# The Vanishing of Iwasawa's $\mu$-Invariant Implies the Weak Leopoldt Conjecture

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Abstract. Let $K$ denote a number field containing a primitive $p$-th root of unity; if $p = 2$, then we assume $K$ to be totally imaginary. If $K_\infty/K$ is a $\mathbb{Z}_p$-extension such that no prime above $p$ splits completely in $K_\infty/K$, then the vanishing of Iwasawa's invariant $\mu(K_\infty/K)$ implies that the weak Leopoldt Conjecture holds for $K_\infty/K$. This is actually known due to a result of Ueda, which appears to have been forgotten. We present an elementary proof which is based on a reflection formula from class field theory.

In the second part of the article, we prove a generalisation in the context of non-commutative Iwasawa theory: we consider admissible $p$-adic Lie extensions of number fields, and we derive a variant for fine Selmer groups of Galois representations over admissible $p$-adic Lie extensions.

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## 1 Introduction

This note is about two famous conjectures in Iwasawa theory and their dependencies. Throughout the article, we fix a rational prime $p$ (which may be even) and a number field $K$. Let $\overline{\mathcal{O}_K^\times}$ be the closure of the group of units $\mathcal{O}_K^\times$ of $K$ embedded diagonally into the product $\prod_{v|p} \mathcal{O}_v^\times$ of the local units $\mathcal{O}_v^\times \subseteq K_v$, where $K_v$ denotes the $v$-adic completion of $K$. *Leopoldt's conjecture* predicts that

$$\mathrm{rank}_{\mathbb{Z}_p}(\overline{\mathcal{O}_K^\times}) = \mathrm{rank}_{\mathbb{Z}}(\mathcal{O}_K^\times) = r_1(K) + r_2(K) - 1,$$

i.e., $\mathbb{Z}$-linearly independent units should remain $p$-adically independent (here $r_1(K)$ and $r_2(K)$ denote the numbers of real and pairs of complex places of $K$). The number $\delta(K) := \operatorname{rank}_{\mathbb{Z}}(\mathcal{O}_K^\times) - \operatorname{rank}_{\mathbb{Z}_p}(\overline{\mathcal{O}_K^\times}) \geq 0$ is called the *Leopoldt defect* of $K$. The *weak Leopoldt Conjecture* holds for a $\mathbb{Z}_p$-extension $K_\infty = \bigcup_n K_n$ of $K$ if and only if the sequence of the Leopoldt defects $\delta(K_n)$ of the intermediate fields $K_n$ remains bounded as $n \to \infty$.

If Leopoldt's Conjecture holds for the number field $K$, then weak Leopoldt holds for every $\mathbb{Z}_p$-extension $K_\infty$ of $K$ (see [NSW08, Corollary 10.3.23]). Leopoldt's Conjecture is known to hold for *totally abelian* $K$, and also for abelian extensions of imaginary quadratic number fields, by results of Baker and Brumer (cf. [Bak66] and [Bru67]). Moreover, the weak Leopoldt Conjecture holds for the *cyclotomic* $\mathbb{Z}_p$*-extension* $K_\infty^c$ of *any* number field $K$ (see [NSW08, Theorem 10.3.25]).

There is another famous conjecture about cyclotomic $\mathbb{Z}_p$-extensions which goes back to Iwasawa, namely the conjectured vanishing of the $\mu$-invariant. This is related to the growth of the ideal class groups of the $K_n$: if $p^{e_n}$ is the exact power of $p$ dividing the class number of $K_n$, $n \in \mathbb{N}$, then

$$e_n = \mu(K_\infty^c/K) \cdot p^n + O(n),$$

where $\mu(K_\infty^c/K) \geq 0$ does not depend on $n$. An analogous asymptotic growth formula for the class numbers holds true in any $\mathbb{Z}_p$-extension $K_\infty$ of $K$, and we call the corresponding integer $\mu(K_\infty/K)$ the *$\mu$-invariant* of $K_\infty/K$. In fact, $\mu(K_\infty/K)$ is a module-theoretic invariant of the projective limit

$$A(K_\infty) = \varprojlim_n A(K_n),$$

where $A(K_n)$ denotes the $p$-primary subgroup of the ideal class group of $K_n$, $n \in \mathbb{N}$ (see Section 2.4 for more details). It is known that the $\mu$-invariant vanishes for the cyclotomic $\mathbb{Z}_p$-extension $K_\infty^c$ of any totally abelian base field $K$, by the results of Ferrero-Washington and Sinnott (see [FW79] and [Sin84]).

Due to work of Babaĭcev (see [Bab82, Corollary 3]), it is known that for an arbitrary number field $K$, the weak Leopoldt Conjecture is true for a dense subset of $\mathbb{Z}_p$-extensions of $K$ with regard to a natural topology on the set of $\mathbb{Z}_p$-extensions of $K$ which has been introduced by Greenberg in [Gre73]. Under the *hypothesis* that $\mu(K_\infty^c/K) = 0$ for the cyclotomic $\mathbb{Z}_p$-extension $K_\infty^c$ of $K$, a similar result is valid for the $\mu$-invariants, i.e. in this case the $\mu$-invariants vanish for a dense subset of $\mathbb{Z}_p$-extensions of $K$ (see [Bab82, Theorem 4]).

It is known that there exist non-cyclotomic $\mathbb{Z}_p$-extensions of number fields having a non-trivial (and in fact arbitrarily large) $\mu$-invariant (see [Iwa73, Theorem 1]). On the other hand, no example of a $\mathbb{Z}_p$-extension $K_\infty/K$ is known where the weak Leopoldt conjecture fails (as mentioned above, such a $\mathbb{Z}_p$-extension would also yield a counterexample to Leopoldt's conjecture).

Heuristically, the above known results suggest that the Iwasawa $\mu$-invariant conjecture might be the harder problem. In this paper, we first give a proof of the following

Theorem 1.1. *Let $K$ be a number field, and let $K_\infty$ be a $\mathbb{Z}_p$-extension of $K$. If $p = 2$, then we assume that $r_1(K) = 0$, i.e. that $K$ is totally imaginary. We denote by $K'$ the finite extension of $K$ which is obtained by adjoining to $K$ a primitive $p$-th root of unity. Then the following holds:*

*(a) Suppose first that $K' = K$, and that no prime of $K$ dividing $p$ splits completely in $K_\infty$. If $\mu(K_\infty/K) = 0$, then the weak Leopoldt Conjecture holds for $K_\infty/K$.*

*(b) Suppose that no prime of $K$ dividing $p$ splits completely in $K'/K$, and consider $K'_\infty = K' \cdot K_\infty$. If the $\mu$-invariant of the $\mathbb{Z}_p$-extension $K'_\infty/K'$ vanishes, then the weak Leopoldt conjecture holds for $K_\infty/K$.*

In fact, the first part of this result is not new and has first been proven, to the author's knowledge, by Ueda in [Ued86], based on previous work of Bertrandias and Payan (see [BP72]). Actually Ueda proved that the vanishing of the $\mu$-invariant of a $\mathbb{Z}_p$-extension $K_\infty/K$ in which no prime above $p$ is completely split implies a condition which is equivalent to the weak Leopoldt conjecture for $K_\infty/K$ by work of Greenberg (see [Gre78, p. 90 and Proposition 1]). Ueda's result is only for number fields $K$ containing a primitive $p$-th root of unity – this condition will be essential also for our proof of (a).

It seems, however, that Ueda's result is not very widely known in the community – the author is aware of only one article, namely the paper [NQD88] of Nguyen Quang Do, citing Ueda's result. The main purpose of the first part of this note, besides popularising the result, is to present a proof which appears to be more direct and elementary than Ueda's. The major ingredient in our proof of Theorem 1.1 is a reflection formula from class field theory which is due to Gras (see [Gra03] and Theorem 2.2 below). For (b), a more general version of Gras's original result (from [Gra82]) is used. For details, we refer to Section 2.

Let $\mathbb{L}_\infty/K$ be a $\mathbb{Z}_p^2$-extension which contains the cyclotomic $\mathbb{Z}_p$-extension $K_\infty^c$. If the $\mu$-invariant of the cyclotomic $\mathbb{Z}_p$-extension (or the $\mu$-invariant of any other $\mathbb{Z}_p$-extension of $K$ in which no prime above $p$ splits completely) vanishes, then there exist at most finitely many $\mathbb{Z}_p$-extensions of $K$ inside of $\mathbb{L}_\infty$ whose $\mu$-invariants might be non-zero (this can be derived from Babaĭcev's denseness results which have been mentioned above). In fact, the possible number of exceptional $\mathbb{Z}_p$-extensions of $K$ has been bounded explicitly by Bloom and Gerth (see [BG81, Theorem 1]). By the results of Babaĭcev, there is also a finite number (if any) of $\mathbb{Z}_p$-extensions $K_\infty \subseteq \mathbb{L}_\infty$ for which the weak Leopoldt conjecture might fail. It seems, however, that no explicit upper bound is known for this number of exceptional $\mathbb{Z}_p$-extensions. We will derive such an upper bound by combining the results of Bloom and Gerth with Theorem 1.1 (see Corollary 2.9 for details).

