Crystalline Lifts and a Variant of the Steinberg–Winter Theorem

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Abstract. Let $K/\mathbb{Q}_p$ be a finite extension. For all irreducible representations $\bar{\rho} : G_K \to G(\bar{\mathbb{F}}_p)$ valued in a general reductive group G, we construct crystalline lifts of $\bar{\rho}$ which are Hodge–Tate regular. We also discuss rationality questions. We prove a variant of the Steinberg–Winter theorem along the way.

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1 Introduction

Fix a connected split reductive group G.

1.1

It is often desirable to describe automorphisms of a reductive group in root-theoretic terms. When we are concerned with finite order automorphisms of a reductive group over a characteristic 0 field, the following theorem suffices

Theorem (Steinberg-Winter, [20, Theorem 7.5]). *Let M be a linear algebraic group over a field k. Let $\bar{k}$ be the algebraic closure of k. Let $F_M : M \to M$ be an automorphism of M which can be realized as conjugation by an element $g \in G(k)$ after an embedding $M \hookrightarrow G$.*
If g is a semi-simple element, then F_M fixes a maximal torus of $M_{\bar{k}}$.

However, in prime characteristic, finite order automorphisms are not always semi-simple, and we need a new criterion for the existence of maximal tori fixed by F_M.

Note that the semi-simplicity of F_M implies that the subgroup Γ_{F_M} generated by g and $Z_M(M)^\circ$ (the neutral component of the center of M) is a G-completely reducible subgroup of $G(\bar{k})$. We conjecture that the G-complete reducibility of Γ_{F_M} is sufficient for (and, to a certain degree, characterizes) the existence of an F_M-fixed maximal torus of M.

The notion of G-complete reducibility was introduced by Serre. In his Moursund Lectures [18], a subgroup $\Gamma \subset G(\bar{k})$ is defined to be *G-completely reducible* if for any parabolic subgroup P of $G_{\bar{k}}$ containing Γ, a Levi subgroup of P also contains Γ. Similarly, we say a subgroup $\Gamma \subset G(\bar{k})$ is *G-irreducible* if it is not contained in any proper parabolic subgroup of G.

We prove the following:

Theorem 1 (3, 1). *Let M be a connected reductive group over a field k. Let $\bar{k}$ be the algebraic closure of k. Let $F_M : M \to M$ be an automorphism of M which can be realized as conjugation by an element $g \in G(k)$ after an embedding $M \hookrightarrow G$.*

Let Γ_{F_M} be the subgroup of G generated by g and $Z_M(M)^\circ$. If either

- *Γ_{F_M} is G-irreducible or*
- *Γ_{F_M} is G-completely reducible, $\operatorname{rk} M = \operatorname{rk} G$, $\operatorname{char} k \neq 2$ or 3, and M has connected center.*

then F_M fixes a maximal torus of $M_{\bar{k}}$.

which suffices for our application to the theory of Galois representations.

We believe our new method can be used to establish a stronger form of Steinberg-Winter by developing the theory of G-complete reducibility for (possibly disconnected) linear algebraic groups using dynamic methods (2.1). We don't pursue this because we don't want to digress too much. We do explain how this can possibly be done.

1.2

Let $K/\mathbb{Q}_p$ be a finite extension. We are interested in the following question:

Question 1. *Let $\bar{\rho} : G_K \to G(\bar{\mathbb{F}}_p)$ be a group homomorphism. Does there exist a crystalline representation $\rho : G_K \to G(\bar{\mathbb{Z}}_p)$ such that $\rho \equiv \bar{\rho}$?*

Question 1 is raised in [3] for $G = \mathrm{GL}_N$, where they used the machinery of p-adic Hodge theory to study general torsion Galois representations; it also has global applications such as constructing geometric Galois representations (see, for example, [7]) which conjecturally correspond to algebraic automorphic forms.

Any characteristic p representation of G_K is an extension of G-completely reducible representations. To construct crystalline lifts of general characteristic p representations of G_K, it is a common strategy (for example [6] for GL_N and [13] for G_2 and classical groups) to first construct lifts of G-completely reducible representations, and then try to lift the extension class.
In this paper, we carry out the first steps of the above strategy. We prove the following theorem:

Theorem 2 (5, 6, 2). *Let κ be the residue field of K. Let K^{ur} be the maximal unramified extension of K in a fixed algebraic closure. Let $\mathbb{F}/\kappa$ be a finite extension of degree f.*
Let $\bar{\rho}: G_K \to G(\mathbb{F})$ be a group homomorphism whose image is a G-completely reducible subgroup. Assume G is split.

- *There exists a characteristic 0 lift $\rho: G_K \to G(W(\bar{\mathbb{F}}_p))$ of $\bar{\rho}$;*
- *There exists a Hodge-Tate regular crystalline lift $\rho: G_K \to G(K^{\mathrm{ur}})$ of $\bar{\rho}$.*

We discuss Hodge-Tate theory of Galois representations valued in general reductive groups in section 5.

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2 A variant of Steinberg-Winter theorem

The key tool in this section is dynamic methods.

2.1 Dynamic methods

We review [5, Section 4.1]. Let X be a scheme over a base scheme S, and fix a $\mathbb{G}_m$-action $m: \mathbb{G}_m \times X \to X$ on X. For each $x \in X(S)$, we say

$$\lim_{t\to 0} m(t,x) \quad \text{exists},$$

if the morphism $\mathbb{G}_m \to X$, $t \mapsto m(t,x)$ extends a a morphism $\mathbb{A}^1 \to X$. If the limit exists, the origin $0 \in \mathbb{A}^1(S)$ maps to a unique element $\alpha \in X(S)$; we write $\lim_{t\to 0} m(t,x) = \alpha$.

Let λ be a cocharacter of a reductive group G over a field k. Define the following functor on the category of k-algebras

$$P_G(\lambda)(A) = \{g \in G(A) | \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists.}\}$$

where A is a general k-algebra.
Define

$$U_G(\lambda)(A) = \{g \in G(A) | \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} = 1\},$$

and denote by $Z_G(\lambda)$ the centralizer of λ in G.
Since G is a reductive group over a field, $P_G(\lambda)$ is a parabolic subgroup of G, $U_G(\lambda)$ is the unipotent radical of $P_G(\lambda)$, and $Z_G(\lambda)$ is a Levi subgroup of $P_G(\lambda)$.
The following proposition is the first application of dynamic methods in this section, and motivates us to consider G-compete reducibility in Steinberg-Winter type questions.

Proposition 1. *Let M be a connected reductive group over a field k. Let $\bar{k}$ be the algebraic closure of k. Let $F_M : M \to M$ be an automorphism of M which can be realized as conjugation by an element $g \in G(k)$ after an embedding $M \hookrightarrow G$.*
If g is semisimple, then g and $Z_M(M)^\circ$ generate a G-completely reducible subgroup.

Proof. Let $P \subset G_{\bar{k}}$ be a parabolic subgroup of G which contains both g and $Z_M(M)^\circ$. We want to show a Levi subgroup of P also contains both g and $Z_M(M)^\circ$.
Put $L := Z_P(Z_M(M)^\circ)$, the centralizer of $Z_M(M)^\circ$ in P. Note that conjugation by g fixes L. We claim L contains a maximal torus of G. Since $Z_M(M)^\circ$ is a torus, it is contained in a maximal torus of P. A maximal torus of P is also a maximal torus of G. Any maximal torus containing $Z_M(M)^\circ$ is in the centralizer of $Z_M(M)^\circ$ because of commutativity of tori.
By Steinberg-Winter, there exists a maximal torus $T \subset L$ which is fixed by g. By the previous paragraph, T is also a maximal torus of G.
By dynamic methods, there exists a cocharacter $\lambda : \mathbb{G}_m \to T$ such that $P = P_G(\lambda)$. The two cocharacters $\lambda, g\lambda g^{-1} : \mathbb{G}_m \to T$ lie in the same maximal torus, and can be regarded as elements of the cocharacter lattice $X_*(G,T)$. Since $g \in P$, $g(P_G(\lambda))g^{-1} = P_G(g\lambda g^{-1}) = P_G(\lambda)$. So $\lambda, g\lambda g^{-1} \in X_*(G,T)$ are in the same Weyl chamber. Since $g \in N_G(T)$, $g\lambda g^{-1}$ and λ are in the same Weyl orbit, and thus we must have $\lambda = g\lambda g^{-1}$. So $g \in Z_G(\lambda)$ and $Z_M(M)^\circ \subset T \subset Z_G(\lambda)$. Since $Z_G(\lambda)$ is a Levi subgroup of P, we are done. □

2.2 A generalization of dynamic methods

Dynamic methods allow us to prove theorems over general base schemes by doing mathematical analysis. To do so, we need to generalize the functors $P_G(-)$.

Let $f : \mathbb{G}_m \to G$ be a *k-scheme morphism*. Define the following functor on the category of k-algebras

$$P_G(f)(A) = \{g \in G(A) | \lim_{t\to 0} f(t)gf(t)^{-1} \text{ exists.}\}$$

where A is a general k-algebra. We call f a *fake cocharacter*. Here "a limit exists" means the scheme morphism $\mathbb{G}_m \to G$, defined by $t \mapsto f(t)gf(t)^{-1}$, extends to a scheme morphism $\mathbb{A}^1 \to G$. Note that $P_G(f)$ is not representable in general. We define similarly $U_G(f)$.

Lemma 1. *Let G be a connected reductive group over a field k. Let $\lambda, \mu : \mathbb{G}_m \to G$ be cocharacters of G. Assume $P_G(\lambda) = P_G(\mu) =: B$ is a Borel subgroup of G. Let U be the unipotent radical of B.*
(1) The functor $P_G(\mu\lambda)$ is representable by a Borel subgroup. In fact, we have $P_G(\mu\lambda) = P_G(\mu) = P_G(\lambda)$.
(2) The limit

$$\lim_{t\to 0} \lambda(t)\mu(t)\lambda(t)^{-1}\mu(t)^{-1}$$

exists in the sense of subsection 2.1, and lies in U.
(3) Let u be an element of U. The limit

$$\lim_{t\to 0} \lambda(t)u\mu(t)u^{-1}\lambda(t)^{-1}\mu(t)^{-1}$$

exists in the sense of subsection 2.1 and lies in U.
(4) Now assume λ is a product of cocharacters $\lambda_1, \ldots, \lambda_s$ such that $P_G(\lambda_i) = B$ for all i. Then $P_G(\lambda) = B$, and the limits in (2) and (3) still exist and lie in U. Moreover, for any embedding $G \hookrightarrow H$ of connected reductive groups, $P_H(\lambda)$ is representable by a parabolic subgroup of H.

