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On the Spectral Representation of Holomorphic Functions on Some Domain

By

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§ 0. Introduction

The investigations of this paper will be concerned with the inverse Fourier-Laplace transform of functions holomorphic in some domain. On this subject the representation of entire functions has been obtained by Paley-Wiener-Schwartz and Eskin (cf.; [2], [7] p. 238). However as for the problem of functions holomorphic in bounded domains, it seems to the author that only the case of a tubular cone has been studied (cf.; [1], [6] Chapter VI Theorem 5, [7] Chapter V § 26).

Among these works, Schwartz' theorem characterizes a class of holomorphic functions whose spectral functions[†)] $f(x)$ possess the following properties:

$$(1) \qquad \operatorname{supp} f \subset [0, \infty]$$

$$(2) \qquad e^{-(x,\xi)} f(x) \in \mathscr{S}'(\boldsymbol{R}^1) \text{ for some } \xi \in \boldsymbol{R}^1.$$

Since a distribution in $\mathscr{S}'$ is represented in the form of a finite sum of derivatives of continuous functions of power increase, we can say that Schwartz' theorem essentially treats about a spectral function $f(x)$ in $\boldsymbol{R}^1$ which satisfies the following properties:

$$(1') \qquad f(x) = \left(\frac{d}{dx} + \xi\right)^k f_0(x), \ \operatorname{supp} f_0 \subset [0, \infty]$$

for some integer $k \geqq 0$ and constant $\xi \in \boldsymbol{R}^1$;

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†) We call the spectral function the inverse Fourier-Laplace transform of functions holomorphic in some domain.

$$e^{(x,\xi)} f_0(x) \tag{2'}$$

is a continuous function of power increase.

In this paper, we shall first prove that the Schwartz theorem can be generalized to the case where the spectral function $g(\lambda)$ satisfies the following conditions:

$$g(\lambda) = \left(\frac{\partial}{\partial\lambda} - y_0\right)^\rho g_0(\lambda), \text{ supp } g_0 \subset \bar{V}^*; \tag{3}$$

(4) $g_0(\lambda)$ is a continuous function and, for any $\varepsilon > 0$, there exists a constant $K_\varepsilon > 0$ satisfying the inequality

$$|g_0(\lambda)| \leqq K_\varepsilon \exp(\lambda, y_0 + \varepsilon e).$$

Here V is an affinely homogeneous convex cone in $\boldsymbol{R}^n$, $\bar{V}^*$ is the closed dual cone of V, ρ is a multi-index and $\left(\frac{\partial}{\partial\lambda}\right)^\rho$ is a Riemann-Liouville operator associated with the cone V (see [5], Proposition 1. 1, p. 202). Since the support of the fundamental solution of $\left(\frac{\partial}{\partial\lambda}\right)^\rho$ is contained in the closed dual cone $\bar{V}^*$ ([5], Theorem 2. 2 p. 216), $\left(\frac{\partial}{\partial\lambda}\right)^\rho$ turns out to be a hyperbolic differential operator. Secondly we shall consider the case of the Riemann-Liouville operator $\left(\frac{\partial}{\partial\lambda} - F\left(\frac{\partial}{\partial\zeta}, \frac{\partial}{\partial\zeta}\right)\right)^\rho$ ([5] Proposition 1. 1, p. 202) associated with the real Siegel domain

$$D = \{(\lambda, \xi) \in \boldsymbol{R}^{n+m} : \lambda - F(\xi, \xi) \in V\},$$

where $F(\ .\ ,\ .\ )$ is a homogeneous V-positive symmetric bilinear form on $\boldsymbol{R}^m$ with values in $\boldsymbol{R}^n$. Our result of this case (a main result) characterizes a class of holomorphic functions whose spectral function $f(\lambda, \zeta)$ satisfies the following properties:

$$f(\lambda, \zeta) = \left(\frac{\partial}{\partial\lambda} - F\left(\frac{\partial}{\partial\zeta}, \frac{\partial}{\partial\zeta}\right)\right)^\rho f_0(\lambda, \zeta), \tag{5}$$

where $f_0(\lambda, \zeta)$ is continuous in $(\lambda, \zeta) \in \boldsymbol{R}^n \times \boldsymbol{C}^m$, entire in $\zeta \in \boldsymbol{C}^m$ and is of support in $\bar{V}^* \times \boldsymbol{C}^m$;

(6) for any $\varepsilon > 0$ there exists a constant $K_\varepsilon > 0$ to satisfy the inequality

$$|f_0(\lambda, \zeta)| \leqq K_\varepsilon \lambda_*^{-q^*/2} \exp\Big\{-\Big(\frac{1}{4}-\varepsilon\Big) \operatorname{sp} F(\hat{\lambda}^{*-1}\xi, \hat{\lambda}^{*-1}\xi) + \Big(\frac{1}{4}+\varepsilon\Big) \operatorname{sp} F(\hat{\lambda}^{*-1}\eta, \hat{\lambda}^{*-1}\eta) + \varepsilon \operatorname{sp} \lambda\Big\}, \ \zeta = \xi + i\eta \in \boldsymbol{C}^m.$$