In the second part of the article, we prove generalisations of Theorem 1.1 in two aspects. In Section 3, we prove an analogue of Theorem 1.1 for admissible uniform $p$-adic Lie extensions – we refer to Section 3 for the details, in particular for the definition of the $\mu$-invariant and for an appropriate notion of the weak Leopoldt conjecture in this setting; here we just say that the validity of the weak Leopoldt conjecture will be formulated in terms of the triviality of a certain second cohomology group. We will prove the following

Theorem 1.2. *Let $K_\infty/K$ be an admissible $p$-adic Lie extension. We assume that $K$ contains a primitive $p$-th root of unity. If $p = 2$, then we assume that $K$ is totally imaginary. Suppose that no prime of $K$ dividing $p$ is completely split in $K_\infty/K$.*
*If the $\mu$-invariant of $K_\infty/K$ vanishes, then the weak Leopoldt conjecture holds for $K_\infty/K$.*

One can derive a variant of this theorem in the style of Theorem 1.1(b), which works if $K$ does not contain a primitive $p$-th root of unity and allows primes of $K$ above $p$ to split completely in $K_\infty$ if they do not split in $K'/K$.
In the last section, we consider $p$-adic representations $V$ of the absolute Galois group of $K$. Let $T \subseteq V$ be a Galois invariant lattice, suppose that the number field $K$ contains a primitive $p$-th root of unity and that $p^{-1}T/T = (V/T)[p]$ is defined over $K$. Then it follows from work of Perrin-Riou (see [PR95, Proposition B.2]) that the vanishing of the $\mu$-invariant of the cyclotomic $\mathbb{Z}_p$-extension $K_\infty^c$ of $K$ implies the weak Leopoldt conjecture for $V$ over $K_\infty^c$, which again is formulated in terms of a certain second cohomology group (see also Theorem 4.3 below). We will prove the following generalisation (again, we formulate the result only for one of the two possible settings):

Theorem 1.3. *Suppose that $p \neq 2$. Let $K_\infty/K$ be an admissible $p$-extension, and let $V$ be a $p$-adic Galois representation with Galois invariant lattice $T$. We assume that there exist only finitely many primes at which $V$ is ramified, that $p^{-1}T/T$ is defined over $K$, and that $K$ contains a primitive $p$-th root of unity.*
*If the $\mu$-invariant of $K_\infty/K$ vanishes and if no prime of $K$ above $p$ and no prime at which $V$ is ramified is totally split in $K_\infty$, then the weak Leopoldt conjecture holds for $V/T$ over $K_\infty$.*

We remark that of course Theorems 1.1 and 1.2 (for $p \neq 2$) are special cases of Theorem 1.3. On the other hand, Theorem 1.2 is used in our proof of Theorem 1.3 – therefore the proof of Theorem 1.2 given in Section 3 can not be omitted.

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## 2 The classical conjectures for $\mathbb{Z}_p$-extensions

This section consists of six subsections, the first four of which are preliminary in nature. In Subsection 2.5 we give a proof of Theorem 1.1. The last subsection contains some applications and further remarks on Theorem 1.1.

### 2.1 The reflection formula

In this section, we will use the following notation. Let $S$ and $T$ denote two finite disjoint sets of finite primes of $K$.

Definition 2.1. We denote by $A_S^T(K)$ the Galois group of the maximal abelian pro-$p$-extension of $K$ which is unramified outside the primes contained in $S$, and in which every prime contained in $T$ is totally split. For brevity, we will call this field the maximal abelian pro-$p$-extension of $K$ which is $S$-ramified and $T$-split.
If the base field is clear, then we abbreviate $A_S^T(K)$ to $A_S^T$. In the special case $S = \emptyset$ or $T = \emptyset$, we omit the corresponding sub- or superscript. In particular, if $S = T = \emptyset$, then we obtain the Galois group $A = A_\emptyset^\emptyset$ of the $p$-Hilbert class field $H(K)$ over $K$.

It follows from class field theory that $A = A_\emptyset^\emptyset$ is isomorphic to the $p$-primary subgroup of the ideal class group of $K$. In what follows, we will identify these two groups without further notice. If $S_p$ denotes the set of primes of $K$ dividing $p$, then we abbreviate $A_\emptyset^{S_p}$ to $A'$.
For any finite abelian group $G$, we denote by $\mathrm{rank}_p(G)$ the dimension of $G/pG$ over the field $\mathbb{F}_p$ with $p$ elements.
The basic ingredient of our proof of Theorem 1.1 is the following

Theorem 2.2 (Reflection formula). *Fix $K$, $S$ and $T$. If $p = 2$, then we assume that $r_1(K) = 0$. If $K$ contains a primitive $p$-th root of unity, and if the set $S_p$ of primes of $K$ dividing $p$ is contained in $S \cup T$, then*

$$\mathrm{rank}_p(A_T^S) \;=\; \mathrm{rank}_p(A_S^T) + |T| - |S| + \sum_{v \in T_p} [K_v : \mathbb{Q}_p] - r_2(K)\,, \qquad (1)$$

*where $T_p = T \cap S_p$.*

*Proof.* See [Gra03, Theorem I.4.6(ii) and pp. 153ff.]. ☐

We denote by $M_p(K)$ the maximal abelian $S_p$-ramified pro-$p$-extension of $K$, i.e., $\mathrm{Gal}(M_p(K)/K) = A_{S_p}^\emptyset$.

Theorem 2.3. *Let $K$ be a number field containing a primitive $p$-th root of unity. If $p = 2$, then we assume that $r_1(K) = 0$. Let $l = |S_p|$ be the number of primes of $K$ dividing $p$. Then*

$$\mathrm{rank}_p(\mathrm{Gal}(M_p(K)/K)) \;=\; r_2(K) + l + \mathrm{rank}_p(A')\,.$$

Note: we consider $p$-ranks, and not $\mathbb{Z}_p$-ranks, as in Leopoldt's Conjecture.

*Proof.* Consider the following sets $S$ and $T$: $S = S_p = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_l\}$ contains the primes of $K$ dividing $p$, and $T = \emptyset$. Then the reflection formula (1) implies that

$$\mathrm{rank}_p(A') = \mathrm{rank}_p(A_{\emptyset}^{S}) = \mathrm{rank}_p(A_S^{\emptyset}) - l - r_2(K)$$

and therefore

$$\mathrm{rank}_p(A_S^{\emptyset}) = \mathrm{rank}_p(A') + l + r_2(K).$$

□

Remark 2.4. Theorem 2.3 will yield a straightforward proof of Theorem 1.1 (see Section 2.5 below). In fact, the same argument (for $p \neq 2$) goes back to work of Bertrandias and Payan (see [BP72, Proposition 2.4]), which was the major ingredient in the proof of Theorem 1.1 given by Ueda (cf. [Ued86, Proposition 1.1]). In Section 2.3, we describe a generalisation of Theorem 2.3 due to Gras (see [Gra82]), which covers also number fields $K$ which do not contain a primitive $p$-th root of unity.
Our proof of Theorem 2.3 is based on Gras's reflection formula from (idèlic) class field theory. On the contrary, the proofs in [BP72] and [Gra82] use Kummer theory and appear to be more technical.

Remark 2.5. The $p$-rank of $\mathrm{Gal}(M_p(K)/K)$ can be computed numerically in a quite efficient manner, due to further results of Gras. For any $n \in \mathbb{N}$, we denote by $K(p^n)$ the ray class field of $K$ of conductor $p^n$, and we let $A_{p^n}(K) = \mathrm{Gal}(K(p^n)/K)$. Then it follows from [Gra19, Theorem 2.1 and Corollary 2.2] that

$$\mathrm{rank}_p(A_{p^n}) = \mathrm{rank}_p(\mathrm{Gal}(M_p(K)/K))$$

for each $n \geq 2 + \varepsilon(K)$, where $\varepsilon(K) = 1$ if $p = 2$, and $\varepsilon(K) = 0$ otherwise.