Proof. (1) Since all maximal tori in B are conjugate to each other, there exists an element $x \in U_G(\lambda) = U_G(\mu)$ such that conjugation by x maps the maximal torus containing λ to the maximal torus containing μ. In particular, $(x\lambda x^{-1})\mu = \mu(x\lambda x^{-1})$. Write ξ for $x\lambda x^{-1}$. We have

$$\begin{aligned}
&\lim_{t\to 0} \mu(t)\lambda(t)g\lambda(t)^{-1}\mu(t)^{-1} = \lim_{t\to 0} \mu(t)x^{-1}\xi(t)xgx^{-1}\xi(t)^{-1}x\mu(t)^{-1} \\
&= \lim_{t\to 0}(\mu(t)x^{-1}\mu(t)^{-1}) \cdot (\mu(t)\xi(t)xgx^{-1}\xi(t)^{-1}\mu(t)^{-1}) \cdot (\mu(t)x\mu(t)^{-1}) \\
&= \lim_{t\to 0} \mu(t)\xi(t)xgx^{-1}\xi(t)^{-1}\mu(t)^{-1}
\end{aligned}$$

Note that the last step is because $x \in U_G(\mu)$, and $\lim_{t\to 0} \mu(t)x\mu(t)^{-1} = 1$. So we have $P_G(\mu\lambda) = x^{-1}P_G(\mu\xi)x$. Since $\mu\xi$ is a genuine cocharacter, $P_G(\mu\xi)$ is representable by a parabolic.
Since $\mu\xi = \xi\mu$, we can regard μ and ξ as elements in a cocharacter lattice $X_*(G,T)$ where T is a maximal torus containing μ and ξ. Since $P_G(\mu) = P_G(\lambda) = P_G(\xi)$, μ and ξ lie in the (interior of the) same Weyl chamber. The cocharacter $\mu\xi$ is the sum of μ and ξ in the cocharacter lattice $X_*(G,T)$, and

lies in the same Weyl chamber. So $P_G(\mu\xi) = P_G(\mu) = P_G(\lambda)$. Since $x \in B$, we have $P_G(\mu\lambda) = x^{-1}P_G(\mu\xi)x = P_G(\mu) = P_G(\lambda)$.
(2) Since all maximal tori of B are conjugate to each other, there exists an element $g \in U$ such that $gZ_G(\lambda)g^{-1} = Z_G(\mu)$. Write $\xi := g\lambda g^{-1}$, and we have $\xi\mu = \mu\xi$. By part (1), $P_G(\xi\mu) = B$. By the dynamic description of the Borel B, the limits

$$\lim_{t\to 0} \xi(t)g\xi(t)^{-1} = 1,$$
$$\lim_{t\to 0} \xi(t)\mu(t)g^{-1}\mu(t)^{-1}\xi(t)^{-1} = 1, \text{ and}$$
$$\lim_{t\to 0} \mu(t)g\mu(t)^{-1} = 1$$

all exist. The expression

$$\begin{aligned}\lambda(t)\mu(t)\lambda(t)^{-1}\mu(t)^{-1} &= g^{-1}\xi(t)g\mu(t)g^{-1}\xi(t)^{-1}g\mu(t)^{-1}\\ &= g^{-1}\cdot(\xi(t)g\xi(t)^{-1})\cdot(\xi(t)\mu(t)g^{-1}\mu(t)^{-1}\xi(t)^{-1})\cdot(\mu(t)g\mu(t)^{-1})\end{aligned}$$

has a limit as $t \to 0$.
(3) We have

$$\begin{aligned}&\lambda(t)u\mu(t)u^{-1}\lambda(t)^{-1}\mu(t)^{-1}\\ =&(\lambda(t)u\mu(t)u^{-1}\lambda(t)^{-1}u\mu(t)^{-1}u^{-1})(u\mu(t)u^{-1}\mu(t)^{-1}).\end{aligned}$$

So (3) follows from (2).
(4) The method is the same but notations are more complicated. We define inductively cocharacters $\xi_i$that commute with each other, and elements u_i of U. Our induction assumption is $P_G(\lambda_1\cdots\lambda_j) = P_G(\xi_1\cdots\xi_j) = B$ for all $j < s$. Define $\xi_1 := \lambda_1$ and $u_1 := 1$. Let u_i be an element of U such that $\xi_i := u_i\lambda_i u_i^{-1}$ commutes with $\xi_1\cdots\xi_{i-1}$. Write ζ_j for $\xi_1\xi_2\cdots\xi_j$, and write v_j for u_j/u_{j-1} (set $u_0 = 1$). We have, for $g \in G$,

$$\begin{aligned}\lambda(t)g\lambda(t)^{-1} =&(\zeta_1(t)v_2\zeta_1(t)^{-1})(\zeta_2(t)v_3\zeta_2(t)^{-1})\cdots\\ &(\zeta_s(t)u_sgu_s^{-1}\zeta_s(t)^{-1})\\ &(\zeta_{s-1}(t)v_s\zeta_{s-1}(t)^{-1})^{-1}\cdots(\zeta_1(t)v_2\zeta_1(t)^{-1})^{-1} \qquad (\dagger)\end{aligned}$$

which has a limit if and only if $g \in B$. Similarly,

$$\begin{aligned}&\lambda(t)\mu(t)\lambda(t)^{-1}\mu(t)^{-1}\\ =&(\zeta_1(t)v_2\zeta_1(t)^{-1})(\zeta_2(t)v_3\zeta_2(t)^{-1})\cdots\\ &(\zeta_s(t)u_s\mu(t)u_s^{-1}\zeta_s(t)^{-1}\mu(t)^{-1}\\ &\mu(t)(\zeta_{s-1}(t)v_s\zeta_{s-1}(t)^{-1})^{-1}\cdots(\zeta_1(t)v_2\zeta_1(t)^{-1})^{-1}\mu(t)^{-1}\end{aligned}$$

By (1), $P_G(\mu\zeta_j) = B$ for all j, and therefore each of the factors

$$\mu(t)(\zeta_j(t)v_{j+1}\zeta_j(t)^{-1})^{-1}\mu(t)^{-1}$$

admits a limit 1. So $\lambda(t)\mu(t)\lambda(t)^{-1}\mu(t)^{-1}$ admits a limit in U by (3).
Next we consider the "moreover" part. (†) holds for $g \in H$ as well. So $P_H(\lambda) = u_s^{-1} P_H(\zeta_s) u_s$ is a parabolic subgroup of H. □

Lemma 2. *Let $F : M \to M$ be an automorphism of a connected reductive group. Let $B \subset M$ be a Borel subgroup fixed by F, with unipotent radical U. There exists a cocharacter μ of M, a positive integer d and an element u of U such that $\mu = uF^d(\mu)u^{-1}$ and $B = P_M(\mu)$.*

Proof. By replacing M by its derived subgroup, we can and do assume M is semi-simple. Let μ be a cocharacter of M such that $B = P_M(\mu)$.
Let $i \geq 0$ be an integer. There exists a maximal torus T_i of B such that $F^i(\mu) \subset T_i$. Since all maximal tori of B are conjugate by an element of U, there exists an element u_i of U such that $T_0 = u_i T_i u_i^{-1}$.
So $u_i^{-1} F^i(\mu) u_i \subset T_0$, and we can regard it as an element x_i of the cocharacter lattice $X_*(M, T_0)$. Since μ is a regular cocharacter, its centralizer $Z_M(\mu)$ is a maximal torus of M, and thus is just T_0. Since automorphisms of M send the centralizers to the centralizers, $u_i^{-1} F^i u_i : M \to M$ fixes T_0. Recall that $\mathrm{Aut}(M) \subset \mathrm{Inn}(M) \rtimes \mathrm{Aut}(\mathrm{Dynkin}(\Phi(M, T_0)))$, that is, after fixing a pinning, an automorphism of M comes from an automorphism of its Dynkin diagram. Since $u_i^{-1} F^i u_i$ fixes T_0 and B, it induces an isomorphism of the Dynkin diagram of M and thus induces an isometry of the coroot lattice of M. Since M is semi-simple, its coroot lattice and its cocharacter lattice span the same $\mathbb{R}$-vector space, and thus $u_i^{-1} F^i u_i$ induces an isometry of $X_*(M, T_0) \otimes_{\mathbb{Z}} \mathbb{R}$. In particular, the set $\{x_i\}$ is bounded and thus finite. So $x_{i_0} = x_{i_0+d}$ for some $i_0 \geq 0$ and $d > 0$. We have $u_{i_0}^{-1} F^{i_0}(\mu) u_{i_0} = u_{i_0+d}^{-1} F^{i_0+d}(\mu) u_{i_0+d}$. Thus $\mu = u_{i_0} u_{i_0+d}^{-1} F^d(\mu) u_{i_0+d} u_{i_0}^{-1}$. □

Recall a subgroup $\Gamma \subset G(\bar{k})$ is said to be G-irreducible if Γ is not contained in any proper parabolic subgroup of $G(\bar{k})$.

Theorem 3. *Let M be a connected reductive group over a field k. Let $\bar{k}$ be the algebraic closure of k. Let $F_M : M \to M$ be an automorphism of M which can be realized as conjugation by an element $g \in G(k)$ after an embedding $M \hookrightarrow G$. If g and $Z_M(M)$ generate a G-irreducible subgroup, then M is a torus.*

Proof. One of the key ingredients is the results of Steinberg on endormorphisms of linear algebraic groups. By [20, Theorem 7.2], any automorphism of a linear algebraic group fixes a Borel subgroup. Let $B_M \subset M$ be a Borel fixed by F_M. There exists a cocharacter $\lambda : \mathbb{G}_m \to M$ such that $B_M = P_M(\lambda)$. Let U_M be the unipotent radical of B_M. By the previous lemma, there exists $d > 0$ and an element u of U_M such that $F_M^d(\lambda) = u\lambda u^{-1}$. Consider the fake cocharacter $\mu : \mathbb{G}_m \to M$, defined by

$$\mu := F_M^{d-1}(\lambda) F_M^{d-2}(\lambda) \cdots F_M(\lambda)\lambda.$$

Note that $F_M(\mu) = (u\lambda u^{-1})\mu\lambda^{-1}$.
By Lemma 1, we have

(i) $P_G(\mu)$ is representable by a parabolic subgroup of G;

(ii) $P_M(\mu) = P_M(\lambda) = M \cap P_G(M)$.

Claim $g \in P_G(\mu)$.

Proof. We verify this using the definition of $P_G(\mu)$. We have

$$\begin{aligned}\lim_{t\to 0}\mu(t)g\mu(t)^{-1} &= \lim_{t\to 0}\mu(t)g\mu(t)^{-1}g^{-1}g\\ &= \lim_{t\to 0}\mu(t)F_M(\mu)(t)^{-1}g\\ &= \lim_{t\to 0}\mu(t)\lambda(t)\mu(t)^{-1}u\lambda(t)^{-1}u^{-1}g\\ &= \lim_{t\to 0}(\mu(t)\lambda(t)\mu(t)^{-1}\lambda(t)^{-1})(\lambda(t)u\lambda(t)^{-1})u^{-1}g\end{aligned}$$

The claim follows from Lemma 1 (4). □

Note that since μ is valued in M, $Z_M(M) \subset Z_G(\mu)$.
Let Γ be the subgroup of G generated by $Z_M(M)^\circ$ and g. As a consequence of the claim, we have $\Gamma \subset P_G(\mu)$. By Lemma 1 (1), $P_M(\mu) = P_M(\lambda)$ is a Borel subgroup of M. By the dynamic description of Borel subgroups, we have $P_G(\mu)\cap M = P_M(\mu)$. So $P_G(\mu)$ is a proper parabolic subgroup of G if $P_M(\mu)$ is a proper parabolic subgroup of M. Since Γ is assumed to be G-irreducible, we must have $P_M(\mu) = M$. Since $M = P_M(\mu) = B_M$ is chosen to be a Borel subgroup of M, M is forced to be a torus. □

Corollary 1. *Let M be a connected reductive group over a field k. Let $\bar{k}$ be the algebraic closure of k. Let $F_M : M \to M$ be an automorphism of M which can be realized as conjugation by an element $g \in G(k)$ after an embedding $M \hookrightarrow G$.*
Assume

(i) g and $Z_M(M)^\circ$ generate a G-completely reducible subgroup;

(ii) $\operatorname{rk} M = \operatorname{rk} G$ and $\operatorname{char} k \neq 2, 3$; and

(iii) M has a connected center.

Then F_M fixes a maximal torus T of $M_{\bar{k}}$.

Proof. Let Γ be the subgroup of G generated by $Z_M(M)$ and g. If Γ is G-irreducible, we are done because of Theorem 3. So we assume there exists a proper parabolic subgroup P of $G_{\bar{k}}$ such that $\Gamma \subset P$.
By Borel-de Siebenthal theory (see [16] or [10, Theorem 0.1]), when k $\neq 2, 3$, $\operatorname{rk} M = \operatorname{rk} G$ implies $M = Z_G(Z_M(M))^\circ$.
We will prove a slightly stronger version of the corollary. We claim F_M fixes a maximal torus of $M_{\bar{k}}$ assuming (i), (ii), and

(iii') There exists a torus Z of M such that $M = Z_G(Z)^\circ$.