Inequalities (4) and (6) can estimate the fundamental solution of the operators $\left(\frac{\partial}{\partial\lambda} - y_0\right)^\rho$ and $\left(\frac{\partial}{\partial\lambda} - F\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\xi}\right)\right)^\sigma$, respectively, where the vectors $\rho = (\rho_1, \ldots, \rho_l)$ and $\sigma = (\sigma_1, \ldots, \sigma_l)$ satisfy the conditions $\rho_i, \sigma_i > -d_i > 0$ $(i = 1, \ldots, l)$ for a fixed vector $d = (d_1, \ldots, d_l)$. Also the support of $f_0(\lambda)$ [resp. $g_0(\lambda, \zeta)$] is contained in the closed dual cone $\bar{V}^*$ [resp. $\bar{V}^* \times \boldsymbol{C}^m$] which is equal to the support of the fundamental solution of the operator $\left(\frac{\partial}{\partial\lambda} - y_0\right)^\rho$ $\left[\text{resp. } \left(\frac{\partial}{\partial\lambda} - F\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\xi}\right)\right)^\rho\right]$. Therefore the spectral function $g(\lambda)$ [resp. $f(\lambda, \zeta)$] is considered to approximate the fundamental solution of the operator $\left(\frac{\partial}{\partial\lambda} - y_0\right)^\rho$ $\left[\text{resp. } \left(\frac{\partial}{\partial\lambda} - F\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\xi}\right)\right)^\rho\right]$. Thus we call holomorphic functions satisfying the conditions (3) and (4) [resp. (5) and (6)] "*V-hyperbolic*" [resp. "*D-parabolic*"].

Let us enumerate symbols and notations used in this paper (as for the details of these symbols and notations, see [4], [5]). Let V be an affinely homogeneous convex cone of rank l which does not contain straight lines in $\boldsymbol{R}^m$ and V^* be a dual cone of V with respect to the scalar product $(. , .)$. Since it is possible to transfer to V the structure of T-algebra ([8], Definition 3, p. 380), we fix a point e in V to satisfy the condition $(e, x) = \operatorname{sp} x$ ([5], p. 22, (2.13)) and define the dual vector x^* by sp $(x^*\lambda) = (x, \lambda)$. We denote by $\Gamma_{V^*}(\rho)$ the gamma function of the cone V^*([5], Definition 2.2, p. 22). The symbol x^ρ[resp. x_*^ρ] is meant by a compound power function of V [resp. V^*], where ρ is a multi-index ([5], p. 20 (2.3)). Put $\rho^* = (\rho_l, \ldots, \rho_1)$ for $\rho = (\rho_1, \ldots, \rho_l)$. Then we have $x = (x^*)_*^{\rho^*}$ ([5], p. 23 (2.26)). The vector ρ for which x^ρ becomes a polynomial are called V-integral ([5], Definition 3.2, p. 37). $F(,)$ denotes a homogeneous V-positive symmetric bilinear form on $\boldsymbol{R}^m$ with values in $\boldsymbol{R}^n$([4], p. 199, (1.1)$\sim$(1.4)) and also $F(,)$ is used in case where it is naturally extended on $\boldsymbol{C}^m$ with values in $\boldsymbol{C}^n$. The vectors $d = (d_i)$ ([5], Proposition 2.2, p. 20), $n = (n_i)$ ([5], p. 14, (1.16)), $q = (q_i)$

([4], p. 201, (1.16)), $\hat{\lambda}$ and $\tilde{\lambda}$ ([4], p. 212, (2.3), (2.4)) are proper symbols associated with the cone V and the bilinear form $F(\ ,\ )$.

§ 1. The Case of "V-hyperbolic" Holomorphic Functions

In this section we prove the following theorem which generalizes Schwartz' theorem [6] and characterizes the "*V-hyperbolicity*" of holomorphic functions.

Theorem 1. *Let $h(z)$ be a holomorphic function in the tubular cone*

$$T=\{z\in \boldsymbol{C}^n:\ \operatorname{Im} z\in V+y_0,\ y_0 \text{ is fixed}\}.$$

