Algebraic Intermediate Hyperbolicities

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Abstract. We extend Lang's conjectures to the setting of intermediate hyperbolicity and prove two new results motivated by these conjectures. More precisely, we first extend the notion of algebraic hyperbolicity (originally introduced by Demailly) to the setting of intermediate hyperbolicity and show that this property holds if the appropriate exterior power of the cotangent bundle is ample. Then, we prove that this intermediate algebraic hyperbolicity implies the finiteness of the group of birational automorphisms and of the set of surjective maps from a given projective variety. Our work answers the algebraic analogue of a question of Kobayashi on analytic hyperbolicity.

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1 Introduction

This paper is concerned with Lang's conjectures on hyperbolic varieties. Lang's conjectures relate different notions of hyperbolicity for X (see [Lan86] and [Jav20, §12] for a summary of his conjectures) from complex analysis to algebraic geometry and number theory. The aim of this paper is to prove several results motivated by these conjectures, and to extend Lang's conjectures on hyperbolic varieties to the more general setting of "intermediate hyperbolicity" (see Section 3).

In the complex analytic setting, intermediate forms of hyperbolicity were first introduced by Eisenmann [Eis70], and mostly studied in the extremal cases of hyperbolicity and measure-hyperbolicity. In particular, for every integer

$1 \leq p \leq \dim X$, Eisenmann defines what it means for X to be p-analytically hyperbolic.

One of Lang's conjectures predicts that projective varieties of general type should be measure-hyperbolic. This conjecture was proved for surfaces by Green–Griffiths [GG80], but remains widely open in larger dimensions. Nevertheless, Lang's conjecture predicts that measure-hyperbolic projective varieties should enjoy the same properties as varieties of general type.

For example, by the work of Kobayashi–Ochiai (see e.g. [Kob98, §7]), it is well-known that a normal projective variety X of general type satisfies the following two remarkable properties:

Finiteness For every normal projective variety Y, the set of dominant maps from Y to X is finite.

Extension For every normal projective variety Y with $\dim(Y) \geq \dim(X)$ and every proper closed subset $A \subsetneq Y$, every holomorphic map from $Y \setminus A$ to X extends to a meromorphic map from Y to X.

These two properties hold for (analytically) hyperbolic varieties, but they have not been proved yet in the intermediate setting. We note that both properties above are discussed in [Ete] where a finiteness result for automorphisms groups of (pseudo) intermediate hyperbolic manifolds is established.

This paper is mainly concerned with "finiteness properties" for varieties which satisfy some form of intermediate hyperbolicity in the geometric setting. More precisely, in the geometric setting, Demailly introduced an algebraic analogue of the notion of hyperbolicity commonly referred to as *algebraic hyperbolicity* (see [JK20a, Dem97]). In this work, we extend Demailly's notion of algebraic hyperbolicity to the intermediate setting (see Section 2.1), and prove the expected finiteness properties (see Section 4). More precisely, we prove the following:

Theorem 1.1 (Main Theorem I). *Let $p \geq 1$ be an integer and let X be a projective pseudo p-algebraically bounded (resp. pseudo p-algebraically hyperbolic) variety over $\mathbb{C}$. Then the group of birational automorphisms* $\mathrm{Bir}(X)$ *is finite, and for every projective variety Y, the set* $\mathrm{Sur}(Y, X)$ *of surjective morphisms $Y \to X$ is finite.*

Theorem 1.1 provides an "intermediate" version of the finiteness of $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$ proven for a pseudo-bounded projective variety X in [JK20a, JX22]. Also, it generalizes Matsumura's finiteness theorem that a variety of general type has only finitely many automorphisms, and provides algebraic analogues of arithmetic finiteness results proven in [Jav21].

Kobayashi asked about analytic analogues of Matsumura's finiteness theorem for $\mathrm{Aut}(X)$. In fact, in [Kob93] Kobayashi asked whether a $\dim(X)$-analytically hyperbolic projective variety has only finitely many automorphisms. The answer is expected to be positive, as such a variety is expected to be of general type (see [Ete] for recent progress on Kobayashi's question). Our result (Theorem 1.1) can be interpreted as providing a positive answer to the algebraic analogue of Kobayashi's question.

Verifying that a given variety is hyperbolic (in the intermediate setting or classical setting) is often very complicated (in both the geometric and analytic setting). In the analytic setting, there is a natural condition on the positivity of the exterior algebra of the cotangent bundle to ensure intermediate hyperbolicity. (For our definition of ampleness modulo a proper closed subset we refer the reader to Section 5.)

Theorem 1.2 (Kobayashi, Carlson, Noguchi). *Let X be a smooth projective variety over $\mathbb{C}$, and let $\Delta \subset X$ be a proper closed subset. If $\bigwedge^p \Omega^1_X$ is ample modulo Δ, then X is p-analytically hyperbolic modulo Δ.*

This result was proved by Kobayashi when $p = 1$ and $\Delta = \emptyset$ [Kob98, §3]. It is due to Carlson [Car72] when $p \geq 1$ and $\Delta = \emptyset$. The general statement ($p \geq 1$ and Δ possibly non-empty) was proven by Noguchi [Nog77] (see also [Dem97]). In the final section of this paper, we prove that the conclusion of Kobayashi-Carlson-Noguchi's theorem also holds when we replace "analytic hyperbolicity " with "algebraic hyperbolicity", where the intermediate notion of p-algebraic hyperbolicity is defined precisely in Definition 2.2.

Theorem 1.3 (Main Theorem II). *Let X be a smooth projective variety, and let $\Delta \subset X$ be a proper closed subset. If $\bigwedge^p \Omega^1_X$ is ample modulo Δ, then X is p-algebraically hyperbolic modulo Δ.*

We stress that the positivity hypothesis in this statement is much stronger than the conclusion (both in the analytic and algebraic setting). Indeed, by the seminal work of Brotbek [Bro17], a general hypersurface of large enough degree in $\mathbb{P}^n$ is hyperbolic (and thus algebraically hyperbolic by Demailly's theorem [Dem97, Theorem 2.1]). However, the cotangent bundle of such a hypersurface is far from being ample.

Note that if we combine our Theorems 1.1 and 1.3 (in the case $p = \dim(X)$), we recover a slightly weaker form of Kobayashi–Ochiai finiteness Theorem: see Corollary 4.10 for a precise statement.

This paper is only concerned with finiteness properties in the intermediate setting. However, as noted above, it is also natural to study extension properties of holomorphic maps (leading to notions closely related to Kobayashi hyperbolicity [JK20b]). We take the opportunity to discuss implications towards extension properties of the positivity hypotheses in the above statements. To do so, let X be a smooth projective variety such that $\bigwedge^p \Omega_X$ is ample modulo Δ. If $\Delta = \emptyset$,then it was shown in [Car72] (resp. [Kwa83]) that non-degenerate (see Definition 2.1 below) holomorphic (resp. meromorphic) maps $Y \setminus A \to X$, where Y is a complex manifold of dimension p and $A \subsetneq Y$ is a proper closed subset, extend meromorphically. For Δ arbitrary, this extension result (for meromorphic maps) was proved by Noguchi in [Nog77]. It is expected that such extension results hold under the much weaker hypothesis of p-algebraic or p-analytic hyperbolicity. For $p = 1$, it is known only in the analytic case [Kob98, §6]. For $p > 1$, it is unknown in both the analytic and algebraic cases (see [Den20] or [Ete] where the notion of *intermediate Picard hyperbolicity* encapsulates these extension properties).

