Cusps and q-Expansion Principles for Modular Curves at Infinite Level

Ben Heuer

Received: January 24, 2020
Revised: August 4, 2021

Communicated by Don Blasius

Abstract. We develop an analytic theory of cusps for Scholze's p-adic modular curves at infinite level in terms of perfectoid parameter spaces for Tate curves. As an application, we describe a canonical tilting isomorphism between an anticanonical overconvergent neighbourhood of the ordinary locus of the modular curve at level $\Gamma_1(p^\infty)$ and the analogous locus of an infinite level perfected Igusa variety. We also prove various q-expansion principles for functions on modular curves at infinite level, namely that the properties of extending to the cusps, vanishing, coming from finite level, and being bounded, can all be detected on q-expansions.

2020 Mathematics Subject Classification: 11G18, 14G35, 14G22
Keywords and Phrases: modular curve, infinite level, cusps, perfectoid

1 Introduction

1.1 Cusps of modular curves at infinite level

Let p be a prime and let K be a perfectoid field extension of $\mathbb{Q}_p$. Throughout we shall assume that K contains all p^n-th unit roots for all $n \in \mathbb{N}$. Let $N \geq 3$ be coprime to p and let $\mathcal{X}^*$ be the compactified modular curve over K of some rigidifying tame level Γ^p such that $\Gamma(N) \subseteq \Gamma^p \subseteq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. Here we consider $\mathcal{X}^*$ as an analytic adic space.

The first goal of this article is to give a detailed analytic description of the geometry at the cusps in the inverse system of modular curves with varying level structures at p, as well as for the modular curves at infinite level introduced by Scholze in [Sch15]. In doing so, we aim to complement results on the boundary of infinite level Siegel varieties for GSp_{2g} for $g \geq 2$ from [Sch15, §3.2.5], proved there using machinery like Hartog's extension principle and a perfectoid version of Riemann's Hebbarkeitssatz: Due to the assumption that the codimension of the boundary is ≥ 2, these tools do not apply in the elliptic case. Instead, here one can get a much more explicit description with more elementary means.

The way we study the boundary in the elliptic case is in terms of adic analytic parameter spaces for Tate curves. Fix a cusp x of $\mathcal{X}^*$, this is in general a point defined over a finite field extension $K \subseteq L_x \subseteq K[\zeta_N]$ depending on x, and it corresponds to a Γ^p-level structure on the Tate curve $\mathrm{T}(q^{e_x})$ over $\mathcal{O}_{L_x}(\!(q)\!)$ for some $1 \leq e_x \leq N$. The analytic Tate curve parameter space in this situation is simply the adic open unit disc $\mathcal{D}_x$ over L_x, and there is a canonical open immersion $\mathcal{D}_x \hookrightarrow \mathcal{X}^*$ that sends the origin to x.

In order to state our main result, let us recall the tower of anticanonical moduli spaces from [Sch15, §3]: Away from the cusps, the modular curve $\mathcal{X}^*$ is the moduli space of elliptic curves E together with a Γ^p-level structure. Let $\mathcal{X}^*_{\Gamma_0(p)} \to \mathcal{X}^*$ be the finite flat cover that relatively represents (away from the cusps) the data of a cyclic subgroup scheme of rank p of $E[p]$. By the theory of the canonical subgroup, for small enough $\epsilon > 0$, the ϵ-overconvergent neighbourhood of the ordinary locus $\mathcal{X}^*_{\Gamma_0(p)}(\epsilon) \subseteq \mathcal{X}^*_{\Gamma_0(p)}$ decomposes into two disjoint open components: the canonical locus $\mathcal{X}^*_{\Gamma_0(p)}(\epsilon)_c$ and the anticanonical locus $\mathcal{X}^*_{\Gamma_0(p)}(\epsilon)_a$. In order to understand the cusps of the perfectoid modular curve at infinite level, we first study the tower of compactified modular curves of higher level at p relatively over $\mathcal{X}^*_{\Gamma_0(p)}(\epsilon)_a$. Specifically, for any $n \in \mathbb{N}$, the pullback of $\mathcal{X}^*_{\Gamma_0(p)}(\epsilon)_a \subseteq \mathcal{X}^*_{\Gamma_0(p)}$ defines a tower

$$\mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_1(p^n)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a \to \mathcal{X}^*(\epsilon)$$

of anticanonical loci. Here the rightmost map is finite flat and totally ramified at the cusps, whereas $\mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$ is finite étale and Galois with group

$$\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z}) := \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})\}.$$

The cusps of these moduli spaces of higher finite level can be described using analogous parameter spaces for Tate curves: As we shall discuss in detail in §2, it is essentially an adic analytic version of the classical calculus of cusps of Katz–Mazur [KM85] that for any cusp x of $\mathcal{X}^*$, there are Cartesian diagrams of adic spaces of topologically finite type over K

$$\begin{array}{ccccccc} \underline{\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})} \times \mathcal{D}_{n,x} & \to & \underline{(\mathbb{Z}/p^n\mathbb{Z})^\times} \times \mathcal{D}_{n,x} & \longrightarrow & \mathcal{D}_{n,x} & \longrightarrow & \mathcal{D}_x \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_1(p^n)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a & \to & \mathcal{X}^*(\epsilon), \end{array} \quad (1)$$

where the top left map is $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \mapsto d$ on the first factor and the identity in the second, and $\mathcal{D}_{n,x}$ is the open unit disc in the variable q^{1/p^n} over L_x, that is,

$$\mathcal{D}_{n,x} = \text{ open subspace of } \mathrm{Spa}(L_x\langle q^{1/p^n}\rangle, \mathcal{O}_{L_x}\langle q^{1/p^n}\rangle) \text{ where } \lim_{m\to\infty} |q|^m = 0.$$

In the limit $n \to \infty$, these open discs become parameter spaces for Tate curves with infinite Γ_0-level structure at p, given by perfectoid open unit discs

$$\mathcal{D}_{\infty,x} = \text{ open subspace of } \operatorname{Spa}(L_x\langle q^{1/p^\infty}\rangle, \mathcal{O}_{L_x}\langle q^{1/p^\infty}\rangle) \text{ where } \lim_{m\to\infty} |q|^m = 0.$$

We then get the following description of the cusps at infinite level: Let

$$\Gamma_0(p^\infty) := \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\} \subseteq \mathrm{GL}_2(\mathbb{Z}_p)$$

and let $\underline{\Gamma_0(p^\infty)}$ be the associated profinite perfectoid space.

Theorem 1.1. *Let x be any cusp of $\mathcal{X}^*$.*

1. *There is a Cartesian diagram of perfectoid spaces over K*

$$\begin{array}{ccccccc} \underline{\Gamma_0(p^\infty)} \times \mathcal{D}_{\infty,x} & \longrightarrow & \underline{\mathbb{Z}_p^\times} \times \mathcal{D}_{\infty,x} & \longrightarrow & \mathcal{D}_{\infty,x} & \longrightarrow & \mathcal{D}_x \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*(\epsilon). \end{array}$$

2. *Define a right action of $\underline{\mathbb{Z}_p}$ on $\underline{\mathrm{GL}_2(\mathbb{Z}_p)} \times \mathcal{D}_{\infty,x}$ by letting $h \in \underline{\mathbb{Z}_p}$ act via*

$$(\gamma, q^{1/p^n}) \cdot h := (\gamma \left(\begin{smallmatrix} 1 & 0 \\ h & 1 \end{smallmatrix}\right), q^{1/p^n} \zeta_{p^n}^{h/e_x}).$$

   Let $\mathrm{GL}_2(\mathbb{Z}_p)$ act on the left via the first factor. Then there is a Cartesian diagram

$$\begin{array}{ccc} (\underline{\mathrm{GL}_2(\mathbb{Z}_p)} \times \mathcal{D}_{\infty,x})/\underline{\mathbb{Z}_p} & \longrightarrow & \mathcal{D}_x \\ \downarrow & & \downarrow \\ \mathcal{X}^*_{\Gamma(p^\infty)} & \longrightarrow & \mathcal{X}^* \end{array}$$

   for which the left map is a $\mathrm{GL}_2(\mathbb{Z}_p)$-equivariant open immersion.

3. *The Hodge–Tate period map $\pi_{\mathrm{HT}} \colon \mathcal{X}^*_{\Gamma(p^\infty)} \to \mathbb{P}^1$ restricts on the open subspace described in 2. to the map*

$$(\underline{\mathrm{GL}_2(\mathbb{Z}_p)} \times \mathcal{D}_{\infty,x})/\underline{\mathbb{Z}_p} \to \underline{\mathbb{P}^1(\mathbb{Z}_p)}, \quad \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), q\right) \mapsto (b:d).$$

We refer to Theorem 3.17 and Theorem 3.22 for slightly more precise statements. In other words, the part of the boundary of $\mathcal{X}^*_{\Gamma(p^\infty)}$ lying over x is given by the closed profinite subspace

$$\underline{\mathrm{GL}_2(\mathbb{Z}_p)/\left(\begin{smallmatrix} 1 & 0 \\ \mathbb{Z}_p & 1 \end{smallmatrix}\right)} \hookrightarrow \mathcal{X}^*_{\Gamma(p^\infty)}$$

and has an open neighbourhood given by a Tate curve parameter space $(\underline{\mathrm{GL}_2(\mathbb{Z}_p)} \times \mathcal{D}_{\infty,x})/\underline{\mathbb{Z}_p}$.

We will prove part 1 of the theorem step by step in §3.1-3.3. We then deduce 2 from 1 via $\mathrm{GL}_2(\mathbb{Z}_p)$-translations. For this one needs to describe the action of the larger group

$$\Gamma_0(p) := \{\left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right) | c \in p\mathbb{Z}_p\} \subseteq \mathrm{GL}_2(\mathbb{Z}_p)$$

on the Tate curve parameter space $\underline{\Gamma_0(p^\infty)} \times \mathcal{D}_{\infty,x} \hookrightarrow \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$, which also takes into account isomorphisms of Tate curves of the form $q \mapsto \zeta^h_{p^n} q$ for $h \in \mathbb{Z}_p$. We will do so in §3.4.

Remark 1.2. We note that Pilloni–Stroh [PS16] in their construction of perfectoid tilde-limits of toroidal compactifications of Siegel moduli varieties also describe the boundary of $\mathcal{X}^*_{\Gamma(p^\infty)}$: More precisely, the second part of Theorem 1.1 also follows from [PS16, Proposition A.14]. While their proposition is much more general, the above description is arguably slightly more explicit. We will also identify the canonical and anticanonical subspaces.

On the way, we discuss in §2 some aspects of modular curves as analytic adic space that are not visible in the rigid setting as treated in [Con06]. For example, in the adic setting there is also a larger quasi-compact analytic Tate curve parameter space $\overline{\mathcal{D}}_x \to \mathcal{X}^*$ given by

$$\overline{\mathcal{D}}_x := \mathrm{Spa}(\mathcal{O}_{L_x}[\![q]\!][\tfrac{1}{p}], \mathcal{O}_{L_x}[\![q]\!])$$

where $\mathcal{O}_{L_x}[\![q]\!]$ is endowed with the p-adic topology (rather than the (p,q)-adic one). This gives rise at infinite level to a map $\overline{\mathcal{D}}_{\infty,x} \to \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$ where

$$\overline{\mathcal{D}}_{\infty,x} := \mathrm{Spa}(\mathcal{O}_{L_x}[\![q^{1/p^\infty}]\!]_p[\tfrac{1}{p}], \mathcal{O}_{L_x}[\![q^{1/p^\infty}]\!]_p) \sim \varprojlim_{q\mapsto q^p} \overline{\mathcal{D}}_x.$$

Here $\mathcal{O}_{L_x}[\![q^{1/p^\infty}]\!]_p$ is the p-adic completion of $\varinjlim_n \mathcal{O}_{L_x}[\![q^{1/p^n}]\!]$.
While these are no longer open immersions, they are sometimes useful, for example because in contrast to $\mathcal{D}_x$, the spaces $\overline{\mathcal{D}}_x$ for all x together with the good reduction locus $\mathcal{X}_{\mathrm{gd}}$ cover the adic space $\mathcal{X}^*$. More precisely, we have $\overline{\mathcal{D}} \setminus \mathcal{D} = \mathrm{Spa}(\mathcal{O}_{L_x}\langle\!\langle q\rangle\!\rangle[\tfrac{1}{p}], \mathcal{O}_{L_x}\langle\!\langle q\rangle\!\rangle^+)$ where $\mathcal{O}_{L_x}\langle\!\langle q\rangle\!\rangle$ is the p-adic completion of $\mathcal{O}_{L_x}[\![q]\!][q^{-1}]$, a local ring, and $\mathcal{O}_L\langle\!\langle q\rangle\!\rangle^+$ is a certain valuation subring of rank 2. The image of this rank 2 point in $\mathcal{X}^*$ is a closed point that is neither contained in $\mathcal{X}_{\mathrm{gd}}$ nor in $\mathcal{D}_x$. We discuss this situation in more detail in §2.4.

1.2 First applications

The main reason why we are interested in an explicit description of the cusps of $\mathcal{X}^*_{\Gamma(p^\infty)}$ is that many constructions involving modular curves require a separate treatment of the boundary: For example, one often defines morphisms from modular curves using moduli interpretations, and then in a second step extends to the compactifications. At finite level this is usually done by normalisation. While at infinite level, one still has moduli interpretations (see Lemma 3.16), it is less clear how to carry out the "normalisation" step. This can in practice be done using the explicit description in Theorem 1.1: For the sake of illustrating the argument, one could apply this to see that π_{HT} extends to the cusps using Theorem 1.1.3 (but here the extension is clear as π_{HT} is a priori known to be locally constant near the cusps). A more serious instance of such an extension argument appears in §1.4.

A second reason why the boundary sometimes requires additional attention is that the morphism $q : \mathcal{X}^*_{\Gamma(p^\infty)} \to \mathcal{X}^*$ is a pro-finite-étale Galois torsor, except for ramification at the boundary, as is evident from (1). For example, Chojecki–Hansen–Johansson [CHJ17] have recently given a beautiful construction of sheaves of p-adic modular forms by writing down an explicit 1-cocycle that defines a descent datum for the morphism $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_c \to \mathcal{X}^*(\epsilon)$ using that this is a $\Gamma_0(p)$-torsor away from the boundary. However, over the cusps the morphism is no longer Galois (and indeed the boundary is ignored in [CHJ17]). As explained in detail in [BHW, §3], Theorem 1.1 makes it easy to see how to extend to the cusps: The point where Theorem 1.1 is used in this context is the statement that

$$(q_*\mathcal{O}_{\mathcal{X}^*_{\Gamma(p^\infty)}})^{\mathrm{GL}_2(\mathbb{Z}_p)} = \mathcal{O}_{\mathcal{X}^*},$$

i.e. $\mathrm{GL}_2(\mathbb{Z}_p)$-equivariant functions on $\mathcal{X}^*_{\Gamma(p^\infty)}$ come from $\mathcal{X}^*$. Away from the cusps, where q is a $\mathrm{GL}_2(\mathbb{Z}_p)$-torsor, this is a formal consequence of the fact that the (completed) structure sheaf on the pro-étale site of $\mathcal{X}^*$ is a sheaf. That the identity extends over the boundary can then be checked on Tate curve parameter spaces, where it is immediate from Theorem 1.1.2.

Besides of such extension arguments, the q-expansions of functions on $\mathcal{X}^*_{\Gamma(p^\infty)}$ that one obtains by restriction to Tate curve parameter spaces are of independent interest in the context of p-adic modular forms, see [Heu19, §2.3 and Theorem 5.3.3] and also §1.5 below.

1.3 Cusps of perfectoid modular curves in characteristic p

There are natural analogues of the above descriptions for modular curves in characteristic p, which we treat in §4: We shall work over the perfectoid field $K^\flat$ and choose $\varpi^\flat \in K^\flat$ with $|\varpi^\flat| = |p|$. Let $\mathcal{X}'^*$ be the compactified modular curve of level Γ^p over $K^\flat$, considered as an analytic adic space. Then, again, for every cusp x of $\mathcal{X}'^*$, there is a Tate curve parameter space $\mathcal{D}'_x \hookrightarrow \mathcal{X}'^*$ where now $\mathcal{D}'_x$ is the adic open unit disc over a finite extension $L^\flat_x \subseteq K^\flat[\zeta_N]$ (the notation as the tilt of an extension of K depending on x will be justified later). Recall that over any overconvergent neighbourhood $\mathcal{X}^*(\epsilon)$ of the locus of ordinary reduction, there is a finite étale Igusa curve $\mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon) \to \mathcal{X}'^*(\epsilon)$. In the limit over the relative Frobenius morphism, and over $n \to \infty$, these give rise to a pro-étale morphism of perfectoid spaces over $\mathcal{X}'^*(\epsilon)$

$$\mathcal{X}'^*_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}} \to \mathcal{X}'^*(\epsilon)^{\mathrm{perf}}.$$

Let now $\mathcal{D}'_{\infty,x}$ denote the perfectoid open unit disc over $L^\flat_x$, which is the perfection of $\mathcal{D}'_x$. Then we have the following analogue of Theorem 1.1 in characteristic p:

Theorem 1.3. *For every cusp x of $\mathcal{X}'^*$, there are Cartesian diagrams*

$$\begin{array}{ccccc} \underline{\mathbb{Z}_p^\times} \times \mathcal{D}'_{\infty,x} & \longrightarrow & \mathcal{D}'_{\infty,x} & \longrightarrow & \mathcal{D}'_x \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{X}'^*_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}} & \longrightarrow & \mathcal{X}'^*(\epsilon)^{\mathrm{perf}} & \longrightarrow & \mathcal{X}'^*(\epsilon). \end{array}$$

We then compare this diagram to the situation in characteristic 0 via tilting:

1.4 A tilting isomorphism at level $\Gamma_1(p^\infty)$

In [Sch15, Corollary 3.2.19], Scholze identifies the tilt of $\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$ by proving that there is a canonical isomorphism

$$\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)^\flat_a \xrightarrow{\sim} \mathcal{X}'^*(\epsilon)^{\mathrm{perf}} \tag{2}$$

of perfectoid spaces over $K^\flat$. For Siegel spaces parametrising abelian varieties of dimension $g \geq 2$, he then extends this tilting isomorphism to level $\Gamma_1(p^\infty)$. Using Tate curve parameter spaces, we complement this result in §5 by treating the case $g = 1$ of elliptic curves. Moreover, we work out the precise situation at the cusps: It follows from (2) that the cusps of $\mathcal{X}^*$ (which can be identified with those of $\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$) and the cusps of $\mathcal{X}'^*$ (which can be identified with those of $\mathcal{X}'^*(\epsilon)^{\mathrm{perf}}$) can be identified via tilting, and the same is true for the field extensions L_x and $L_x^\flat$. Using these identifications, we have:

Theorem 1.4. *1. There is a canonical isomorphism*

$$\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)^\flat_a \xrightarrow{\sim} \mathcal{X}'^*_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}$$

which is $\mathbb{Z}_p^\times$-equivariant and makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)^\flat_a & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)^\flat_a \\ \wr\Vert & & \wr\Vert \\ \mathcal{X}'^*_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}} & \longrightarrow & \mathcal{X}'^*(\epsilon)^{\mathrm{perf}}. \end{array}$$

2. For any cusp x of $\mathcal{X}^*$ with corresponding cusp $x^\flat$ of $\mathcal{X}'^*$, the diagram

$$\begin{array}{ccc} \underline{\mathbb{Z}_p^\times} \times \mathcal{D}^\flat_{\infty,x} & \hookrightarrow & \mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)^\flat_a \\ \wr\Vert & & \wr\Vert \\ \underline{\mathbb{Z}_p^\times} \times \mathcal{D}'_{\infty,x^\flat} & \hookrightarrow & \mathcal{X}'^*_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}} \end{array}$$

commutes, where the left map is given by the canonical identification $\mathcal{D}^\flat_{\infty,x} \cong \mathcal{D}'_{\infty,x^\flat}$.

We are interested in this result because of an application to p-adic modular forms: In [Heu19], we use Theorem 1.4 to give a perfectoid perspective on the t-adic modular forms at the boundary of weight space as introduced by Andreatta–Iovita–Pilloni [AIP18].

1.5 q-expansion principles

As a second application, Tate curve parameter spaces give a way to talk about q-expansions of functions on modular curves: For any $f \in \mathcal{O}(\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a)$, we may define the q-expansion of f at a cusp $x \in \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ to be its restriction to the associated locally closed subspace $\mathcal{D}_{\infty,x} \hookrightarrow \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$, i.e. the image under

$$\mathcal{O}(\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a) \to \mathcal{O}(\mathcal{D}_{\infty,x})$$

which will automatically lie in $\mathcal{O}_K[\![q^{1/p^\infty}]\!][\frac{1}{p}] \subseteq \mathcal{O}(\mathcal{D}_{\infty,x})$. One has analogous definitions for other infinite level modular curves, or open subspaces thereof, as well as for profinite families of cusps. Such q-expansions can be useful when working with modular curves at infinite level, as they often allow one to extend constructions which are a priori defined only away from the cusps (or even just on the good reduction locus), for instance maps defined using moduli functors, to the compactifications. For example:

Lemma 1.5. *A function f on the uncompactified modular curve $\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a$ extends to a function on $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ if and only if at every cusp x of $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$, the q-expansion of f is already contained in $\mathcal{O}_{L_x}[\![q^{1/p^\infty}]\!][\frac{1}{p}] \subseteq \mathcal{O}_{L_x}\langle\!\langle q^{1/p^\infty}\rangle\!\rangle[\frac{1}{p}]$. Any such extension is unique.*

This is what we mean when we say that q-expansions can be used as a replacement for Hartog's extension principle in the elliptic case of $g = 1$.
As the final goal of this article, we show in §6 that in the spirit of Katz' q-expansion principle for modular forms [Kat73, Theorem 1.6.1], one can use Tate curve parameter spaces to prove various q-expansion principles for functions on perfectoid modular curves.

Proposition 1.6 (q-expansion principle I). *Let $\mathcal{C}$ be a collection of cusps of $\mathcal{X}^*$ such that each connected component of $\mathcal{X}^*$ contains at least one $x \in \mathcal{C}$. Then restriction of functions to the Tate curve parameter spaces associated to $\mathcal{C}$ defines injective maps*

$$\begin{aligned}
\mathcal{O}(\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a) &\hookrightarrow \textstyle\prod_{x\in\mathcal{C}} \mathcal{O}_{L_x}[\![q^{1/p^\infty}]\!][\frac{1}{p}],\\
\mathcal{O}(\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a) &\hookrightarrow \textstyle\prod_{x\in\mathcal{C}} \mathrm{Map}_{\mathrm{cts}}(\mathbb{Z}_p^\times, \mathcal{O}_{L_x}[\![q^{1/p^\infty}]\!])[\frac{1}{p}],\\
\mathcal{O}(\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a) &\hookrightarrow \textstyle\prod_{x\in\mathcal{C}} \mathrm{Map}_{\mathrm{cts}}(\Gamma_0(p^\infty), \mathcal{O}_{L_x}[\![q^{1/p^\infty}]\!])[\frac{1}{p}],\\
\mathcal{O}(\mathcal{X}'^*(\epsilon)^{\mathrm{perf}}) &\hookrightarrow \textstyle\prod_{x\in\mathcal{C}} \mathcal{O}_{L_x^\flat}[\![q^{1/p^\infty}]\!][\frac{1}{\varpi^\flat}],\\
\mathcal{O}(\mathcal{X}'^*_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}) &\hookrightarrow \textstyle\prod_{x\in\mathcal{C}} \mathrm{Map}_{\mathrm{cts}}(\mathbb{Z}_p^\times, \mathcal{O}_{L_x^\flat}[\![q^{1/p^\infty}]\!])[\frac{1}{\varpi^\flat}].
\end{aligned}$$

As mentioned in §1.2, p-adic modular forms can be described as functions on $\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$ satisfying a certain equivariance property, see [CHJ17], [BHW]. Proposition 1.6 may thus be seen as a generalisation of its classical version from modular forms to more general functions.
Similarly, one can detect on q-expansions whether a function comes from some finite level:

Proposition 1.7 (q-expansion principle II). *Let $f \in \mathcal{O}(\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a)$. Then for any $n \in \mathbb{Z}_{\geq 0}$, the following are equivalent:*

1. *f comes via pullback from $\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$, i.e. f is already contained in $\mathcal{O}(\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a) \subseteq \mathcal{O}(\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a)$.*
2. *The q-expansion of f at every cusp x is already contained in $\mathcal{O}_{L_x}[\![q^{1/p^n}]\!][\frac{1}{p}] \subseteq \mathcal{O}_{L_x}[\![q^{1/p^\infty}]\!][\frac{1}{p}]$.*
3. *On each connected component of $\mathcal{X}^*$, there is at least one cusp x at which the q-expansion of f is already contained in $\mathcal{O}_{L_x}[\![q^{1/p^n}]\!][\frac{1}{p}] \subseteq \mathcal{O}_{L_x}[\![q^{1/p^\infty}]\!][\frac{1}{p}]$.*

The analogous statements for $\mathcal{X}'^*(\epsilon)^{\mathrm{perf}}$ are also true.

For $\epsilon = 0$, i.e. on the ordinary locus, one can see on q-expansions whether a function is integral, i.e. bounded by 1. We note that this fails for $\epsilon > 0$.