Since Γ is G-completely reducible, there exists a Levi subgroup $L \subset P$ such that $\Gamma \subset L$. Note that $(M \cap L)^\circ = Z_L(Z)^\circ$, which is a reductive subgroup (see [10, Lemma 0.2(1)]) of L fixed by g. We claim $(M \cap L)^\circ$ is of maximal rank. Let S be any maximal torus of L containing Z. Since S is commutative and connected, we have $S \subset Z_L(Z)^\circ = (M \cap L)^\circ$. Thus $\operatorname{rk}(M \cap L)^\circ = \operatorname{rk} S = \operatorname{rk} L = \operatorname{rk} G$.
We apply induction on the dimension of G. Since $Z_L(Z)^\circ = (M \cap L)^\circ$, assumption (iii') is satisfied by $(M \cap L)^\circ$; assumption (i) is also satisfied because L is a Levi subgroup of G. Since $\dim L < \dim G$, by induction there exists a maximal torus T of $(M \cap L)^\circ$ which is fixed by F_M. Since $\operatorname{rk}(M \cap L)^\circ = \operatorname{rk} G$, T is also a maximal torus of G. □

2.3

We explain how our methods can possibly be used to establish a stronger form of Steinberg-Winter, at least for groups having connected center. Dynamic methods are very well behaved for disconnected linear algebraic groups. We similarly define G-complete reducibility for general linear algebraic groups by replacing parabolics by pseudo-parabolics. Let $F : M \to M$ be an automorphism which can be realized as conjugation by an element g of G after an embedding $M \hookrightarrow G$. Let H be the scheme-theoretic closure of the (abstract) group generated by M and g. Note that H is a disconnected reductive group, and $\operatorname{rk} H = \operatorname{rk} M$. Let Γ be the subgroup of H generated by $Z_M(M)^\circ$ and g. We expect that the H-complete reducibility of Γ implies the existence of a fixed maximal torus.

3 The structure of G-completely reducible mod ϖ Galois representations

In this section, we give a complete description of all G-completely reducible mod ϖ Galois representations valued in split reductive groups.
The first step is to show G-complete reducibility implies tame ramification, reducing the classification of mod ϖ Galois representations to the question of classification of (certain) solvable subgroups of derived length 2 of reductive groups.

Lemma 3. *Let P_K be the wild inertia of G_K. If $\bar\rho : G_K \to G(\bar{\mathbb{F}}_p)$ is G-completely reducible, $\bar\rho(P_K) = \{\mathrm{id}\}$.*

Proof. Let $P_K \subset G_K$ be the wild inertia. The image $\bar\rho(P_K) \subset G(\bar{\mathbb{F}}_p)$ is a p-group, and thus consists of unipotent elements. By [2, Corollaire 3.9], there exists a parabolic subgroup P of $G_{\bar{\mathbb{F}}_p}$ with unipotent radical $R_u(P)$ such that

- $\bar\rho(P_K) \subset R_u(P)(\bar{\mathbb{F}}_p)$, and
- $N(\bar\rho(P_K)) \subset P(\bar{\mathbb{F}}_p)$;

here $N(\bar{\rho}(P_K))$ is the normalizer of $\bar{\rho}(P_K)$. Since P_K is a normal subgroup of G_K, $\bar{\rho}(G_K) \subset N(\bar{\rho}(P_K)) \subset P(\bar{\mathbb{F}}_p)$. Since $\bar{\rho}$ is G-completely reducible, $\bar{\rho}(G_K)$ is contained in a Levi subgroup L of P. So $\bar{\rho}(P_K) \subset L(\bar{\mathbb{F}}_p) \cap R_u(P)(\bar{\mathbb{F}}_p) = \{\mathrm{id}\}$. □

Definition 1. *We say $\bar{\rho} : G_K \to G(\bar{\mathbb{F}}_p)$ is quasi-semisimple if there exists a maximal torus T of $G(\bar{\mathbb{F}}_p)$ such that $\bar{\rho}(I_K) \subset T(\bar{\mathbb{F}}_p)$ and $\bar{\rho}(G_K) \subset N_G(T(\bar{\mathbb{F}}_p))$.*

Theorem 4. *If $\bar{\rho} : G_K \to G(\bar{\mathbb{F}}_p)$ is G-completely reducible, then $\bar{\rho}$ is quasi-semisimple.*
Moreover, if $\bar{\rho}$ is G-irreducible, there exists a unique maximal torus T of $G(\bar{\mathbb{F}}_p)$ containing $\bar{\rho}(I_K)$. Consequently, if $\bar{\rho}(G_K) \subset G(\mathbb{F})$, T has a model defined over the ring of Witt vectors $W(\mathbb{F})$.

Proof. By induction on the dimension of G, we can reduce the general case to the case where $\bar{\rho}$ is G-irreducible. Recall that $\bar{\rho}$ is G-irreducible if it does not factor through any proper parabolic of G. If $\bar{\rho}$ does factor through a proper parabolic of G, the G-complete reducibility forces $\bar{\rho}$ to factor through a proper Levi subgroup of G, which is a reductive group of strictly smaller dimension.

So we assume $\bar{\rho}$ is G-irreducible in the rest of the proof. By Lemma 3, $\bar{\rho}(I_K)$ is a finite cyclic group generated by elements of order prime to p. Write M for $Z^\circ_{G(\bar{\mathbb{F}}_p)}(\bar{\rho}(I_K))$, the neutral component of the centralizer of $\bar{\rho}(I_K)$ in G. Since $\bar{\rho}(I_K)$ consists of semi-simple elements of $G(\mathbb{F})$, M is a reductive subgroup of G. Let $\Phi_K \in G_K$ be a topological generator of G_K/I_K. Since I_K is a normal subgroup of G_K, the conjugation by $\bar{\rho}(\Phi_K)$ action induces an automorphism of M, which we denote by $F_M : M \to M$.

Next we show $\bar{\rho}(I_K) \subset Z_M(M)$. Since G is connected, a semisimple element of G is contained in a maximal torus. Since $\bar{\rho}(I_K)$ is a cyclic group consisting of semi-simple elements, there exists a maximal torus T containing $\bar{\rho}(I_K)$. Since a torus is connected, we have $T \subset M$, and thus $\bar{\rho}(I_K) \subset M(\bar{\mathbb{F}}_p)$. It is immediate from the definition of M that $\bar{\rho}(I_K) \subset Z_M(M)$.

By Theorem 3, M is a torus. Let T be any maximal torus of G containing $\bar{\rho}(I_K)$. Since T is commutative and connected, we have $T \subset Z_G(\bar{\rho}(I_K))^\circ = M$. So M is the unique maximal torus containing $\bar{\rho}(I_K)$. Now consider the "moreover" part. For $\sigma \in \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F})$, $\sigma(M)$ is also a maximal torus containing $\bar{\rho}(I_K)$. So $\sigma(M) = M$, and thus by Galois descent M is defined over $\mathbb{F}$. By [5, B.3.5], T has a model over $W(\mathbb{F})$. □

3.1 Example

We illustrate the technical proof using a very concrete example. Let $G = \mathrm{GL}_4$. Let $\bar{\rho} : G_K \to \mathrm{GL}_4(\bar{\mathbb{F}}_p)$ be a semi-simple Galois representation. We decompose $V = V_{\chi_1} \oplus V_{\chi_2}$ into I_K-isotropic subspaces. Here $\chi_1, \chi_2 : I_K \to \bar{\mathbb{F}}_p^\times$ are distinct characters such that for $v \in V_{\chi_i}$ and $\sigma \in I_K$, $\bar{\rho}(\sigma)v = \chi_i(\sigma)v$, $i = 1, 2$.

$$\bar{\rho}|_{I_K} = \begin{bmatrix} \chi_1 & & & \\ & \chi_1 & & \\ & & \chi_2 & \\ & & & \chi_2 \end{bmatrix}$$

There are two possibilities: either both V_i are $\bar{\rho}(\Phi_K)$-stable, or $\bar{\rho}(\Phi_K)$ sends V_i to V_{3-i}, $i = 1, 2$. The first case is simple: $V = V_{\chi_1} \oplus V_{\chi_2}$ as a G_K-module. Now we consider the latter case. By Steinberg's theorem [20, Theorem 7.2], we can assume $\bar{\rho}(\Phi_K)$ fixes a Borel

$$P_M = \begin{bmatrix} * & * & & \\ & * & & \\ & & * & * \\ & & & * \end{bmatrix}$$

of $M = \mathrm{GL}_2 \times \mathrm{GL}_2 \subset \mathrm{GL}_4$ and thus we must have

$$\bar{\rho}(\Phi_K) = \begin{bmatrix} & & a & b \\ & & & c \\ d & f & & \\ & e & & \end{bmatrix}$$

for some $a, b, c, d, e, f \in \bar{\mathbb{F}}_p$. The Borel subgroup P_M is of shape $P_M(\lambda)$ for

$$\lambda(t) = \begin{bmatrix} t^\alpha & * & & \\ & t^\beta & & \\ & & t^\gamma & * \\ & & & t^\delta \end{bmatrix}$$

for $\alpha > \beta$ and $\gamma > \delta$. We have

$$\bar{\rho}(\Phi_K)\lambda(t)\bar{\rho}(\Phi_K)^{-1} = \begin{bmatrix} t^\gamma & * & & \\ & t^\delta & & \\ & & t^\alpha & * \\ & & & t^\beta \end{bmatrix}$$

and thus

$$\bar{\rho}(\Phi_K)\lambda(t)\bar{\rho}(\Phi_K)^{-1}\lambda(t) = \begin{bmatrix} t^{\alpha+\gamma} & * & & \\ & t^{\beta+\delta} & & \\ & & t^{\alpha+\gamma} & * \\ & & & t^{\beta+\delta} \end{bmatrix}$$

Since $\alpha + \gamma > \beta + \delta$, we have

$$P_{\mathrm{GL}_4}(\bar{\rho}(\Phi_K)\lambda\bar{\rho}(\Phi_K)^{-1}\lambda) = \begin{bmatrix} * & * & * & * \\ & * & & * \\ * & * & * & * \\ & * & & * \end{bmatrix}$$

and finally we observe $\bar{\rho}(\Phi_K) \in P_{\mathrm{GL}_4}(\bar{\rho}(\Phi_K)\lambda\bar{\rho}(\Phi_K)^{-1}\lambda)$. In general, by Lemma 2, there exists an integer d such that $\prod_{i=d-1}^{0} \bar{\rho}(\Phi_K)^i\lambda(t)\bar{\rho}(\Phi_K)^{-i}$ gives the desired parabolic.

4 Crystalline lifts of irreducible mod ϖ Galois representations

Write κ for the residue field of K. Fix a coefficient field E with ring of integers $\mathcal{O}$ and uniformizer ϖ. Write $\mathbb{F}$ for the residue field $\mathcal{O}/\varpi$. Assume $\kappa \subset \mathbb{F}$. Let $\Phi_K \in G_K$ be a (lift of a) topological generator of G_K/I_K. Fix an algebraic closure $\bar{K}$ of K.

In this section, we assume G is a split group since we are primarily interested in Galois representations valued in L-groups. The L-group of a connected reductive group is split, albeit possibly disconnected.