1.1 Outline of paper

In Section 2, we introduce the relevant definitions of (pseudo) intermediate algebraic hyperbolicity and intermediate algebraic boundedness, and we state and prove basic results concerning these notions. In particular, our definitions and terminologies are (a posteriori) motivated and explained by Proposition 2.9. In Section 3, we state conjectures relating several intermediate notions of hyperbolicity. We take the opportunity to introduce several *new* arithmetic notions of intermediate hyperbolicity which will be the topic of future works. In Section 4, we prove our finiteness results (Theorem 1.1) for pseudo p-algebraically bounded (resp. pseudo p-algebraically hyperbolic) varieties. Finally, in Section 5, we prove our criteria ensuring pseudo p-algebraic hyperbolicity (Theorem 1.3).

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Conventions

Throughout this paper, we will let k denote an algebraically closed field of characteristic zero. A *variety over k* is a finite type separated integral (i.e. irreducible and reduced) scheme over k. Unless stated otherwise explicitly (e.g. if we want to restrict ourselves to complex projective varieties as in Theorem 1.1), all our varieties will be defined over k. (We will explicitly mention the base field in Section 3 as the notions there depend *a priori* on the base field.)

2 Algebraic hyperbolicity and algebraic boundedness

2.1 Definitions

We first introduce a notion of intermediate hyperbolicity inspired by Demailly's notion of algebraic hyperbolicity [Dem97] (see also [Jav21, Jav20, JK20a, JX22, Rou10]). In order to state it concisely, we adopt the following convenient terminology:

Definition 2.1 (Non-degenerate maps). A rational map $f : Y \dashrightarrow X$ between varieties is said to be *non-degenerate* if its image $f(Y)$ is of dimension $\min(\dim(X), \dim(Y))$.

Note that, if $\dim(Y) \leq \dim(X)$, a rational map $Y \dashrightarrow X$ of projective varieties is non-degenerate if and only if it is generically finite onto its image. If $\dim(Y) > \dim(X)$, a rational map $Y \dashrightarrow X$ is non-degenerate if and only if it is dominant.

Definition 2.2 (p-algebraic hyperbolicity modulo Δ). Let X be a projective variety, let Δ be a closed subset of X, and let p be a positive integer. We say that X is *p-algebraically hyperbolic modulo* Δ if, for every ample line bundle L on X, there is a real number $\alpha = \alpha(X, \Delta, L)$ such that, for every smooth projective p-dimensional variety Y with ample canonical bundle ω_Y and every non-degenerate rational map $f : Y \dashrightarrow X$ whose image is not included in Δ, the following inequality is satisfied

$$(f^*L) \cdot (K_Y)^{p-1} \leq \alpha \cdot K_Y^p.$$

Following Lang's terminology, we will say that X is *pseudo-p-algebraically hyperbolic* if there is a proper closed subset $\Delta \subsetneq X$ such that X is p-algebraically hyperbolic modulo Δ. Also, we will say that X is *p-algebraically hyperbolic* if X is p-algebraically hyperbolic modulo the empty subset.

Note that the pull-back by f of the line bundle L is well-defined: since Y is smooth, the rational map f is well-defined outside a closed subset F of codimension at least 2, and one has the equality of the Picard groups $\mathrm{Pic}(Y \setminus F) = \mathrm{Pic}(Y)$. Note also that a projective variety X is 1-algebraically hyperbolic modulo the emptyset if and only if it is algebraically hyperbolic in Demailly's sense [JK20a]. More generally, a projective variety X is pseudo-1-algebraically hyperbolic if and only if it is pseudo-algebraically hyperbolic [Jav20, §9].

Observe that we restrict to varieties Y with ample canonical bundle. This is analogous to the fact that one may test the algebraic hyperbolicity of a projective variety on maps from curves of genus at least two (see the introduction of [JK20a] for a detailed explanation). Let us also emphasize on the fact that the constant α appearing in the definition is *independent* of Y: see Definition 2.3 below where this hypothesis is relaxed.

The algebraic hyperbolicity of a projective variety implies that moduli spaces of maps from varieties are "bounded" (i.e. have only finitely many connected components). We refer to [JK20a] for precise statements. The following definition extends the notion of boundedness introduced in [JK20a] (see also [Jav20, JX22]).

Definition 2.3 (p-algebraically bounded modulo Δ). Let X be a projective variety, let Δ be a closed subset of X, and let p be a positive integer. We say that X is *p-algebraically bounded modulo* Δ if, for every ample line bundle L on X, every smooth projective p-dimensional variety Y with $\dim Y = p$, and every ample line bundle A on Y, there is a real number $C > 0$ such that, for every non-degenerate rational map $f : Y \dashrightarrow X$ whose image is not included in Δ, the following inequality is satisfied

$$f^*L \cdot A^{p-1} \leq C.$$

Following Lang's terminology as before, we will say that X is *pseudo-p-algebraically bounded* if there is a proper closed subset $\Delta \subsetneq X$ such that X is p-algebraically bounded modulo Δ. Also, we say that X is *p-algebraically bounded* if it is p-algebraically bounded modulo the empty subset. With this terminology at hand, a projective variety is 1-algebraically bounded over k if and only if it is bounded in the sense of [Jav20, §10].

Remark 2.4. Voisin introduced an algebraic analogue of (analytic) measure-hyperbolicity; see [Voi03, Definition 2.20]. Indeed, she defines a variety X to be *algebraically measure hyperbolic* if, for every ample line bundle L on X, there exists a constant $A > 0$ such that, for any covering family of curves $\pi : \mathcal{C} \to B$ (with generic fiber C of genus g) and every dominant map $\phi : \mathcal{C} \to X$ non-constant on the fibers, one has $2g - 2 \geq A \cdot \deg \phi_C^* L$. Conjecturally, if X is a smooth projective variety, then X is $\dim X$-algebraically hyperbolic if and only if it is algebraically measure hyperbolic (in Voisin's sense). For example, by [Voi03, Lemma 2.19], a variety of general type is algebraically measure-hyperbolic, and in this paper we prove the similar (a priori different) statement that a variety of general type is $\dim X$-algebraically hyperbolic; see Theorem 1.3 in the case $p = \dim(X)$.

The notion of intermediate boundedness is weaker than the notion of intermediate algebraic hyperbolicity (see Proposition 2.5), but we conjecture that it is equivalent (see Section 3). Both notions allow for strings of implications (see Proposition 2.6 and 2.7 for precise statements). However, we emphasize that for intermediate algebraic boundedness this statement is only proven under the additional assumption that the base field k is *uncountable*.
These two intermediate notions of pseudo-hyperbolicity turn out to be strong enough to force certain finiteness properties, as we show in Section 4.