Suppose that, for any $\varepsilon>0$, there exist a constant $K_\varepsilon>0$ and a V-integral vector ρ_0 satisfying

$$|h(z)|\leqq K_\varepsilon|(-iz-y_0)^{\rho_0}| \tag{1.1}$$

in the closed domain

$$T_\varepsilon=\{z\in \boldsymbol{C}^n:\ \operatorname{Im} z-y_0-\varepsilon e\in \bar{V}\},$$

where $\bar{V}$ is the closure of V and e is the identity. Then the spectral function† *$g(\lambda)$ of $h(z)$ is represented for some V-integral vector ρ_1 as*

$$g(\lambda)=\left(\frac{\partial}{\partial\lambda}-y_0\right)^{\rho_1} g_0(\lambda), \tag{1.2}$$

where $g_0(\lambda)$ is a continuous function with support in the closed dual cone $\bar{V}^*$ such that, for any $\varepsilon>0$, there exists a constant $K'_\varepsilon>0$ satisfying

$$|g_0(\lambda)|\leqq K'_\varepsilon \exp(\lambda,\ y_0+e). \tag{1.3}$$

Conversely, if $g(\lambda)$ satisfies these conditions for a V-integral vector ρ_1 and a fixed vector $y_0\in \boldsymbol{R}^m$, then the Fourier-Laplace transform $h(z)$ of $g(\lambda)$ is holomorphic in the tubular domain T and satisfies inequality (1) *for a constant $K_\varepsilon>0$ and a V-integral vector ρ_0.*

Proof. We prepare an equality to use in the proof. Since we

†) As for the definition of the spectral function of the holomorphic function in the tubular cone, see [7], p. 230.

have

$$(1.4)\qquad \int_{V^*}\exp\,(i(z,\ \lambda))\lambda_*^{-\rho^*+d^*}d\lambda$$
$$=\int_{V^*}\exp\,(-\mathrm{sp}\,(-iz)^*\lambda)\lambda_*^{-\rho^*+d^*}d\lambda$$
$$=(-iz)^{\rho}\int_{V^*}\exp\,(-\mathrm{sp}\,\lambda)\lambda_*^{-\rho^*+d^*}d\lambda$$
$$=\Gamma_{V^*}(-\rho^*)(-iz)^{\rho},\ \mathrm{Im}\,z\in V,\ \mathrm{Re}\ \rho_i<-\frac{n_i}{2},$$
$$i=1,\ldots,\ l,$$

the Parseval-Plancherel formula gives

$$(1.5)\qquad \int_{R^n+iy}|(-iz-y_0)^{\rho}|^2dx$$
$$=(2\pi)^n|\Gamma_{V^*}(-\rho^*)|^{-2}\int_{V^*}\exp\{-2(y-y_0,\ \lambda)\}\lambda^{-2\mathrm{Re}\rho^*+2d^*}d\lambda$$
$$=(2\pi)^n|\Gamma_{V^*}(-\rho^*)|^{-2}\Gamma_{V^*}(-2\mathrm{Re}p^*+d^*)(2y-2y_0)^{2\mathrm{Re}\rho-d},$$

where $2\ \mathrm{Re}\ \rho_i<d_i-\frac{n_i}{2}$ $(i=1,\ldots,\ l)$ and $y\in\boldsymbol{R}^n$ is chosen so that $z=x+iy\in T$.

Now let $h(z)$ be a holomorphic function in T satisfying (1.1). Then, for a sufficiently large V-integral vector, the spectral function $g(\lambda)$ of $h(z)$ is expressed as follows:

$$(1.6)\qquad g(\lambda)=\left(\frac{\partial}{\partial\lambda}-y_0\right)^{\rho_1}\int_{R^n+iy}e^{-i(\lambda,z)}h(z)(-iz-y_0)^{-\rho_1}dx$$
$$(z=x+iy\in T).$$

We set

$$g_0(\lambda)=\int_{R^n+iy}e^{-i(\lambda,z)}h(z)(-iz-y_0)^{-\rho_1}dx.$$

Then in virtue of the Cauchy theorem the function $g_0(\lambda)$ is independent of the plane of integration in the tubular cone T and we obtain from (1.1) and (1.5)

$$(1.7)\qquad |g_0(\lambda)|\leqq K_\varepsilon\exp(\lambda,\ y)\int_{R^n+iy}|(-iz-y_0)^{\rho_0-\rho_1}|dx$$
$$=K_\varepsilon\exp(\lambda,\ y)(2\pi)^n|\Gamma_{V^*}(-(\rho_0^*-\rho_1^*)/2)|^{-2}$$
$$\times\Gamma_{V^*}(-\mathrm{Re}\ \rho_0^*+\mathrm{Re}\ \rho_1^*+d^*)(2y-2y_0)^{\mathrm{Re}\ \rho_0-\mathrm{Re}\ \rho_1-d}$$

if y is chosen so that $z=x+iy\in T_\varepsilon$. Setting $y=y_0+\varepsilon e$ in (1. 7), we obtain inequality (1. 3). If $\lambda\notin \bar{V}^*$, there exists $y_1\in V+y_0$ satisfying $(\lambda, y_1-y_0)\lneqq 0$. Therefore if we set $y=t(y_1-y_0)+y_0$, then $y\in V+y_0$ for any $t>0$. Letting $t\to+\infty$ in (1. 7), we see that the right side of (1. 7) vanishes. Hence supp $g\subset\bar{V}^*$. Since inequality (1. 1) and equality (1. 5) gives

$$(1.8)\qquad \begin{aligned}&|e^{-i(\lambda,z)}h(z)/(-iz-y_0)^{\rho_1}|\\ &\leqq K_\varepsilon e^{(\lambda,y)}|(-iz-y_0)^{\rho_0-\rho_1}|\in L^1,\\ &z\in T_\varepsilon,\end{aligned}$$

the continuity of $g_0(\lambda)$ follows from Lebegue's convergence theorem.