## 2.2 An alternative approach to Theorem 2.3

In the following subsection we describe a second approach for deriving (a part of) Theorem 2.3 from the reflection formula, since this will yield an interesting application below. We choose $S = S_p$ as above, and we let $T = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_l\}$ contain exactly $l = |S|$ different prime ideals of $K$ which are coprime with $p$ and satisfy the following condition: if $M \subseteq M_p(K)$ denotes the maximal *$p$-elementary* extension of $K$ which is unramified outside $p$, then $\mathfrak{q}_1, \ldots, \mathfrak{q}_l$ are totally split in the finite extension $M/K$. Note that the existence of infinitely many primes of $K$ with these properties follows from Chebotarev's Theorem.
The reflection formula then implies that

$$\mathrm{rank}_p(A_T^S) = \mathrm{rank}_p(A_S^T) + l - l - r_2(K)$$

and therefore

$$\operatorname{rank}_p(A_S^T) \;=\; \operatorname{rank}_p(A_T^S) + r_2(K)\,. \tag{2}$$

Recall that $A_T^S$ corresponds to the maximal $p$-abelian $\{\mathfrak{q}_1, \dots, \mathfrak{q}_l\}$-ramified (and $S_p$-split) extension of $K$. We claim that the right hand side of (2) is bounded by

$$r_2(K) + l + \operatorname{rank}_p(A')\,. \tag{3}$$

In order to prove this claim, we use the following notation. As in the Introduction, we denote by $\mathcal{O}_K$ the ring of integers of $K$. For any prime $v$ of $K$, we let $\mathcal{O}_v^\times$ denote the group of local units in the $v$-adic completion of $K$. Moreover, we denote by $\overline{\mathcal{O}_K^\times} \subseteq \prod_{v \in S} \mathcal{O}_v^\times$ the $p$-adic closure of the group $\mathcal{O}_K^\times$, embedded diagonally into $\prod_{v \in S} \mathcal{O}_v^\times$. For any abelian group $G$, we denote by $G[p^\infty]$ the (pro-) $p$-primary subgroup of $G$.
We recall that we have an exact sequence

$$((\prod_{i=1}^{l} \mathcal{O}_{\mathfrak{q}_i}^\times)/\overline{\mathcal{O}_K^\times})[p^\infty] \longrightarrow A_T^S \longrightarrow A' \longrightarrow 0 \tag{4}$$

from class field theory. For every $i \in \{1, \dots, l\}$, the pro-$p$-part of the group of local units $\mathcal{O}_{\mathfrak{q}_i}^\times$ is finite cyclic. This proves the above claim in view of the sub-additivity of the $p$-rank (cf., for example, [KM22, Lemma 3.2]).
On the other hand, by the choice of the $\mathfrak{q}_i$, the $p$-rank of the Galois group $A_S^T$ on the left hand side of (2) corresponds to rank of the Galois group of the maximal $p$-abelian $S_p$-ramified extension $M_p(K)$ of $K$ (the $\mathfrak{q}_i$ are totally split in the first layer of this extension). Therefore it follows from (3) that

$$\operatorname{rank}_p(\operatorname{Gal}(M_p(K)/K)) \leq r_2(K) + l + \operatorname{rank}_p(A'),$$

which is one half of the statement of Theorem 2.3.
Up to now, we seemingly did not gain any advantage by using this second approach. But there might be a chance to improve on the above inequality. Recall that for each $\mathfrak{q}_i$, the maximal $\mathfrak{q}_i$-ramified abelian $p$-extension $M_{\mathfrak{q}_i}(K)$ of $K$ is finite and cyclic over $H(K)$ (this follows from an exact sequence similar to (4)). Suppose now that some primes of $K$ dividing $p$ do not split at all in $M_{\mathfrak{q}_i}(K)$ for certain $i$, and let $r \leq l$ be the number of such $i$. Then the above approach implies that

$$\operatorname{rank}_p(\operatorname{Gal}(M_p(K)/K)) \leq r_2(K) + l - r + \operatorname{rank}_p(A'),$$

i.e., we save a rank of $r$ in our estimate.
However, since the statement of Theorem 2.3 is actually an equality, $r$ must be zero. In other words, this modified approach, together with Theorem 2.3, proves the following corollary, which very much deserves to be called a reflection-type result.

COROLLARY 2.6. *Let $K$ be as in Theorem 2.3. If $\mathfrak{q}$ denotes any prime of $K$ which does not divide $p$ and which is completely split in the maximal $p$-elementary extension of $K$ contained in $M_p(K)$, then each prime of $K$ dividing $p$ has to split in $M_\mathfrak{q}(K)/H(K)$, where $M_\mathfrak{q}(K)$ denotes the maximal $p$-abelian $\mathfrak{q}$-ramified extension of $K$.*

## 2.3 GENERALISATION

In [Gra82], Gras proved a variant of Theorem 2.3 which works also if $K$ does not contain the group $\mu_p$ of $p$-th roots of unity. This variant will be used in the proof of Theorem 1.1(b).

THEOREM 2.7. *Let $K$ be a number field, let $K' = K(\mu_p)$ and let $g$ be the number of primes $v \in S_p(K)$ which are totally split in $K'/K$. If $p = 2$, then we assume that $r_1(K) = 0$. Moreover, we let $\nu = 1$ if $\mu_p \subseteq K$ (i.e. if $K' = K$) and $\nu = 0$ otherwise. Then*

$$\operatorname{rank}_p(\operatorname{Gal}(M_p(K)/K)) = r_2(K) + 1 + g - \nu + \operatorname{rank}_p(\varepsilon_\chi(A')^{(K')}), \qquad (5)$$

*where $\chi$ generates the group of characters of $\operatorname{Gal}(K'/K)$ and $\varepsilon_\chi$ means the idempotent of $\mathbb{Z}_p[\operatorname{Gal}(K'/K)]$.*

*Proof.* This follows from [Gra82, Théorème I.2]. □

In particular, in the case $\mu_p \subseteq K$, i.e. $K' = K$, we recover Theorem 2.3. Note that this condition is of course satisfied if $p = 2$. It is worth mentioning that Theorem 2.7 allows us to prove variants of Theorem 1.1 also for $\mathbb{Z}_p$-extensions $K_\infty = \bigcup_n K_n$ in which some of the primes above $p$ are totally split – provided that these primes do not split totally in $(K'K_n)/K_n$, $n \in \mathbb{N}$ (e.g. because they are ramified in $K'/K$), and for the price of considering the $\mu$-invariant of the $\mathbb{Z}_p$-extension $K'_\infty := K_\infty \cdot K'$ of $K'$ (see also part (b) of Theorem 1.1).

## 2.4 IWASAWA THEORY OF $\mathbb{Z}_p$-EXTENSIONS

Let $K_\infty/K$ be a $\mathbb{Z}_p$-extension, i.e., $\operatorname{Gal}(K_\infty/K) \cong \mathbb{Z}_p$. We write $K_\infty = \bigcup_n K_n$, with $\operatorname{Gal}(K_n/K) \cong \mathbb{Z}/p^n\mathbb{Z}$ for each $n \in \mathbb{N}$ (this determines the $K_n$ uniquely).

DEFINITION 2.8. The *weak Leopoldt conjecture* holds for $K_\infty/K$ if and only if the sequence of Leopoldt defects $\delta(K_n)$, defined as in the Introduction, is bounded as $n \to \infty$.

For each $n \in \mathbb{N}$, we let $A_n = A(K_n)$ and $A'_n = A'(K_n)$ for brevity. By a famous theorem of Iwasawa (see [Iwa59]), there exist integers $\mu(K_\infty/K) \geq 0$, $\lambda(K_\infty/K) \geq 0$ and $\nu(K_\infty/K)$ such that

$$v_p(|A_n|) = \mu(K_\infty/K) \cdot p^n + \lambda(K_\infty/K) \cdot n + \nu(K_\infty/K) \qquad (6)$$

for each sufficiently large $n \in \mathbb{N}$.

By the *$\mu$-invariant conjecture* of Iwasawa, $\mu(K^c_\infty/K)$ should be zero for the *cyclotomic* $\mathbb{Z}_p$-extension $K^c_\infty$ of $K$, which is defined as $K^c_\infty = \bigcup_n K(\mu_{p^n})$ if $K$ contains a primitive $p$-th root of unity (here, for any $n \in \mathbb{N}$, $\mu_{p^n}$ denotes the group of $p^n$-th roots of unity in some fixed algebraic closure of $K$). For general $K$, $K^c_\infty$ is a certain subextension of $\bigcup_n K(\mu_{p^n})$. It turns out that $\mu(K_\infty/K)$ can be non-zero for non-cyclotomic $\mathbb{Z}_p$-extensions $K_\infty/K$ (see [Iwa73, Theorem 1]).

Before we prove Theorem 1.1 from the Introduction, we say a few more words about Iwasawa modules.

If $\Gamma := \mathrm{Gal}(K_\infty/K) \cong \mathbb{Z}_p$ is topologically generated by an element $\gamma$, then the completed group ring $\mathbb{Z}_p[[\Gamma]]$ can be identified with the ring $\Lambda := \mathbb{Z}_p[[T]]$ of formal power series in one variable – the non-canonical isomorphism is induced by mapping $\gamma$ to $T+1$ (for this and the following facts, we refer to [Was97, Chapter 13]). A *pseudo-isomorphism* of finitely generated $\Lambda$-modules is a $\Lambda$-module homomorphism having finite kernel and cokernel. It is well-known that every finitely generated $\Lambda$-module $A$ is pseudo-isomorphic to a so-called *elementary* $\Lambda$-module $E_A$ of the form

$$E_A = \Lambda^r \oplus \bigoplus_{i=1}^{s} \Lambda/(p^{n_i}) \oplus \bigoplus_{j=1}^{t} \Lambda/(g_j^{m_j}).$$

Here $r = \mathrm{rank}_\Lambda(A)$, $n_i, m_j \in \mathbb{N}$ for every $i$ and $j$ and the $g_j$ are irreducible and so-called *distinguished* polynomials (i.e. they are monic, and each coefficient besides the leading term is divisible by $p$). One defines the *Iwasawa invariants* of $A$ as $\mu(A) = \sum_{i=1}^s n_i$ and $\lambda(A) = \sum_{j=1}^t \deg(g_j^{m_j})$.