2.2 Elementary properties

In this section, we prove a few basic facts concerning intermediate algebraic hyperbolicity and algebraic boundedness, justifying in particular our definitions. Let us first record the relationship between intermediate algebraic hyperbolicity and intermediate algebraic boundedness in the following proposition:

Proposition 2.5. *Let $\Delta \subset X$ be a proper closed subset of a projective variety X. If X is p-algebraically hyperbolic modulo Δ, then X is p-algebraically bounded modulo Δ.*

Proof. Let Y be a smooth projective variety of dimension p, and consider a ramified cover $\psi : \tilde{Y} \to Y$ with $\tilde{Y}$ smooth and $K_{\tilde{Y}}$ ample (see e.g. [Laz04a]). Let R be the ramification divisor, so that

$$K_{\tilde{Y}} = \psi^* K_Y + R.$$

Let $f : Y \dashrightarrow X$ be a non-degenerate rational map, whose image is not included in Δ. Fix an ample line bundle L on X and an ample line bundle A on Y. By our hypothesis on X, there exists a constant α independent of f and Y such that the following inequality is satisfied

$$(\psi \circ f)^* L \cdot K_{\tilde{Y}}^{p-1} \leq \alpha K_{\tilde{Y}}^{p}.$$

Let m be an integer such that $mK_{\tilde{Y}} - \psi^* A$ is ample, and note that m depends on Y but not on f. Then one has the following inequality:

$$(\psi \circ f)^* L \cdot (\psi^* A)^{p-1} \leq \alpha m^{p-1} K_{\tilde{Y}}^{p}.$$

Using the projection formula, one deduces that X is indeed p-algebraically bounded, as the bound on the right does not depend on the rational map f. □

We now prove that the notions of intermediate algebraic hyperbolicity and intermediate algebraic boundedness form a string of implications, starting with the case of algebraic boundedness:

Proposition 2.6. *Assume that k is uncountable. Let $\Delta \subset X$ be a closed subset of a projective variety X, and let $p \in \mathbb{N}_{\geq 1}$. If X is p-algebraically bounded modulo Δ, then X is $(p+1)$-algebraically bounded modulo Δ.*

Proof. We argue by contradiction. Thus, let L be an ample line bundle on X, let Y be a smooth projective variety of dimension $p+1$, and let A be a very ample line bundle on Y such that there exists a sequence of non-degenerate rational maps $f_i : Y \dashrightarrow X$ with $f_i(Y) \not\subset \Delta$ and with $f_i^* L \cdot A^p$ tending to infinity as i tends to infinity. As k is uncountable, one can choose a smooth ample divisor $H \subset Y$ in the linear system $|A|$ such that, for any $i \in \mathbb{N}_{\geq 1}$, the rational map $(f_i)_{|H} : H \dashrightarrow X$ is a well-defined non-degenerate map whose image is not included in Δ. Since $f_i^* L \cdot A^p = (f_i)_{|H}^* L \cdot A_{|H}^{p-1}$, this contradicts the p-algebraic boundedness of X. □

Replacing "algebraic boundedness" by "algebraic hyperbolicity", we also have the following (without any uncountability hypothesis on the base field).

Proposition 2.7. *Let $\Delta \subset X$ be a closed subset of a projective variety, and let $p \in \mathbb{N}_{\geq 1}$. If X is p-algebraically hyperbolic modulo Δ, then X is $(p+1)$-algebraically hyperbolic modulo Δ.*

Proof. Let L be an ample line on X, and suppose that X is p-algebraically hyperbolic modulo Δ. To prove the proposition, it suffices to show that there exists a constant α such that, for any smooth projective canonically polarized variety Y of dimension $(p+1)$ and any non-degenerate rational map $f : Y \dashrightarrow X$ whose image is not included in Δ, one has the following inequality

$$f^*L \cdot K_Y^p \leq \alpha K_Y^{p+1}.$$

Let $f : Y \dashrightarrow X$ be a rational map as above. By the work of Demailly and Angehrn-Siu (see e.g. [Laz04b][10.2]), there exists a natural number $m \in \mathbb{N}$ *independent of* Y such that mK_Y is very ample. Let $H \in |mK_Y|$ be a smooth hypersurface such that $f_{|H} : H \dashrightarrow X$ remains a non-degenerate rational map whose image is not included in Δ. By the adjunction formula, one has the following equality of linear equivalence classes of divisors

$$K_H = (m+1)(K_Y)_{|H}.$$

Since X is p-algebraically hyperbolic modulo Δ, there exists a constant β, independent of $f_{|H}$ and H, such that the following inequality

$$f_{|H}^* L \cdot K_H^{p-1} \leq \beta K_H^p$$

holds. This inequality can be rewritten as follows

$$f^*L \cdot K_Y^p \leq \beta(m+1)K_Y^{p+1},$$

where m and β are independent of f and Y. This concludes the proof. □

We end this section with an application of the work of Kollár and Matsusaka [KM83], which justifies our definition and terminology of intermediate algebraic boundedness. Before doing so, let us recall the notion of Hilbert polynomials of rational maps. Let $f : Y \dashrightarrow X$ be a rational map, let A be an ample line bundle on Y, and let L be an ample line bundle on X. Let $\mathrm{Graph}(f)$ be the closure of the graph of f. Then, the *Hilbert polynomial of* f *with respect to* A *and* L is the Hilbert polynomial of the projective variety $\mathrm{Graph}(f)$ computed with respect to the ample line bundle $A \boxtimes L_{|\,\mathrm{Graph}(f)}$. This is the unique polynomial $P_f \in \mathbb{Z}[X]$ such that, for every large enough integer m, the following equality is satisfied

$$\begin{aligned} P_f(m) &= \chi\left(\mathrm{Graph}(f), \left(A \boxtimes L_{|\,\mathrm{Graph}(f)}\right)^m\right) \\ &= \dim \mathrm{H}^0\left(\mathrm{Graph}(f), \left(A \boxtimes L_{|\,\mathrm{Graph}(f)}\right)^m\right), \end{aligned}$$

where the last equality follows from the ampleness of $A \boxtimes L_{|\,\mathrm{Graph}(f)}$. In the case where the map $f : Y \dashrightarrow X$ is actually a morphism, one easily sees that the Hilbert polynomial of f computed with respect to A and L is also the Hilbert polynomial of Y computed with respect to the ample line bundle $A \otimes f^*L$. As we show now, this practical way of interpreting Hilbert polynomials of morphisms can be carried over to rational maps, provided Y is smooth.

Lemma 2.8. *Let $f : Y \dashrightarrow X$ be a rational map of projective varieties. Assume that Y is smooth and let P_f be the Hilbert polynomial of f with respect to A and L. Then, for any $m \gg 1$, one has the following equality:*

$$P_f(m) = \chi(Y, (A \otimes f^*L)^m) = \dim \mathrm{H}^0(Y, (A \otimes f^*L)^m).$$

Proof. Since Y is smooth, the rational map f is well-defined outside a closed subset F of codimension at least 2, so that one has the equality of the Picard groups $\mathrm{Pic}(Y \setminus F) = \mathrm{Pic}(Y)$. Therefore, the pull-back by f of the line bundle L is well-defined.

Let $\pi_1 : \mathrm{Graph}(f) \to Y$ (resp. $\pi_2 : \mathrm{Graph}(f) \to X$) be the first (resp. second) projection, and observe that the pull-back f^*L is nef. Indeed, let $m \in \mathbb{N}_{\geq 1}$ be such that L^m is globally generated, and let $y \in Y$. Pick any $x \in f(y) := \pi_2(\pi_1^{-1}(y))$, and take $s \in \mathrm{H}^0(X, L^m)$ such that $s(x) \neq 0$. Then the section $f^*s \in \mathrm{H}^0(Y, f^*L^m)$ induced by f does not vanish at y. Therefore, the line bundle $A \otimes f^*L$ is ample. To conclude, it is now enough to show the equality

$$\dim \mathrm{H}^0(\mathrm{Graph}(f), (A \boxtimes L)^m_{|\,\mathrm{Graph}(f)}) = \dim \mathrm{H}^0(Y, (A \otimes f^*L)^m)$$

for any $m \in \mathbb{N}$. Observe that one has the following equality of (isomorphism classes of) line bundles on $\mathrm{Graph}(f)$:

$$\pi_1^*(A \otimes f^*L) = (A \boxtimes L)_{|\,\mathrm{Graph}(f)}\,.$$

Therefore, there is a natural injective map of vector spaces induced by π_1

$$\begin{array}{ccc} \mathrm{H}^0(Y, (A \otimes f^*L)^m) & \longrightarrow & \mathrm{H}^0(\mathrm{Graph}(f), (A \boxtimes L)^m_{|\,\mathrm{Graph}(f)}) \\ s & \longmapsto & s \circ \pi_1 \end{array}.$$