Proposition 1.8 (q-expansion principle III). *For $f \in \mathcal{O}(\mathcal{X}^*_{\Gamma_0(p^\infty)}(0)_a)$, the following are equivalent:*

1. *f is integral, i.e. it is already contained in $\mathcal{O}^+(\mathcal{X}^*_{\Gamma_0(p^\infty)}(0)_a)$.*
2. *The q-expansion of f at every cusp x is already contained in $\mathcal{O}_{L_x}[\![q^{1/p^\infty}]\!]$.*
3. *On each connected component of $\mathcal{X}^*$, there is at least one cusp x at which the q-expansion of f is already contained in $\mathcal{O}_{L_x}[\![q^{1/p^\infty}]\!]$.*

The analogous statements for $\mathcal{X}^*_{\Gamma_1(p^\infty)}(0)_a$, $\mathcal{X}^*_{\Gamma(p^\infty)}(0)_a$, $\mathcal{X}'^*(0)^{\mathrm{perf}}$ and $\mathcal{X}'^*_{\mathrm{Ig}(p^\infty)}(0)^{\mathrm{perf}}$ are also true when we replace $\mathcal{O}_{L_x}[\![q^{1/p^\infty}]\!]$ by the respective algebra from Proposition 1.6.

Finally, there is also a version of q-expansions for the good reduction locus, which uses instead the quasi-compact Tate curve parameter space $\overline{\mathcal{D}}$ mentioned before §1.2 (see also Definition 2.15).

Proposition 1.9 (q-expansion principle IV). *Let $\mathcal{C}$ be a collection of cusps of $\mathcal{X}^*$ such that each connected component contains at least one $x \in \mathcal{C}$. Then a function on the good reduction locus $\mathcal{X}_{\mathrm{gd}}(\epsilon)$ extends to all of $\mathcal{X}^*(\epsilon)$ if and only if its q-expansion with respect to*

$$\overline{\mathcal{D}}(|q| \geq 1) \to \mathcal{X}_{\mathrm{gd}}(\epsilon)$$

at each $x \in \mathcal{C}$ is already contained in $\mathcal{O}_{L_x}[\![q]\!][\frac{1}{p}] \subseteq \mathcal{O}_{L_x}\langle\!\langle q \rangle\!\rangle[\frac{1}{p}]$. In this case, the extension is unique. The analogous statements for $\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$, $\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a$, $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$, $\mathcal{X}'^*(\epsilon)$, $\mathcal{X}'^*(\epsilon)^{\mathrm{perf}}$ and $\mathcal{X}'^*_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}$ are also true.

Acknowledgements

We would like to thank Johannes Anschütz, Kevin Buzzard, Ana Caraiani, Vincent Pilloni and Peter Scholze for helpful discussions. This work was carried out while the author was supported by the Engineering and Physical Sciences Research Council [EP/L015234/1], the EPSRC Centre for Doctoral Training in Geometry and Number Theory (The London School of Geometry and Number Theory), University College London. The author was funded by Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy – EXC-2047/1 – 390685813, as well as by DFG Project-ID 444845124 – TRR 326. The author was moreover supported by DFG via the Leibniz-Preis of Peter Scholze.

2 Adic analytic theory of cusps at finite level

2.1 Recollections on the classical theory of cusps

We start by recalling from [KM85, §8.6-8.11] some basic facts about cusps of modular curves that we will use freely throughout, and fix some notation and conventions:
Let $N \geq 3$ and let R be an excellent Noetherian regular $\mathbb{Z}[1/N]$-algebra, for instance $R = \mathbb{Z}[1/N]$. Let X_R be the modular curve of some rigidifying level $\Gamma(N) \subseteq \Gamma \subseteq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ over R. By definition, the compactification X^*_R of X_R is then the normalisation in $\mathbb{P}^1_R$ of the finite flat j-invariant $j\colon X_R \to \mathbb{A}^1_R$. The divisor of cusps is defined as the closed subscheme

$$\partial X^*_R := (X^*_R \backslash X_R)^{\mathrm{red}} = j^{-1}(\infty)^{\mathrm{red}} \hookrightarrow X^*_R,$$

which is finite étale over $\mathrm{Spec}(R)$. When we refer to "a cusp" we shall mean by this a (not necessarily geometrically) connected component of ∂X^*_R.
We recall from [KM85, §8.11] that the divisor of cusps can be computed explicitly using the Tate curve $\mathrm{T}(q)$: This is an elliptic curve over $\mathbb{Z}(\!(q)\!)$ of j-invariant $1/q + 744 + \dots$, which we may base–change to $\mathrm{T}(q)_{R(\!(q)\!)} \to \mathrm{Spec}(R(\!(q)\!))$. Then we have:

Proposition 2.1 ([KM85, Theorem 8.11.10]). *The completion $\hat{\partial} X^*_R$ of X^*_R along ∂X^*_R is the normalisation of $R[\![q]\!]$ in the finite flat scheme over $R(\!(q)\!)$ that represents Γ-level structures of $\mathrm{T}(q)_{R(\!(q)\!)}$. Via the j-invariant, $\hat{\partial} X^*_R$ is finite over the completion $R[\![q]\!]$ of $\mathbb{P}^1_R$ at ∞.*

To say more concretely what ∂X^*_R looks like, recall that $\mathrm{T}(q)[N]$ is canonically an extension

$$0 \to \mu_N \to \mathrm{T}(q)[N] \to \underline{\mathbb{Z}/N\mathbb{Z}} \to 0$$

over $\mathbb{Z}(\!(q)\!)$ which becomes split over $\mathbb{Z}(\!(q^{1/N})\!)$. Consequently, the Γ-level structures of $\mathrm{T}(q)_{R(\!(q)\!)}$ are defined over various subrings of $R[\zeta_N](\!(q^{1/N})\!)$. In particular, each component of $\hat{\partial} X^*_R$ will be of the form $\mathrm{Spf}(R[\zeta_d][\![q^{1/e}]\!])$ for some $d|N$ and $e|N$.

Notation 2.2. In order to simplify notation, we wish to reduce the amount of N-th roots of q throughout. Therefore, we shall by convention renormalise this ring of definition at each cusp to be a subring of the form $R[\zeta_d]((q))$ for some $d|N$, by passing from $\mathrm{T}(q)$ to $\mathrm{T}(q^e)$.
This means that depending on the cusp, the completion of the j-invariant at the cusp is now given by a map of the form $\mathrm{Spf}(R[\zeta_d][[q]]) \to \mathrm{Spf}(R[[q]])$ that sends $q \mapsto q^e$. If R' is another Noetherian excellent regular $\mathbb{Z}[1/N]$-algebra, then $\partial X^*_{R'}$, $\hat{\partial} X^*_{R'}$, etc. agree with the base-changes via $R \to R'$ by [KM85, Proposition 8.6.6]. Therefore, more generally, for any $\mathbb{Z}[1/N]$-algebra S we may simply define X_S, X^*_S, ∂X^*_S, $\hat{\partial} X^*_S$, etc. by base-change.

2.2 The analytic setup

We now switch to a p-adic analytic situation and recall the setup from [Sch15]. Let p be a prime and let K be a perfectoid field extension of $\mathbb{Q}_p$ like in the introduction. We denote by $\mathfrak{m}$ the maximal ideal of the ring of integers $\mathcal{O}_K$ and by k the residue field. We fix a complete algebraically closed extension C of K and assume that K contains all p-power roots of unity in C. We moreover fix a pseudo-uniformiser $\varpi \in K$ with $|\varpi| = |p|$ such that ϖ contains arbitrary p-power roots in K. This is possible since K is perfectoid.
Let $N \geq 3$ be coprime to p and let $\Gamma(N) \subseteq \Gamma^p \subseteq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ be a rigidifying tame level. Let $X := X_K$ be the modular curve of level Γ^p over K and let $X^* := X^*_K$ be its compactification. We denote by $\mathfrak{X}$ and $\mathfrak{X}^*$ the respective p-adic completions of $X_{\mathcal{O}_K}$ and $X^*_{\mathcal{O}_K}$. Let $\mathcal{X}$ and $\mathcal{X}^*$ be the respective adic analytifications of X and X^*. This is the only way in which we deviate from the notation in [Sch15], where $\mathcal{X}$ denotes the good reduction locus, which we shall instead denote by $\mathcal{X}_{\mathrm{gd}} \subseteq \mathcal{X} \subseteq \mathcal{X}^*$.
For any of the classical level structures $\Gamma = \Gamma_0(p^n), \Gamma_1(p^n), \Gamma(p^n)$, $n \in \mathbb{N}$, we denote by $X_\Gamma \to X$ the representing moduli scheme. These all have compactifications $X^*_\Gamma \to X^*$, and we have associated adic spaces $\mathcal{X}_\Gamma \to \mathcal{X}$ and $\mathcal{X}^*_\Gamma \to \mathcal{X}^*$. The uncompactified spaces have a natural moduli interpretation in the category **Adic** of (sheafy) adic spaces over $(K, \mathcal{O}_K)$:

Lemma 2.3. *Let S be a (sheafy) adic space over* $\mathrm{Spa}(K, \mathcal{O}_K)$. *Then*

$$\mathrm{Hom}_{\mathbf{Adic}}(S, \mathcal{X}_\Gamma) = X_\Gamma(\mathcal{O}_S(S)).$$

In particular, the S-points of $\mathcal{X}_\Gamma$ are in functorial correspondence with isomorphism classes of elliptic curves over $\mathcal{O}_S(S)$ with tame level structure Γ^p and level structure Γ at p.

Proof. The scheme X_Γ over K is an affine curve [KM85, Corollary 4.7.2]. Let **LRS** be the category of locally ringed spaces over K, then by the universal property of the analytification $\mathcal{X}_\Gamma = X^{\mathrm{an}}_\Gamma$:

$$\mathrm{Hom}_{\mathbf{Adic}}(S, \mathcal{X}_\Gamma) = \mathrm{Hom}_{\mathbf{LRS}}(S, X_\Gamma) = X_\Gamma(\mathcal{O}_S(S))$$

where the last step is the adjunction of Spec and global sections for locally ringed spaces. □

Let $0 \leq \epsilon < \frac{1}{2}$ be such that $|p|^\epsilon \in |K|$. Using local trivialisations of the Hodge bundle and lifts Ha of the Hasse invariant one defines an open subspace $\mathcal{X}^*(\epsilon) \subseteq \mathcal{X}^*$ cut out by the condition that $|\mathrm{Ha}| \geq |p|^\epsilon$. This has a canonical integral model $\mathfrak{X}^*(\epsilon) \to \mathfrak{X}^*$, for example by [Sch15, Lemma 3.2.13]. In general, for any morphism $S \to \mathcal{X}^*$ we shall write

$$S(\epsilon) := S \times_{\mathcal{X}^*} \mathcal{X}^*(\epsilon).$$

In particular, for any of the classical level structures $\Gamma = \Gamma_0(p^n), \Gamma_1(p^n), \Gamma(p^n)$, the modular curve $\mathcal{X}^*_\Gamma \to \mathcal{X}^*$ restricts to a morphism $\mathcal{X}^*_\Gamma(\epsilon) \to \mathcal{X}^*(\epsilon)$. We note that the open subspace $\mathcal{X}^*(0)$ is precisely the ordinary locus of $\mathcal{X}^*$.

Definition 2.4. We shall say that an elliptic curve E is ϵ-nearly ordinary if $|\mathrm{Ha}(E)| \geq |p|^\epsilon$.

By the theory of the canonical subgroup, the forgetful morphism $\mathcal{X}^*_{\Gamma_0(p)}(\epsilon) \to \mathcal{X}^*(\epsilon)$ has a canonical section. We denote by $\mathcal{X}^*_{\Gamma_0(p)}(\epsilon)_c$ the image of this section, that is the open and closed component of $\mathcal{X}^*_{\Gamma_0(p)}(\epsilon)$ that parametrises the $\Gamma_0(p)$-structure given by the canonical subgroup. This is called the canonical locus. We denote its complement by $\mathcal{X}^*_{\Gamma_0(p)}(\epsilon)_a$ and call it the anticanonical locus. For any adic space $S \to \mathcal{X}^*_{\Gamma_0(p)}$ we denote by

$$S(\epsilon)_a := S \times_{\mathcal{X}^*_{\Gamma_0(p)}} \mathcal{X}^*_{\Gamma_0(p)}(\epsilon)_a$$

the open subspace that lies over the anticanonical locus.

Definition 2.5. For any adic space $S \to \mathcal{X}(\epsilon)$ corresponding to an ϵ-nearly ordinary elliptic curve E over $\mathcal{O}_S(S)$, we shall call a Γ-level structure anticanonical if it corresponds to a point of $\mathcal{X}_\Gamma(\epsilon)_a \subseteq \mathcal{X}_\Gamma$. For instance, a $\Gamma_0(p^n)$-level structure is a locally free subgroup scheme $G_n \subseteq E[p^n]$, étale-locally cyclic of rank p^n, and it is anticanonical if $G_n \cap C_1 = 0$ inside $E[p^n]$, where $C_1 \subseteq E[p]$ denotes the canonical subgroup. Similarly, a $\Gamma(p^n)$-level structure, given by an isomorphism of group schemes $\alpha\colon (\mathbb{Z}/p^n\mathbb{Z})^2 \to E[p^n]$ is anticanonical if the subgroup scheme generated by $\alpha(1,0)$ is anticanonical.

For any $n \in \mathbb{N}$, the transformation of moduli functors that sends an elliptic curve E together with an anticanonical $\Gamma_0(p^n)$-structure G_n to E/G_n induces an isomorphism

$$\mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a \xrightarrow{\sim} \mathcal{X}(p^{-n}\epsilon) \qquad (3)$$

that is called the Atkin–Lehner isomorphism. The inverse is given by sending E with its canonical subgroup C_n of rank p^n to the data of E/C_n with $\Gamma_0(p^n)_a$-structure $E[p^n]/C_n$. The Atkin–Lehner isomorphism uniquely extends to the cusps for all n, and for varying n the resulting isomorphisms fit into a commutative diagram of towers

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \mathcal{X}^*_{\Gamma_0(p^2)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*(\epsilon) \\
& \wr\vert & & \wr\vert & & \Vert \\
\cdots \longrightarrow & \mathcal{X}^*(p^{-2}\epsilon) & \xrightarrow{\phi} & \mathcal{X}^*(p^{-1}\epsilon) & \xrightarrow{\phi} & \mathcal{X}^*(\epsilon)
\end{array}$$

where in the bottom row, the morphism ϕ is the "Frobenius lift" defined in terms of moduli by sending E to E/C_1. The resulting tower is called the "anticanonical tower".

It is a crucial intermediate result in [Sch15] that the anticanonical tower "becomes perfectoid in the inverse limit". More precisely:

Theorem 2.6 ([Sch15, Corollary 3.2.19]). *There is an affinoid perfectoid tilde-limit*

$$\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a \sim \varprojlim_{n\in\mathbb{N}} \mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a.$$

Since the forgetful morphisms $\mathcal{X}^*_{\Gamma_1(p^n)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$ are finite étale $(\mathbb{Z}/p^n\mathbb{Z})^\times$-torsors, even over the cusps, one immediately deduces that in the inverse limit these give rise to an affinoid perfectoid space $\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a \sim \varprojlim_n \mathcal{X}^*_{\Gamma_1(p^n)}(\epsilon)_a$ together with a forgetful map

$$\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$$

that is a pro-étale $\mathbb{Z}_p^\times$-torsor. Similarly, for full level $\Gamma(p^n)$, one obtains an affinoid perfectoid space $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ together with a forgetful map $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$ that is a pro-étale $\Gamma_0(p^\infty)$-torsor, where we set:

Definition 2.7. For any $m \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, let

$$\Gamma_0(p^m) = \{\left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}_p) \mid c \equiv 0 \bmod p^m\}.$$

All in all, we have a tower of morphisms

$$\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a \longrightarrow \mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a \longrightarrow \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a \longrightarrow \mathcal{X}^*_{\Gamma_0(p)}(\epsilon)_a$$

which is a pro-étale $\Gamma_0(p)$-torsor away from the boundary, but not globally: One reason is that there is ramification over the cusps in $\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_0(p)}(\epsilon)_a$, but we note that the tower is still no torsor on the quasi-pro-étale site, as we will see on q-expansions.

2.3 Analytic Tate curve parameter spaces at tame level

In this subsection, we recall the universal analytic Tate curves at the cusps, as developed by [Con06]. The main technical difference is that we work with analytic adic spaces instead of rigid spaces. In particular, instead of the generalisation of Berthelot's functor constructed in §3 of *loc. cit.*, we may use the adic generic fibre functor.

For now, we shall focus on the adic analytic modular curve $\mathcal{X}^*$ over K. We remark that everything in this section and the next will also work for $\mathcal{X}^*_{\Gamma_0(p^n)}$, except for the additional phenomenon of ramification at the cusps. In order to separate the discussion of these two topics, and to simplify the exposition, we shall therefore focus on $\mathcal{X}^*$ for now.

As discussed in §2.1, the subscheme $\partial X^* \subseteq X^*$ decomposes into a union of points of the form $x\colon \mathrm{Spec}(L) \to X^*$ where $K \subseteq L \subseteq K[\zeta_N]$ is a subfield depending on x. For example, if $\Gamma^p = \Gamma(N)$, we have $L = K[\zeta_N]$ at every cusp. We now switch to an analytic setup:

DEFINITION 2.8. By a cusp of $\mathcal{X}^*$ we shall mean a connected (but not necessarily geometrically connected) component of $(\mathcal{X}^*\backslash\mathcal{X})^{\mathrm{red}}$. Given a fixed cusp x, we shall denote by $L = L_x \subseteq K[\zeta_N]$ the coefficient field of definition of the corresponding Tate curve. We have $L = K[\zeta_d]$ for some $d|N$. Let $\mathfrak{m}_L$ be the maximal ideal of $\mathcal{O}_L$ and let k_L be the residue field.

From now on and for the rest of this section, let us fix a cusp $x \in \mathcal{X}^*$. To simplify notation, we will write $L = L_x$ and $e = e_x$. We note that the cusp x will decompose into $[L : K]$ disjoint L-points after base-changing $\mathcal{X}^*$ from K to L. By Proposition 2.1, the completion of $X^*_{\mathcal{O}_K}$ along x is canonically of the form

$$\mathrm{Spf}(\mathcal{O}_L[\![q]\!]) \to X^*_{\mathcal{O}_K} \tag{4}$$

where $\mathcal{O}_L[\![q]\!]$ carries the q-adic topology. Here we recall from Notation 2.2 that we have renormalised parameters from $q^{1/e}$ to q, so that the universal Tate curve is $\mathrm{T}(q^e)$. Upon p-adic completion, this becomes a morphism

$$\mathrm{Spf}(\mathcal{O}_L[\![q]\!]) \to \mathfrak{X}^*$$

where now $\mathcal{O}_L[\![q]\!]$ is endowed with the (p,q)-adic topology. We note that this morphism restricts to $\mathrm{Spf}(\mathcal{O}_L[\![q]\!]) \to \mathfrak{X}^*(0)$ since the supersingular locus is disjoint from the cusps.

On the adic generic fibre, we obtain a morphism of analytic adic spaces over K

$$\mathcal{D} := \mathrm{Spf}(\mathcal{O}_L[\![q]\!])^{\mathrm{ad}}_{\eta} \to \mathcal{X}^*.$$

Here $\mathcal{D}$ is the open unit disc over K, a topologically finite type but non-quasicompact, non-affinoid analytic adic space. We emphasize that in general, this depends on x, as $L = L_x$ does. If the cusp x is not clear from the context, we shall therefore denote this space by $\mathcal{D}_x$.

The global functions on $\mathcal{D}$ are given by $\mathcal{O}^+(\mathcal{D}) = \mathcal{O}_L[\![q]\!]$ and

$$\mathcal{O}(\mathcal{D}) = \Big\{ \textstyle\sum_{n\geq 0} a_n q^n \in L[\![q]\!] \text{ such that } |a_n| z^n \to 0 \text{ for all } 0 \leq z < 1 \Big\}.$$

More classically, if we regarded $\mathcal{D}$ as a rigid space, it would be associated to the formal scheme $\mathrm{Spf}(\mathcal{O}_L[\![q]\!])$ via Conrad's non-Noetherian generalisation of Berthelot's rigid generic fibre construction, [Con06, Theorem 3.1.5].

Lemma 2.9. *The map $\mathcal{D} \hookrightarrow \mathcal{X}^*$ is an open immersion that sends the origin to the cusp.*

Remark 2.10. This is part of [Con06, Theorem 3.2.8] for $\Gamma^p = \Gamma_1(N)$, and in general follows from [Con06, Theorem 3.2.6]. These moreover give a moduli interpretation in terms of analytic generalised elliptic curves, as well as a universal analytic generalised Tate curve over $\mathcal{D}$.

Proof. In an affine open formal neighbourhood $\mathrm{Spf}(A) \subseteq \mathfrak{X}^*(0)$ of the cusp, x is cut out by a principal ideal (f) for some non-zero-divisor f. The completion along the cusp is then $A[\![T]\!]/(T-f)$. The adic generic fibre is thus the union of the spaces $\mathrm{Spa}(A\langle f^n/p\rangle[1/p])$ for $n \in \mathbb{N}$, and each of these is the open subspace of $\mathrm{Spa}(A[1/p]) \subseteq \mathcal{X}^*$ defined by the condition $|f| \leq |p|^{1/p^n}$. □

Lemma 2.11. *The morphism of locally ringed spaces $\mathcal{D} \to \mathrm{Spec}(\mathbb{Z}_p[\![q]\!] \otimes_{\mathbb{Z}_p} \mathcal{O}_L)$ induced by the inclusion $\mathbb{Z}_p[\![q]\!] \hookrightarrow \mathcal{O}^+(\mathcal{D})$ fits into a commutative diagram of ringed spaces:*

$$\begin{array}{ccc} \mathrm{Spec}(\mathbb{Z}_p[\![q]\!] \otimes_{\mathbb{Z}_p} \mathcal{O}_L) & \longrightarrow & X^*_{\mathcal{O}_K} \\ \uparrow & & \uparrow \\ \mathcal{D} & \hookrightarrow & \mathcal{X}^*. \end{array}$$

Proof. Let $R := \mathbb{Z}_p[\zeta_d]$ and $R_0 := \mathbb{Z}_p[\zeta_{d_0}]$ where d_0 is the largest divisor of d such that K contains a primitive d_0-th root of unity. The adification of the p-adic completion of $f\colon \mathrm{Spf}(R[\![q]\!]) \to \mathrm{Spec}(R[\![q]\!]) \to X^*_{R_0}$ fits into a commutative diagram of ringed spaces

$$\begin{array}{ccccc} \mathrm{Spec}(R[\![q]\!]) & \xrightarrow{f} & X^*_{R_0} & \longrightarrow & \mathrm{Spec}(R_0) \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{Spa}(R[\![q]\!], R[\![q]\!]) & \xrightarrow{\hat{f}} & \mathfrak{X}^{*\mathrm{ad}}_{R_0} & \longrightarrow & \mathrm{Spa}(R_0, R_0). \end{array}$$

The lemma follows upon base-changing to $\mathrm{Spa}(K, \mathcal{O}_K) \to \mathrm{Spec}(\mathcal{O}_K)$. □

We thus have the following moduli interpretation of $\mathring{\mathcal{D}} := \mathcal{D}(q \neq 0) \subseteq \mathcal{D}$:

Lemma 2.12. *Let S be a (sheafy) adic space over K and let $\varphi\colon S \to \mathcal{X}$ be a morphism corresponding to an elliptic curve E over $\mathcal{O}_S(S)$ with Γ^p-level structure α_N. Then φ factors through the punctured open unit disc $\mathring{\mathcal{D}} \to \mathcal{X}$ at the cusp x if and only if $E \cong \mathrm{T}(q_E)$ is a Tate curve for some $q_E \in S$, the Γ^p-structure α_N corresponds to x under Proposition 2.1, and q_E is locally topologically nilpotent on S, i.e. $v_z(q_E)$ is cofinal in the value group for all $z \in S$.*

Proof. If $\varphi\colon S \to \mathcal{X}^* \to X^*$ factors through $\mathring{\mathcal{D}} \hookrightarrow \mathcal{X}$, then by Lemma 2.11 it factors through the map $\operatorname{Spec}(\mathcal{O}_L(\!(q)\!)) \to X^*$. Consequently, E is a Tate curve and we obtain a parameter $q_E \in \mathcal{O}_S(S)$ as the image of $q \in \mathcal{O}_L(\!(q)\!) \subseteq \mathcal{O}(\mathcal{D})$ on global sections. This is locally topologically nilpotent because $q \in \mathcal{O}(\mathcal{D})$ is locally topologically nilpotent.
Conversely, assume that E is a Tate curve such that $q_E \in \mathcal{O}(S)$ is topologically nilpotent, with Γ^p-level structure associated to x. The latter condition implies that $\mathcal{O}(S)$ is naturally an L-algebra. It therefore suffices to consider the case that $S = \operatorname{Spa}(B, B^+)$ is an affinoid adic space over $\operatorname{Spa}(L, \mathcal{O}_L)$. The condition that q_E is topologically nilpotent implies that for any x there is n such that $|q_E(x)|^n \leq |\varpi|$. Since S is affinoid and thus quasicompact, we can find n that works for all $x \in S$. Similarly, since E is a Tate curve, $q_E \in B$ is a unit and we thus have $0 < |q_E(x)|$ for all $x \in S$. Again by compactness, we can find m such that $|\varpi|^m \leq |q_E|$. But then $q_E^n/\varpi, \varpi^m/q_E \in B^+$ and there is a natural morphism of affinoids

$$(L\langle q, q^n/\varpi, \varpi^m/q\rangle, \mathcal{O}_L\langle q, q^n/\varpi, \varpi^m/q\rangle) \to (B, B^+), \quad q \mapsto q_E$$

through which the map $\mathcal{O}_L(\!(q)\!) \to B$ defining the Tate curve structure factors. Since the algebra on the left defines an affinoid open of $\mathring{\mathcal{D}}$, this gives the desired factorisation. □

Definition 2.13. Consider $\mathcal{O}_L(\!(q)\!)$ with the p-adic topology, that is we suppress the topology coming from q. Let $\mathcal{O}_L\langle\!\langle q\rangle\!\rangle := \mathcal{O}_L(\!(q)\!)^\wedge$ be the p-adic completion. Explicitly,

$$\mathcal{O}_L\langle\!\langle q\rangle\!\rangle = \big\{ \textstyle\sum_{n\in\mathbb{Z}} a_n q^n \in \mathcal{O}_L[\![q^{\pm 1}]\!] \mid a_n \to 0 \text{ for } n \to -\infty \big\}.$$

This is a discrete valuation ring with maximal ideal (p) and residue field $k_L(\!(q)\!)$.