4.1

Let T be a maximal torus of G. More precisely, T is a smooth group scheme over $\operatorname{Spec}\mathcal{O}$ such that $T_{\bar{k}} \subset G_{\bar{k}}$ is a maximal torus for all geometric points $\mathcal{O} \to \bar{k}$. Write $W(G,T)$ for the Weyl group scheme $N_G(T)/T$. Write $\mathcal{O}_{\bar{K}}$ for the ring of integers in $\bar{K}$. Note that we have a commutative diagram

$$\begin{tikzcd} W(G,T)(\mathcal{O}) \arrow[r, hook] \arrow[d] & W(G,T)(\mathcal{O}_{\bar{K}}) \arrow[r, equal] \arrow[d, equal] & W(G,T)(\bar{K}) \\ W(G,T)(\mathbb{F}) \arrow[r, hook] & W(G,T)(\bar{\mathbb{F}}_p) & \end{tikzcd}$$

and as a consequence, the map $W(G,T)(\mathcal{O}) \to W(G,T)(\mathbb{F})$ is injective. On the other hand, since $N_G(T)$ is a smooth group scheme over $\operatorname{Spec}\mathcal{O}$, $W(G,T)(\mathcal{O}) \to W(G,T)(\mathbb{F})$ is also surjective. We will identify $W(G,T)(\mathcal{O})$ with $W(G,T)(\mathbb{F})$ and write it as $W(G,T)$. It will be clear from the context if the notation $W(G,T)$ denotes a set or a group scheme.

Write $M_{T,\mathrm{cris}}$ for the set of representations $I_K \to T(\mathcal{O})$, which can be extended to a crystalline representation $G_{K'} \to T(\mathcal{O})$ for some finite unramified extension K'/K inside $\bar{K}$. Since the union of two finite unramified extensions inside $\bar{K}$ is still a finite unramified extension, $M_{T,\mathrm{cris}}$ is an abelian group.

The abelian group $M_{T,\mathrm{cris}}$ has a $\mathbb{Z}[W(G,T)]$-module structure, defined by $wv := (\sigma \mapsto wv(\sigma)w^{-1})$, for $w \in W(G,T)$ and $v \in M_{T,\mathrm{cris}}$.

The abelian group $M_{T,\mathrm{cris}}$ also has a $\mathbb{Z}[G_K/I_K]$-module structure, defined by $\alpha v := (\sigma \mapsto v(\alpha^{-1}\sigma\alpha))$ for $\alpha \in G_K$ and $v \in M_{T,\mathrm{cris}}$.

The following lemma is clear.

Lemma and Definition The $\mathbb{Z}[W(G,T)]$-structure and the $\mathbb{Z}[G_K/I_K]$-structure on $M_{T,\mathrm{cris}}$ commute with each other. Therefore $M_{T,\mathrm{cris}}$ is a $\mathbb{Z}[W(G,T)] \otimes \mathbb{Z}[G_K/I_K]$-module.

Similarly, write $M_{T,\mathbb{F}}$ for the abelian group of mod ϖ representations $I_K \to T(\mathbb{F})$. The abelian group $M_{T,\mathbb{F}}$ has a $\mathbb{Z}[W(G,T)]\otimes\mathbb{Z}[G_K/I_K]$-module structure.

Lemma 4. *Write $\zeta : N_G(T) \to W(G,T)$ for the quotient map.*
(1) Let w be an element of $N_G(T)(\mathcal{O})$ of finite order. An element $v \in M_{T,\mathrm{cris}}$ extends to a continuous representation $\rho : G_K \to N_G(T)(\mathcal{O})$ by setting $\rho(\Phi_K) = w^{-1}$ and $\rho|_{I_K} = v$ if and only if

$$v \in \ker(M_{T,\mathrm{cris}} \xrightarrow{\zeta(w)\otimes 1 - 1\otimes\Phi_K} M_{T,\mathrm{cris}}).$$

(2) Let $\bar{w}$ be an element of $N_G(T)(\mathbb{F})$. An element $v \in M_{T,\mathbb{F}}$ extends to a representation $\bar{\rho} : G_K \to N_G(T)(\mathbb{F})$ by setting $\bar{\rho}(\Phi_K) = \bar{w}^{-1}$ and $\bar{\rho}|_{I_K} = v$ if and only if

$$v \in \ker(M_{T,\mathbb{F}} \xrightarrow{\zeta(\bar{w})\otimes 1 - 1\otimes\Phi_K} M_{T,\mathbb{F}}).$$

Proof. (1) Since w is of finite order, it suffices to show $v \in M_{T,\mathrm{cris}}$ extends to a representation $\rho : W_K \to N_G(T)(\mathcal{O})$ of the Weil group $W_K \cong I_K \rtimes \mathbb{Z}$ by setting $\rho(\Phi_K) = w^{-1}$ and $\rho|_{I_K} = v$ if and only if $v \in \ker(M_{T,\mathrm{cris}} \xrightarrow{\zeta(w)\otimes 1 - 1\otimes\Phi_K} M_{T,\mathrm{cris}})$. If v is extendable to ρ, then for all $\sigma \in I_K$

$$\rho(\Phi_K^{-1}\sigma\Phi_K) = w\rho(\sigma)w^{-1};$$

the left hand side restricted to I_K is $(1\otimes\Phi_K)v$, and the right hand side restricted to I_K is $(\zeta(w)\otimes 1)v$. So $(1\otimes\Phi_K)v = (\zeta(w)\otimes 1)v$. Conversely, if $(1\otimes\Phi_K)v = (\zeta(w)\otimes 1)v$, then $v(\Phi_K^{-1}\sigma\Phi_K) = wv(\sigma)w^{-1}$ for all $\sigma \in I_K$. Define $\rho(\sigma\Phi^n) := v(\sigma)w^{-n}$ for all $\sigma \in I_K$ and $n \in \mathbb{Z}$. It is clear ρ is well-defined on W_F, and extends to G_F uniquely by continuity.
(2) is similar to (1). ☐

Definition 2. *For an element of the Weyl group $w \in W(G,T) = W(G,T)(\mathbb{F}) = W(G,T)(\mathcal{O})$, define*

$$M_{T,w,\mathrm{cris}} := \ker(M_{T,\mathrm{cris}} \xrightarrow{w\otimes 1 - 1\otimes\Phi_K} M_{T,\mathrm{cris}}),\ \text{and}$$

$$M_{T,w,\mathbb{F}} := \ker(M_{T,\mathbb{F}} \xrightarrow{w\otimes 1 - 1\otimes\Phi_K} M_{T,\mathbb{F}}).$$

The following simple lemma is essentially how we construct crystalline lifts.

Lemma 5. *Let $\mathbb{Z}[X]$ be the polynomial ring. Let $a(X), b(X) \in \mathbb{Z}[X]$ be two polynomials. Let n and N be integers. Assume $a(n)b(n) = 0$.*
Let $\widetilde{M}$ be a $\mathbb{Z}[X]/(a(X)b(X) - N)$-module. Write M for $\widetilde{M} \otimes_{\mathbb{Z}} \mathbb{Z}/N$.
If $\widetilde{M}$ has a torsion-free, finitely generated underlying abelian group, the sequence

$$0 \to a(X)M \to M \xrightarrow{\cdot b(X)} b(X)M \to 0$$

is short exact.

Proof. Pick $\bar{v} \in \ker(M \to b(X)M)$. Let $v \in \widetilde{M}$ be a lifting of $\bar{v}$. We have $b(X)v \mapsto 0$ in M. Since $M = \widetilde{M} \otimes \mathbb{Z}/N$, $b(X)v = Nu$ for some $u \in \widetilde{M}$. Multiply both sides by $a(X)$, we get $a(X)b(X)v = Nv = Na(X)u$. Since $\widetilde{M}$ is $\mathbb{Z}$-torsion-free, we have $v = a(X)u$, as desired. □

Proposition 2. *If $w^{[\mathbb{F}:\kappa]} = 1$ and E contains K, the map*

$$M_{T,w,\mathrm{cris}} \to M_{T,w,\mathbb{F}}$$

is surjective.

Proof. Write $f := [\mathbb{F} : \kappa]$. Let K_f be the unramified extension of K of degree f. We single out a $\mathbb{Z}[W(G,T)] \otimes \mathbb{Z}[G_K/I_K]$-submodule $M^0_{T,\mathrm{cris}} \subset M_{T,\mathrm{cris}}$ which consists of elements that can be extended to a representation $G_{K_f} \to T(\mathcal{O})$. Note that $M^0_{T,\mathrm{cris}} \to M_{T,\mathbb{F}}$ is surjective because the fundamental character of niveau f admits a crystalline lift, namely, the Lubin-Tate character of the field K_f. Put $M^0_{T,w,\mathrm{cris}} := M^0_{T,\mathrm{cris}} \cap M_{T,w,\mathrm{cris}}$.
Note that on both $M^0_{T,\mathrm{cris}}$ and $M_{T,\mathbb{F}}$, we have $(w \otimes 1)^f = (1 \otimes \Phi_K)^f = \mathrm{id}$, where Φ_K is the fixed topological generator of G_K/I_K.
Put

$$\Xi := \sum_{i=0}^{f-1} w^i \otimes \Phi_K^{f-1-i}.$$

Commutativity of $w \otimes 1$ and $1 \otimes \Phi_K$ implies $(w \otimes 1 - 1 \otimes \Phi_K)\Xi = (w \otimes 1)^f - (1 \otimes \Phi_K)^f$. In particular, the inclusion $\Xi M^0_{T,\mathrm{cris}} \to M^0_{T,\mathrm{cris}}$ factors through $M^0_{T,w,\mathrm{cris}}$ (which is the arrow at the top of the diagram below).
Consider the commutative diagram

$$\begin{tikzcd}
\Xi M^0_{T,\mathrm{cris}} \arrow[r] \arrow[d] & M^0_{T,w,\mathrm{cris}} \arrow[r] \arrow[d] & M^0_{T,\mathrm{cris}} \arrow[r, "w\otimes 1 - 1\otimes\Phi_K"] \arrow[d] & M^0_{T,\mathrm{cris}} \arrow[d] \\
\Xi M_{T,\mathbb{F}} \arrow[r] \arrow[d] & M_{T,w,\mathbb{F}} \arrow[r] & M_{T,\mathbb{F}} \arrow[r, "w\otimes 1 - 1\otimes\Phi_K"] & M_{T,\mathbb{F}} \\
0 & & &
\end{tikzcd}$$

It is clear that $\Xi M^0_{T,\mathrm{cris}} \to \Xi M_{T,\mathbb{F}}$ is surjective. So it suffices to show

$$\Xi M_{T,\mathbb{F}} \hookrightarrow M_{T,w,\mathbb{F}}$$

is surjective.
Let $\bar{\chi} : I_K \to \mathbb{F}^\times$ be a fundamental character of niveau f. Note that $\bar{\chi}$ generates the abelian group $M_{\mathbb{G}_m,\mathbb{F}}$. Indeed, there is an abelian group isomorphism $\iota_{\bar{\chi}} : \mathbb{Z}/(q^f - 1) \xrightarrow{\cong} M_{\mathbb{G}_m,\mathbb{F}}$, sending 1 to $\bar{\chi}$. We have $M_{T,\mathbb{F}} \cong M_{\mathbb{G}_m,\mathbb{F}} \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathrm{GrpSch}}(\mathbb{G}_m, T)$. Note that the Weyl group element w acts on $\mathrm{Hom}_{\mathrm{GrpSch}}(\mathbb{G}_m, T)$ via conjugation $v \mapsto wvw^{-1}$.
We specialize Lemma 5 as follows:

- Set $\widetilde{M} = \mathrm{Hom}_{\mathrm{GrpSch}}(\mathbb{G}_m, T)$, and regard it as a $\mathbb{Z}[X]$-module where X acts by w;
- Set $M = M_{T,\mathbb{F}}$, and regard M as a $\mathbb{Z}[X]$-module via $X \mapsto w \otimes 1$;
- Set $N = q^f - 1$;
- Set $n = q$;
- Set $a(X) = \sum_{i=0}^{f-1-i} X^i q^{f-1-i}$;
- Set $b(X) = q - X$;

We can identify M with $\widetilde{M} \otimes_{\mathbb{Z}} \mathbb{Z}/(q^f - 1)$ via the map $\iota_{\widetilde{\chi}} : \mathbb{Z}/(q^f - 1) \xrightarrow{\cong} M_{\mathbb{G}_m,\mathbb{F}}$. Here are a few things to check:

(i) $\widetilde{M}$ is finitely generated and torsion-free over $\mathbb{Z}$.