In the other direction, any global section $\tilde{s} \in \mathrm{H}^0(\mathrm{Graph}(f), (A \boxtimes L)^m_{|\,\mathrm{Graph}(f)})$ induces a global section $s \in \mathrm{H}^0(Y \setminus F, (A \otimes f^*L)^m)$, and Riemann's extension theorem (or an algebraic variant of it) allows to conclude. ☐

The terminology *algebraic boundedness* now comes from the following classic application of the work of Kollár and Matsusaka [KM83] (see [Laz04b][Thm 6.3.29]):

Proposition 2.9. *Let X be a projective variety and let $\Delta \subset X$ be a proper closed subset of X. Suppose that X is p-algebraically bounded modulo Δ. Then, for any smooth projective variety Y of dimension p, the coefficients of Hilbert polynomial of non-degenerate rational morphisms from Y to X whose image is not included in Δ are uniformly bounded.*

Proof. Let A be an ample line bundle on Y, let L be an ample line bundle on X, and let $f : Y \dashrightarrow X$ be a non-degenerate rational map whose image is not included in Δ. By Lemma 2.8, it suffices to to bound, independently of f, the coefficients of the Hilbert polynomial of Y computed with respect to

$A \otimes f^*L$. By the work of Kollár and Matsusaka [KM83], it suffices to bound the intersection numbers $(A+f^*L)^p$ and $(A+f^*L)^{p-1} \cdot K_Y$ independently of f. Recall that the hypothesis of p-algebraic boundedness gives us a constant C independent of f such that the following inequality

$$f^*L \cdot A^{p-1} \leq C$$

holds. By the numerical criterion of bigness (see e.g. [Laz04a][Thm 2.2.15]), one deduces that for $r > p\frac{C}{A^p}$, the line bundle $rA - f^*L$ is big. In particular, a large enough multiple of this line bundle is effective. One now shows by induction on $i \geq 1$ the existence of a constant C_i independent of f such that the following inequality holds

$$(f^*L)^i A^{p-i} \leq C_i.$$

We know that it is satisfied for $i = 1$. To conclude, we argue by induction and suppose that it is satisfied for $1 \leq i < p$. Since the line bundle f^*L is nef (see e.g. the proof of Lemma 2.8), the line bundle $f^*L + A$ is ample. From the effectivity of a large enough multiple of $rA - f^*L$, one then deduces the following inequality

$$(rA - f^*L) \cdot (f^*L + A)^i \cdot A^{p-i-1} \geq 0.$$

This in turn implies the following inequality

$$(f^*L)^{i+1} \cdot A^{p-i-1} \leq (rA - f^*L) \cdot \left(\sum_{k=0}^{i-1} \binom{k}{i} (f^*L)^k \cdot A^{p-1-k} \right) + rf^*L^i \cdot A^{p-i}.$$

By the induction hypothesis, each term appearing in the (developed) sum on the right can be bounded by a constant independent of f, so that the induction step is complete.
Clearly, the above implies that one can bound $(f^*L + A)^p$ independently of f. As for $(f^*L + A)^{p-1} \cdot K_Y$, one picks m such that $mA - K_Y$ is ample (with m obviously independent of f). Then,

$$(f^*L + A)^{p-1} \cdot K_Y \leq m(f^*L + A)^{p-1} \cdot A,$$

so that the above allows to conclude. ☐

3 Lang's intermediate algebraic and arithmetic conjectures

In this section, we pursue the intermediate p-hyperbolicity analogues of Lang's conjectures in the algebraic and arithmetic setting (thereby leaving out the complex-analytic analogues which are discussed in [Ete]). We will build on Lang's original conjectures [Lan86] and the extensions of his conjectures summarized in [Jav20, §12].

Throughout this section, let k be an algebraically closed field of characteristic zero. Given a proper scheme X over k, we refer to [Jav20, §7] for the definition of pseudo-Mordellicity, to [Jav20, §9] for the definition of pseudo-algebraic hyperbolicity, and to [Jav20, §10] for the definition of pseudo-boundedness. The notions of p-algebraic hyperbolicity and p-algebraic boundedness are defined in the first Section 2.1 (see Definition 2.2 and Definition 2.3). To state the general conjecture for varieties of general type, we will need one additional definition. To state this definition, we refer to [Jav20, §3] for the notion of a model.

DEFINITION 3.1 (p-Mordellicity modulo Δ). Let $p \geq 0$ be an integer, let X be a proper variety over k, and let $\Delta \subset X$ be a closed subset. Then, we say that X is *p-Mordellic modulo Δ over k* if, for every finitely generated subfield $K \subset k$, every model $\mathcal{X}$ for X over K, and every p-dimensional smooth projective geometrically connected variety Y over K, there are only finitely many rational maps $f : Y \dashrightarrow \mathcal{X}$ such that the image of f_k in X is of dimension at least p and not included in Δ is finite.

DEFINITION 3.2. Let $p \geq 0$. A proper variety X over k is *p-Mordellic over k* if X is p-Mordellic modulo the empty subset over k.

DEFINITION 3.3. Let $p \geq 0$. A proper variety X over k is *pseudo-p-Mordellic over k* if there is a proper closed subset $\Delta \subsetneq X$ such that X is p-Mordellic modulo Δ over k.

Note that X is Mordellic over k (as defined in [Jav20, §3]) if and only if X is 0-Mordellic modulo the empty subset over k. Indeed, a 0-dimensional smooth projective geometrically connected variety Y over K is isomorphic to $\operatorname{Spec} K$, so that the set of morphisms $Y \to \mathcal{X}$ equals the set of K-rational points of $\mathcal{X}$. Note also that the fact that we allow for $p = 0$ in the above definition is an artifact of the arithmetic setting; finiteness of "points" is a reasonable property to impose (and study) over finitely generated fields of characteristic zero.
Let X be a projective variety over k, and let $\Delta \subset X$ be a closed subset. Assume that for every algebraically closed field extension L/k of finite transcendence degree, the variety X_L is Mordellic modulo Δ_L over L. Then, for every p, the variety X is p-Mordellic modulo Δ over k. Indeed, the set of rational maps $f : Y \dashrightarrow \mathcal{X}$ with $f(Y) \not\subset \Delta$ equals the set of $K(Y)$-rational points of $\mathcal{X} \setminus \Delta$, and the latter is finite by the Mordellicity assumption on the varieties X_L.
For varieties of general type, we expect all notions (including the intermediate ones) of pseudo-hyperbolicity to coincide. The following conjecture provides a precise statement.

CONJECTURE 3.4 (Lang's intermediate pseudo-conjectures). *Let X be a projective variety over k, and let $p \geq 1$ be an integer. Then the following are equivalent.*

1. *The variety X is of general type.*
2. *There is a proper closed subset $\Delta \subsetneq X$ such that every subvariety $Y \subset X$ of dimension at least p with $Y \not\subset \Delta$ is of general type.*

3. *The variety X is pseudo-Mordellic over k.*

4. *The variety X is pseudo-p-Mordellic over k.*

5. *The variety X is pseudo-algebraically hyperbolic over k.*

6. *The variety X is pseudo-p-algebraically hyperbolic over k.*

7. *The variety X is pseudo-bounded over k.*

8. *The variety X is pseudo-p-algebraically bounded over k.*

9. *The variety X is* $\dim(X)$*-Mordellic over k.*

10. *The variety X is* $\dim(X)$*-algebraically-hyperbolic over k.*

11. *The variety X is* $\dim(X)$*-algebraically-bounded over k.*

Note that (1) is independent of p, so that part of this conjecture is reduntant. For example, (5) is equivalent to (6) with $p = 1$. Nonetheless, we chose to present the conjecture in this way to facilitate discussing known results in the following remark.