Remark 2.14. The adic space $S = \operatorname{Spa}(\mathcal{O}_L\langle\!\langle q\rangle\!\rangle[\frac{1}{p}], \mathcal{O}_L\langle\!\langle q\rangle\!\rangle)$ consists of a single point. As we have suppressed the q-adic topology, it is clear that q is not topologically nilpotent on S. We conclude that the map

$$S \to \operatorname{Spec} \mathcal{O}_L(\!(q)\!) \to X^*$$

does not factor through $\mathring{\mathcal{D}} \hookrightarrow \mathcal{X}^*$ even though it corresponds to a Tate curve. The point is that this Tate curve has good reduction: Concretely, this means that it is already an elliptic curve over $\mathcal{O}_L(\!(q)\!)$. Its reduction mod $\mathfrak{m}_L$ is simply the universal Tate curve $\mathrm{T}(q)$ over $k_L(\!(q)\!)$. It therefore gives rise to a point in the good reduction locus $\mathcal{X}_{\mathrm{gd}} \subseteq \mathcal{X} \subseteq \mathcal{X}^*$.

We can enlarge the Tate curve parameter space $\mathcal{D}$ so that it includes the above example:

Definition 2.15. Let $\overline{\mathcal{D}} = \operatorname{Spa}(\mathcal{O}_L[\![q]\!][\frac{1}{p}], \mathcal{O}_L[\![q]\!])$ where, contrary to our usual convention, $\mathcal{O}_L[\![q]\!]$ is endowed with the p-adic topology. We let

$$\mathcal{O}_L[\![q^{1/p^\infty}]\!]_p := (\varinjlim_m \mathcal{O}_L[\![q^{1/p^m}]\!])^\wedge$$

where, crucially, the completion is the p-adic one. This is a perfectoid $\mathcal{O}_L$-algebra. If we use the (p,q)-adic topology instead, we obtain a (p,q)-adically complete ring $\mathcal{O}_L[\![q^{1/p^\infty}]\!]$. There is a natural inclusion $\mathcal{O}_L[\![q^{1/p^\infty}]\!]_p \hookrightarrow \mathcal{O}_L[\![q^{1/p^\infty}]\!]$, but this is not an isomorphism: For example, $\sum_{n\in\mathbb{N}} q^{n+\frac{1}{p^n}}$ defines an element contained in the codomain but not in the image.
As before, we emphasize that $\overline{\mathcal{D}}$ depends on our chosen cusp x. If this cusp is not clear from the context, we shall also write $\overline{\mathcal{D}}_x$ for the parameter space associated to x.

Lemma 2.16. *1. The Huber pair $(\mathcal{O}_L[\![q]\!][\frac{1}{p}], \mathcal{O}_L[\![q]\!])$, where $\mathcal{O}_L[\![q]\!]$ is endowed with the p-adic topology, is sous-perfectoid in the sense of [SW20, §6.3], and thus sheafy.*

2. We have an open immersion $\mathcal{D} = \cup_m \overline{\mathcal{D}}(|q| \leq |p|^{1/p^m}) \hookrightarrow \overline{\mathcal{D}}$.

3. We have an open immersion $\mathrm{Spa}(\mathcal{O}_L\langle\!\langle q\rangle\!\rangle[\frac{1}{p}], \mathcal{O}_L\langle\!\langle q\rangle\!\rangle) = \overline{\mathcal{D}}(|q| \geq 1) \hookrightarrow \overline{\mathcal{D}}$.

Parts 1. and 2. of the lemma say that we may think of $\overline{\mathcal{D}}$ as a compactification of $\mathcal{D}$, albeit a non-standard one. As discussed in Example 2.20 below, $\mathcal{D}$ and $\overline{\mathcal{D}}(|q| \geq 1)$ form a disjoint open cover of $\overline{\mathcal{D}}$ up to one additional point of rank 2 which lies in between them.

Proof. The traces $\mathcal{O}_L[\![q^{1/p^n}]\!] \to \mathcal{O}_L[\![q]\!]$ defined by $q^m \mapsto q^m$ for $m \in \mathbb{Z}_{\geq 0}$ and $q^{i/p^n} \mapsto 0$ for $0 < i < p^n$ give rise in the p-adically completed limit to an $\mathcal{O}_L[\![q]\!]$-linear section $\mathcal{O}_L[\![q^{1/p^\infty}]\!]_p \to \mathcal{O}_L[\![q]\!]$. This shows the first part. Part (3) is clear as $\overline{\mathcal{D}}(|q| \geq 1)$ is formed by adjoining q^{-1} and completing p-adically.
Part (2) follows from $\mathcal{D} = \cup_m \mathcal{D}(|q|^m \leq |p|)$ and

$$\mathcal{O}(\overline{\mathcal{D}}(|q|^m \leq |p|)) = \mathcal{O}_L[\![q]\!]\langle \tfrac{q^m}{p}\rangle[\tfrac{1}{p}] = \mathcal{O}_L\langle \tfrac{q^m}{p}\rangle[\tfrac{1}{p}] = \mathcal{O}(\mathcal{D}(|q|^m \leq |p|)). \qquad \square$$

Lemma 2.17. *For every cusp of $\mathcal{X}^*$ there is a natural morphism $\overline{\mathcal{D}} \to \mathcal{X}^*(\epsilon)$, extending the morphism $\mathcal{D} \hookrightarrow \mathcal{X}^*(\epsilon)$. The fibre of the good reduction locus is precisely $\overline{\mathcal{D}}(|q| \geq 1)$.*

Proof. For the first part, we take the morphism $\mathrm{Spec}(\mathcal{O}_L[\![q]\!]) \to X^*_{\mathcal{O}_K}$ from Lemma 2.11, complete p-adically and pass to the generic fibre.
To see the second part, we note that morphisms $\mathrm{Spa}(R, R^+) \to \mathcal{X}_{\mathrm{gd}}$ correspond to elliptic curves over R^+. The locus of $\overline{\mathcal{D}}$ where the Tate curve is defined over $\mathcal{O}^+$ is precisely that where q is in $\mathcal{O}^{+,\times}$. Equivalently, as we have $|q| \leq 1$ on $\overline{\mathcal{D}}$, this means that $|q| \geq 1$. $\square$

Remark 2.18. 1. The map $\overline{\mathcal{D}} \to \mathcal{X}^*(\epsilon)$ is no longer an open immersion. Indeed, $\overline{\mathcal{D}}$ is not even of locally topologically finite type over K. We therefore do not have a rigid analogue of $\overline{\mathcal{D}}$, and can only describe this map in the setting of adic spaces.

2. Here and in the following, by base-change we could replace $\mathcal{O}_L[\![q]\!]$ by the smaller ring $\mathbb{Z}_p[\![q]\!]\hat{\otimes}_{\mathbb{Z}_p}\mathcal{O}_L$ where the completion is p-adic. But we will not need this.

3. If we form the union over all cusps $\mathcal{C}$, we get two maps

$$(i)\ \bigsqcup_{x\in\mathcal{C}}\mathcal{D}_x\times\mathcal{X}_{\mathrm{gd}}\to\mathcal{X},\qquad (ii)\ \bigsqcup_{x\in\mathcal{C}}\overline{\mathcal{D}}_x\times\mathcal{X}_{\mathrm{gd}}\to\mathcal{X}$$

of which (ii) is a cover (for example in the v-topology), in contrast to (i) whose image misses some points of rank 2. This is what we discuss next.

2.4 The points of the adic space $\mathcal{X}^*$

Our next goal is to see how much of the adic space $\mathcal{X}^*$ is captured by the Tate curve parameter spaces $\mathcal{D}$ in conjunction with the good reduction locus. The answer is that they give everything except for a finite set of higher rank points whose moduli interpretation in terms of Tate curves we shall describe.
Indeed, since the loci $\mathcal{X}_{\mathrm{gd}}$ and $\mathcal{D}$ in $\mathcal{X}^*$ are the preimages of a cover by an open and a closed set of the special fibre X_k under the specialisation map $|\mathcal{X}^*|\to|X_k|$, it follows from the adaptation of [dJ95, Lemma 7.2.5], to the setting of [Con06] that if we worked with rigid spaces, then $\mathcal{D}$ and $\mathcal{X}$ would cover $\mathcal{X}^*$ set-theoretically, but not admissibly so [dJ95, §7.5.1].

PROPOSITION 2.19. *Let $z\in\mathcal{X}^*$ be any point, then we are in either of the following cases:*

(a) $z\in\mathcal{X}^*$ is contained in the good reduction locus $\mathcal{X}_{\mathrm{gd}}$,

(b) $z\in\mathcal{D}_x\hookrightarrow\mathcal{X}^*$ is contained in a Tate curve parameter space around a cusp x of $\mathcal{X}^*$,

(c) $z\in\mathcal{X}^*\backslash\mathcal{X}_{\mathrm{gd}}$ is of rank >1 and its unique height 1 vertical generisation z' is in $\mathcal{X}_{\mathrm{gd}}$.

When we denote by j the global function on $\mathcal{X}$ induced by the j-invariant $j\colon\mathcal{X}\to\mathbb{A}^{1,\mathrm{an}}$, then the above are respectively equivalent to

(a') $|j(z)|\leq 1$,

(b') $|j(z)|>1$ and its inverse is cofinal in the value group,

(c') $|j(z)|>1$ and its unique rank 1 generisation z' satisfies $|j(z')|=1$.

We note that the analogous description also holds after adding level at p.

Proof. The space $\mathcal{X}^*$ is analytic, hence the valuation v_z is always microbial. This implies that z has a unique generisation z' of height 1, so statements (c) and (c') make sense.
The case of the cusps is clear, so we may assume that $z\in\mathcal{X}$.

We start by proving that (a) and (a') are equivalent. Recall that $X_{\mathcal{O}_K}$ is the preimage of $\mathbb{A}^1_{\mathcal{O}_K}$ under the morphism of $\mathcal{O}_K$-schemes $j\colon X^*_{\mathcal{O}_K} \to \mathbb{P}^1_{\mathcal{O}_K}$. Upon formal completion and passing to the adic generic fibre, j becomes j^{an} while $\mathbb{A}^1_{\mathcal{O}_K} \subseteq \mathbb{P}^1_{\mathcal{O}_K}$ is sent to the open disc $B_1(0) \subseteq \mathbb{A}^{1,\mathrm{an}}_K \subseteq \mathbb{P}^{1,\mathrm{an}}_K$ defined by $|j(z)| \leq 1$. Since the adic generic fibre of the completion of $X_{\mathcal{O}_K} \subseteq X^*_{\mathcal{O}_K}$ is $\mathcal{X}_{\mathrm{gd}} \subseteq \mathcal{X}^*$, this shows that $\mathcal{X}_{\mathrm{gd}}$ is precisely the preimage of $B_1(0)$ under $j^{\mathrm{an}}\colon \mathcal{X} = X^{\mathrm{an}} \to \mathbb{A}^{1,\mathrm{an}}$.

Next, let us prove that (b) and $(b$') are equivalent. We can always find a morphism

$$r_z\colon \mathrm{Spa}(C, C^+) \to \mathcal{X}$$

where (C, C^+) is a complete algebraically closed non-archimedean field, such that z is in the image of r_z. It thus suffices to show that r_z factors through some $\mathcal{D} \hookrightarrow \mathcal{X}^*$. By Lemma 2.12 it suffices to show that $(b$') holds if and only if the elliptic curve E over C that r_z represents is a Tate curve with nilpotent parameter $q_E \in C$.

The image of j in C is precisely the j-invariant j_E of E. Since in a non-archimedean field the elements with cofinal valuation are precisely the topologically nilpotent ones, condition $(b$') is equivalent to $j_E \neq 0$ and j_E^{-1} being topologically nilpotent. We can now argue like in the classical case of p-adic fields to see that this is equivalent to E being a Tate curve with q_E topologically nilpotent: Assume the latter, then $j_E = q_E^{-1} + 744 + 196884q + \dots \neq 0$ has valuation $|j_E| = |1/q_E|$ in C and thus j_E satisfies $(b$'). To see the converse, recall that in the formal Laurent series ring $\mathbb{Z}(\!(q)\!)$, the formula $j(q) = q^{-1} + 744 + \dots$ reverses to

$$q(j^{-1}) = j^{-1} + 744j^{-2} + 750420j^{-3} + \dots \in \mathbb{Z}[\![j^{-1}]\!].$$

If now j_E^{-1} is topologically nilpotent, the above series converges in C and we obtain a topologically nilpotent element $q_E \in C^\times$ with $j_E = 1/q_E + 744 + \dots = j(q_E)$. The Tate curve $\mathrm{T}(q_E)_C$ over C thus has the same j-invariant as E, and since C is algebraically closed we conclude that $E \cong \mathrm{T}(q_E)_C$. This shows that (b) and $(b$') are equivalent.

Next, let us show that $(c$') holds if and only if $(a$') and $(b$') don't hold. Recall that we always have a unique height 1 vertical generisation z'. Clearly $|j(z)| \neq 0$ if and only if $|j(z')| \neq 0$, and if in this case $|j(z)|^{-1}$ is cofinal then $|j(z')|^{-1}$ is cofinal. This implies that (b') and (c') can't hold at the same time. On the other hand, if $|j(z)| > 1$, then either $|j(z')| = 1$, or $|j(z')| > 1$ in which case $|j(z')|^{-1} < 1$ is cofinal because $v_{z'}$ has height 1. This shows that if $|j(z)| > 1$ then we are either in case $(b$') or in $(c$').

Since clearly (c) implies that (a) and (b) do not hold, it remains to prove $(c$') implies (c). But this follows from applying the equivalence of (a) and (a') first to z and then to z'. □

Example 2.20. Let us work out an example for an elliptic curve corresponding to a point of type (c): Let x be a cusp and $L = L_x$. Let $\mathbb{R}^\times_{>0} \times \gamma^{\mathbb{Z}}$ be the totally ordered group for which γ is such that $z < \gamma^n < 1$ for all $n \in \mathbb{Z}_{\geq 1}$ and all $z \in \mathbb{R}_{<1}$. Equip the field $F = \mathcal{O}_L\langle\langle q\rangle\rangle[\frac{1}{p}]$ with the valuation

$$v_{1^-}: F \to (\mathbb{R}^\times_{>0} \times \gamma^{\mathbb{Z}}) \cup \{0\}, \quad \sum a_n q^n \mapsto \max_{n\in\mathbb{Z}} |a_n|\gamma^n.$$

Recall that $\mathfrak{m}_L \subseteq \mathcal{O}_L$ is the maximal ideal. The valuation subring of F defined by v_{1^-} is

$$F^+ = \left\{ \sum_{n\gg-\infty}^{\infty} a_n q^n \in \mathcal{O}_L\langle\langle q\rangle\rangle \,\middle|\, a_n \in \mathfrak{m}_L \text{ for all } n < 0 \right\}.$$

Indeed, we have $v_{1^-}(\sum_{n\gg-\infty}^{\infty} a_n q^n) \leq 1$ if and only if $|a_n|\gamma^n \leq 1$ for all n. For $n \geq 0$ we have $|a_n|\gamma^n \leq 1$ if and only if $|a_n| \leq 1$, that is $a_n \in \mathcal{O}_L$. For $n < 0$, on the other hand, $|\gamma|^n$ is "infinitesimally" bigger than 1, so that $|a_n|\gamma^n \leq 1$ if and only if $|a_n| < 1$, that is $a_n \in \mathfrak{m}_L$.

The Tate curve $\mathrm{T}(q^e)$ over F for $e = e_x$, equipped with the Γ^p-structure corresponding to x, gives rise to a map

$$\varphi\colon \operatorname{Spa}(F, F^+) \to \mathcal{X}^*.$$

We claim that φ sends v_{1^-} neither into $\mathcal{X}_{\mathrm{gd}}$ nor into any of the Tate curve parameter spaces $\mathcal{D} \subseteq \mathcal{X}^*$. Indeed, the j-invariant of $\mathrm{T}(q^e)$ is

$$j = q^{-e} + 744 + q^e(\dots) \notin F^+ \tag{5}$$

which is not contained in F^+ by the above description. This shows that $\mathrm{T}(q^e)$ does not extend to an elliptic curve over F^+. On the other hand, q is not locally topologically nilpotent in L. Thus, by Lemma 2.12, v_{1^-} does not land in the open subspace $\mathcal{D}_x \hookrightarrow \mathcal{X}^*$.

Proposition 2.19 explains this as follows: The unique rank 1 vertical generisation of v_{1^-} is

$$v\colon F \to \mathbb{R}_{\geq 0}, \quad \sum a_n q^n \mapsto \max_{n\in\mathbb{Z}} |a_n|$$

with larger valuation ring $\mathcal{O}_L\langle\langle q\rangle\rangle \supseteq F^+$. We see from equation (5) that

$$|j(v_{1^-})| = \gamma^{-e} < 1, \text{ while } |j(v)| = 1.$$

This shows that φ sends v_{1^-} to one of the points of type (c) in Proposition 2.19, while its generisation v goes to the point of $\mathcal{X}_{\mathrm{gd}}$ defined in Remark 2.14.

2.5 Tate curve parameter spaces at level $\Gamma_0(p^n)$

Next, we discuss the behaviour of the Tate curve parameter spaces in the anticanonical tower. For this we first recall the situation at the cusps on the level of schemes:

Consider the forgetful morphism $f\colon X^*_{\Gamma_0(p)} \to X^*$. Over each cusp of X^* there are precisely two cusps of $X^*_{\Gamma_0(p)}$: One is called the étale cusp, it corresponds to the $\Gamma_0(p)$-level structure $\mu_p \subseteq \mathrm{T}(q)[p]$ on the Tate curve. The other is the ramified cusp, it corresponds to the level structure $\langle q^{1/p}\rangle \subseteq \mathrm{T}(q)[p]$. In particular, this latter level structure is only defined over $\mathbb{Z}(\!(q^{1/p})\!)$, and by Proposition 2.1 the completion at this cusp is given by

$$\mathrm{Spf}(\mathcal{O}_L[\![q^{1/p}]\!]) \to X^*_{\Gamma_0(p)}.$$

The names reflect that the map $X^*_{\Gamma_0(p)} \to X^*$ is étale at the one sort of cusps, but is ramified at the other: Over the étale cusp the map induced on completions is the identity

$$\mathcal{O}_L[\![q]\!] \to \mathcal{O}_L[\![q]\!],$$

whereas over the ramified cusp it is the inclusion

$$\mathcal{O}_L[\![q]\!] \to \mathcal{O}_L[\![q^{1/p}]\!].$$

For higher level structures $\Gamma_0(p^n)$, the curve $X^*_{\Gamma_0(p^n)} \to X^*$ has more cusps of different degrees of ramification p^i with $i \in \{0,\dots,n\}$, and corresponding completions of the form $\mathrm{Spf}(\mathcal{O}_L[\![q^{1/p^i}]\!]) \to X^*_{\Gamma_0(p^n)}$. There is, however, exactly one étale cusp, corresponding to the $\Gamma_0(p^n)$-level structure $\mu_{p^n} \subseteq \mathrm{T}(q)[p^n]$, and exactly one purely ramified one, corresponding to $\langle q^{1/p^n}\rangle$. Relatively with respect to $X^*_{\Gamma_0(p^n)} \to X^*_{\Gamma_0(p)}$, the purely ramified cusps lie over the ramified cusps of $X^*_{\Gamma_0(p)}$, while all other cusps of $X^*_{\Gamma_0(p^n)}$ lie over the étale cusps of $X^*_{\Gamma_0(p)}$.

All constructions from the last two sections now go through with the same proofs for $\mathcal{X}^*$ replaced by $\mathcal{X}^*_{\Gamma_0(p^n)}$, and $\mathcal{O}_L[\![q]\!]$ replaced by $\mathcal{O}_L[\![q^{1/p^i}]\!]$ where i depends on the cusp:

Definition 2.21. For any $i \in \mathbb{Z}_{\geq 0}$, we write

$$\mathcal{D}_i := \mathrm{Spa}(L\langle q^{1/p^i}\rangle, \mathcal{O}_L\langle q^{1/p^i}\rangle)(|q|<1)$$

for the open unit disc over L in the parameter q^{1/p^i}. Here by the condition on q we mean that we take the union over subspaces where $|q| \leq |p|^{1/n}$, i.e. we do not include points of rank >1 where q is infinitesimally smaller than 1. In other words, this is the open locus where q is locally topologically nilpotent. We write $\mathring{\mathcal{D}}_i$ for the open subspace obtained by removing the origin, i.e. the point defined by $q^{1/p^i} \mapsto 0$. When we want to emphasize the dependence on the field $L = L_x$ determined by a cusp $x \in \mathcal{X}^*$, we write these as $\mathcal{D}_{i,x}$ and $\mathring{\mathcal{D}}_{i,x}$.

Then like in Lemma 2.9, we have for any cusp x of $\mathcal{X}^*$ and any cusp of $\mathcal{X}^*_{\Gamma_0(p^n)}$ over x of ramification degree p^i a canonical open immersion

$$\mathcal{D}_{i,x} \hookrightarrow \mathcal{X}^*_{\Gamma_0(p^n)}$$

that sends the origin to x.

From now on until the next chapter, we shall focus exclusively on the anti-canonical locus $\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$. Here the ramification is very easy to describe, by the following proposition:

Proposition 2.22. *Fix a cusp $x \in \mathcal{X}^*$.*

1. *The cusps of $\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$ are precisely the purely ramified cusps of $\mathcal{X}^*_{\Gamma_0(p^n)}$. In particular, there is a canonical open immersion $\mathcal{D}_{n,x} \hookrightarrow \mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$ over x.*
2. *The forgetful map $\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_0(p^{n-1})}(\epsilon)_a$ gives a bijection between the respective cusps. The associated Tate curve parameter spaces fit into Cartesian diagrams*

$$\begin{array}{ccc} \mathcal{D}_{n,x} & \longrightarrow & \mathcal{D}_{n-1,x} \\ \downarrow & & \downarrow \\ \mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p^{n-1})}(\epsilon)_a, \end{array}$$

where $\mathcal{D}_n \to \mathcal{D}_{n-1}$ is the canonical finite flat map which sends $q \mapsto q$.

3. *For any adic space S over $(K, \mathcal{O}_K)$, the S-points of $\mathring{\mathcal{D}}_{n,x} \hookrightarrow \mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a$ correspond functorially to Tate curves $\mathrm{T}(q)$ over $\mathcal{O}(S)$ with topologically nilpotent parameter $q \in \mathcal{O}(S)$, a Γ^p-structure corresponding to x and a choice of p^n-th root q^{1/p^n} of q defining a $\Gamma_0(p^n)$-structure $\langle q^{1/p^n} \rangle \subseteq \mathrm{T}(q)$.*

Proof. Since the canonical subgroup of the Tate curve is given by $\mu_p \subseteq \mathrm{T}(q)[p]$, the cusps of $\mathcal{X}^*_{\Gamma_0(p^n)}$ contained in $\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$ are precisely the ones over the ramified cusps in $X^*_{\Gamma_0(p)}$. But the cusps of $X^*_{\Gamma_0(p^n)}$ over the ramified cusps of $X^*_{\Gamma_0(p)}$ are precisely the purely ramified ones. This proves (1). Part (3) follows immediately.

The diagram in (2) commutes because by construction, it is the generic fibre of a commutative diagram of formal schemes. Since the morphisms are open immersions, it suffices to check that it is Cartesian on the level of points, where it follows from (3) and Lemma 2.12. □

2.6 Tate curve parameter spaces of $\mathcal{X}^*_{\Gamma_1(p^n)}(\epsilon)_a$

We now pass to higher level at p and describe Tate curve parameter spaces of the form $\mathcal{D}_n \hookrightarrow \mathcal{X}^*_{\Gamma_1(p^n)}(\epsilon)_a$. We note that the integral theory of cusps for $\Gamma_1(p^n)$ is slightly complicated in general, see §4.2 of [Con07] for a thorough discussion. However, over the anticanonical locus, the story is very simple:

Lemma 2.23. *Let x be a cusp of $\mathcal{X}^*$. Then there are $\mathbb{Z}_p^\times$-equivariant Cartesian squares*

$$\begin{array}{ccccc} \underline{(\mathbb{Z}/p^{n+1}\mathbb{Z})^\times} \times \mathcal{D}_{n+1,x} & \longrightarrow & \underline{(\mathbb{Z}/p^n\mathbb{Z})^\times} \times \mathcal{D}_{n,x} & \longrightarrow & \mathcal{D}_{n,x} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{X}^*_{\Gamma_1(p^{n+1})}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_1(p^n)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a, \end{array}$$

in which the morphism on the top left is given by the reduction $(a, q) \mapsto (\overline{a}, q)$.