(ii) $(a(X)b(X) - q^f + 1)$ kills $\widetilde{M}$;

(iii) $\widetilde{M} \otimes_{\mathbb{Z}} \mathbb{Z}/(q^f - 1) \cong M$ as abelian groups;

(iv) $a(q)b(q) = 0$.

Items (i), (iii) and (iv) are clear. For item (ii), notice that $a(X)b(X) = q^f - X^f$. Since we assumed $w^f = 1$, $a(X)b(X) = q^f - 1$. ☐

The goal of the rest of this section is to prove the following theorem:

THEOREM 5. *Let κ be the residue field of K. Let $\mathbb{F}/\kappa$ be a finite extension. Let K^{ur} be the maximal unramified extension of K with ring of integers $\mathcal{O}_{K_{\mathrm{ur}}}$. Let $\bar{\rho} : G_K \to G(\mathbb{F})$ be a quasi-semisimple (see Definition 1) representation.*
(1) There exists a crystalline representation $\rho : G_K \to G(\mathcal{O}_{K^{\mathrm{ur}}})$ lifting $\bar{\rho}$.
(2) Assume G admits a simply-connected derived subgroup and $\bar{\rho}$ is G-irreducible. Let $\mathbb{F}_{\bar{\rho}}$ be the splitting field of $\bar{\rho}|_{I_K}$, that is, the smallest field extension $\mathbb{F}_{\bar{\rho}}$ of $\mathbb{F}$ such that $\bar{\rho}|_{I_K} : I_K \to G(\mathbb{F})$ factors through the $\mathbb{F}_{\bar{\rho}}$-points of a split torus of G. Then ρ can be chosen to have image in $G(\mathcal{O}_{K_{\bar{\rho}}})$ where $K_{\bar{\rho}}$ is the unramified extension of K with residue field $\mathbb{F}_{\bar{\rho}}$.

4.2

The strategy is as follows: the first step is to choose a lift of $\bar{\rho}|_{I_K}$ which admits an extension to the whole Galois group G_K. This is already done in Proposition 2. The second step is to choose a lift of all Frobenius elements. The continuity of the lift is free because we'll only use finite order lifts (modulo the image of I_K) of Frobenius elements.

LEMMA 6. *Assume the special fiber $T_{\mathbb{F}}$ of T is a split torus. There exists a finite subgroup $\widetilde{N} \subset N_G(T)(W(\mathbb{F}))$ such that $\widetilde{N} \to N_G(T)(\mathbb{F})$ is surjective.*

Proof. By [5, B.3.5], T splits if and only if $T_{\mathbb{F}}$ splits. The key ingredient is Tits' theory of extended Weyl groups.
By [21], there exists a subgroup $\widetilde{W} \subset N_G(T)(W(\mathbb{F}))$ which is isomorphic to the extension of the Weyl group $W(G,T)$ by $(\mathbb{Z}/2)^{\otimes l}$ for some $l \geq 0$, and generates the whole Weyl group. Write $[-] : T(\mathbb{F}) \to T(W(\mathbb{F}))$ for the Teichmuller lift.

Fact The Teichmüller lift is the unique p-adic continuous multiplicative section of $T(W(\mathbb{F})) \to T(\mathbb{F})$.

Proof. We include a proof here because it is short. It is well-known for $T = \mathbb{G}_m$. In general, choose a faithful representation $i : T \to \mathrm{GL}_N \subset \mathrm{Mat}_{N\times N}$. Let $s, t : T(\mathbb{F}) \to T(\mathcal{O})$ be two multiplicative sections. We have $i(s(x)) - i(t(x)) \equiv 1 \mod p^f$ for all $x \in T(\mathbb{F})$; $(i(s(x)) - i(t(x)))^{p^{nf}} \equiv 1 \mod p^{(n+1)f}$; and $i(s(x)) - i(t(x)) = i(s(x^{p^{nf}})) - i(t(x^{p^{nf}})) \equiv (i(s(x)) - i(t(x)))^{p^{nf}} \equiv 1 \mod p^{nf}$ for all n. □

For each $w \in \widetilde{W}$ and $x \in T(\mathbb{F})$, $x \mapsto w^{-1}[wxw^{-1}]w$ is a continuous section of $T(W(\mathbb{F})) \to T(\mathbb{F})$ and must be equal to the Techmüller lift. Let $\widetilde{N}$ be the composite $\widetilde{W} \cdot [T(\mathbb{F})]$. Since for all $w, w' \in \widetilde{W}$ and $x, x' \in T(\mathbb{F})$, we have $w[x]w'[x'] = ww'[w'^{-1}xw'x']$, $\widetilde{N}$ is a finite order subgroup of $N_G(T)(W(\mathbb{F}))$, as desired. □

The existence of $\widetilde{N}$ has the following immediate consequence:

Corollary 2. *Let $\bar{\rho} : G_K \to G(\mathbb{F})$ be a G-completely reducible representation. There exists a lift $\rho : G_K \to G(W(\bar{\mathbb{F}}_p))$ of $\bar{\rho}$.*
Indeed, for any lift v of $\bar{\rho}|_{I_K}$ to $G(\mathcal{O}_{K^{\mathrm{ur}}})$ that can be extended to the whole Galois group G_K, there exists a lift $\bar{\rho}$ to $G(\mathcal{O}_{K^{\mathrm{ur}}})$ whose inertia is v.

Proof. We first prove the first paragraph. We are allowed to enlarge the coefficient field $\mathbb{F}$ to make T split. Set $\rho|_{I_K}$ to be the Teichmüller lift of $\bar{\rho}|_{I_K}$. Let $\Phi_K \in G_K$ be a lift of the topological generator of G_K/I_K. Choose an element $n \in \widetilde{N}$ which lifts $\bar{\rho}(\Phi_K)$. Set $\rho(\Phi_K) = n$. Write $n = wt$ where w is an element of Tits' extended Weyl group $\widetilde{W}$ and t lies in the Teichmüller lift of $T(\mathbb{F})$. Let σ be an element of I_K. Write x for $\bar{\rho}(\sigma)$. We have $\rho(\Phi_K\sigma\Phi_K^{-1}) = [\bar{\rho}(\Phi_K\sigma\Phi_K^{-1})] = [wxw^{-1}]$. By the proof of the previous lemma, $[wxw^{-1}] = w[x]w^{-1} = w\rho(\sigma)w^{-1} = n\rho(\sigma)n^{-1}$, and thus ρ extends uniquely to a continuous homomorphism $G_K \to G(W(\mathbb{F}))$.
Now we prove the "indeed" part. It is an immediate consequence of Lemma 4 and Lemma 6. □

Lemma 7. *Let $\bar{\rho} : G_K \to G(\mathbb{F})$ be a G-irreducible Galois representaion. By Theorem 4, there exists a unique maximal torus T of G such that $\bar{\rho}(G_K) \subset N_G(T)(\mathbb{F})$.*
Let κ be the residue field of K. Let $\mathbb{F}_0 \subset \mathbb{F}$ be the smallest subfield of $\mathbb{F}$ containing κ such that $\bar{\rho}(I_K) \subset T(\mathbb{F}_0)$. (Recall that G is a Chevalley group and

has a $\mathbb{Z}$-model.) Let $\Phi_K \in G_K$ be a lift of a topological generator of G_K/I_K. The map $G_K \to N_G(T)(\mathbb{F}) \to W(G,T)(\mathbb{F})$ maps Φ_K to an element w of the Weyl group $W(G,T)(\mathbb{F})$.
If G admits a simply-connected derived subgroup, then $w^{[\mathbb{F}_0:\kappa]} = 1$ in $W(G,T)(\mathbb{F})$.

Proof. Write $f_0 := [\mathbb{F}_0 : \kappa]$. Let $s \in \bar{\rho}(I_K)$ be a generator. By the proof of Theorem 4, $T = Z_G(s)^\circ$ is the connected centralizer of s. Since $\bar{\rho}(I_K) \subset T(\mathbb{F}_0)$, we have $\bar{\rho}(\Phi_K)^{f_0} s \bar{\rho}(\Phi_K)^{-f_0} = s$. So $\bar{\rho}(\Phi_K)^{f_0} \in Z_G(s) \cap N_G(T)$. Since G has a simply-connected derived subgroup, $Z_G(s) = Z_G(s)^\circ$. So $\bar{\rho}(\Phi_K)^{f_0} \in T$, that is, $w^{[\mathbb{F}_0:\kappa]} = 1$ in $W(G,T)(\mathbb{F})$. □

Proof of Theorem 5. (1) We choose a sufficiently large coefficient field E (which is unramified over K) such that the cardinality of the Weyl group $W(G,T)$ divides $[\mathbb{F} : \kappa]$ and $T_{\mathbb{F}}$ splits. The assumption of Proposition 2 is satisfied. So there exists a crystalline lift $v : I_K \to T(\mathcal{O})$ such that $v = \bar{\rho}|_{I_K} \bmod \varpi$. By Lemma 4, v can be extended to G_K.
(2) For ease of notation, replace $\mathbb{F}$ by $\mathbb{F}_{\bar{\rho}}$. Write $\mathcal{O}$ for $\mathcal{O}_{K_{\bar{\rho}}}$. We choose the field $\mathbb{F}_0$ as in Lemma 7. Note that the maximal torus in Lemma 7 is split: let S be a maximal split torus over $\mathbb{F}$ such that $\bar{\rho}(I_K) \subset S(\mathbb{F})$; since $T = Z_G(\bar{\rho}(I_K))^\circ$, we have $T \supset S$; now since G is a split group, we must have $S = T$. Let K_{f_0} be the unramified extension of K of degree $[K_{f_0} : K] = [\mathbb{F}_0 : \kappa]$. Let $\mathcal{O}_0$ be the ring of integers of K_{f_0}. We have $\mathcal{O}_0 \subset \mathcal{O}$. By the previous Lemma, Proposition 2 is applicable, and thus there exists a lift $v : I_K \to T(\mathcal{O}_0)$ such that $v = \bar{\rho}|_{I_K} \bmod \varpi$ and v admits an extension to a representation $G_K \to N_G(T)(\mathcal{O}_0)$. By Corollary 2, v admits an extension to G_K which lifts $\bar{\rho}$.
Fix $\Phi_K \in G_K$, a lift of a topological generator of G_K/I_K. By Lemma 6, we choose a finite order lift $X \in \widetilde{N} \subset N_G(T)(\mathcal{O})$ of $\bar{\rho}(\Phi_K)$. Since the Weyl group scheme is a constant group scheme, any two lifts of $\bar{\rho}(\Phi_K)$ have the same conjugation action on the maximal torus $N_G(T)(\mathcal{O})$, and therefore we can extend v to a representation $G_K \to N_G(T)(\mathcal{O})$ by setting $\Phi_K \mapsto X$. □

5 HODGE-TATE THEORY FOR GALOIS REPRESENTATIONS VALUED IN REDUCTIVE GROUPS

The Hodge-Tate theory for GL_N is reviewed in Appendix A. In this section, we discuss Hodge-Tate theory for general reductive groups, and show G-irreducible mod ϖ Galois representations admit Hodge-Tate regular crystalline lifts.

