mu \subset \mathcal{E}_\nu.$$

Then it is easily checked that this is a partially ordered set satisfying the directed set axiom (6.2). For each $\mu \in \mathfrak{A}$, we let \mathfrak{X}_μ be the linear space consisting of pairs of cp nc kernels (Γ_1, Γ_2)

$$\begin{aligned} \Gamma_1: \Omega_\mu \times \Omega_\mu &\rightarrow \mathcal{L}(\mathcal{L}(\mathcal{N})_{\text{nc}}, \mathcal{L}(\mathcal{E}_\mu)_{\text{nc}}), \\ \Gamma_2: \Omega_\mu \times \Omega_\mu &\rightarrow \mathcal{L}(\mathbb{C}_{\text{nc}}, \mathcal{L}(\mathcal{E}_\mu)_{\text{nc}}). \end{aligned} \quad (6.6)$$

The set of such pairs forms a Banach space in the norm

$$\|(\Gamma_1, \Gamma_2)\| = \sup\{\|\Gamma_1(Z, Z)(I_{\mathcal{N}^n z})\|, \|\Gamma_2(Z, Z)(I_{n z})\|: Z \in \Omega_\mu\}.$$

Just as was the case in Section 6.2 where \mathfrak{X}_μ was defined using only one kernel, the space \mathfrak{X}_μ is a dual Banach space and the weak-* topology is given via pointwise weak-* convergence in $\mathcal{L}(\mathcal{E}_\mu)$ after evaluation at a fixed element T in the pre-dual space.

We define the inverse spectrum $\{\mathbb{K}_\mu: \mu \in \mathfrak{A}\}$ with $\mathbb{K}_\mu \subset \mathfrak{X}_\mu$ as follows. We let \mathbb{K}_μ consist of all cp nc kernel-pairs (Γ_1, Γ_2) on Ω_μ as in (6.6) so that

$$P_{\mathcal{E}_\mu^{nz}} \mathfrak{S}(Z, W)(P)|_{\mathcal{E}_\mu^{nw}} = \Gamma_1(Z, W)(\mathfrak{Q}(Z, W)(P)) + \Gamma_2(Z, W)(P). \quad (6.7)$$

The fact that \mathbb{K}_μ is nonempty for each μ is the content of the proof of the special case in Section 5 with hypotheses (H1) and (H2) assumed; let us note that hypotheses (iii) and (iv) in the statement of Theorem 5.1 are used only to prove the finite-point special case as handled in Section 5 above.

For $\mu, \nu \in \mathfrak{A}$ with $\mu \preceq \nu$, we define the restriction map π'_μ by

$$\pi'_\mu(\Gamma_1, \Gamma_2) = (\pi'_\mu \Gamma_1, \pi'_\mu \Gamma_2)$$

where

$$(\pi_\mu^\nu \Gamma_1)(Z, W)(P) = P_{\mathcal{E}_\mu^{nz}} \Gamma_1(Z, W)(P)|_{\mathcal{E}_\mu^{nw}}$$

for $Z, W \in \Omega_\mu, P \in \mathcal{L}(\mathcal{N})^{nz \times nw}$, and

$$(\pi_\mu^\nu \Gamma_2)(Z, W)(P) = P_{\mathcal{E}_\mu^{nz}} \Gamma_2(Z, W)(P)|_{\mathcal{E}_\mu^{nw}}$$

for $Z, W \in \Omega_\mu, P \in \mathbb{C}^{nz \times nw}$, where here $P_{\mathcal{E}_\mu^{nz}}$ is the orthogonal projection of \mathcal{E}_μ^{nz} onto its subspace \mathcal{E}_μ^{nz} . It is straightforward to check that this projection system $\{\pi_\mu^\nu: \mu \preceq \nu \text{ in } \mathfrak{A}\}$ satisfies the inverse-spectrum axioms (6.2)

The fact that each \mathbb{K}_μ is bounded can be seen from the estimate (5.8) for the special case where the sequence of kernels $\{K_k\}_{k \in \mathbb{N}}$ is taken to be the fixed kernel $P_{\mathcal{E}_\mu^{nz}} \mathfrak{S}(Z, W)(P)|_{\mathcal{E}_\mu^{nw}}$. Just as in the proof of the Arveson extension theorem in Section 6.2, it follows that \mathbb{K}_μ is pre-compact in the weak-* topology on \mathfrak{X}_μ . One can check that the \mathbb{K}_μ is weak-* closed in \mathfrak{X}_μ and hence is itself compact in the weak-* topology inherited from \mathfrak{X}_μ . Hence we are now in a position to apply the theorem of Kurosh to conclude that limit points of this inverse system exist.