Conversely, suppose that a function $g(\lambda)$ is given for some V-integral vector ρ_1 by (1. 2) with $g_0(\lambda)$ satisfying inequality (1. 3). Let us set

$$h(z)=\int_{V^*}e^{i(\lambda,z)}g(\lambda)\,d\lambda.$$

Then we have

$$(1.9)\qquad \begin{aligned}h(z)&=\int_{V^*}e^{i(\lambda,z)}\Big(\frac{\partial}{\partial\lambda}-y_0\Big)^{\rho_1}g_0(\lambda)\,d\lambda\\ &=(-iz-y_0)^{\rho_1}\int_{V^*}e^{i(\lambda,z)}g_0(\lambda)\,d\lambda.\end{aligned}$$

Inequality (1. 3) yields

$$(1.10)\qquad \Big|\int_{V^*}e^{i(\lambda,z)}g_0(\lambda)\;d\lambda\Big|\leqq K'_\varepsilon\int_{V^*}e^{-(\lambda,y-y_0-\varepsilon e)}d\lambda,\qquad z=x+iy.$$

Since the right side of (1. 10) is convergent for $y-y_0-\varepsilon e\in V$, we have inequality (1. 1) in the closed domain T_ε. Let us show that $h(z)$ is a holomorphic function in the tubular cone T. From (1. 3), we see that for $y-y_0-\varepsilon e\in V$

$$(1.11)\qquad |e^{i(\lambda,z)}i\lambda_k g_0(\lambda)|\leqq K'_\varepsilon|\lambda_k|\exp(\lambda,\,-y+y_0+\varepsilon e)\in L^1,\qquad \lambda=(\lambda_1,\dots,\lambda_n).$$

Then in virtue of Lebesque's convergence theorem we have

$$\begin{aligned}(1.12)\qquad &\frac{\partial}{\partial x^k}\int_{V^*} e^{i(\lambda,z)}g_0(\lambda)\,d\lambda \\ &= -i\frac{\partial}{\partial y_k}\int_{V^*} e^{i(\lambda,z)}g_0(\lambda)\,d\lambda \\ &= \int_{V^*} e^{i(\lambda,z)} i\lambda_k g_0(\lambda)\,d\lambda,\end{aligned}$$

where $z=(z_1,\dots,z_n)\in \boldsymbol{C}^n$, $z_k=x_k+iy_k$ and $\lambda=(\lambda_1,\dots,\lambda_n)$ and also each integration of (1.12) is continuous in x_k and y_k $(k=1,\dots,n)$. Since the first equality of (1.12) is the Cauchy-Riemann equation, we conclude that $h(z)$ is holomorphic in the domain T.

Q. E. D.

§ 2. The Case of "D-parabolic" Holomorphic Functions

In order to characterize the "D-*parabolicity*" of holomorphic functions, we prepare a lemma.

Lemma. *Let $h(u)$ be an entire function in $\boldsymbol{C}^m$ which, for any $\varepsilon>0$, satisfies the following:*

$$\begin{aligned}(2.1)\qquad |h(u)| \leqq K_\varepsilon \exp\{&-(1-\varepsilon)(\lambda, F(u_1, u_1))^{p_1/2} \\ &+(1+\varepsilon)(\lambda, F(u_2, u_2))^{p_2/2}\},\end{aligned}$$

where $\lambda\in V^*$, $p_i\gneqq 1$ $(i=1, 2)$ and $u=u_1+iu_2\in\boldsymbol{C}^m$. Then the spectral function $f(\zeta)$ of $h(u)$:

$$f(\zeta)=\int_{R^m+iu_2} e^{-i\,\mathrm{sp}F(\zeta,u)}h(u)\,du_1$$

is entire and for any $\varepsilon>0$ satisfies the inequality

$$\begin{aligned}(2.2)\qquad |f(\zeta)| \leqq K'_\varepsilon \lambda_*^{-q^*/2}\exp\{&-(p_2'^{-1}p_2^{-p_2'/p_2}-\varepsilon) \\ &\times(\mathrm{sp}\,F(\hat\lambda^{*-1}\xi, \hat\lambda^{*-1}\xi))^{p_2'/2} \\ &+(p_1'^{-1}p_1^{-p_1'/p_1}+\varepsilon)(\mathrm{sp}\,F(\hat\lambda^{*-1}\eta, \hat\lambda^{*-1}\eta))^{p_1'/2}\},\end{aligned}$$

where $1/p_i+1/p_i'=1$ $(i=1,2)$ and $\zeta=\xi+i\eta\in\boldsymbol{C}^m$.