In the following, we call any finitely generated $\Lambda$-module an *Iwasawa module.* The most classical example is $A(K_\infty) = \varprojlim A_n$, where $K_\infty/K$ is a $\mathbb{Z}_p$-extension as above, $A_n = A(K_n)$, and where the projective limit is taken with respect to the norm maps. It turns out that $\mu(A(K_\infty)) = \mu(K_\infty/K)$ and $\lambda(A(K_\infty)) = \lambda(K_\infty/K)$ are exactly the parameters occurring in the formula (6).

### 2.5 The proof of Theorem 1.1

Now we have everything at hand which is needed for the

*Proof of Theorem 1.1.* It follows from class field theory that

$$\mathrm{rank}_{\mathbb{Z}_p}(\mathrm{Gal}(M_p(K_n)/K_n)) = r_2(K_n) + 1 + \delta(K_n).$$

Theorem 2.3 implies that

$$\mathrm{rank}_p(\mathrm{Gal}(M_p(K_n)/K_n)) = r_2(K_n) + l_n + \mathrm{rank}_p(A'_n)$$

for each $n \in \mathbb{N}$, where $l_n$ denotes the number of primes of $K_n$ above $p$. By hypothesis, the $l_n$ are bounded as $n \to \infty$. Moreover, $\mu(K_\infty/K) = 0$ holds if and only if $\operatorname{rank}_p(A_n)$ is bounded as $n \to \infty$ (see e.g. [Was97, Proposition 13.23]). Part (a) of the theorem follows because $A_n'$ is a quotient of $A_n$ for every $n \in \mathbb{N}$, and

$$\begin{aligned}\delta(K_n) &= \operatorname{rank}_{\mathbb{Z}_p}(\operatorname{Gal}(M_p(K_n)/K_n)) - r_2(K_n) - 1 \\ &\leq \operatorname{rank}_p(\operatorname{Gal}(M_p(K_n)/K_n)) - r_2(K_n) - 1.\end{aligned}$$

Using Gras' generalisation of Theorem 2.3 (cf. Theorem 2.7), we can derive the second part of the theorem as follows. Since $p \nmid [K' : K]$, the primes of $K_n$ above $p$ are not totally split in $K_n'/K_n$ for any $n \in \mathbb{N}$ (here we denote by $K_n' \subseteq K_\infty'$ the corresponding intermediate fields). Since $\mu(K_\infty'/K') = 0$, the formula (5) implies that

$$r_2(K_n) + \delta(K_n) + 1 \leq \operatorname{rank}_p(\operatorname{Gal}(M_p(K_n)/K_n)) = r_2(K_n) + 1 - \nu + O(1).$$

Therefore $\delta(K_n) = O(1)$. □

## 2.6 Applications and further remarks

First we show how to use Theorem 1.1 in order to bound the number of $\mathbb{Z}_p$-extensions of $K$ for which the weak Leopoldt conjecture might fail. We make use of the notation which has been introduced in the preceding subsections. Recall that the $p$-ranks of the groups $A(K_n)$ stabilise if $K_n$ runs over the intermediate fields of a $\mathbb{Z}_p$-extension $K_\infty/K$ with $\mu$-invariant zero. We will denote the stabilised rank by $\operatorname{rank}_p(A(K_\infty))$.

Corollary 2.9. *Let $K$ be a number field, and let $K' = K(\mu_p)$. We assume that no prime of $K$ dividing $p$ splits completely in $K'$. Now let $\mathbb{L}_\infty$ be a $\mathbb{Z}_p^2$-extension of $K$.*
*Suppose that there exists a $\mathbb{Z}_p$-extension $K_\infty \subseteq \mathbb{L}_\infty$ of $K$ such that*
*(a) no prime of $K$ above $p$ splits completely in $K_\infty$, and*
*(b) $\mu(K_\infty'/K') = 0$, where we let $K_\infty' = K' \cdot K_\infty$.*
*Then the weak Leopoldt conjecture holds for all but finitely many $\mathbb{Z}_p$-extensions of $K$ contained in $\mathbb{L}_\infty$, with at most*

$$\operatorname{rank}_p(A(K_\infty')) + |S_p(K_\infty')| - 1 \tag{7}$$

*exceptions. Here $S_p(K_\infty')$ denotes the number of primes of $K_\infty'$ above $p$.*

Note that all the ingredients of the upper bound (7) are finite in view of our hypotheses.

*Proof.* It follows from assumption (a) that no prime of $K'$ above $p$ splits completely in $K_\infty'$. We consider the $\mathbb{Z}_p^2$-extension $\mathbb{L}_\infty' = \mathbb{L}_\infty \cdot K'$ of $K'$. Then [BG81, Theorem 1] implies that the $\mu$-invariant vanishes for all but finitely many $\mathbb{Z}_p$-extensions of $K'$ inside $\mathbb{L}_\infty'$, with at most

$$C := \operatorname{rank}_p(A(K_\infty')) + |S_p(K_\infty')| - 1$$

possible exceptions. By construction, each such $\mathbb{Z}_p$-extension is the composite of $K'$ with a $\mathbb{Z}_p$-extension of $K$. Since no prime of $K$ splits completely in $K'$ by assumption, we may use Theorem 1.1(b) in order to derive that the weak Leopoldt conjecture holds for all but at most $C$ of the $\mathbb{Z}_p$-extensions of $K$ inside $\mathbb{L}_\infty$. □

We mention also a complementary

EXAMPLE 2.10. Suppose that the number field $K$ contains only one prime above $p$, and that a primitive $p$-th root of unity is contained in $K$ (i.e. $K$ fits into the setting of Theorem 1.1(a), rather than into the situation of the last corollary). If there exists a $\mathbb{Z}_p$-extension $K_\infty$ of $K$ such that $\mu(K_\infty/K) = 0$, and if $\mathbb{L}_\infty$ is a $\mathbb{Z}_p^2$-extension of $K$ which contains $K_\infty$, then the weak Leopoldt conjecture holds for all but at most

$$\mathrm{rank}_p(A(K_\infty)) + |S_p(K_\infty)| - 1$$

$\mathbb{Z}_p$-extensions of $K$ which are contained in $\mathbb{L}_\infty$ (note that $|S_p(K_\infty)|$ will be finite because the single prime of $K$ above $p$ ramifies in each $\mathbb{Z}_p$-extension of $K$).
The same argument applies to any $\mathbb{Z}_p^2$-extension $\mathbb{L}_\infty$ of $K$ in which the primes above $p$ are finitely split, provided that $\mu(K_\infty/K) = 0$ for some $K_\infty \subseteq \mathbb{L}_\infty$.

In the remainder of this section, we investigate a weak converse of Theorem 1.1.

REMARK 2.11. Let $T_n \subseteq \mathrm{Gal}(M_p(K_n)/K_n)$ denote the $\mathbb{Z}_p$-torsion submodule, $n \in \mathbb{N}$. The weak Leopoldt conjecture holds for the $\mathbb{Z}_p$-extension $K_\infty/K$ if and only if

$$\mathrm{rank}_{\mathbb{Z}_p}(\mathrm{Gal}(M_p(K_n)/K_n)) - r_2(K_n)$$

remains bounded as $n \to \infty$. If $\mu_p \subseteq K$, and if one knew that also

$$\mathrm{rank}_p(T_n) = \mathrm{rank}_p(\mathrm{Gal}(M_p(K_n)/K_n)) - \mathrm{rank}_{\mathbb{Z}_p}(\mathrm{Gal}(M_p(K_n)/K_n))$$

was bounded, then it would follow that $\mu(K_\infty/K) = 0$, and no prime of $K$ dividing $p$ was totally split in $K_\infty$. Indeed, it follows from Theorem 2.3 that in this case,

$$r_2(K_n) + l_n + \mathrm{rank}_p(A'_n) = \mathrm{rank}_p(\mathrm{Gal}(M_p(K_n)/K_n)) \leq r_2(K_n) + C$$

for some fixed constant $C$ and each $n \in \mathbb{N}$. In particular, $\mathrm{rank}_p(A'_n)$ and also the number $l_n$ of primes of $K_n$ dividing $p$ remain bounded as $n \to \infty$. Since

$$\mathrm{rank}_p(A_n) \leq \mathrm{rank}_p(A'_n) + l_n,$$

it follows that $\mu(K_\infty/K) = 0$.