Let $\Gamma = \{\Gamma_{\mu,1}, \Gamma_{\mu,2}\}$ be any such inverse-system limit point. We define two kernels Γ_1 and Γ_2 defined on all of Ω via the quadratic form

$$\langle \Gamma_i(Z, W)(P)e', e \rangle_{\mathcal{Y}_{nz}} \text{ where } e \in \mathcal{E}^{nz}, e' \in \mathcal{E}^{nw}$$

for points $Z, W \in \Omega, P \in \mathcal{N}^{nz \times nw}$ if $i = 1$ and $P \in \mathbb{C}^{nz \times nw}$ for $i = 2, e \in \mathcal{E}^{nz}$ as follows. Given $Z, W \in \Omega$ and

$$e = \begin{bmatrix} e_1 \\ \vdots \\ e_{nz} \end{bmatrix} \in \mathcal{E}^{nz}, \quad e' = \begin{bmatrix} e'_1 \\ \vdots \\ e'_{nw} \end{bmatrix} \in \mathcal{E}^{nw},$$

choose any $\mu \in \mathfrak{A}$ so that $Z, W \in \Omega_\mu, e_1, \dots, e_{nz}, e'_1, \dots, e'_{nw} \in \mathcal{E}_\mu$ and then define

$$\langle \Gamma_i(Z, W)(P)e', e \rangle_{\mathcal{E}^{nz}} = \langle \Gamma_{\mu,i}(Z, W)(P)e', e \rangle_{\mathcal{E}_\mu^{nz}} \text{ for } i = 1, 2. \tag{6.8}$$

It is a routine check to see that the inverse-system axioms (6.2) satisfied by the system of restriction mappings $\{\pi_\mu^\nu: \mu \preceq \nu \text{ in } \mathfrak{A}\}$ imply that this formula (6.8) is well-defined and uniquely specifies a kernel pair (Γ_1, Γ_2) having all the desired properties yielding a global representation (5.3) for \mathfrak{S} as wanted. \square

REMARK 6.5. NONCOMMUTATIVE INTERPOLATION AND SCHUR-AGLER CLASS REVISITED. The special case of Theorem 5.1 where the kernels \mathfrak{Q} and \mathfrak{S} are assumed to have the special Hermitian-decomposed form

$$\begin{aligned} \mathfrak{Q}(Z, W)(P) &= P \otimes I_{\mathcal{N}} - Q(Z)(P \otimes I_{\mathcal{M}})Q(W)^*, \\ \mathfrak{S}(Z, W)(P) &= a(Z)(P \otimes I_{\mathcal{E}})a(W)^* - b(Z)(P \otimes I_{\mathcal{U}})b(W)^* \end{aligned} \tag{6.9}$$

is a key piece in one of the main results of our paper [9], specifically, the implication (1') \Rightarrow (2) in Theorem 3.1 there. The cone separation argument as presented here in Section 5 largely follows the arguments in [9] but with some improvements in the exposition. There are also some distinctive differences which we would like to point out:

(1) For the case where the kernel Ω has the special form in (6.9), Lemma 4.1 in [9] shows that *any cp nc kernel* $\Gamma_2(Z, W)(P)$ *can be represented as* $\Gamma_1(Z, W)(\Omega(Z, W)(P))$ *for a cp nc kernel* Γ_1 . *Hence the second term* $\Gamma_2(Z, W)(P)$ *in the sought-after cone representation (5.3) can be absorbed into the first term and thus can be dropped in the formula (5.3).* The idea for this lemma in fact goes back to the seminal paper of Agler [1] which handles the

commutative polydisk situation: $\Omega(z, w) = \begin{bmatrix} 1 - z_1 \bar{w}_1 & & \\ & \ddots & \\ & & 1 - z_d \bar{w}_d \end{bmatrix}$.