Conversely, if $f(\zeta)$ satisfies these conditions for certain numbers $p_i>1$ $(i=1,2)$, then its Fourier-Laplace transform $h(u)$:

$$h(u)=\int_{R^m+i\eta} e^{i\mathrm{sp}F(u,\zeta)} f(\zeta)\,d\xi$$

is entire and satisfies inequality (2.1) *for any* $\varepsilon>0$.

Proof. From inequality (2.1) we have

$$(2.3)\qquad |f(\zeta)|\leqq K_\varepsilon \exp\{\mathrm{sp}F(\xi,\ u_2)+(1+\varepsilon)(\lambda,\ F(u_2,\ u_2))^{p_2/2}\}$$
$$\times\int_{R^m}\exp\{\mathrm{sp}F(\eta,\ u_1)-(1-\varepsilon)(\lambda,\ F(u_1,\ u_1))^{p_1/2}\}\,du_1.$$

Put $u_1'=\tilde{\lambda}^* u_1$. Then the integral of the right side of (2.3) becomes

$$(2.4)\qquad \int_{R^m}\exp\{\mathrm{sp}F(\eta,\ u_1)-(1-\varepsilon)(\lambda,\ F(u_1,\ u_1))^{p_1/2}\}\,du_1$$
$$=\int_{R^m}\exp\{\mathrm{sp}F(\eta,\ u_1)-(1-\varepsilon)(\mathrm{sp}F(\tilde{\lambda}^* u_1,\ \tilde{\lambda}^* u_1))^{p_1/2}\}\,du_1$$
$$=\lambda_*^{-q^*/2}\int_{R^m}\exp\{\mathrm{sp}F(\tilde{\lambda}^{*-1}\eta,\ u_1')-(1-\varepsilon)(\mathrm{sp}F(u_1',\ u_1'))^{p_1/2}\}\,du_1'.$$

Putting

$$r=(\mathrm{sp}F(u_1',\ u_1'))^{1/2}\ \text{and}\ s=(\mathrm{sp}F(\hat{\lambda}^{*-1}\eta,\ \hat{\lambda}^{*-1}\eta))^{1/2}$$

and using the Schwarz inequality, we have

$$(2.5)\qquad \lambda_*^{-q^*/2}\int_{R^m}\exp\{\mathrm{sp}F(\hat{\lambda}^{*-1}\eta,\ u_1')-(1-\varepsilon)(\mathrm{sp}F(u_1',\ u_1'))^{p_1/2}\}\,du_1'$$
$$\leqq C_\varepsilon\lambda_*^{-q^*/2}\sup_{0\leqq r<\infty}\{\exp(r\cdot s-(1-\varepsilon)r^{p_1})\}.$$

In virtue of the Young inequality we can estimate the right side of (2.5) as follows:

$$(2.6)\qquad C_\varepsilon\lambda_*^{-q^*/2}\sup_{0\leqq r<\infty}\{\exp(r\cdot s-(1-\varepsilon)r^{p_1})\}$$
$$\leqq C_\varepsilon'\lambda_*^{-q^*/2}\exp\{(p_1'^{-1}p_1^{-p_1'/p_1}+\varepsilon)s^{p_1'}\}.$$

Summing up, we obtain

$$(2.7)\qquad \int_{R^m}\exp\{\mathrm{sp}F(\eta,\ u_1)-(1-\varepsilon)(\lambda,\ F(u_1,\ u_1))^{p_1/2}\}\,du_1$$
$$\leqq C_\varepsilon'\lambda_*^{-q^*/2}\exp\{(p_1'^{-1}p_1^{-p_1'/p_1}+\varepsilon)(\mathrm{sp}F(\hat{\lambda}^{*-1}\eta,\ \hat{\lambda}^{*-1}\eta))^{p_1'/2}\}.$$

Since $h(u)$ is an entire function, the spectral function $f(\zeta)$ is independent of $u_2=\text{Im}\ u$. So we put

$$u_2=-p_2^{-p_2'/p_2}(\text{sp}F(\hat{\lambda}^{*-1}\xi,\ \hat{\lambda}^{*-1}\xi))^{1/2(p_2'/p_2-1)}\tilde{\lambda}^{*-1}\hat{\lambda}^{*-1}\xi$$

in inequality (2.3). Then by setting $t=(\text{sp}F(\hat{\lambda}^{*-1}\xi,\ \hat{\lambda}^{*-1}\xi))^{1/2}$, we have

$$\begin{aligned}(2.8)\qquad &\exp\{\text{sp}F(\xi,\ u_2)+(1+\varepsilon)(\lambda,\ F(u_2,\ u_2))^{p_2/2}\}\\ &=\exp\{-p_2^{-p_2'/p_2}t^{(p_2'/p_2+1)}+(1+\varepsilon)p_2^{-p_2'}t^{p_2'}\}\\ &=\exp\{((1+\varepsilon)p_2^{-p_2'}-p_2^{-p_2'/p_2})t^{p_2'}\}\\ &=\exp\{p_2^{-p_2'/p_2}(p_2^{-p_2'+p_2'/p_2}-1+\varepsilon)t^{p_2'}\}\\ &=\exp\{p_2^{-p_2'/p_2}(p_2^{-1}-1+\varepsilon)t^{p_2'}\}\\ &=\exp\{-(p_2'^{-1}p_2^{-p_2'/p_2}-\varepsilon)(\text{sp}F(\hat{\lambda}^{*-1}\xi,\ \hat{\lambda}^{*-1}\xi))^{p_2'/2}\}.\end{aligned}$$

Inequalities (2.3) and (2.7) and equality (2.8) prove (2.2). The analyticity of $f(\zeta)$ can be proved by a way similar to the proof of Theorem 1.