Remark 3.5 (What do we know about Conjecture 3.4 ?). The following statements hold.

1. Obviously, (2) $\implies$ (1).

2. If $\dim X = 1$, then Conjecture 3.4 holds by Faltings's proof of Mordell's conjecture and the classical finiteness theorem of De Franchis-Severi for Riemann surfaces. More generally, if X is a closed subvariety of an abelian variety, then conjecture 3.4 holds by Faltings's proof of Mordell-Lang [Fal94] and the work of Ueno, Bloch-Ochiai-Kawamata [Kaw80] and Yamanoi [Yam15] on closed subvarieties of abelian varieties.

3. If $\dim X = 2$, then (3) $\implies$ (2), (5) $\implies$ (2), and (7) $\implies$ (2). This is explained in [Jav20]. Some further evidence is given in [Jav].

4. By [BJK], we have that (5) $\implies$ (7).

5. In this paper we prove (6) $\implies$ (8) (Proposition 2.5), and thus (10) $\implies$ (11). We also show that (1) $\implies$ (10) (and thus (1) $\implies$ (11)). See Theorem 1.3.

6. Assuming k is uncountable, we show that (8) $\implies$ (11); see Proposition 2.6.

7. We show that, if $\Lambda^p\Omega^1_X$ is ample modulo some proper closed subset, then X satisfies (1), (2), (6), (8), (10), and (11) (see Theorem 1.3).

8. We show that, assuming X satisfies (10) or (11), then $\mathrm{Bir}_k(X)$ is finite and $\mathrm{Sur}_k(Y, X)$ is finite for every Y (see Theorem 1.1).

Part of the above conjecture already appears in Lang's original paper. For example, the equivalence of (1), and (3) is stated explicitly in [Lan86]. Moreover, the equivalence of (1), (5), and (7) is implicit in Lang's original conjectures (see [Jav20, §12]). However, the other conjectured equivalences are new.

To conclude this section, we push Lang's conjectures further, and state the general conjecture for "intermediate" exceptional loci.

Conjecture 3.6 (Lang's intermediate conjectures for exceptional loci). *Let X be a projective variety over k, let $\Delta \subset X$ be a closed subset, and let p be a positive integer. Then the following statements are equivalent.*

1. *Every subvariety Y of X of dimension at least p with $Y \not\subset \Delta$ is of general type.*
2. *The variety X is Mordellic modulo Δ over k.*
3. *The variety X is p-Mordellic modulo Δ over k.*
4. *The variety X is p-algebraically hyperbolic modulo Δ over k.*
5. *The variety X is p-algebraically bounded modulo Δ over k.*

Note that Conjecture 3.6 (for all X and p) implies Conjecture 3.4 (for all X and p).

Remark 3.7 (What do we know about this conjecture?). By Proposition 2.5, we have that (4) $\implies$ (5). We also show that, if $\Lambda^p\Omega_X$ is ample modulo Δ, then (1), (3), and (4) hold; see Theorem 1.3. Since (1) is stable under field extensions L/k of algebraically closed fields of characteristic zero [BJK, Theorem 1.3], Conjecture 3.6 predicts the same for (2), (3), (4), and (5).

4 Finiteness results for intermediate pseudo bounded and pseudo hyperbolic varieties

In this section, we investigate the finiteness of the sets of surjective morphisms and birational automorphisms for pseudo p-algebraically bounded varieties and pseudo p-algebraically hyperbolic varieties. In the first two parts, we prove such finiteness results for $\dim(X)$-algebraically bounded varieties. In a third and last part, we state the same finiteness results for p-algebraically bounded and p-algebraically hyperbolic varieties: even though the statements are immediate from Proposition 2.6 and Proposition 2.7, we choose to put them in a separate section to emphasize that the hypothesis "k uncountable" is a priori important in the case of intermediate algebraic boundedness, while it is not for intermediate algebraic hyperbolicity.

4.1 Surjective morphisms to $\dim(X)$-algebraically bounded varieties

Crucial to our proofs below is the (obvious) non-uniruledness of a $\dim(X)$-algebraically bounded variety. We record this observation in the following lemma.

Lemma 4.1. *Let X be a projective variety. If X is $\dim(X)$-algebraically bounded, then X is non-uniruled.* □

If X and Y are projective varieties, we let $\underline{\mathrm{Hom}}(Y,X)$ be the scheme parametrizing morphisms from Y to X. If $\mathrm{Hilb}(Y\times X)$ is the Hilbert scheme of $Y\times X$, then the morphism $\underline{\mathrm{Hom}}(Y,X)\to \mathrm{Hilb}(Y\times X)$ mapping a morphism $f:Y\to X$ to its graph $\mathrm{Graph}(f)\subset Y\times X$ is an open immersion [Nit05, Theorem 6.6]. We also let $\underline{\mathrm{Sur}}(Y,X)$ be the open and closed subscheme of $\underline{\mathrm{Hom}}(Y,X)$ parametrizing surjective morphisms from Y to X.
Before stating and proving our results, recall that the scheme $\underline{\mathrm{Sur}}(Y,X)$ is of finite type if and only if it has finitely many connected components. That is, $\underline{\mathrm{Sur}}(Y,X)$ is of finite type if and only if, for every ample line bundle L on X and every ample line bundle A on Y, there is an integer $n\geq 1$ and polynomials $\Phi_1,\ldots,\Phi_n$ in $\mathbb{Q}[t]$ such that, for every surjective morphism $f:Y\to X$, the Hilbert polynomial of f with respect to the ample line bundle $A\boxtimes L_{|\,\mathrm{Graph}(f)}$ lies in the finite set $\{\Phi_1,\ldots,\Phi_n\}$. Adapting the proofs of Proposition 2.6 and Proposition 2.9, we obtain the following lemma.

Lemma 4.2. *Let X be a projective $\dim X$-algebraically bounded variety. Then for every smooth projective variety Y of dimension $\dim(Y)\geq \dim(X)$, the scheme $\underline{\mathrm{Sur}}(Y,X)$ is of finite type.*

Proof. Let $p=\dim Y$. If k is uncountable, the lemma follows immediately from Proposition 2.6 and Proposition 2.9. If k is countable, one can proceed as in Proposition 2.6 and Proposition 2.9 once the following is noticed. First, if $p=\dim(X)$, the lemma follows immediately from Proposition 2.9. Thus, we may suppose that $p>\dim(X)$. In Proposition 2.6, the uncountability of k was used to find a smooth ample divisor H in the linear system $|A|$ such that for any $i\in\mathbb{N}$, the rational maps $(f_i)_{|H}$ remain non-degenerate. Suppose now that the $f_i:Y\to X$ are instead surjective morphisms. It is then clear that for any smooth ample divisor H, the restriction maps $(f_i)_{|H}:H\to X$ remain surjective, as for any $x\in X$, the fibre $Y_x=f^{-1}(\{x\})$ is positive dimensional, and thus intersects the ample divisor H. This allows to argue by induction on $\dim Y$ as in Proposition 2.6. □

To prove the rigidity of surjective morphisms, we will appeal to a theorem of Hwang-Kebekus-Peternell [HKP06]. Their result relates the infinitesimal deformation space of a surjective morphism $Y\to X$ to the infinitesimal automorphisms of a suitable cover of X. For this reason, we investigate first the discreteness of $\mathrm{Aut}_{X/k}$, where $\mathrm{Aut}_{X/k})$ denotes the locally finite type group scheme of automorphisms of X. Interestingly, to prove the rigidity of automorphisms, we will appeal to the boundedness of $\underline{\mathrm{Sur}}(Y, X)$ (for every smooth projective variety Y) proven above.