(2) The fact that the cone \mathcal{C} (5.7) is not only closed and bounded but also pointed (Lemma 5.2) was missed in [9]. The fact that the cone is also pointed leads to a sharper cone-separation principle: *if* \mathfrak{S} *is not in the cone* \mathcal{C} , *then there is a linear functional* ℓ *so that* $\operatorname{Re} \ell(\mathfrak{S}) < 0$ *and* $\operatorname{Re} \ell(K) < 0$ *(not just* $\operatorname{Re}(K) \leq 0$ *for all nonzero* $K \in \mathcal{C}$. Consequently the cone-separation argument in [9] is more involved. The inner product on the spaces \mathcal{H} in [9] is seen to only be a semi-inner product; to form a Hilbert space one has to form equivalence classes. Then it is more complicated to define the operators $M_{\mathcal{X},0}^*$ and $M_{\mathcal{Q},0}^*$ on \mathcal{H} and $\mathcal{H}_{\mathcal{N}}$; to make sense of such formal operators not necessarily well defined on equivalence classes, it is necessary to introduce extra regularization terms depending on a parameter $\rho < 1$ to form an approximate cone \mathcal{C}_ρ , and then eventually let ρ tend up to 1.

In this more complicated setting, it should be explained that the definition of $\widehat{M}_{\mathcal{X},0}^*$ involves restriction to an invariant subspace followed by a quotient map, and in the end one only knows that $M_{\mathcal{X},0}$ is similar to a point in the bi-full envelope $[\Omega]_{\text{bi-full}}$ of Ω (points corresponding to a semi-invariant subspace \mathcal{N} of a point $\widehat{Z}^{(0)}$ in $[\Omega]_{\text{nc}}$). In the present exposition in Section 5, there is no quotient operation involved in the definition of $\widehat{M}_{\mathcal{X},0}$ and it turns out that $M_{\mathcal{X},0}$ is in $[\Omega]_{\text{full}}$ after all.

(3) The extension to the general case of the result for the special case where Ω consists of only finitely many points and $\dim \mathcal{E} < \infty$ as sketched here is exactly the same as the extension from the special case to the general case as carried out in [9] for the case where Ω has the special form (6.9).

REMARK 6.6. Theorem 5.1 is somewhat unsatisfying since the kernel decomposition (5.3) holds only for $Z, W \in \Omega$ even if we assume that the kernel \mathfrak{S} is given on all of Ξ , unlike the case for Positivstellensätze as appearing in classical commutative algebraic geometry (see e.g. [13, 22] as well as [16] for the matrix-valued case). To get a version of Theorem 5.1 closer to the form of a classical Positivstellensatz, let us consider the following variation of Problem C:

PROBLEM C': Given nc Hermitian kernels Ω and \mathfrak{S}

$$\Omega: \Xi \times \Xi \rightarrow \mathcal{L}(\mathbb{C}_{\text{nc}}, \mathcal{L}(\mathcal{N})_{\text{nc}}), \quad \mathfrak{S}: \Xi \times \Xi \rightarrow \mathcal{L}(\mathbb{C}_{\text{nc}}, \mathcal{L}(\mathcal{E})_{\text{nc}})$$

(for auxiliary coefficient Hilbert spaces \mathcal{N} and \mathcal{E}) such that

$$Z \in \Xi_n \text{ and } \Omega(Z, Z)(I_n) \succ 0 \Rightarrow \mathfrak{S}(Z, Z)(I_{\mathcal{E}^n}) \succeq 0,$$

find cp nc kernels

$$\Gamma_1: \Xi \times \Xi \rightarrow \mathcal{L}(\mathcal{L}(\mathcal{N})_{\text{nc}}, \mathcal{L}(\mathcal{E})_{\text{nc}}), \quad \Gamma_2: \Xi \times \Xi \rightarrow \mathcal{L}(\mathbb{C}_{\text{nc}}, \mathcal{L}(\mathcal{E})_{\text{nc}})$$

so that

$$\mathfrak{S}(Z, W)(P) = \Gamma_1(\Omega(Z, W)(P)) + \Gamma_2(Z, W)(P). \tag{6.10}$$