On the other hand, the above argument makes it plausible that a completely split prime above $p$ might force either weak Leopoldt to be wrong for this specific $\mathbb{Z}_p$-extension, or $\mu(K_\infty/K) > 0$. In fact, Babaĭcev conjectured that $\mu(K_\infty/K) > 0$ can happen only if there exist sufficiently many primes of $K$ which ramify in $K/\mathbb{Q}$ and split completely in $K_\infty$ (see [Bab82, Conjecture on p. 11]). This conjecture is supported also by the recent work of Hajir and Maire (see [HM19, Section 4]).
In this paper, we cannot prove any result in the direction of Babaĭcev's conjecture. However, since the above results are related to this conjecture, we make the connections more explicit. First, we can derive from the arguments of Remark 2.11 the following

Corollary 2.12. *Suppose that $\mu_p \subseteq K$. Let $K_\infty/K$ be a $\mathbb{Z}_p$-extension such that at least one prime of $K$ above $p$ splits completely in $K_\infty$. If the weak Leopoldt conjecture holds for $K_\infty/K$, then* $\operatorname{rank}_p(T_n) \to \infty$, $n \to \infty$.

Now consider the projective limits

$$X(K_\infty) = \varprojlim X_n \quad \text{and} \quad T(K_\infty) = \varprojlim T_n,$$

where $X_n = \operatorname{Gal}(M_p(K_n)/K_n)$ and where $T_n \subseteq X_n$ denotes the $\mathbb{Z}_p$-torsion subgroup, as above. Here the projective limits are induced by the norm maps (cf. [Gra82, Subsection I.3]). In the next section, we will prove a result which implies the following lemma (see Lemma 3.9 below).

Lemma 2.13. *Let $K_\infty/K$ be a $\mathbb{Z}_p$-extension, and let $X(K_\infty)$ and $T(K_\infty)$ be defined as above. Then*

$$\mu(X(K_\infty)) = \mu(T(K_\infty)).$$

Moreover, we will show that if no prime of $K$ above $p$ is totally split in $K_\infty$, then $\mu(X(K_\infty)) > 0$ implies that also $\mu(A(K_\infty)) > 0$ (see Corollary 3.10 below). However, we can *not* conclude that $\mu(A(K_\infty)) > 0$ in the situation of Corollary 2.12, since in that setting there were totally split primes above $p$. This is why we cannot derive results about Babaĭcev's conjecture.

## 3 Admissible $p$-adic Lie extensions

The aim of this section is the proof of Theorem 1.2 from the Introduction: a generalisation of Theorem 1.1 to the context of more general, non-necessarily abelian, Galois extensions of number fields.
Let $K_\infty/K$ be a normal extension such that $G := \operatorname{Gal}(K_\infty/K)$ is a uniform $p$-adic Lie group of dimension $l$ (see [DdSMS99, Definition 4.1]). By a theorem of Lazard (see [DdSMS99, Theorem 8.34]), every compact $p$-adic Lie group contains an open normal uniform subgroup. Therefore the consideration of uniform $p$-extensions is fairly general.

Although a uniform pro-$p$-group $G$ in general will not be commutative, it turns out that the subsets $G_n := G^{p^n}$ of $p^n$-th powers of elements of $G$ yield canonical and normal subgroups of $G$ (see [DdSMS99, Lemma 2.4 and Theorem 3.6]). Moreover, a uniform $p$-group $G$ does not contain any $p$-torsion elements (see [DdSMS99, Theorem 4.5]). For every $n \in \mathbb{N}$, we denote by $K_n \subseteq K_\infty$ the subfield fixed by $G_n$. If $G$ has dimension $l$, then $[K_n : K] = p^{nl}$ for every $n \in \mathbb{N}$, and $K_\infty = \bigcup_n K_n$.

If $G$ is a uniform $p$-group, then the completed group ring $\mathbb{Z}_p[[G]]$ is a Noetherian domain (see [Neu88] and [Ven02, Theorem 3.26]). For any finitely generated $\mathbb{Z}_p[[G]]$-module $A$, we define (following Howson, see [How02, equation (33)]) the *$\mu$-invariant* of $A$ as

$$\mu(A) = \sum_{i \geq 0} \operatorname{rank}_{\mathbb{F}_p[[G]]}(p^i A[p^\infty]/p^{i+1}A[p^\infty]),$$

where $\mathbb{F}_p \cong \mathbb{Z}_p/p\mathbb{Z}_p$ denotes the field with $p$ elements, and where $A[p^\infty] \subseteq A$ denotes the submodule of $p$-power torsion elements. If $G \cong \mathbb{Z}_p$, then this definition reduces to the classical definition of the Iwasawa $\mu$-invariant from the previous section (see [KM22, Lemma 2.4]).

For any normal algebraic extension $M/L$ of fields we will write the $i$-th homology and cohomology groups of a $\mathrm{Gal}(M/L)$-module $A$ as $H_i(M/L, A)$ and $H^i(M/L, A)$.

In this article, an *admissible* $p$-extension $K_\infty/K$ will be defined as follows. (Note that the corresponding notion is used in different ways in the literature. For example, some authors assume that an admissible extension always contains the cyclotomic $\mathbb{Z}_p$-extension $K_\infty^c$ of $K$. We will work without this assumption.)

Definition 3.1. An *admissible* $p$-extension $K_\infty/K$ is a (non-trivial) uniform $p$-adic Lie extension such that only finitely many primes of $K$ ramify in $K_\infty$.

Let $K_\infty/K$ be an admissible $p$-extension. We fix a finite set $S$ of primes of $K$ which contains the set $S_p$ of primes above $p$, the infinite primes and the set of primes which ramify in $K_\infty/K$. For every $n \in \mathbb{N}$, we define the *Leopoldt defect* of $K_n$ (with respect to $S$) as

$$\delta_S(K_n) = \operatorname{rank}_{\mathbb{Z}_p}(H_2(K_S/K_n, \mathbb{Z}_p)).$$

Here $K_S$ denotes the maximal algebraic extension of $K$ (contained in some fixed algebraic closure $\overline{K}$ of $K$) which is unramified outside of $S$. Note that $K_S/K$ is not assumed to be abelian. It follows from [NSW08, Corollary 10.3.7] that this definition does not depend on the choice of the finite set $S$ (as long as $S$ contains $S_p$, the set $S_\infty$ of infinite primes and the ramified primes) and that

$$\operatorname{rank}_{\mathbb{Z}_p}(\mathrm{Gal}(M_S(K_n)/K_n)) = r_2(K_n) + 1 + \delta_S(K_n), \tag{8}$$

where $M_S(K_n) \subseteq K_S$ denotes the maximal subextension which is abelian over $K_n$. In particular, if $S = S_p$ and $K_\infty/K$ is a $\mathbb{Z}_p$-extension, then $\delta_S(K_n)$

coincides with the Leopoldt defect defined in the Introduction (we will always assume $K$ to be totally imaginary if $p = 2$).

Definition 3.2. We say that the *weak Leopoldt conjecture* (with respect to $S$) holds for $K_\infty/K$ if

$$H^2(K_S/K_\infty, \mathbb{Q}_p/\mathbb{Z}_p) = 0.$$

Now we prove the relation to the Leopoldt defects which will be used below. In particular, this will show that in the case of a $\mathbb{Z}_p$-extension $K_\infty/K$, the assertion $H^2(K_S/K_\infty, \mathbb{Q}_p/\mathbb{Z}_p) = 0$ is equivalent to the Leopoldt defects $\delta_S(K_n)$ being bounded as $n \to \infty$. Therefore the above formulation of the weak Leopoldt conjecture coincides with the one given in Section 2.4 in this special case.

Lemma 3.3. *Let $K_\infty/K$ be an admissible $p$-extension of dimension $l$, with intermediate fields $K_n \subseteq K_\infty$, $n \in \mathbb{N}$, and fix a finite set $S$ as above. Then the following assertions are equivalent.*
*(i)* $\delta_S(K_n) = O(p^{(l-1)n})$,
*(ii)* $H^2(K_S/K_\infty, \mathbb{Q}_p/\mathbb{Z}_p) = 0$,
*(iii)* $\delta_S(K_n) = O(1)$.

*Proof.* We generalise the proof of [NSW08, Theorem 10.3.22], which treats the equivalence of the two assertions (ii) and (iii) in the case of a $\mathbb{Z}_p$-extension $K_\infty/K$.
For every $n \in \mathbb{N}$, we let $\Gamma_n = \mathrm{Gal}(K_\infty/K_n)$ and $\mathcal{G}_n = \mathrm{Gal}(K_S/K_n)$. Then we have exact sequences

$$0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{G}_n \longrightarrow \Gamma_n \longrightarrow 0, \tag{9}$$

where $\mathcal{H} = \mathrm{Gal}(K_S/K_\infty)$. Recall that the weak Leopoldt conjecture holds for $K_\infty/K$ if and only if $H^2(\mathcal{H}, \mathbb{Q}_p/\mathbb{Z}_p) = 0$. Now we use the Hochschild–Serre spectral sequence in order to bound the cokernel of the canonical restriction map

$$\mathrm{res}_n \colon H^2(\mathcal{G}_n, \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow H^2(\mathcal{H}, \mathbb{Q}_p/\mathbb{Z}_p)^{\Gamma_n}.$$

We first note that the image of $\mathrm{res}_n$ coincides with the kernel of the transgression map

$$t_n \colon H^2(\mathcal{H}, \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow H^3(\Gamma_n, \mathbb{Q}_p/\mathbb{Z}_p),$$

since $\mathcal{H}$ acts trivially on $\mathbb{Q}_p/\mathbb{Z}_p$ (see [Rat79, Theorem 1.1]).
Now we use the following