Proof. As the morphisms in the bottom row are finite étale Galois torsors for the groups $\mathbb{Z}/p^n\mathbb{Z}$ and $\mathbb{Z}/p^{n+1}\mathbb{Z}$, it suffices to produce a section $\mathcal{D}_{n,x} \to \mathcal{X}^*_{\Gamma_1(p^n)}$. By Proposition 2.22, the cusp of $\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$ over x corresponds to the choice of $\langle q^{1/p^n}\rangle \subseteq \mathrm{T}(q)$ as a $\Gamma_0(p^n)$-structure. This can be lifted canonically to the $\Gamma_1(p^n)$-structure given by the p^n-th root q^{1/p^n} in $\mathcal{O}(\mathcal{D}_{n,x})$. Upon normalisation, we thus get a canonical lift

$$\begin{array}{ccc} & & \operatorname{Spec}(\mathbb{Z}_p[\![q^{1/p^n}]\!] \otimes L_x) \\ & \swarrow & \downarrow \\ X^*_{\Gamma_1(p^n)} & \longrightarrow & X^*_{\Gamma_0(p^n)}. \end{array}$$

The factorisation $\mathcal{D}_{n,x} \to \operatorname{Spec}(\mathbb{Z}_p[\![q^{1/p^n}]\!] \otimes L_x) \to X^*_{\Gamma_0(p^n)}$ from the analogue for level $\Gamma_0(p^n)$ of Lemma 2.11 together with the universal property of the analytification now give rise to the desired section

$$\begin{array}{ccc} & & \mathcal{D}_{n,x} \\ & \swarrow & \downarrow \\ \mathcal{X}^*_{\Gamma_1(p^n)} & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p^n)}. \end{array}$$

This shows that the right square is Cartesian. That the outer square is Cartesian follows in combination with Proposition 2.22. Consequently, the left square is also Cartesian. □

2.7 Tate curve parameter spaces of $\mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a$

Next, we look at what happens with the cusps in the transition

$$\mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_1(p^n)}(\epsilon)_a.$$

Let us fix notation for the left action of $\mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$ on $\mathcal{X}^*_{\Gamma(p^n)}$ in terms of moduli: For any $\gamma \in \mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$ it is given by sending a trivialisation $\alpha : (\mathbb{Z}/p^n\mathbb{Z})^2 \xrightarrow{\sim} E[p^n]$ to

$$(\mathbb{Z}/p^n\mathbb{Z})^2 \xrightarrow{\gamma^\vee} (\mathbb{Z}/p^n\mathbb{Z})^2 \xrightarrow{\alpha} E[p^n]$$

where $\gamma^\vee = \det(\gamma)\gamma^{-1}$. Here the inverse is necessary to indeed obtain a left action, and the twist by $\det(\gamma)$ ensures that the action on the fibres of the map $\mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_1(p^n)}(\epsilon)_a$ is given by matrices of the form $\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)$ rather than $\left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right)$.

Definition 2.24. For $0 \leq m \leq n \in \mathbb{N}$, we denote by $\Gamma_0(p^m, \mathbb{Z}/p^n\mathbb{Z}) \subseteq \mathrm{GL}_2(p^m, \mathbb{Z}/p^n\mathbb{Z})$ the subgroup of matrices of the form $\left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right)$ with $c \equiv 0 \bmod p^m$.

The forgetful map $X^*_{\Gamma(p^n)} \to X^*_{\Gamma_0(p)}$ is given by reducing $(\mathbb{Z}/p^n\mathbb{Z})^2 \xrightarrow{\sim} E[p^n]$ mod p to $(\mathbb{Z}/p\mathbb{Z})^2 \xrightarrow{\sim} E[p]$ and sending it to the subgroup generated by $(1,0)$. Consequently, the action of $\Gamma_0(p, \mathbb{Z}/p^n\mathbb{Z})$ leaves the forgetful morphism $X^*_{\Gamma(p^n)} \to X^*_{\Gamma_0(p)}$ invariant. We see from this that the action of $\Gamma_0(p, \mathbb{Z}/p^n\mathbb{Z})$ restricts to an action on $\mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a \subseteq \mathcal{X}^*_{\Gamma(p^n)}$.
From now on, let us fix a compatible choice $(\zeta_{p^n})_{n\in\mathbb{N}}$ of p-power roots of unity in K.

Lemma 2.25. *Let x be a cusp of $\mathcal{X}^*$.*

1. *Depending on our chosen primitive p^n-root of unity ζ_{p^n}, there is a canonical Cartesian diagram*

$$\begin{array}{ccc} \underline{\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})} \times \mathcal{D}_{n,x} & \longrightarrow & \mathcal{D}_{n,x} \\ \downarrow & & \downarrow \\ \mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a. \end{array}$$

where the left map is $\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})$-equivariant for the action via the first factor.

2. *Let x_γ be the cusp of $\mathcal{X}^*_{\Gamma(p^n)}(0)_a$ over x determined by $\gamma = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})$. Then for any honest adic space S over K, the S-points of $x_\gamma\colon \mathcal{D}_{n,x} \hookrightarrow \mathcal{X}_{\Gamma(p^n)}(\epsilon)_a$ correspond functorially to Tate curves $E = \mathrm{T}(q_E)$ with topologically nilpotent parameter $q_E \in \mathcal{O}(S)$, a Γ^p-structure corresponding to x, and the basis $(q_E^{d/p^n}, q_E^{-b/p^n}\zeta_{p^n}^a)$ of $E[p^n]$, where q_E^{1/p^n} is the p^n-th root of q_E determined by x.*

3. *The reduction $\pi : \Gamma_0(p^{n+1}, \mathbb{Z}/p^{n+1}\mathbb{Z}) \to \Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})$ gives a Cartesian diagram*

$$\begin{array}{ccc} \underline{\Gamma_0(p^{n+1}, \mathbb{Z}/p^{n+1}\mathbb{Z})} \times \mathcal{D}_{n+1,x} & \xrightarrow{(\gamma,q)\mapsto(\pi(\gamma),q)} & \underline{\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})} \times \mathcal{D}_{n,x} \\ \downarrow & & \downarrow \\ \mathcal{X}^*_{\Gamma(p^{n+1})}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a. \end{array}$$

Proof. 1. Arguing as in Lemma 2.23, it suffices to produce a splitting

$$\begin{array}{ccc} & & \mathrm{Spec}(\mathbb{Z}_p[\![q^{1/p^n}]\!] \otimes L_x) \\ & \swarrow & \downarrow \\ X^*_{\Gamma(p^n)} & \longrightarrow & X^*_{\Gamma_1(p^n)} \end{array}$$

which we construct as follows: Consider the Tate curve $T = \mathrm{T}(q^e)_R$ over $R = \mathbb{Z}_p(\!(q^{1/p^n})\!) \otimes \mathcal{O}_L$ with its Weil pairing $e_{p^n}\colon T[p^n] \times T[p^n] \to \mu_{p^n}$.

Specialising at $q^{e/p^n} \in T[p^n]$, we obtain an isomorphism

$$e(q^{e/p^n}, -) \colon C_n \to \mu_{p^n}$$

where C_n is the canonical subgroup of T. The preimage of ζ_{p^n} gives the second basis vector of $T[p^n]$ of an anticanonical $\Gamma(p^n)$-structure on $T[p^n]$, defining the desired lift. By analytification we then obtain the map $g \colon \mathcal{D}_{n,x} \to \mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a$ which gives the Cartesian diagram as $\mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$ is Galois with group $\underline{\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})}$.

2. We have just seen that the cusp label $1 \in \Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})$ corresponds via $q^e \mapsto q_E$ to the isomorphism $\alpha \colon (\mathbb{Z}/p^n\mathbb{Z})^2 \to E[p^n]$ defined by the ordered basis $(q_E^{1/p^n}, \zeta_{p^n})$. For the general case, we use that the action of $\gamma = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ is given by

$$\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \xmapsto{\gamma^\vee} \left(\begin{smallmatrix} d \\ 0 \end{smallmatrix}\right) \xmapsto{\alpha} q_E^{d/p^n},$$

$$\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \xmapsto{\gamma^\vee} \left(\begin{smallmatrix} -b \\ a \end{smallmatrix}\right) \xmapsto{\alpha} q_E^{-b/p^n}\zeta_{p^n}^a.$$

3. This follows from (1) and Lemma 2.22 as $X^*_{\Gamma(p^n)} \to X^*_{\Gamma(p^{n-1})}$ is equivariant for π. □

All in all, we get the following description of Tate curve parameter spaces at finite level:

PROPOSITION 2.26. *Let x be any cusp of $\mathcal{X}^*$. Then depending on our choice of $\zeta_{p^n} \in K$, there is a canonical tower of Cartesian squares*

$$\begin{array}{ccccccc}
\underline{\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})} \times \mathcal{D}_{n,x} & \longrightarrow & \underline{(\mathbb{Z}/p^n\mathbb{Z})^\times} \times \mathcal{D}_{n,x} & \longrightarrow & \mathcal{D}_{n,x} & \longrightarrow & \mathcal{D}_x \\
\big\downarrow{\scriptstyle\varphi} & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_1(p^n)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*(\epsilon).
\end{array}$$

where the top left map sends $(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right), q) \mapsto (d, q)$.

Proof. The square on the right is Proposition 2.22.(2). The square in the middle is Lemma 2.23. The Cartesian diagram on the left exists as a consequence of Lemma 2.25.1 combined with the fact that $X^*_{\Gamma(p^n)} \to X^*_{\Gamma_1(p^n)}$ is equivariant with respect to the map $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \mapsto d$. □

Lemma 2.25 describes the $\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})$-action at the cusps of $\mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a$, but there is also an action of the larger group $\Gamma_0(p, \mathbb{Z}/p^n\mathbb{Z})$. While the action of $\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})$ just permutes the different copies of $\mathcal{D}_{n,x}$ over x, the action of $\Gamma_0(p, \mathbb{Z}/p^n\mathbb{Z})$ in general has a non-trivial effect on each of these Tate curve parameter spaces, because it also takes into account isomorphisms of Tate curves of the form $q^{1/p^n} \mapsto \zeta_{p^n} q^{1/p^n}$, as we shall now discuss.

Proposition 2.27. *Over any cusp x of $\mathcal{X}^*$, the $\Gamma_0(p,\mathbb{Z}/p^n\mathbb{Z})$-action on $\mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a$ restricts to an action on $\varphi\colon \underline{\Gamma_0(p^n,\mathbb{Z}/p^n\mathbb{Z})}\times\mathcal{D}_{n,x}\hookrightarrow\mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a$ that can be described as follows: Equip $\underline{\Gamma_0(p,\mathbb{Z}/p^n\mathbb{Z})}\times\mathcal{D}_{n,x}$ with the right action by $p\mathbb{Z}/p^n\mathbb{Z}$ via*

$$(\gamma,q)\cdot h := (\gamma\left(\begin{smallmatrix}1&0\\h&1\end{smallmatrix}\right), \zeta_{p^n}^{h/e_x}q)$$

for $h\in p\mathbb{Z}/p^n\mathbb{Z}$, then we have a natural identification

$$(\underline{\Gamma_0(p,\mathbb{Z}/p^n\mathbb{Z})}\times\mathcal{D}_{n,x})/(p\mathbb{Z}/p^n\mathbb{Z}) = \underline{\Gamma_0(p^n,\mathbb{Z}/p^n\mathbb{Z})}\times\mathcal{D}_{n,x}$$

and the left action of $\Gamma_0(p,\mathbb{Z}/p^n\mathbb{Z})$ is the natural left action via the first factor. Explicitly, for any $\gamma_1\in\Gamma_0(p,\mathbb{Z}/p^n\mathbb{Z})$, the action is given by

$$\begin{aligned}\gamma_1\colon \underline{\Gamma_0(p^n,\mathbb{Z}/p^n\mathbb{Z})}\times\mathcal{D}_{n,x} &\xrightarrow{\sim} \underline{\Gamma_0(p^n,\mathbb{Z}/p^n\mathbb{Z})}\times\mathcal{D}_{n,x}\\ \gamma_2, q^{1/p^n} &\mapsto \begin{pmatrix}\det(\gamma_3)/d_3 & b_3\\ 0 & d_3\end{pmatrix}, \zeta_{p^n}^{-\frac{c_3}{d_3e_x}}q^{1/p^n}\end{aligned}$$

where $\gamma_3=\left(\begin{smallmatrix}a_3&b_3\\c_3&d_3\end{smallmatrix}\right):=\gamma_1\cdot\gamma_2$.

Here we recall that e_x was introduced in Notation 2.2.

Proof. Recall that the reason why the pullback of $\mathcal{D}_x\hookrightarrow\mathcal{X}^*_{\Gamma_0(p)}$ to $\mathcal{X}^*_{\Gamma(p^n)}$ is of the form $\underline{\Gamma_0(p^n,\mathbb{Z}/p^n\mathbb{Z})}\times\mathcal{D}_{n,x}$ even though $\mathcal{X}_{\Gamma(p^n)}\to\mathcal{X}_{\Gamma_0(p)}$ has larger Galois group $\Gamma_0(p,\mathbb{Z}/p^n\mathbb{Z})$ is that in the step from $\mathcal{X}^*$ to $\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$, the isomorphism

$$\psi_h\colon\mathcal{D}_{n,x}\to\mathcal{D}_{n,x},\quad q^{1/p^n}\mapsto\zeta_{p^n}^h q^{1/p^n}$$

for any $h\in\mathbb{Z}/p^n\mathbb{Z}$ induces an isomorphism of moduli for the universal Tate curve $T=\mathrm{T}(q^{e_x})$ over $\mathcal{D}_x$ that sends the anti-canonical $\Gamma_0(p^n)$-level structure $\langle q^{e_x/p^n}\rangle$ to $\langle\zeta_{p^n}^{he_x}q^{e_x/p^n}\rangle$. For the action of $\Gamma_0(p,\mathbb{Z}/p^n\mathbb{Z})$, this means the following:

Consider the Tate curve parameter space $\varphi(1)\colon\mathcal{D}_{n,x}\hookrightarrow\mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a$ at $1\in\Gamma_0(p,\mathbb{Z}/p^n\mathbb{Z})$, associated to the isomorphism $\alpha\colon(\mathbb{Z}/p^n\mathbb{Z})^2\to T[p^n]$ defined by the ordered basis $(q^{e_x/p^n},\zeta_{p^n})$. Then the action of $\gamma=\left(\begin{smallmatrix}1&0\\h&1\end{smallmatrix}\right)$ sends this to the isomorphism $\alpha\circ\gamma^\vee$ defined by $(1,0)\mapsto\zeta_{p^n}^{-h}q^{e_x/p^n}$ and $(0,1)\mapsto\zeta_{p^n}$. The isomorphism ψ_{-h/e_x} identifies this with the basis $(q^{1/p^n},\zeta_{p^n})$:

$$\begin{array}{ccccccc} q^{1/p^n} & \qquad & \langle q^{e_x/p^n}\rangle & \longrightarrow & \mathring{\mathcal{D}}_{n,x} & \xhookrightarrow{\varphi(1)} & \mathcal{X}_{\Gamma(p^n)}(\epsilon)_a\\ \downarrow & & \downarrow & & \downarrow{\scriptstyle\psi_{-h/e_x}} & & \downarrow{\scriptstyle\gamma_1}\\ \zeta_{p^n}^{-h/e_x}q^{1/p^n} & & \langle\zeta_{p^n}^{-h}q^{e_x/p^n}\rangle & \longrightarrow & \mathring{\mathcal{D}}_{n,x} & \xhookrightarrow{\varphi(1)} & \mathcal{X}_{\Gamma(p^n)}(\epsilon)_a. \end{array}$$

The action of $\left(\begin{smallmatrix}1&0\\h&1\end{smallmatrix}\right)$ on the component $\{1\}\times\mathcal{D}_{n,x}$ of $\underline{\Gamma_0(p^n,\mathbb{Z}/p^n\mathbb{Z})}\times\mathcal{D}_{n,x}$ defined by $1\in\Gamma_0(p^n,\mathbb{Z}/p^n\mathbb{Z})$ is thus given by ψ_{-h/e_x}.

In general, in order to describe the action of γ_1 on the component $\{\gamma_2\} \times \mathcal{D}_{n,x}$, it suffices to compute the action of $\gamma_3 = \gamma_1 \cdot \gamma_2$ on $\{1\} \times \mathcal{D}_{n,x}$, since we have a commutative diagram

$$\begin{array}{ccc} \{1\} \times \mathcal{D}_{n,x} & \dashrightarrow & \{1\} \times \mathcal{D}_{n,x} \\ \downarrow{\scriptstyle \gamma_2} & \searrow^{\gamma_3} & \vdots \\ \{\gamma_2\} \times \mathcal{D}_{n,x} & \xrightarrow{\gamma_1} & \{\gamma_3\} \times \mathcal{D}_{n,x}. \end{array}$$

One can now decompose γ_3 into the actions which we have already computed, using

$$\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} \det(\gamma)/d & b \\ 0 & d \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ c/d & 1 \end{smallmatrix}\right). \tag{6}$$

Applying this to γ_3, we get the desired action by equivariance of φ under $\Gamma_0(p, \mathbb{Z}/p^n\mathbb{Z})$. ☐

3 Adic theory of cusps at infinite level

We now pass to infinite level, starting with $\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a \sim \varprojlim \mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$. We first note:

Lemma 3.1. *Let (R, R^+) be a perfectoid K-algebra. Then the set $\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a(R, R^+)$ is in functorial bijection with isomorphism classes of triples $(E, \alpha_N, (G_n)_{n\in\mathbb{N}})$ of elliptic curves E over R that are ϵ-nearly ordinary, together with a Γ^p-structure α_N and a collection of anticanonical cyclic subgroups $G_n \subseteq E[p^n]$ of order p^n for all n that are compatible in the sense that $G_n = G_{n+1}[p^n]$. Equivalently, one could view $G = (G_n)_{n\in\mathbb{N}}$ as a p-divisible subgroup of $E[p^\infty]$ of height 1 such that G_1 is anticanonical.*

Proof. Since (R, R^+) is perfectoid, one has

$$\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a(R, R^+) = \varprojlim_{n\in\mathbb{N}} \mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a(R, R^+)$$

by [SW13, Prop 2.4.5]. The statement thus follows from Lemma 2.3. ☐

Definition 3.2. We shall call the p-divisible group G an anticanonical $\Gamma_0(p^\infty)$-structure.

We wish to study the cusps at infinite level, by which we mean the following:

Definition 3.3. By the cusps of $\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$, $\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a$ etc. we mean the preimage of the divisor of cusps of $\mathcal{X}^*$ endowed with its induced reduced structure.

We note that at infinite level, the cusps are in general a profinite Zariski-closed subspace of the respective infinite level modular curve. However, at level $\Gamma_0(p^\infty)$, we will see that the map $\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a \to \mathcal{X}^*$ isomorphically identifies the cusps of $\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$ and those of $\mathcal{X}^*$.

3.1 The perfectoid Tate curve parameter space at level $\Gamma_0(p^\infty)$

We start our discussion by having a closer look at the cusps in the anticanonical tower.
As before, we fix a cusp x of $\mathcal{X}^*$ and let L be the field of definition of the associated Tate curve, like in Def. 4.5. In particular, if K contains a primitive N-th root of unity, we simply have $L = K$. By Lemma 2.22, there is for x a tower of Cartesian squares

$$\begin{array}{ccccccc} \dots & \longrightarrow & \mathcal{D}_2 & \longrightarrow & \mathcal{D}_1 & \longrightarrow & \mathcal{D} \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p^2)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*(\epsilon). \end{array} \tag{7}$$

To describe the limit of this tower, we let

$$\mathcal{D}_\infty := \mathrm{Spa}(L\langle q^{1/p^\infty}\rangle, \mathcal{O}_L\langle q^{1/p^\infty}\rangle)(|q| < 1)$$

be the open locus of the closed perfectoid disc where q is locally topologically nilpotent, with notation as in Def. 2.21. Explicitly, this is the union of discs where $|q| < |p|^{1/n}$ for some n.

Lemma 3.4. *1. The perfectoid space $\mathcal{D}_\infty$ is the tilde-limit $\mathcal{D}_\infty \sim \varprojlim_{n\in\mathbb{N}} \mathcal{D}_n$.*

2. Denote by $\mathcal{O}_L[\![q^{1/p^\infty}]\!]$ the (p,q)-adic completion of $\varinjlim_{n\in\mathbb{N}} \mathcal{O}_L[\![q^{1/p^n}]\!]$. Then $\mathcal{D}_\infty$ is the adic generic fibre of the formal scheme $\mathrm{Spf}(\mathcal{O}_L[\![q^{1/p^\infty}]\!])$.

3. The global sections of $\mathcal{D}_\infty$ are given by $\mathcal{O}^+(\mathcal{D}_\infty) = \mathcal{O}_L[\![q^{1/p^\infty}]\!]$ and

$$\mathcal{O}(\mathcal{D}_\infty) = \left\{ \sum_{n\in\mathbb{Z}[\frac{1}{p}]_{\geq 0}} a_n q^n \in L[\![q^{1/p^\infty}]\!] \;\middle|\; \begin{array}{c} |a_n| z^n \xrightarrow{n\to\infty} 0 \text{ for all } z \in [0,1), \\ |a_n| \to 0 \text{ on bounded intervals} \end{array} \right\}$$

where the second condition means that for any $\delta > 0$ and for any bounded interval $I \subseteq \mathbb{Z}[\frac{1}{p}]_{\geq 0}$ there are only finitely many $n \in I$ such that $|a_n| > \delta$.

Proof. It is clear on global sections that

$$\mathrm{Spa}(L\langle q^{1/p^\infty}\rangle, \mathcal{O}_L\langle q^{1/p^\infty}\rangle) \sim \varprojlim_{n\in\mathbb{N}} \mathrm{Spa}(L\langle q^{1/p^n}\rangle, \mathcal{O}_L\langle q^{1/p^n}\rangle).$$

The first part follows from [SW13, Proposition 2.4.3] since $\mathcal{D}$ is the restriction to the open subspace defined by the union of $|q| \leq |p|^{1/p^n}$ for all $n \in \mathbb{N}$. More explicitly, this means that $\mathcal{D}_\infty$ is given by glueing the affinoid perfectoid unit discs of increasing radii < 1 given by

$$\mathcal{D}_\infty(|q|^{p^m} \leq |\varpi|) = \mathrm{Spa}(L\langle (q/\varpi^{1/p^m})^{1/p^\infty}\rangle, \mathcal{O}_L\langle (q/\varpi^{1/p^m})^{1/p^\infty}\rangle)$$

for all $m \in \mathbb{N}$. These can be obtained by rescaling the perfectoid unit disc. Computing the intersection of their respective global functions gives $\mathcal{O}(\mathcal{D}_\infty)$ and $\mathcal{O}^+(\mathcal{D}_\infty)$.

Part (2) is not just a formal consequence as tilde-limits do not necessarily commute with taking generic fibres. But it follows from the construction: Let

$$S = \mathrm{Spa}(\mathcal{O}_L[\![q^{1/p^\infty}]\!], \mathcal{O}_L[\![q^{1/p^\infty}]\!])$$

and consider the subspaces $S(|q|^{p^m} \leq |\varpi| \neq 0)$ which are rational because (q^n, ϖ) is open. As usual, one shows that since $\mathcal{O}_L[\![q^{1/p^\infty}]\!]$ has ideal of definition (q, ϖ), the element $|q(x)|$ must be cofinal in the value group for any $x \in S$. This shows that

$$S^{\mathrm{ad}}_\eta = S(|\varpi| \neq 0) = \cup_{m\in\mathbb{N}} S(|q|^{p^m} \leq |\varpi| \neq 0).$$

Let $B^+_m = \mathcal{O}^+(S(|q|^{p^m} \leq |\varpi| \neq 0))$, then as $q^{p^m}/\varpi \in B^+_m$, we have $(q, \varpi)^{p^m} \subseteq (\varpi)$ and the ring B^+_m thus has the ϖ-adic topology. From this one deduces that $B^+_m = \mathcal{O}_L\langle (q/\varpi^{1/m})^{1/p^\infty}\rangle$ and thus the spaces $S(|q|^{p^m} \leq |\varpi| \neq 0)$ and $\mathcal{D}_\infty(|q|^{p^m} \leq |\varpi| \neq 0)$ coincide. □

Remark 3.5. Note that $\mathcal{D}_\infty$ is not affinoid, even though it is the generic fibre of an affine formal scheme, as we did not equip $\mathcal{O}_L[\![q^{1/p^\infty}]\!]$ with the p-adic topology.

Definition 3.6. The origin in $\mathcal{D}_\infty$ is the closed point $x\colon \mathrm{Spa}(L, \mathcal{O}_L) \to \mathcal{D}_\infty$ where $q = 0$. By removing this point, we obtain a space $\mathring{\mathcal{D}}_\infty := \mathcal{D}_\infty \backslash \{x\}$ that satisfies $\mathring{\mathcal{D}}_\infty \sim \varprojlim \mathring{\mathcal{D}}_n$.

Definition 3.7. Let $\overline{\mathcal{D}}_\infty := \mathrm{Spa}(\mathcal{O}_L[\![q^{1/p^\infty}]\!]_p[\frac{1}{p}], \mathcal{O}_L[\![q^{1/p^\infty}]\!]_p)$ with the p-adic topology on $\mathcal{O}_L[\![q^{1/p^\infty}]\!]_p$ (see Def. 2.15). Then it is clear from the definition that $\overline{\mathcal{D}}_\infty \sim \varprojlim_{q\mapsto q^p} \overline{\mathcal{D}}$.

We are now ready to discuss cusps at infinite level and the corresponding Tate curve parameter spaces.

Proposition 3.8. *Fix a cusp x of $\mathcal{X}^*$, with corresponding cusps in the anti-canonical tower.*

1. *The open immersions $\mathcal{D}_n \hookrightarrow \mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$ over x in the limit $n \to \infty$ give rise to an open immersion $\mathcal{D}_\infty \hookrightarrow \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$ that fits into a Cartesian diagram*

$$\begin{array}{ccccc} \mathcal{D}_\infty & \hookrightarrow & \overline{\mathcal{D}}_\infty & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{D} & \hookrightarrow & \overline{\mathcal{D}} & \hookrightarrow & \mathcal{X}^*(\epsilon). \end{array}$$

2. *Consider the restriction* $\mathring{\mathcal{D}}_\infty \hookrightarrow \mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$. *For any perfectoid* K-*algebra* (R, R^+),

$$\mathring{\mathcal{D}}_\infty(R, R^+) \subseteq \mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a(R, R^+)$$

is in functorial bijection with isomorphism classes of triples $(E, \alpha_N, (q_E^{1/p^n})_{n\in\mathbb{N}})$ *where* $E \cong \mathrm{T}(q_E)$ *is a Tate curve over* R *for some topologically nilpotent unit* $q_E \in R$, *where* α_N *is a* Γ^p-*level structure, and* $(q_E^{1/p^n})_{n\in\mathbb{N}}$ *is a compatible system of* p^n-*th roots of* q_E, *defining an anticanonical* $\Gamma_0(p^\infty)$-*structure on* E.