5.1 FIRST PROPERTIES OF HODGE-TATE COCHARACTERS

DEFINITION 3. *Fix an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$. Let $K, E \subset \overline{\mathbb{Q}_p}$ be finite extensions of $\mathbb{Q}_p$. The field E will serve as the coefficient field. To define colabeled Hodge-Tate gradings, we assume K is a subfield of E and therefore G_E as a subgroup of G_K.*

Let $\mathbb{C} := \mathbb{C}_K$ be the completed algebraic closure of K. Let $\sigma : E \hookrightarrow \mathbb{C}$ be an embedding. Let (ρ, V) be a Hodge-Tate representation of G_K. Then one can define the σ-colabeled Hodge-Tate grading on $\mathbb{C}\otimes_{\sigma,E}V$ by setting the i-th graded piece to be

$$\mathrm{Im}((\mathbb{C}(i) \otimes_{\sigma,E} V)^{G_E} \otimes_E \mathbb{C}(-i) \to \mathbb{C} \otimes_{\sigma,E} V)$$

which is compatible with tensor product and duality.
Let G be a reductive group over E. A G-valued representation is Hodge-Tate if for all representations $G \to \mathrm{GL}(V)$, V is a Hodge-Tate G_K-module. Let $\rho : G_K \to G(E)$ be a Hodge-Tate G-valued representation. Consider $G(\sigma) \circ \rho : G_K \to G(\mathbb{C})$. By Tannakian theory, there is a cocharacter $\mathcal{HT}(\rho)^\sigma : \mathbb{G}_m \to G_{\mathbb{C}}$, such that for any faithful representation $i : G \to \mathrm{GL}_N$, the composition $i(\mathcal{HT}(\rho)^\sigma)$ recovers the Hodge-Tate grading on $i(G(\sigma) \circ \rho) : G_K \to \mathrm{GL}_N(\mathbb{C})$. Set $\mathcal{HT}(\rho) := (\mathcal{HT}(\rho)^\sigma)_{\sigma:E\hookrightarrow\mathbb{C}} \in \prod_{E\hookrightarrow\mathbb{C}} X_*(G_{\mathbb{C}})$. We call $\mathcal{HT}(\rho)$ the co-labeled Hodge-Tate cocharacter of ρ.

The formation of co-labeled Hodge-Tate cocharacters is clearly functorial in G.

Lemma 8. *Let $f : G \to H$ be a morphism of reductive groups over E. If $\rho : G_K \to G(E)$ is a Hodge-Tate representation, we have $\mathcal{HT}(f \circ \rho) = f(\mathcal{HT}(\rho))$.*

Proof. It follows immediately from Tannakian theory. ☐

5.1.1 Regular cocharacter

Let H be a reductive group with maximal torus S. A cocharacter $x \in X_*(H, S)$ is said to be *regular* if it is not killed by any root of H (with respect to S).
We say ρ is *Hodge-Tate regular* if for all $\sigma : E \hookrightarrow \mathbb{C}$, the cocharacter $\mathcal{HT}(\rho)^\sigma$ of $G_{\mathbb{C}}$ is regular.
When $G = \mathrm{GL}_N$, we can also define *labeled Hodge-Tate weights* (see Appendix A). It turns out labeled Hodge-Tate regularity is equivalent to colabeled Hodge-Tate regularity. So our definition coincides with the usual notion of Hodge-Tate regularity in the literature.

Lemma 9. *Assume $G = \mathrm{GL}_N$. Assume E admits an embedding of the Galois closure of K. Then ρ is Hodge-Tate regular if and only if the labeled Hodge-Tate weight $\mathbf{k} = (k_\tau)_{\tau:K\hookrightarrow E}$ is regular in the sense that each $k_\tau \in \mathbb{Z}^N$ contains distinct numbers.*

Proof. It follows from Proposition 3. ☐

Lemma 10. *Let K'/K be a finite field extension such that $K' \subset E$. Let $\rho : G_K \to G(E)$ be a Hodge-Tate G-valued representation. We have $\mathcal{HT}(\rho|_{G_{K'}}) = \mathcal{HT}(\rho)$.*

Proof. Note that the Definition 3 only makes use of G_E and does not depend on K. ☐

Lemma 11. *Let $\rho_1, \rho_2 : G_K \to G(E)$ be two Hodge-Tate representations whose image is abelian and consists of semisimple elements. If $\rho_1\rho_2 = \rho_2\rho_1$, then $\mathcal{HT}(\rho_1\rho_2) = \mathcal{HT}(\rho_1)\mathcal{HT}(\rho_2)$.*

Proof. By the previous lemma, it is harmless to shrink G_K and thus we can assume ρ_1, ρ_2 both factor through a maximal torus T of G. By descent, we can assume T is split. Write $i : T \hookrightarrow G$ for the embedding of the maximal torus T. Let $t_1, t_2 : G_K \to T(E)$ be representations such that $i(t_1) = \rho_1$ and $i(t_2) = \rho_2$. We have $\mathcal{HT}(\rho_1) = i(\mathcal{HT}(t_1))$ and $\mathcal{HT}(\rho_2) = i(\mathcal{HT}(t_2))$ by functoriality (Lemma 8). So it suffices to show $\mathcal{HT}(t_1t_2) = \mathcal{HT}(t_1)\mathcal{HT}(t_2)$. Since T is a split torus, the general case follows from the special case $T = \mathbb{G}_m$. The Hodge-Tate cocharacter of $t_1t_2 : G_K \to \mathbb{G}_m(E)$ is completely decided by the Hodge-Tate weight of t_1t_2. The lemma follows because the Hodge-Tate weight of t_1t_2 is the sum of the Hodge-Tate weight of t_1 and the Hodge-Tate weight of t_2. $\square$

We use the following lemma to construct Hodge-Tate regular cocharacters.

Lemma 12. *Assume $E = K_f$ is the unramified extension of K of degree f inside the fixed algebraic closure $\overline{K} := \overline{\mathbb{Q}_p}$ of K. Fix a maximal split torus T of G. Write $i : T \to G$ for the embedding.*
For each colabel $\sigma_0 : K_f \hookrightarrow \mathbb{C}$, and each cocharacter $\lambda \in X_*(G(\mathbb{C}), T(\mathbb{C}))$, there exists a crystalline representation $t : G_{K_f} \to T(K_f)$ such that

$$\mathcal{HT}(i(t))^\sigma = \begin{cases} \lambda & \text{if} \quad \sigma = \sigma_0, \\ \text{the trivial cocharacter} & \text{if otherwise.} \end{cases}$$

Proof. Let $\chi_{\mathrm{LT}} : G_{K_f} \to \mathcal{O}^*_{K_f}$ be a Lubin-Tate character. Choose an isomorphism $T \cong \mathbb{G}_m^{\times r}$, $r = \operatorname{rk} T$.
The field K_f is a subfield of $\overline{K}$ by its choice. The composite $K_f \hookrightarrow \overline{K} \hookrightarrow \mathbb{C}$ defines a canonical embedding of K_f in $\mathbb{C}$. Since K_f/K is a Galois extension, there exists a unique $\iota \in \mathrm{Gal}(K_f/K)$ such that $\sigma_0 \circ \iota$ is the canonical embedding $K_f \hookrightarrow \mathbb{C}_K$.
Put $t = \iota(\chi_{\mathrm{LT}}^{h_1}, \cdots, \chi_{\mathrm{LT}}^{h_r})$, $h_1, \cdots, h_r \in \mathbb{Z}$. By Lemma 8 and Lemma 18, $\mathcal{HT}(i(t))^\sigma$ is the trivial cocharacter if $\sigma \neq \sigma_0$. Since the co-labeled Hodge-Tate weights of the Lubin-Tate character is $(1, 0 \cdots, 0)$, if we let the tuple $(h_1, \cdots, h_r)$ range over all $\mathbb{Z}^r$, then $\mathcal{HT}(i(\iota(\chi_{\mathrm{LT}}^{h_1}, \cdots, \chi_{\mathrm{LT}}^{h_r})))^{\sigma_0}$ ranges over all cocharacters in $X_*(G(\mathbb{C}), T(\mathbb{C}))$. So we can choose $(h_1, \cdots, h_r)$ so that $\mathcal{HT}(i(\iota(\chi_{\mathrm{LT}}^{h_1}, \cdots, \chi_{\mathrm{LT}}^{h_r})))^{\sigma_0} = \lambda$. $\square$

5.2 Hodge-Tate regular lifts of quasi-semisimple mod ϖ Galois representations

In many applications, we need Hodge-Tate regular crystalline representations. For example, crystalline deformation rings of regular Hodge-Tate weights have the largest dimension, which is exploited in the work [6].

The following lemma shows as long as a crystalline lift exists, Hodge-Tate regular lifts also exist.
We will specialize to the case where $E = K_f$, the unramified extension of K of degree f.

5.2.1 Local class field theory

Let $\mathrm{Art}_K : K^\times \to G_K^{\mathrm{ab}}$ be the local Artin map, which we normalize so that a uniformizer corresponds to a geometric Frobenius element.
Note that Art_K induces an isomorphism

$$\mathrm{Art}_K^{-1} : \mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{ur}}) \xrightarrow{\cong} \mathcal{O}_K^\times$$

See the paragraph after the proof of [12, 6.2] for a reference. Denote by r_K the induced map $I_K \to \mathcal{O}_K^\times$.

Theorem [12, 6.11] Let $\sigma : K \to K'$ be an isomorphism of fields. Then the following diagram is commutative:

$$\begin{tikzcd} K^\times \arrow[r, "\mathrm{Art}_K"] \arrow[d, "\sigma"] & G_K^{\mathrm{ab}} \arrow[d, "\sigma^*"] \\ K'^\times \arrow[r, "\mathrm{Art}_{K'}"] & G_{K'}^{\mathrm{ab}} \end{tikzcd}$$

Here $\sigma^* : \tau \mapsto \sigma\tau\sigma^{-1}$.

Corollary 3. *Let $\sigma : K \to K$ be a continuous field automorphism. Then $r_K(\sigma\tau\sigma^{-1}) = \sigma(r_K(\tau))$ for all $\tau \in I_K$.*

Proof. It is an immediate consequence of Theorem 5.2.1. □

Theorem 6. *Let $\bar{\rho} : G_K \to G(\mathbb{F})$ be a G-completely reducible representation. Let κ be the residue field of K. Assume $\kappa \subset \mathbb{F}$.*
(1) There exists a Hodge-Tate regular crystalline lift $\rho : G_K \to G(\mathcal{O}_{K_f})$ for some positive integer f.
(2) If G has a simply connected derived subgroup and $\mathbb{F}$ is the splitting field of $\bar{\rho}|_{I_K}$ (see Theorem 5), then f can be taken as $[\mathbb{F} : \kappa]$.