Lemma 4.3. *Let X be a projective $\dim(X)$-algebraically bounded variety over k. Then $\mathrm{Aut}_{X/k}$ is zero-dimensional.*

Proof. Since X is non-uniruled (Lemma 4.1), the connected component $A := \mathrm{Aut}^0_{X/k}$ of the identity of $\mathrm{Aut}_{X/k}$ is an abelian variety. For a in A and x in X, let $a \cdot x$ denote the action of A on X. Let $\psi : \tilde{X} \to X$ be a resolution of singularities of X. Consider the sequence of surjective morphisms $f_n : A \times \tilde{X} \to X$ given by

$$f_n(a, \tilde{x}) = (na) \cdot \psi(x).$$

Since the degree of the (finite étale) morphism $[n] : A \times X \to A \times X$ equals $n^{2 \dim A}$ and thus increases with n, one sees that the Hilbert polynomials of the morphisms $f_n : A \times \tilde{X} \to X$ are pairwise distinct. In particular, the scheme $\underline{\mathrm{Sur}}(A \times \tilde{X}, X)$ is not of finite type. Since $A \times \tilde{X}$ is smooth, this contradicts Lemma 4.2. □

Remark 4.4. The intermediate Lang conjectures (Conjecture 3.4) predict that a projective $\dim(X)$-algebraically bounded variety is of general type. (The converse statement that a projective variety of general type is $\dim(X)$-algebraically bounded follows from Theorem 1.3.) In the following proposition, we show that a $\dim(X)$-algebraically bounded variety is not dominated by a family of abelian varieties. This gives more "evidence" for Conjecture 3.4 since Lang has also predicted that a variety not of general type is covered by the images of non-constant rational maps from abelian varieties. As observed by Voisin [Voi03], this implies that a variety which is not of general type is dominated by a family of abelian varieties.

Proposition 4.5. *Let X be a projective variety, let B be a variety and let $\mathcal{A} \to B$ be an abelian scheme. Suppose that there is a dominant rational map $f : \mathcal{A} \dashrightarrow X$ of varieties. Then X is not $\dim(X)$-algebraically bounded.*

Proof. (We adapt the proof of Lemma 4.3.) Let $\overline{\mathcal{A}}$ be a projective compactification of $\mathcal{A}$. Consider the sequence of rational maps $f_n : \mathcal{A} \dashrightarrow X$, where f_n is defined to be the composition of f with the multiplication-by-n map on $\mathcal{A} \to B$. Note that $f_n : \overline{\mathcal{A}} \dashrightarrow X$ is a dominant rational map, and that the Hilbert polynomials of the f_n are pairwise distinct as n runs over all natural numbers. This shows that X is not $\dim(X)$-algebraically bounded, as required. □

We now record the basic fact that a finite surjective cover of a p-algebraically bounded projective variety remains p-algebraically bounded.

Lemma 4.6. *Let $Z \to X$ be a finite surjective morphism of projective varieties. If X is p-algebraically bounded, then Z is p-algebraically bounded.*

Proof. This is a straightforward consequence of the projection formula, and the fact that the pull-back of an ample by a finite morphism remains ample. □

We now prove the desired finiteness of surjective morphisms $Y \to X$, assuming X is $\dim(X)$-algebraically bounded.

Theorem 4.7. *If X is a $\dim(X)$-algebraically bounded projective variety and Y is a projective variety, then $\underline{\mathrm{Sur}}(Y,X)$ is finite.*

Proof. Observe first that one can always suppose that Y is smooth. Indeed, if one takes $\tilde{Y} \to Y$ a resolution of singularities of Y, and if one knows that $\underline{\mathrm{Sur}}(\tilde{Y},X)$ is finite, then one immediately deduces the finiteness of $\mathrm{Sur}(Y,X)$. Furthermore, by Lemma 4.6, one can always suppose that X is normal: indeed, every surjective morphism from Y smooth to X factors uniquely through the normalization $\tilde{X} \to X$, where the normalization map is a surjective and finite morphism.
By Lemma 4.2, the scheme $\underline{\mathrm{Sur}}(Y,X)$ is of finite type. Thus, it suffices to show that $\underline{\mathrm{Sur}}(Y,X)$ is zero-dimensional. To do so, let $f: Y \to X$ be a surjective morphism. As X is non-uniruled (Lemma 4.1) and normal, by the theorem of Hwang-Kebekus-Peternell [HKP06], there is a finite surjective morphism $Z \to X$ and a morphism $Y \to Z$ such that $f: Y \to X$ factors as

$$Y \to Z \to X,$$

with $\mathrm{Aut}^0(Z)$ surjecting onto the connected component of f in $\underline{\mathrm{Sur}}(Y,X)$. Now, as $Z \to X$ is finite surjective (so that $\dim X = \dim Z$) and X is $\dim X$-algebraically bounded, it follows from Lemma 4.6 that Z is $\dim Z$-algebraically bounded. In particular, it follows from Lemma 4.3 that $\mathrm{Aut}^0(Z)$ is trivial. As $\mathrm{Aut}^0(Z)$ surjects onto the connected component of f in $\underline{\mathrm{Sur}}(Y,X)$, it follows that the latter is trivial, which finishes the proof. □

4.2 Birational selfmaps of $\dim(X)$-algebraically bounded varieties

Let X be a projective integral variety over k. We define $\mathrm{Bir}_{X/k}$ to be the subscheme of the Hilbert scheme $\mathrm{Hilb}(X \times X)$ parametrizing, roughly speaking, closed subschemes $Z \subset X \times X$ such that Z is integral and both projections $Z \to X$ are birational (see [Han87, Definition 1.8] for a more precise formulation). Note that the (abstract) group $\mathrm{Bir}(X)$ of birational selfmaps $X \dashrightarrow X$ is in bijection with the set of k-points of the k-scheme $\mathrm{Bir}_{X/k}$.
Hanamura shows that the scheme $\mathrm{Bir}_{X/k}$ can be endowed with the structure of a group scheme structure, assuming that X is a (terminal) minimal model (see [Han87, §3]). We will use a slight extension of his result.
In fact, in [PS14, §4] Prokhorov and Shramov show that a proper non-uniruled integral variety over k has a pseudo-minimal model. Hanamura's main result on the scheme $\mathrm{Bir}_{X/k}$ for minimal models X is easily seen to extend to pseudo-minimal models by following his proof closely. Indeed, Hanamura's proof relies

on the fact that on a minimal model every pseudo-automorphism is an automorphism. This property also holds for pseudo-minimal models by [PS14, Corollary 4.7]. In particular, Hanamura's work gives the following statement (see [Han87, §3]).

Theorem 4.8 (Hanamura). *If X is a pseudo-minimal model over k, then* $\mathrm{Bir}_{X/k}$ *can be endowed with the structure of a group scheme (over k) such that* $\mathrm{Bir}^0_{X/k}$ *is isomorphic to* $\mathrm{Aut}^0_{X/k}$.

We use Hanamura's structure result to prove the following finiteness result.