Exactly as before, we also write $\mathcal{D}_{\infty,x} \hookrightarrow \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$ for the open immersion in the proposition if we wish to emphasize the cusp x we are working over.

Proof. The map $\mathcal{D}_\infty \to \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$ is induced from Proposition 2.22 and Proposition 3.4 by the universal property of the perfectoid tilde-limit. The outer square in part (1) is now Cartesian using [SW13, Prop 2.4.3], and the fact that the squares in diagram (7) are Cartesian.
It is clear that the left square is Cartesian. To show that the right square is as well, it now suffices to prove this away from the cusps, where it follows from the relative moduli interpretation: Giving an anticanonical $\Gamma_0(p^\infty)$-level structure on $\mathrm{T}(q)$ over some $\mathcal{O}_L[\![q]\!]$-algebra where q is invertible corresponds to giving a system of p^n-th roots of q.
The moduli interpretation of $\mathring{\mathcal{D}}_\infty$ follows from diagram (7) and Corollary 2.22. ☐

3.2 TATE CURVE PARAMETER SPACES OF $\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a$

Next, we discuss Tate curve parameter spaces for the pro-étale map $\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$. This is just a matter of pulling back the descriptions from finite level: Let

$$\mathcal{X}^*_{\Gamma_1(p^n)\cap\Gamma_0(p^\infty)}(\epsilon)_a := \mathcal{X}^*_{\Gamma_1(p^n)}(\epsilon)_a \times_{\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a} \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a.$$

LEMMA 3.9. *Let* x *be any cusp of* $\mathcal{X}^*$. *Let* $n \in \mathbb{Z}_{\geq 0}$.

1. *The map* $\mathcal{X}^*_{\Gamma_1(p^n)\cap\Gamma_0(p^\infty)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_1(p^n)}(\epsilon)_a$ *restricts to an isomorphism on the cusps.*

2. *There are canonical Cartesian cubes*

$$\begin{array}{ccccc}
& \underline{(\mathbb{Z}/p^{n+1}\mathbb{Z})^\times} \times \mathcal{D}_{n+1} & \to & \underline{(\mathbb{Z}/p^{n}\mathbb{Z})^\times} \times \mathcal{D}_{n} & \longrightarrow \ \mathcal{D}_n \\
\underline{(\mathbb{Z}/p^{n+1}\mathbb{Z})^\times} \times \mathcal{D}_\infty \nearrow & \downarrow \! \uparrow & \to \ \underline{(\mathbb{Z}/p^{n}\mathbb{Z})^\times} \times \mathcal{D}_\infty \nearrow & \downarrow \! \uparrow \longrightarrow \ \mathcal{D}_\infty \nearrow & \downarrow \\
\downarrow & \mathcal{X}^*_{\Gamma_1(p^{n+1})}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_1(p^{n})}(\epsilon)_a & \longrightarrow \ \mathcal{X}^*_{\Gamma_0(p^{n})}(\epsilon)_a \\
\mathcal{X}^*_{\Gamma_1(p^{n+1})\cap\Gamma_0(p^\infty)}(\epsilon)_a \nearrow & \rightarrowtail & \mathcal{X}^*_{\Gamma_1(p^{n})\cap\Gamma_0(p^\infty)}(\epsilon)_a \nearrow & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p^{\infty})}(\epsilon)_a. \nearrow
\end{array}$$

Proof. In part (2), the bottom faces are Cartesian by definition, the back faces are Cartesian by Lemma 2.23, the rightmost square is Cartesian by Proposition 3.8, and the top faces are clearly also Cartesian. Thus all other faces are Cartesian. Part (1) follows immediately. □

We now take the limit $n \to \infty$ to get Tate curve parameter spaces for $\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a$: In doing so, we need to account for the fact that in the inverse limit, the divisor of cusps becomes a profinite set of points, rather than just a disjoint union of closed points.

Definition 3.10. Let S be a profinite set, and let $(S_i)_{i\in I}$ be a system of finite sets with $S = \varprojlim_{i\in I} S_i$. Then we define $\underline{S}$ to be the unique perfectoid tilde-limit $\underline{S} \sim \varprojlim_{i\in I} \underline{S_i}$. This is independent of the choice of S_i: Explicitly, $\underline{S}$ is the affinoid perfectoid space

$$\underline{S} = \mathrm{Spa}(\mathrm{Map}_{\mathrm{cts}}(S, K), \mathrm{Map}_{\mathrm{cts}}(S, \mathcal{O}_K)).$$

Proposition 3.11. *Let x be a cusp of $\mathcal{X}^*$ with Tate curve parameter space $\mathcal{D}_{\infty,x} \hookrightarrow \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$. Then in the limit, the canonical open immersions $\underline{(\mathbb{Z}/p^n\mathbb{Z})^\times} \times \mathcal{D}_{\infty,x} \hookrightarrow \mathcal{X}^*_{\Gamma_1(p^n)\cap\Gamma_0(p^\infty)}(\epsilon)_a$ give rise to a $\mathbb{Z}_p^\times$-equivariant open immersion $\underline{\mathbb{Z}_p^\times} \times \mathcal{D}_{\infty,x} \hookrightarrow \mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a$ that fits into a Cartesian diagram*

$$\begin{array}{ccc} \underline{\mathbb{Z}_p^\times} \times \mathcal{D}_{\infty,x} & \longrightarrow & \mathcal{D}_{\infty,x} \\ \downarrow & & \downarrow \\ \mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a. \end{array}$$

Remark 3.12. Upon specialisation to the origin $\mathrm{Spa}(L, \mathcal{O}_L) \to \mathcal{D}_{\infty,x}$, this shows that the subspace of cusps of $\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a$ over x can be identified with $(\underline{\mathbb{Z}_p^\times})_L$, the base-change of the profinite adic space $\underline{\mathbb{Z}_p^\times}$ to L. In particular, for any $a \in \mathbb{Z}_p^\times$, specialisation at a gives rise to a locally closed immersion $\mathcal{D}_{\infty,x} \hookrightarrow \mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a$ but in contrast to the case of $\Gamma_0(p^\infty)$, this is not going to be an open immersion due to the non-trivial topology on the cusps.

Proof. This follows in the inverse limit over the front of the cubes in Lemma 3.9.2, since

$$\underline{\mathbb{Z}_p^\times} \times \mathcal{D}_{\infty,x} \sim \varprojlim \underline{(\mathbb{Z}/p^n\mathbb{Z})^\times} \times \mathcal{D}_{\infty,x}. \tag{8}$$

That this holds is easy to verify, see for example [BGH+22, Lemma 2.8]. □

3.3 Tate curve parameter spaces of $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$

Next, we look at the Tate curve parameter spaces in the pro-étale map $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a$. As before, we do so by looking at the limit of the finite morphisms

$$\mathcal{X}^*_{\Gamma(p^n)\cap\Gamma_0(p^\infty)} := \mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a \times_{\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a} \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a.$$

Combining the moduli descriptions of $\mathcal{X}_{\Gamma(p^n)}$ and Lemma 3.1, we see:

Lemma 3.13. *Let (R, R^+) be a perfectoid K-algebra. Then $\mathcal{X}_{\Gamma(p^n)\cap\Gamma_0(p^\infty)}(\epsilon)_a(R, R^+)$ is in functorial bijection with isomorphism classes of tuples $(E, \alpha_N, G, \beta_n)$ with E, α_N, G as in Lemma 3.1 and β_n an isomorphism $(\mathbb{Z}/p^n\mathbb{Z})^2 \to E[p^n]$ such that $\beta_n(1,0)$ generates G_n.*

We have the following description of the cusps of $\mathcal{X}^*_{\Gamma(p^m)\cap\Gamma_0(p^\infty)}(\epsilon)_a$:

Lemma 3.14. *Let x be a cusp of $\mathcal{X}^*$.*

1. *The map $\mathcal{X}^*_{\Gamma(p^m)\cap\Gamma_0(p^\infty)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma(p^m)}(\epsilon)_a$ induces an isomorphism on cusps. The cusps of $\mathcal{X}^*_{\Gamma(p^m)\cap\Gamma_0(p^\infty)}(\epsilon)_a$ over x are thus parametrised by $\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})$ where $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ corresponds to the ordered basis $(q^{d/p^n}, \zeta_{p^n}^a q^{-b/p^n})$ of $\mathrm{T}(q)[p^n]$.*
2. *There is a Cartesian diagram*

$$\begin{array}{ccc} \underline{\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})} \times \mathcal{D}_{\infty,x} & \longrightarrow & \underline{(\mathbb{Z}/p^n\mathbb{Z})^\times} \times \mathcal{D}_{\infty,x} \\ \big\downarrow & & \big\downarrow \\ \mathcal{X}^*_{\Gamma(p^n)\cap\Gamma_0(p^\infty)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_1(p^n)\cap\Gamma_0(p^\infty)}(\epsilon)_a \end{array}$$

where the top left map is given by $(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right), q) \mapsto (d, q)$.

Proof. Part (1) follows from Lemma 2.25 and Proposition 3.8 exactly like in Lemma 3.9. Part (2) follows from a similar Cartesian cube using the left square in Proposition 2.26 and Lemma 3.9. □

Definition 3.15. Let $\Gamma_0(p^\infty) = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ 0 & \mathbb{Z}_p^\times \end{pmatrix}$ be the subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$ of upper triangular matrices. This is a profinite group with $\Gamma_0(p^\infty) = \varprojlim_n \Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})$.

Lemma 3.16. *Let (R, R^+) be a perfectoid K-algebra. Then $\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a(R, R^+)$ is in functorial bijection with isomorphism classes of triples (E, α_N, β) of an ϵ-nearly ordinary elliptic curve E over R, together with a Γ^p-structure α_N, and an isomorphism of p-divisible groups $\beta\colon (\mathbb{Q}_p/\mathbb{Z}_p)^2 \to E[p^\infty]$ over R (or equivalently an isomorphism $\mathbb{Z}_p^2 \to T_pE(R)$) such that the restriction of β to the first factor is an anti-canonical $\Gamma_1(p^\infty)$-structure.*

Proof. This is an immediate consequence of Lemma 3.13 and [SW13, Proposition 2.4.5]. □

We are now ready to give the main result of this section, which summarises the discussion so far and moreover describes the cusps of $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$. For the statement, let us recall that for any n, the Tate curve $\mathrm{T}(q)$ over $\mathcal{D}_\infty$ has an anticanonical ordered basis for $\mathrm{T}(q)[p^n]$ given by $(q^{1/p^n}, \zeta_{p^n})$. In particular, an anticanonical ordered basis of the Tate module $T_p\mathrm{T}(q)$ is given by the compatible system $(q^{1/p^n})_{n\in\mathbb{N}}$ that we denote by q^{1/p^∞}, and the chosen compatible system $(\zeta_{p^n})_{n\in\mathbb{N}}$, that we now denote by ζ_{p^∞}.

Theorem 3.17. *Let x be a cusp of $\mathcal{X}^*$ with corresponding morphism $\mathcal{D}_x \hookrightarrow \mathcal{X}^*$.*

1. *We have a tower of Cartesian squares*

$$\begin{array}{ccccccc} \underline{\Gamma_0(p^\infty)} \times \mathcal{D}_{\infty,x} & \longrightarrow & \underline{\mathbb{Z}_p^\times} \times \mathcal{D}_{\infty,x} & \longrightarrow & \mathcal{D}_{\infty,x} & \longrightarrow & \mathcal{D}_x \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*(\epsilon). \end{array}$$

   where the top left map is given by $\left(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right), q\right) \mapsto (d, q)$.

2. *For any $\gamma = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \Gamma_0(p^\infty)$, the cusp of $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ obtained by specialising $\underline{\Gamma_0(p^\infty)} \times \mathcal{D}_{\infty,x} \hookrightarrow \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ at γ is the one corresponding to the isomorphism $\mathbb{Z}_p^2 \to T_p\mathrm{T}(q)$ defined by the ordered basis $(q^{d/p^\infty}, \zeta_{p^\infty}^a q^{-b/p^\infty})$ of $T_p\mathrm{T}(q)$.*

Proof. 1. The only statement we have not yet proved is that the left square is Cartesian. But this follows from Lemma 3.14.(2) in the limit $n \to \infty$. Here we use that

$$\underline{\Gamma_0(p^\infty)} \times \mathcal{D}_{\infty,x} \sim \varprojlim_{n\in\mathbb{N}} \underline{\Gamma_0(p^n, \mathbb{Z}/p^n)} \times \mathcal{D}_{n,x}$$

as well as the analogous statement from (8), which hold by the same argument.

2. This follows from Lemma 3.14.(1) in the limit. □

We note the following easy consequence. The analogue of this for Siegel moduli spaces for dimension $g > 1$ is shown in the proof of [Sch15, Lemma 3.2.35].

Corollary 3.18. *For any $n \in \mathbb{N} \cup \{\infty\}$, our choice of ζ_{p^n} induces a canonical isomorphism*

$$\mathcal{X}^*_{\Gamma(p^n)}(0)_a = \bigsqcup_{\Gamma(p^n)/\Gamma_1(p^n)} \mathcal{X}^*_{\Gamma_1(p^n)}(0)_a.$$

Proof. For $n = \infty$, there is away from the cusps a canonical section induced on moduli interpretations as follows: Let E be any ordinary elliptic curve over an adic space S over K, let C be its canonical subgroup and let G be an anticanonical $\Gamma_0(p^\infty)$-level structure on E. Then $T_pE = T_pC \times T_pG$ and there is a canonical isomorphism $T_pC = T_pG^\vee$ induced by the Weil pairing. Thus any trivialisation of T_pG induces a trivialisation of T_pE. On Tate curve parameter spaces, one checks that this splitting is given by the map

$$\underline{\mathbb{Z}_p^\times} \times \mathring{\mathcal{D}}_\infty \to \underline{\Gamma_0(p^\infty)} \times \mathring{\mathcal{D}}_\infty, \quad (a, q) \mapsto \left(\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right), q\right).$$

This clearly extends over the puncture. Similarly for $n < \infty$. □

3.4 The action of $\Gamma_0(p)$ on the cusps of $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$

Next, we discuss the full action of $\Gamma_0(p)$ on the Tate curve parameter spaces at infinite level.

Since the full $\mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$-action on each $\mathcal{X}^*_{\Gamma(p^n)}$ restricts to a $\Gamma_0(p,\mathbb{Z}/p^n\mathbb{Z})$-action on $\mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a$ as discussed in Proposition 2.27, we see that the $\mathrm{GL}_2(\mathbb{Z}_p)$-action on $\mathcal{X}^*_{\Gamma(p^\infty)}$ restricts to an action of $\Gamma_0(p) = \varprojlim_n \Gamma_0(p,\mathbb{Z}/p^n\mathbb{Z})$ on $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$. In order to describe this explicitly, it is convenient to work in the category $\mathbf{Perf}_K$ of perfectoid spaces over K.

Proposition 3.19. *Over any cusp x of $\mathcal{X}^*$, the $\Gamma_0(p)$-action on $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ restricts to an action on $\underline{\Gamma_0(p^\infty)} \times \mathcal{D}_\infty \hookrightarrow \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ where it can be described as follows: Equip $\underline{\Gamma_0(p)} \times \mathcal{D}_\infty$ with the right action by $\underline{p\mathbb{Z}_p}$ via $(\gamma, q^{1/p^m}) \cdot h := (\gamma \left(\begin{smallmatrix} 1 & 0 \\ h & 1 \end{smallmatrix}\right), \zeta_{p^m}^{h/e_x} q^{1/p^m})$ for $h \in \underline{p\mathbb{Z}_p}$, then*

$$(\underline{\Gamma_0(p)} \times \mathcal{D}_\infty)/\underline{p\mathbb{Z}_p} = \underline{\Gamma_0(p^\infty)} \times \mathcal{D}_\infty$$

as sheaves on $\mathbf{Perf}_K$ and the left action of $\Gamma_0(p)$ is the one induced by letting $\Gamma_0(p)$ act on the first factor of $\underline{\Gamma_0(p)} \times \mathcal{D}_\infty$. Explicitly, in terms of any $\gamma_1 \in \Gamma_0(p)$, the action is given by

$$\begin{aligned} \gamma_1 : \underline{\Gamma_0(p^\infty)} \times \mathcal{D}_\infty &\xrightarrow{\sim} \underline{\Gamma_0(p^\infty)} \times \mathcal{D}_\infty \\ \gamma_2, q^{1/p^m} &\mapsto \begin{pmatrix} \det(\gamma_3)/d_3 & b_3 \\ 0 & d_3 \end{pmatrix}, \zeta_{p^m}^{-\frac{c_3}{d_3 e_x}} q^{1/p^m}. \end{aligned}$$

where $\gamma_3 = \left(\begin{smallmatrix} a_3 & b_3 \\ c_3 & d_3 \end{smallmatrix}\right) := \gamma_1 \cdot \gamma_2$.

Proof. That the action restricts to an action on $\underline{\Gamma_0(p^\infty)} \times \mathcal{D}_\infty$ is a consequence of Proposition 2.27 in the limit over n. The same argument gives the explicit formula.

It remains to verify the isomorphism of sheaves. For this we check that the following diagram commutes, where to ease notation, let $\Gamma_m := \Gamma_0(p^m, \mathbb{Z}/p^m\mathbb{Z})$ and $\Gamma'_m := \Gamma_0(p, \mathbb{Z}/p^m\mathbb{Z})$:

$$\begin{array}{ccccccc} \underline{\Gamma'_{n+1}} \times \mathcal{D}_{n+1} & \to & \underline{\Gamma_{n+1}} \times \mathcal{D}_{n+1}, & \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), q^{1/p^{n+1}} & \mapsto & \begin{pmatrix} \det(\gamma)/d & b \\ 0 & d \end{pmatrix}, \zeta_{p^{n+1}}^{-\frac{c}{de_x}} q^{1/p^{n+1}} \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ \underline{\Gamma'_n} \times \mathcal{D}_n & \longrightarrow & \underline{\Gamma_n} \times \mathcal{D}_n, & \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), q^{1/p^n} & \longmapsto & \begin{pmatrix} \det(\gamma)/d & b \\ 0 & d \end{pmatrix}, \zeta_{p^n}^{-\frac{c}{de_x}} q^{1/p^n} \end{array}$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ (we emphasize that on the right we describe the maps in terms of *points* rather than *functions*). This diagram is $p\mathbb{Z}_p$-equivariant when we endow the spaces on the left with the $p\mathbb{Z}_p$-actions from Proposition 2.27, and the spaces on the right with the trivial $p\mathbb{Z}_p$-action. The diagram is moreover equivariant for the $\Gamma_0(p)$-action on the left via the natural reduction maps. In the limit we therefore obtain a $p\mathbb{Z}_p$-invariant morphism

$$\underline{\Gamma_0(p)} \times \mathcal{D}_\infty \to \underline{\Gamma_0(p^\infty)} \times \mathcal{D}_\infty,$$

equivariant for the $\Gamma_0(p)$-action via the first factor on the left, and the action described in the proposition on the right. This induces a morphism of sheaves

$$(\underline{\Gamma_0(p)} \times \mathcal{D}_\infty)/\underline{p\mathbb{Z}_p} \to \underline{\Gamma_0(p^\infty)} \times \mathcal{D}_\infty.$$

On the other hand, the inclusion $\Gamma_0(p^\infty) \subseteq \Gamma_0(p)$ defines an inverse of this map. □

3.5 The Hodge–Tate period map on Tate curve parameter spaces

Next, we give an explicit description of the restriction of Scholze's Hodge–Tate period map π_{HT} [Sch15, §3.3] to the Tate curve parameter spaces.
Recall that over the ordinary locus, the kernel of the Hodge–Tate map $T_pE \to \omega_E$ is the Tate module T_pC of the canonical p-divisible subgroup, and thus the Hodge–Tate filtration is given by $T_pC \subseteq T_pE$. In particular, this means that $\pi_{\mathrm{HT}}(\mathcal{X}^*_{\Gamma(p^\infty)}(0)) \subseteq \mathbb{P}^1(\mathbb{Z}_p)$.
When we further restrict to the anticanonical locus, the image lies in the points of the form $(a:1) \in \mathbb{P}^1(\mathbb{Z}_p)$ with $a \in \mathbb{Z}_p$. Denote by $B_1(0) \subseteq \mathbb{P}^1(\mathbb{Z}_p)$ the ball of radius 1 inside the canonical chart $\mathbb{A}^1 \subseteq \mathbb{P}^1$ around $(0:1)$, then the Hodge–Tate period map thus restricts to

$$\pi_{\mathrm{HT}}(\mathcal{X}^*_{\Gamma(p^\infty)}(0)_a) \subseteq B_1(0) \subseteq \mathbb{P}^1(\mathbb{Z}_p).$$

Proposition 3.20. *Let x be a cusp of $\mathcal{X}^*$. Then the Hodge–Tate period map $\pi_{\mathrm{HT}}\colon \mathcal{X}^*_{\Gamma(p^\infty)} \to \mathbb{P}^1$ restricts on $(\underline{\Gamma_0(p)} \times \mathcal{D}_{\infty,x})/\underline{p\mathbb{Z}_p} \hookrightarrow \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ to the map*

$$(\underline{\Gamma_0(p)} \times \mathcal{D}_{\infty,x})/\underline{p\mathbb{Z}_p} \to \underline{\mathbb{P}^1(\mathbb{Z}_p)} \subseteq \mathbb{P}^1, \quad \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), q\right) \mapsto (b:d).$$

We deduce this from the following lemma:

Lemma 3.21. *Let $f\colon \mathcal{D}_\infty \to \mathbb{A}^1_K$ be a function such that f is constant on $(C, \mathcal{O}_C)$-points with value $a \in L$. Then the corresponding $f \in \mathcal{O}(\mathcal{D}_\infty)$ is the constant $a \in L \subseteq \mathcal{O}(\mathcal{D}_\infty)$.*

Proof. It suffices to prove this for the spaces $\mathcal{D}_\infty(|q| \leq \varpi^n)$. After rescaling, we are reduced to showing the lemma for $\mathcal{D}_\infty$ replaced by $\mathrm{Spa}(L\langle q^{1/p^\infty}\rangle, \mathcal{O}_L\langle q^{1/p^\infty}\rangle)$. One can now argue like in the classical proof of the maximum principle: We can regard f as a function

$$f \in L\langle q^{1/p^\infty}\rangle, \quad f = \sum_{m \in \mathbb{Z}[\frac{1}{p}]_{\geq 0}} a_m q^m.$$

We need to prove that if $f((x^{1/p^i})_{i\in\mathbb{N}}) = a$ for all $(x^{1/p^i})_{i\in\mathbb{N}} \in \varprojlim_{x\mapsto x^p} \mathcal{O}_C$ then $f = a$.