Proof. Write $i : T \hookrightarrow G$ for the embedding of the maximal torus T.
We will show that as long as a crystalline lift exists, a Hodge-Tate regular crystalline also exists with the same coefficient field. The existence of crystalline lifts is Theorem 5.
We keep notations used in the proof of Proposition 2. We set $\mathcal{O} := \mathcal{O}_{K_f}$. Recall that $\Xi := \sum_{i=0}^{f-1} w^i \otimes \Phi_K^{f-1-i}$, where $\Phi_K \in G_K/I_K$ is a generator of G_K/I_K, and $w \in W(G,T)$ is the Weyl group element which corresponds to $\bar{\rho}(\Phi_K)^{-1}$. Recall that the submodule $M^0_{T,\mathrm{cris}} \subset M_{T,\mathrm{cris}}$ consists of representations $I_K \to T(\mathcal{O})$

which are extendable to G_{K_f}. For each element of $u \in M^0_{T,\mathrm{cris}}$, choose an extension $t_u : G_{K_f} \to T(\mathcal{O})$. The Hodge-Tate cocharacter $\mathcal{HT}(i(t_u))$ does not depend on the choice of t_u. It makes sense to write $\mathcal{HT}(u)$ for $\mathcal{HT}(i(t_u))$ (where t_u is any choice of extension).
In the proof of Proposition 2, we've shown that there exists $v \in \Xi M^0_{T,\mathrm{cris}} \subset M^0_{T,w,\mathrm{cris}}$ which is a lift of $\bar{\rho}|_{I_K}$.
Fix a colabel $\sigma_0 : K_f \hookrightarrow \mathbb{C}$. By Lemma 12, there exists a crystalline representation $t : G_{K_f} \to T(\mathcal{O})$ such that $\mathcal{HT}(i(t))^\sigma$ is a regular cocharacter in $X_*(G(\mathbb{C}), T(\mathbb{C}))$ if $\sigma = \sigma_0$, and is the trivial cocharacter if $\sigma \neq \sigma_0$.
The restriction $t|_{I_K}$ defines an element $v_0 \in M^0_{T,\mathrm{cris}}$. By Lemma 8, we have

$$\mathcal{HT}((w \otimes 1)v_0) = w\mathcal{HT}(v_0)w^{-1}.$$

By Lemma 18 and Corollary 3, we have

$$\mathcal{HT}((1 \otimes \Phi_K)v_0)^\sigma = \mathcal{HT}(v_0)^{\sigma\circ\Phi_K^{-1}}.$$

Summing up, we have

$$\begin{aligned}
\mathcal{HT}(\Xi v_0)^{\sigma_0\circ\Phi_K^{-1-i+f}} &= \mathcal{HT}(\sum_{j=0}^{f-1} w^j \otimes \Phi_K^{f-1-j} v_0)^{\sigma_0\circ\Phi_K^{-1-i+f}} \\
&= \prod_{j=0}^{f-1} \mathcal{HT}(w^j \otimes \Phi_K^{f-1-j} v_0)^{\sigma_0\circ\Phi_K^{-1-i+f}} \\
&= \prod_{j=0}^{f-1} w^j \mathcal{HT}(1 \otimes \Phi_K^{f-1-j} v_0)^{\sigma_0\circ\Phi_K^{-1-i+f}} w^{-j} \\
&= \prod_{j=0}^{f-1} w^j \mathcal{HT}(v_0)^{\sigma_0\circ\Phi_K^{-1-i+f}\circ\Phi_K^{1+j-f}} w^{-j} \\
&= w^i \mathcal{HT}(v_0)^{\sigma_0} w^{-i}
\end{aligned}$$

By Definition 5.1.1, Ξv_0 is Hodge-Tate regular.
Let C be a very large positive integer. Write N for the cardinality of $\mathbb{F}^\times$. Define $v' := v + CN\Xi v_0$. Since $M_{T,\mathbb{F}}$ is N-torsion, v' is a lift of $\bar{\rho}|_{I_K}$. We have $\mathcal{HT}(v') = \mathcal{HT}(v)\mathcal{HT}(\Xi v_0)^{CN}$. Since $\mathcal{HT}(\Xi v_0)$ is a regular cocharacter, $\mathcal{HT}(v')$ is also a regular cocharacter if $C \gg 0$.
Since $\Xi M^0_{T,\mathrm{cris}} \subset M^0_{T,w,\mathrm{cris}}$, we have $v + \Xi v_0 \in M_{T,w,\mathrm{cris}}$. By Corollary 2, v' extends to a representation $G_K \to G(\mathcal{O})$ which is a crystalline representation lifting $\bar{\rho}$. □

A Appendix: Hodge-Tate theory with coefficients

Let $K/\mathbb{Q}_p$, $E/\mathbb{Q}_p$ be finite extensions. Assume E admits an embedding of the Galois closure of K. Fix an embedding $K \hookrightarrow E$. Let V be a finite dimensional E-vector space. Let $\rho : G_K \to \mathrm{GL}(V)$ be a continuous representation. Assume ρ is Hodge-Tate. Let $\mathbb{C} := \mathbb{C}_K$ be the completed algebraic

closure of K. Let $\mathbb{B}_{\mathrm{HT}} := \bigoplus_{n\in\mathbb{Z}} \mathbb{C}(n)$ be the Hodge-Tate period ring. Then $\mathbb{B}_{\mathrm{HT}} \otimes V := \mathbb{B}_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V$ is a $\mathbb{C}\otimes E$-module with G_K-action.
Let σ be an embedding $E \hookrightarrow \mathbb{C}$. Define

$$\begin{aligned} V_\sigma &:= \{\sum x_i \otimes y_i \in \mathbb{B}_{\mathrm{HT}} \otimes V | \sum \sigma(a)x_i \otimes y_i = \sum x_i \otimes ay_i \text{ for all } a \in E\} \\ &= \bigcap_{a\in E} \mathrm{Ker}(l_{1\otimes a} - l_{\sigma(a)\otimes 1}) \qquad \text{(where } l_x \text{ is scalar multiplication by } x) \end{aligned}$$

It is easy to see that

LEMMA 13. *Let $L_\sigma \subset \mathbb{C}$ be the subfield generated by K and $\sigma(E)$.*

(i) V_σ is a G_{L_σ}-stable $\mathbb{C}\otimes E$-submodule of $\mathbb{B}_{\mathrm{HT}} \otimes V$;

(ii) V_σ is isomorphic to $\mathbb{B}_{\mathrm{HT}} \otimes_{\sigma,E} V$ as a G_{L_σ}-semi-linear $\mathbb{C}$-module;

(iii) $\mathbb{B}_{\mathrm{HT}} \otimes V = \bigoplus_{\sigma:E\hookrightarrow\mathbb{C}} V_\sigma$.

Let L be the Galois closure of L_σ in $\mathbb{C}$. Write $D_\sigma(V) := V_\sigma^{G_L}$. By (iii),

$$\bigoplus_{\sigma:E\hookrightarrow\mathbb{C}} D_\sigma(V) = (\mathbb{B}_{\mathrm{HT}} \otimes V)^{G_L} = D_{\mathrm{HT}}(V) \otimes_K L$$

The Hodge-Tate grading on $D_{\mathrm{HT}}(V)$ induces a grading on each of $D_\sigma(V)$. So $D_\sigma(V_\sigma)$ is a graded L-vector space. We denote by $\mathrm{HT}^\sigma(V)$ the multiset of integers n in which n occurs with multiplicity $\dim_L \mathrm{gr}^n D_\sigma(V_\sigma)$, and call it the *$\sigma$-co-labeled Hodge-Tate weights of V*. [1]

A.1 Labeled Hodge-Tate weights

Let $\tau : K \hookrightarrow E$ be an embedding. Define

$$\begin{aligned} \tilde{V}_\tau &:= \{\sum x_i \otimes y_i \in \mathbb{B}_{\mathrm{HT}} \otimes V | \sum ax_i \otimes y_i = \sum x_i \otimes \tau(a)y_i \text{ for all } a \in K\} \\ &= \bigcap_{a\in K} \mathrm{Ker}(l_{a\otimes 1} - l_{1\otimes\tau(a)}) \end{aligned}$$

LEMMA 14. *We have*

$$\tilde{V}_\tau = \bigoplus_{\sigma:E\hookrightarrow\mathbb{C}, \sigma|_{\tau K}=\tau^{-1}} V_\sigma$$

Proof. Unravel the definitions. ☐

While V_σ is only G_{L_σ}-stable, $\tilde{V}_\tau$ is G_K-stable! Write $\tilde{D}_\tau(V) := (\tilde{V}_\tau)^{G_K}$. We want to remind readers the usual definition of τ-labeled Hodge-Tate weights (for example, the definition in [9, 1.1]).

[1] This is a non-standard terminology.

Definition 4. *The multiset* $\mathrm{HT}_\tau(V)$ *is as follows: an integer* n *appears with multiplicity*

$$\dim_E \mathrm{gr}^n(D_{\mathrm{HT}}(V) \otimes_{E\otimes_{\mathbb{Q}_p}K,\tau} E)$$

Lemma 15. *We have* $\dim_E \mathrm{gr}^n(D_{\mathrm{HT}}(V) \otimes_{E\otimes_{\mathbb{Q}_p}K,\tau} E) = \dim_E \mathrm{gr}^n(\tilde{D}_\tau(V))$.

Proof. It is easy to see (by unravelling the definitions) that the natural map

$$\tilde{V}_\tau^{G_K} \hookrightarrow D_{\mathrm{HT}}(V) \twoheadrightarrow D_{\mathrm{HT}}(V) \otimes_{E\otimes_{\mathbb{Q}_p}K,\tau} E$$

is injective, and E-linear. So it must be an E-isomorphism because of the direct sum decomposition. □

When we divide a multiset by an integer s, we divide the multiplicity of all members of the multiset by s. For example $\frac{1}{2}\{1,1,2,2,2,2\} = \{1,2,2\}$.

Proposition 3. *We have* $\mathrm{HT}_\tau(V) = \frac{1}{[E:K]} \bigcup_{\sigma:E\hookrightarrow\mathbb{C},\sigma|_{\tau K}=\tau^{-1}} \mathrm{HT}^\sigma(V)$.

Proof. Let L be as before. We have

$$\tilde{D}_\tau(V) \otimes_K L \;=\; \tilde{V}_\tau^{G_L} = \bigoplus_{\sigma:E\hookrightarrow\mathbb{C},\sigma|_{\tau K}=\tau^{-1}} V_\sigma^{G_L} = \bigoplus_{\sigma:E\hookrightarrow\mathbb{C},\sigma|_{\tau K}=\tau^{-1}} D_\sigma(V)$$

as graded modules. So

$$\dim_E(\tilde{D}_\tau(V)) = \frac{1}{[E:K]}\dim_K(\tilde{D}_\tau(V)) = \frac{1}{[E:K]}\dim_L(\tilde{D}_\tau(V)\otimes_K L)$$
$$= \frac{1}{[E:K]} \sum_{\sigma:E\hookrightarrow\mathbb{C},\sigma|_{\tau K}=\tau^{-1}} \dim_L D_\sigma(V)$$

□

Thus the multiset of τ-labeled Hodge-Tate weights is the average of certain multisets of σ-co-labeled Hodge-Tate weights.

A.2 Galois twist

The following is a convenient observation.

Lemma 16. *Let* K*,* E *be arbitrary finite extensions of* $\mathbb{Q}_p$*. Let* L/E *be a field extension. Let* $\sigma : E \hookrightarrow \mathbb{C}$ *be an embedding. Let* $\tilde{\sigma} : L \hookrightarrow \mathbb{C}$ *be an embedding extending* σ*. Let* K'/K *be a finite extension. Then*

(1) $\mathrm{HT}^\sigma(\mathrm{Res}^{G_{K'}}_{G_K} V) = \mathrm{HT}^\sigma(V)$*;*

(2) $\mathrm{HT}^\sigma(V) = \mathrm{HT}^{\tilde{\sigma}}(V \otimes_E L)$*.*

Assume moreover that E *admits an embedding of the Galois closure of* K*. Let* $\tau : K \hookrightarrow E$ *be an embedding. Then*

(3) $\mathrm{HT}_\tau(V) = \mathrm{HT}_\tau(V \otimes_E L)$.

Proof. (1), (3): unravel definitions; (2): $\mathbb{B}_{\mathrm{HT}} \otimes_{L,\tilde{\sigma}} (V \otimes_E L) = (\mathbb{B}_{\mathrm{HT}} \otimes_{L,\tilde{\sigma}} L) \otimes_E V = \mathbb{B}_{\mathrm{HT}} \otimes_{E,\sigma} V$. □

Corollary 4. *Assume E contains the Galois closure of K. Let $\theta \in \mathrm{Aut}(E/\mathbb{Q}_p)$. Let $\tau : K \hookrightarrow E$ be an embedding. Then*
(1) $\mathrm{HT}^\sigma(V \otimes_{E,\theta} E) = \mathrm{HT}^{\sigma\circ\theta}(V)$.
(2) $\mathrm{HT}_\tau(V \otimes_{E,\theta} E) = \mathrm{HT}_{\theta^{-1}\circ\tau}(V)$.