The result of Theorem 5.1 is that Problem C' has an affirmative solution under some additional hypotheses, but in a weaker form. The weaker form is that we get cp kernels Γ_1, Γ_2 defined only for $Z, W \in \mathbb{P}_\Omega$ (the set where $\Omega(Z, Z)(I) \succ 0$), and then the decomposition (5.3) or (6.10) is guaranteed to hold only for $Z, W \in \mathbb{P}_\Omega$. To apply Theorem 5.1 to arrive at this result, we take $\Omega = \mathbb{P}_\Omega$ in which case also $\Omega' = \mathbb{P}_\Omega$. Then hypotheses (i) and (ii) in Theorem 5.1 are satisfied, but the decomposability assumptions (iii) and (iv) do not appear to hold in general (see Theorem 3.5 and the ensuing discussion). On the other hand, the hereditary case of Theorem 1.2 in the paper of Helton-McCullough [17] gives an affirmative answer to Conjecture C' under the assumptions that $\mathcal{V} = \mathbb{C}^d, \Xi = \mathcal{V}_{\text{nc}}$, the coefficient Hilbert spaces \mathcal{N} and \mathcal{E} are finite-dimensional, and that Ω and \mathfrak{S} are matrix-polynomial nc kernels, i.e.,

$$\Omega(Z, W)(P) = q_1(Z)Pq_2(W)^*, \quad \mathfrak{S}(Z, W)(P) = s_1(Z)Ps_2(W)^*$$

for nc matrix polynomials q_1, q_2, s_1, s_2 of appropriate sizes, and also the positivity set \mathbb{P}_Ω should be bounded in norm (the *Archimedean hypothesis*); here we make use of the observations of Section 3.6 of [9] that the general kernel decomposition (6.10) is equivalent to the special case with $P = I$ and $Z = W$ as long as (6.10) holds over a set Ξ which is finitely open. This Positivstellensatz has now been extended to the case where Ω and \mathfrak{S} are *rational matrix Hermitian kernels* (i.e., the functions q_1, q_2, s_1, s_2 as above are nc rational matrix functions with intersection of domains equal to some generic nc subset \mathcal{D} of \mathbb{C}_{nc}^d). Furthermore, in case the Ω -positivity set \mathbb{P}_Ω is convex, results of Helton-Klep-McCullough [18] provide an explicit finite algorithm for actually computing the Kolmogorov decompositions for the kernels Γ_1 and Γ_2 in the kernel decomposition (6.10).

REFERENCES

[1] J. Agler, *On the representation of certain holomorphic functions defined on a polydisc*, in Topics in operator theory: Ernst D. Hellinger memorial volume, 17–66, Operator Theory: Advances and Applications, 48. Birkhäuser, Basel, 1990.

- [2] J. Agler and J.E. McCarthy, *Pick interpolation and Hilbert function spaces*, Graduate Studies in Mathematics, 44. Amer. Math. Soc., Providence, RI, 2002.
- [3] D. Alpay, *Some remarks on reproducing kernel Krein spaces*, Rocky Mountain J. Math. 22 (1992) no. 4, 1243–1258.
- [4] D. Alpay, A. Dijksma, J. Rovnyak, and H. de Snoo, *Schur functions, operator colligations and reproducing kernel Pontryagin spaces*, Operator Theory: Advances and Applications, 96. Birkhäuser, Basel, 1997.
- [5] N.K. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. 68 (1950), 337–404.
- [6] A.V. Arkhangel'skiĭ and L.S. Pontryagin (eds.), *General Topology I*. Springer, Berlin, 1990.
- [7] W.B. Arveson, *Subalgebras of C^* -algebras, I*, Acta Math. 123 (1969), 141–224.
- [8] J.A. Ball, G. Marx, and V. Vinnikov, *Noncommutative reproducing kernel Hilbert spaces*, J. Funct. Anal. 271 (2016), 1844–1920.
- [9] J.A. Ball, G. Marx, and V. Vinnikov, *Interpolation and transfer-function realization for the noncommutative Schur-Agler class*, Operator theory in different settings and related applications, 23–116. Operator Theory: Advances and Applications, 262. Birkhäuser, Cham, 2018.
- [10] S.D. Barreto, B.V.R. Bhat, V. Liebscher, and M. Skeide, *Type I product systems of Hilbert modules*, J. Funct. Analysis 212 (2004), 121–181.
- [11] T. Bhattacharyya, M.A. Dritschel, and C.T. Todd, *Completely bounded kernels*, Acta Sci. Math. (Szeged) 79 (2013), 191–217.
- [12] D.P. Blecher and C. le Merdy, *Operator algebras and their modules: An operator space approach*, London Mathematical Society Monographs. New Series, 30. Oxford Science Publications. Oxford University Press, 2004.
- [13] J. Bochnack, M. Coste, M.-F. Roy, *Real algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, 36. Springer, Berlin, 1998.
- [14] R.G. Douglas, *Banach algebra techniques in operator theory*, second edition, Graduate Texts in Mathematics, 179. Springer, New York, 1998.
- [15] E.G. Effros and Z.-J. Ruan, *Operator spaces*, London Mathematical Society Monographs. New Series, 23. The Clarendon Press, Oxford University Press, New York, 2000.