Lemma 3.4. *For any $k \geq 1$, we have $v_p(|H^k(\Gamma_n, \mathbb{Q}_p/\mathbb{Z}_p)[p]|) = O(1)$, where the implicit constant may depend only on $k$, but not on $n$.*

*Proof.* For $k = 1$, this follows from [LM15, Lemma 3.2]; we describe a similar argument which works for arbitrary $k$.
First note that, as in the proof of [LM15, Lemma 3.2], we have a surjection

$$H^k(\Gamma_n, (\mathbb{Q}_p/\mathbb{Z}_p)[p]) \twoheadrightarrow H^k(\Gamma_n, \mathbb{Q}_p/\mathbb{Z}_p)[p]$$

for each $k$ and $n$, which results from the long exact sequence for the short exact sequence

$$0 \longrightarrow (\mathbb{Q}_p/\mathbb{Z}_p)[p] \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\cdot p} \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow 0.$$

Since $(\mathbb{Q}_p/\mathbb{Z}_p)[p] \cong \mathbb{Z}/p\mathbb{Z}$, it suffices to bound the cardinalities

$$h_k(\Gamma_n) := v_p(|H^k(\Gamma_n, \mathbb{Z}/p\mathbb{Z})|).$$

But $\Gamma_n \cong \mathbb{Z}_p^l$ acts trivially on $\mathbb{Z}/p\mathbb{Z}$ for each $n \in \mathbb{N}$. This implies that $h_k(\Gamma_n)$ does not depend on $n$. □

REMARK 3.5. More generally, suppose that $M$ denotes a discrete $\Gamma = \Gamma_0$-module which is cofinitely generated over $\mathbb{Z}_p$. Then the arguments from the proof of [LM15, Lemma 3.2] can be used in order to show that

$$v_p(|H^k(\Gamma_n, M)[p]|) = O(1).$$

Indeed, most of the proof of [LM15, Lemma 3.2] can be used without changes. For the first part on a *finite* $\Gamma_n$-module $M$, we suppose that $M$ is annihilated by some power $p^j$ of $p$. Fix $k \geq 1$, and let $2 \leq i \leq j$ be arbitrary. Then the exact sequence

$$0 \longrightarrow p^{i-1}M/p^iM \longrightarrow M/p^iM \longrightarrow M/p^{i-1}M \longrightarrow 0$$

yields an exact sequence

$$H^k(\Gamma_n, p^{i-1}M/p^i) \longrightarrow H^k(\Gamma_n, M/p^i) \longrightarrow H^k(\Gamma_n, M/p^{i-1}) \longrightarrow H^{k+1}(\Gamma_n, p^{i-1}M/p^i).$$

Now the first assertion of [LM15, Lemma 3.1] implies that

$$v_p(|H^k(\Gamma_n, p^{i-1}M/p^i|)) \leq \text{rank}_p(M) \cdot h_k(\Gamma_n)$$

for each $i$, where $h_k(\Gamma_n)$ is defined as in the proof of Lemma 3.4. Therefore we may derive from the above exact sequence, [KM22, Lemma 5.2] and [LM15, Lemma 3.2], via an induction on $k$, that

$$v_p(|H^k(\Gamma_n, M)|) \leq \text{rank}_p(M) \cdot (j \cdot h_k(\Gamma_n) + (j-1) \cdot h_{k+1}(\Gamma_n))$$

for each $k$ and $n$. The right hand side of this inequality does not depend on $n$. The case of an arbitrary cofinitely generated $\mathbb{Z}_p$-module $M$ can now be handled exactly as in the proof of [LM15, Lemma 3.2].

We return to the proof of Lemma 3.3. Since $\text{coker}(\text{res}_n) \cong \text{im}(t_n)$, we may conclude that $\text{rank}_{\mathbb{Z}_p}(\text{coker}(\text{res}_n)^\vee) = O(1)$.
Now suppose that $\delta_S(K_n) = \text{rank}_{\mathbb{Z}_p}(H_2(\mathcal{G}_n, \mathbb{Z}_p)) = O(p^{(l-1)n})$. Then

$$\text{rank}_{\mathbb{Z}_p}(H_2(\mathcal{H}, \mathbb{Z}_p)_{\Gamma_n}) = O(p^{(l-1)n})$$

by the above, because

$$H_2(\mathcal{G}_n, \mathbb{Z}_p)^\vee \cong H^2(\mathcal{G}_n, \mathbb{Q}_p/\mathbb{Z}_p)$$

and

$$(H_2(\mathcal{H}, \mathbb{Z}_p)_{\Gamma_n})^\vee \cong H^2(\mathcal{H}, \mathbb{Q}_p/\mathbb{Z}_p)^{\Gamma_n}$$

(see [NSW08, Theorem 2.6.9]). By a theorem of Harris (see [Har00, Theorem 1.10]), we have

$$\mathrm{rank}_{\mathbb{Z}_p}(H_2(\mathcal{H}, \mathbb{Z}_p)_{G_n}) = \mathrm{rank}_{\mathbb{Z}_p[[G]]}(H_2(\mathcal{H}, \mathbb{Z}_p)) \cdot p^{ln} + O(p^{(l-1)n}).$$

But $\mathrm{rank}_{\mathbb{Z}_p}(H_2(\mathcal{H}, \mathbb{Z}_p)_{G_n}) \leq \mathrm{rank}_{\mathbb{Z}_p}(H_2(\mathcal{H}, \mathbb{Z}_p)_{\Gamma_n})$. This implies that $H_2(\mathcal{H}, \mathbb{Z}_p)$ must be a torsion $\mathbb{Z}_p[[G]]$-module. On the other hand, $H_2(\mathcal{H}, \mathbb{Z}_p)$ is $\mathbb{Z}_p[[G]]$-torsion-free since it is a submodule of a projective (and thus free) $\mathbb{Z}_p[[G]]$-module, by [NSW08, Proposition 5.6.7 and Corollary 5.2.20]. Therefore $H_2(\mathcal{H}, \mathbb{Z}_p) = 0$.
We have shown that $(i) \Longrightarrow (ii)$. Now suppose that $(ii)$ holds, i.e. that

$$H_2(\mathcal{H}, \mathbb{Z}_p) = 0.$$

Then it follows from the exact sequence (9) that we have an injection

$$H_2(\mathcal{G}_n, \mathbb{Z}_p) \hookrightarrow H_2(\Gamma_n, \mathbb{Z}_p)$$

for each $n \in \mathbb{N}$. Since $H_2(\Gamma_n, \mathbb{Z}_p)^\vee \cong H^2(\Gamma_n, \mathbb{Q}_p/\mathbb{Z}_p)$, it follows from Lemma 3.4 that

$$\mathrm{rank}_p(H_2(\Gamma_n, \mathbb{Z}_p)) = O(1).$$

In view of the above embeddings, we may conclude that $\delta_S(K_n) = O(1)$.
Finally, it is obvious that statement $(iii)$ implies $(i)$. □

Remark 3.6. Using some of the arguments from the previous proof, one can also show the following fact.
Let $K_\infty/K$ be an admissible $p$-extension. If $L_\infty/K_\infty$ denotes either a finite extension or a $\mathbb{Z}_p^l$-extension such that $L_\infty/K$ is again admissible, then the validity of the weak Leopoldt conjecture for $K_\infty/K$ implies that it also holds for $L_\infty/K$. Indeed, as in the above proof the cokernel of the restriction map

$$\mathrm{res}\colon H^2(K_S/K_\infty, \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow H^2(K_S/L_\infty, \mathbb{Q}_p/\mathbb{Z}_p)$$

is isomorphic to a subgroup of $H^3(L_\infty/K_\infty, \mathbb{Q}_p/\mathbb{Z}_p)$. If $L_\infty/K_\infty$ is either finite or a $\mathbb{Z}_p^l$-extension, then the latter is cofinitely generated as a $\mathbb{Z}_p$-module, i.e. it is a torsion $\mathbb{Z}_p[[G]]$-module (where $G = \mathrm{Gal}(K_\infty/K)$ as usual). Now suppose that the weak Leopoldt conjecture holds for $K_\infty$. Then $H_2(K_S/L_\infty, \mathbb{Z}_p)$ is a torsion $\mathbb{Z}_p[[G]]$-module, and [NSW08, Proposition 5.6.7 and Corollary 5.2.20] imply that it is in fact trivial. This proves the above claim.

The following theorem restates Theorem 1.2 and makes its assertion more precise.