Conversely, suppose that $f(\zeta)$ is an entire function satisfying inequality (2.2) for any $\varepsilon>0$. Then the Fourier-Laplace transform $h(u)$ of $f(\zeta)$:

$$h(u)=\int_{R^m+i\eta}e^{i\text{sp}F(u,\zeta)}f(\zeta)\,d\xi\qquad \zeta=\xi+i\eta,$$

can be estimated as follows:

$$\begin{aligned}(2.9)\qquad |h(u)|\leqq K_\varepsilon &\exp\{-\text{sp}F(u_1,\ \eta)\\ &+(p_1'^{-1}p_1^{-p_1'/p_1}+\varepsilon)(\text{sp}F(\hat{\lambda}^{*-1}\eta,\hat{\lambda}^{*-1}\eta))^{p_1'/2}\}\\ &\times\int_{R^m}\lambda_*^{-q^*/2}\exp\{-\text{sp}F(u_2,\ \xi)-(p_2'^{-1}p_2^{-p_2'/p_2}-\varepsilon)\\ &\times(\text{sp}F(\hat{\lambda}^{*-1}\xi,\ \hat{\lambda}^{*-1}\xi))^{p_2'/2}\}\,d\xi.\end{aligned}$$

Putting $\xi'=\hat{\lambda}^{*-1}\xi$ in the integral of (2.9), we have

$$\begin{aligned}(2.10)\qquad &\int_{R^m}\lambda_*^{-q^*/2}\exp\{-\text{sp}F(u_2,\ \xi)\\ &\qquad -(p_2'^{-1}p_2^{-p_2'/p_2}-\varepsilon)(\text{sp}F(\hat{\lambda}^{*-1}\xi,\ \hat{\lambda}^{*-1}\xi))^{p_2'/2}\}\,d\xi\\ &=\int_{R^m}\exp\{-\text{sp}F(\tilde{\lambda}^{*}u_2,\ \xi')\\ &\qquad -(p_2'^{-1}p_2^{-p_2'/p_2}-\varepsilon)(\text{sp}F(\xi',\ \xi'))^{p_2'/2}\}\,d\xi'.\end{aligned}$$

Further putting

$$a=(\mathrm{sp}F(\tilde{\lambda}^* u_2,\ \tilde{\lambda}^* u_2))^{1/2} \text{ and } b=(\mathrm{sp}F(\xi',\ \xi'))^{1/2},$$

and using the Schwarz inequality, we obtain

$$(2.11)\qquad \int_{R^m} \exp\{-\mathrm{sp}F(\tilde{\lambda}^* u_2,\ \xi') - (p_2'^{-1} p_2^{-p_2'/p_2} - \varepsilon)(\mathrm{sp}F(\xi',\ \xi'))^{p_2'/2}\}\, d\xi' \leqq C_\varepsilon \sup_{0\leqq b<\infty} \{\exp(a\cdot b - (p_2'^{-1} p_2^{-p_2'/p_2} - \varepsilon) b^{p_2'})\}.$$

The Young inequality yields the inequality

$$(2.12)\qquad C_\varepsilon \sup_{0\leqq b<\infty} \{\exp(a\cdot b - (p_2'^{-1} p_2^{-p_2'/p_2} - \varepsilon) b^{p_2'})\} \leqq C_\varepsilon' \exp\{(1+\varepsilon) a^{p_2}\}$$

Consequently, we obtain

$$(2.13)\qquad \int_{R^m} \lambda_*^{-q^*/2} \exp\{-\mathrm{sp}F(u_2,\ \xi) - (p_2'^{-1} p_2^{-p_2'/p_2} - \varepsilon)(\mathrm{sp}F(\hat{\lambda}^{*-1}\xi,\ \hat{\lambda}^{*-1}\xi))^{p_2'/2}\}\, d\xi \leqq C_\varepsilon' \exp\{(1+\varepsilon)(\lambda,\ F(u_2,\ u_2))^{p_2/2}\}.$$