After subtracting by $a = a_0$, we may assume that $f(x) = 0$ for all $x \in \mathcal{O}_C$. Suppose $f \neq 0$. The convergence condition on coefficients ensures that $\sup_{m\in\mathbb{Z}[\frac{1}{p}]} |a_m| > 0$ is attained and after renormalising we may assume that $|f| = \max_{m\in\mathbb{Z}[\frac{1}{p}]} |a_m| = 1$. Consider the reduction

$$r \colon \mathcal{O}_L\langle q^{1/p^\infty}\rangle \to k_L[q^{1/p^n}|n \in \mathbb{N}]$$

modulo $\mathfrak{m}_L \subseteq \mathcal{O}_L$. After replacing $q \mapsto q^{p^m}$ we may assume that $r(f) \in k_L[q]$. As $\mathcal{O}_C$ is perfectoid, the projection $\varprojlim \mathcal{O}_C \to \mathcal{O}_C \to k_C$ to the residue field is surjective, and the assumption on f now implies that $r(f)$ is a non-zero polynomial in $k_C[q]$ which evaluates to 0 on all $q \in k_C$, a contradiction as k_C is infinite. ☐

Proof of Proposition 3.20. We use Lemma 3.21 to see that for any $\gamma \in \Gamma_0(p^\infty)$, the map

$$\mathcal{D}_\infty \xrightarrow{q\mapsto(\gamma,q)} \underline{\Gamma_0(p^\infty)} \times \mathcal{D}_{\infty,x} \hookrightarrow \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a \xrightarrow{\pi_{\mathrm{HT}}} \mathbb{P}^1$$

is constant with image b/d. To check this on $(C, \mathcal{O}_C)$-points, we use the moduli description:

On the ordinary locus, π_{HT} sends any isomorphism $\mathbb{Z}_p^2 \to T_pE$ to the point of $\mathbb{P}^1(\mathbb{Z}_p)$ defined by the line $T_pC \subseteq T_pE$ where C is the canonical p-divisible subgroup. By Theorem 3.17.(2), any $(C, \mathcal{O}_C)$-point of $\mathcal{D}_\infty \xrightarrow{q\mapsto(\gamma,q)} \underline{\Gamma_0(p^\infty)} \times \mathcal{D}_\infty$ corresponds to a Tate curve $E = \mathrm{T}(q_E)$ with an ordered basis of T_pE given by $(e_1, e_2) = (q_E^{d/p^\infty}, \zeta_{p^\infty}^a q_E^{-b/p^\infty})$. Then (using additive notation on T_pE)

$$be_1 + de_2 = q_E^{bd/p^\infty} \zeta_{p^\infty}^{ad} q_E^{-db/p^\infty} = \zeta_{p^\infty}^{ad}$$

which spans the line $\langle\zeta_{p^\infty}\rangle = T_pC \subseteq T_pE$. Consequently, the image of (γ, q) under π_{HT} is

$$\pi_{\mathrm{HT}}(\gamma, q) = (b : d) = (b/d : 1) \in \underline{(\mathbb{Z}_p^\times : 1)} \subseteq \mathbb{P}^1(\mathbb{Z}_p).$$

We conclude from this that the function $f \in \mathrm{Map}_{\mathrm{cts}}(\Gamma_0(p^\infty), \mathcal{O}(\mathcal{D}_\infty))$ defined by the restriction $\pi_{\mathrm{HT}} \colon \underline{\Gamma_0(p^\infty)} \times \mathcal{D}_\infty \to B(0)$ evaluates at γ to $f(\gamma) = b/d$. Since this is true for all $\gamma \in \Gamma_0(p^\infty)$, we see that f is given by a function in

$$\mathrm{Map}_{\mathrm{cts}}(\Gamma_0(p^\infty), \mathbb{Z}_p^\times) \subseteq \mathrm{Map}_{\mathrm{cts}}(\Gamma_0(p^\infty), \mathcal{O}(\mathcal{D}_\infty)).$$

We conclude that π_{HT} has the desired description

$$\underline{\Gamma_0(p^\infty)} \times \mathcal{D}_\infty \xrightarrow{(\gamma,q)\mapsto b/d} \underline{\mathbb{Z}_p^\times} = \underline{(\mathbb{Z}_p^\times : 1)} \subseteq \mathbb{P}^1(\mathbb{Z}_p) \hookrightarrow \mathbb{P}^1, \quad (\gamma, q) \mapsto (b/d : 1).$$

☐

3.6 Tate curve parameter spaces of the modular curve at infinite level

Finally, as a consequence of the above, we can now consider the entire modular curve $\mathcal{X}^*_{\Gamma(p^\infty)}$. Recall that by the very construction in [Sch15], this is the space $\mathrm{GL}_2(\mathbb{Q}_p)\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ defined by glueing translates of $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$. This allows us to deduce parts (2)-(3) of Theorem 1.1:

Theorem 3.22. *Let $x \in \mathcal{X}^*$ be any cusp. Define a right action of $\underline{\mathbb{Z}_p}$ on $\underline{\mathrm{GL}_2(\mathbb{Z}_p)} \times \mathcal{D}_{\infty,x}$ by letting $h \in \underline{\mathbb{Z}_p}$ act as*

$$(\gamma, q^{1/p^n}) \cdot h := (\gamma \left(\begin{smallmatrix} 1 & 0 \\ h & 1 \end{smallmatrix}\right), q^{1/p^n}\zeta_{p^n}^{h/e_x}).$$

Then the sheaf quotient $(\underline{\mathrm{GL}_2(\mathbb{Z}_p)} \times \mathcal{D}_{\infty,x})/\underline{\mathbb{Z}_p}$ on $\mathbf{Perf}_K$ is represented by a perfectoid space, and there is a Cartesian diagram

$$\begin{array}{ccc} (\underline{\mathrm{GL}_2(\mathbb{Z}_p)} \times \mathcal{D}_{\infty,x})/\underline{\mathbb{Z}_p} & \longrightarrow & \mathcal{D}_x \\ \big\downarrow & & \big\downarrow \\ \mathcal{X}^*_{\Gamma(p^\infty)} & \longrightarrow & \mathcal{X}^* \end{array}$$

where the top map is induced by the projection from the second factor. The left map is $\mathrm{GL}_2(\mathbb{Z}_p)$-equivariant for the left action on $(\underline{\mathrm{GL}_2(\mathbb{Z}_p)} \times \mathcal{D}_{\infty,x})/\underline{\mathbb{Z}_p}$ via the first factor. Under this description, the fibres of the canonical and anticanonical locus are precisely

$$\left(\underline{\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}} \times \mathcal{D}_{\infty,x}\right)/\underline{\mathbb{Z}_p} \quad \hookrightarrow \quad \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a, \tag{9}$$

$$\left(\underline{\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p^\times \\ \mathbb{Z}_p^\times & p\mathbb{Z}_p \end{pmatrix}} \times \mathcal{D}_{\infty,x}\right)/\underline{\mathbb{Z}_p} \quad \hookrightarrow \quad \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_c. \tag{10}$$

Proof. We need to translate the $\Gamma_0(p)$-equivariant open immersion from Prop 3.19

$$(\Gamma_0(p) \times \mathcal{D}_{\infty,x})/\underline{p\mathbb{Z}_p} \hookrightarrow \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$$

according to the $\mathrm{GL}_2(\mathbb{Z}_p)$-action on the right hand side.
We first rewrite the left hand side: We have

$$\Gamma_0(p) \left(\begin{smallmatrix} 1 & 0 \\ \mathbb{Z}_p & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p^\times \end{smallmatrix}\right),$$

and by extending the $p\mathbb{Z}_p$-action to a $\mathbb{Z}_p$-action in the natural way, we get the equivalent description of anticanonical Tate curve parameter spaces stated in (9) in the theorem.
Next, we note that we may without loss of generality replace $\mathcal{X}^*_{\Gamma(p^\infty)}$ by

$$\mathcal{X}^*_{\Gamma(p^\infty)}(0) = \mathcal{X}^*_{\Gamma(p^\infty)}(0)_a \sqcup \mathcal{X}^*_{\Gamma(p^\infty)}(0)_c.$$

To simplify the discussion of translates, we introduce an auxiliary open subspace

$$\mathcal{X}^*_{\Gamma(p^\infty)}(0)^c_a \subseteq \mathcal{X}^*_{\Gamma(p^\infty)}(0)_a.$$

parametrising isomorphisms $\alpha\colon \mathbb{Z}_p^2 \to T_pE$ such that $\alpha(0,1) \bmod p$ generates the canonical subgroup ("first basis vector anticanonical, second canonical"). More precisely, this subspace can be constructed as follows: According to Corollary 3.18, there is a canonical splitting

$$\mathcal{X}_{\Gamma_1(p)}(0)_a \to \mathcal{X}_{\Gamma(p)}(0)_a$$

that identifies the image with a component of $\mathcal{X}_{\Gamma(p)}(0)_a$. Let $\mathcal{X}^*_{\Gamma(p)}(0)^c_a$ be the finite union of the $\left(\begin{smallmatrix}(\mathbb{Z}/p\mathbb{Z})^\times & 0\\ 0 & 1\end{smallmatrix}\right)$-translates of the image. Then $\mathcal{X}^*_{\Gamma(p^\infty)}(0)^c_a$ is defined as the pullback

$$\begin{array}{ccc}\mathcal{X}^*_{\Gamma(p^\infty)}(0)^c_a & \longrightarrow & \mathcal{X}^*_{\Gamma(p)}(0)^c_a\\ \cap & & \cap\\ \mathcal{X}^*_{\Gamma(p^\infty)}(0)_a & \longrightarrow & \mathcal{X}^*_{\Gamma(p)}(\epsilon)_a.\end{array}$$

It is clear from this definition that $\mathcal{X}^*_{\Gamma(p^\infty)}(0)^c_a$ defines an open and closed subspace of $\mathcal{X}^*_{\Gamma(p^\infty)}(0)_a$. By tracing the Tate curve parameter spaces through the construction, we moreover see that their fibre over $\mathcal{X}^*_{\Gamma(p^\infty)}(0)^c_a$ is

$$\left(\underline{\begin{pmatrix}\mathbb{Z}_p^\times & p\mathbb{Z}_p\\ \mathbb{Z}_p & \mathbb{Z}_p^\times\end{pmatrix}} \times \mathcal{D}_{\infty,x}\right)/\underline{\mathbb{Z}_p} \hookrightarrow \mathcal{X}^*_{\Gamma(p^\infty)}(0)^c_a \tag{11}$$

We can now identify $\mathcal{X}^*_{\Gamma(p^\infty)}(0)_a$ with the finite union of translates

$$\left(\begin{smallmatrix}1 & \mathbb{Z}_p\\ 0 & 1\end{smallmatrix}\right)\mathcal{X}^*_{\Gamma(p^\infty)}(0)^c_a = \mathcal{X}^*_{\Gamma(p^\infty)}(0)_a. \tag{12}$$

Indeed, away from the cusps this follows on moduli functors using Lemma 3.16, whereas over the cusps it follows from the above explicit description using that

$$\left(\begin{smallmatrix}1 & \mathbb{Z}_p\\ 0 & 1\end{smallmatrix}\right)\begin{pmatrix}\mathbb{Z}_p^\times & p\mathbb{Z}_p\\ \mathbb{Z}_p & \mathbb{Z}_p^\times\end{pmatrix} = \begin{pmatrix}\mathbb{Z}_p & \mathbb{Z}_p\\ \mathbb{Z}_p & \mathbb{Z}_p^\times\end{pmatrix}.$$

On the other hand, inside $\mathcal{X}^*_{\Gamma(p^\infty)}$ we have an identification

$$\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)\mathcal{X}^*_{\Gamma(p^\infty)}(0)^c_a = \mathcal{X}^*_{\Gamma(p^\infty)}(0)_c. \tag{13}$$

Indeed, one can first check this for $\mathcal{X}^*_{\Gamma(p)}(0)_c$ on moduli functors, extend to compactifications, and then pull back to infinite level. On Tate curve parameter spaces, the identity

$$\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)\begin{pmatrix}\mathbb{Z}_p^\times & p\mathbb{Z}_p\\ \mathbb{Z}_p & \mathbb{Z}_p^\times\end{pmatrix} = \begin{pmatrix}\mathbb{Z}_p & \mathbb{Z}_p^\times\\ \mathbb{Z}_p^\times & p\mathbb{Z}_p\end{pmatrix}$$

therefore gives the desired open immersion onto a neighbourhood of the cusps over x

$$\left(\underline{\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p^\times \\ \mathbb{Z}_p^\times & p\mathbb{Z}_p \end{pmatrix}} \times \mathcal{D}_{\infty,x}\right)/\mathbb{Z}_p \hookrightarrow \mathcal{X}^*_{\Gamma(p^\infty)}(0)_c.$$

Taking the disjoint union of the morphisms in (9) and (10), we get the desired description.

To check $\mathrm{GL}_2(\mathbb{Z}_p)$-equivariance, we note that every $\gamma \in \mathrm{GL}_2(\mathbb{Z}_p)$ can be decomposed into $\gamma_1 \cdot \gamma_2$ where $\gamma_2 \in \begin{pmatrix} \mathbb{Z}_p^\times & p\mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$ and either $\gamma_1 \in \left(\begin{smallmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$ or $\gamma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Since the open immersion in (11) is $\begin{pmatrix} \mathbb{Z}_p^\times & p\mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$-equivariant, it thus suffices to check this for γ_1, for which this follows from equivariance in the anticanonical case (12), and glueing in the canonical case (13). □

Proof of Theorem 1.1.(3). This follows from Proposition 3.20 by $\mathrm{GL}_2(\mathbb{Z}_p)$-equivariance of π_{HT}. □

4 Modular curves in characteristic p

We now switch to moduli spaces in characteristic p. We start by recalling the general setup: Let R be any $\mathbb{F}_p$-algebra. Recall from §2 that X_R denotes the modular curve over R of tame level Γ^p, and X_R^* denotes its compactification. We write $X_{R,\mathrm{ord}} \subseteq X_R$ for the affine open subscheme where the Hasse invariant Ha is invertible. Similarly, one defines $X^*_{R,\mathrm{ord}} \subseteq X_R^*$ which is also an affine open subspace.

In this section, we consider analytic modular curves over the perfectoid field $K^\flat$, the tilt of K. We fix a pseudo-uniformiser $\varpi^\flat$ such that $\varpi^{\flat\sharp} = \varpi$. Following the notational conventions in [Sch15], we shall denote modular curves over $K^\flat$ with a prime, e.g. $X' := X_{K^\flat}$ and $X'^* := X^*_{K^\flat}$, to distinguish them from the modular curves over K.

Let $\mathfrak{X}'$ be the $\varpi^\flat$-adic completion of $X_{\mathcal{O}_{K^\flat}}$ and let $\mathcal{X}'$ be the analytification of X' over $\mathrm{Spa}(K^\flat, \mathcal{O}_{K^\flat})$. We analogously define $\mathfrak{X}'^*$ and $\mathcal{X}'^*$. Like in characteristic 0, for $0 \leq \epsilon < 1/2$ such that $|\varpi^\flat|^\epsilon \in |K^\flat|$, we denote by $\mathcal{X}'^*(\epsilon)$ the open subspace of $\mathcal{X}'^*$ where $|\mathrm{Ha}| \geq |\varpi^\flat|^\epsilon$. Like before, this has a canonical formal model $\mathfrak{X}'^*(\epsilon) \to \mathfrak{X}'^*$. For any adic space $\mathcal{Y} \to \mathcal{X}'^*$ we write $\mathcal{Y}(\epsilon) := \mathcal{Y} \times_{\mathcal{X}'^*} \mathcal{X}'^*(\epsilon)$. Finally, let $\mathcal{X}'^*_{\mathrm{ord}}$ be the analytification of $X'^*_{\mathrm{ord}} = X^*_{K^\flat,\mathrm{ord}}$.

Remark 4.1. We recall that while the elliptic curves parametrised by $\mathcal{X}'(\epsilon)$ may have good supersingular *reduction*, the condition on the Hasse invariant ensures that *generically*, these elliptic curves are always ordinary. In other words, $\mathcal{X}'(\epsilon) \subseteq \mathcal{X}'_{\mathrm{ord}}$ even for $\epsilon > 0$.

4.1 Igusa curves

In characteristic p, one has the Igusa moduli problem:

Definition 4.2 ([KM85], Def. 12.3.1). Let S be a scheme of characteristic p and let E be an elliptic curve over S. Consider the Verschiebung morphism $V^n\colon E^{(p^n)} \to E$. An Igusa structure on E of level p^n is a group homomorphism $\phi\colon \mathbb{Z}/p^n\mathbb{Z} \to E^{(p^n)}(S)$ that is a Drinfeld generator of $\ker V^n$. This means that the Cartier divisor

$$\sum_{a\in\mathbb{Z}/p^n\mathbb{Z}} [\phi(a)] \subseteq E^{(p^n)}$$

coincides with $\ker V^n$.

The Igusa problem $[\mathrm{Ig}(p^n)]$ is the moduli problem defined by the functor sending $E|S$ to the set of Igusa structures on E of level p^n. If $E|S$ is ordinary, the group scheme $\ker V^n$ is étale and naturally isomorphic to the Cartier dual $C_n^\vee$ of the canonical subgroup $C_n \subseteq E[p^n]$. In particular, in this situation, an $\mathrm{Ig}(p^n)$-structure is the same as an isomorphism of group schemes

$$\underline{\mathbb{Z}/p^n\mathbb{Z}} \xrightarrow{\sim} C_n^\vee,$$

or equivalently, an isomorphism of the Cartier duals $\mu_{p^n} \xrightarrow{\sim} C_n$.

For any $n \geq 0$, the Igusa problem $[\mathrm{Ig}(p^n)]$ is relatively representable, finite and flat of degree $\varphi(p^n)$ over the stack $\mathbf{Ell}|R$ of elliptic curves over R by [KM85, Theorem 12.6.1]. In particular, the simultaneous moduli problem $[\mathrm{Ig}(p^n), \Gamma^p]$ is representable by a moduli scheme $X_{R,\mathrm{Ig}(p^n)}$ over R. The forgetful map $X_{R,\mathrm{Ig}(p^n)} \to X_R$ is finite and flat, and is an étale $(\mathbb{Z}/p^n\mathbb{Z})^\times$-torsor over the ordinary locus $X_{R,\mathrm{ord}} \subseteq X_R$. One defines by normalisation a compactification $X^*_{R,\mathrm{Ig}(p^n)}$. The morphism $X_{R,\mathrm{Ig}(p^n)} \to X_R$ then extends to

$$X^*_{R,\mathrm{Ig}(p^n)} \to X^*_R$$

which is still finite étale Galois with group $(\mathbb{Z}/p^n\mathbb{Z})^\times$ over the ordinary locus. For any map $\mathrm{Spec}(R'(\!(q)\!)) \to X_R$ corresponding to a choice of Γ^p-structure on $\mathrm{T}(q^e)$ over some cyclotomic extension R' of R and for some $1 \leq e \leq N$, the canonical isomorphism

$$\mu_{p^n} \xrightarrow{\sim} C_n(\mathrm{T}(q^e)) \subseteq \mathrm{T}(q^e)[p^n]$$

induces a canonical lifting to a map $\mathrm{Spec}(R'(\!(q)\!)) \to X_{R,\mathrm{Ig}(p^n)}$. In particular, over any cusp x of X^*_R, the subscheme of cusps of $X^*_{R,\mathrm{Ig}(p^n)}$ consists of $\varphi(p^n)$ disjoint copies of x.

4.2 Tate curve parameter spaces in the Igusa tower

Returning to our analytic setting over $K^\flat$, we let $X'^*_{\mathrm{Ig}(p^n)} := X^*_{K^\flat,\mathrm{Ig}(p^n)}$. We write $\mathfrak{X}^*_{\mathrm{Ig}(p^n)}$ for the $\varpi^\flat$-adic completion of $X^*_{\mathcal{O}_{K^\flat},\mathrm{Ig}(p^n)}$ and we write $\mathcal{X}'^*_{\mathrm{Ig}(p^n)}$ for the analytification of $X'^*_{\mathrm{Ig}(p^n)}$. We then get an open subspace $\mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon)$. Since $\mathcal{X}'^*(\epsilon) \subseteq \mathcal{X}'^*_{\mathrm{ord}}$, the morphism $\mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon) \to \mathcal{X}'^*(\epsilon)$ is a finite étale $(\mathbb{Z}/p^n\mathbb{Z})^\times$-torsor. Like in Lemma 2.3, one can use that $X'_{\mathrm{Ig}(p^n)}$ is affine to show that these spaces represent the obvious adic moduli functors away from the cusps.

Definition 4.3. The inverse system of natural forgetful morphisms

$$\cdots \to \mathcal{X}'^{*}_{\mathrm{Ig}(p^{n+1})}(\epsilon) \to \mathcal{X}'^{*}_{\mathrm{Ig}(p^{n})}(\epsilon) \to \cdots \to \mathcal{X}'^{*}(\epsilon)$$

is called the Igusa tower. Note that all transition maps in this inverse system are finite étale.

Question 4.4. For $\epsilon = 0$, one can show that this system has a sous-perfectoid (but not perfectoid) tilde limit $\mathcal{X}'^{*}_{\mathrm{Ig}(p^{\infty})}(0) \sim \varprojlim_{n\in\mathbb{N}} \mathcal{X}'^{*}_{\mathrm{Ig}(p^{n})}(0)$. Is this still true for $\epsilon > 0$?

Definition 4.5. As in characteristic 0, by a cusp we shall mean a (not necessarily geometrically) connected component of the closed subscheme $X'^{*}_{\mathrm{Ig}(p^{n})} \backslash X'_{\mathrm{Ig}(p^{n})} \subseteq X'^{*}_{\mathrm{Ig}(p^{n})}$ with its induced reduced structure.

Given a fixed cusp x of $X'^{*}_{\mathrm{Ig}(p^{n})}$, we denote by $L_x \subseteq K^{\flat}[\zeta_N]$ the field of definition of the associated Tate curve. Then the completion of $X'^{*}_{\mathcal{O}_{K^{\flat}},\mathrm{Ig}(p^{n})}$ along the integral extension of x is canonically of the form $\mathrm{Spf}(\mathcal{O}_{L_x}[\![q]\!]) \to X'^{*}_{\mathrm{Ig}(p^{n})}$. Upon ϖ-adic completion this becomes

$$\mathrm{Spf}(\mathcal{O}_{L_x}[\![q]\!]) \to \mathfrak{X}'^{*}_{\mathrm{Ig}(p^{n})}$$

where $\mathcal{O}_{L_x}[\![q]\!]$ carries the $(\varpi^{\flat}, q)$-adic topology. Denote by

$$\mathcal{D}' \to \mathcal{X}'^{*}_{\mathrm{Ig}(p^{n})}$$

the adic generic fibre, a morphism of adic spaces over $\mathrm{Spa}(K^{\flat}, \mathcal{O}_{K^{\flat}})$. Then like before, $\mathcal{D}'$ is the open unit disc over L_x in the variable q. Exactly like in Lemma 2.9 one sees:

Lemma 4.6. *The morphism $\mathcal{D}' \hookrightarrow \mathcal{X}'^{*}_{\mathrm{Ig}(p^{n})}$ is an open immersion.*

If we want to indicate the dependence on the cusp x, we shall also call this $\mathcal{D}'_x \hookrightarrow \mathcal{X}'^{*}_{\mathrm{Ig}(p^{n})}$.
The following lemma explains how the above individual descriptions fit together for different cusps of $\mathcal{X}'^{*}_{\mathrm{Ig}(p^{n})}$ lying over the same cusp of $\mathcal{X}'^{*}$.

Lemma 4.7. *Let x be a cusp of $\mathcal{X}'^{*}$. Then there are Cartesian diagrams*

$$\begin{array}{ccccc}
\underline{\mathbb{Z}/p^{n+1}\mathbb{Z}} \times \mathcal{D}'_x & \longrightarrow & \underline{\mathbb{Z}/p^{n}\mathbb{Z}} \times \mathcal{D}'_x & \longrightarrow & \mathcal{D}'_x \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathcal{X}'^{*}_{\mathrm{Ig}(p^{n+1})}(\epsilon) & \longrightarrow & \mathcal{X}'^{*}_{\mathrm{Ig}(p^{n})}(\epsilon) & \longrightarrow & \mathcal{X}'^{*}(\epsilon).
\end{array}$$

Proof. Using the canonical lift described in §4.1, this can be seen exactly like in Lemma 2.23, based on the straightforward analogue of Lemma 2.11 in this setting. □

Like in the p-adic case, there is also a larger, quasi-compact Tate curve parameter space:

Definition 4.8. Let $\overline{\mathcal{D}}' = \overline{\mathcal{D}}'_x = \mathrm{Spa}(\mathcal{O}_{L_x}[\![q]\!][\frac{1}{\varpi^\flat}], \mathcal{O}_{L_x}[\![q]\!])$ where $\mathcal{O}_{L_x}[\![q]\!]$ is endowed with the $\varpi^\flat$-adic topology. Like in Lemma 2.16, one sees that this is a sousperfectoid adic space with an open immersion $\mathcal{D}' = \cup_m \overline{\mathcal{D}}'(|q|^m \leq |\varpi^\flat|) \hookrightarrow \overline{\mathcal{D}}'$.

Lemma 4.9. *For any cusp of $\mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon)$, the map $\mathcal{D}' \hookrightarrow \mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon)$ extends uniquely to a natural map $\overline{\mathcal{D}}' \to \mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon)$. The fibre of the good reduction locus is $\overline{\mathcal{D}}'(|q| \geq 1)$.*

Proof. Exactly like in Lemma 2.17. ☐

Lemma 4.10. *Let $n \in \mathbb{Z}_{\geq 0}$.*

1. *For any cusp x of $\mathcal{X}'^*_{\mathrm{Ig}(p^n)}$, the following square is Cartesian:*

$$\begin{array}{ccccc} \mathcal{D}' & \hookrightarrow & \overline{\mathcal{D}}' & \longrightarrow & \mathcal{X}'^*_{\mathrm{Ig}(p^n)}(p^{-1}\epsilon) \\ \downarrow{\scriptstyle q\mapsto q^p} & & \downarrow{\scriptstyle q\mapsto q^p} & & \downarrow{\scriptstyle F_{\mathrm{rel}}} \\ \mathcal{D}' & \hookrightarrow & \overline{\mathcal{D}}' & \longrightarrow & \mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon) \end{array}$$

2. *The following square is Cartesian:*

$$\begin{array}{ccc} \mathcal{X}'^*_{\mathrm{Ig}(p^{n+1})}(p^{-1}\epsilon) & \longrightarrow & \mathcal{X}'^*_{\mathrm{Ig}(p^n)}(p^{-1}\epsilon) \\ \downarrow{\scriptstyle F_{\mathrm{rel}}} & & \downarrow{\scriptstyle F_{\mathrm{rel}}} \\ \mathcal{X}'^*_{\mathrm{Ig}(p^{n+1})}(\epsilon) & \longrightarrow & \mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon) \end{array}$$

Proof. It is clear that the diagrams commute by functoriality of the relative Frobenius morphism. The second diagram is Cartesian because the bottom map is étale.

To see that the first diagram is Cartesian, we first consider the outer square. For this it suffices to check this on $(C^\flat, C^{\flat+})$-points because the horizontal compositions are open immersions. It is clear that the cusps correspond under F_{rel}, and $q \mapsto q^p$ sends the origin to the origin. Away from the cusps, we can check on moduli interpretations that the square is Cartesian: The desired statement follows as F_{rel} sends $\mathrm{T}(q)$ to $\mathrm{T}(q^p)$, and the $\mathrm{Ig}(p^n)$-structure $\langle q\rangle \subseteq \mathrm{T}(q)^{(p^n)} = \mathrm{T}(q^{p^n})$ to $\langle q^p\rangle \subseteq \mathrm{T}(q^p)^{(p^n)} = \mathrm{T}(q^{p^{n+1}})$.

This argument extends to $\overline{\mathcal{D}}'$, which (away from x) we may regard as the moduli space of Tate curves $\mathrm{T}(q)$ with level structure associated to x over adic spaces S over $\mathcal{O}_{L_x}[\![q]\!][\frac{1}{p}]$. Lifts of maps $S \to \overline{\mathcal{D}}'$ along the right vertical morphism correspond to Tate curves $\mathrm{T}(q)^{(p^{-1})} = \mathrm{T}(q^{1/p})$ over S whose base change along F_{rel} is $\mathrm{T}(q)$, and thus to p-th roots of $q \in \mathcal{O}^+(S)$. ☐

4.3 Perfections of Igusa curves

In this section, we discuss perfectoid Igusa curves and their Tate curve parameter spaces. We first recall the perfection functor in characteristic p:

Definition 4.11 ([Sch15, Def. 3.2.18]). Let $\mathcal{Y}$ be an analytic adic space over $(K^\flat, \mathcal{O}_K^\flat)$. Then there is a perfectoid space $\mathcal{Y}^{\mathrm{perf}}$ over $(K^\flat, \mathcal{O}_K^\flat)$ such that $\mathcal{Y}^{\mathrm{perf}} \sim \varprojlim_{F_{\mathrm{rel}}} \mathcal{Y}$ where we identify $\mathcal{Y}^{(p)}$ with $\mathcal{Y}$ using that $K^\flat$ is perfect. We call $\mathcal{Y}^{\mathrm{perf}}$ the perfection of $\mathcal{Y}$. The formation $\mathcal{Y} \mapsto \mathcal{Y}^{\mathrm{perf}}$ is functorial and defines a right adjoint to the forgetful functor from perfectoid spaces over $K^\flat$ to analytic adic spaces over $K^\flat$.