- [16] A. Grinshpan, D. S. Kaliuzhnyi-Verbovetskyi, V. Vinnikov, and H. J. Woerdeman, *Matrix-valued Hermitian Positivstellensatz, lurking contractions, and contractive determinantal representations of stable polynomials*, in: Operator theory, functions spaces, and applications (eds. T. Eisner, B. Jacob, A. Ran, and H. Zwart), 123–136, Operator Theory: Advances and Applications, 255. Birkhäuser, Basel, 2016.
- [17] J.W. Helton and S.A. McCullough, *A Positivstellensatz for non-commutative polynomials*, Trans. Amer. Math. Soc. 356 (2004), no. 9, 3721–3737.
- [18] J.W. Helton, I. Klep, and S. McCullough, *The convex Positivstellensatz in a free algebra*, Adv. Math. 231 (2012), 516–534.
- [19] J. Jahn, *Vector optimization: Theory, applications, and extensions*. Springer, Berlin, 2004.
- [20] D. S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov, *Foundations of noncommutative function theory*, Mathematical Surveys and Monographs, 199. Amer. Math. Soc., Providence, 2014.
- [21] A. Kurosch, *Kombinatorischer Aufbau der bikompakten topologischen Räume*, Compositio Math. 2 (1935), 471–476.
- [22] M. Marshall, *Positive polynomials and sums of squares*, Mathematical Surveys and Monographs, 146. Amer. Math. Soc., Providence, 2008.
- [23] G. Marx, *Noncommutative kernels*, Ph.D. Dissertation submitted to Virginia Tech, May 2017, available at <https://vtechworks.lib.vt.edu/handle/10919/78353>.
- [24] J.E. Pascoe, *An elementary method to compute the algebra generated by some given matrices and its dimension*, Linear Algebra Appl. 571 (2017), 132–142.
- [25] J.E. Pascoe, *Positivstellensätze for noncommutative rational expressions*, Proc. Amer. Math. Soc. 146 (2018), no. 3, 933–937.
- [26] V. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics, 78. Cambridge University Press, Cambridge, 2002.
- [27] M. Rosenblum and J. Rovnyak, *Hardy classes and operator theory*. Oxford Univ. Press, 1985 (reprinted by Dover, 1997).
- [28] W. Rudin, *Functional analysis. Second edition*. McGraw-Hill, 1991.
- [29] L. Schwartz, *Sous-espaces hilbertiens d'espaces vectoriels topologiques et noyaux associés (noyaux reproduisants)*, J. Analyse Math. 13 (1964), 115–256.

- [30] M. Takesaki, *Theory of operator algebras I*, Encyclopaedia of Mathematical Sciences, 124. Operator Algebras and Non-Commutative Geometry, V. Springer, Berlin, 1979.
- [31] G. Wittstock, *Ein operatorwertiger Hahn-Banach Satz*, J. Funct. Anal. 40 (1981), 127–150.
- [32] G. Wittstock, *Extensions of completely bounded C^* -module homomorphisms*, Proc. Int. Conf., Neptun/Romania 1980, II, 238–250, Monogr. Stud. Math., 18. Pitman, Boston, MA, 1984.

Joseph A. Ball
Department of Mathematics
Virginia Tech
Blacksburg, VA 24061-0123
USA
joball@math.vt.edu

Gregory Marx
Department of Mathematics
Virginia Tech
Blacksburg, VA 24061-0123
USA
marxg@vt.edu

Victor Vinnikov
Department of Mathematics
Ben-Gurion University of the Negev
84105 Beer-Sheva
Israel
vinnikov@math.bgu.ac.il