Proposition 4.9. *Let X be a projective variety. If X is* $\dim(X)$*-algebraically bounded, then* $\mathrm{Bir}(X)$ *is finite.*

Proof. As X is $\dim(X)$-algebraically bounded, the variety X is non-uniruled (Lemma 4.1). Therefore, by the work of Prokhorov-Shramov [PS14, §4], the projective variety X has a pseudo-minimal model. Let Y be a pseudo-minimal model for X. Note that $\mathrm{Bir}(X) = \mathrm{Bir}(Y)$ (since X and Y are birational). Now, by Theorem 4.8, the scheme $\mathrm{Bir}_{Y/k}$ can be endowed with the structure of a group scheme in such a way that $\mathrm{Bir}^0_{Y/k} = \mathrm{Aut}^0_{Y/k}$. Since X is $\dim(X)$-algebraically bounded and Y is birational to X, it follows that Y is also $\dim(X)$-algebraically bounded. In particular, $\mathrm{Aut}_{Y/k}$ is zero-dimensional (Lemma 4.3) and thus Hanamura's group scheme $\mathrm{Bir}_{Y/k}$ is zero-dimensional. Finally, to conclude that $\mathrm{Bir}(X)$ is finite, it suffices to show that $\mathrm{Bir}_{Y/k}$ is of finite type. This boundedness statement is a straightforward consequence of the definition of $\mathrm{Bir}(Y)$ and the fact that Y is $\dim(Y)$-algebraically bounded. □

Note that the special case of Theorem 1.3 with $p = \dim(X)$ combined with Theorem 4.7 and Proposition 4.9 gives an alternative proof of (a slightly weaker form of) Kobayachi-Ochiai finiteness theorem.

Corollary 4.10 (Kobayachi–Ochiai). *Let X be a smooth projective variety of general type. Then for any projective variety Y, the set of surjective morphisms* $\mathrm{Sur}(Y, X)$ *is finite. Furthermore, the set of bimeromorphisms* $\mathrm{Bir}(X)$ *is also finite.*

4.3 Finiteness results for intermediate pseudo-algebraically bounded and pseudo-algebraically hyperbolic varieties

Using Proposition 2.6, Theorem 4.7 and Proposition 4.9, we obtain immediately the following:

Theorem 4.11. *Assume that k is uncountable. Let X be a projective pseudo-p-algebraically bounded variety with $1 \leq p \leq \dim(X)$. Then* $\mathrm{Bir}(X)$ *is finite and, for any projective variety Y, the set of surjective morphisms* $\mathrm{Sur}(Y, X)$ *is finite.* □

In particular, this gives Theorem 1.1 as $\mathbb{C}$ is uncountable. Combining Proposition 2.6, Proposition 2.7, Theorem 4.7 and Proposition 4.9, we obtain the following (in which the uncountability hypothesis has been dropped):

Theorem 4.12. *Let X be a projective pseudo-p-algebraically hyperbolic variety with $1 \leq p \leq \dim(X)$. Then $\mathrm{Bir}(X)$ is finite and, for any projective variety Y, the set $\mathrm{Sur}(Y, X)$ is finite.* □

5 From positivity modulo Δ to intermediate algebraic hyperbolicity

We now prove the criterion for intermediate algebraic hyperbolicity, which is the algebraic analogue of Demailly's Theorem 1.2. In order to state it properly, recall the following definition of ampleness modulo a closed subset (we refer the reader to Lazarsfeld's books for a definition of the augmented base locus [Laz04a, Laz04b]):

Definition 5.1. Let X be a projective variety, let $\Delta \subset X$ be a closed subset, let E be a vector bundle on X, and let $p : \mathbb{P}(E^\vee) \to X$ be the natural projection. Then E is *ample modulo* Δ if the augmented base locus $B^+(\mathcal{O}_{\mathbb{P}(E^\vee)}(1))$ is included in $p^{-1}(\Delta)$.

In other words, E is ample modulo Δ, if and only if for any ample line bundle A on $\mathbb{P}(E^\vee)$, there is an integer $m \geq 1$ such that the base locus of $\mathcal{O}_{\mathbb{P}(E^\vee)}(m) \otimes A^{-1}$ is included in $p^{-1}(\Delta)$.
Our last Theorem now reads as follows:

Theorem 5.2. *Let X be a smooth projective variety, and let $\Delta \subset X$ be a proper closed subset. If $\bigwedge^p \Omega^1_X$ is ample modulo Δ, then X is p-algebraically hyperbolic modulo Δ.*

Proof. Fix an ample line bundle L on X, and let Y be a smooth projective p-dimensional variety with K_Y ample. Let $f : Y \dashrightarrow X$ be a non-degenerate rational map such that $f(Y) \not\subset \Delta$. To prove the theorem, it suffices to show that there exists a constant α independent of f and Y such that

$$f^*L \cdot K_Y^{p-1} \leq \alpha K_Y^p.$$

Let $\pi_X : \mathbb{P}(\bigwedge^p T_X) \to X$ be the natural projection onto X, and let $\mathcal{O}_{\mathbb{P}(\bigwedge^p T_X)}(1)$ be the dual of the tautological line bundle on $\mathbb{P}(\bigwedge^p T_X)$. As the rational map f is non-degenerate, it induces via its differential a rational map

$$\tilde{f} \colon \left(\begin{array}{ccc} Y \simeq \mathbb{P}(\bigwedge^p T_Y) & \dashrightarrow & \mathbb{P}\big(\bigwedge^p T_X\big) \\ (y, [v_1 \wedge \ldots \wedge v_p]) & \mapsto & \big(f(y), [df_y(v_1) \wedge \ldots \wedge df_y(v_p)]\big) \end{array} \right).$$

Observe that $\tilde{f}$ is well-defined outside a closed subset F of codimension at least two. Indeed, writing $\tilde{f}$ in trivializations, one sees that the indeterminacy locus of $\tilde{f}$ comes either from the indeterminacy locus of f, which has codimension at least two as Y is smooth, or from the indeterminacy locus of a rational map from Y into the projective space $\mathbb{P}\left(\bigwedge^p k^{\dim(X)}\right)$, which is also of codimension at least two by smoothness of Y. In particular, as $\mathrm{Pic}(Y) \simeq \mathrm{Pic}(Y \setminus F)$ by smoothness of Y, the pull-back by $\tilde{f}$ of any line bundle on $\mathbb{P}(\bigwedge^p TX)$ is well-defined.

Note that the rational map f also induces the following non-trivial morphism of line bundles on $Y \setminus F$:

$$\overline{f}\colon \left| \begin{array}{rcl} (K_Y^\vee)_{|Y\setminus F} & \longrightarrow & \left(\tilde{f}^*\mathcal{O}_{\mathbb{P}(\bigwedge^p T_X)}(-1)\right)_{|Y\setminus F} \\ \big(y, v_1 \wedge \ldots \wedge v_p\big) & \longmapsto & \big(y, df_y(v_1) \wedge \ldots \wedge df_y(v_p)\big). \end{array} \right. .$$

It extends to a non-trivial morphism of line bundles on Y by Riemann's extension theorem (or an algebraic variant of it). In particular, since

$$\mathrm{Hom}(K_Y^\vee, \tilde{f}^*\mathcal{O}_{\mathbb{P}(\bigwedge^p T_X)}(-1)) \simeq \mathrm{H}^0(Y, K_Y \otimes \tilde{f}^*\mathcal{O}_{\mathbb{P}(\bigwedge^p T_X)}(-1)),$$

one deduces that the divisor $K_Y \otimes \tilde{f}^*\mathcal{O}_{\mathbb{P}(\bigwedge^p T_X)}(-1)$ is effective. On the other hand, as $\bigwedge^p \Omega_X^1$ is ample modulo Δ, there exists an integer $m > 0$ such that

$$\mathrm{Bs}\left(\mathcal{O}_{\mathbb{P}(\bigwedge^p TX)}(m) \otimes \pi_X^* L^{-1}\right) \subset \pi_X^{-1}(\Delta).$$

Note that m is independent of f and Y. Since $f(Y) \not\subset \Delta$ (by assumption), the pull-back

$$\tilde{f}^*\big(\mathcal{O}_{\mathbb{P}(\bigwedge^p TX)}(m) \otimes \pi_X^* L^{-1}\big) = \tilde{f}^*\mathcal{O}_{\mathbb{P}(\bigwedge^p TX)}(m) \otimes f^* L^{-1}$$

remains effective. Define $E := f^*L^{-1} \otimes K_Y^{\otimes m}$, and note that the following equality

$$f^*L + E = mK_Y \tag{5.1}$$

holds. By the above, E is an effective divisor, as it the sum of two effective divisors.