In the case of $\mathcal{Y} = \mathcal{X}'^*(\epsilon)$, we can first take the inverse limit $\mathfrak{X}'^*(\epsilon)^{\mathrm{perf}} := \varprojlim_{F_{\mathrm{rel}}} \mathfrak{X}'^*(p^{-n}\epsilon)$ in the category of formal schemes. Its generic fibre is then the tilde limit

$$\mathcal{X}'^*(\epsilon)^{\mathrm{perf}} = \mathfrak{X}'^*(\epsilon)^{\mathrm{perf}}_\eta \sim \varprojlim_{F_{\mathrm{rel}}} (\mathfrak{X}'^*(p^{-n}\epsilon))^{\mathrm{ad}}_\eta$$

by [SW13, Proposition 2.4.2] and it is clear on any affine open formal subscheme of $\mathfrak{X}'^*(\epsilon)$ that this space is perfectoid. The analogous construction also works for $\mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon)^{\mathrm{perf}}$.

Lemma 4.12. *Let (R, R^+) be a perfectoid $K^\flat$-algebra. Then $\mathcal{X}'_{\mathrm{Ig}(p^n)}(\epsilon)^{\mathrm{perf}}(R, R^+)$ is in functorial bijection with isomorphism classes of triples (E, α_N, β_n) of elliptic curves E over R with ϵ-nearly ordinary reduction, with a Γ^p-structure α_N and an isomorphism of group schemes*

$$\beta_n \colon \underline{\mathbb{Z}/p^n\mathbb{Z}} \xrightarrow{\sim} \ker(V : E^{(p^n)} \to E) \cong \ker(V : E \to E^{(p^{-n})}) \subseteq E[p^n].$$

Proof. By adjunction, we have $\mathcal{X}'_{\mathrm{Ig}(p^n)}(\epsilon)^{\mathrm{perf}}(R, R^+) = \mathcal{X}'_{\mathrm{Ig}(p^n)}(\epsilon)(R, R^+)$. □

Lemma 4.13. *For any cusp x of $\mathcal{X}'^*_{\mathrm{Ig}(p^n)}$, the perfection of the corresponding Tate curve parameter space $\mathcal{D}' \hookrightarrow \mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon)$ fits into a Cartesian diagram*

$$\begin{array}{ccccc}
\mathcal{D}'_\infty & \hookrightarrow & \overline{\mathcal{D}}'_\infty & \hookrightarrow & \mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon)^{\mathrm{perf}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{D}' & \hookrightarrow & \overline{\mathcal{D}}' & \longrightarrow & \mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon).
\end{array}$$

Here we set $\overline{\mathcal{D}}'_\infty := \overline{\mathcal{D}}'^{\mathrm{perf}}$, and the space $\mathcal{D}'_\infty := \mathcal{D}'^{\mathrm{perf}}$ can be canonically identified with the open subspace of the perfectoid unit disc $\mathrm{Spa}(K^\flat\langle q^{1/p^\infty}\rangle, \mathcal{O}_{K^\flat}\langle q^{1/p^\infty}\rangle)$ where q is locally topologically nilpotent, defined as the union of open subspaces where $|q| \leq |\varpi^\flat|^{1/n}$ for $n \in \mathbb{N}$.

Proof. This follows in the limit over the Cartesian diagrams from Lemma 4.10.(1). □

Lemma 4.14. *The following diagram is Cartesian:*

$$\begin{CD}
\mathcal{X}'^{*}_{\mathrm{Ig}(p^n)}(\epsilon)^{\mathrm{perf}} @>>> \mathcal{X}'^{*}(\epsilon)^{\mathrm{perf}} \\
@VVV @VVV \\
\mathcal{X}'^{*}_{\mathrm{Ig}(p^n)}(\epsilon) @>>> \mathcal{X}'^{*}(\epsilon).
\end{CD}$$

Proof. This follows from Lemma 4.10.(2) in the limit over F_{rel} because perfectoid tilde-limits commute with fibre products by the respective universal properties. ☐

Definition 4.15. Consider the tower of affinoid perfectoid spaces with finite étale maps

$$\cdots \to \mathcal{X}'^{*}_{\mathrm{Ig}(p^{n+1})}(\epsilon)^{\mathrm{perf}} \to \mathcal{X}'^{*}_{\mathrm{Ig}(p^n)}(\epsilon)^{\mathrm{perf}} \to \cdots \to \mathcal{X}'^{*}(\epsilon)^{\mathrm{perf}}.$$

We denote by $\mathcal{X}'^{*}_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}$ the unique affinoid perfectoid tilde-limit of this system.

Proposition 4.16. *Let x be any cusp of $\mathcal{X}'^{*}$. Then there are natural Cartesian diagrams*

$$(1)\quad \begin{CD}
\underline{(\mathbb{Z}/p^n\mathbb{Z})^\times} \times \mathcal{D}'_\infty @>>> \mathcal{D}'_\infty \\
@VVV @VVV \\
\mathcal{X}'^{*}_{\mathrm{Ig}(p^n)}(\epsilon)^{\mathrm{perf}} @>>> \mathcal{X}'^{*}(\epsilon)^{\mathrm{perf}},
\end{CD}
\qquad (2)\quad \begin{CD}
\underline{\mathbb{Z}_p^\times} \times \mathcal{D}'_\infty @>>> \mathcal{D}'_\infty \\
@VVV @VVV \\
\mathcal{X}'^{*}_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}} @>>> \mathcal{X}'^{*}(\epsilon)^{\mathrm{perf}}.
\end{CD}$$

Proof. Part (1) follows from Lemma 4.7, Lemma 4.13 and Lemma 4.14 using the Cartesian cube that these three squares span. Part (2) follows in the inverse limit $n \to \infty$. ☐

5 Tilting isomorphisms for modular curves

5.1 The tilting isomorphism at level $\Gamma_0(p^\infty)$

While so far we have studied modular curves in characteristic 0 and p separately, we now compare the two worlds via tilting. This is possible based on the following result:

Theorem 5.1 ([Sch15, Corollary 3.2.19]). *There is a canonical isomorphism*

$$\mathcal{X}^{*}_{\Gamma_0(p^\infty)}(\epsilon)^{\flat}_{a} \xrightarrow{\sim} \mathcal{X}'^{*}(\epsilon)^{\mathrm{perf}}.$$

Let us recall how this is proved: Via $\mathcal{O}_K/p \cong \mathcal{O}_{K^\flat}/\varpi^\flat$ we have an identification of the reductions $\mathfrak{X}^*/p = \mathfrak{X}'^*/\varpi^\flat$ which by explicit inspection extends to a natural isomorphism

$$\mathfrak{X}^*(\epsilon)/p \cong \mathfrak{X}'^*(\epsilon)/\varpi^\flat. \tag{14}$$

The morphism $\mathcal{X}^*_{\Gamma_0(p^{n+1})}(\epsilon) \to \mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)$ gets identified via the Atkin–Lehner isomorphism (3) with a map $\mathcal{X}^*(p^{-(n+1)}\epsilon) \to \mathcal{X}^*(p^{-n}\epsilon)$ that has a formal model $\phi\colon \mathfrak{X}^*(p^{-(n+1)}\epsilon) \to \mathfrak{X}^*(p^{-n}\epsilon)$. One can then prove that mod $p^{1-\delta}$ where $\delta := \frac{p+1}{p}\epsilon$, the map ϕ gets identified with F_{rel} in the sense that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{X}^*(p^{-(n+1)}\epsilon)/p^{1-\delta} & \xrightarrow{\phi} & \mathfrak{X}^*(p^{-n}\epsilon)/p^{1-\delta} \\ \wr\| & & \wr\| \\ \mathfrak{X}'^*(p^{-(n+1)}\epsilon)/\varpi^{\flat(1-\delta)} & \xrightarrow{F_{\mathrm{rel}}} & \mathfrak{X}'^*(p^{-n}\epsilon)/\varpi^{\flat(1-\delta)}. \end{array}$$

In the inverse limit, this gives the result by [Sch12, Theorem 5.2].
The following lemma says that the isomorphism of Theorem 5.1 identifies the cusps:

Lemma 5.2. *The cusps of $\mathcal{X}^*$ and $\mathcal{X}'^*$ correspond via tilting: For each cusp x of $\mathcal{X}^*$, considered as a finite étale adic space over K, its tilt can be canonically identified with a cusp $x^\flat$ of $\mathcal{X}'^*$. In particular, $(L_x)^\flat = L_{x^\flat}$ for the fields of definition.*

Proof. The cusp x can be described as a closed immersion $\mathrm{Spa}(L_x) \hookrightarrow \mathcal{X}'^*$. It has a canonical formal model $\mathrm{Spf}(\mathcal{O}_{L_x}) \to \mathfrak{X}'^*$ that reduces mod p to a morphism $\mathrm{Spec}(\mathcal{O}_{L_x}/p) \to X^*_{\mathcal{O}_K/p}$ which in turn can be interpreted as cusp of $X^*_{\mathcal{O}_{K^\flat/\varpi^\flat}}$. Lifting to $\mathfrak{X}'^*$ and taking generic fibre gives a cusp $x^\flat$ defined by a closed point

$$\mathrm{Spa}((L_x)^\flat) \hookrightarrow \mathcal{X}'^*.$$

It is clear from this construction that this can be identified with the tilt of x via the equivalence of étale sites. Reversing this argument shows that this defines a bijection on cusps. □

Proposition 5.3. *Let x be any cusp of $\mathcal{X}^*$. Then the canonical isomorphism of $K^\flat$-algebras*

$$L_x[\![q^{1/p^\infty}]\!]^\flat = L_{x^\flat}[\![q^{1/p^\infty}]\!]$$

defines isomorphisms $\overline{\mathcal{D}}^\flat_{\infty,x} \cong \overline{\mathcal{D}}'_{\infty,x^\flat}$ and $\mathcal{D}^\flat_{\infty,x} \cong \mathcal{D}'_{\infty,x^\flat}$ that fit into a commutative diagram

$$\begin{array}{ccccc} \mathcal{D}^\flat_{\infty,x} & \hookrightarrow & \overline{\mathcal{D}}^\flat_{\infty,x} & \hookrightarrow & \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)^\flat_a \\ \wr\| & & \wr\| & & \wr\| \\ \mathcal{D}'_{\infty,x^\flat} & \hookrightarrow & \overline{\mathcal{D}}'_{\infty,x^\flat} & \hookrightarrow & \mathcal{X}'^*(\epsilon)^{\mathrm{perf}}. \end{array}$$

Proof. For the proof we use that $\overline{\mathcal{D}}_\infty$ has a very simple p-adic formal model. Here and in the following, let us for simplicity drop the additional x and $x^\flat$ in the index.

We can without loss of generality assume $\epsilon = 0$. Using the identifications

$$\mathcal{D}^\flat_\infty = \cup_m \overline{\mathcal{D}}_\infty(|q|^m \leq |\varpi|)^\flat = \cup_m \overline{\mathcal{D}}'_\infty(|q|^m \leq |\varpi^\flat|) = \mathcal{D}'_\infty,$$

it is clear that the left square commutes. It therefore suffices to consider the right square.

Recall that the morphism $\overline{\mathcal{D}} \to \mathcal{X}^*(0)$ arises as the adic generic fibre of the morphism $\overline{\mathfrak{D}} \to \mathfrak{X}^*(0)$ where $\overline{\mathfrak{D}} := \mathrm{Spf}(\mathcal{O}_L[\![q]\!])$ is endowed with the p-adic topology. Similarly, $\overline{\mathcal{D}}' \to \mathcal{X}'^*(0)$ is the adic generic fibre of $\overline{\mathfrak{D}}' \to \mathfrak{X}'^*(0)$ where $\overline{\mathfrak{D}}' := \mathrm{Spf}(\mathcal{O}_{L^\flat}[\![q]\!])$. The reductions mod ϖ and $\varpi^\flat$ of these formal models can be canonically identified with the map

$$\mathrm{Spec}((\mathcal{O}_L/p)[\![q]\!]) \to \mathfrak{X}^*(0)/p = \mathfrak{X}'^*(0)/\varpi^\flat$$

associated to the Tate curve for the corresponding cusp of $X^*_{\mathcal{O}_K/p}$.

In the limit over ϕ and F_{rel}, these identifications therefore fit into a commutative diagram

$$\begin{array}{ccc} \varprojlim_{q\mapsto q^p} \overline{\mathfrak{D}}/p & \longrightarrow & \varprojlim_\phi \mathfrak{X}^*(0)/p \\ \wr\| & & \wr\| \\ \varprojlim_{q\mapsto q^p} \overline{\mathfrak{D}}'/\varpi^\flat & \longrightarrow & \mathfrak{X}'^*(0)^{\mathrm{perf}}/\varpi^\flat. \end{array}$$

As these are perfectoid schemes over $\mathcal{O}^a_K/p$ (or $\mathcal{O}^a_{K^\flat}/\varpi^\flat$) with corresponding perfectoid spaces $\overline{\mathcal{D}}_\infty \to \mathcal{X}^*_{\Gamma_0(p^\infty)}(0)_a$ and $\overline{\mathcal{D}}'_\infty \to \mathcal{X}'^*(0)^{\mathrm{perf}}$ via [Sch12, Theorem 5.2], this gives the desired identification of the tilts. □

Remark 5.4. The correspondence of moduli of Tate curves implicit in Proposition 5.3 can be made explicit as follows: Let (R, R^+) be a perfectoid K-algebra, then Theorem 5.1 gives a correspondence

$$\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a(R, R^+) = \mathcal{X}'(\epsilon)^{\mathrm{perf}}(R^\flat, R^{\flat+}) = \mathcal{X}'(\epsilon)(R^\flat, R^{\flat+})$$

of elliptic curves with extra data. If now E_q is a Tate curve with parameter $q \in R$, equipped with $\Gamma_0(p^\infty)$-structure $(q^{1/p^n})_{n\in\mathbb{N}}$ and Γ^p-structure, corresponding to a point in $\mathcal{D}_\infty(R, R^+)$, then via $\mathcal{D}_\infty(R, R^+) = \mathcal{D}'_\infty(R^\flat, R^{\flat+})$, this corresponds to the Tate curve $E_{q'}$ with parameter

$$q' := (q^{1/p^n})_{n\in\mathbb{N}} \in \varprojlim_{q\mapsto q^p} R^\times = R^{\flat\times}.$$

One can moreover identify the Γ^p-structure of $E_{q'}$ using that $E_q[N]^\flat = E_{q'}[N]$.

5.2 The tilting isomorphism at level $\Gamma_1(p^\infty)$

We now extend the tilting isomorphism of Theorem 5.1 to level $\Gamma_1(p^\infty)$ by proving the following theorem stated in the introduction:

Theorem 5.5. *1. There is a canonical isomorphism*

$$\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)^\flat_a \xrightarrow{\sim} \mathcal{X}'^*_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}$$

which is $\mathbb{Z}_p^\times$-equivariant and makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)^\flat_a & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)^\flat_a \\ \wr\| & & \wr\| \\ \mathcal{X}'^*_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}} & \longrightarrow & \mathcal{X}'^*(\epsilon)^{\mathrm{perf}}. \end{array}$$

2. The cusps of $\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a$ and $\mathcal{X}'^*_{\mathrm{Ig}(p^\infty)}(\epsilon)$ correspond via the isomorphism in (1). Moreover, for any cusp x of $\mathcal{X}^*$, the following diagram commutes:

$$\begin{array}{ccc} \underline{\mathbb{Z}_p^\times} \times \mathcal{D}^\flat_{\infty,x} & \hookrightarrow & \mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)^\flat_a \\ \wr\| & & \wr\| \\ \underline{\mathbb{Z}_p^\times} \times \mathcal{D}'_{\infty,x^\flat} & \hookrightarrow & \mathcal{X}'^*_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}, \end{array}$$

where the left map is given by the canonical identification $\mathcal{D}^\flat_{\infty,x} \cong \mathcal{D}'_{\infty,x^\flat}$.

For the proof, we use the universal anticanonical subgroup at infinite level:

Definition 5.6. For any $n \in \mathbb{Z}_{\geq 1}$, we denote by $G_n \to \mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$ the universal anticanonical subgroup of rank n. This can be defined via pullback from finite level $\mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a$, and is a finite étale morphism of perfectoid spaces.
Let $\mathcal{E}' \to \mathcal{X}'$ be the analytification of the universal elliptic curve over X', and write $\mathcal{E}'(\epsilon) \to \mathcal{X}'(\epsilon)$ for the pullback. We denote by $G'_n \to \mathcal{X}'(\epsilon)^{\mathrm{perf}}$ the finite étale morphism of perfectoid spaces given by the perfection of $\ker V^n \subseteq \mathcal{E}'(\epsilon)^{(p^n)}$.

Lemma 5.7. *There is a natural isomorphism making the following diagram commutative:*

$$\begin{array}{ccc} G_n^\flat & \xrightarrow{\sim} & G'_n \\ \downarrow & & \downarrow \\ \mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)^\flat_a & \cong & \mathcal{X}'(\epsilon)^{\mathrm{perf}}, \end{array}$$

This lemma is a slight extension of [Sch15, Lemma 3.2.26], from the good reduction locus to the whole uncompactified modular curve (we reiterate that [Sch15] writes $\mathcal{X}$ for the good reduction locus, whereas we use this symbol to denote the whole open modular curve).

Proof. It suffices to see this locally on $\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$. The case of good reduction is [Sch15, Lemma 3.2.26]. It therefore suffices to prove the lemma over the ordinary locus $\mathcal{X}_{\Gamma_0(p^\infty)}(0)_a$.

Over $\mathfrak{X}^*(0)$, the universal semi-abelian scheme has a canonical subgroup, a finite flat group scheme $C_n \to \mathfrak{X}^*(0)$. Via the Atkin–Lehner isomorphism $\mathcal{X}^*(0) \xrightarrow{\sim} \mathcal{X}^*_{\Gamma_0(p^n)}(0)_a$, the adic generic fibre of its dual $(C_n^\vee)^{\mathrm{ad}}_\eta$ can be identified over $\mathcal{X}_{\Gamma_0(p^n)}(0)_a$ with the universal anticanonical subgroup over $\mathcal{X}_{\Gamma_0(p^n)}(0)_a$. Similarly, over $\mathfrak{X}'^*(0)$, we have a canonical subgroup $C'_n \to \mathfrak{X}^*(0)$, and the dual $(C_n'^\vee)^{\mathrm{ad}}_\eta$ restricted to $\mathcal{X}'(0)$ can be identified with the kernel of Verschiebung of $\mathcal{E}'(0)$ over $\mathcal{X}'(0)$.
It follows from these descriptions that after pullback we have identifications

$$G_n = \big(C_n^\vee \times_{\mathfrak{X}^*(0)} \varprojlim_{\phi} \mathfrak{X}^*(0)\big)^{\mathrm{ad}}_\eta \text{ restricted to } \mathcal{X}_{\Gamma_0(p^\infty)}(0)_a,$$

$$G'_n = \big(C_n'^\vee \times_{\mathfrak{X}'^*(0)} \mathfrak{X}'^*(0)^{\mathrm{perf}}\big)^{\mathrm{ad}}_\eta \text{ restricted to } \mathcal{X}'(0)^{\mathrm{perf}}.$$

To prove the lemma, it therefore suffices to prove that the formal models on the right hand side can be identified after reduction to $\mathcal{O}_K/p = \mathcal{O}_{K^\flat}/\varpi^\flat$, for which it suffices to prove that $C_n^\vee/p = C_n'^\vee/\varpi^\flat$ on $\mathfrak{X}^*(0)/p = \mathfrak{X}'^*(0)/\varpi^\flat$. But over the ordinary locus, $C_n/p = C'_n/\varpi^\flat$ are both the kernel of Frobenius, and Cartier duals commute with base change. □

We can now complete the proof of Theorem 5.5 stated in the introduction:

Proof of Theorem 5.5. We start by proving that for any $n \in \mathbb{Z}_{\geq 1}$, there is a natural isomorphism

$$\begin{array}{ccc} \mathcal{X}^*_{\Gamma_1(p^n)\cap\Gamma_0(p^\infty)}(\epsilon)^\flat_a & \xrightarrow{\sim} & \mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon)^{\mathrm{perf}} \\ \downarrow & & \downarrow \\ \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)^\flat_a & = & \mathcal{X}'^*(\epsilon)^{\mathrm{perf}} \end{array} \tag{15}$$

making the diagram commute. Part (1) of the theorem then follows in the limit $n \to \infty$.
Away from the cusps, the desired isomorphism is induced by the natural isomorphism from Lemma 5.7, using the moduli interpretations in Lemma 3.13 and Lemma 4.12.
We need to extend this over the cusps. One way of doing this is to give the vertical maps in the diagram a relative moduli interpretation that extends to the cusps. More in the spirit of our arguments so far, we shall instead give a more explicit proof using Tate curve parameter spaces, which also has the added benefit that it yields part (2).
To this end, fix a cusp x of $\mathcal{X}^*$. By Proposition 5.3, the isomorphism $\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)^\flat_a \to \mathcal{X}'^*(\epsilon)^{\mathrm{perf}}$ restricts to the canonical isomorphism $\mathcal{D}^\flat_\infty = \mathcal{D}'_\infty$ over x. Using the description of the Tate curve parameter spaces in $\mathcal{X}^*_{\Gamma_1(p^m)\cap\Gamma_0(p^\infty)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$ from Lemma 3.9 and similarly in $\mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon)^{\mathrm{perf}} \to \mathcal{X}'^*(\epsilon)^{\mathrm{perf}}$ from Proposition 4.16.(1), it now suffices to prove

that the isomorphism $G_n^\flat = G'_n$ over the Tate curve parameter spaces becomes the natural map

$$\begin{array}{ccc} (\underline{\mathbb{Z}/p^n\mathbb{Z}} \times \mathring{\mathcal{D}}_\infty)^\flat & \xrightarrow{\sim} & \underline{\mathbb{Z}/p^n\mathbb{Z}} \times \mathring{\mathcal{D}}'_\infty \\ \downarrow & & \downarrow \\ \mathring{\mathcal{D}}^\flat_\infty & \xrightarrow{\sim} & \mathring{\mathcal{D}}'_\infty. \end{array} \tag{16}$$

It is then clear that the diagram extends uniquely over the cusps.
To see this, we note that the restriction of G_n to $\mathring{\mathcal{D}}_\infty$ is indeed canonically isomorphic to $\underline{\mathbb{Z}/p^n\mathbb{Z}} \times \mathring{\mathcal{D}}_\infty$ due to the canonical section given by the element q^{1/p^n} of the anticanonical subgroup $\langle q^{1/p^n}\rangle \subseteq \mathrm{T}(q)[p^n]$. Similarly, G'_n is isomorphic to $\underline{\mathbb{Z}/p^n\mathbb{Z}} \times \mathring{\mathcal{D}}'_\infty$ on $\mathring{\mathcal{D}}'_\infty \to \mathcal{X}'(\epsilon)^{\mathrm{perf}}$. By considering the dual trivialisations of the respective canonical subgroups, it follows from the construction in the proof of Lemma 5.7 that these isomorphisms are compatible with tilting and make diagram (16) commute, as desired.
Part (2) now follows from diagram (16) in the limit $n \to \infty$. □

6 q-expansion principles

As a second application, in this section, we prove various q-expansion principles for functions on the infinite level spaces $\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$, $\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a$, $\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a$, etc., based on our discussion of cusps in §2-§4.

6.1 Detecting vanishing

We begin with the proof of q-expansion principle I, Proposition 1.6 in the introduction, recalled below. On the way, we also prove principles III and IV. We focus on the case of characteristic 0, the case of characteristic p is completely analogous.

Proposition 6.1. *Let $\mathcal{C}$ be a collection of cusps of $\mathcal{X}^*$ such that each connected component of $\mathcal{X}^*$ contains at least one $x \in \mathcal{C}$. Let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and let Γ be one of $\Gamma_0(p^n), \Gamma_1(p^n), \Gamma(p^n)$. Define $\mathcal{D}_{\mathcal{C},\Gamma}$ as the pullback*

$$\begin{array}{ccc} \mathcal{D}_{\mathcal{C},\Gamma} & \hookrightarrow & \mathcal{X}^*_\Gamma(\epsilon)_a \\ \downarrow & & \downarrow \\ \bigsqcup_{x\in\mathcal{C}} \mathcal{D}_x & \hookrightarrow & \mathcal{X}^*(\epsilon). \end{array}$$

Then the map $\mathcal{O}(\mathcal{X}^*_\Gamma(\epsilon)_a) \to \mathcal{O}(\mathcal{D}_{\mathcal{C},\Gamma})$ is injective.

This is an analogue of saying that for any affine irreducible integral variety over K, completion at any K-point gives rise to an injection on functions, which

is a consequence of Krull's Intersection Theorem. As this requires Noetherianess, we first reduce to the Noetherian situation using that all of the above spaces have natural models over $\mathbb{Z}_p$.
The proof is in two steps: We first consider $\mathcal{X}^*_{\Gamma_0(p^\infty)}(0)_a$ where it is easy to reduce to the Noetherian case. In a second step, we then show that restriction of functions from $\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a$ to $\mathcal{X}^*_{\Gamma_1(p^\infty)}(0)_a$ is injective, which is a straightforward computation on power series. We start with the case of $\epsilon = 0$. On the way we will also see Proposition 1.8.