On the other hand the function $h(u)$ is independent of the plane of integration $\zeta=\xi+i\eta$ (where $\eta=$constant). Therefore by setting

$$\eta = p_1(\lambda,\ F(u_1,\ u_1))^{1/2(p_1/p_1'-1)} \hat{\lambda}^* \tilde{\lambda}^* u_1 \text{ and } c=(\lambda,\ F(u_1,\ u_1))^{1/2},$$

we have

$$\begin{aligned}(2.14)\qquad &\exp\{-\mathrm{sp}F(u_1,\ \eta) + (p_1'^{-1} p_1^{-p_1'/p_1} + \varepsilon) \times (\mathrm{sp}F(\hat{\lambda}^{*-1}\eta,\ \hat{\lambda}^{*-1}\eta))^{p_1'}\} \\ &= \exp\{-p_1 c^{p_1/p_1'+1} + (p_1'^{-1} p_1^{-p_1'/p_1 + p_1'} + \varepsilon) c^{p_1}\} \\ &= \exp\{(p_1'^{-1} p_1 - p_1 + \varepsilon) c^{p_1}\} \\ &= \exp\{-(1-\varepsilon)(\lambda,\ F(u_1,\ u_1))^{p_1/2}\}.\end{aligned}$$

Thus inequality (2.1) follows from (2.9), (2.13) and (2.14). The analyticity of $f(\zeta)$ can be proved by a way similar to the proof of Theorem 1. Q. E. D.

Now we can state the main result concerning the "*D-parabolicity*" of holomorphic functions.

Theorem 2. *Let $h(z, u)$ be a holomorphic function in the domain*

$$D=\{(z, u)\in \boldsymbol{C}^{n+m}: \operatorname{Im} z+F(u_1, u_1)-F(u_2, u_2)\in V, \\ u=u_1+iu_2\in \boldsymbol{C}^m\}.$$

Suppose, for any $\varepsilon>0$, there exist a constant $C_\varepsilon>0$, a V-integral vector ρ_0 and integers $k_i>0$ $(i=1, 2)$ such that

$$(2.15)\qquad |h(z, u)|\leqq C_\varepsilon\{(1+\mathrm{sp}F(u_1, u_1)+\mathrm{sp}F(u_2, u_2))^{k_1} \\ \times|(-iz+F(u, u))^{\rho_0}|+(1+\mathrm{sp}F(u_1, u_1)+\mathrm{sp}F(u_2, u_2))^{k_2}\}$$

in the closed domain

$$D_\varepsilon=\{(z, u)\in \boldsymbol{C}^{n+m}: \operatorname{Im} z+F(u_1, u_1)-F(u_2, u_2)-\varepsilon e\in \bar{V}\}.$$

Then, for some V-integral vector ρ_1, the spectral function

$$f(\lambda, \zeta)=\int_{\boldsymbol{R}^m+iu_2}\int_{\boldsymbol{R}^m+iy} e^{-i(\lambda, z)-i\mathrm{sp}F(\zeta, u)}h(z, u)\,dx\,du_1 \\ z=x+iy\in \boldsymbol{C}^n,\ u=u_1+iu_2\in \boldsymbol{C}^m,$$

is represented as

$$(2.16)\qquad f(\lambda, \zeta)=\left(\frac{\partial}{\partial\lambda}-F\left(\frac{\partial}{\partial\zeta}, \frac{\partial}{\partial\zeta}\right)\right)^{\rho_1} f_0(\lambda, \zeta),$$

where the function $f_0(\lambda, \zeta)$ is continuous in $(\lambda, \zeta)\in \boldsymbol{R}^n\times \boldsymbol{C}^m$, entire in $\zeta\in \boldsymbol{C}^m$ and is of support in $\bar{V}^*\times \boldsymbol{C}^m$, and satisfies in $V^*\times \boldsymbol{C}^m$ the inequality

$$(2.17)\qquad |f_0(\lambda, \zeta)|\leqq K_\varepsilon \lambda_*^{-q^*/2}\exp\left\{-\left(\frac{1}{4}-\varepsilon\right)\mathrm{sp}\,F(\hat{\lambda}^{*-1}\xi, \hat{\lambda}^{*-1}\xi) \\ +\left(\frac{1}{4}+\varepsilon\right)\mathrm{sp}\,F(\hat{\lambda}^{*-1}\eta, \hat{\lambda}^{*-1}\eta)+\varepsilon\,\mathrm{sp}\,\lambda\right\}, \\ \zeta=\xi+i\eta\in \boldsymbol{C}^m.$$

Conversely, if a function $f(\lambda, \zeta)$ satisfies these conditions (2.16) *and* (2.17) *for any $\varepsilon>0$ and some V-integral ρ_1, the Fourier-Laplace transform*

$$h(z, u)=\int_{\boldsymbol{R}^m+i\eta}\int_{V^*} e^{i(\lambda, z)+i\,\mathrm{sp}\,F(u, \zeta)}f(\lambda, \zeta)\,d\lambda\, d\xi$$

is holomorphic in the domain D and satisfies the inequality

$$(2.18)\qquad |h(z, u)| \leqq C'_{\varepsilon} |(-iz+F(u, u))^{\rho_1}|$$

in the closed domain

$$D'_{\varepsilon} = \{(z, u) \in \boldsymbol{C}^{n+m} : \operatorname{Im} z + (1-\varepsilon)F(u_1, u_1) - (1+\varepsilon)F(u_2, u_2) - \varepsilon e \in \bar{V}\} \subset D \quad \textit{for any } \varepsilon > 0.$$