THEOREM 3.7. *Let $K_\infty/K$ be an admissible $p$-adic Lie extension with Galois group $G$. We assume that $K$ contains a primitive $p$-th root of unity. If $p = 2$, then we assume that $K$ is totally imaginary. Suppose that no prime of $S$ is completely split in $K_\infty/K$.*
*If $\mu(K_\infty/K) = 0$, then $H^2(K_S/K_\infty, \mathbb{Q}_p/\mathbb{Z}_p) = 0$, i.e. the weak Leopoldt conjecture (with respect to $S$) holds for $K_\infty/K$.*

*Proof.* Since no prime above $p$ is completely split in $K_\infty/K$, it follows from [OV03, Theorem 6.1] that the projective limit $A(K_\infty) = \varprojlim_n A(K_n)$ is a torsion $\mathbb{Z}_p[[G]]$-module. As $\mu(K_\infty/K) = 0$ by assumption, it follows from a result of Perbet (see [Per11, Theorem 2.1]) that

$$\mathrm{rank}_p(A(K_n)) = O(p^{(l-1)n}).$$

Moreover, $|S(K_n)| = O(p^{(l-1)n})$ by [Lim21, Lemma 4.1] (this is basically an application of [DdSMS99, Chapter 4, Exercise 14]), since the primes are not completely split in $K_\infty/K$ by assumption. As above, let $M_S(K_n)$ denote the maximal abelian $S$-ramified extension of $K_n$, and let $A_{\langle S\rangle}(K_n)$ denote the quotient of $A(K_n)$ by the subgroup which is generated by the primes in $S(K_n)$. Then Theorem 2.2 (cf. also the proof of Theorem 2.3) implies that

$$\begin{aligned}\mathrm{rank}_p(\mathrm{Gal}(M_S(K_n)/K_n)) &= r_2(K_n) + \mathrm{rank}_p(A_{\langle S\rangle}(K_n)) + |S(K_n)| \\ &= r_2(K_n) + O(p^{(l-1)n}).\end{aligned}$$

On the other hand, it follows from (8) that

$$\mathrm{rank}_p(\mathrm{Gal}(M_S(K_n)/K_n)) = r_2(K_n) + 1 + \delta_S(K_n) + \mathrm{rank}_p(T_S(K_n)),$$

where $T_S(K_n) \subseteq X_S(K_n) := \mathrm{Gal}(M_S(K_n)/K_n)$ denotes the $\mathbb{Z}_p$-torsion submodule, as in Section 2. It follows that both the Leopoldt defects and the $p$-ranks of $T_S(K_n)$ are $O(p^{(l-1)n})$. Now we apply Lemma 3.3 in order to conclude the proof of the theorem. □

REMARK 3.8. Using Theorem 2.7 instead of Theorem 2.3 in the above proof, one can prove a variant of this theorem which allows primes of $K$ to split completely in $K_\infty$ – as long as they do not split in $K(\mu_p)/K$. In this setting, if $\mu(K'_\infty/K') = 0$, then $H^2(K_S/K_\infty, \mathbb{Q}_p/\mathbb{Z}_p) = 0$.

We fix an admissible $p$-adic Lie extension $K_\infty/K$ and a finite set $S$ which contains the primes above $p$, the infinite primes and the primes which ramify in $K_\infty/K$. Now we compare the $\mu$-invariants of the three Iwasawa modules

$$X_S(K_\infty) = \varprojlim X_S(K_n), \quad T_S(K_\infty) = \varprojlim T_S(K_n)$$

and $A(K_\infty) = \varprojlim A(K_n)$.

LEMMA 3.9. $\mu(X_S(K_\infty)) = \mu(T_S(K_\infty))$.

*Proof.* Fix an integer $n$, write $X_n = X_S(K_n)$ and $T_n = T_S(K_n)$ for brevity, and let $\mathrm{pr}_n \colon X_S(K_\infty) \longrightarrow X_n$ and $\mathrm{pr}_n \colon T_S(K_\infty) \longrightarrow T_n$ be the canonical projections.
Let $x = (x_n)_n \in X$ be an element which is annihilated by a power $p^k$ of $p$. For every $n \in \mathbb{N}$, the element $x_n = \mathrm{pr}_n(x) \in X_n$ is then annihilated by $p^k$, i.e. $x_n \in T_n$. This shows that $X[p^\infty] \subseteq T$, i.e. $X[p^\infty] = T[p^\infty]$. The assertion of the lemma now follows from the definition of the $\mu$-invariants. □

Corollary 3.10. *Let $K_\infty/K$ be as in Theorem 3.7.*
*If $\mu(A(K_\infty)) = 0$, then $\mu_S(X(K_\infty)) = \mu_S(T(K_\infty)) = 0$.*

Note: in the special case where $K_\infty = K_\infty^c$ is the cyclotomic $\mathbb{Z}_p$-extension of $K$, this result is well-known (see [NSW08, Corollaries 11.3.6, 11.3.16 and 11.3.17]).

*Proof.* We have already seen in Lemma 3.9 that $\mu(X_S(K_\infty)) = \mu(T_S(K_\infty))$. It follows from (8) and from the proof of Theorem 3.7 that the vanishing of $\mu(A(K_\infty))$ implies that

$$\mathrm{rank}_{\mathbb{Z}_p}(X_S(K_n)) = r_2(K_n) + O(p^{(l-1)n}) = r_2(K)p^n + O(p^{(l-1)n}).$$

Therefore $\mathrm{rank}_{\mathbb{Z}_p[[G]]}(X_S(K_\infty)) = r_2(K)$ by [Har00, Theorem 1.10] and [Per11, Proposition 3.2]. Since moreover $\mathrm{rank}_p(X_S(K_n)) = r_2(K)p^n + O(p^{(l-1)n})$, it follows from [Per11, Theorem 2.1]) that $\mu(X_S(K_\infty)) = 0$. □

## 4 Galois representations

In this section, we assume that $p \neq 2$ for simplicity. Let $V$ be a finite dimensional $\mathbb{Q}_p$-vector space with a continuous action of the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$ of $K$ (for some fixed algebraic closure $\overline{K}$ of $K$), let $T \subseteq V$ be a Galois stable lattice, and write $A = V/T$. Then $A$ is isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^d$ for some $d \geq 1$, which we call the *dimension* of the representation $V$. Note that each element of the $\mathbb{Z}_p$-module $A$ is annihilated by some power of $p$.
We denote by $S_{ram}(A)$ the set of primes of $K$ where the representation $V$ is ramified (i.e. the set of primes $v$ of $K$ such that the inertia subgroup $I_v \subseteq \mathrm{Gal}(\overline{K}/K)$ acts non-trivially on $V$). We will throughout this section consider only representations $V$ such that the corresponding ramification set is *finite*.
Now let $K_\infty/K$ be an admissible $p$-adic Lie extension, $K_\infty = \bigcup_n K_n$. Let $S$ be a finite set of primes of $K$ which contains $S_p$, $S_{ram}(A)$ and the set of primes which ramify in $K_\infty$. Following Lim and Sujatha (see [LS18, Section 3]), we give the following

Definition 4.1. For each $n \in \mathbb{N}$, we define the ($p$-part of the) *fine Selmer group of $A$ over $K_n$* (with respect to $S$) as

$$\mathrm{Sel}_{A,0,S}(K_n) = \ker\left( H^1(K_S/K_n, A) \longrightarrow \prod_{v \in S(K_n)} H^1(K_{n,v}, A) \right),$$

where the local cohomology means the Galois cohomology of the absolute Galois group of the completion $K_{n,v}$ of $K_n$ at $v$. Here $S(K_n)$ is the set of all primes of $K_n$ dividing some prime in $S$, and $K_S$ denotes the maximal algebraic $S$-ramified pro-$p$-extension of $K$ (recall the convention $H^1(K_S/K_n, A) = H^1(\mathrm{Gal}(K_S/K_n), A)$ from Section 3).

Let $\mathrm{Sel}_{A,0,S}(K_\infty) = \varinjlim_n \mathrm{Sel}_{A,0,S}(K_n)$ (we take the injective limit with respect to the restriction maps). If $K_\infty$ contains the cyclotomic $\mathbb{Z}_p$-extension $K_\infty^c$ of $K$, then the definition of the fine Selmer group over $K_\infty$ does not depend on our choice of $S$ by a result of Sujatha and Witte (see [SW18, Section 3]). Finally, we consider the Pontryagin duals

$$Y_{A,S}^{(K_n)} = \mathrm{Sel}_{A,0,S}(K_n)^\vee := \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Sel}_{A,0,S}(K_n), \mathbb{Q}_p/\mathbb{Z}_p)$$

and

$$Y_{A,S}^{(K_\infty)} = \varprojlim_n Y_{A,S}^{(K_n)} = \mathrm{Sel}_{A,0,S}(K_\infty)^\vee.$$

It can be shown that $Y_{A,S}^{(K_\infty)}$ is a finitely generated $\mathbb{Z}_p[[G]]$-module, where we recall that $G = \mathrm{Gal}(K_\infty/K)$.