Proof of Proposition 6.1 for $\epsilon = 0$. The case of $\Gamma = \Gamma(p^n)$ reduces to the one of $\Gamma_1(p^n)$ by Corollary 3.18.
We first consider the case of $n < \infty$. Then the case of $\Gamma = \Gamma_0(p^n)$ further reduces to the case of tame level via the Atkin–Lehner isomorphism $\mathcal{X}^*(0) \cong \mathcal{X}^*_{\Gamma_0(p^n)}(0)_a$. We are therefore left with the case of $\Gamma_1(p^n)$ for $n \in \mathbb{Z}_{\geq 0}$ (the case of tame level being $n = 0$).
The space $\mathcal{X}^*_{\Gamma_1(p^n)}(0)_a$ has an affine formal model $\mathfrak{X}^*_{\Gamma_1(p^n)}(0)_a = \mathrm{Spf}(R)$ for some complete $\mathbb{Z}_p$-algebra R. Let $\mathcal{C}'$ be the pullback of $\mathcal{C}$ to $\mathfrak{X}^*_{\Gamma_1(p^n)}(0)_a$ and let

$$\sqcup_{x\in\mathcal{C}'} \mathrm{Spf}\, \mathcal{O}_{L_x}[\![q]\!] \to \mathfrak{X}^*_{\Gamma_1(p^n)}(0)_a$$

be the completion along $\mathcal{C}'$. It suffices to show that the map on global sections

$$\varphi : R \to \prod_{x\in\mathcal{C}'} \mathcal{O}_{L_x}[\![q]\!]$$

is injective. As these are complete $\mathcal{O}_K$-algebras, it suffices to see that the reduction

$$R/p \to \prod_{c\in\mathcal{C}'} \mathcal{O}_{L_x}/p[\![q]\!] \tag{17}$$

is injective. But this reduction can be interpreted as the completion of $X^*_{\mathcal{O}_K/p,\mathrm{Ig}(p^n),\mathrm{ord}}$ at the divisor of cusps $\mathcal{C}'$. By base change from $\mathbb{F}_p$ to $\mathcal{O}_K/p$, we can now reduce to showing that for $Y := X^*_{\mathbb{F}_p,\mathrm{Ig}(p^n),\mathrm{ord}}$, completion at $\mathcal{C}'$ defines an injection

$$\mathcal{O}(Y) \to \prod_{x\in\mathcal{C}'} \mathbb{F}_p(x)[\![q]\!]$$

where $\mathbb{F}_p(x) \subseteq \mathbb{F}_p[\zeta_N]$ is the coefficient field of definition of the level structure on the Tate curve corresponding to the cusp $x \in \mathcal{C}'$.
Since Y is a smooth affine curve over $\mathbb{F}_p$, and by considering each connected component separately, the desired injectivity follows as for an integral Noetherian ring A, completion at any maximal ideal $\mathfrak{m} \subseteq A$ gives an injection $A \to \hat{A}_\mathfrak{m}$ by Krull's intersection theorem.
The case of $n = \infty$ can be deduced in the limit: As the natural restriction map

$$\mathcal{O}^+(\overline{\mathcal{D}}_\infty) \hookrightarrow \mathcal{O}^+(\mathcal{D}_\infty)$$

is injective (see Def. 2.15), it suffices to prove the statement for $\overline{\mathcal{D}}_\infty \sim \varprojlim \overline{\mathcal{D}}_n$, while conversely it is clear from $\mathcal{O}^+(\overline{\mathcal{D}}_n) = \mathcal{O}^+(\mathcal{D}_n)$ for $n < \infty$ and the first part that the corresponding result at finite level holds for $\mathcal{D}_n$ replaced by $\overline{\mathcal{D}}_n$. For any $m \in \mathbb{N}$, let $\mathfrak{Y}_m = \mathfrak{X}^*_{\Gamma_0(p^m)}(0)_a$ or $\mathfrak{Y}_m = \mathfrak{X}^*_{\Gamma_1(p^m)}(0)_a$. Then $\mathfrak{Y} = \varprojlim \mathfrak{Y}_m$ is a formal model of $\mathcal{X}^*_\Gamma(0)_a$. To see the result it suffices to prove that the natural maps

$$\mathcal{O}(\mathfrak{X}^*_{\Gamma_0(p^m)}(0)_a) \to \prod_{x\in\mathcal{C}} \mathcal{O}(\mathfrak{D}_{\infty,x})$$
$$\mathcal{O}(\mathfrak{X}^*_{\Gamma_1(p^m)}(0)_a) \to \prod_{x\in\mathcal{C}} \mathrm{Map}_{\mathrm{cts}}(\mathbb{Z}_p^\times, \mathcal{O}(\mathfrak{D}_{\infty,x})).$$

are injective. By completeness, it suffices to prove this on the reduction mod ϖ. But here it follows in the direct limit over $m \to \infty$ from the case of finite level. □

The proof of Proposition 6.1 in general is completed by the following two lemmas:

Lemma 6.2. *Let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and let $\mathcal{Y} \to \mathcal{X}^*$ be one of $\mathcal{X}^*_{\Gamma_0(p^n)}$, $\mathcal{X}^*_{\Gamma_1(p^n)}$, $\mathcal{X}^*_{\Gamma(p^n)}$. Then the open immersion $\mathcal{Y}(0) \to \mathcal{Y}(\epsilon)$ defines on sections an injection $\mathcal{O}(\mathcal{Y}(\epsilon)) \to \mathcal{O}(\mathcal{Y}(0))$.*

Proof. It suffices to prove this locally. Let $\mathcal{Y}_{\mathrm{gd}}(\epsilon) := \mathcal{Y}(\epsilon) \times_{\mathcal{X}} \mathcal{X}_{\mathrm{gd}}$. Then since $\mathcal{Y}(0) \to \mathcal{Y}(\epsilon)$ is an open immersion, and $\mathcal{Y}(0)$ and $\mathcal{Y}_{\mathrm{gd}}(\epsilon)$ cover all of $\mathcal{Y}(\epsilon)$, it suffices to prove the statement for $\mathcal{Y}_{\mathrm{gd}}(\epsilon)$. Let thus $\mathfrak{Y}$ be one of $\mathfrak{X}_{\Gamma_0(p^n)}$, $\mathfrak{X}_{\Gamma_1(p^n)}$, $\mathfrak{X}_{\Gamma(p^n)}$, each for any $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. It suffices to prove that for any affine open $\mathfrak{U} = \mathrm{Spf}(R) \subseteq \mathfrak{Y}$ where the Hodge bundle ω is trivial, the natural map $\mathfrak{Y}(0) \to \mathfrak{Y}(\epsilon)$ induces an injection $\mathcal{O}(\mathfrak{Y}(\epsilon)|_{\mathfrak{U}}) \to \mathcal{O}(\mathfrak{Y}(0)|_{\mathfrak{U}})$. We have $\mathfrak{Y}(\epsilon)|_{\mathfrak{U}} = \mathrm{Spf}(S)$ where $S = R\langle X\rangle/(X\mathrm{Ha} - p^\epsilon)$, and $\mathfrak{Y}(0)|_{\mathfrak{U}} = \mathrm{Spf}(R\langle \mathrm{Ha}^{-1}\rangle)$. Since Ha is a non-zero-divisor on R/p^n, Lemma 6.3 below now gives the desired statement. □

Lemma 6.3. *Let A be any ring, let $0 \neq \varpi \in A$ be a non-zero-divisor and let $H \in A$ be such that its image in A/ϖ is a non-zero-divisor. Endow A with the ϖ-adic topology. Then*

$$\varphi\colon A\langle X\rangle/(XH - \varpi) \xrightarrow{X\mapsto \varpi X} A\langle X\rangle/(XH-1)$$

is injective.

Proof. We first note that the assumption on $H \in A$ implies that H is a non-zero-divisor in any A/ϖ^n. Suppose $f = \sum a_n X^n$ is in the kernel of $A\langle X\rangle \to A\langle X\rangle/(XH - \varpi) \xrightarrow{\varphi} A\langle X\rangle/(XH-1)$. Then there is $g = \sum b_n X^n \in A\langle X\rangle$ such that

$$f(\varpi X) = \sum a_n \varpi^n X^n = (XH-1)g = (XH-1)\sum b_n X^n.$$

Reducing mod ϖ^m, we see that

$$a_0 + \cdots + a_{m-1}\varpi^{m-1}X^{m-1} \equiv (XH-1)\sum b_n X^n \bmod \varpi^m$$

By comparing degrees as polynomials in $A/\varpi^m[X]$, we conclude from H being a non-zero-divisor mod ϖ^m that $\deg(\sum b_n X^n \bmod \varpi^m) < m-1$, thus $b_k \equiv 0 \bmod \varpi^m$ for $k \geq m-1$.
Consequently, there are elements $c_m = b_m/\varpi^{m+1} \in A$ for all m and in $A[\![X]\!]$ we have

$$f' := (XH - \varpi)\sum c_m X^m \overset{X\mapsto\varpi X}{\longmapsto} (XH-1)\sum b_m X^m.$$

Thus $f'(\varpi X) = f(\varpi X)$ in $A[\![X]\!]$ which implies $f' = f$ since ϖ is a non-zero-divisor.
It remains to see that $\sum c_m X^m$ converges in $A\langle X\rangle$: Since $f \in A\langle X\rangle$, for every $k \in \mathbb{N}$ there is an N_k such that $v(a_m) \geq k$ for all $m \geq N_k$, where v is the ϖ-adic valuation. In particular, we then have $v(\varpi^m a_m) \geq k+m$ for all $m \geq N_k$. Consequently, for all $m \geq N_k$

$$a_0 + \cdots + a_{m-1}\varpi^{m-1}X^{m-1} \equiv (XH-1)\sum_{n\in\mathbb{N}} b_n X^n \bmod \varpi^{m+k}.$$

This shows that $v(b_{m-1}) \geq m+k$, and thus $v(c_m) \geq k$ for all $m \geq N_k$. Thus $\sum c_m X^m \in A\langle X\rangle$ as desired. We conclude that f is already in $(XH-\varpi)A\langle X\rangle$. Thus φ is injective. ☐

We can extract from this argument a proof of q-expansion principle III in the introduction:

PROPOSITION 6.4 (q-expansion principle III). *The following are equivalent for $f \in \mathcal{O}(\mathcal{X}^*_{\Gamma_0(p^\infty)}(0)_a)$:*

1. *f is integral, i.e. it is already contained in $\mathcal{O}^+(\mathcal{X}^*_{\Gamma_0(p^\infty)}(0)_a)$.*
2. *The q-expansion of f at every cusp x is already contained in $\mathcal{O}_{L_x}[\![q^{1/p^\infty}]\!]$.*
3. *On each connected component of $\mathcal{X}^*$, there is at least one cusp x at which the q-expansion of f is already contained in $\mathcal{O}_{L_x}[\![q^{1/p^\infty}]\!]$.*

Equivalently, the natural map

$$\varphi\colon \mathcal{O}^+(\mathcal{X}^*_{\Gamma_0(p^\infty)}(0)_a)/p \to \prod_x (\mathcal{O}_{L_x}/p)[\![q^{1/p^\infty}]\!]$$

is injective. The analogous statements for $\mathcal{X}^*_{\Gamma_1(p^\infty)}(0)_a$, $\mathcal{X}^*_{\Gamma(p^\infty)}(0)_a$, $\mathcal{X}'^*(0)^{\mathrm{perf}}$ and $\mathcal{X}'^*_{\mathrm{Ig}(p^\infty)}(0)^{\mathrm{perf}}$ are also true when we replace $\mathcal{O}_{L_x}[\![q^{1/p^\infty}]\!]$ by the respective algebra from Proposition 1.6.

Proof. It is clear from $\mathcal{O}^+(\mathcal{D}_{\infty,x}) = \mathcal{O}_{L_x}[\![q^{1/p^\infty}]\!]$ that (1) implies (2) implies (3). To prove that (3) implies (1), it suffices to see that $\mathcal{O}^+(\mathcal{X}^*_{\Gamma_0(p^\infty)}(0)_a)/p \to \prod_x(\mathcal{O}_{L_x}/p)[\![q^{1/p^\infty}]\!]$ is injective. We have already seen in (17) in the proof of Proposition 6.1 that

$$\mathcal{O}(\mathfrak{X}^*_{\Gamma_0(p^n)}(0)_a)/p \hookrightarrow \prod_{x\in\mathcal{C}}(\mathcal{O}_{L_x}/p)[\![q^{1/p^n}]\!]$$

is injective for any $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Since by [Heu19, Proposition 4.1.3], we have

$$\mathcal{O}^+(\mathcal{X}^*_{\Gamma_0(p^\infty)}(0)_a) = \mathcal{O}(\mathfrak{X}^*_{\Gamma_0(p^\infty)}(0)_a),$$

this gives the desired statement in the case of $\Gamma_0(p^\infty)$.
By the same argument, the cases of $\mathcal{X}^*_{\Gamma_1(p^\infty)}(0)_a$, $\mathcal{X}^*_{\Gamma(p^\infty)}(0)_a$, $\mathcal{X}'^*(0)^{\mathrm{perf}}$ and $\mathcal{X}'^*_{\mathrm{Ig}(p^\infty)}(0)^{\mathrm{perf}}$ also follow from (17) in the limit $n \to \infty$ using instead [Heu19, Lemma A.2.2.3]. ☐

We can also use the lemmas for the proof of q-expansion principle IV:

Proposition 6.5 (q-expansion principle IV). *Let $\mathcal{C}$ be a collection of cusps of $\mathcal{X}^*$ such that each connected component contains at least one $x \in \mathcal{C}$. Then a function on the good reduction locus $\mathcal{X}_{\mathrm{gd}}(\epsilon)$ extends to all of $\mathcal{X}^*(\epsilon)$ if and only if its q-expansion with respect to $\overline{\mathcal{D}}(|q| \geq 1) \to \mathcal{X}_{\mathrm{gd}}(\epsilon)$ at each $x \in \mathcal{C}$ is already in $\mathcal{O}_{L_x}[\![q]\!][\frac{1}{p}] \subseteq \mathcal{O}_{L_x}\langle\!\langle q\rangle\!\rangle[\frac{1}{p}]$. In this case, the extension is unique. The analogous statements for $\mathcal{X}^*_{\Gamma_0(p^\infty)}(0)_a$, $\mathcal{X}^*_{\Gamma_1(p^\infty)}(0)_a$, $\mathcal{X}^*_{\Gamma(p^\infty)}(0)_a$, $\mathcal{X}'^*(0)$, $\mathcal{X}'^*(0)^{\mathrm{perf}}$ and $\mathcal{X}'^*_{\mathrm{Ig}(p^\infty)}(0)^{\mathrm{perf}}$ are also true.*

Proof. As before, one can reduce to the case of finite level. For simplicity, let us treat $\mathcal{X}_{\mathrm{gd}}$, the other cases are similar. By Lemma 6.2, we can reduce to $\epsilon = 0$. We then need to prove that the following sequence is left exact:

$$0 \to \mathcal{O}(\mathfrak{X}^*(0)) \to \mathcal{O}(\mathfrak{X}(0)) \times \textstyle\prod_{x\in\mathcal{C}} \mathcal{O}_{L_x}[\![q]\!] \xrightarrow{(f,g)\mapsto f-g} \textstyle\prod_{x\in\mathcal{C}} \mathcal{O}_{L_x}\langle\!\langle q\rangle\!\rangle.$$

It suffices to prove that this is true mod ϖ^n for all n. By tensoring with the flat $\mathbb{F}_p$-algebra $\mathcal{O}_K/\varpi^n$, the statement then follows from the following sequence being left-exact:

$$0 \to \mathcal{O}(X^*_{\mathbb{F}_p,\mathrm{ord}}) \to \mathcal{O}(X_{\mathbb{F}_p,\mathrm{ord}}) \times \textstyle\prod_{x\in\mathcal{C}} \mathbb{F}[\![q]\!] \xrightarrow{(f,g)\mapsto f-g} \textstyle\prod_{x\in\mathcal{C}} \mathbb{F}(\!(q)\!),$$

where $\mathbb{F} := \mathbb{F}_p(x)$ depends on x. This holds as $X^*_{\mathbb{F}}$ is the normalisation of $j\colon X_{\mathbb{F}} \to \mathbb{A}^1_{\mathbb{F}}$ in $\mathbb{P}^1_{\mathbb{F}}$, and thus a function f extends to the cusp x if and only if it is finite over the completion $\mathbb{F}_p[\![q]\!]$ of $\mathbb{P}^1_{\mathbb{F}_p}$ at ∞. ☐

6.2 Tate traces and detecting the level

While the transition from $\Gamma_0(p^\infty)$ to $\Gamma(p^\infty)$ is controlled by the Galois action, the transition from $\Gamma_0(p)$ to $\Gamma_0(p^\infty)$ is controlled by normalised Tate traces, as discussed in [Sch15, §3.2.4]. We now briefly recall what these are and then use them to deduce the last remaining q-expansion principle:

Proposition 6.6 ([Sch15, Corollary 3.2.23]). *Let $0 \leq n \leq k \in \mathbb{N}$. Then the normalised traces*

$$\mathrm{tr}_{k,n} \colon \mathcal{O}_{\mathfrak{X}^*_{\Gamma_0(p^k)}(\epsilon)_a} \to \mathcal{O}_{\mathfrak{X}^*_{\Gamma_0(p^n)}(\epsilon)_a}[\tfrac{1}{p}]$$

on $\mathfrak{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$ of the finite flat forgetful map $\mathfrak{X}^*_{\Gamma_0(p^k)}(\epsilon)_a \to \mathfrak{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$ give rise for $k \to \infty$ to compatible continuous $\mathcal{O}_{\mathfrak{X}^*_{\Gamma_0(p^n)}(\epsilon)_a}$-linear morphisms with bounded image

$$\mathrm{tr}_n \colon \mathcal{O}_{\mathfrak{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a} \to \mathcal{O}_{\mathfrak{X}^*_{\Gamma_0(p^n)}(\epsilon)_a}[\tfrac{1}{p}].$$

Proof. Via the Atkin–Lehner isomorphism $\mathfrak{X}^*_{\Gamma_0(p^n)}(\epsilon)_a \cong \mathfrak{X}^*(p^{-n}\epsilon)$, this is the statement of [Sch15, Corollary 3.2.23], except that we use the compactified $\mathfrak{X}^*$ instead of $\mathfrak{X}$: This is possible since in contrast to the higher dimensional Siegel moduli spaces, the minimal compactification of the modular curve $\mathfrak{X}^*$ is a smooth formal scheme, and thus Corollary 3.2.22 applies over all of $\mathfrak{X}^*$, not just over $\mathfrak{X}$, which means that the proof of 3.2.23 goes through for $\mathfrak{X}^*$. □

Definition 6.7. Taking global sections and inverting p, the trace tr_n defines a linear map

$$\mathrm{tr}_n \colon \mathcal{O}(\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a) \to \mathcal{O}(\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a).$$

Proposition 6.8. *Let x be any cusp of $\mathcal{X}^*$, with corresponding Tate curve parameter space $\mathcal{D}_{n,x} \hookrightarrow \mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$. Then the normalised Tate trace fits into a commutative diagram*

$$\begin{array}{ccc} \mathcal{O}(\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a) & \xrightarrow{\mathrm{tr}_n} & \mathcal{O}(\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a) \\ \downarrow & & \downarrow \\ \mathcal{O}(\mathcal{D}_{\infty,x}) & \xrightarrow{\mathrm{tr}_n} & \mathcal{O}(\mathcal{D}_{n,x}), \end{array} \qquad \sum_{m\in\mathbb{Z}[\frac{1}{p}]_{\geq 0}} a_m q^m \mapsto \sum_{m\in\frac{1}{p^n}\mathbb{Z}_{\geq 0}} a_m q^m$$

where the bottom map is given by forgetting all coefficients a_m for $m \notin \frac{1}{p^n}\mathbb{Z}_{\geq 0}$.

Proof. Let us treat the case of $n = 0$, the other cases are completely analogous. By continuity, tr_n is uniquely determined by the normalised traces $\mathrm{tr}_{k,0}$. By Lemma 2.22, this is on q-expansions the trace of the inclusion $\mathcal{O}_L[\![q]\!] \to \mathcal{O}_L[\![q^{1/p^k}]\!]$. Since after inverting q, this map becomes Galois with automorphisms $q^{1/p^k} \mapsto q^{1/p^k}\zeta^d_{p^k}$ for $d \in \mathbb{Z}/p^k\mathbb{Z}$, we compute

$$\mathrm{tr}_{k,0}\left(\sum_{i=0}^{\infty} a_{\frac{i}{p^k}} q^{\frac{i}{p^k}}\right) = \frac{1}{p^k}\sum_{i=0}^{\infty} a_{\frac{i}{p^k}}(1+\zeta^i_{p^k}+\dots+\zeta^{(p^k-1)i}_{p^k})q^{\frac{i}{p^k}} = \sum_{i=0}^{\infty} a_i q^i$$

as $1+\zeta^i_{p^k}+\dots+\zeta^{(p^k-1)i}_{p^k} = 0$ unless $p^k | i$, when it is $= p^k$, giving the desired description. □

Proposition 6.9 (q-expansion principle II). *Let $f \in \mathcal{O}(\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a)$. Then for any $n \in \mathbb{Z}_{\geq 0}$, the following are equivalent:*

1. *f comes via pullback from $\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$, i.e. f is already contained in $\mathcal{O}(\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a) \subseteq \mathcal{O}(\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a)$.*
2. *The q-expansion of f at every cusp x is already contained in $\mathcal{O}_{L_x}[\![q^{1/p^n}]\!][\frac{1}{p}] \subseteq \mathcal{O}_{L_x}[\![q^{1/p^\infty}]\!][\frac{1}{p}]$.*
3. *On each connected component of $\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$, there is at least one cusp x at which the q-expansion of f is already contained in $\mathcal{O}_{L_x}[\![q^{1/p^n}]\!][\frac{1}{p}] \subseteq \mathcal{O}_{L_x}[\![q^{1/p^\infty}]\!][\frac{1}{p}]$.*

The analogous statements for $\mathcal{X}'^*(\epsilon)^{\mathrm{perf}} \to \mathcal{X}'^*(\epsilon)$ are also true.

Proof. It suffices to prove that (3) implies (1). Clearly f is in $\mathcal{O}(\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a)$ if and only if $\mathrm{tr}_n(f) = f$. By Proposition 6.1, this can be checked on q-expansions on each component. By Proposition 6.8, we have $\mathrm{tr}_n(f) = f$ if and only if the q-expansion at each x is in $\mathcal{O}_L[\![q^{1/p^n}]\!][\frac{1}{p}]$.
The case of $\mathcal{X}'^*(\epsilon)^{\mathrm{perf}}$ is completely analogous, by replacing the normalised Tate traces of [Sch15] with those of [AIP18, §6.3]. □

References

[AIP18] Fabrizio Andreatta, Adrian Iovita, and Vincent Pilloni. Le halo spectral. *Ann. Sci. Éc. Norm. Supér. (4)* 51(3):603–655, 2018.

[BGH+22] Clifford Blakestad, Damián Gvirtz, Ben Heuer, Daria Shchedrina, Koji Shimizu, Peter Wear, and Zijian Yao. Perfectoid covers of abelian varieties. *Math. Res. Lett.* 29(3):631–662, 2022.

[BHW] Christopher Birkbeck, Ben Heuer, and Chris Williams. Overconvergent Hilbert modular forms via perfectoid modular varieties. Preprint, 2019. *To appear in Ann. Inst. Fourier (Grenoble).* https://arxiv.org/abs/1902.03985.

[CHJ17] Przemysław Chojecki, David Hansen, and Christian Johansson. Overconvergent modular forms and perfectoid Shimura curves. *Doc. Math.* 22:191–262, 2017.

[Con06] Brian Conrad. Modular curves and rigid-analytic spaces. *Pure Appl. Math. Q.* 2(1, Special Issue: In honor of John H. Coates. Part 1):29–110, 2006.

[Con07] Brian Conrad. Arithmetic moduli of generalized elliptic curves. *J. Inst. Math. Jussieu* 6(2):209–278, 2007.

[dJ95] Aise Johan de Jong. Crystalline Dieudonné module theory via formal and rigid geometry. *Inst. Hautes Études Sci. Publ. Math.* 82:5–96, 1995.

[Heu19] Ben Heuer. *Perfectoid geometry of p-adic modular forms.* PhD thesis, King's College London, 2019. https://kclpure.kcl.ac.uk/portal/en/theses/perfectoid-geometry-of-padic-modular-forms(30325fef-65bb-4482-a813-e8fb23de2b96).html.

[Kat73] Nicholas M. Katz. p-adic properties of modular schemes and modular forms. In *Modular functions of one variable, III*, 69–190. Lecture Notes in Math., 350. Springer, Berlin, 1973.

[KM85] Nicholas M. Katz and Barry Mazur. *Arithmetic moduli of elliptic curves. Annals of Mathematics Studies*, 108. Princeton University Press, Princeton, NJ, 1985.

[PS16] Vincent Pilloni and Benoît Stroh. Cohomologie cohérente et représentations Galoisiennes. *Ann. Math. Qué.* 40(1):167–202, 2016.

[Sch12] Peter Scholze. Perfectoid spaces. *Publ. Math. Inst. Hautes Études Sci.* 116:245–313, 2012.

[Sch15] Peter Scholze. On torsion in the cohomology of locally symmetric varieties. *Ann. of Math. (2)* 182(3):945–1066, 2015.

[SW13] Peter Scholze and Jared Weinstein. Moduli of p-divisible groups. *Camb. J. Math.* 1(2):145–237, 2013.

[SW20] Peter Scholze and Jared Weinstein. *Berkeley lectures on p-adic geometry.* Annals of Mathematics Studies, 207. Princeton University Press, Princeton, NJ, 2020.

Ben Heuer
Institut für Mathematik
Johann Wolfgang Goethe-Universität
Robert-Mayer-Str. 6-8
60325 Frankfurt am Main
Germany
heuer@math.uni-frankfurt.de