Proof. (1) It is a special case of Lemma 16(2).
(2) By Proposition 3,

$$\begin{aligned}
\mathrm{HT}_\tau(V \otimes_{E,\theta} E) &= \frac{1}{[E:K]} \sum_{\sigma : E \hookrightarrow \mathbb{C}, \sigma|_{\tau K} = \tau^{-1}} \mathrm{HT}^\sigma(V \otimes_{E,\theta} E) \\
&= \frac{1}{[E:K]} \sum_{\sigma : E \hookrightarrow \mathbb{C}, \sigma|_{\tau K} = \tau^{-1}} \mathrm{HT}^{\sigma\circ\theta}(V) \\
&= \frac{1}{[E:K]} \sum_{\sigma : E \hookrightarrow \mathbb{C}, \sigma\circ\theta^{-1}|_{\tau K} = \tau^{-1}} \mathrm{HT}^{\sigma}(V) \\
&= \mathrm{HT}_{\theta^{-1}\circ\tau}(V) \quad \square
\end{aligned}$$

A.3 Lubin-Tate characters

In this subsection, we want to rewrite some results of [17, III.A.1-III.A.5] using the language we just developed.

Remark Proposition B.2 of [4, Appendix B] contains a result more general than this subsection.

A.3.1

Note that *the cyclotomic character has Hodge-Tate weight* -1.

A.3.2 Lubin-Tate characters of Galois extensions of $\mathbb{Q}_p$

We start with the simpliest case. Let $E = K/\mathbb{Q}_p$ be a finite Galois extension. Let π be a uniformizer of K. Let F_π be the Lubin-Tate formal group associated to K and π. Let $\chi_K := \chi_{K,\pi} : G_K \to \mathcal{O}_E^\times$ be the Tate module of F_π, as is the notation of [17]. Then $\chi_K|_{I_K} = r_K^{\otimes -1}$ (see subsection 3). (So r_K is crystalline.)

Lemma Let $\sigma_1 \in \mathrm{Gal}(K/\mathbb{Q}_p)$. Then a σ-co-labeled Hodge-Tate weight of $\sigma_1 \circ \chi_K$ is -1 if $\sigma = \sigma_1^{-1}$, and 0 if otherwise.

Proof. See [17, Thm 2, III.A.5] and [17, Prop III.A.4]. Note that

- Serre's K and E are reversed,
- Galois hypothesis is required by [17, III.A.3(b)],
- Serre's W_σ is our $\mathrm{gr}^0 V_\sigma$.

$\square$

Lemma 17. *Now suppose $E = K/\mathbb{Q}_p$ is not necessarily Galois. A σ-co-labeled Hodge-Tate weight of χ_K is -1 if $\sigma = \mathrm{id}$* [2]*, and 0 if otherwise.*

Proof. Choose a Galois closure L of K over $\mathbb{Q}_p$. Consider

$$\begin{tikzcd} G_L^{\mathrm{ab}} \arrow[r] \arrow[d, "\mathrm{rec}"] & G_K^{\mathrm{ab}} \arrow[d, "\mathrm{rec}"] \\ L^\times \arrow[r, "N_{L/K}"] & K^\times \end{tikzcd}$$

By local class field theory, $\chi_K|_{G_L} = N_{L/K} \circ \chi_L = \prod_{\sigma \in \mathrm{Gal}(L/K)} \sigma \circ \chi_L$. By Lemma A.3.2, for $\tau \in \mathrm{Gal}(L/\mathbb{Q}_p)$,

$$\mathrm{HT}^\tau(\chi_K|_{G_L}) = \begin{cases} -1 & \text{if } \tau \text{ fixes } K \\ 0 & \text{if otherwise} \end{cases}$$

Now apply Lemma 16(1), (2) to conclude. $\square$

Lemma 18. *Let $K/\mathbb{Q}_p$ be a finite extension, and let $E/\mathbb{Q}_p$ be a finite extension admitting $\iota : K \hookrightarrow E$.*
(1) For each $\sigma : E \hookrightarrow \mathbb{C}$, the σ-co-labeled Hodge-Tate weight of $\iota \circ \chi_K$ is -1 if $\sigma \circ \iota = \mathrm{id}_K$, and 0 if otherwise.
(2) Suppose further E admits an embedding of the normal closure of K. Then for each $\sigma : K \to E$, the σ-labeled Hodge-Tate weight of $\iota \circ \chi_K$ is -1 if $\sigma = \iota$, and 0 if otherwise.

Proof. (1) We have

$$\begin{aligned} \mathrm{HT}^\sigma(\iota \circ \chi_K) &= \mathrm{HT}^{\sigma\circ\iota}(\chi_K) && \text{By Lemma 16(2)} \\ &= \begin{cases} -1 & \text{if } \sigma \circ \iota = \mathrm{id}_K \\ 0 & \text{if otherwise} \end{cases} && \text{By Lemma 17} \end{aligned}$$

(2) Follows from Proposition 3 and (1). $\square$

Lemma 19. *Let $K/\mathbb{Q}_p$ be a finite extension. Let L/K be an unramified extension in $\mathbb{C}$. Let L' be the Galois closure of L over $\mathbb{Q}_p$. Let $\iota : K \hookrightarrow L'$ be the tautological embedding. Let $\Phi_K \in G_K$ be a lift of a topological generator of G_K/I_K. Let $d = [L : K]$. Then*

[2] More precisely the tautological embedding of E in $\mathbb{C}$

(1) Let $\sigma : L \hookrightarrow \mathbb{C}$. Then

$$\mathrm{HT}^{\sigma}(\mathrm{Ind}_{G_L}^{G_K}(\chi_L)) = \mathrm{HT}^{\sigma}(\chi_L) \cup \mathrm{HT}^{\sigma}(\Phi_K \circ \chi_L) \cup \cdots \cup \mathrm{HT}^{\sigma}(\Phi_K^{d-1} \circ \chi_L)$$

(2) Let $\tau : K \hookrightarrow \mathbb{C}$. Then

$$\mathrm{HT}_{\tau}(\mathrm{Ind}_{G_L}^{G_K}(\iota \circ \chi_L)) = \begin{cases} \{0, \ldots, 0, -1\} & \text{if } \tau \text{ is the canonical embedding, and} \\ \{0, \ldots, 0, 0\} & \text{if otherwise.} \end{cases}$$

Proof. (1) Follows from Lemma 16 and Corollary 3.
(2) We have

$$\mathrm{HT}_{\tau}(\mathrm{Ind}_{G_L}^{G_K}(\iota \circ \chi_L)) = \frac{1}{[L' : K]} \bigcup_{\tilde{\sigma}: L' \hookrightarrow \mathbb{C}, \tilde{\sigma}|_{\tau K} \circ \tau = id} \mathrm{HT}^{\tilde{\sigma}}(\mathrm{Ind}_{G_L}^{G_K}(\iota \circ \chi_L))$$

$$= \frac{1}{[L' : K]} \bigcup_{\tilde{\sigma}: L' \hookrightarrow \mathbb{C}, \tilde{\sigma}|_{\tau K} \circ \tau = id, \sigma := \tilde{\sigma}|_L} \mathrm{HT}^{\sigma}(\mathrm{Ind}_{G_L}^{G_K}(\chi_L))$$

$$= \frac{1}{[L : K]} \bigcup_{\sigma: L \hookrightarrow \mathbb{C}, \sigma|_{\tau K} \circ \tau = id} \mathrm{HT}^{\sigma}(\mathrm{Ind}_{G_L}^{G_K}(\chi_L))$$

$$= \frac{1}{[L : K]} \bigcup_{\sigma: L \hookrightarrow \mathbb{C}, \sigma|_{\tau K} \circ \tau = id} \mathrm{HT}^{\sigma}(\chi_L) \cup \mathrm{HT}^{\sigma}(\Phi_K \circ \chi_L) \cup \cdots \cup \mathrm{HT}^{\sigma}(\Phi_K^{d-1} \circ \chi_L)$$

$$= \frac{1}{[L : K]} \bigcup_{\sigma: L \hookrightarrow \mathbb{C}, \sigma|_{\tau K} \circ \tau = id} \bigcup_{k=0}^{d-1} \delta_{\sigma, \iota \circ \Phi_K^k}$$

Here $\delta_{X,Y}$ is $\{-1\}$ if $X = Y$ and is $\{0\}$ if otherwise. Since Φ_K^k is the identity when restricted on K, the last line is 0 unless τ is the canonical embedding; in this case, the last line becomes $\frac{1}{[L:K]} \bigcup_{j=0}^{d-1} \bigcup_{k=0}^{d-1} \delta_{\iota \circ \Phi_K^j, \iota \circ \Phi_K^k} = \{0, \ldots, 0, -1\}$. ∎

References

[1] Borel, A. and Tits, J. Groupes réductifs. *Inst. Hautes Études Sci. Publ. Math.* 27, 55–150 (1965), http://www.numdam.org/item?id=PMIHES_1965__27__55_0.

[2] Borel, A. and Tits, J. Éléments unipotent et sous-groupes paraboliques de groupes réductifs. I. *Invent. Math.* 12, 95–104 (1971).

[3] Caruso, X. and Liu, T. Some bounds for ramification of p^n-torsion semi-stable representations. *J. Algebra* 325, 70–96 (2011), https://doi.org/10.1016/j.jalgebra.2010.10.005.

[4] Conrad, B. Lifting global representations with local properties (2011), http://virtualmath1.stanford.edu/~conrad/papers/locchar.pdf.

[5] Conrad, B. Reductive group schemes. *Autour des schémas en groupes. Vol. I.* 42/43, 93–444 (2014).

[6] Emerton, M. and Gee, T. Moduli stacks of étale (Φ, Γ)-modules, and the existence of crystalline lifts. Annals of Mathematics Studies, 215. Princeton University Press, Princeton, NJ (2023).

[7] Fakhruddin, N., Khare, C., and Patrikis, S. Relative deformation theory, relative Selmer groups, and lifting irreducible Galois representations. *Duke Math. J.* 170, 3505–3599 (2021), https://doi.org/10.1215/00127094-2021-0003.

[8] Gee, T., Herzig, F., Liu, T., and Savitt, D. Potentially crystalline lifts of certain prescribed types. *Doc. Math.* 22, 397–422 (2017).

[9] Gee, T. and Kisin, M. The Breuil-Mézard conjecture for potentially Barsotti-Tate representations. *Forum Math. Pi* 2, e1, 56 p. (2014), https://doi.org/10.1017/fmp.2014.1.

[10] Gille, P. The Borel-de Siebenthal's theorem (2010), http://math.univ-lyon1.fr/homes-www/gille/prenotes/bds.pdf.

[11] Iwahori, N. and Matsumoto, H. On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups. *Publications Mathématiques de L'I.H.E.S.* 25, 5–48 (1965).

[12] Iwasawa, K. Local class field theory. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York (1986).

[13] Lin, Z. Lyndon-Demuškin method and crystalline lifts of G_2-valued Galois representations. Preprint (2020), https://arxiv.org/abs/2011.08773.

[14] Milne, J. Reductive groups (2018), https://www.jmilne.org/math/CourseNotes/RG.pdf.

[15] Muller, A. Relèvements cristallins de représentations galoisiennes (2013), https://tel.archives-ouvertes.fr/tel-00873407/document.

[16] Pepin Lehalleur, S. Subgroups of maximal rank of reductive groups. *Autour des schémas en groupes. École d'Été "Schémas en groupes". Volume III*, 147–172, Panor. Synthèses, 47. Soc. Math. France, Paris (2015).

[17] Serre, J. Abelian l-adic representations and elliptic curves. With the collaboration of Willem Kuyk and John Labute. Advanced Book Classics. Addison-Wesley Publishing Company, Redwood City, CA (1989).

[18] Serre, J. 1998 Moursund lectures (1998), https://arxiv.org/abs/math/0305257.

[19] Springer, T. Linear algebraic groups. Birkhäuser, Basel (1998).

[20] Steinberg, R. Endomorphism of linear algebraic groups. *Memoirs of the American Mathematical Society*, 80. American Mathematical Society, Providence, RI (1968).

[21] Tits, J. Normalisateurs de tores. I. Groupes de Coxeter étendus. *J. Algebra* 4, 96–116 (1966), https://doi.org/10.1016/0021-8693(66)90053-6.

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