DEFINITION 4.2. Fix $S$ as above. We say that the *weak Leopoldt conjecture for $A$ over $K_\infty$* (with respect to $S$) holds if

$$H^2(K_S/K_\infty, A) = 0.$$

For example, suppose that $E$ is an elliptic curve defined over $K$, and consider the Tate module $T = \varprojlim_n E[p^n]$. We let $V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Then $A := V/T \cong E[p^\infty]$, and it is known that the weak Leopoldt conjecture holds for $A$ over each admissible $p$-adic Lie extension $K_\infty/K$ such that $X_S(K_\infty) = \mathrm{Gal}(M_S(K_\infty)/K_\infty)$ acts trivially on $E[p^\infty]$ (see [CS05, Lemma 2.4] or the proof of [OV02, Corollary 4.8]).
Moreover, it has been proved by Coates and Sujatha (see [CS05, Lemma 3.1]) and, more generally, by Lim (see [Lim17, Lemma 7.1]) that the weak Leopoldt conjecture (with respect to $S$) holds for $A$ over $K_\infty$ if and only if $Y_{A,S}^{(K_\infty)}$ is a $\mathbb{Z}_p[[G]]$-torsion module.
In fact, if $K_\infty = K_\infty^c$ is the cyclotomic $\mathbb{Z}_p$-extension of $K$, then one conjectures that even more is true: $Y_{A,S}^{(K_\infty^c)}$ should be finitely generated over $\mathbb{Z}_p$, i.e. $\mathbb{Z}_p[[\mathrm{Gal}(K_\infty^c/K)]]$-torsion with trivial $\mu$-invariant. This conjecture has been studied first by Coates and Sujatha in the special case of elliptic curves, see [CS05, Conjecture A]; it has been generalised to and studied in much more general settings (see, for example, [LS18], [JS11], [Jha12], [Ari14] or [Mat18]).
In this context, there exists a result similar to Theorem 1.1, which is due to Perrin-Riou:

THEOREM 4.3. *Let $V$ be a $p$-adic Galois representation as above, and write $A = V/T$. Let $K' = K(A[p], \mu_p)$, where $\mu_p$ denotes the group of $p$-th roots of* *unity. Let further* $K_\infty^c$ *denote the cyclotomic* $\mathbb{Z}_p$*-extension of* $K$*, and let*

$$(K'_\infty)^c = K' \cdot K_\infty^c.$$

*If* $\mu((K'_\infty)^c/K') = 0$*, then the weak Leopoldt conjecture holds for* $A$ *over* $K_\infty^c$ *(and also over* $(K'_\infty)^c$*).*

*Proof.* See [PR95, Proposition B.2]. □

We will now derive a generalisation of this result, in particular proving Theorem 1.3 from the Introduction. In the following, $S$ will always be a finite set of primes of $K$ which contains the primes of $K$ above $p$, the primes which ramify in $K_\infty$ and the primes where $V$ is ramified.

Theorem 4.4. *Let* $K_\infty/K$ *be an admissible* $p$*-adic Lie extension of dimension* $l$*, and let* $V$ *be a* $p$*-adic Galois representation with Galois invariant lattice* $T$*. Let* $A = V/T$ *and* $K' = K(A[p], \mu_p)$ *be as in Theorem 4.3, and write* $K'_\infty = K' \cdot K_\infty$*.*
*If* $\mu(K'_\infty/K') = 0$ *and no prime of* $S(K')$ *is totally split in* $K'_\infty$*, then the weak Leopoldt conjecture holds for* $A$ *over* $K_\infty$*, i.e.* $H^2(K_S/K_\infty, V/T) = 0$*.*

*Proof.* We mimic the proof of [PR95, Proposition B.2], first noting that it suffices to prove the weak Leopoldt conjecture for $V$ over $K'_\infty/K'$. Indeed, since $K'/K$ is a finite extension, the cohomology group $H^2(K'_\infty/K_\infty, A^{\mathrm{Gal}(K_S/K'_\infty)})$ is finite. The image of

$$\mathrm{inf}\colon H^2(K'_\infty/K_\infty, A^{\mathrm{Gal}(K_S/K'_\infty)}) \longrightarrow H^2(K_S/K_\infty, A)$$

is contained in the kernel of

$$\mathrm{res}\colon H^2(K_S/K_\infty, A) \longrightarrow H^2(K_S/K'_\infty, A)$$

(see [DHW12, Corollary 5.9]). Moreover, the quotient ker(res)/im(inf) is isomorphic to a subgroup of

$$H^1(K'_\infty/K_\infty, H^1(K_S/K'_\infty, A))$$

(see [DHW12, Theorem 6.7 and Lemma 6.8]). Therefore ker(res) has a finite exponent.
On the other hand, note that $H^3(K_S/K_\infty, A) \cong \varinjlim_n H^3(K_S/K_n, A) = 0$ because the cohomological dimension of $\mathrm{Gal}(K_S/K_n)$ is equal to 2 for each $n \in \mathbb{N}$ by [NSW08, Proposition 10.11.3]. Therefore the group

$$H^2(K_S/K_\infty, A) \cong H_2(K_S/K_\infty, \mathbb{Z}_p^d)^\vee$$

is divisible. Since ker(res) has finite exponent, it follows from the above that the nullity of the $H^2(K_S/K'_\infty, A)$ implies that also $H^2(K_S/K_\infty, A)$ is trivial.

By the above, we may without loss of generality assume that $K' = K$. We will show below that $H^2(K_S/K_\infty, p^{-1}T/T) = 0$ – this proves the theorem because the exact sequence

$$0 \longrightarrow p^{-1}T/T \longrightarrow V/T \xrightarrow{\cdot p} V/T \longrightarrow 0$$

induces a canonical surjection

$$H^2(K_S/K_\infty, p^{-1}T/T) \twoheadrightarrow H^2(K_S/K_\infty, A)[p]$$

(recall that $A = V/T$). Since $p^{-1}T/T = A[p]$ is defined over $K \subseteq K_\infty$ by assumption, it is sufficient to show that

$$H^2(K_S/K_\infty, p^{-1}\mathbb{Z}_p/\mathbb{Z}_p) = 0.$$

As in the proof of [PR95, Proposition B.2], the exact sequence

$$0 \longrightarrow p^{-1}\mathbb{Z}_p/\mathbb{Z}_p \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\cdot p} \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow 0$$

induces an exact sequence

$$X \longrightarrow H^2(K_S/K_\infty, p^{-1}\mathbb{Z}_p/\mathbb{Z}_p) \longrightarrow H^2(K_S/K_\infty, \mathbb{Q}_p/\mathbb{Z}_p)[p],$$

where we write $X = H^1(K_S/K_\infty, \mathbb{Q}_p/\mathbb{Z}_p)/pH^1(K_S/K_\infty, \mathbb{Q}_p/\mathbb{Z}_p)$ for the moment. But the last term in this sequence is 0 by Theorem 3.7. Moreover, the first term $X$ is (by [NSW08, Theorem 2.6.9]) the Pontryagin dual of

$$H_1(K_S/K_\infty, \mathbb{Z}_p)[p] = \mathrm{Gal}(K_S/K_\infty)^{\mathrm{ab}}[p],$$

where we denote by $\mathrm{Gal}(K_S/K_\infty)^{\mathrm{ab}} \cong X_S(K_\infty)$ the largest abelian quotient of $\mathrm{Gal}(K_S/K_\infty)$. Since $H^2(K_S/K_\infty, \mathbb{Q}_p/\mathbb{Z}_p) = 0$ by Theorem 3.7, it follows from [OV03, Theorem 4.7] and [SS12, Theorem 6.2] that $H_1(K_S/K_\infty, \mathbb{Z}_p)$ does not contain any non-trivial pseudo-null submodules. Since $\mu(K_\infty/K) = 0$ by assumption, it follows from Corollary 3.10 that also $\mu(X_S(K_\infty)) = 0$. Therefore [Ven02, Remark 3.33] implies that $X_S(K_\infty)[p]$ is pseudo-null. But we have seen above that $H_1(K_S/K_\infty, \mathbb{Z}_p)$ does not contain any non-trivial pseudo-null submodules. Summarising, we have shown that

$$H_1(K_S/K_\infty, \mathbb{Z}_p)[p] \cong X_S(K_\infty)[p] = \{0\}.$$

This concludes the proof of the theorem. ☐

Let us finally mention an important special case: let $A$ be a fixed abelian variety defined over $K$. Let $A[p^\infty]$ be the group of $p$-power torsion points (with coordinates in some fixed algebraic closure $\overline{K}$ of $K$). If $T = T_p(A) = \varprojlim_n A[p^n]$ denotes the $p$-adic Tate module of $A$ and $V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, then $V/T$ is isomorphic to $A[p^\infty]$. We denote by $S_{br}$ the finite set of primes of $K$ where $A$ has bad reduction; these are exactly the primes where the $\mathrm{Gal}(\overline{K}/K)$-representation $V$ is ramified.

Corollary 4.5. *Let $K_\infty$ be an admissible $p$-adic Lie extension, $p \neq 2$, and let $A$ be an abelian variety defined over $K$. We assume that the finite set $S$ contains $S_p \cup S_{br}$ and the primes which ramify in $K_\infty$. Let $K' = K(A[p], \mu_p)$ and $K'_\infty = K' \cdot K_\infty$.*
*If $\mu(K'_\infty/K') = 0$ and no prime of $S(K')$ is totally split in $K'_\infty$, then the weak Leopoldt conjecture holds for $A$ over $K_\infty$, i.e. $H^2(\mathrm{Gal}(K_S/K_\infty), A[p^\infty]) = 0$.*

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