One now concludes the proof as follows. Take $r \in \mathbb{N}$ such that rK_Y is very ample, and let H be a general complete intersection of $p-1$ hypersurfaces in the linear system $|rK_Y|$. Then one has the following equality

$$(f^*L) \cdot K_Y^{p-1} = \frac{1}{r^{p-1}} \deg(f^*L_{|H}).$$

For a general H, the restricted divisor $E_{|H}$ remains effective, so that it follows from (5.1) that

$$(f^*L) \cdot K_Y^{p-1} \leq \frac{1}{r^{p-1}} \deg\big((mK_Y)_{|H}\big) = mK_Y^p.$$

This finishes the proof, as m is independent of f and Y. ☐

References

[BJK] R. van Bommel, A. Javanpeykar, and L. Kamenova. Boundedness in families of projective varieties. Preprint, 2019. https://arxiv.org/abs/1907.11225.

[Bro17] D. Brotbek. On the hyperbolicity of general hypersurfaces. *Publ. Math. Inst. Hautes Études Sci.* 126:1–34, 2017.

[Car72] J. A. Carlson. Some degeneracy theorems for entire functions with values in an algebraic variety. *Trans. Amer. Math. Soc.* 168:273–301, 1972.

[Dem97] J.-P. Demailly. Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials. In *Algebraic geometry, Santa Cruz, CA, 1995*, 285–360. *Proc. Sympos. Pure Math.*, 62. Amer. Math. Soc., Providence, RI, 1997.

[Den20] Y. Deng. Big Picard theorem and algebraic hyperbolicity for varieties admitting a variation of Hodge structures. Preprint, 2020. https://arxiv.org/abs/2001.04426.

[Eis70] D. A. Eisenman. *Intrinsic measures on complex manifolds and holomorphic mappings*. Memoirs of the American Mathematical Society, 96. American Mathematical Soc., Providence, RI, 1970.

[Ete] A. Etesse. Complex-analytic intermediate hyperbolicity, and finiteness properties. Preprint, 2020. *Communications in Contemporary Mathematics*, to appear. https://arxiv.org/abs/2011.12583.

[Fal94] G. Faltings. The general case of S. Lang's conjecture. In *Barsotti symposium in algebraic geometry (Abano Terme, 1991)*, 175–182. *Perspect. Math.*, 15. Academic Press, San Diego, CA, 1994.

[GG80] M. Green and P. Griffiths. Two applications of algebraic geometry to entire holomorphic mappings. In *The Chern symposium 1979 (Proc. Internat. Sympos., Berkeley, CA, 1979)*, 41–74. Springer, New York-Berlin, 1980.

[Han87] M. Hanamura. On the birational automorphism groups of algebraic varieties. *Compositio Math.* 63(1):123–142, 1987.

[HKP06] J.-M. Hwang, S. Kebekus, and T. Peternell. Holomorphic maps onto varieties of non-negative Kodaira dimension. *J. Algebraic Geom.* 15(3):551–561, 2006.

[Jav] A. Javanpeykar. Finiteness of non-constant maps over a number field. Preprint, 2021. *Mathematical Research Letters*, to appear. https://arxiv.org/abs/2112.11408.

[Jav20] A. Javanpeykar. The Lang-Vojta conjectures on projective pseudo-hyperbolic varieties. In *Arithmetic geometry of logarithmic pairs and hyperbolicity of moduli spaces—hyperbolicity in Montréal*. CRM Short Courses, 135–196. Springer, Cham, 2020.

[Jav21] A. Javanpeykar. Arithmetic hyperbolicity: automorphisms and persistence. *Math. Ann.* 381(1-2):439–457, 2021.

[JK20a] A. Javanpeykar and L. Kamenova. Demailly's notion of algebraic hyperbolicity: geometricity, boundedness, moduli of maps. *Math. Z.* 296(3-4):1645–1672, 2020.

[JK20b] A. Javanpeykar and R. Kucharczyk. Algebraicity of analytic maps to a hyperbolic variety. *Math. Nachr.* 293(8):1490–1504, 2020.

[JX22] A. Javanpeykar and J. Xie. Finiteness properties of pseudo-hyperbolic varieties. *Int. Math. Res. Not. IMRN* (3):1601–1643, 2022.

[Kaw80] Y. Kawamata. On Bloch's conjecture. *Invent. Math.* 57(1):97–100, 1980.

[KM83] J. Kollár and T. Matsusaka. Riemann-Roch type inequalities. *Amer. J. Math.* 105(1):229–252, 1983.

[Kob93] S. Kobayashi. Some problems in hyperbolic complex analysis. In *Complex geometry (Osaka, 1990)*, 113–120. *Lecture Notes in Pure and Appl. Math.*, 143. Dekker, New York, 1993.

[Kob98] S. Kobayashi. *Hyperbolic complex spaces. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, 318. Springer, Berlin, 1998.

[Kwa83] Myung H. Kwack. Meromorphic mappings into compact complex manifolds with a Grauert positive bundle of q-forms. *Proc. Amer. Math. Soc.* 87(4):699–703, 1983.

[Lan86] S. Lang. Hyperbolic and Diophantine analysis. *Bull. Amer. Math. Soc. (N.S.)* 14(2):159–205, 1986.

[Laz04a] R. Lazarsfeld. *Positivity in algebraic geometry. I. Classical setting: line bundles and linear series. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*, 48. Springer, Berlin, 2004.

[Laz04b] R. Lazarsfeld. *Positivity in algebraic geometry. II. Positivity for vector bundles, and multiplier ideals. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*, 49. Springer, Berlin, 2004.

[Nit05] N. Nitsure. Construction of Hilbert and Quot schemes. In *Fundamental algebraic geometry*, 105–137. *Math. Surveys Monogr.*, 123. Amer. Math. Soc., Providence, RI, 2005.

[Nog77] J. Noguchi. Meromorphic mappings into a compact complex space. *Hiroshima Math. J.* 7(2):411–425, 1977.

[PS14] Y. Prokhorov and C. Shramov. Jordan property for groups of birational selfmaps. *Compos. Math.* 150(12):2054–2072, 2014.

[Rou10] E. Rousseau. Hyperbolicity of geometric orbifolds. *Trans. Amer. Math. Soc.* 362(7):3799–3826, 2010.

[Voi03] C. Voisin. On some problems of Kobayashi and Lang; algebraic approaches. In *Current developments in mathematics, 2003*, 53–125. Int. Press, Somerville, MA, 2003.

[Yam15] K. Yamanoi. Holomorphic curves in algebraic varieties of maximal Albanese dimension. *Internat. J. Math.* 26(6):1541006, 45 p., 2015.

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