Proof. We denote by $g(\lambda, u)$ the spectral function of $h(z, u)$ with respect to z:

$$g(\lambda, u) = \int_{R^n+iy} e^{-i(\lambda, z)} h(z, u) dx \qquad (z = x+iy)$$

where, for any fixed $u \in \boldsymbol{C}^m$, y is chosen so that $(z, u) \in D$. Then for any V-integral vector ρ_1,

$$(2.19)\quad g(\lambda, u) = \left(\frac{\partial}{\partial \lambda} + F(u, u)\right)^{\rho_1} \times \int_{R^n+iy} e^{-i(\lambda, z)} h(z, u)(-iz+F(u, u))^{-\rho_1} dx \quad (z, u) \in D.$$

Put

$$(2.20)\qquad g_0(\lambda, u) = \int_{R^n+iy} e^{-i(\lambda, z)} h(z, u)(-iz+F(u, u))^{-\rho_1} dx.$$

Then by Theorem 1 we see that $g_0(\lambda, u)$ is continuous in $\lambda \in \boldsymbol{R}^n$ and the support of $g_0(\lambda, u)$ is contained in $\bar{V}^* \times \boldsymbol{C}^m$. Since, for any fixed $u \in \boldsymbol{C}^m$, the plane of integration of (2.20) is chosen so that $(z, u) \in D$, $g_0(\lambda, u)$ is an entire function of $u \in \boldsymbol{C}^m$. Since the integrand $h(z, u)(-iz+F(u, u))^{-\rho_1}$ of (2.20) is holomorphic in the domain D, we see that $g_0(\lambda, u)$ is independent of y. Therefore it follows from (1.5), (2.15) and (2.20) that for a sufficiently large V-integral vector ρ_1 and $y = -F(u_1, u_1) + F(u_2, u_2) + \varepsilon e$ $(\varepsilon > 0)$,

$$\begin{aligned}(2.21)\qquad |g_0(\lambda, u)| &\leqq C'_{\varepsilon}(1+\operatorname{sp} F(u_1, u_1) + \operatorname{sp} F(u_2, u_2))^{k_3} \\ &\quad \times \exp(\lambda, -F(u_1, u_1) + F(u_2, u_2) + \varepsilon e) \\ &\leqq C''_{\varepsilon} \exp\{-(1-\varepsilon)(\lambda, F(u_1, u_1)) \\ &\quad + (1+\varepsilon)(\lambda, F(u_2, u_2)) + \varepsilon \operatorname{sp} \lambda\}.\end{aligned}$$

Now if we apply Lemma with $p_1 = p_2 = 2$ to the function

$$f_0(\lambda,\ \zeta)=\int_{R^m+iu_2} e^{-i\ \mathrm{sp}\ F(\zeta,u)}g_0(\lambda,\ u)\,du_1 \tag{2.22}$$

then we see from (2. 2) that $f_0(\lambda,\ \zeta)$ satisfies inequality (2. 17) and is an entire function of $\zeta\in \boldsymbol{C}^n$. The continuity of the function $f_0(\lambda,\ \zeta)$ with respect to $\lambda\in\boldsymbol{R}^n$ is obvious. We have from (2. 19)

$$\begin{aligned} f(\lambda,\ \zeta) &=\int_{R^n+iu_2} e^{-i\ \mathrm{sp}\ F(\zeta,u)}g(\lambda,\ u)\,du_1 \\ &=\left(\frac{\partial}{\partial\lambda}-F\left(\frac{\partial}{\partial\zeta},\ \frac{\partial}{\partial\zeta}\right)\right)^{\rho_1} f_0(\lambda,\ \zeta). \end{aligned}$$

Conversely, if the function $f(\lambda,\ \zeta)$ satisfies conditions (2. 16) and (2. 17) for any $\varepsilon>0$ and some V-integral vector ρ_1, it follows from Lemma with $p_1=p_2=2$ that $g_0(\lambda,\ u)$ in (2. 22) is continuous in $\lambda\in\boldsymbol{R}^n$ and entire in $\zeta\in\boldsymbol{C}^m$, and satisfies the inequality

$$\begin{aligned} |g_0(\lambda,\ u)|\leqq K'_\varepsilon \exp\{&-(1-\varepsilon)(\lambda,\ F(u_1,\ u_1)) \\ &+(1+\varepsilon)(\lambda,\ F(u_2,\ u_2))+\varepsilon\,\mathrm{sp}\,\lambda\}. \end{aligned} \tag{2.23}$$

Then, by use of Theorem 1, the Fourier-Laplace transform $h(z,\ u)$ of $f(\lambda,\ \zeta)$ is holomorphic in the domain D and satisfies (2. 18) in the domain D'_ε. Q. E. D.

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